13

Circle and Tangent

13.01. Introduction

In the previous chapter we have studied various concepts of circles such as chords, angle made by arcs, cyclic quadrilateral etc. In this chapter we will study a line and a circle of their various positions on a plane and their corresponding properties which appears ?

13.02. Secant and Tangent Line

On a white paper draw a circle and a line. Now compare this figure with the following figures. Definately one shape will appear similar to the figures given below means that a line and a circle drawing together is possible as there figures below given figure 13.01.

Let us consider the three figures :

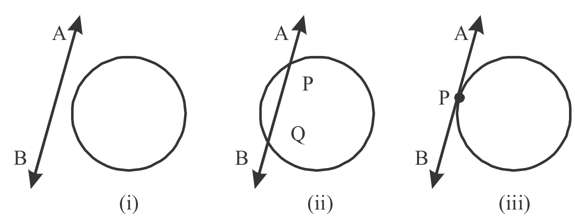

Fig. 13.01

1. In the fig. 13.01 (i). The line is outside the circle so means line and circle are separate figures on the same plane. There is no relation between them.
2. fig. 13.01 (ii) AB is secant for the circle. If a line intersects a circle at two points then it is called secant line.
3. In fig. 13.01 (iii) AB line is tangent for the circle . Here the line AB is passing by touching the circle at point P or we can say in other words that AB line which is intersecting a circle at one point only. Here point P will be the tangent point of the circle and line AB. *It means the line which intersects the circle at a point only is known as a tangemt line.*

To understand the concept of tangent, let us perform the following activity :

Activity :

On a drawing board or on a wood table, put a plain paper and fix two pins A and B. Now keeping normal stress tie up a black colour thread on these two points and draw a circle on the other side of another paper. See fig. 13.02 (i).

Now shift the circle drawn on paper in such away that the thread appear like it is dividing it in two parts to the circle. Name these points as P and Q. Keeping the paper on which circle is drawn stable and fix a pin on point P. In this way the paper can move with respect to point P. See figure 13.02 (ii). Now move the paper on which circle is drawn and observe this process. We observe the following :

(i) The distance between P and Q is reducing with moving means in every situation the chord length decreases form its initial length. See fig. 13.02 (iii).

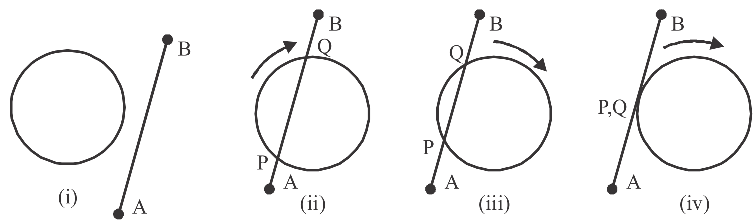

Fig. 13.02

(ii) When point Q approaches to point P or when both coincides then the length of chord becomes zero. The line looks like intersecting at one point. See fig. 13.02 (iv). In this setuation line AB touches the circle at point P.

(iii) If circle is moved more in the same direction then we observe that the length of the chord increases up to a certain limit and after that it decreases and again we get the results as described in (i) and (ii).

Repeat the same process in opposite direction you will get the same result. After this activity we can say that

The secant which is the chord of a circle, whose both piercing ends coincided in special conditions then it is converted into tangent line or we can say that at a point of circle one and only one tangent can exists.

Activity :

On a plain paper that draw a circle and its secant PQ. Now draw parallel lines to PQ. You will observe after the few secants the length of chords is reducing continuously. In one condition the measurment of chord cut by secant tends zero.

It means piercing lines P_1Q_1 and P_2Q_2 on both side becomes tangent . See fig. 13.03. It is clear from this experiment that.

There can not be more then two tangents parallel to the secant or there may exists only two parallel tangents on any circle.

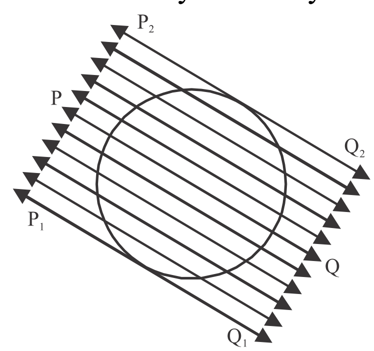

Fig. 13.03

Activity :

Draw a circle with the help of the compass . Draw many radii in the circle with the help of scale and paste it on the card board and cut it along the circumference of the circle.Thus a circular wheel is ready . Now fin a pin on the centre of this wheel and roll this wheel on the ground with respect to the centre of the circle. What will you observe? You will observe that ***at the line of rolling wheel are radii of circle appear perpendicular with respect to the horizontal.*** See fig. 13.04.

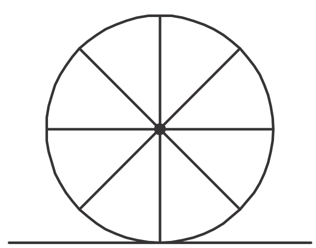

Fig. 13.04

Theorem 13.1

Tangent drawn form any point to a circle, is perpendicular on the line (radius) which joins the centre.

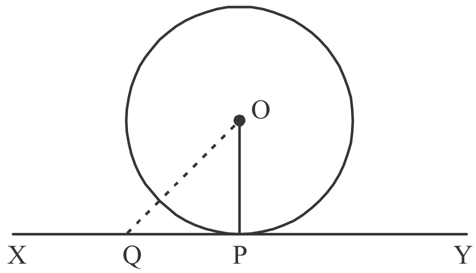

Fig. 13.05

Given : O centre of the circle. Tangent XY touches the circle at point P and OP is radius.

To prove : $OP \perp XY$

Construction : Take a point Q on XY and join OQ.

Proof : $\because$ Every point of tangent will be outside the circle except the point of contact.

$\therefore$ OP < OQ (The distance of point outside the circle is more than the radius). It means OP (radius) will be the smallest in distance form the points situated on XY. But as we know that perpendicular is the smallest in all distances of a straight line.

Hence $OP \perp XY$ **Hence Proved.**

Theorem 13.2 (Converse of Theorem 13.1)

If as a line drawn form any point situated on a circle is perpendiuclar to the radius then it is tangent.

Given : O is the centre of circle and OP is radius and $OP \perp XY$

To prove : XY is a tangent on point P.

Construction : Join O and Q which is a point on XY.

Proof : $\because$ $OP \perp XY$

$\therefore$ $OP < OQ$

(The perpendicular drawn form any point to a line is smallest in all line segments that join this line.) Since all points including Q lying on XY are outside the circle. Hence XY is a tangent.

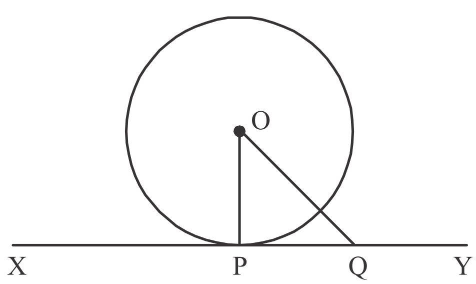

Fig. 13.06

So, perpendicular drawn from a point on any line is the smallest segment among all the line segments drawn from that point to all points on the line.

Hence Proved.

13.03. Number of tangents which can be drawn from any point on a circle

In the preceding section we have studied about secant and tangent. How many tangents can be drawn from inside and outside points of circle and what is the relation between these tangent lines? Let us solve this problem with the help of the following figures.

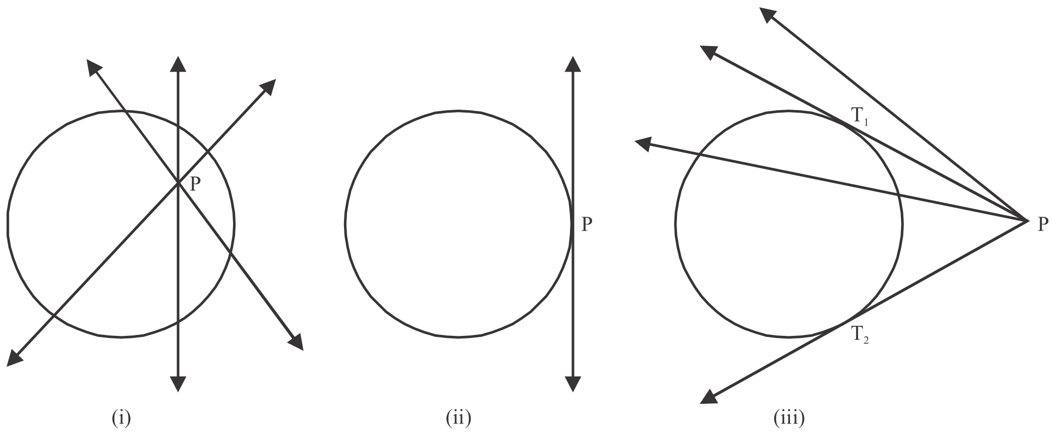

Fig. 13.07

A circle is situated in a plane and we want to choose a point then we will choose it from inside the circle, on the circle and outside the circle. Out of there three situations, we will choose any one situation. Now, we will think on all points one by one.

(i) When point P is situated inside the circle? Then all the lines passing through the points P, are secant lines and number of tangents is zero. See fig. 13.07 (i) It means in such condition the number of tangent will be zero.

(ii) When point P is situated on the circle then we have learnt in the last section that only one tangent can be drawn form a point which is on the circle. See fig, 13.07 (ii).

(iii) When point P is situated outside the circle, only two tangent lines can be drawn. Remaining lines may be either secant or outside the circle. See fig. 13.07 (iii). In the fig. two tangent PT_1 and PT_2 are appearing from point P. Can you tell what is the relation between them? Are these tangents really equal? Let us prove this concept with the help of following theorems.

Theorem 13.3 :

The tangents drawn form an exterior point to a circle are equal.

Given : Two tangents PT_1 and PT_2 are drawn form a point P to a circle with centre O.

To prove : $PT_1 = PT_2$

Construction : Join O to T_1, T_2 and P

Proof : In ΔOPT_1 and ΔOPT_2

$$\angle OT_1P = \angle OT_2P = 90°$$

(Tangent and radius are perpendicular by theorem 13.1)

$$OT_1 = OT_2 \qquad \text{(radius of same circle)}$$

$$OP = OP \qquad \text{(common)}$$

According to R.H.S. congruence

$$\Delta OPT_1 \cong \Delta OPT_2$$

Hence, $PT_1 = PT_2$ **Hence Proved**

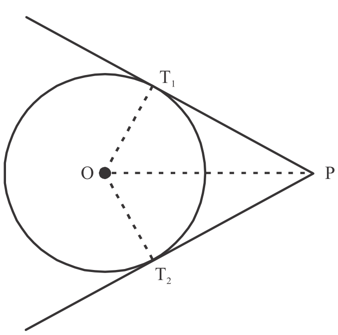

Fig. 13.08

Illustrative Examples

Example 1. Find the length of the tangent, if the distance between tangent point and centre is 13 cm and radius of the circle is 5 cm.

Solution : Since $OQ^2 = OP^2 + PQ^2$ (in right Δ OPQ)

(In right ΔOPQ)

$\Rightarrow$ $PQ^2 = OQ^2 - OP^2$

$= 13^2 - 5^2 = 169 - 25 = 144$

$\Rightarrow$ $PQ = \sqrt{144} = 12$

So, length of tangent is 12 cm

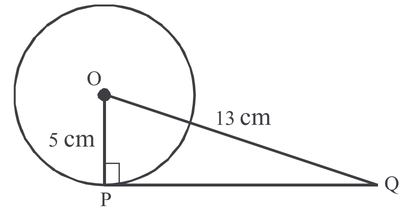

Fig. 13.09

Example 2. In two concentric circles if the chord of larger circle touches the smaller one then prove that the point of contact bisects the chord.

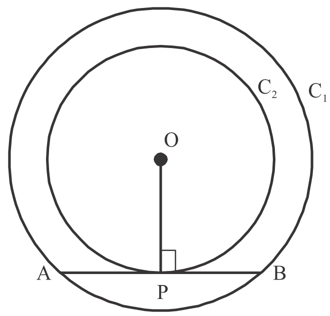

Fig. 13.10

Solution : Given : *AB* is a chord of larger circle C_1 which touches the smaller circle at point *P*.

To prove : $AP = PB$

Proof : *AB* touches the circle C_2 at point *P*.

So, $OP \perp AB$ (According to theorm 13.1)

Since *O* is also centre of circle C_1 and *AB* is chord of circle C_1, so (as per class IX theorem) perpendicular drawn form centre of the circle to the chord, then it bisects the chord.

Hence,

$AP = PB$ **Hence proved**

Example 3. A circle touches the side BC of ΔABC at P externally and AB and AC are produced they meet to the circle at Q and R respectively then prove that $AQ = \frac{1}{2}$ (Perimeter of ΔABC)

Solution : Given : ΔABC, side *BC* touches the circle at point *P* and by producing side *AB* and *AC* they touches the circle at point *Q* and *R*, respectively.

To prove : $AQ = \frac{1}{2}$ (Perimeter of ΔABC)

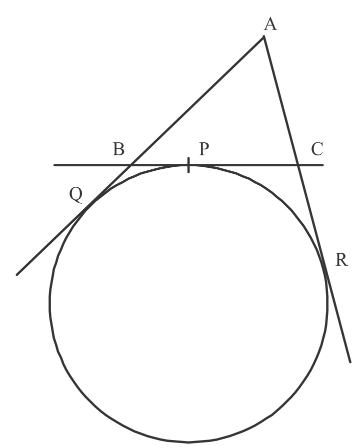

Fig. 13.11

Proof : AQ = AR (According to theorem 13.2) . . . (1)

Similarly BQ = BP . . . (2)

CP = CR . . . (3)

$AQ + AR = [AB + BQ] + [AC + CR]$

$\Rightarrow$ $= [AB + BP] + [AC + CP]$

$= AB + (BP + CP) + AC$

$\Rightarrow$ $2AQ = AB + BC + AC$ from equation (1)

$\Rightarrow$ $AQ = \frac{1}{2}[AB + BC + AC]$

$\Rightarrow$ $AQ = \frac{1}{2}$ (Perimeter of ΔABC) **Hence Proved**

Example 4. The side AB, BC and CA of ΔABC touches a circle of radius 4 cm at points L, M and N respectively. If $AN = 6$ cm, $CN = 8$ cm then find the perimeter of ABC.

Solution : Let $'O'$ be the centre of the circle inscribed in the ΔABC.

means $OL = OM = ON = 4$ cm

Let $BL = x$ cm

$\Rightarrow$ $BL = BM = x$ (See fig. 13.12)

$\because$ $AN = AL = 6$ cm

Similarly, CN = CM = 8 cm

$BC = (x+8)$ cm $= a$ and $AB = (x+6)$ cm $= c$

and AC = 6 + 8 = 14 cm = b

According to Hiron's formula.

$$2s = a + b + c$$

$\Rightarrow$ $2s = x + 8 + 14 + x + 6$

$\Rightarrow$ $2s = 2x + 28$

$\Rightarrow$ $s = x + 14$

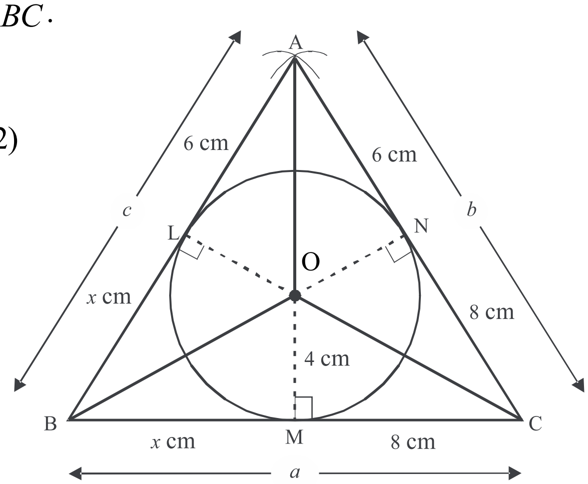

Fig. 13.12

Area of ΔABC $= \sqrt{s(s-a)(s-b)(s-c)}$

$$= \sqrt{(x+14)(x+14-x-8)(x+14-14)(x+14-x-6)}$$

$$\sqrt{(x+14)\times 6\times x\times 8} = \sqrt{48x(x+14)} \quad \ldots(1)$$

and Area of ΔABC = Area of ΔAOB + Area of ΔBOC + Area of ΔAOC

$$= \frac{1}{2}AB\times OL + \frac{1}{2}BC\times OM + \frac{1}{2}AC\times ON$$

$$= \frac{1}{2}(x+6)\times 4 + \frac{1}{2}(x+8)\times 4 + \frac{1}{2}\times 14\times 4$$

$$= 2(x+6) + 2(x+8) + 28$$

$$= 2x + 12 + 2x + 16 + 28$$

$$= 4x + 56 \quad \ldots(2)$$

From equation (1) and (2)

$$\sqrt{48x(x+14)} = 4x + 56$$

$$4\sqrt{3x(x+14)} = 4(x+14)$$

$\Rightarrow$ $\sqrt{3x(x+14)} = (x+14)$

Squaring both sides

$$3x(x+14) = (x+14)^2$$

$$3x = x + 14$$

$$3x - x = 14$$

$$x = 7$$

So, AB = 6 + 7 = 13 cm

BC = 7 + 8 = 15 cm

Hence, perimeter of $\Delta ABC = (13 + 15 + 14) = 42$ cm.

Exercise 13.1

1. Write true or false. Also write the reason of your answer.

   (i) Tangent of a circle is the line which intersect the circle at two points.

   (ii) A tangent XY, touches a circle at point P whose center is O and Q is any other point on the tangent, then $OP = OQ$.

   (iii) LM and XY two tangents at the points P and Q on the circle if PQ is diameter then $LM \| XY$.

   (iv) $'O'$ is the centre of circle situated on the other circle whose centre is A. If circle having centre $'O'$ passes form A and B such that AOB are in one line then the tangents drawn from B will pass through the intersecting points of both circles.

2. Fill in the blanks :

   (i) .... tangents can be drawn form a point situated on circle.

   (ii) A line which intersects the circle at two points is known as ....

   (iii) A circle can have .... parallel tangents.

   (iv) The common point of tangent and circle is known as ....

3. Two concentric circles have radius 5 cm and 3 cm respectively. Find the length of the chord of circle which touches the smaller circle.

4. The length of tangent is 4 cm drawn form any point, 10 cm away form the centre of the circle then what will be the radius of that circle?

5. A circle with centre at $'O'$ touches the four side of a quadrilateral $ABCD$ internally in such a way that it divides AB in 3 : 1 and $AB = 8$ cm then find the radius of the circle where $OA = 10$ cm.

6. A circle touches the all sides of a quadrilateral. Prove that the angle made by opposite sides at the centre are supplementry.

7. In fig. 13.13 centre of a circle is O and the tangents drawn form a point P are PA and PB which touches the circle at A and B respectively then prove that OP is the bisector of line AB.

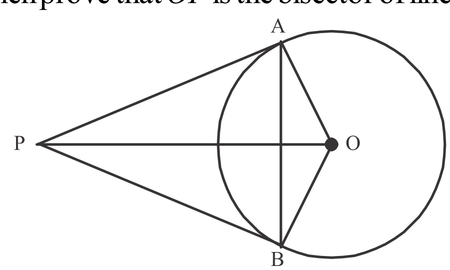

Fig. 13.13

8. In figure 13.13, O is the centre of the circle and from a exterior point $'P'$ two tangents PA and PB are drawn to the circle at A and B respectively then prove that $PAOB$ is a cyclic quadrilateral.

So far, we have learnt about tangents to the circles and solved many problems related to tangents and the circle. If a chord is drawn from a point of contact of the tangent to the circle then we will discuss segments made by the chord. By doing so we may come across some more informations. Let us try to understand these problems.

13.04. Angles of Alternate Segment of Circle

In fig. 13.14, chord AB of any circle, is drawn form tangent point A on the tangent PAQ which makes $\angle BAP$ and $\angle BAQ$ with PAQ.

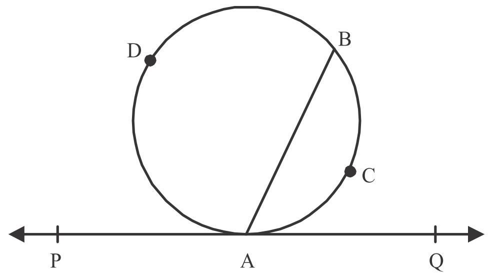

Fig. 13.14

The chord AB divide the circle into two segments ADB and ACB. The segments ADB and ACB of the circle are alternate segment of $\angle BAQ$ and $\angle BAP$.

Theorem 13.4

If a chord is drawn form a point of contact A, of the tangent PQ, of the circle then angle made by this chord with the tangent are equal to the respective alternate angles made by segments with this chord .

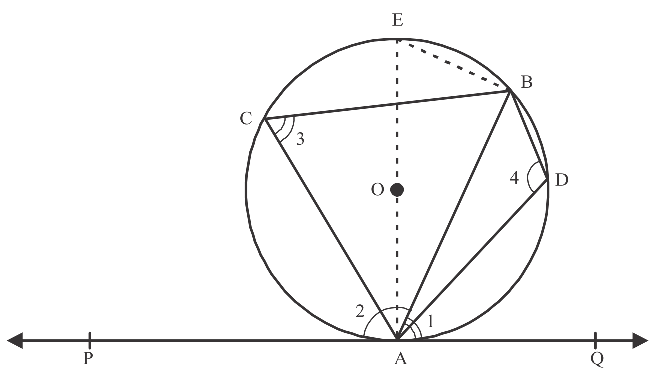

Fig. 13.15

Given : PQ is a tangent to the circle at point A, chord AB make $\angle 1$ and $\angle 2$ respectively with this tangent. $\angle 3$ and $\angle 4$ respective are alternate angles of $\angle 1$ and $\angle 2$ made in alterante segments at point C and D.

To be prove : $\angle 1 = \angle 3$ and $\angle 2 = \angle 4$

Construction : Draw diameter AOE and join EB

Proof : In $\Delta\, AEB$

$\angle ABE = 90°$ (angle made by semi circle)

$\therefore \quad \angle AEB + \angle EAB = 90°$ . . . (1)

$\because \quad \angle EAP = 90°$ (diameter is perpendicular to the tangent)

$\therefore \quad \angle EAB + \angle 1 = 90°$ . . . (2)

From equation (1) and (2)

$\angle EAB + \angle 1 = \angle AEB + \angle EAB$

$\Rightarrow \quad \angle 1 = \angle AEB$ . . . (3)

$\because \quad \angle AEB = \angle 3$ (angles are equal in same circle segment) . . . (4)

From eqaution (3) and (4)

$\angle 1 = \angle 3$ . . . (5)

Again $\quad \angle 1 + \angle 2 = 180°$ (linear pair)

$\angle 3 + \angle 4 = 180°$

(opposite angles of cyclic quadrilateral are supplimentary)

$\angle 1 + \angle 2 = \angle 3 + \angle 4$

But $\angle 1 = \angle 3$ (from equation 5)

$\angle 2 = \angle 4$ **Hence Proved**

Theorem 13.5 (Converse of 13.4)

If a line is drawn at one end of a chord of a circle in such a way that angle made with the chord are equal to alternate angle made by the chord in segment then this line is tangent to the circle.

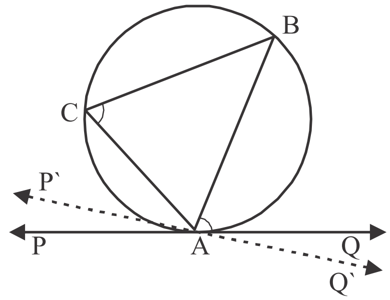

Fig. 13.16

Given : AB is a chord of any circle and PAQ is a line such that $\angle BAQ = \angle ACB$

Where, C is any point in alternate segment.

To prove : PAQ is a tangent.

Solution : $\angle BAQ = \angle ACB$ given. . . (1)

Let instead of PAQ line $P'AQ'$ touches the circle at point A.

So, $\angle BAQ' = \angle ACB$ (by theorem) . . . (2)

From equation (1) and (2)

$\angle BAQ = \angle BAQ'$ . . . (3)

According to the figure $\angle BAQ' = \angle BAQ + \angle QAQ'$ . . . (4)

i.e. $\angle BAQ = \angle BAQ + \angle QAQ'$

hence, $\angle QAQ' = 0$

This is only possible when PAQ and $P'AQ'$ coincides with each other

It means PAQ is a tangent to the circle at point A. **Hence Proved**

Illustrative Examples

Example 1. Write true or false and give reason of your answer.

(i) According to fig. 13.17, $\angle A = 70°$, where PQ touches to the circle at point C.

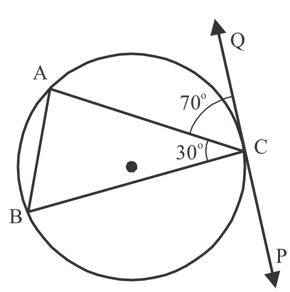

Fig. 13.17

Solution : False, since $\angle A$ is the alternate segment angle of $\angle PCB's$.

So, $\angle A = \angle PCB$

$\angle PCB = 180° - (70° + 30°) = 80°$

$\angle A = 80°$

Example 2. In figure 13.18, PQ is a tangent of a circle whose centre is O which touches the circle at point R. If $\angle TRQ = 30°$ then find $\angle SOR$ and $\angle RTO$.

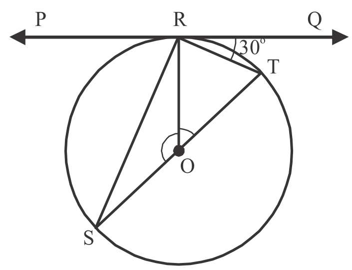

Fig. 13.18

Solution: Since diameter of the circle is SOT

$\therefore$ $\angle SRT = 90°$

and by chord RT , $\angle$ RST is alternate segment of $\angle$ TR Q

So $\angle RST = \angle TRQ = 30°$

But ΔORS is an Isosceles triangle. Radii of circle are OS = OR

$\angle RST = \angle SRO = 30°$

$\therefore$ $\angle SOR = 180° - (30° + 30°) = 180° - 60° = 120°$

and $\angle ORT = \angle SRT - \angle SRO$

$= 90° - 30° = 60°$

Now, in ΔORT

OR = OT (radii of a circle)

$\angle RTO = \angle ORT = 60°$

Hence, $\angle SOR = 120°$

Example 3. In figure 13.19 PQ and RS are tangents at point A and C respectively if $\angle ABC = 60°$ and $\angle BAP = 40°$ then find the value of $\angle BCR$.

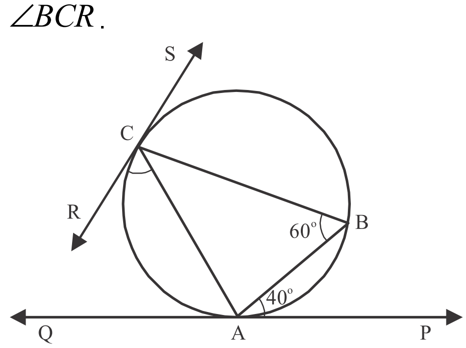

Fig. 13.29

Solution: Tangent PQ and chord AB passes through point A

$\therefore$ $\angle ACB = \angle BAP = 40°$ (By thorem 13.4) . . . (1)

Similary tangent line CR and chord AC passes through point C

$\therefore \qquad \angle ACR = \angle ABC = 60°$ ...(2)

Adding equation (1) and (2)

$$\angle ACB + \angle ACR = 40 + 60 = 100°$$

or $\qquad \angle BCR = 100°$

Example 4. In figure 13.20, M is middle point of the segment AB, taking AM, MB and AB as diameter of semi circles has been drawn in one side. Taking O as centre and r radius of a circle is drawn in such a way that it touches the all the three circles, then porve that $r = \frac{1}{6}AB$.

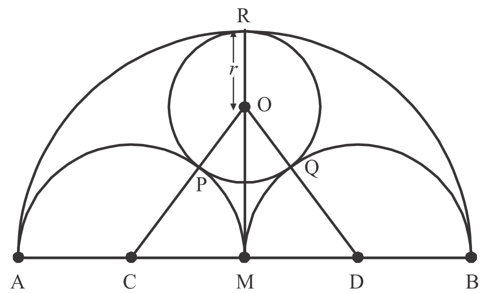

Fig. 13.20

Solution: Given: In figure 13.20, according to the question semicircles are drawn by taking C, M, D and O as centre.

To prove : $r = \frac{1}{6}AB$

Proof : Let AB = a then $AM = \frac{a}{2}$

But $\qquad$ AC = CM = MD = DM = CP = DQ are radii of equal semicircles

$\therefore \qquad CM = MD = CP = DQ = \frac{a}{4}$ ...(1)

Now, $\qquad OC = OD = \left(\frac{a}{4} + r\right)$ ...(2)

$$OM = (MR - OR) = \left(\frac{a}{2} - r\right)$$ ...(3)

Since ΔOCD is an isoscles triangle whose side OC = OD and M is the mid point of CB

$\therefore \qquad OM \perp CD$

Now, in right triangle OMC, $\quad OC^2 = CM^2 + OM^2$

So, from equation (1), (2) and (3)

$$\left(\frac{a}{4} + r\right)^2 = \left(\frac{a}{4}\right)^2 + \left(\frac{a}{2} - r\right)^2$$

$$\Rightarrow \quad \frac{a^2}{16} + r^2 + \frac{1}{2}ra = \frac{a^2}{16} + \frac{a^2}{4} + r^2 - ra$$

$$\Rightarrow \quad \frac{1}{2}ra + ra = \frac{a^2}{4}$$

$\Rightarrow \quad \frac{3}{2}ra = \frac{a^2}{4}$

$\Rightarrow \quad a(6r-a)=0$ but $a \neq 0$

$\therefore \quad 6r = a$, $r = \frac{1}{6}a$ or $r = \frac{1}{6}AB$ Hence proved

Exercise 13.2

1. According to figure 13.21 answer the following questions:

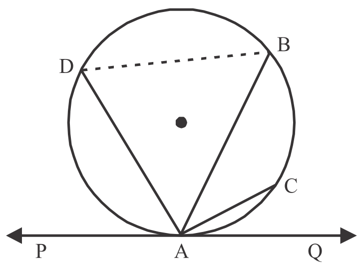

Fig. 13.21

(i) $\angle BAQ$ is an alternate segment of circle.

(ii) $\angle DAP$ is an alternate segment of circle.

(iii) If B is joined with C then $\angle ACB$ is equal to which angle?

(iv) $\angle ABD$ and $\angle ADB$ is equal to which angles.

2. According to figure 13.22 if $\angle BAC = 80^\circ$ then find the value of $\angle BCP$

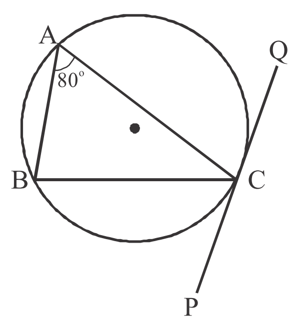

Fig. 13.22

3. According to figure 13.23, PQ and XY are parallel tangents. If $\angle QRT = 30^\circ$ then find the value of $\angle TSY$.

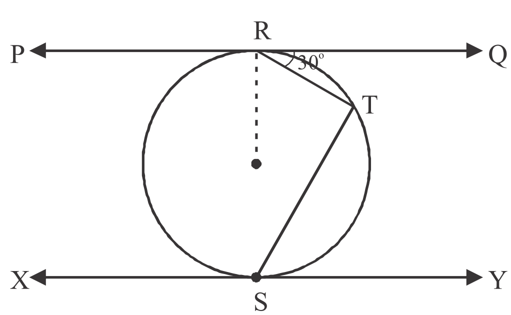

Fig. 13.23

4. Figure 13.24, in a cyclic quadrilateral ABCD diagonal AC bisects the angle C. Then prove that diagonal BD is parallel to the tangent PQ of a circle which passes through the points A, B, C and D.

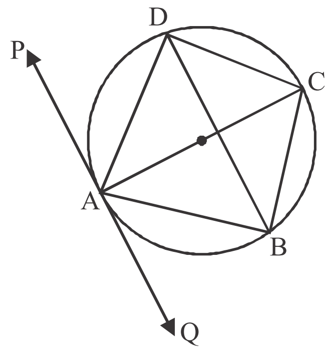

Fig. 13.24

Answer

Exercise 13.1

1. (i) **False :** Tangent intersects the cirlce at one and only one point .
   (ii) **False :** Because OP is perpendicular to the tangent and perpendicular is the smallest among the distances.
   (iii) **True :** Tangent is perpendicular to its diameter.
   (iv) **True :** Because AOB is a diameter and angle made on semi circle is always a right angle.
2. (i) one (ii) Secent (iii) Two (iv) Tangent
3. 8 cm 4. $2\sqrt{21}$ cm 5. 8 cm

Exercise 13.2

1. (i) ADB, (ii) ACBD, (iii) $\angle BAP$, (iv) $\angle DAP$ and $\angle BAQ$
2. 80° 3. 60°