EXERCISE 6.4 (APPROXIMATIONS)

Q.No. 1 Using differentials, find the approximate value of each of following up to 3 places of decimal.

(i) $\sqrt{25.3}$

Here Let $y = f(x) = \sqrt{x}$, $x = 25$, $\Delta x = 0.3$

Then $f(x+\Delta x) = \sqrt{x+\Delta x} = \sqrt{25.3}$

So that $\Delta y = f(x+\Delta x) - f(x) = \sqrt{x+\Delta x} - \sqrt{x} = \sqrt{25.3} - 5$

$\Rightarrow \sqrt{25.3} = \Delta y + 5$

Now dy is approximately equal to Δy and is given by

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{2\sqrt{x}} \times (0.3) = \frac{1}{2\times 5} \times 0.3 = 0.03$$

$\therefore \sqrt{25.3} = 0.03 + 5 = 5.03$.

(ii) $\sqrt{49.5}$

Let $y = f(x) = \sqrt{x}$; $x = 49$, $\Delta x = 0.5$

Then $y + \Delta y = f(x+\Delta x) = \sqrt{x+\Delta x} = \sqrt{49.5}$

$\therefore \Delta y = f(x+\Delta x) - f(x) = \sqrt{x+\Delta x} - \sqrt{x} = \sqrt{49.5} - 7$

$\Rightarrow \sqrt{49.5} = \Delta y + 7$.

Now dy is approximately equal to Δy and is given by

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{2\sqrt{x}} \times \Delta x = \frac{1}{2\times 7} \times 0.5 = 0.036$$

$\therefore \sqrt{49.5} = \Delta y + 7 = 0.036 + 7 = 7.036$.

(iii) $\sqrt{0.6}$

Let $y = f(x) = \sqrt{x}$ so that $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

and let $x = 0.64$ and $\Delta x = -0.04$

Then $y+\Delta y = f(x+\Delta x)$

$\Rightarrow \Delta y = f(x+\Delta x) - y = f(x+\Delta x) - f(x) = \sqrt{0.6} - 0.8$

$\therefore \sqrt{0.6} = \Delta y + 0.8$

Now dy is approx. equal to Δy and is given by

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{2\sqrt{x}} \times (\Delta x) = \frac{1}{2\sqrt{0.64}} \times (-0.04)$$

$$= -\frac{0.04}{2\times 0.8} = -0.025$$

$\therefore \sqrt{0.6} = \Delta y + 0.8 = -0.025 + 0.8 = 0.775.$

(iv) $(0.009)^{1/3}$

Let $y = f(x) = (x)^{1/3}$ so that $\frac{dy}{dx} = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$

Let $x = 0.008$ and $\Delta x = 0.001$

Then $\Delta y = f(x+\Delta x) - f(x) = (0.009)^{1/3} - (0.008)^{1/3}$

$\therefore (0.009)^{1/3} = \Delta y + (0.008)^{3} = \Delta y + 0.2$

Now dy is approx equal to Δy and is given by

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \left(\frac{1}{3x^{2/3}}\right)\Delta x = \frac{1}{3(0.008)^{2/3}} \times 0.001$$

$$= \frac{0.001}{3\times\left[(0.008)^{1/3}\right]^2} = \frac{0.001}{3\times(0.2)^2} = \frac{0.001}{3\times 0.04} = 0.008$$

$\therefore (0.009)^{1/3} = \Delta y + 0.2 = 0.2 + 0.008 = 0.208$

(v) $(0.999)^{\frac{1}{10}}$.

Let $y = f(x) = (x)^{\frac{1}{10}}$ so that $\frac{dy}{dx} = \frac{1}{10} \cdot x^{-\frac{9}{10}} = \frac{1}{10x^{9/10}}$

Let $x = 1$ then $f(x) = (1)^{1/10} = 1$

$\Delta x = -0.001.$

Then $\Delta y = f(x+\Delta x) - f(x) = (0.999)^{\frac{1}{10}} - 1$

$\therefore (0.999)^{\frac{1}{10}} = \Delta y + 1$ ---(1)

Now dy is approx. equal to Δy and is given by

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{10\,x^{\frac{9}{10}}}(-0.001) = \frac{1}{10\times 1^{9/10}}\times(-0.001)$$

$$= -0.0001$$

$\therefore$ from (1) $(0.999)^{\frac{1}{10}} = \Delta y + 1 = -0.0001 + 1 = 0.9999$

(vi) $(15)^{\frac{1}{4}}$

Let $y = f(x) = x^{1/4}$ so that $\frac{dy}{dx} = \frac{1}{4}x^{\frac{1}{4}-1} = \frac{1}{4x^{3/4}}$

Take $x = 16$ $\Delta x = -1$

$f(x) = (16)^{\frac{1}{4}} = (2^4)^{1/4} = 2$

$f(x+\Delta x) = (15)^{1/4}$.

$\therefore$ $\Delta y = f(x+\Delta x) - f(x) = (15)^{1/4} - 2$

$\Rightarrow$ $(15)^{\frac{1}{4}} = \Delta y + 2$ ---- (1)

Now dy is approx. equal to Δy and is given by

$$dy = \left(\frac{dy}{dx}\right)\times\Delta x = \frac{1}{4x^{3/4}}(-1) = \frac{-1}{4(16)^{3/4}} = \frac{-1}{32}$$

$$= -0.031.$$

$\therefore$ from (1) $(15)^{\frac{1}{4}} = \Delta y + 2 = -0.031 + 2 = 1.969.$

(vii) $(26)^{1/3}$.

Let $y = f(x) = x^{1/3}$ so that $\frac{dy}{dx} = \frac{1}{3}x^{\frac{1}{3}-1} = \frac{1}{3x^{2/3}}$

Take $x = 27$ and $\Delta x = -1$

Then $f(x) = x^{1/3} \Rightarrow f(27) = (27)^{1/3} = (3^3)^{1/3} = 3$

$f(x+\Delta x) = f(27-1) = f(26) = (26)^{1/3}$.

$\Delta y = f(x+\Delta x) - f(x) = (26)^{1/3} - 3$

$\Rightarrow$ $(26)^{1/3} = \Delta y + 3$ ---- (1)

Now dy is approx. equal to Δy and is given by

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{3x^{2/3}}(-1) = \frac{-1}{3(27)^{2/3}} = \frac{-1}{27} = -0.037$$

$\therefore$ From (1) $(26)^{1/3} = \Delta y + 3 = -0.037 + 3 = 2.963$.

(viii) $(255)^{1/4}$

Let $y = f(x) = x^{1/4}$ so that $\frac{dy}{dx} = \frac{1}{4}x^{\frac{1}{4}-1} = \frac{1}{4x^{3/4}}$.

Take $x = 256$ and $\Delta x = -1$

Then $f(x) = f(256) = (256)^{1/4} = (4^4)^{1/4} = 4$

$f(x+\Delta x) = f(256-1) = f(255) = (255)^{1/4}$.

Now $\Delta y = f(x+\Delta x) - f(x) = (255)^{1/4} - 4$

$\therefore$ $(255)^{1/4} = \Delta y + 4$ ---(i)

Now dy is approx. equal to Δy and is given by.

$$\Delta y = dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{4x^{3/4}}(-1) = \frac{-1}{4(4^4)^{3/4}} = \frac{-1}{256} = -0.004$$

$\therefore$ From (i) $(255)^{1/4} = \Delta y + 4 = -0.004 + 4 = 3.996$

(ix) $(82)^{1/4}$.

Let $y = f(x) = x^{1/4}$ so that $\frac{dy}{dx} = \frac{1}{4}x^{\frac{1}{4}-1} = \frac{1}{4x^{3/4}}$.

Take $x = 81$ and $\Delta x = 1$

Then $f(x) = f(81) = (81)^{1/4} = (3^4)^{1/4} = 3$.

$f(x+\Delta x) = f(81+1) = f(82) = (82)^{1/4}$.

Now $\Delta y = f(x+\Delta x) - f(x) = (82)^{1/4} - 3$.

$\Rightarrow$ $(82)^{1/4} = \Delta y + 3$ ---(1)

Now as dy is very small and approx. equal to Δy and is given by

$$\Delta y = dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{4x^{3/4}}(1) = \frac{1}{4(81)^{3/4}} = \frac{1}{108} = 0.009$$

$\therefore$ From (1) $(82)^{1/4} = \Delta y + 3 = 0.009 + 3 = 3.009$

(X) $(401)^{1/2}$

Let $y = f(x) = (x)^{1/2}$ so that $\frac{dy}{dx} = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2x^{1/2}}$

Take $x = 400$ and $\Delta x = 1$

Then $f(x) = f(400) = (400)^{1/2} = 20$

$f(x+\Delta x) = f(400+1) = f(401) = (401)^{1/2}$

Now. $\Delta y = f(x+\Delta x) - f(x) = (401)^{1/2} - 20$

$\Rightarrow (401)^{1/2} = \Delta y + 20$ ---(1)

Now as dy is very small and approx. equal to Δy and is given by

$$\Delta y = dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{2x^{1/2}}(1) = \frac{1}{2(400)^{1/2}} = \frac{1}{40} = 0.025$$

$\therefore$ From (1) $(401)^{\frac{1}{2}} = \Delta y + 20 = 0.025 + 20 = 20.025$.

(XI) $(0.0037)^{\frac{1}{2}}$

Let $y = f(x) = x^{1/2}$ so that $\frac{dy}{dx} = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2x^{1/2}}$

Take $x = 0.0036$ and $\Delta x = 0.0001$

Then $f(x) = f(0.0036) = (0.0036)^{1/2} = 0.06$.

$f(x+\Delta x) = f(0.0036+0.0001) = f(0.0037) = (0.0037)^{1/2}$

Now $\Delta y = f(x+\Delta x) - f(x) = (0.0037)^{1/2} - 0.06$

$\therefore (0.0037)^{1/2} = \Delta y + 0.06$ ---(1)

Now dy is very small and approx. equal to Δy and is given by.

$$\Delta y = dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{2x^{1/2}}(0.0001) = \frac{0.0001}{2(0.0036)^{1/2}} = \frac{0.0001}{0.12} = 0.0008$$

$\therefore$ From (1) $(0.0037)^{1/2} = \Delta y + 0.06 = 0.0008 + 0.06 = 0.0608$

(xii) $(26.57)^{\frac{1}{3}}$

Let $y = f(x) = x^{1/3}$ so that $\frac{dy}{dx} = \frac{1}{3}x^{1/3-1} = \frac{1}{3x^{2/3}}$

Take $x = 27$ and $\Delta x = -0.43$

Then $f(x) = f(27) = (27)^{1/3} = (3^3)^{1/3} = 3$

$f(x+\Delta x) = f(27-0.43) = f(26.57) = (26.57)^{1/3}$

$\therefore \Delta y = f(x+\Delta x) - f(x) = (26.57)^{1/3} - 3$

$\Rightarrow (26.57)^{1/3} = \Delta y + 3$ ....(1)

Now dy is approx. equal to Δy and is given by.

$\Delta y = dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{3x^{2/3}}(-0.43) = \frac{-0.43}{3(3)^2} = \frac{-0.43}{27} = -0.016$

$\therefore$ From (1) $(26.57)^{\frac{1}{3}} = \Delta y + 3 = -0.016 + 3 = 2.984$.

(xiii) $(81.5)^{1/4}$

Let $y = f(x) = x^{1/4}$ so that $\frac{dy}{dx} = \frac{1}{4}x^{\frac{1}{4}-1} = \frac{1}{4x^{3/4}}$

Take $x = 81$ and $\Delta x = 0.5$

Then $f(x) = f(81) = (81)^{1/4} = (3^4)^{1/4} = 3$

$f(x+\Delta x) = f(81+0.5) = f(81.5) = (81.5)^{1/4}$

$\therefore \Delta y = f(x+\Delta x) - f(x) = (81.5)^{1/4} - 3$

$\Rightarrow (81.5)^{\frac{1}{4}} = \Delta y + 3$ ......(1)

Now dy is approx. equal to Δy and is given by.

$\Delta y = dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{4(x)^{3/4}}(0.5) = \frac{0.5}{4(3)^3} = 0.0046$

$\therefore$ From (1) $(81.5)^{\frac{1}{4}} = \Delta y + 3 = 0.0046 + 3 = 3.0046$

(xiv) $(3.968)^{3/2}$

Let $y = f(x) = (x)^{3/2}$ so that $\frac{dy}{dx} = \frac{3}{2}\cdot(x)^{3/2-1} = \frac{3}{2}x^{1/2}$

Take $x = 4$ and $\Delta x = -0.032$

Then $f(x) = f(4) = (4)^{3/2} = (2^2)^{3/2} = 2^3 = 8$

$f(x+\Delta x) = f(4-0.032) = f(3.968) = (3.968)^{\frac{3}{2}}$

$\Delta y = f(x+\Delta x) - f(x) = (3.968)^{3/2} - 8$

$\therefore (3.968)^{3/2} = \Delta y + 8$ ----(1)

Now dy is approx. equal to Δy and is given by.

$\Delta y = dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{3}{2}x^{1/2} \times (-0.032) = \frac{3}{2}(4)^{1/2}(-0.032) = 0.096$

$\therefore$ From (1) $(3.968)^{3/2} = \Delta y + 8 = (-0.096) + 8 = 7.904$.

(XV) $(32.15)^{\frac{1}{5}}$

Let $y = f(x) = x^{\frac{1}{5}}$ so that $\frac{dy}{dx} = \frac{1}{5}x^{\frac{1}{5}-1} = \frac{1}{5x^{4/5}}$

Take $x = 32$ and $\Delta x = 0.15$

Then $f(x) = f(32) = (32)^{1/5} = (2^5)^{\frac{1}{5}} = 2^{5\times\frac{1}{5}} = 2$

$f(x+\Delta x) = f(32+0.15) = f(32.15) = (32.15)^{1/5}$

$\therefore \Delta y = f(x+\Delta x) - f(x) = (32.15)^{\frac{1}{5}} - 2$

$\therefore (32.15)^{\frac{1}{5}} = \Delta y + 2$ ---(1)

Now dy is approx. equal to Δy and is given by.

$\Delta y = dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{5x^{4/5}}(0.15) = \frac{0.15}{5(32)^{4/5}} = \frac{0.15}{5(2^4)} = 0.0019$

$\therefore$ from (1) $(32.15)^{\frac{1}{5}} = \Delta y + 2 = 0.0019 + 2 = 2.0019$

QNo 2: Find the approximate value of $f(2.01)$ where $f(x) = 4x^2+5x+2$.

Sol

Given $f(x) = 4x^2+5x+2$

$\Rightarrow f'(x) = 8x+5$

Also $f(x+\Delta x) \simeq f(x) + \Delta x\, f'(x)$

Taking $x = 2$ and $\Delta x = 0.01$, we get

$f(2.01) = f(2) + (0.01) f'(2)$

$= (4(2)^2 + 5(2) + 2) + \frac{1}{100}(8(2)+5)$

$= 28.21$

$\Rightarrow f(2.01) \simeq 28.21$

Q No 3. Find the approximate value $f(5.001)$ where $f(x) = x^3 - 7x^2 + 15$

Sol.

Given $f(x) = x^3 - 7x^2 + 15$

$\Rightarrow f'(x) = 3x^2 - 14x$

Also $f(x+\Delta x) \simeq f(x) + \Delta x\, f'(x)$

$\therefore f(x+\Delta x) \simeq (x^3 - 7x^2 + 15) + \Delta x\,(3x^2 - 14x)$

Taking $x = 5$ and $\Delta x = 0.001$

$f(5.001) \simeq (5^3 - 7(5)^2 + 15) + (0.001)(3\times 5^2 - 14\times 5)$

$= (125 - 175 + 15) + \frac{1}{1000}(5) = (-35) + (0.005) = -34.995$

Q No 4: Find the approximate change in the volume V of a cube of side x metres caused by increasing the side by 1%

Sol.

Volume of cube $V = x^3$

$\Rightarrow \frac{dV}{dx} = 3x^2$

Hence $\Delta V \simeq 3x^2 \Delta x$

$= (3x^2)\left(\frac{x}{100}\right) \quad \left[\because \Delta x = 1\% \text{ of } x = \frac{x}{100}\right]$

$= \frac{3x^3}{100}$

Change in volume $= \frac{3}{100}x^3$

$\Rightarrow$ Percentage change in volume $= \frac{\Delta V}{V} = \frac{\frac{3x^3}{100}}{x^3} = \frac{3}{100} = 3\%$

Q No 5. Find the approximate change in the surface area of cube of side x metres caused by decreasing the side by 1%

Sol.

Surface area S of a cube is given by $S = 6x^2$

$\Rightarrow \frac{dS}{dx} = 12x.$

$\therefore \Delta S \simeq 12x\,\Delta x = 12x\left(-\frac{x}{100}\right) \quad [\because \Delta x = -1\% \text{ of } x]$

$= -\frac{12x^2}{100}$

$\therefore$ Change in surface area $\simeq -\frac{12x^2}{100}$

$\Rightarrow$ Percentage change in surface area

$= \frac{\Delta S}{S} = -\frac{12x^2}{100\times(6x^2)} = -\frac{2}{100} = -2\%$

Q.No 6 If the radius of a sphere is measured as 7 m with an error of 0.02 m, then find the approximate error in calculating its volume.

Sol:

Volume V of a sphere of radius r is given by.

$$V = \frac{4}{3}\pi r^3$$

$$\Rightarrow \frac{dV}{dr} = \left(\frac{4}{3}\pi\right)(3r^2) = 4\pi r^2$$

Hence $\Delta V = (4\pi r^2)\Delta r$

$$= 4\pi(7)^2(\pm 0.02) = \pm 3.92\pi \ m^3.$$

$\therefore$ Error in Calculating volume $= \pm 3.92\pi \ m^3$.

Q.No 7 If the radius of a sphere is measured as 9 m with an error of 0.03 m. then find the approximate error in calculating surface area

Sol..

Surface Area S of sphere of radius r is given by.

$$S = 4\pi r^2$$

$$\Rightarrow \frac{dS}{dr} = 4\pi(2r) = 8\pi r$$

Hence $\Delta S = (8\pi r)\Delta r = 8\pi(9)(\pm 0.03) = \pm 2.16 m^2$.

$\therefore$ Error in Calculating the Surface Area

$$= \pm 2.16 m^2.$$

Q.No 8! If $f(x) = 3x^2+15x+5$, then the approximate value of $f(3.02)$ is

Sol: (A) 47.66 (B) 57.66 (C) 67.66 (D) 77.66

Sol

Given $f(x) = 3x^2+15x+5$

$$\Rightarrow f'(x) = 6x+15$$

As $f(x+\Delta x) \simeq f(x) + \Delta x f'(x)$ [$\because$ From def.]

$$\Rightarrow f(x+\Delta x) \simeq (3x^2+15x+5) + \Delta x(6x+15)$$

Taking $x = 3$ and $\Delta x = 0.02$ we get

$$f(x+\Delta x) = f(3.02) = [3(3)^2+15(3)+5] + (0.02)(6\times 3+5) = 77+0.66$$

$\therefore$ Correct option is D.

Q.No 9: The approximate change is the volume of a cube of x metres caused by increasing the side by 3% is

(A) $0.06x^3 m^3$ (B) $0.6x^3 m^3$ (C) $0.09x^3 m^3$ (D) $0.9x^3 m^3$.

Sol.

Volume V of cube is given by

$$V = x^3$$

$$\Rightarrow \frac{dV}{dx} = 3x^2$$

Hence $\Delta V \simeq (3x^2)\Delta x$

$$= 3x^2\left(\frac{3x}{100}\right) \quad \left[\because \Delta x = 3\% \text{ of } x = \frac{3x}{100}\right]$$

$$= \frac{9x^3}{100} = 0.09x^3 m^3$$

$\therefore$ Approximate change in Volume $= 0.09x^3 m^3$.

$\therefore$ Correct option is (C).

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