Chapter 4

# INDEFINITE INTEGRALS. BASIC METHODS OF INTEGRATION

## § 4.1. Direct Integration and the Method of Expansion

*Direct integration* consists in using the following table of integrals:

(1) $\int u^n\,du = \frac{u^{n+1}}{n+1} + C \quad (n \neq -1)$;

(2) $\int \frac{du}{u} = \ln|u| + C$;

(3) $\int a^u\,du = \frac{1}{\ln a} a^u + C$; $\quad \int e^u\,du = e^u + C$;

(4) $\int \cos u\,du = \sin u + C$; $\quad \int \sin u\,du = -\cos u + C$;

(5) $\int \cosh u\,du = \sinh u + C$; $\quad \int \sinh u\,du = \cosh u + C$;

(6) $\int \frac{du}{\cos^2 u} = \tan u + C$; $\quad \int \frac{du}{\sin^2 u} = -\cot u + C$;

(7) $\int \frac{du}{u^2+a^2} = \frac{1}{a}\operatorname{arc\,tan}\frac{u}{a} + C = -\frac{1}{a}\operatorname{arc\,cot}\frac{u}{a} + C_1 \quad (a > 0)$;

(8) $\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a} + C = -\arccos\frac{u}{a} + C_1 \quad (a > 0)$;

(9) $\int \frac{du}{\sqrt{u^2 \pm a^2}} = \ln\left(u + \sqrt{u^2 \pm a^2}\right) + C$;

(10) $\int \frac{du}{u^2-a^2} = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + C$.

In all these formulas the variable $u$ is either an independent variable or a differentiable function of some variable. If

$$\int f(u)\,du = F(u) + C,$$

then

$$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + C.$$

*The method of expansion* consists in expanding the integrand into a linear combination of simpler functions and using the linearity property of the integral:

$$\int \sum_{i=1}^{n} a_i f_i(x)\,dx = \sum_{i=1}^{n} a_i \int f_i(x)\,dx \quad \left(\sum_{i=1}^{n} |a_i| > 0\right).$$

**4.1.1.** Find the integral $I = \int \frac{x^2+5x-1}{\sqrt{x}}\,dx.$

*Solution.*

$$I = \int \frac{x^2+5x-1}{\sqrt{x}}\,dx = \int (x^{3/2} + 5x^{1/2} - x^{-1/2})\,dx =$$
$$= \int x^{3/2}\,dx + 5\int x^{1/2}\,dx - \int x^{-1/2}\,dx =$$
$$= \frac{2x^{5/2}}{5} + C_1 + \frac{5\cdot 2}{3} x^{3/2} + C_2 - 2x^{1/2} + C_3 =$$
$$= 2\sqrt{x}\left(\frac{x^2}{5} + \frac{5x}{3} - 1\right) + C.$$

*Note.* There is no need to introduce an arbitrary constant after calculating each integral (as is done in the above example). By combining all arbitrary constants we get a single arbitrary constant, denoted by letter $C$, which is added to the final answer.

**4.1.2.** $I = \int \frac{6x^3+x^2-2x+1}{2x-1}\,dx.$

**4.1.3.** $I = \int \frac{dx}{\sin^2 x \cos^2 x}.$

*Solution.* Transform the integrand in the following way:

$$\frac{1}{\sin^2 x \cos^2 x} = \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} = \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}.$$

Hence,

$$I = \int \frac{dx}{\cos^2 x} + \int \frac{dx}{\sin^2 x} = \tan x - \cot x + C.$$

**4.1.4.** $I = \int \tan^2 x\,dx.$

*Solution.* Since $\tan^2 x = \sec^2 x - 1$, then

$$I = \int \tan^2 x\,dx = \int \frac{dx}{\cos^2 x} - \int dx = \tan x - x + C.$$

**4.1.5.** $I = \int (x^2+5)^3\,dx.$

*Solution.* Expanding the integrand by the binomial formula, we find

$$I = \int (x^6 + 15x^4 + 75x^2 + 125)\,dx = \frac{x^7}{7} + \frac{15x^5}{5} + \frac{75x^3}{3} + 125x + C.$$

**4.1.6.** $I = \int (3x+5)^{17}\,dx.$

*Solution.* Here it is not expedient to raise the binomial to the 17th power, since $u = 3x + 5$ is a linear function.

Proceeding from the tabular integral

$$\int u^{17}\,du = \frac{u^{18}}{18} + C,$$

we get

$$I = \frac{1}{3}\cdot\frac{(3x+5)^{18}}{18} + C.$$

**4.1.7.** $I = \int \frac{dx}{\sqrt{x+1} - \sqrt{x}}$.

**4.1.8.** $I = \int \cos(\pi x + 1)\,dx$.

*Solution.* Proceeding from the tabular integral (4)

$$\int \cos u\,du = \sin u + C,$$

we obtain

$$I = \frac{1}{\pi}\sin(\pi x + 1) + C.$$

**4.1.9.** $I = \int \cos 4x \cos 7x\,dx$.

*Solution.* When calculating such integrals it is advisable to use the trigonometric product formulas. Here

$$\cos 4x \cos 7x = \frac{1}{2}(\cos 3x + \cos 11x)$$

and therefore

$$I = \frac{1}{2}\int \cos 3x\,dx + \frac{1}{2}\int \cos 11x\,dx = \frac{1}{6}\sin 3x + \frac{1}{22}\sin 11x + C.$$

*Note.* When solving such problems it is expedient to use the following trigonometric identities:

$$\sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x];$$

$$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x];$$

$$\cos mx \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x].$$

**4.1.10.** $I = \int \cos x \cos 2x \cos 5x\,dx$.

*Solution.* We have

$$(\cos x \cos 2x)\cos 5x = \frac{1}{2}(\cos x + \cos 3x)\cos 5x =$$

$$= \frac{1}{4}[\cos 4x + \cos 6x] + \frac{1}{4}(\cos 2x + \cos 8x).$$

Thus,

$$I=\frac{1}{4}\left[\int \cos 2x\,dx+\int \cos 4x\,dx+\int \cos 6x\,dx+\int \cos 8x\,dx\right]=$$
$$=\frac{1}{8}\sin 2x+\frac{1}{16}\sin 4x+\frac{1}{24}\sin 6x+\frac{1}{32}\sin 8x+C.$$

**4.1.11.** $I=\int \sin^2 3x\,dx.$

*Solution.* Since $\sin^2 3x=\frac{1-\cos 6x}{2}$, then

$$I=\frac{1}{2}\int (1-\cos 6x)\,dx=\frac{1}{2}x-\frac{1}{12}\sin 6x+C.$$

**4.1.12.** $I=\int \cosh^2 (8x+5)\,dx.$

*Solution.* Since $\cosh^2 u=\frac{\cosh 2u+1}{2}$, then

$$I=\frac{1}{2}\int [1+\cosh (16x+10)]\,dx=\frac{1}{2}x+\frac{1}{32}\sinh (16x+10)+C.$$

**4.1.13.** $I=\int \frac{dx}{x^2+4x+5}.$

*Solution.* $I=\int \frac{dx}{x^2+4x+5}=\int \frac{dx}{(x+2)^2+1}=\arctan (x+2)+C.$

**4.1.14.** $I=\int \frac{dx}{4x^2+25}.$

**4.1.15.** $I=\int \frac{dx}{x^2+x+1}.$

**4.1.16.** $I=\int \frac{dx}{\sqrt{4-9x^2}}.$

*Solution.* $I=\int \frac{dx}{\sqrt{4-9x^2}}=\frac{1}{3}\int \frac{dx}{\sqrt{4/9-x^2}}=\frac{1}{3}\arcsin \frac{3x}{2}+C.$

**4.1.17.** $I=\int \frac{dx}{\sqrt{5-x^2-4x}}.$

*Solution.* $I=\int \frac{dx}{\sqrt{5-x^2-4x}}=\int \frac{dx}{\sqrt{9-(x+2)^2}}=\arcsin \frac{x+2}{3}+C.$

**4.1.18.** $I=\int \frac{dx}{\sqrt{x^2+6x+1}}.$

**4.1.19.** $I=\int \frac{dx}{4-x^2-4x}.$

*Solution.*

$$I=\int \frac{dx}{4-x^2-4x}=\int \frac{dx}{8-(x+2)^2}=\frac{1}{4\sqrt{2}}\ln\left|\frac{2\sqrt{2}+x+2}{2\sqrt{2}-(x+2)}\right|+C.$$

**4.1.20.** $I=\int \frac{dx}{10x^2-7}.$

**4.1.21.** Evaluate the following integrals:

(a) $\int \frac{dx}{x^2-6x+13}$; (b) $\int \frac{x-1}{\sqrt[3]{x^2}}\,dx$;

(c) $\int \frac{3-2\cot^2 x}{\cos^2 x}\,dx$; (d) $\int \frac{2+3x^2}{x^2(1+x^2)}\,dx$.

**4.1.22.** Integrate:

(a) $\int \frac{\sqrt{1-x^2}+\sqrt{1+x^2}}{\sqrt{1-x^4}}\,dx$; (b) $\int \frac{\cos 2x}{\cos x-\sin x}\,dx$;

(c) $\int \frac{2^{x+1}-5^{x-1}}{10^x}\,dx$; (d) $\int (\sin 5x-\sin 5\alpha)\,dx$.

## § 4.2. Integration by Substitution

The *method of substitution* (or *change of variable*) consists in substituting $\varphi(t)$ for $x$ where $\varphi(t)$ is a continuously differentiable function. On substituting we have:

$$\int f(x)\,dx=\int f[\varphi(t)]\,\varphi'(t)\,dt,$$

and after integration we return to the old variable by inverse substitution $t=\varphi^{-1}(x)$.

The indicated formula is also used in the reverse direction:

$$\int f[\varphi(t)]\,\varphi'(t)\,dt=\int f(x)\,dx, \text{ where } x=\varphi(t).$$

**4.2.1.** $I=\int x\sqrt{x-5}\,dx$.

*Solution.* Make the substitution

$$\sqrt{x-5}=t.$$

Whence

$$x-5=t^2,\quad x=t^2+5,\quad dx=2t\,dt.$$

Substituting into the integral we get

$$I=\int (t^2+5)\,t\cdot 2t\,dt=2\int (t^4+5t^2)\,dt=2\,\frac{t^5}{5}+\frac{10t^3}{3}+C.$$

Now return to the initial variable $x$:

$$I=\frac{2(x-5)^{5/2}}{5}+\frac{10(x-5)^{3/2}}{3}+C.$$

**4.2.2.** $I=\int \frac{dx}{1+e^x}$.

*Solution.* Let us make the substitution $1+e^x=t$. Whence

$$e^x=t-1,\ x=\ln(t-1),\quad dx=dt/(t-1).$$

Substituting into the integral, we get

$$I=\int \frac{dx}{1+e^x}=\int \frac{dt}{t(t-1)}.$$

But

$$\frac{1}{t(t-1)}=\frac{1}{t-1}-\frac{1}{t},$$

therefore

$$I=\int \frac{dt}{t-1}-\int \frac{dt}{t}=\ln|t-1|-\ln|t|+C.$$

Coming back to the variable $x$, we obtain

$$I=\ln \frac{e^x}{1+e^x}+C=x-\ln(1+e^x)+C.$$

*Note*. This integral can be calculated in a simpler way by multiplying both the numerator and denominator by $e^{-x}$:

$$\int \frac{e^{-x}}{e^{-x}+1}\,dx=-\int \frac{-e^{-x}}{e^{-x}+1}\,dx=-\ln(e^{-x}+1)+C=$$
$$=-\ln\frac{e^x+1}{e^x}=x-\ln(e^x+1)+C.$$

**4.2.3.** $I=\int \frac{x^2+3}{\sqrt{(2x-5)^3}}\,dx.$

**4.2.4.** $I=\int \frac{(x^2-1)\,dx}{(x^4+3x^2+1)\arctan\frac{x^2+1}{x}}.$

*Solution*. Transform the integrand

$$I=\int \frac{(1-1/x^2)\,dx}{[(x+1/x)^2+1]\arctan(x+1/x)}.$$

Make the substitution $x+\frac{1}{x}=t$; differentiating, we get

$$\left(1-\frac{1}{x^2}\right)dx=dt.$$

Whence

$$I=\int \frac{dt}{(t^2+1)\arctan t}.$$

Make one more substitution: $\arctan t=u$. Then

$$\frac{dt}{t^2+1}=du$$

and

$$I=\int \frac{du}{u}=\ln|u|+C.$$

Returning first to $t$, and then to $x$, we have

$$I=\ln|\arctan t|+C=\ln\left|\arctan\left(x+\frac{1}{x}\right)\right|+C.$$

**4.2.5.** $I=\int\frac{\sqrt{a^2-x^2}}{x^4}\,dx.$

*Solution.* Make the substitution:

$$x=\frac{1}{t};\quad dx=-\frac{dt}{t^2}.$$

Hence,

$$I=-\int\frac{\sqrt{a^2-1/t^2}}{(1/t^4)\,t^2}\,dt=-\int t\sqrt{a^2t^2-1}\,dt.$$

Now make one more substitution: $\sqrt{a^2t^2-1}=z$. Then $2a^2t\,dt=2z\,dz$ and

$$I=-\frac{1}{a^2}\int z^2\,dz=-\frac{1}{3a^2}z^3+C.$$

Returning to $t$ and then to $x$, we obtain

$$I=-\frac{(a^2-x^2)^{3/2}}{3a^2x^3}+C.$$

**4.2.6.** $I=\int\frac{dx}{a^2\sin^2x+b^2\cos^2x}.$

*Solution.*

$$I=\int\frac{dx}{a^2\sin^2x+b^2\cos^2x}=\frac{1}{b^2}\int\frac{1}{\frac{a^2}{b^2}\tan^2x+1}\cdot\frac{dx}{\cos^2x}.$$

Make the substitution $\frac{a}{b}\tan x=t$; $dt=\frac{a}{b}\frac{dx}{\cos^2x}$. Then

$$I=\frac{1}{ab}\int\frac{dt}{1+t^2}=\frac{1}{ab}\arctan t+C.$$

Returning to $x$, we obtain

$$I=\frac{1}{ab}\arctan\left(\frac{a}{b}\tan x\right)+C.$$

**4.2.7.** $I=\int\sqrt[3]{1+3\sin x}\cos x\,dx.$

*Solution.* Make the substitution $1+3\sin x=t$, $3\cos x\,dx=dt$. Then

$$I=\frac{1}{3}\int\sqrt[3]{t}\,dt=\frac{1}{3}\int t^{1/3}dt=\frac{1}{3}\cdot\frac{3}{4}t^{4/3}+C=\frac{(1+3\sin x)^{4/3}}{4}+C.$$

**4.2.8.** $I=\int\frac{\sin x\,dx}{\sqrt{\cos x}}.$

**4.2.9.** $I=\int \frac{dx}{(\arccos x)^5 \sqrt{1-x^2}}$.

*Solution.* Make the substitution: $\arccos x = t$; $-\frac{dx}{\sqrt{1-x^2}}=dt$. Then

$$I=-\int \frac{dt}{t^5}=-\int t^{-5}\,dt=\frac{1}{4}t^{-4}+C=\frac{1}{4\arccos^4 x}+C.$$

**4.2.10.** $I=\int \frac{x^2+1}{\sqrt[3]{x^3+3x+1}}\,dx$.

**4.2.11.** $I=\int \frac{\sin 2x}{1+\sin^2 x}\,dx$.

*Solution.* Make the substitution:

$$1+\sin^2 x = t;\ \ 2\sin x\cos x\,dx=\sin 2x\,dx=dt.$$

Then

$$I=\int \frac{dt}{t}=\ln t+C=\ln(1+\sin^2 x)+C.$$

**4.2.12.** $I=\int \frac{1+\ln x}{3+x\ln x}\,dx$.

*Solution.* Substitute

$$3+x\ln x=t,\qquad (1+\ln x)\,dx=dt$$

and get

$$I=\int \frac{dt}{t}=\ln|t|+C=\ln|3+x\ln x|+C.$$

**4.2.13.** Evaluate the following integrals:

(a) $\int \frac{\sqrt[3]{1+\ln x}}{x}\,dx$; (b) $\int \frac{dx}{x\ln x}$;

(c) $\int \frac{x\,dx}{\sqrt{3-x^4}}$; (d) $\int \frac{x^{n-1}}{x^{2n}+a^2}\,dx$;

(e) $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$; (f) $\int \left(\ln x+\frac{1}{\ln x}\right)\frac{dx}{x}$.

**4.2.14.** Find the following integrals:

(a) $\int x^2\sqrt[3]{1-x}\,dx$; (b) $\int \frac{\ln x\,dx}{x\sqrt{1+\ln x}}$;

(c) $\int \cos^5 x\sqrt{\sin x}\,dx$; (d) $\int \frac{x^5}{\sqrt{1-x^2}}\,dx$.

## § 4.3. Integration by Parts

The formula

$$\int u\,dv=uv-\int v\,du$$

is known as the formula for *integration by parts*, where $u$ and $v$ are differentiable functions of $x$.

To use this formula the integrand should be reduced to the product of two factors: one function and the differential of another function. If the integrand is the product of a logarithmic or an inverse trigonometric function and a polynomial, then $u$ is usually taken to be either the logarithmic or the inverse trigonometric function. But if the integrand is the product of a trigonometric or an exponential function and an algebraic one, then $u$ usually denotes the algebraic function.

**4.3.1.** $I = \int \arctan x \cdot dx.$

*Solution.* Let us put here

$$u = \arctan x, \qquad dv = dx,$$

whence

$$du = \frac{dx}{1+x^2}; \qquad v = x;$$

$$I = \int \arctan x\, dx = x \arctan x - \int \frac{x\, dx}{1+x^2} = x \arctan x - \frac{1}{2} \ln(1+x^2) + C.$$

**4.3.2.** $I = \int \arcsin x\, dx.$

**4.3.3.** $I = \int x \cos x\, dx.$

*Solution.* Let us put

$$u = x; \qquad dv = \cos x\, dx,$$

whence

$$du = dx; \qquad v = \sin x,$$

$$I = \int x \cos x\, dx = x \sin x - \int \sin x\, dx = x \sin x + \cos x + C.$$

We will show now what would result from an unsuitable choice of the multipliers $u$ and $dv$.

In the integral $\int x \cos x\, dx$ let us put

$$u = \cos x; \qquad dv = x\, dx,$$

whence

$$du = -\sin x\, dx; \qquad v = \frac{1}{2} x^2.$$

In this case

$$I = \frac{1}{2} x^2 \cos x + \frac{1}{2} \int x^2 \sin x\, dx.$$

As is obvious, the integral has become more complicated.

**4.3.4.** $I = \int x^3 \ln x\, dx.$

*Solution.* Let us put

$$u=\ln x;\quad dv=x^3\,dx,$$

whence

$$du=\frac{dx}{x};\quad v=\frac{1}{4}x^4,$$

$$I=\frac{1}{4}x^4\ln x-\frac{1}{4}\int x^4\frac{dx}{x}=\frac{1}{4}x^4\ln x-\frac{1}{4}\int x^3\,dx=\frac{1}{4}x^4\ln x-\frac{1}{16}x^4+C.$$

**4.3.5.** $I=\int (x^2-2x+5)\,e^{-x}\,dx.$

*Solution.* Let us put

$$u=x^2-2x+5;\quad dv=e^{-x}\,dx,$$

whence

$$du=(2x-2)\,dx;\quad v=-e^{-x};$$

$$I=\int (x^2-2x+5)\,e^{-x}\,dx=-e^{-x}(x^2-2x+5)+2\int (x-1)e^{-x}\,dx.$$

We again integrate the last integral by parts. Put

$$x-1=u;\qquad dv=e^{-x}\,dx,$$

whence

$$du=dx;\ v=-e^{-x}.$$

$$I_1=2\int (x-1)\,e^{-x}\,dx=-2e^{-x}(x-1)+2\int e^{-x}\,dx=-2xe^{-x}+C.$$

Finally we get

$$I=-e^{-x}(x^2-2x+5)-2xe^{-x}+C=-e^{-x}(x^2+5)+C.$$

*Note.* As a result of calculation of integrals of the form $\int P(x)\,e^{ax}\,dx$ we obtain a function of the form $Q(x)\,e^{ax}$, where $Q(x)$ is a polynomial of the same degree as the polynomial $P(x)$.

This circumstance allows us to calculate the integrals of the indicated type using the method of indefinite coefficients, the essence of which is explained by the following example.

**4.3.6.** Applying the method of indefinite coefficients, evaluate

$$I=\int (3x^3-17)\,e^{2x}\,dx.$$

*Solution.* $\int (3x^3-17)\,e^{2x}\,dx=(Ax^3+Bx^2+Dx+E)\,e^{2x}+C.$

Differentiating the right and the left sides, we obtain

$$(3x^3-17)\,e^{2x}=2\,(Ax^3+Bx^2+Dx+E)\,e^{2x}+e^{2x}(3Ax^2+2Bx+D).$$

Cancelling $e^{2x}$, we have

$$3x^3-17\equiv 2Ax^3+(2B+3A)\,x^2+(2D+2B)\,x+(2E+D).$$

Equating the coefficients at the equal powers of $x$ in the left and right sides of this identity, we get

$$3=2A; \qquad 0=2B+3A;$$
$$0=2D+2B; \qquad -17=2E+D.$$

Solving the system, we obtain

$$A=\frac{3}{2}; \quad B=-\frac{9}{4}; \quad D=\frac{9}{4}; \quad E=-\frac{77}{8}.$$

Hence,

$$\int (3x^3-17)\, e^{2x}\, dx = \left(\frac{3}{2}x^3-\frac{9}{4}x^2+\frac{9}{4}x-\frac{77}{8}\right) e^{2x}+C.$$

**4.3.7.** Integrate:

$$I=\int (x^3+1)\cos x\, dx.$$

*Solution.* Let us put

$$u=x^3+1; \quad dv=\cos x\, dx,$$

whence

$$du=3x^2\, dx; \quad v=\sin x.$$

$$I=(x^3+1)\sin x-3\int x^2 \sin x\, dx=(x^3+1)\sin x-3I_1,$$

where $I_1=\int x^2 \sin x\, dx$.

Integrating by parts again, we get

$$I_1=-x^2\cos x+2I_2,$$

where $I_2=\int x\cos x\, dx$.

Integrating by parts again, we obtain

$$I_2=x\sin x+\cos x+C.$$

Finally, we have:

$$I=\int (x^3+1)\cos x\, dx=(x^3+1)\sin x+3x^2\cos x-6x\sin x-6\cos x+C=$$
$$=(x^3-6x+1)\sin x+(3x^2-6)\cos x+C.$$

*Note.* The method of indefinite coefficients may also be applied to integrals of the form

$$\int P(x)\sin ax\, dx, \quad \int P(x)\cos ax\, dx.$$

**4.3.8.** $I=\int (x^2+3x+5)\cos 2x\, dx.$

*Solution.* Let us put

$$\int (x^2+3x+5)\cos 2x\, dx=$$
$$=(A_0x^2+A_1x+A_2)\cos 2x+(B_0x^2+B_1x+B_2)\sin 2x+C.$$

Differentiate both sides of the identity:

$$(x^2+3x+5)\cos 2x = -2(A_0x^2+A_1x+A_2)\sin 2x +$$
$$+(2A_0x+A_1)\cos 2x+2(B_0x^2+B_1x+B_2)\cos 2x+(2B_0x+B_1)\sin 2x=$$
$$=[2B_0x^2+(2B_1+2A_0)x+(A_1+2B_2)]\cos 2x+$$
$$+[-2A_0x^2+(2B_0-2A_1)x+(B_1-2A_2)]\sin 2x.$$

Equating the coefficients at equal powers of $x$ in the multipliers $\cos 2x$ and $\sin 2x$, we get a system of equations:

$$2B_0=1; \qquad 2(B_1+A_0)=3; \qquad A_1+2B_2=5;$$
$$-2A_0=0; \qquad 2(B_0-A_1)=0; \qquad B_1-2A_2=0.$$

Solving the system, we find

$$A_0=0;\ B_0=\frac{1}{2};\ A_1=\frac{1}{2};\ B_1=\frac{3}{2};\ A_2=\frac{3}{4};\ B_2=\frac{9}{4}.$$

Thus,

$$\int (x^2+3x+5)\cos 2x\,dx=\left(\frac{x}{2}+\frac{3}{4}\right)\cos 2x+\left(\frac{1}{2}x^2+\frac{3}{2}x+\frac{9}{4}\right)\sin^2 x+C.$$

**4.3.9.** $I=\int (3x^2+6x+5)\arctan x\,dx.$

*Solution.* Let us put

$$u=\arctan x; \qquad dv=(3x^2+6x+5)\,dx,$$

whence

$$du=\frac{dx}{1+x^2}; \quad v=x^3+3x^2+5x.$$

Hence,

$$I=(x^3+3x^2+5x)\arctan x-\int\frac{x^3+3x^2+5x}{1+x^2}dx.$$

Single out the integral part under the last integral by dividing the numerator by the denominator:

$$I_1=\int\frac{x^3+3x^2+5x}{1+x^2}dx=\int (x+3)\,dx+\int\frac{4x-3}{x^2+1}dx=$$
$$=\frac{x^2}{2}+3x+2\int\frac{2x\,dx}{x^2+1}-3\int\frac{dx}{x^2+1}=\frac{x^2}{2}+3x+2\ln(x^2+1)-3\arctan x+C.$$

Substituting the value of $I_1$, we finally get

$$I=(x^3+3x^2+5x+3)\arctan x-x^2/2-3x-2\ln(x^2+1)+C.$$

**4.3.10.** Find the integral

$$I=\int e^{5x}\cos 4x\,dx.$$

*Solution.* Let us put

$$e^{5x}=u; \quad \cos 4x\,dx=dv,$$

whence

$$5e^{5x}\,dx = du;\quad v = \frac{1}{4}\sin 4x.$$

Hence,

$$I = \frac{1}{4}e^{5x}\sin 4x - \frac{5}{4}\int e^{5x}\sin 4x\,dx.$$

Integrating by parts again, we obtain

$$I_1 = \int e^{5x}\sin 4x\,dx = -\frac{1}{4}e^{5x}\cos 4x + \frac{5}{4}\int e^{5x}\cos 4x\,dx.$$

Thus,

$$I = \frac{1}{4}e^{5x}\sin 4x - \frac{5}{4}\left(-\frac{1}{4}e^{5x}\cos 4x + \frac{5}{4}\int e^{5x}\cos 4x\,dx\right),$$

i. e.

$$I = \frac{1}{4}e^{5x}\left(\sin 4x + \frac{5}{4}\cos 4x\right) - \frac{25}{16}I.$$

Whence

$$I = \frac{4}{41}e^{5x}\left(\sin 4x + \frac{5}{4}\cos 4x\right) + C.$$

**4.3.11.** $I = \int \cos(\ln x)\,dx.$

*Solution.* Let us put

$$u = \cos(\ln x);\quad dv = dx,$$

whence

$$du = -\sin(\ln x)\frac{dx}{x};\quad v = x.$$

Hence,

$$I = \int \cos(\ln x)\,dx = x\cos(\ln x) + \int \sin(\ln x)\,dx.$$

Integrate by parts once again

$$u = \sin(\ln x);\qquad dv = dx,$$

whence

$$du = \cos(\ln x)\frac{dx}{x};\quad v = x.$$

Hence,

$$I_1 = \int \sin(\ln x)\,dx = x\sin(\ln x) - \int \cos(\ln x)\,dx.$$

Thus

$$I = \int \cos(\ln x)\,dx = x\cos(\ln x) + x\sin(\ln x) - I.$$

Hence

$$I = \frac{x}{2}[\cos(\ln x) + \sin(\ln x)] + C.$$

**4.3.12.** $I=\int x\ln\left(1+\frac{1}{x}\right)dx.$

*Solution.* Let us transform the integrand

$$\ln\left(1+\frac{1}{x}\right)=\ln\frac{x+1}{x}=\ln(x+1)-\ln x.$$

Hence

$$I=\int x\ln(x+1)\,dx-\int x\ln x\,dx=I_1-I_2.$$

Let us integrate $I_1$ and $I_2$ by parts. Put

$$u=\ln(x+1);\quad dv=x\,dx,$$

whence

$$du=\frac{dx}{1+x};\quad v=\frac{1}{2}(x^2-1).$$

Hence

$$I_1=\int x\ln(x+1)\,dx=\frac{1}{2}(x^2-1)\ln(x+1)-\frac{1}{2}\int\frac{(x^2-1)\,dx}{1+x}=$$
$$=\frac{x^2-1}{2}\ln(x+1)-\frac{1}{2}\int(x-1)\,dx=\frac{x^2-1}{2}\ln(x+1)-\frac{1}{4}x^2+\frac{1}{2}x+C.$$

Analogously,

$$I_2=\int x\ln x\,dx=\frac{x^2}{2}\ln x-\frac{1}{4}x^2+C.$$

Finally we have

$$I=\int x\ln\left(1+\frac{1}{x}\right)dx=\frac{1}{2}(x^2-1)\ln(x+1)-\frac{x^2}{2}\ln x+\frac{x}{2}+C.$$

**4.3.13.** $I=\int\frac{\sqrt{x^2+1}\,[\ln(x^2+1)-2\ln x]}{x^4}dx.$

*Solution.* First apply the substitution

$$1+\frac{1}{x^2}=t.$$

Then

$$dt=-\frac{2dx}{x^3}\quad\text{or}\quad\frac{dx}{x^3}=-\frac{1}{2}dt.$$

Hence,

$$I=\int\sqrt{1+\frac{1}{x^2}}\ln\frac{x^2+1}{x^2}\cdot\frac{dx}{x^3}=-\frac{1}{2}\int\sqrt{t}\ln t\,dt.$$

The obtained integral is easily evaluated by parts. Let us put

$$u=\ln t;\quad dv=\sqrt{t}\,dt.$$

Then

$$du=\frac{dt}{t};\quad v=\frac{2}{3}t\sqrt{t}.$$

Whence

$$-\frac{1}{2}\int \sqrt{t}\ln t\,dt = -\frac{1}{2}\left[\frac{2}{3}t\sqrt{t}\ln t - \frac{2}{3}\int \sqrt{t}\,dt\right] =$$
$$= -\frac{1}{2}\left[\frac{2}{3}t\sqrt{t}\ln t - \frac{4}{9}t\sqrt{t}\right] + C.$$

Returning to $x$, we obtain

$$I = -\frac{1}{2}\left[\frac{2}{3}\left(1+\frac{1}{x^2}\right)^{3/2}\ln\left(1+\frac{1}{x^2}\right) - \frac{4}{9}\left(1+\frac{1}{x^2}\right)^{3/2}\right] + C =$$
$$= \frac{(x^2+1)\sqrt{x^2+1}}{9x^3}\left[2 - 3\ln\left(1+\frac{1}{x^2}\right)\right] + C.$$

**4.3.14.** $I = \int \sin x \ln \tan x\,dx.$

**4.3.15.** $I = \int \ln(\sqrt{1-x}+\sqrt{1+x})\,dx.$

*Solution.* Let us put

$$u = \ln(\sqrt{1-x}+\sqrt{1+x}); \quad dv = dx,$$

whence

$$du = \frac{1}{\sqrt{1-x}+\sqrt{1+x}}\left(-\frac{1}{2\sqrt{1-x}} + \frac{1}{2\sqrt{1+x}}\right)dx =$$
$$= \frac{1}{2}\cdot\frac{\sqrt{1-x}-\sqrt{1+x}}{\sqrt{1-x}+\sqrt{1+x}}\cdot\frac{dx}{\sqrt{1-x^2}} = \frac{1}{2}\cdot\frac{\sqrt{1-x^2}-1}{x\sqrt{1-x^2}}\,dx;$$
$$v = x.$$

Hence,

$$I = x\ln(\sqrt{1-x}+\sqrt{1+x}) - \frac{1}{2}\int x\frac{\sqrt{1-x^2}-1}{x\sqrt{1-x^2}}\,dx =$$
$$= x\ln(\sqrt{1-x}+\sqrt{1+x}) - \frac{1}{2}\int dx + \frac{1}{2}\int\frac{dx}{\sqrt{1-x^2}} =$$
$$= x\ln(\sqrt{1-x}+\sqrt{1+x}) - \frac{1}{2}x + \frac{1}{2}\arcsin x + C.$$

*Note.* In calculating a number of integrals we had to use the method of integration by parts several times in succession. The result could be obtained more rapidly and in a more concise form by using the so-called *generalized formula for integration by parts* (or the *formula for multiple integration by parts*):

$$\int u(x)v(x)\,dx = u(x)v_1(x) - u'(x)v_2(x) + u''(x)v_3(x) - \ldots$$
$$\ldots + (-1)^{n-1}u^{(n-1)}(x)v_n(x) - (-1)^{n-1}\int u^{(n)}(x)v_n(x)\,dx,$$

where

$$v_1(x) = \int v(x)\,dx;\quad v_2(x) = \int v_1(x)\,dx;\ \ldots;\ v_n(x) = \int v_{n-1}(x)\,dx.$$

Here, of course, we assume that all derivatives and integrals appearing in this formula exist.

The use of the generalized formula for integration by parts is especially advantageous when calculating the integral $\int P_n(x)\varphi(x)\,dx$, where $P_n(x)$ is a polynomial of degree $n$, and the factor $\varphi(x)$ is such that it can be integrated successively $n+1$ times. For example,

$$\int P_n(x)e^{kx}\,dx = P_n(x)\frac{e^{kx}}{k} - P_n'(x)\frac{e^{kx}}{k^2} + \ldots + (-1)^n P_n^{(n)}(x)\frac{e^{kx}}{k^{n+1}} + C = e^{kx}\left[\frac{P_n(x)}{k} - \frac{1}{k^2}P_n'(x) + \ldots + \frac{(-1)^n}{k^{n+1}}P_n^{(n)}(x)\right] + C.$$

**4.3.16.** Applying the generalized formula for integration by parts, find the following integrals:

(a) $\int (x^3-2x^2+3x-1)\cos 2x\,dx$,

(b) $\int (2x^3+3x^2-8x+1)\sqrt{2x+6}\,dx$.

*Solution.*

(a) $$\int (x^3-2x^2+3x-1)\cos 2x\,dx = (x^3-2x^2+3x-1)\frac{\sin 2x}{2} - (3x^2-4x+3)\left(-\frac{\cos 2x}{4}\right) + (6x-4)\left(-\frac{\sin 2x}{8}\right) - 6\frac{\cos 2x}{16} + C = \frac{\sin 2x}{4}(2x^3-4x^2+3x) + \frac{\cos 2x}{8}(6x^2-8x+3) + C;$$

(b) $$\int (2x^3+3x^2-8x+1)\sqrt{2x+6}\,dx = (2x^3+3x^2-8x+1)\frac{(2x+6)^{3/2}}{3} - (6x^2+6x-8)\frac{(2x+6)^{5/2}}{3\cdot 5} + (12x+6)\frac{(2x+6)^{7/2}}{3\cdot 5\cdot 7} - 12\frac{(2x+6)^{9/2}}{3\cdot 5\cdot 7\cdot 9} + C = \frac{\sqrt{2x+6}}{5\cdot 7\cdot 9}(2x+6)(70x^3-45x^2-396x+897) + C.$$

Evaluate the following integrals:

**4.3.17.** $\int \ln(x+\sqrt{1+x^2})\,dx$.

**4.3.18.** $\int \sqrt[3]{x}(\ln x)^2\,dx$.

**4.3.19.** $\int \frac{\arcsin x\,dx}{\sqrt{1+x}}$.

**4.3.20.** $\int \frac{x\cos x\,dx}{\sin^3 x}$.

**4.3.21.** $\int 3^x \cos x\,dx$.

**4.3.22.** $\int (x^3 - 2x^2 + 5) e^{3x}\, dx.$

**4.3.23.** $\int (1 + x^2)^2 \cos x\, dx.$

**4.3.24.** $\int (x^2 + 2x - 1) \sin 3x\, dx.$

**4.3.25.** $\int (x^2 - 2x + 3) \ln x\, dx.$

**4.3.26.** $\int x^3 \operatorname{arc\,tan} x\, dx.$

**4.3.27.** $\int x^2 \operatorname{arc\,cos} x\, dx.$

**4.3.28.** Applying the formula for multiple integration by parts, calculate the following integrals:

(a) $\int (3x^2 + x - 2) \sin^2 (3x + 1)\, dx$; (b) $\int \frac{x^2 - 7x + 1}{\sqrt[3]{2x+1}}\, dx.$

## § 4.4. Reduction Formulas

Reduction formulas make it possible to reduce an integral depending on the index $n > 0$, called the order of the integral, to an integral of the same type with a smaller index.

**4.4.1.** Integrating by parts, derive reduction formulas for calculating the following integrals:

(a) $I_n = \int \frac{dx}{(x^2 + a^2)^n}$; (b) $I_{n,-m} = \int \frac{\sin^n x}{\cos^m x}\, dx$;

(c) $I_n = \int (a^2 - x^2)^n\, dx.$

*Solution.* (a) We integrate by parts. Let us put

$$u = \frac{1}{(x^2 + a^2)^n}, \quad dv = dx,$$

whence

$$du = -\frac{2n\, x\, dx}{(x^2 + a^2)^{n+1}}, \quad v = x.$$

Hence,

$$I_n = \frac{x}{(x^2 + a^2)^n} + 2n \int \frac{x^2}{(x^2 + a^2)^{n+1}}\, dx =$$

$$= \frac{x}{(x^2 + a^2)^n} + 2n \int \frac{(x^2 + a^2) - a^2}{(x^2 + a^2)^{n+1}}\, dx = \frac{x}{(x^2 + a^2)^n} + 2nI_n - 2na^2 I_{n+1},$$

whence

$$I_{n+1} = \frac{1}{2na^2} \cdot \frac{x}{(x^2 + a^2)^n} + \frac{2n-1}{2n} \cdot \frac{1}{a^2} I_n.$$

The obtained formula reduces the calculation of the integral $I_{n+1}$ to the calculation of the integral $I_n$ and, consequently, allows us to calculate completely an integral with a natural index, since

$$I_1=\int \frac{dx}{x^2+a^2}=\frac{1}{a}\arctan\frac{x}{a}+C.$$

For instance, putting $n=1$, we obtain

$$I_2=\int \frac{dx}{(x^2+a^2)^2}=\frac{1}{2a^2}\cdot\frac{x}{x^2+a^2}+\frac{1}{2a^2}I_1=\frac{1}{2a^2}\frac{x}{x^2+a^2}+\frac{1}{2a^3}\arctan\frac{x}{a}+C;$$

putting $n=2$, we get

$$I_3=\int \frac{dx}{(x^2+a^2)^3}=\frac{1}{4a^2}\cdot\frac{x}{(x^2+a^2)^2}+\frac{3}{4a^2}I_2=$$

$$=\frac{1}{4a^2}\cdot\frac{x}{(x^2+a^2)^2}+\frac{3}{8a^4}\cdot\frac{x}{x^2+a^2}+\frac{3}{8a^5}\arctan\frac{x}{a}+C.$$

(b) Let us apply the method of integration by parts, putting

$$u=\sin^{n-1}x;\quad dv=\frac{\sin x}{\cos^m x}dx,$$

whence

$$du=(n-1)\sin^{n-2}x\cos x\,dx;\ v=\frac{1}{(m-1)\cos^{m-1}x}\quad (m\neq 1).$$

Hence,

$$I_{n,\,-m}=\frac{\sin^{n-1}x}{(m-1)\cos^{m-1}x}-\frac{n-1}{m-1}\int\frac{\sin^{n-2}x\,dx}{\cos^{m-2}x}=$$

$$=\frac{\sin^{n-1}x}{(m-1)\cos^{m-1}x}-\frac{n-1}{m-1}I_{n-2,\,2-m}\quad (m\neq 1).$$

(c) Integrate by parts, putting

$$u=(a^2-x^2)^n;\quad dv=dx,$$

whence

$$du=-2nx(a^2-x^2)^{n-1}dx;\quad v=x.$$

Hence

$$I_n=x(a^2-x^2)^n+2n\int x^2(a^2-x^2)^{n-1}dx=$$

$$=x(a^2-x^2)^n+2n\int (x^2-a^2+a^2)(a^2-x^2)^{n-1}dx=$$

$$=x(a^2-x^2)^n-2nI_n+2na^2I_{n-1}.$$

Wherefrom, reducing the similar terms, we obtain

$$(1+2n)I_n=x(a^2-x^2)^n+2na^2I_{n-1}.$$

Hence,

$$I_n=\frac{x(a^2-x^2)^n}{2n+1}+\frac{2na^2}{2n+1}I_{n-1}.$$

For instance, noting that

$$I_{-1/2}=\int \frac{dx}{\sqrt{a^2-x^2}}=\arcsin\frac{x}{a}+C,$$

we can find successively

$$I_{1/2}=\int \sqrt{a^2-x^2}\,dx=\frac{x}{2}(a^2-x^2)^{1/2}+\frac{a^2}{2}I_{-1/2}=$$
$$=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\arcsin\frac{x}{a}+C,$$

$$I_{3/2}=\int (a^2-x^2)^{3/2}\,dx=\frac{x}{4}(a^2-x^2)^{3/2}+\frac{3}{4}a^2 I_{1/2},$$ and so on.

**4.4.2.** Applying integration by parts, derive the following reduction formulas:

(a) $I_n=\int (\ln x)^n\,dx=x(\ln x)^n-nI_{n-1}$;

(b) $I_n=\int x^{\alpha}(\ln x)^n\,dx=\frac{x^{\alpha+1}(\ln x)^n}{\alpha+1}-\frac{n}{\alpha+1}I_{n-1}\quad(\alpha\neq-1)$;

(c) $I_n=\int x^n e^x\,dx=x^n e^x-nI_{n-1}$;

(d) $I_n=\int e^{\alpha x}\sin^n x\,dx=$

$$=\frac{e^{\alpha x}}{\alpha^2+n^2}\sin^{n-1}x(\alpha\sin x-n\cos x)+\frac{n(n-1)}{\alpha^2+n^2}I_{n-2}.$$

**4.4.3.** Derive the reduction formula for the integration of $I_n=\int\frac{dx}{\sin^n x}$ and use it for calculating the integral $I_3=\int\frac{dx}{\sin^3 x}$.

**4.4.4.** Derive the reduction formulas for the following integrals:

(a) $I_n=\int \tan^n x\,dx$; (b) $I_n=\int \cot^n x\,dx$;

(c) $I_n=\int\frac{x^n\,dx}{\sqrt{x^2+a}}$.