CHAPTER

01

Polynomials

Introduction

In the expressions $3x^2+7x-2$, $x^2-\frac{1}{2}x+3$ and $y^3-\sqrt{2}y^2+3y-7$ the exponents of each of the variables (expressed in letters) are whole numbers. These type of expressions are known as polynomials. You have learnt addition, subtraction and multiplication of polynomials in Class-IX. Let us consider the operations of addition, subtraction and multiplication once again.

1. Add $x+3$ and $x+4$

Solution: Addition of $(x+3)$ and $(x+4)$ i.e. $(x+3)+(x+4)$

$$= x+3+x+4$$
$$= (x+x)+(3+4)$$
$$= 2x+7$$

2. Subtract x^2+x-2 from the polynomial $2x^2+3x+5$

Solution: Subtract x^2+x-2 from $2x^2+3x+5$ i.e. $\left(2x^2+3x+5\right)-\left(x^2+x-2\right)$

$$= 2x^2+3x+5-x^2-x+2$$
$$= \left(2x^2-x^2\right)+\left(3x-x\right)+\left(5+2\right)$$
$$= x^2+2x+7$$

3. Multiply $(x+5)$ to $(x-7)$

Solution : Multiply $(x+5)$ to $(x-7)$ i.e. $(x+5)(x-7)$

$$= x(x-7)+5(x-7)$$
$$= x^2-2x-35$$

Try These

1. Add the polynomials $2x - 7$ and $5x + 9$.
2. Subtract $x^2 + 3x - 4$ from the polynomial $3x^2 + 2x - 3$.
3. Multiply the polynomials $x^2 + 2x - 3$ and $x^2 + x - 2$.

Can We Divide a Polynomial

Notice that the terms with the same exponent are put together while adding and subtracting. Exponents are added in case of multiplication. We have carried out addition, subtraction and multiplication of polynomials and know how to do these operations.

Can we do division of polynomials in the same way as we do addition, subtraction and multiplication of polynomials? How do we manage terms and exponents during division? Before finding the answer to these questions let us see why we need to divide polynomials.

Look at the situation given below:

1. A car travels a distance of x *km* in 4 *hours*. Find the speed of the car.

Solution: Distance travel led by the car = x *km*

and time taken to travel this distance = 4 *hours*

$$\because \quad \text{Speed} = \frac{\text{Distance}}{\text{Time}}$$

$$\therefore \quad \text{Speed} = \frac{x}{4} \ km/hour$$

This division is easy because it involves division of a polynomial having one term by a polynomial having one constant.

2. If the area of a rectangle is $40x^2$ m^2 and length of one of its side is $10x$ meter then what is breadth of the rectangle?

Solution: Area of rectangle $= 40x^2$ *Sq.mt.*

Length of rectangle $= 10x$ meter

$\because$ Area of rectangle = length × breadth

$$40x^2 = 10x \times \text{breadth}$$

$$\therefore \quad \text{breadth} = \frac{40x^2}{10x}$$

$$= \frac{4 \times 10 \times x \times x}{10x}$$

$$= 4x \text{ meter}$$

Here, quotient and divided both are monomials and remainder is also a monomial. Now, we divide a binomial polynomial by a monomial.

3. Divide the polynomial $18x^2 + 9x$ by $3x$.

Solution: To divide $18x^2 + 9x$ by $3x$ we can write in the following way:

$$\frac{18x^2}{3x} + \frac{9x}{3x}$$

$$= 6x + 3$$

Try These

1. Divide $2x^3 + 12x + 6$ by $2x$.
2. A bus travels a distance of y *km* in 5 hours. Find the speed of the bus.
3. Area of a rectangular garden is $65x^2$ square meter and the breadth of that garden is $5x$ meter. Then, find the length of the garden.
4. The length of the base a right angle triangle is $2x$ units and its area is $4x^2 + 4$. Find the length of the perpendicular of the triangle.

The process of division followed in the above examples can also be used to solve practical problems. Let us see some examples.

Example-1. We have a line segment AB whose length is $8x$ units and we have to divide it into two eqaul parts. How will you tell the length of each part?

Solution: Suppose C is any point on line segment AB which divides AB into two equal parts.

We can write this in the following way:-

$AB = AC + BC$

Now, since C divides line segment AB into two equal parts

So, $AC = BC$

$\therefore AB = AC + AC$

$$8x = 2AC$$

or $$AC = \frac{8x}{2}$$

$$AC = \frac{2 \times 4x}{2}$$

$$AC = 4x$$

Hence, the lengths of both equal parts of the line segment are $4x$ units each.

Division by Polynomial with Multiple Terms

While dividing a multiple term polynomial by a one term polynomial, we first write each term separately. Let us see how.

Factorise polynomial $18x^2 + 9x$ and divide it by $3x$.

To divide $18x^2 + 9x$ by $3x$, we can write in the following way:-

$$\frac{18x^2 + 9x}{3x}$$

$$= \frac{9 \times 2 \times x \times x + 9 \times x}{3x}$$

$$= \frac{9x(2x+1)}{3x}$$

$$= 3(2x+1)$$

$$= 6x + 3$$

See one more example

Factorise the polynomial $4x^4 + 12x^3 + 8x^2$ and divide it by $4x^2$

To divide $4x^4 + 12x^3 + 8x^2$ by $4x^2$ we can write in the following way:

$$\frac{4x^4 + 12x^3 + 8x^2}{4x^2}$$

$$= \frac{4x^2 \times x^2 + 3x \times 4x^2 + 2 \times 4x^2}{4x^2}$$

$$= \frac{4x^2(x^2 + 3x + 2)}{4x^2}$$

$$= x^2 + 3x + 2$$

Division of a Polynomial by Factorisation

Now we will learn division of a polynomial by factorisation method.

If we have to divide polynomial $2x^2 + 5x - 3$ by $(x - 2)$, then can we apply the above method of division?

To divide $2x^2 + 5x - 3$ by $(x - 2)$ we can write in the following way:-

$$\frac{2x^2 + 5x - 3}{x - 2}$$

But here we cannot find any common factor in numerator and denominator and we cannot determine the quotient. In this situation, we can use long division method of division.

In algebra you know that the meaning of division of 25 by 4 is

$\frac{25}{4}$ i.e.

$$\begin{array}{r|l|l} & \text{Dividend} & \\ \text{Divisor } 4 & \quad 25 & 6 \text{ Quotient} \\ & \underline{-24} & \\ & \quad 1 & \\ & \text{Remainder} & \end{array}$$

Here $25 = 4 \times 6 + 1$

This is the same as

Dividend = Divisor × Quotient + Remainder

Similarly, by dividing dividend by divisor we will get quotient and remainder. The remainder will be zero in the case of complete division.

Example-2. Divide the polynomial $2x^2 + 5x - 3$ by the polynomial $x - 2$.

Solution: Here, polynomial $2x^2 + 5x - 3$ is dividend and $(x - 2)$ is divisor.

$$\begin{array}{c|r|c} & \text{Dividend} & \\ \text{Divisor} & 2x^2 + 5x - 3 & \text{Quotient} \\ (x-2) & -(2x^2 - 4x) & (2x+9) \\ \hline & 9x - 3 & \\ & -(9x - 18) & \\ \hline & +15 & \\ & \text{Remainder} & \end{array}$$

Here, we found that the quotient is $2x + 9$ and remainder is 15.

So, the process of divison is completed using the following steps:-

Step-1. Write the dividend and divisor in the descending order of their degrees.

Step-2. Divide the first term of dividend by the first term of the divisor

Here $\frac{2x^2}{x} = 2x$

This, is the first term of the quotient

Step-3. We will multiply the divisor with this quotient and will subtract the product from dividend

$$(x-2)2x = 2x^2 - 4x$$

$$\begin{array}{r} 2x^2 + 5x - 3 \\ -2x^2 + 4x \\ \hline 9x - 3 \end{array}$$

Step-4. Divide the the first term of result of subtraction by first term of divisor.

i.e. $\frac{9x}{x} = 9$ This is the second term of quotient.

Step-5. We will again multiply this quotient with the divisor.

i.e. $9 \times (x-2) = 9x - 18$

Now we will subtract $9x - 18$ from $9x - 3$.

$$\begin{array}{r} 9x - 3 \\ -(9x - 18) \end{array} \quad \text{Or} \quad \begin{array}{r} 9x - 3 \\ -9x + 18 \\ \hline +15 \end{array}$$

We will repeat this proces till the remainder becomes zero or the degree of the remainder becomes less than the power of the variables of the divisor. The remainder is 15 in this example and its power is less than the power of the variable in $(2x + 9)$.

Brief representation of this division is:-

$$(2x^2 + 5x - 3) = (x-2)(2x+9) + 15$$

i.e. Dividend = Divisor × Quotient + Reminder

Example-3. Divide the polynomial $5x - 11 - 12x^2 + 2x^3$ by the polynomial $x - 5$.

Solution: Here, dividend is $5x - 11 - 12x^2 + 2x^3$ and divisor is $x - 5$.

The power of x in divisor is in descending order and we will also have to write the power of x of dividend in descending order.

When we write the power in descending order, the dividend will be $2x^3 - 12x^2 + 5x - 11$.

Now,

$$
\begin{array}{r|r|l}
(x-5) & 2x^3 - 12x^2 + 5x - 11 & 2x^2 - 2x - 5 \\
 & -(2x^3 - 10x^2) & \\
\cline{2-2}
 & -2x^2 + 5x - 11 & \\
 & -(-2x^2 + 10x) & \\
\cline{2-2}
 & -5x - 11 & \\
 & -(-5x + 25) & \\
\cline{2-2}
 & -36 &
\end{array}
$$

For $2x^3$ we will take quotient $2x^2$

Now we will take $-2x$ for $-2x^2$

Now we will take -5 for $-5x$.

Now we can't divide any further.

This is the remainder.

Here, quotient = $2x^2 - 2x - 5$

Remainder = -36

Example-4. Divide the polynomial $2x^3 - 3x^2 - x + 3$ by the polynomial $2x^2 = 4x + 3$.

Solution: Here, dividend is $2x^3 - 3x^2 - x + 3$ and divisor is $2x^3 - 4x + 3$.

Now,

$$
\begin{array}{r|r|l}
2x^2 - 4x + 3 & 2x^3 - 3x^2 - x + 3 & x + \frac{1}{2} \\
 & -(2x^3 - 4x^2 + 3x) & \\
\cline{2-2}
 & x^2 - 4x + 3 & \\
 & -\left(x^2 - 2x + \frac{3}{2}\right) & \\
\cline{2-2}
 & -2x + \left(3 - \frac{3}{2}\right) &
\end{array}
$$

Power of remainder is less than the power of dividend and divisor

or $-2x + \frac{3}{2}$. So Remainder $= -2x + \frac{3}{2}$ and quotient $= x + \frac{1}{2}$

Example-5. Divide the polynomial $2x^3 + 4x - 3$ by the polynomial $x - 2$.

Solution: Here, dividend is $2x^3 + 4x - 3$ which we can write as $2x^3 + 0.x^2 + 4x - 3$ and divisor is $x - 2$.

Now,

$$
\begin{array}{r|r|l}
(x-2) & 2x^3+0.x^2+4x-3 & 2x^2+4x+12 \\
 & 2x^3-4x^2 & \\
 & (-)\quad(+) & \\ \cline{2-2}
 & 4x^2+4x-3 & \\
 & 4x^2-8x & \\
 & (-)\quad(+) & \\ \cline{2-2}
 & 12x-3 & \\
 & 12x-24 & \\
 & (-)\quad(+) & \\ \cline{2-2}
 & 21 &
\end{array}
$$

Quotient and remainder are also polynomials.

Quotient $= 2x^2+4x+12$

Remainder $= 21$

Example-6. If divisor $=3x+1$, quotient $=2x-1$, remainder is 4 then find the dividend.

Solution: $\because$ Dividend $=$ Divisor $\times$ Quotient $+$ Remainder

$$=(3x+1)\times(2x-1)+4$$

$$=3x(2x-1)+1(2x-1)+4$$

$$=6x^2-3x+2x-1+4$$

Dividend $=6x^2-x+3$

Example-7. Prove that on dividing $(2x^3+x^2-5x+2)$ by $(x+2)$ the remainder is zero.

Solution:

$$
\begin{array}{r|r|l}
(x+2) & 2x^3+x^2-5x+2 & 2x^2-3x+1 \\
 & 2x^3+4x^2 & \\
 & (-)\quad(-) & \\ \cline{2-2}
 & -3x^2-5x+2 & \\
 & -3x^2-6x & \\
 & (+)\quad(+) & \\ \cline{2-2}
 & x+2 & \\
 & x+2 & \\
 & (-)\quad(-) & \\ \cline{2-2}
 & 0 &
\end{array}
$$

Clearly, remainder is zero.

Try These

Write the polynomial $x^2+2xy+y^2$ in the form of factors and divide by $x+y$.

Example-8. Divide the polynomial $a^3 - 3a^2b + 3ab^2 - b^3$ by the polynomial $a - b$

Solution : Here, dividend $= a^3 - 3a^2b + 3ab^2 - b^3$ and divisor $= a - b$.

$$
\begin{array}{r|l|l}
a-b & a^3 - 3a^2b + 3ab^2 - b^3 & a^2 - 2ab + b^2 \\
 & a^3 - a^2b & \\
 & (-)\ \ (+) & \\
\hline
 & \quad -2a^2b + 3ab^2 - b^3 & \\
 & \quad -2a^2b + 2ab^2 & \\
 & \quad (+)\ \ \ (-) & \\
\hline
 & \qquad\qquad ab^2 - b^3 & \\
 & \qquad\qquad ab^2 - b^3 & \\
 & \qquad\qquad (-)\ \ (+) & \\
\hline
 & \qquad\qquad\qquad 0 &
\end{array}
$$

Exercise - 1

1. Find the quotient and remainder on dividing polynomial $x^2 - x + 1$ by $x + 1$.
2. Find the quotient and remainder on dividing $6x^2 - 5x + 1$ by $2x - 1$.
3. Find the quotient and remainder on dividing $2y^3 + 4y^2 + 3y + 1$ by $y + 1$
4. Find the quotient and remainder on dividing $x^5 + 5x + 3x^2 + 5x^3 + 3$ by $4x + x^2 + 2$.
5. Find the quotient and remainder on dividing $x^2 - 2xy + y^2$ by $x - y$.
6. Divide polynomial a by polynomial $a - b$.
7. If divisor $= 3x^2 - 2x + 2$, quotient $= x + 1$, remainder $= 3$ then what is the dividend.
8. If divisor $= 4x - 7$, quotient $= x + 1$, remainder $= 0$ then what is the dividend.
9. Prove that on dividing the polynomial $4x^3 + 3x^2 + 2x - 9$ by $x - 1$ the remainder is zero.
10. Verify whether on dividing polynomial $x^2 - 5x + 3$ by $x - 3$ the remainder is zero or not.
11. If the area of a rectangle is $45x^2 + 30x$ square meter and its breadth is $15x$ meter then what will be the length?
12. A line segment AB whose length is $28x$ unit is to be divided into two equal parts. What will be the length of each part?

Remainder Theorem

Now, we go through various examples of division again. Do you find any important points?

We can say that "If we divide a polynomial $f(x)$ by $(x-a)$ then remainder is $f(a)$." This is remainder theorem. Meaning of $f(a)$ is the value of f when $x=a$.

Proof: $\because$ Dividend = Divisor × Quotient + Remainder

Now, $f(x)=(x-a)q(x)+r$

Value of $f(x)$ is as follows when $x=a$

$$f(a)=(a-a).q(a)+r$$

$$f(a)=0.q(a)+r$$

$$f(a)=0+r$$

Or $f(a)=r$

Since r is called remainder therefore here remainder $=f(a)$

We divide $f(x)$ by $(x-a)$ and find that remainder is $f(x)$.

Therefore, we can say that if we divide a polynomial $f(x)$ by $(x-a)$ then remainder is $f(a)$.

Try These

If divisor of $f(x)$ is $x+a$ then find remainder.

(i) $f(x)=2x-a$ (ii) $f(x)=x^2-a^2$ (iii) $f(x)=x^2-2x+1$

Now, by using remainder theorem we can find remainder without doing division if we know dividend and divisor.

Example-9. Find the remainder when we divide dividend $p(x)=3x^4-x^3+30x-1$ by the following:

(a) $x+1$ (b) $2x-1$

Solution: (a) Dividend $p(x)=3x^4-x^3+30x-1$

and divisor is $g(x)=x+1$

Then, remainder = ?

We know by the remainder theorem that remainder $r = p(a)$ when division is given by $x-a$.

$\because$ Here divisor is $x+1$ and remainder $r = p(-1)$

On putting $x=-1$ in $p(x)$

Remainder $= p(-1)$

$= 3(-1)^4 - (-1)^3 + 30(-1) - 1$

$= 3+1-30-1$

Remainder $= -27$

When divisor is $x-a$ then remainder $r = f(a)$ but when divisor is $x+a$ then remainder is $r = f(-a)$.

Solution: (b) Dividend $p(x) = 3x^4 - x^3 + 30x - 1$

and divisor $g(x) = 2x-1$

and remainder $= p(a)$

Here, we will write $2\left(x-\frac{1}{2}\right)$ as $2x-1$. Now $\frac{1}{2}$ is appearing in place of a.

So, remainder $= p\left(\frac{1}{2}\right)$

$= 3\left(\frac{1}{2}\right)^4 - \left(\frac{1}{2}\right)^3 + 30\times\frac{1}{2} - 1$

$= 3\times\frac{1}{16} - \frac{1}{8} + 15 - 1$

$= \frac{3}{16} - \frac{1}{8} + 14$

$= \frac{3-2}{16} + 14$

$= \frac{1}{16} + 14$

Remainder $= 14\frac{1}{16}$

Example-10. Find remainder by using remainder theorem if dividend and divisor are $p(x) = 2x^2 - 3x + 6$, $g(x) = x-2$ respectively.

Solution: Here, dividend $p(x) = 2x^2 - 3x + 6$

and divisor $g(x) = x - 2$

Then by remainder theorem

remainder $r = p(2)$

$$= 2(2)^2 - 3(2) + 6$$

$$r = 8$$

Example-11. When a polynomial $f(x)$ is divided by $x^2 - 4$ then we get remainder $5x + 6$. If the same polynomial is divided by $x - 2$ then what will be the remainder?

Solution : Here dividend is $= f(x)$ and divisors are $x^2 - 4$ and $x - 2$. When $f(x)$ is divided by $x^2 - 4$ then remainder is $5x + 6$. We can write this in the following way:-

$\because$ Dividend = Divisor × Quotient + Remainder.

$$f(x) = (x^2 - 4) \times q(x) + (5x + 6)$$

Now, we know that once dividend and divisor are known then we can find remainder with the help of remainder theorem.

Since, $x - 2$ is a divisor of $f(x)$.

So, by remainder theorem,

Remainder $= f(2)$

$$= (2^2 - 4) \times q(2) + (5 \times 2 + 6)$$

$$= (4 - 4) \times q(2) + 10 + 6$$

$$= 0 \times q(2) + 16$$

Remainder $= 16$

So, when $f(x)$ is divided by $x - 2$ then remainder will be 16.

Think and Discuss

1. In the above examples can we find the remainder if another divisor $x + 2$ is used in place of second divisor $(x - 2)$? If yes, then find the remainder.
2. Can you see any particular or unique relationship between the divisors in the above examples? Find this particular relationship with the help of your friends. If there is no relationship between the divisors then can we still find the remainder? Try to get the answer by taking a example.

The Factor Theorem

What does it mean when we are dividing a polynomial dividend by another polynomial and remainder comes as zero. Do we find a new relationship between dividend and divisor when remainder becomes zero?

Now we well try to understand the realationship of dividend and divisor by taking an example of algebra where remainder becomes zero, then use this to find relationship in polynomials.

We take 25 as a dividend and 5 as divisor and then see what will be the quotient and remainder.

$$\begin{array}{lr|r|l} & & \text{Dividend} & \\ \text{Divisor} & 5 & 25 & 5 \;\; \text{Quotient} \\ & & -25 & \\ \hline & & 0 & \\ & & \text{Remainder} & \end{array}$$

$\because$ Dividend = Quotient × Divisor + Remainder.

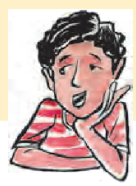
$$25 = 5 \times 5 + 0$$

$$25 = 5 \times 5$$

By this relationship we can say that divisor 5 is a factor of dividend 5.

Try These

Divide 15 by 3 and write it in the above form and check whether there is also some relationship.



Does the same kind of relation apear in the division of polynomials?

Example-12. What will be the quotient and remainder if the polynomial $x^2 - 16$ is divided by polynomial $x - 4$.

Solution :

$$\begin{array}{c|r|l} \text{Divisor} & \text{Dividend} & \text{Quotient} \\ (x-4) & x^2 - 0.x - 16 & x+4 \\ & -(x^2 - 4x) & \\ \hline & 4x - 16 & \\ & -(4x - 16) & \\ \hline & 0 & \\ & \text{Remainder} & \end{array}$$

Clearly, quotient is $x+4$ and remainder is 0.

Now, write this in the following form

$\because$ Dividend = Divisor × Quotient + Remainder

$$x^2-16=(x-4)(x+4)+0$$

$x^2 - 16 = (x - 4)(x + 4)$. In the above example, we can see that product of $(x - 4)$ and $(x - 4)$ is $x^2 - 16$. This means that the divisor $(x - 4)$ is a factor of $x^2 - 16$. But we can say this only when the remainder is zero.

Find out if $(x + 4)$ is a factor of $x^2 - 16$.

We can say that a divisor is a factor of dividend if remainder is zero on dividing dividend by the divisor. This statement is said to be the simplest form of the remainder theorem. In a way, factor theorem is expanded form of remainder theorem.

Proof of factor theorem

This statement can be written in the following manner in the form of a theorem. To write this in the form of a theorem, we need to prove it first.

Theorem : If $x = a$, such that a is zero of polynomial $f(x)$ where remainder $f(a)=0$ then $(x-a)$, is a factor of $f(x)$

If on dividing $f(x)$ by $(x-a)$, the remainder $f(a)=0$, then $(x-a)$ is a factor of $f(x)$.

Proof : We can write the relationship between dividend, divisor, quotient and remainder in the following way .

i.e. $f(x)=g(x).q(x)+r(x)$

By the remainder theorem we know that if $f(x)$ is divided by $(x-a)$ then remainder is $f(a)$.

$\because$ Dividend = Dividsor × Quotient + Remainder

i.e. $f(x)=(x-a).q(x)+f(a)$

Now, if remainder $f(a)=0$

Then, $f(x)=(x-a).q(x)$

Clearly, $(x-a)$ is a factor of $f(x)$.

The values of x for which the value of polynomial $f(x)$ is zero are called zeroes of the polynomial.

The converse of this theorem is also true therefore, if divisor is a factor of a dividend, then remainder is zero.

Converse : Remainder is zero if $(x-a)$ is a factor of $f(x)$.

Proof : Since $(x-a)$ is a factor of $f(x)$.

i.e. $x=a$ is a zero of $f(x)$

$f(x)=(x-a).q(x)$

On putting $x=a$ in

$f(a)=(a-a).q(a)$

$f(a)=0$

Clearly if $(x-a)$ is a factor of $f(x)$ then remainder $f(a)$ is zero.

1. If there are two factors $(x-a)$, $(x-b)$ of a polynomial, then $f(x)=(x-a)(x-b).q(x)$
2. If there are three factors $(x-a)$, $(x-b)$, $(x-c)$ of a polynomial then $f(x)=(x-a)(x-b)(x-c).q(x)$

We can tell whether a divisor is a factor of dividend polynomial or not without doing division with the help of factor theorem. You will understand the importance of factor theorem more by the following example:-

Example-13. Is $(x-2)$, a factor of polynomial $p(x)=x^3-3x^2+4x-4$?

Solution : If $(x-2)$, is a factor of polynomial $p(x)=x^3-3x^2+4x-4$, then on putting $x=2$ remainder should be zero.

Putting $p(x)$ in $x=2$

$$p(2)=(2)^3-3(2)^2+4(2)-4$$
$$=8-3\times4+8-4$$
$$=8-12+4$$
$$=12-12$$
$$p(2)=0$$

Clearly, $p(2)=0$ so $(x-2)$ is a factor of $p(x)$

Example-14. Is $(x-a)$ a factor of the polynomial $p(x)=x^3-ax^2+5x-5a$?

Solution : If on putting $x=a$ in polynomial $p(x)=x^3-ax^2+5x-5a$, $p(a)=0$

then we can say that $(x-a)$ is a factor $p(x)$.

On putting $x=a$

$$p(a)=a^3-a.a^2+5a-5a$$

$$=a^3-a^3+0$$

$$p(a)=0$$

Clearly, $p(a)=0$ so $(x-a)$ is a factor of $p(x)$

Example-15. Find the value of k if $(x-1)$ is a factor of $p(x)=x^2+x+k$

Solution : Since $(x-1)$ is a factor of x^2+x+k . Then we can say by the converse of factor theorm that on putting $x=1$, the remainder $p(1)$ will be zero

So, $p(1)=0$

$$1^2+1+k=0$$

$$1+1+k=0$$

$$2+k=0$$

$$k=-2$$

Exercise - 2

1. If $p(x)=x^3+3x^2-5x+8$ is divided by the following, then find the remainder with the help of remainder theorem.

(i) $x+1$ (ii) $2x-1$ (iii) $x+2$ (iv) $x-4$ (v) $x+\frac{1}{3}$

2. Verify in the following whether $g(x)$ is a factor of $p(x)$.

(i) $g(x)=x-3$; $p(x)=x^3-4x^2+x+6$

(ii) $g(x)=x+1$; $p(x)=2x^3+x^2-2x+1$

(iii) $g(x)=x-2$; $p(x)=x^4-x^3-x^2-x-2$

(iv) $g(x)=x-1$; $p(x)=x^3+5x^2-5x+1$

(v) $g(y)=y+4$; $p(y)=y^2+2y-1$

3. Find the value of a when $g(x)$ is a factor of $p(x)$.

(i) $g(x) = x+1$; $p(x) = x^2 + ax + 2$

(ii) $g(x) = x-1$; $p(x) = ax^2 - 5x + 3$

(iii) $g(x) = x+2$; $p(x) = 2x^2 + 6x + a$

(iv) $g(t) = t-3$; $p(t) = t^2 + 2at - 2a + 3$

(v) $g(y) = y+5$; $p(y) = y^2 - 2y + a$

4. If a polynomial $f(x)$ is divided by $x^2 - 9$ then remainder is $3x+2$, what will be the remainder when the same polynomial is divided by $(x-3)$?

5. If a polynomial $f(x)$ is divided by $x^2 - 16$ then remainder is $5x+3$, what will be the remainder when the same polynomial is divided by $(x+4)$?

Factoring Polynomials

You have already seen that if a polynomial is divided by any other polynomial and the remainder is zero then we can say that the divisor polynomial is a factor of the dividend polynomial. If we can't find the factors of a polynomials using the factor theorem then how can we find the factors of a polynomial? On the basis of the type of the polynomial, we can get its factors. We will discuss here factors of linear polynomials and quadratic polynomials.

Factoring of a number means spliting it into its prime factors such that we can get the same number on multipling those factors.

Let us think about the factors of 6

$6 = 2 \times 3$

Here 6 is written in the form of its prime factors 2 and 3 whose product is 6.

Similarly, 12 can be written as

$$12 = 2 \times 2 \times 3$$

Similarly, when we talk about factoring polynomials it means splitting a polynomial into simple polynomials in such a way that we get the same polynomial by multiplying the factors.

There are some methods of factoring polynomials, for example, sometimes we identify common factors for factorising it and sometimes by using the following identities.

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

$$a^2 - b^2 = (a+b)(a-b)$$

To get factors by taking common factors

It is possible to find factors by common factors method only if in each term of the polynomial that factor is present. We can understand this by some examples which are given below:

Example-16. Find the factors of $12x + 4x^2$.

Solution : $12x + 4x^2 = 4 \times 3 \times x + 4 \times x \times x$ (Polynomial $4x$ is present in both terms)

$$= 4x(3+x)$$

Example-17. Find the factors of $ab + ac + a^2$.

Solution : $ab + ac + a^2 = a(b+c+a)$ (a is in all three terms)

$$= a(a+b+c)$$

Example-18. Find the factors of $2x^3 + 4x$.

Solution : $2x^3 + 4x = 2 \times x \times x^2 + 2 \times 2 \times x$

$$= 2x(x^2+2)$$

Factorisation by using identities

Were you able to find the factors of $x^2 - 4$, $x^2 + 6x + 9$, $x^2 + 5x + 6$ by taking common factors.

Let us consider the following examples $x^2 - 4$, $x^2 + 6x + 9$ and $x^2 + 5x + 6$. There are no common terms in each of these polynomials. We cannot find factors of these polynomials by taking a common polynomial. Then what should we do?

$x^2 - 4$ can be factorized as shown below:

$x^2 - 4 = x^2 - 2^2$ $\because$ Identity $a^2 - b^2 = (a+b)(a-b)$

$$= (x+2)(x-2)$$

Can we write $x^2 + 6x + 9$ in the form of an identity?

Yes, we can write $x^2 + 6x + 9$ in the form of $(a+b)^2 = a^2 + 2ab + b^2$

$$x^2+6x+9 = x^2+2\times 3x+3^2$$
$$=(x+3)^2$$
$$=(x+3)(x+3)$$

Try These

1. Factorise x^2-16
2. Factorise $4x^2-20x+25$

Find factors by splitting the middle term of a polynomial which is in the form $ax^2 + bx + c$

We again consider the factorisation of x^2+5x+6. Can we find factors of this by writing this in the form of an identity?

You will find that you are not able to write this polynomial in the form of an identity.

We need to split the middle term into two parts for this type of polynomial. The two parts should be such that their sum is equal to middle term and their product is equal to product of first and last term of the polynomial.

Now, we factorse x^2+5x+6

$$x^2+5x+6 = x^2+(2+3)x+2\times 3$$
$$= x^2+2x+3x+2\times 3$$
$$=(x^2+2x)+(3x+2\times 3)$$
$$= x(x+2)+3(x+2)$$
$$=(x+2)(x+3)$$

To understand this method we use the following expression:

$$(x+a)(x+b) = x^2+(a+b)x+ab$$
$$=1.x^2+(a+b)x+ab$$

The expression which is product of $(x+a)$ and $(x+b)$ can be written in the form of Ax^2+Bx+C. Then we will see that here, $A=1$, $B=a+b$ and $C=ab$.

To find the factors of any polynomial in the form Ax^2+Bx+C coefficient A of first term x^2 is multiplied with the last term C. Then we try to find two factors of the product obtained whose sum is equal to the coefficient B of the middle term x.

Let us understand this by looking at the following example:-

Example-19. Factorise the polynomial x^2+3x+2.

Solution : On comparing polynomial x^2+3x+2 with Ax^2+Bx+C

$A=1$, $B=3$, $C=2$

Since, $A\times C=1\times 2=2$,

Following are the possible factors of 2

1×2 | $(-1)\times(-2)$

Factors $1+2=3$ but $(-1)+(-2)=-3$ which means that 1×2 is the only correct factorization of 2 where sum is 3 (which is equal to B).

So, $x^2+3x+2=x^2+(1+2)x+1\times 2$

$$=x^2+1.x+2.x+1\times 2$$

$$=(x^2+1.x)+(2.x+1\times 2)$$

$$=x(x+1)+2(x+1)$$

$=(x+1)(x+2)$ are the required factors.

Example-20. Factorise the polynomial $6x^2-5x-6$.

Solution : On comparing polynomial $6x^2-5x-6$ with Ax^2+Bx+C

$A=6$, $B=-5$, $C=-6$

Since, $A\times C=6\times(-6)=-36$

Possible factors of -36 are:-

-1×36	$1\times(-36)$
-2×18	$2\times(-18)$
-3×12	$3\times(-12)$
-4×9	$\boxed{4\times(-9)}$
-6×6	$6\times(-6)$

Clearly in the above example, 4 and –9 are the factors of $AC = -36$, as sum of 4 and –9 is –5 which is equal to the middle term B.

So, $6x^2 - 5x - 6 = 6x^2 + (4-9)x - 6$

$$= 6x^2 + 4x - 9x - 6$$

$$= (6x^2 + 4x) - 1(9x + 6)$$

$$= 2x(3x+2) - 3(3x+2)$$

$= (3x+2)(2x-3)$ are the required factors.

Example-21. Factorise the polynomial $14x^2 + 19x - 3$.

Solution : On comparing polynomial $14x^2 + 19x - 3$ with $Ax^2 + Bx + C$

$A = 14$, $B = 19$, $C = -3$

Since, $A \times C = 14 \times (-3) = -42$

Following are the possible factor of –42.

-1×42	$1 \times (-42)$
$\boxed{-2 \times 21}$	$2 \times (-21)$
-3×14	$3 \times (-14)$
-6×7	$6 \times (-7)$

Clearly, in the above,–2 and 21 are the factors of $A \times C = -42$ as sum of – 2 and 21 is $-2 + 21 = 19$ which is equal to the middle term B.

So, $14x^2 + 19x - 3 = 14x^2 + (-2+21)x - 3$

$$= 14x^2 - 2x + 21x - 3$$

$$= (14x^2 - 2x) + (21x - 3)$$

$$= 2x(7x-1) + 3(7x-1)$$

$= (7x-1)(2x+3)$ are the required factors.

Example-22. Factorise the polynomial $4\sqrt{3}x^2 + 5x - 2\sqrt{3}$.

Solution : On comparing the polynomial $4\sqrt{3}x^2 + 5x - 2\sqrt{3}$ with $Ax^2 + Bx + C$

$A = 4\sqrt{3}$, $B = 5$, $C = -2\sqrt{3}$

So, $A \times C = 4\sqrt{3} \times \left(-2\sqrt{3}\right) = -8 \times 3 = -24$

Following are the possible factors of –24.

-1×24	$1 \times (-24)$
-2×12	$2 \times (-12)$
$\boxed{-3 \times 8}$	$3 \times (-8)$
-6×4	$6 \times (-4)$

Clearly, –3 and 8 are the factors of $A \times C = -24$, since sum of –3 and 8 is $-3 + 8 = 5$, which is equal to middle term B.

So, $$4\sqrt{3}x^2 + 5x - 2\sqrt{3} = 4\sqrt{3}x^2 + (-3+8)x - 2\sqrt{3}$$
$$= 4\sqrt{3}x^2 - 3x + 8x - 2\sqrt{3}$$
$$= \left(4\sqrt{3}x^2 - 3x\right) + \left(8x - 2\sqrt{3}\right)$$
$$= \left(4\sqrt{3}x^2 - \sqrt{3}\sqrt{3}x\right) + 2\left(4x - \sqrt{3}\right)$$
$$= \sqrt{3}x\left(4x - \sqrt{3}\right) + 2\left(4x - \sqrt{3}\right)$$
$$= \left(4x - \sqrt{3}\right)\left(\sqrt{3}x + 2\right)$$

Think and Discuss

Is it possible for a quadratic polynomial to have more than two factors? Go through the examples of this chapter. Work with your friends to form quadratic polynomials and verify if it is possible to get more than two factors for any of them.

Exercise-3

Factorize thefollowing polynomials by splitting the middle term:-

(1)	$x^2 - 3x - 4$	(2)	$x^2 + 2x + 1$	(3)	$x^2 + x - 12$
(4)	$x^2 - 8x + 15$	(5)	$t^2 - 4t - 21$	(6)	$-y^2 + 35y + 156$
(7)	$7x^2 - 2x - 5$	(8)	$12x^2 - 24x + 12$	(9)	$6x^2 - 7x - 3$
(10)	$14y^2 + 19y - 3$	(11)	$\sqrt{3}y^2 + 9y + 6\sqrt{3}$	(12)	$144x^2 + 24x + 1$

Value and Zeroes of a Qudratic Polynomial

Consider the polynomial $p(x) = x^2 - 6x + 9$. If we put $x = 1$ in the polynomial

Then, $p(1) = (1)^2 - 6(1) + 9$

$= 1 - 6 + 9$

$= 4$

On putting $x = 1$ we get 4 as the value of $p(1)$. This is the value of the polynomial for x=1. We can also say that the value of $p(x)$ at $x = 1$ is 4. Similarly, we can get the values of $p(-1)$, $p(2)$ etc.

We see that when $x = 3$

then $p(3) = 3^2 - 6(3) + 9$

$= 9 - 18 + 9$

$= 0$

The value of the polynomial is 0 for x=3. So, we can say that 3 is a zero of the polynomial.

Now, we will find the zeroes of the polynomial in the following examples.

Example-23. Find the zeroes of the polynomial $x^2 - 3x - 4$.

Solution : Let $p(x) = x^2 - 3x - 4$

Here, we need to find a value of x for which the value of polynomial is zero.

If $x = 1$

then $p(1) = (1)^2 - 3(1) - 4$

$= 1 - 3 - 4$

$= -6$

If $x = -1$

then $p(-1) = (-1)^2 - 3(-1) - 4$

$= 1 + 3 - 4$

$= 0$

On putting $x = -1$ the value of the polynomial is zero so -1 is a zero of the polynomial. Are there any more zeroes for this polynomial? To know this we can try to put more values of x. But we can use the factors of a polynomial to more easily find all its zeroes.

Find the factors of the polynomial $x^2 - 3x - 4$

$$\begin{aligned} x^2 - 3x - 4 &= x^2 - 4x + x - 4 \\ &= (x^2 - 4x) + 1(x - 4) \\ &= x(x-4) + 1(x-4) \\ &= (x-4)(x+1) \end{aligned}$$

The zero of this polynomial is that value of x for which the value of polynomial is zero.

So, $\quad x^2 - 3x - 4 = 0$

$\Rightarrow \quad (x-4)(x+1) = 0$

$\Rightarrow \quad (x-4) = 0 \quad$ and $\quad (x+1) = 0$

$\Rightarrow \quad x - 4 = 0 \quad$ and $\quad x + 1 = 0$

$\Rightarrow \quad x = 4 \quad$ and $\quad x = -1$

Here we see that for both $x = -1$ and $y = 4$ the value of the polynomial is zero.

So -1 and 4 are zeroes of the polynomial.

In the above example -1 and 4 are zeroes of the polynomial while $(x-4)$ and $(x+1)$ are factors of the polynomial. We see that on making the factors of the polynomial zero we get zeroes of the polynomial. So if we know the factors of a polynomial then we can get zeroes of the polynomial. Can we find the factors if zeroes of the polynomial are given?

Try These

1. Find factors and zeroes of the polynomial $x^2 - 9$.
2. What will be the factors of a polynomial if its zeroes are 4 and -1.

Relationship between Zeroes and Coefficients of a Polynomial

Zeroes of a polynomial $x^2 - 5x + 6$ are 3 and 2, then its the factors are $(x-3)$ and $(x-2)$.

i.e. $x^2 - 5x + 6 = 1.(x-3)(x-2)$

Now, think about the factors and zeroes of the polynomial $4x^2 - 4x + 1$

$$4x^2 - 4x + 1 = 4x^2 - 2x - 2x + 1$$
$$= \left(4x^2 - 2x\right) - 1\left(2x - 1\right)$$
$$= 2x\left(2x - 1\right) - 1\left(2x - 1\right)$$
$$= \left(2x - 1\right)\left(2x - 1\right)$$
$$= 2\left(x - \frac{1}{2}\right) \times 2\left(x - \frac{1}{2}\right)$$
$$= 4\left(x - \frac{1}{2}\right)\left(x - \frac{1}{2}\right)$$

So, the factors of $4x^2 - 4x + 1$ are $4\left(x - \frac{1}{2}\right)\left(x - \frac{1}{2}\right)$ and clearly the zeroes of the polynomial are $\frac{1}{2}, \frac{1}{2}$.

Do you see any important point (pattern) in the factors of $x^2 - 5x + 6$ and $4x^2 - 4x + 1$. The coefficient of x^2 in $x^2 - 5x + 6$ is seen as one of its factors. Similarly, 4 which is the coefficient of x^2 in polynomial $4x^2 - 4x + 1$ is a factor of this polynomial. This means that the polynomial $ax^2 + bx + c$ whose zeroes are α and β and where a, b, c are real numbers where $a \neq 0$ can be written in the following way:-

$$ax^2 + bx + c = k\left(x - \alpha\right)\left(x - \beta\right) \;\; ; \;\; k = a$$

Where k is a real number and $k \neq 0$

Again $ax^2 + bx + c = kx^2 - k\left(\alpha + \beta\right)x + k\alpha\beta$ (on multiplication)

On comparing the coefficients of x^2, x and the constant terms on both sides of this equation

$$a = k; \quad b = -k\left(\alpha + \beta\right); \quad c = k\alpha\beta$$

$$\frac{b}{-k} = \alpha + \beta \qquad ; \quad \frac{c}{k} = \alpha\beta$$

$$\frac{b}{-a} = \alpha + \beta$$

So $\alpha + \beta = \frac{-b}{a}$ and $\alpha\beta = \frac{c}{a}$ (On multiplying the numerator and denominator by –1)

We can say that in a quadratic polynomial $ax^2 + bx + c$

Sum of zeroes, $\alpha + \beta = \dfrac{\text{Coefficient of } -x}{\text{Coefficient of } x^2}$

and product of zeroes $\alpha\beta = \dfrac{\text{Constant term}}{\text{Coefficient of } x^2}$

Let us understand the realtionship between zeroes and coefficient of polynomials through some examples.

Example-24. Find the sum and product of the zeroes of the polynomial $6x^2 + 13x + 7$.

Solution : On comparing the polynomial $6x^2 + 13x + 7$ with $ax^2 + bx + c$

$a = 6$, $b = 13$, $c = 7$

$\because$ Sum of zeroes $= \dfrac{-b}{a}$

$\therefore$ Sum of zeroes $= \dfrac{-13}{6}$

$\because$ Product of zeroes $= \dfrac{c}{a}$

$\therefore$ Product of zeroes $= \dfrac{7}{6}$

Example-25. Find the sum and product of the zeroes of the polynomial $4x^2 + 4\sqrt{3}x + 3$ in the form $ax^2 + bx + c$.

Solution : On comparing the polynomial $4x^2 + 4\sqrt{3}x + 3$ with $ax^2 + bx + c$

$\because$ Sum of zeroes $= \dfrac{-b}{a}$

$\therefore$ Sum of zeroes $= \dfrac{-\left(4\sqrt{3}\right)}{4}$

$= -\sqrt{3}$

$\because$ Product of zeroes $= \dfrac{c}{a}$

$\therefore$ Product of zeroes $= \dfrac{3}{4}$

Think and Discuss

Can we find the polynomial if its zeroes are given? Form a polynomial using any two zeroes.

Exercise - 4

1. Zeroes of some polynomials of the form $ax^2 + bx + c$ are given below : Find the factors of the polynomials.

   (i) $(3, 4)$ (ii) $(-2, -3)$ (iii) $\left(\frac{1}{2}, \frac{-1}{2}\right)$

   (iv) $(15, 17)$ (v) $(-18, 12)$

2. Find the sum and product of zeroes of the following polynomials.

   (i) $x^2 + 10x + 24$ (iv) $-5x^2 + 3x + 4$

   (ii) $2x^2 - 7x - 9$ (v) $x^2 + x - 12$

   (iii) $x^2 + 11x + 30$

What We Have Learnt

1. The process of division in polynomials is slightly different from division in arithmetic. Here, we have to be careful about the exponents of the variables.
2. For division of polynomials we have to write the dividend and divisor in descending order of their exponents.
3. Long division method is also used to divide polynomials.
4. In long division method, we keep repeating the process of division till we get zero as the remainder.
5. In division of polynomials, the quotient and remainder are also polynomials.
6. If a polynomial $f(x)$ is divided by $(x - a)$ then remainder is $f(a)$. This is called the remainder theorem.
7. $(x-a)$ is a factor of polynomial $f(x)$, if $f(a) = 0$. Also, if $(x - a)$ is a factor of $f(x)$, then $f(a) = 0$.
8. Quadratic polynomials have two zeroes.

ANSWER KEY

Exercise - 1

1. Quotient $= x-2$, Remainder $= 3$ 2. Quotient $= 3x-1$, Remainder $= 0$
3. Quotient $= 2y^2+2y+1$, Remainder $= 0$
4. Quotient $= x^3-4x^2+19x-65$, Remainder $= 227x+133$
5. Quotient $= x-y$, Remainder $= 0$ 6. Quotient $= 1$, Remainder $= b$
7. $3x^3+x^2+5$ 8. $4x^2-3x-7$ 10. Remainder is not zero
11. $(3x+2)$ meter   12. $14x$ meter

Exercise - 2

1. (i) 15 (ii) $\frac{51}{8}$ (iii) 22 (iv) 100 (v) $\frac{269}{27}$
2. (i) $(x-3)$ is a factor of given polynomial.
   (ii) $(x+1)$ is not a factor of given polynomial.
   (iii) $(x-2)$ is a factor of given polynomial.
   (iv) $(x-1)$ is not a factor of given polynomial.
   (v) $(y+4)$ is not a factor of given polynomial.
3. (i) $a=3$ (ii) $a=2$ (iii) $a=4$ (iv) $a=-3$ (v) $a=-35$
4. Remainder $= 11$ 5. Remainder $= -17$

Exercise - 3

1. $(x-4)(x+1)$ 2. $(x+1)(x+1)$ 3. $(x+4)(x-3)$
4. $(x-5)(x-3)$ 5. $(t-7)(t+3)$ 6. $-(y-39)(y+4)$
7. $(7x+5)(x-1)$ 8. $12(x-1)(x-1)$ 9. $(2x-3)(3x+1)$
10. $(2y+3)(7y-1)$ 11. $\left(y+2\sqrt{3}\right)\left(\sqrt{3}y+3\right)$ 12. $(12x+1)^2$

Exercise - 4

1. (i) $(x-3)(x-4)$ (ii) $(x+2)(x+3)$ (iii) $\left(x-\frac{1}{2}\right)\left(x+\frac{1}{2}\right)$
   (iv) $(x-15)(x-17)$ (v) $(x+18)(x-12)$
2. (i) $-10, 24$ (ii) $\frac{7}{2}, -\frac{9}{2}$ (iii) $-11, 30$
   (iv) $\frac{3}{5}, -\frac{4}{5}$ (v) $-1, -12$