EXERCISE 7.7.

Some special types of standard Results based on technique of Integration by parts:

(i) $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + c$

(ii) $\int \sqrt{x^2+a^2}\,dx = \frac{1}{2}x\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + c$

(iii) $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

(i) $I = \int \sqrt{x^2-a^2}\,dx = \int \sqrt{x^2-a^2} \times 1\,dx$

$$= \sqrt{x^2-a^2}\int 1\cdot dx - \int\left(\frac{d}{dx}\left(\sqrt{x^2-a^2}\right)\int 1\cdot dx\right)dx$$

$$= x\sqrt{x^2-a^2} - \int \frac{1}{2}(x^2-a^2)^{\frac{1}{2}-1}(2x)\cdot x\,dx$$

$$= x\sqrt{x^2-a^2} - \int \frac{x^2}{\sqrt{x^2-a^2}}\,dx.$$

$$= x\sqrt{x^2-a^2} - \int \frac{x^2-a^2+a^2}{\sqrt{x^2-a^2}}\,dx.$$

$$= x\sqrt{x^2-a^2} - \int \sqrt{x^2-a^2} - a^2\int \frac{1}{\sqrt{x^2-a^2}}\,dx$$

$$\therefore I = x\sqrt{x^2-a^2} - I - a^2\log\left|x+\sqrt{x^2-a^2}\right| + c'$$

$$\Rightarrow 2I = x\sqrt{x^2-a^2} - a^2\log\left|x+\sqrt{x^2-a^2}\right| + c'$$

$$\therefore I = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + c$$

where $c = \frac{c'}{2}$.

Similarly we can prove (ii) and (iii)

Q No 1. $\int \sqrt{4-x^2}\, dx$.

Sol. $\int \sqrt{4-x^2}\, dx = \int \sqrt{(2)^2-(x)^2}\, dx$.

$$= \frac{x}{2}\sqrt{2^2-x^2} + \frac{2^2}{2}\sin^{-1}\frac{x}{2} + c \quad (\text{Using (iii)})$$

$$= \frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\frac{x}{2} + C$$

Q No 2 $\int \sqrt{1-4x^2}\, dx$

Sol. $\int \sqrt{1-4x^2}\, dx = \int \sqrt{4\left(\frac{1}{4}-x^2\right)}\, dx = \int 2\sqrt{\left(\frac{1}{2}\right)^2-(x)^2}\, dx$

$$= 2\left[\frac{x}{2}\sqrt{\left(\frac{1}{2}\right)^2-(x)^2} + \frac{\left(\frac{1}{2}\right)^2}{2}\sin^{-1}\frac{x}{\frac{1}{2}}\right] + c \quad (\text{Using iii})$$

$$= \frac{x}{2}\sqrt{1-4x^2} + \frac{1}{4}\sin^{-1}2x + C$$

Q No. 4. $\int \sqrt{x^2+4x+1}\, dx = \int \sqrt{x^2+4x+4-4+1}\, dx$.

$$= \int \sqrt{(x+2)^2-(\sqrt{3})^2}\, dx = \frac{x+2}{2}\sqrt{(x+2)^2-(\sqrt{3})^2} - \frac{(\sqrt{3})^2}{2}\log\left|(x+2)+\sqrt{(x+2)^2-(\sqrt{3})^2}\right| + c$$

$$= \frac{x+2}{2}\sqrt{x^2+4x+1} - \frac{3}{2}\log\left|x+2+\sqrt{x^2+4x+1}\right| + C \quad (\text{Using (i)})$$

Q No. 3. $\int \sqrt{x^2+4x+6}\, dx = \int \sqrt{x^2+4x+4+2}\, dx = \int \sqrt{(x+2)^2+(\sqrt{2})^2}\, dx$

$$= \frac{x+2}{2}\sqrt{(x+2)^2+(\sqrt{2})^2} + \frac{(\sqrt{2})^2}{2}\log\left(x+2+\sqrt{(x+2)^2+(\sqrt{2})^2}\right) + C$$

(Using ii)

Q No. 5 $\int \sqrt{1-4x-x^2}\, dx = \int \sqrt{1-(x^2+4x)}\, dx$

$$= \int \sqrt{1-(x^2+4x+4-4)}\, dx = \int \sqrt{1+4-(x+2)^2}\, dx$$

$$= \int \sqrt{(\sqrt{5})^2-(x+2)^2}\, dx$$

$$= \frac{x+2}{2}\sqrt{(\sqrt{5})^2-(x+2)^2} + \frac{(\sqrt{5})^2}{2}\sin^{-1}\frac{x+2}{\sqrt{5}} + C$$

$$= \frac{x+2}{2}\sqrt{1-4x-x^2} + \frac{5}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right) + C.$$

Q No 6. $\int \sqrt{x^2+4x-5}\,dx = \int \sqrt{x^2+4x+4-4-5}\,dx$

$$= \int \sqrt{(x+2)^2-(3)^2}\,dx = \frac{x+2}{2}\sqrt{(x+2)^2-3^2} - \frac{3^2}{2}\log\left|x+2+\sqrt{(x+2)^2-3^2}\right| + C$$

$$= \frac{x+2}{2}\sqrt{x^2+4x-5} - \frac{9}{2}\log\left|x+2+\sqrt{x^2+4x-5}\right| + C$$

Q No 7. $\int \sqrt{1+3x-x^2}\,dx = \int \sqrt{1-(x^2-3x)}\,dx = \int \sqrt{1-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)}\,dx$

$$= \int \sqrt{1+\frac{9}{4}-\left(x^2-3x+\frac{9}{4}\right)}\,dx = \int \sqrt{\frac{13}{4}-\left(x-\frac{3}{2}\right)^2}\,dx.$$

$$= \frac{x-\frac{3}{2}}{2}\sqrt{\frac{13}{4}-\left(x-\frac{3}{2}\right)^2} + \frac{\frac{13}{4}}{2}\sin^{-1}\left(\frac{x-3/2}{\sqrt{13}/\sqrt{4}}\right) + C$$

$$= \frac{2x-3}{4}\sqrt{1-x^2+3x} + \frac{13}{8}\sin^{-1}\left(\frac{2x-3}{\sqrt{13}}\right) + C$$

Q No. 8 $\int \sqrt{x^2+3x}\,dx. = \int \sqrt{x^2+3x+\frac{9}{4}-\frac{9}{4}}\,dx = \int \sqrt{\left(x^2+\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2}$

$$= \frac{x+\frac{3}{2}}{2}\sqrt{\left(x+\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2} - \frac{\left(\frac{3}{2}\right)^2}{2}\log\left|x+\frac{3}{2}+\sqrt{\left(x+\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2}\right| + C$$

$$= \frac{2x+3}{4}\sqrt{x^2+3x} - \frac{9}{8}\log\left|\frac{2x+3}{2}+\sqrt{x^2+3x}\right| + C$$

Q No 9. $\int \sqrt{1+\frac{x^2}{9}}\,dx = \int \sqrt{\frac{9+x^2}{9}}\,dx = \int \frac{1}{3}\sqrt{(3)^2+(x)^2}\,dx.$

$$= \frac{1}{3}\left[\frac{x}{2}\sqrt{9+x^2} + \frac{9}{2}\log\left|x+\sqrt{9+x^2}\right|\right] + C.$$

$$= \frac{x}{6}\sqrt{9+x^2} + \frac{3}{2}\log\left|x+\sqrt{9+x^2}\right| + C$$

Choose the correct answer in exercises 10 and 11.

Q No 10 $\int \sqrt{1+x^2}\,dx$ equals.

(A) $\frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\log\left|x+\sqrt{1+x^2}\right| + C$ (B) $\frac{2}{3}(1+x^2)^{3/2} + C$

(C) $\frac{2}{3}x(1+x^2)^{3/2} + C$ (D) $\frac{x^2}{2}\sqrt{1+x^2} + \frac{1}{2}x^2\log\left|x+\sqrt{1+x^2}\right| + C$

Sol. $\int \sqrt{1+x^2}\,dx = \int \sqrt{(1)^2+(x)^2}\,dx = \frac{x}{2}\sqrt{1^2+x^2} + \frac{(1)^2}{2}\log\left|x+\sqrt{1^2+x^2}\right| + c$

$= \frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\log\left|x+\sqrt{1+x^2}\right| + c$

$\therefore$ A is the correct option.

QNo 11 $\int \sqrt{x^2-8x+7}\,dx = \int \sqrt{x^2-8x+16-16+7}\,dx$

$= \int \sqrt{(x-4)^2-(3)^2}\,dx$

$= \frac{x-4}{2}\sqrt{(x-4)^2-3} - \frac{3^2}{2}\log\left|x-4+\sqrt{(x-4)^2-3^2}\right| + c$

$= \frac{x-4}{2}\sqrt{x^2-8x+7} - \frac{9}{2}\log\left|x-4+\sqrt{x^2-8x+7}\right| + c$

$\therefore$ (D) is the Right option.

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