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# Differential Equations

## 12.01 Introduction

Most of the problems in science and engineering are solved by finding how one quantitiy is related or depends upon one or more quantities. In many problems, it is easier to find a relation between the rate of changes in the variables than between the variables themselves. The study of this relationship gives rise to differential equations. Therefore, an equation involving dependent variable, independent variable and derivative of the dependent variable with respect to independent variable is called a differential equation.

Differential equations which involve only one independant variable are called ordinary differential equations. If the differential equation involves more than one independents variable, then it is called a partial different equations. Here we shall confine ourselves to the study of ordinary differential equations only. Now onward, we will use the term 'differential equation' for ordinary differential equation.

For example : $\frac{dy}{dx} = x^2 y, \ \frac{d^2 y}{dx^2} - 5\frac{dy}{dx} + 6y = \sin x$ ,

Where $x$ is independent variable and $y$ is dependent variable.

## 12.02 Order and Degree of a Differential Equation

**Order of differential equation:** Order of a differential equation is defined as the order of the highest order derivative of the dependent variable with respect to the independent variable involved into the given differential equation.

**For example :**

(i) Differential equation $\frac{dy}{dx} = e^x$ is the order one because in this equation the dependent variable $y$ has maximum one differentiation.

(ii) Differential equation $x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + 2y = \sin\theta$, because in this equation the dependent variable $y$ has maximum two times differentiation.

(iii) Differential equation $\left(\frac{dy}{dx}\right)^3 + \frac{dy}{dx} + 3y = 0$ the of order one because the dependent variable $y$ has maximum one differentiation.

**Degree of a Differential Equation :**

The degree of a differential equation is the degree of the highest order derivatives, when differential cofficients are made free from redicals and fractions.

(i) The degree of $\left(\frac{d^3 y}{dx^3}\right)^2 + \frac{dy}{dx} - 3y = 0$ is two because the highest order derivative is $\frac{d^3 y}{dx^3}$ whose power is 2.

(ii) The degree of $\frac{d^2y}{dx^2}+\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{2/3}=0$ is three, because on rationalization it becomes

$\left(\frac{d^2y}{dx^2}\right)^3=-\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^2$ and the power of highest derivative is 3.

(iii) The degree of differential equation $\frac{dy}{dx}=\frac{x^2+y^2}{xy}$ is one.

**Remark :** Order and degree (if defined) of a differential equation are always positive integrals.

## Illustrative Examples

**Example 1.** Find the order and degree of following differential equations.

(i) $\frac{dy}{dx}-\cos x=0$ (ii) $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=e^x$ (iii) $\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^4=\cos x$

(iv) $y=x\frac{dy}{dx}+\frac{a^2}{dy/dx}$ (v) $\frac{d^4y}{dx^4}+\sin\left(\frac{d^3y}{dx^3}\right)=0$

**Solution :**

(i) The highest order derivative of $y$ in this differential equation is $\frac{dy}{dx}$ so its order is 1 and the highest power of $\frac{dy}{dx}$ is 1, so its degree is 1.

(ii) The highest order derivative of $y$ in the given differential equation is $\frac{d^2y}{dx^2}$ so its order is 2 and the highest power of $\frac{d^2y}{dx^2}$ is 1, so its degree is 1.

(iii) The highest order derivative of $y$ in the given differential equation is $\frac{d^2y}{dx^2}$, so its order is 2 and the highest power of $\frac{d^2y}{dx^2}$ is one, so its degree is 1.

(iv) On simplification we see that the given differential equation is $x\left(\frac{dy}{dx}\right)^2+a^2=y\frac{dy}{dx}$, hence order is 1 and degree is 2.

(v) The highest derivative of $y$ in the given differential equation is $\frac{d^4y}{dx^4}$, so its order is 4, also the given differential equation is not a polynomial in context with diffrential coefficients. So the degree of equation is not defined.

## Exercise 12.1

**Find the order and degree of following differential equations.**

1. $\frac{dy}{dx} = \sin 2x + \cos 2x$

2. $\frac{d^2 y}{dx^2} = \sin x + \cos x$

3. $\left(\frac{d^2 y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right) = 0$

4. $\left(\frac{dy}{dx}\right)^3 + \frac{1}{dy/dx} = 2$

5. $a\frac{d^2 y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}$

6. $xdx + ydy = 0$

7. $\left(\frac{d^2 y}{dx^2}\right)^3 + y\left(\frac{dy}{dx}\right)^2 + y^5 = 0$

8. $x\frac{dy}{dx} + \frac{3}{(dy/dx)} = y^2$

## 12.03 Formation of differential equation

If the given family f of curves depends on only one constant parameter then it is represented by an equation of the form

$$f(x,\ y,\ a) = 0 \tag{1}$$

Differentiating equatioin (1) with respect to $x$

$$\phi(x,\ y,\ y',\ a) = 0 \qquad \left[\text{where } y' = \frac{dy}{dx}\right] \tag{2}$$

The required differential equation is then obtained by eliminating a from equation (1) and (2) as

$$f(x,\ y,\ y') = 0$$

This is called the required differential equation of family of curves. Similarly if the given equation has two arbitrary constants then differentiatign twice and by eliminating the arbitrary constants, we get the equation of family of curves.

## Illustrative Examples

**Example 2.** Find the differential equation of family of straight lines which passes through orgin.

**Solution :** The equation of straight line passing through origin is

$$y = mx, \text{ where } m \text{ is arbitary.} \tag{1}$$

On differentiating equation (1)

$$\frac{dy}{dx} = m \tag{2}$$

On eliminating $m$ from (1) and (2)

$$x\frac{dy}{dx} = y \text{, which is the required differential equation.}$$

**Example 3.** Find the differential equation of family of $y = ae^{2x} + be^{-x}$

**Solution :**
$$y = ae^{2x} + be^{-x} \tag{1}$$

Differentiating eq. (1) with respect to $x$

$$\frac{dy}{dx} = 2ae^{2x} - be^{-x} \tag{2}$$

Again differentiating

$$\frac{d^2y}{dx^2} = 4ae^{2x} + be^{-x} \tag{3}$$

From (2) and (3)

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = 2ae^{2x} + 2be^{-x} = 2\left(ae^{2x} + be^{-x}\right)$$

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = 2y. \qquad \text{(From eq. (1))}$$

This is the required differential equation.

**Example 4.** Find the differential equation of family of curves for $y = e^x\left[A\sin x + B\cos x\right]$

**Solution :** $y = e^x\left[A\sin x + B\cos x\right]$ Differentiatign with respect to $x$ (1)

$$\frac{dy}{dx} = e^x[A\sin x + B\cos x] + e^x[A\cos x - B\sin x]$$

$$\Rightarrow \quad \frac{dy}{dx} = y + e^x[A\cos x - B\sin x] \tag{2}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{dy}{dx} + e^x\left[A\cos x - B\sin x\right] + e^x\left[-A\sin x - B\cos x\right]$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} - \frac{dy}{dx} = \frac{dy}{dx} - y - y \qquad \text{(From (2))}$$

or
$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0 .$$

This is the required differential equation.

## Exercise 12.2

1. Find the differential equation of family of curves for $y = ax + \frac{b}{x}$.

2. Find the differential equation of family of curves for $x^2 + y^2 = a^2$.

3. Find the differential equation of family of curves for $y = Ae^{3x} + Be^{5x}$ .

4. Find the differential equation of family of curves for $y = e^x\left[A\cos x + B\sin x\right]$.

5. Find the differential equation of family of curves for $y = a\cos\left(x+b\right)$, where a and b are arbitrary variables.

## 12.04 Solution of a Differential Equation

The solution to the differential equation used in the equation refers to a relationship in the independent and dependent variables which does not contain any differential coefficient and the given differential equation is satisfied for derivative obtained.

The solution of a differential equation is also called its primitive because the differential equation is a relation derived from it.

## General, particular and singular solution

**(i)** **General solution :** In the solution of a differential equation if number of arbitrary constant are equal to the order of it then that solution is called general solution. This is also called total solution or total integral or total primitive.

**For Example :** $y = A\cos x + B\sin x$ is a general solution of differential equation $\frac{d^2y}{dx^2} + y = 0$ because arbitrary variables present in the solution are equal to the order 2 of the equation.

**(ii)** **Particular solution :** The solution of a differential equation obtained by assigning particular values of the arbitrary constants in the general solution is called 'particular solution'.

**For Example :** $y = 3\cos x + 2\sin x$ is a particular solution of differential equation $\frac{d^2y}{dx^2} + y = 0$

**(iii)** **Singular solution :** Singular solutions of a differential equation are those where arbitrary constants are not present and fails to have a particular solution of general solution.

**Remark :** Singular solution is not there in syllabus. Hence we will not discuss it here in detail.

## Illustrative Examples

**Example 5:** Prove that $y = cx + \frac{a}{c}$ is a solution of differential equation $y = x\frac{dy}{dx} + \frac{a}{dy/dx}$.

**Solution :** Given equation is $y = cx + (a/c)$. (1)

differentiating with respect to $x$

$$\frac{dy}{dx} = c \qquad (2)$$

On eliminating $c$ from (1) and (2)

$$y = x\left(\frac{dy}{dx}\right) + \frac{a}{\left(dy/dx\right)}$$

Hence $y = cx + a/c$ is solution of given differential equation.

**Example 6.** Prove that $y = a\sin 2x$ is solution of given differential equation $\frac{d^2y}{dx^2} + 4y = 0$ .

**Solution :** Given equation is $y = a\sin 2x$ . (1)

differentiating with respect to $x$

$$\frac{dy}{dx} = 2a\cos 2x \qquad (2)$$

again differentiating with respect to $x$

$$\frac{d^2y}{dx^2} = -4a\sin 2x \qquad (3)$$

$$\frac{d^2y}{dx^2} + 4a\sin 2x = 0$$

and $$\frac{d^2y}{dx^2} + 4y = 0 \qquad \text{[From Eq. (1)]}$$

Hence $y = a\sin 2x$ is a solution of given differential equation.

**Example 7.** Prove that $y + x + 1 = 0$ is solution of differential equation $(y - x)dy - (y^2 - x^2)dx = 0$.

**Solution :** Given equation is

$\because$ $$y + x + 1 = 0$$

$\therefore$ $$y = -(x+1) \Rightarrow dy = -dx \qquad (1)$$

LHS of given differential equation

$$= (y - x)dy - (y^2 - x^2)dx$$

$$= (y - x)(-dx) - (y - x)(y + x)dx \qquad [\because \text{ From eq. (1)}]$$

$$= -(y - x)(1 + x + y)dx$$

$$= 0$$

$$= \text{RHS}$$

Hence $y + x + 1 = 0$ is a solution of differential equation.

## Exercise 12.3

1. Prove that $y^2 = 4a(x + a)$ is a solution of differential equation $y = \left[1 - \left(\frac{dy}{dx}\right)^2\right] = 2x\frac{dy}{dx}$.

2. Prove that $y = ae^{-2x} + be^{x}$ is a solution of differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$ .

3. Prove that $y=\frac{c-x}{1+cx}$ is a solution of differential equation $\left(1+x^2\right)\frac{dy}{dx}+\left(1+y^2\right)=0$.

4. Prove that $y=a\cos\left(\log x\right)+b\sin\left(\log x\right)$ is a solution of differential equation $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$.

5. Prove that $xy=\log y+c$ is a solution of differential equation $\frac{dy}{dx}=\frac{y^2}{1-xy}\left(xy\neq 1\right)$.

## 12.05 Differential Equation of First Order and First Degree

There exists a dependent variable $x$, an independent variable $y$ and $\frac{dy}{dx}$ in an differential equation of first order and first degree. hence the equation may be written as

$$\frac{dy}{dx}=f\left(x,\ y\right),\ \text{where}\ f\left(x,\ y\right)\ \text{is a function of}\ x\ \text{and}\ y$$

or
$$\frac{dy}{dx}=\frac{f\left(x,\ y\right)}{g\left(x,\ y\right)}$$

or $\quad f(x,\ y)\,dx+g(x,\ y)\,dy=0$

As it is not possible to integrate every function similarly it is not possible to find solution of every differential equation. But if the differential equation is in standard form of any one out of below mentioned then it is possible to have solution of such differential equations.

(A) Differential equation in which variable separation is possible.
(B) Variable separation is possible by substitution.
(C) Homogeneous differential equations.
(D) Differential equation are reducible to homogeneous form.
(E) Linear differential equation.
(F) Differential equation are reducible to linear differential equation.

**Remark :** Apart from above discussed methods in some situation the solution of differential equation is possible by finding integral multiple, but as not a part of syllabus, the studies of such cases is not porvided here.

### (A) Variable separable form

In the equation $M(x,\ y)\,dx+N(x,\ y)\,dy=0$ on separating the variables and writing in the form of

$$f(x)\,dx+g(y)\,dy=0 \qquad (1)$$

here the variables are separated hence on integrating the each term of equation (1) following solution is obtained.

$$\int f(x)\,dx+\int g(y)\,dy=C\ ,\ \text{where}\ C\ \text{is any arbitrary cosntant.}$$

# Illustrative Examples

**Example 8.** Solve $\frac{dy}{dx} = e^{x+y}$ .

**Solution :** Given equation is $\frac{dy}{dx} = e^x \cdot e^y$

now on separating the variables $\quad e^x dx = e^{-y} dy$

integrating both the sides $\quad \int e^x dx = \int e^{-y} dy$

$\Rightarrow \quad e^x = -e^{-y} + C$

or $\quad e^x + e^{-y} = C$ , where $C$ is integral constant.

This is the required solution.

**Example 9.** Solve $\frac{dy}{dx} = \sin x - x$ .

**Solution :** Given equation is $\quad \frac{dy}{dx} = \sin x - x$

on separating the variables, $\quad dy = (\sin x - x)\, dx$

integrating both the sides, $\quad \int dy = \int (\sin x - x)\, dx$

or $\quad y = -\cos x - \frac{x^2}{2} + C$ , where $C$ is integral constant.

This is the required solution.

**Example 10.** Solve $x\cos^2 y dx = y\cos^2 x dy$.

**Solution :** Given equation is $\quad x\cos^2 y dx = y\cos^2 x dy$

or $\quad \frac{dy}{dx} = \frac{x\cos^2 y}{y\cos^2 x} = \frac{x\sec^2 x}{y\sec^2 y}$

On separating the variables

or $\quad y\sec^2 y\, dy = x\sec^2 x\, dx$

integrating both the sides $\quad \int y\sec^2 y\, dy = \int x\sec^2 x\, dx$

on integrating by parts

$y\tan y - \log\sec y = x\tan x - \log\sec x + C$ , where $C$ is integral constant.

This is the required solution.

**Example 8.** Solve: $\frac{dy}{dx}+\sqrt{\frac{1-y^2}{1-x^2}}=0.$

**Solution :** Given equation is $\frac{dy}{dx}=-\sqrt{\frac{1-y^2}{1-x^2}}$

Now on separating the variables $\frac{dx}{\sqrt{1-x^2}}=-\frac{dy}{\sqrt{1-y^2}}$

integrating both the sides $\int\frac{dx}{\sqrt{1-x^2}}=-\int\frac{dy}{\sqrt{1-y^2}}$

$\sin^{-1}x=-\sin^{-1}y+C_1$ (First form) where $C_1$ is integral constant

If we take $C_1$ as $\sin^{-1}C$ then

$$\sin^{-1}x+\sin^{-1}y=\sin^{-1}C$$

by inverse circular formula $\left[\sin^{-1}x+\sin^{-1}y=\sin^{-1}\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\}\right]$

$$\sin^{-1}\left[x\sqrt{1-y^2}+y\sqrt{1-x^2}\right]=\sin^{-1}C$$

or $$x\sqrt{1-y^2}+y\sqrt{1-x^2}=C$$

This is the required solution.

## Exercise 12.4

**Solve the following differential equations.**

1. $\left(e^y+1\right)\cos x dx+e^y\sin x dy=0$
2. $\left(1+x^2\right)dy=\left(1+y^2\right)dx$
3. $(x+1)\frac{dy}{dx}=2xy$
4. $\frac{dy}{dx}=e^{x-y}+x^2e^{-y}$
5. $\left(\sin x+\cos x\right)dy+\left(\cos x-\sin x\right)dx=0$
6. $\frac{dy}{dx}=\frac{3e^{2x}+3e^{4x}}{e^x+e^{-x}}$
7. $\sec^2 x\tan y dy+\sec^2 y\tan x dx=0$
8. $\frac{dy}{dx}=\frac{x\left(2\log x+1\right)}{\sin y+y\cos y}$
9. $\left(1+\cos x\right)dy=\left(1-\cos x\right)dx$
10. $\sqrt{1-x^6}dy=x^2dx$

## (B) Differential equation reducable to variable separable

In this method the given differential equation may be reduced to variable separable form by suitable substitution and by getting its solution and again substituting required solution can be obtained. Following examples will explain the above method.

## Illustrative Examples

**Example 12.** Solve $\frac{dy}{dx}=(4x+y+1)^2$.

**Solution :** Let $4x+y+1=t$

On differentiating with respect to $x$,

$$4+\frac{dy}{dx}=\frac{dt}{dx}\Rightarrow\frac{dy}{dx}=\frac{dt}{dx}-4$$

by substitution in given equation

$$\frac{dt}{dx}-4=t^2$$

$$\Rightarrow\quad \frac{dt}{dx}=t^2+4$$

$$\Rightarrow\quad \frac{1}{t^2+4}dt=dx \quad \text{(seperation of variables)}$$

on integration $$\int\frac{1}{t^2+(2)^2}dt=\int dx$$

or $$\frac{1}{2}\tan^{-1}(t/2)=x+C\text{, where C is integral cosntant}$$

or $$\tan^{-1}t/2=2x+2C$$

or $$t=2\tan(2x+C_1),\quad \text{where } C_1=2C$$

putting the vlaue of $t$ the desired solution is

$$4x+y+1=2\tan(2x+C_1)$$

**Example 13.** Solve: $(x-y)^2\frac{dy}{dx}=a^2$.

**Solution :** On writing the given equation in the following form

$$\frac{dy}{dx}=\frac{a^2}{(x-y)^2} \tag{1}$$

Let $$x-y=t\Rightarrow 1-\frac{dy}{dx}=\frac{dt}{dx}$$

So from eq. (1) $$1-\frac{dt}{dx}=\frac{a^2}{t^2}$$

on simplification $$\frac{dt}{dx}=1-\frac{a^2}{t^2}=\frac{t^2-a^2}{t^2}$$

so $$dx = \left[1 + \frac{a^2}{(t^2 - a^2)}\right] dt$$

On integration $$\int dx = \int \left[1 + \frac{a^2}{t^2 - a^2}\right] dt$$

or $$x = t + a^2 \frac{1}{2a} \log\left(\frac{t-a}{t+a}\right) + C,$$ where $C$ is integral constant.

putting the value of $t$ the required solution is

$$y = \frac{a}{2} \log\left\{\frac{x-y-a}{x-y+a}\right\} + C.$$

**Example 14.** Solve: $\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$.

**Solution :** Let $x + y = t$, on differentiating with respect to $x$

$$1 + \frac{dy}{dx} = \frac{dt}{dx}$$

$\Rightarrow$ $$\frac{dy}{dx} = \frac{dt}{dx} - 1$$

on substitution in given equation

$$\frac{dt}{dx} - 1 = \sin t + \cos t$$

or $$\frac{dt}{dx} = 1 + \sin t + \cos t$$

or $$\frac{dt}{(\sin t + \cos t + 1)} = dx$$ [separation of variable]

or $$\frac{(1/2)\sec^2(t/2)}{1 + \tan\ (t/2)} dt = dx$$

on integration $$\int \frac{(1/2)\sec^2(t/2)}{1 + \tan(t/2)} dt = \int dx$$

$$\log\left[1 + \tan\frac{t}{2}\right] = x + C,$$ where $C$ is integral constant

or $$\log\left[1 + \tan\frac{(x+y)}{2}\right] = x + C.$$ [$\because$ on putting $t = x + y$ ]

**Example 15.** Solve $\left[\dfrac{x+y-a}{x+y-b}\right]\dfrac{dy}{dx}=\dfrac{x+y+a}{x+y+b}$

**Solution :** From the given equation

$$\frac{dy}{dx}=\frac{(x+y+a)(x+y-b)}{(x+y-a)(x+y+b)} \quad (1)$$

Let $\quad x+y=t \Rightarrow 1+\dfrac{dy}{dx}=\dfrac{dt}{dx} \quad$ (on differentiation)

So, $\quad \dfrac{dt}{dx}=\dfrac{(t+a)(t-b)}{(t-a)(t+b)}+1$

on simplifying $\quad \dfrac{dt}{dx}=\dfrac{2(t^2-ab)}{(t-a)(t+b)}$

or $\quad 2dx=\left[1+\dfrac{t(b-a)}{t^2-ab}\right]dt$

on integration $\quad \displaystyle\int 2dx=\int\left[1+\frac{t(b-a)}{t^2-ab}\right]dt$

$$2x=t+\frac{b-a}{2}\log\left(t^2-ab\right)+C,$$ where $C$ is the integral constant

on putting the vlaue of $t$, the requried solution is

$$x-y=\frac{b-a}{2}\log\left[(x+y)^2-ab\right]+C.$$

## Exercise 12.5

Solve the following differential equations.

1. $(x+y)^2\dfrac{dy}{dx}=a^2$
2. $\dfrac{dy}{dx}=\dfrac{1}{x+y+1}$
3. $\cos(x+y)dy=dx$
4. $e^{x+y}=1+\dfrac{dy}{dx}$
5. $(x+y)(dx-dy)=dx+dy$
6. $\dfrac{dy}{dx}=\dfrac{x+y+1}{x+y}$
7. $x+y=\sin^{-1}\left(\dfrac{dy}{dx}\right)$
8. $\dfrac{dy}{dx}=\dfrac{1}{x-y}+1$
9. $\dfrac{dy}{dx}=\sec(x+y)$
10. $\dfrac{dy}{dx}=\dfrac{(x-y)+3}{2(x-y)+5}$

## (C) Homogeneous differential equation

Differential equation $f(x,\ y)\,dx + g(x,\ y)\,dy = 0$ is called homogeneous differential equation if it could be expressed in following form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right) \qquad (1)$$

i.e. in $f(x,\ y)$ and $g(x,\ y)$ the sum of degrees of $x$ and $y$ in every term always remains same.

Let, $$y = vx \qquad (2)$$

differentiating with respect to $x$

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \qquad (3)$$

Using (2) and (3) in (1)

$$v + x\frac{dv}{dx} = F(v)$$

or $$x\frac{dv}{dx} = F(v) - v$$

or $$\frac{1}{F(v)-v}\,dv = \frac{dx}{x} \qquad \text{[separationof variable]}$$

on integration $$\int \frac{1}{F(v)-v}\,dv = \int \frac{1}{x}\,dx = \log x + C$$, where $C$ is integral constant.

On solving LHS and putting $v = \frac{y}{x}$, gives the required solution of differential equation.

**Remark :** If the homogeneous differential equation is of the form $\frac{dx}{dy} = f(x,\ y)$, where $f(x,\ y)$ is a homogeneous function of degree zero, then put $x = vy$ and find $\frac{dx}{dy}$ and put the value of $\frac{dx}{dy} = f(x,\ y)$ and find the general solution of differeneation equation.

## Illustrative Examples

**Example 16.** Solve, $\frac{dy}{dx} = \frac{3xy + y^2}{3x^2}$

**Solution :** Given equation $$\frac{dy}{dx} = \frac{3xy + y^2}{3x^2} \qquad (1)$$

Given equation is homogeneous differential equation

so let $$y = vx \qquad (2)$$

$\Rightarrow$ $$\frac{dy}{dx} = v + \frac{xdv}{dx} \qquad (3)$$

using equation (2) and (3) in (1)

$$v+x\frac{dv}{dx}=\frac{3vx^2+v^2x^2}{3x^2}=\frac{3v+v^2}{3}$$

or $$x\frac{dv}{dx}=\frac{3v+v^2}{3}-v=\frac{v^2}{3}$$

or $$\frac{1}{v^2}dv=\frac{1}{3x}dx$$ [on separating the varaibles]

or $$-\frac{1}{v}=\frac{1}{3}\log|x|+C,$$ where $C$ is integral constant

or $$-\frac{x}{y}=\frac{1}{3}\log|x|+C.$$ $\left[\because v=\frac{y}{x}\right]$

this is the required solution.

**Example 17.** Solve : $\frac{dy}{dx}=\frac{y}{x}+\tan\left(\frac{y}{x}\right)$.

**Solution :** $$\frac{dy}{dx}=\frac{y}{x}+\tan\left(\frac{y}{x}\right) \qquad (1)$$

This given equation is homogeneous differential equation

So, let $y=vx$

$\Rightarrow$ $$\frac{dy}{dx}=v+x\frac{dv}{dx}$$

now from (1) $$v+x\frac{dv}{dx}=v+\tan v$$

or $$\frac{1}{x}dx=\cot v\,dv$$ [by separating the variables]

on integrating $\log|x|=\log\sin v+\log C$, where $\log C$ is integral constant.

or $$x=C\sin v$$

on putting the value of $v$ required solution is

$$x=C\sin\left(\frac{y}{x}\right).$$

**Example 18.** Solve $x\sin\left(\frac{y}{x}\right)\frac{dy}{dx}=y\sin\left(\frac{y}{x}\right)-x$

**Solution :** From the given equation

$$\frac{dy}{dx}=\frac{y\sin(y/x)-x}{x\sin(y/x)} \qquad (1)$$

Given equation is homogeneous differential equation

So, let $$y = vx \qquad (2)$$

$\therefore$ $$\frac{dy}{dx} = v + x\frac{dv}{dx} \qquad (3)$$

so by eq. (1) $$v + x\frac{dv}{dx} = \frac{v\sin v - 1}{\sin v}$$

or $$v + x\frac{dv}{dx} = v - \operatorname{cosec} v$$

or $$\frac{1}{x}dx = -\sin v dv \qquad \text{[by separating the variables]}$$

$$\log(x/c) = \cos v, \text{ where } C \text{ is integral constant}$$

or $$x = Ce^{\cos v}$$

on putting the vlaue of $v$ required solution is

$$x = ce^{\cos(y/x)}$$

**Example 19.** Solve : $x\dfrac{dy}{dx} = y(\log y - \log x + 1)$

**Solution :** From given equation $$\frac{dy}{dx} = \frac{y}{x}\left[\log\frac{y}{x} + 1\right] \qquad (1)$$

equation (1) is homogeneous equation

So, let $$y = vx \qquad (2)$$

$\therefore$ $$\frac{dy}{dx} = v + x\frac{dv}{dx} \qquad (3)$$

using equation (2) and (3) in equation (1)

$$v + x\frac{dv}{dx} = v(\log v + 1)$$

or $$x\frac{dv}{dx} = v\log v$$

or $$\frac{1}{v\log v}dv = \frac{1}{x}dx \qquad \text{[by separating the variables]}$$

on integration $$\int\frac{(1/v)}{\log v}dv = \int\frac{1}{x}dx$$

or $$\log(\log v) = \log x + \log C, \text{ where } \log C \text{ is integral constant}$$

or $$\log v = Cx$$

or $$\log\frac{y}{x}=Cx \qquad [\because\ v=y/x]$$

This is the required solution

## Exercise 12.6

Solve the following differential equations.

1. $x^2ydx-\left(x^3+y^3\right)dy=0$
2. $\dfrac{dy}{dx}=\dfrac{y}{x}+\sin\left(\dfrac{y}{x}\right)$
3. $x\dfrac{dy}{dx}+\dfrac{y^2}{x}=y$
4. $x\sin\left[\dfrac{y}{x}\right]\dfrac{dy}{dx}=y\sin\left[\dfrac{y}{x}\right]-x$
5. $xdy-ydx=\sqrt{x^2+y^2}\,dx$
6. $\left(x^2+y^2\right)dy=2xydx$
7. $\left(1+e^{x/y}\right)dx+e^{x/y}\left(1-\dfrac{x}{y}\right)dy=0$
8. $\left(3xy+y^2\right)dx+\left(x^2+xy\right)dy=0$
9. $x^2\dfrac{dy}{dx}=x^2+xy+y^2$
10. $x\left(x-y\right)dy=y\left(x+y\right)dx$

## (D) Differential Equation Reducible to Homogeneous Form

When differential equation is of the form $\dfrac{dy}{dx}=\dfrac{ax+by+c}{a'x+b'y+c'}$, where $\dfrac{a}{a'}\neq\dfrac{b}{b'}$ (1)

where c and c' are constants then this may be reduced to a homogeneous eq. by substitution $x=X+h$ and $y=Y+k$ we may get the required solutions

so, let $X=x-h$ ; $Y=y-k$

$dx=dX$ ; $dy=dY$

so by eq. (1) $$\frac{dY}{dX}=\frac{a(X+h)+b(Y+k)+c}{a'(X+h)+b'(Y+k)+c'}$$

or $$\frac{dY}{dX}=\frac{(aX+bY)+(ah+bk+c)}{(a'X+b'Y)+(a'h+b'k+c')} \qquad (2)$$

In order to make equation (2) a homogeneous, the constants $h$ and $k$ are selected such that

$$\left.\begin{aligned}ah+bk+c=0\\ a'h+b'k+c'=0\end{aligned}\right\} \qquad (3)$$

on solving them the values of $h$ and $k$ are found now using equation (3) in equation (2)

$$\frac{dY}{dX}=\frac{aX+bY}{a'X+b'Y} \qquad (4)$$

which is homogeneous, hence solve (4) by homogeneous method and at last put $X=x-h$ and $Y=y-k$ and get the required solutions.

**Remark :** The above methods fails when $\frac{a}{b}=\frac{a'}{b'}$ because then the vlaues of $h$ and $k$ will be either infinite or not defined, in such case let $\frac{a}{a'}=\frac{b}{b'}=\frac{1}{m}$ then the equation (1) will be of form.

$$\frac{dy}{dx}=\frac{ax+by+c}{m[ax+by]+c'} \tag{5}$$

Now solving eq. (5) by substitution $ax+by=v$

$$\frac{dv}{dx}=a+b\left(\frac{v+c}{mv+c'}\right)$$

which can be solved by method of separation of variables.

## Illustrative Examples

**Example 20.** Solve : $\frac{dy}{dx}=\frac{7x-3y-7}{7y-3x+3}$.

**Solution :** Given equation is reducible to homogeneous differential equation because $\frac{a}{a'}\neq\frac{b}{b'}$

so put $$x=X+h,\ y=Y+k$$

$$\frac{dY}{dX}=\frac{7X-3Y+(7h-3k-7)}{-3X+7Y+(7K-3h+3)} \tag{1}$$

Select $h$ and $k$ such that

$$7h-3k-7=0$$

and $$7k-3h+3=0$$

on solving these, $h=1$ and $k=0$

So, from equation (1) $$\frac{dY}{dX}=\frac{7X-3Y}{-3X+7Y} \tag{2}$$

which is homogeneous, so put $Y=vX$

$$\frac{dY}{dX}=v+X\frac{dv}{dX}$$

so, from (2) $$v+X\frac{dv}{dx}=\frac{7-3v}{-3+7v}$$

$\Rightarrow$ $$X\frac{dv}{dX}=\frac{7-3v}{-3+7v}-v$$

or $$-7\frac{dX}{X}=\frac{7v-3}{v^2-1}dv \quad \text{[separation of variable]}$$

or $$-7\frac{dX}{X}=\frac{7}{2}\left(\frac{2v}{v^2-1}\right)dv-\frac{3}{v^2-1}dv$$

On integration $-7\log X = \frac{7}{2}\log\left(v^2-1\right) - \frac{3}{2}\log\left(\frac{v-1}{v+1}\right) - \log C$, where log C is integral constant

$\therefore$
$$\log X^7 + \log\frac{(v^2-1)^{7/2}(v+1)^{3/2}}{(v-1)^{3/2}} = \log C$$

$$\log\left[(v+1)^5(v-1)^2\right]X^7 = \log C$$

putting the value of $v$
$$\log\left[\left(\frac{Y}{X}+1\right)^5\left(\frac{Y}{X}-1\right)^2\right]X^7 = \log C$$

or
$$\left(Y+X\right)^5\left(Y-X\right)^2 = C$$

now put
$$X = x-1 \text{ and } Y = y$$

$$\left(y+x-1\right)^5\left(y-x+1\right)^2 = C$$

This is the required solution.

**Example 21.** Solve : $\frac{dy}{dx} = \frac{x+y+1}{x+y-1}$.

**Solution :** The given differential eq. is not reducible to homogeneous form because here $\frac{a}{a'} = \frac{b}{b'}$

So, to solve such equation we will substitute.

$$x+y=v$$

or
$$1+\frac{dy}{dx} = \frac{dv}{dx}$$

so
$$\frac{dy}{dx} = \frac{dv}{dx} - 1 = \frac{v+1}{v-1} \qquad \text{[From given eq.]}$$

or
$$\frac{dv}{dx} = \frac{2v}{v-1}$$

or
$$2dx = \frac{\left(v-1\right)}{v}dv$$

or
$$2dx = \left(1-\frac{1}{v}\right)dv$$

on integration,
$$\int 2dx = \int\left(1-\frac{1}{v}\right)dv$$

$2x = v - \log v + C$, where C is integral constant

on putting the value of $v$,
$$2x = x+y-\log\left(x+y\right)+C$$

or, $$x - y + \log(x+y) = C$$

This is the required solution.

**Example 22.** Solve $\dfrac{dy}{dx} = \dfrac{x+y+1}{2x+2y+3}$.

**Solution :** Given equation $\dfrac{dy}{dx} = \dfrac{x+y+1}{2x+2y+3}$, $\dfrac{a}{a'} = \dfrac{b}{c'}$ is of the form

so let $x + y = v \Rightarrow \dfrac{dy}{dx} = \dfrac{dv}{dx} - 1$

or $$\frac{dv}{dx} - 1 = \frac{v+1}{2v+3}$$

or $$\frac{dv}{dx} = \frac{v+1}{2v+3} + 1 = \frac{3v+4}{2v+3}$$

or $$\frac{2v+3}{3v+4} dv = dx \qquad \text{[Separation of variable]}$$

on integration $$\int \left[ \frac{2}{3} + \frac{1}{3}\left( \frac{1}{3v+4} \right) \right] dv = \int dx$$

$$\frac{2}{3}v + \frac{1}{9}\log(3v+4) = x + C$$, where C is integral constant.

$$6v + \log(3v+4) = 9x + C_1 \qquad \text{(where, } C_1 = 9\,C\text{)}$$

or $$6(x+y) + \log(3x+3y+4) = 9x + C_1 \qquad \text{(on putting the value of } v\text{)}$$

or $$6y - 3x + \log(3x+3y+4) = C_1$$

This is the required solution.

## Exericse 12.7

Solve the following differential equations.

1. $\dfrac{dy}{dx} + \dfrac{3x+2y-5}{2x+3y-5} = 0$
2. $\dfrac{dy}{dx} = \dfrac{x-y+3}{2x+2y+5}$
3. $(2x+y+1)dx + (4x+2y-1)dy = 0$
4. $\dfrac{dy}{dx} = \dfrac{1-3x-3y}{2(x+y)}$
5. $\dfrac{dy}{dx} = \dfrac{6x-2y-7}{2x+3y-6}$

## (E) Linear Differential Equation

A differential equation in the form

$$\frac{dy}{dx} + Py = Q, \qquad (1)$$

Where P and Q are cosntants or functions of $x$ only, is known as first order linear differential equation. Another form of first order linear differential equation is

$$\frac{dx}{dy} + p_1 x = Q_1 \qquad (2)$$

where $P_1$ and $Q_1$ are constants or functions of $y$ only.

**Solution of linear differential equation (1) :** Multiplying both sides of (1) by $e^{\int P dx}$

$$e^{\int P dx}\left[\frac{dy}{dx} + Py\right] = e^{\int P dx} Q$$

or $$\frac{d}{dx}\left[ye^{\int P dx}\right] = e^{\int P dx} Q$$

integrating both the sides

$$y \cdot e^{\int P dx} = \int Q e^{\int P dx} dy + C$$, where C is integral constant.

or $$y = e^{-\int P dx}\{\int Q e^{\int P dx} dx + C\}$$

Whcih is the required solution of (1).

**Remarks:**

(i) $e^{\int P dx}$ is called as integrating factor of eq. (1), which is abreviated as I.F. .

(ii) Before solving the differential equation the coefficient of derivative should be always one.

(iii) In linear differential eq $\left(\frac{dx}{dy} + P_1 x = Q_1\right)$ the integrating factor is $e^{\int P_1 dy}$ and its solution is given by

$$x = e^{-\int P_1 dy}\left\{\int Q_1 e^{\int P_1 dy} dy + C\right\}$$

## Illustrative Examples

**Example 23.** Solve $\left(1 - x^2\right)\frac{dy}{dx} - xy = 1$.

**Solution :** On writing the given equation in standard form

$$\frac{dy}{dx} + \left(-\frac{x}{(1-x^2)}\right)y = \frac{1}{(1-x^2)}$$

here $$P = -\frac{x}{\left(1-x^2\right)},\ Q = \frac{1}{\left(1-x^2\right)}$$

So integrating factor $$\text{(I.F.)} = e^{\int P dx} = e^{-\frac{1}{2}\int \frac{2x}{1-x^2} dx} = e^{\frac{1}{2}\log\left(1-x^2\right)} = \sqrt{1-x^2}$$

so, solution will be $\quad y\,(\text{I.F.}) = \int (\text{I.F.})\,Q\,dx + C$, where C is integral constant

$$\therefore \quad y\sqrt{1-x^2} = \int \sqrt{1-x^2}\cdot\frac{1}{(1-x^2)}\,dx$$

$$= \int \frac{1}{\sqrt{1-x^2}}\,dx$$

or $\quad y\sqrt{1-x^2} = \sin^{-1}x + C$.

This is the required solution.

**Example 24.** Solve : $\sec x\dfrac{dy}{dx} = y + \sin x$.

**Solution :** On writing the given equation in standard form

$$\frac{dy}{dx} - y\cos x = \sin x\cos x,$$

here $\quad P = -\cos x,\ Q = \sin x\cos x$

So integrating factor $\quad (\text{I.F.}) = e^{\int P dx} = e^{-\int \cos dx} = e^{-\sin x}$

so, solution is $\quad y\cdot e^{-\sin x} = \int \sin x\cos x e^{-\sin x}dx + C$, where C is integral constant

$$= \int te^{-t}dt + C \qquad [\text{here } t = \sin x,\ \therefore dt = \cos x dx\,]$$

$$= -e^{-t}(1+t) + C \qquad [\text{integration by parts}]$$

$$= -e^{-\sin x}(1+\sin x) + C \qquad (\because\ t = \sin x)$$

or $\quad y = Ce^{\sin x} - (1+\sin x)$

This is the required solution.

**Example 25.** Solve : $x\log x\dfrac{dy}{dx} + y = 2\log x$

**Solution :** On writing the given equation in standard form

$$\frac{dy}{dx} + \frac{y}{x\log x} = \frac{2}{x},$$

where $\quad P = \dfrac{1}{x\log x},\ Q = \dfrac{2}{x}$

Integrating factor $\quad (\text{I.F.}) = e^{\int P dx} = e^{\int \frac{1}{x\log x}dx} = e^{\log(\log x)} = \log x$

Integrating factor $\quad y\log x = \int \dfrac{2}{x}\log x dx + C$, Where C is integral constant

$$= 2\frac{(\log x)^2}{2} + C$$

or $$y = (\log x) + \frac{C}{(\log x)}.$$

This is the required solution.

**Example 26.** Solve : $(1+y^2)dx = (\tan^{-1} y - x)dy$ .

**Solution :** From given equation

$$\frac{dx}{dy} + \frac{1}{(1+y^2)}x = \frac{\tan^{-1} y}{1+y^2},$$

here $$P_1 = \frac{1}{1+y^2},\ Q_1 = \frac{\tan^{-1} y}{1+y^2}$$

so integrating factor $$(\text{I.F.}) = e^{\int P_1 dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

so, the solution is $$xe^{\tan^{-1} y} = \int e^{\tan^{-1} y}\left(\frac{\tan^{-1} y}{1+y^2}\right)dy + C$$, where C is integral constant

$$= \int te^t dt + C \qquad [\text{where } \tan^{-1} y = t\,]$$

$$= (t-1)e^t + C$$

on putting the vlaue of $t$, the required solution of equation is

$$x = (\tan^{-1} y - 1) + ce^{-\tan^{-1} y}.$$

## Exercise 12.8

Solve the following differential equations.

1. $\frac{dy}{dx} + 2y = 4x$
2. $\cos^2 x\frac{dy}{dx} + y = \tan x$
3. $(1+x^2)\frac{dy}{dx} + 2yx = 4x^2$
4. $(2x - 10y^3)\frac{dy}{dx} + y = 0$
5. $\frac{dy}{dx} + y\cot x = \sin x$
6. $(1-x^2)\frac{dy}{dx} + 2xy = x\sqrt{1-x^2}$
7. $\sin^{-1}\left[\frac{dy}{dx} + \frac{2}{x}y\right] = x$
8. $x\frac{dy}{dx} + 2y = x^2 \log x$
9. $dx + xdy = e^{-y}\sec^2 y dy$
10. $(1+y^2) + \left(x - e^{\tan^{-1} y}\right)\frac{dy}{dx} = 0$

## (F) Differential Equation Reducible to Linear Differential Equation

**Bernoulli's equation**

$$\frac{dy}{dx} + Py = Qy^n \qquad (1)$$

Above equation may be transformed in linear differential equation by dividing the differntial equation by $y^n$ so dividing by $y^n$ to both sides

$$y^{-n}\frac{dy}{dx}+Py^{1-n}=Q \qquad (2)$$

Let $$y^{1-n}=v$$

$$(1-n)\,y^{-n}\frac{dy}{dx}=\frac{dv}{dx}$$

$$y^{-n}\frac{dy}{dx}=\frac{1}{(1-n)}\frac{dv}{dx}$$

putting the above value in equation (2)

$$\frac{1}{(1-n)}\frac{dv}{dx}+Pv=Q$$

or $$\frac{dv}{dx}+(1-n)Pv=(1-n)Q$$

which is a linear differential equation and can be solved by the method discussed in article ($E$).

## Illustrative Examples

**Example 27.** Solve : $x\frac{dy}{dx}+y=x^3y^6$.

**Solution :** On dividing both sides of equation by $xy^6$

$$\frac{1}{y^6}\frac{dy}{dx}+\frac{1}{xy^5}=x^2 \qquad (1)$$

Let $$\frac{1}{y^5}=v \Rightarrow \frac{-5}{y^6}\frac{dy}{dx}=\frac{dv}{dx}$$

so, transformed form of (1) is $-\frac{1}{5}\frac{dv}{dx}+\frac{1}{x}v=x^2$

or $\frac{dv}{dx}-\frac{5}{x}v=-5x^2$, which is linear differential equation (2)

so, integrating factor $$(I.F.)=e^{\int Pdx}=e^{-5\int\frac{1}{x}dx}=e^{-5\log x}=\frac{1}{x^5}$$

so, the solution of equation (2) $$v\frac{1}{x^5}=\int\frac{1}{x^5}(-5x^2)dx+C$$

or $$\frac{v}{x^5}=-5\int x^{-3}dx+C=\frac{5}{2x^2}+C$$

so, putting the value of $v$, the required solution is

$$y^{-5}=\frac{5}{2}x^3+6x^5.$$

**Example 28.** Solve : $\frac{dy}{dx}=\frac{e^y}{x^2}-\frac{1}{x}$.

**Solution :** From given equation $\frac{dy}{dx}+\frac{1}{x}=\frac{e^y}{x^2}$

dividing by $e^y$ $\quad e^{-y}\frac{dy}{dx}+\frac{e^{-y}}{x}=\frac{1}{x^2}$ (1)

Let $\quad e^{-y}=v\Rightarrow -e^{-y}\frac{dy}{dx}=\frac{dv}{dx}$

so, transformed form of (1) is $-\frac{dv}{dx}+\frac{1}{x}v=\frac{1}{x^2}$

or $\quad \frac{dv}{dx}-\frac{1}{x}v=-\frac{1}{x^2}$ (2)

which is linear differential equation.

so integrating factor $\quad (I.F.)=e^{\int P dx}=e^{-\int\frac{1}{x}dx}=e^{-\log x}=\frac{1}{x}$

so, the solution of (2) will be $\quad v\cdot\frac{1}{x}=\int\frac{1}{x}\left(-\frac{1}{x^2}\right)dx$

or $\quad \frac{v}{x}=\frac{1}{2x^2}+C$

on putting the vlaue of $v$ the required solution is

$$2xe^{-y}-1=2x^2C.$$

**Example 29.** Solve : $\frac{dy}{dx}+\left(2x\tan^{-1}y-x^3\right)\left(1+y^2\right)=0$.

**Solution :** Given equation is $\frac{dy}{dx}+\left(2x\tan^{-1}y-x^3\right)\left(1+y^2\right)=0$

or $\quad \frac{1}{\left(1+y^2\right)}\frac{dy}{dx}=-\left(2x\tan^{-1}y-x^3\right)$

or $\quad \frac{1}{\left(1+y^2\right)}\frac{dy}{dx}+2x\tan^{-1}y=x^3$ (1)

Let $\quad \tan^{-1}y=v\Rightarrow\frac{1}{\left(1+y^2\right)}\frac{dy}{dx}=\frac{dv}{dx}$

so, from eq. (1) $$\frac{dv}{dx}+2xv=x^3$$

which is a linear differential equation, where $P=2x,\ Q=x^3$

$\therefore$ Integrating factor $(I.F.)=e^{2\int xdx}=e^{x^2}$

so required solution is
$$v\cdot e^{x^2}=\int x^3e^{x^2}dx+C$$
$$=\frac{1}{2}\int x^2(2x)e^{x^2}dx+C$$
$$=\frac{1}{2}\int te^t dt+C, \qquad \text{[where } t=x^2,\ \therefore dt=2xdx\text{]}$$
$$=\frac{1}{2}e^t(t-1)+C \qquad \text{[Integration by parts]}$$
$$=\frac{1}{2}e^{x^2}(x^2-1)+C, \qquad [\because t=x^2]$$

again substituting the vlaue of $v$
$$(\tan^{-1}y)e^{x^2}=\frac{1}{2}e^{x^2}(x^2-1)+C$$
$$\tan^{-1}y=\frac{1}{2}(x^2-1)+ce^{-x^2}.$$

This is the required solution.

**Example 30.** Find the particular solution of differential equation $\frac{dy}{dx}+2y\tan x=\sin x$ If $x=\pi/3$ and $y=0$ .

**Solution :** Given differential equation is
$$\frac{dy}{dx}+2y\tan x=\sin x \qquad (1)$$

Here $P=2\tan x,\ Q=\sin x$
$$\text{I.F.}=e^{2\int \tan xdx}=e^{2\log\sec x}=e^{\log\sec^2 x}=\sec^2 x$$

General solution of differnetial equation is
$$y\times\text{I.F.}=\int(\text{I.F.})\times Qdx$$

or $$y\cdot\sec^2 x=\int\sec^2 x\times\sin xdx$$

or $$y\cdot\sec^2 x=\int\sec x\tan x\,dx$$

or $$y\cdot\sec^2 x=\sec x+C \qquad (2)$$

when $x = \pi/3,\ y = 0$ put in eq. (2)

$$0 = \sec \pi/3 + C$$

or $$C = -2$$

put $C = -2$ in equation (2)

$$y \sec^2 x = \sec x - 2$$

or $$y = \cos x - 2\cos^2 x$$

Which is the required solution.

## Exercise 12.9

Solve the following differential equations.

1. $\dfrac{dy}{dx} + xy = x^3 y^3$

2. $\dfrac{dy}{dx} = e^{x-y}\left(e^x - e^y\right)$

3. $\dfrac{dy}{dx} - y\tan x = -y^2 \sec x$

4. $\tan x \cos y \dfrac{dy}{dx} + \sin y + e^{\sin x} = 0$

5. $\dfrac{dy}{dx} + x\sin 2y = x^3 \cos^2 y$

6. $\dfrac{dy}{dx} + \dfrac{y}{x}\log y = \dfrac{y}{x^2}(\log y)^2$

7. $(1+x^2)\dfrac{dy}{dx} + 2xy = \dfrac{1}{1+x^2}$ ; where $x = 1,\ y = 0$

## Miscellaneous Exercise 12

1. Solution of $\left(x^2+1\right)\dfrac{dy}{dx} = 1$ is

(a) $y = \cot^{-1} x + C$ (b) $y = \tan^{-1} x + C$ (c) $y = \sin^{-1} x + C$ (d) $y = \cos^{-1} x + C$

2. Solution of $\dfrac{dy}{dx} + 2x = e^{3x}$ is

(a) $y + x^2 = \dfrac{1}{3}e^{3x} + C$ (b) $y - x^2 = \dfrac{1}{3}e^{3x} + C$ (c) $y + x^2 = e^{3x} + C$ (d) $y - x^2 = e^{3x} + C$

3. Solution of $\dfrac{dy}{dx} + \cos x \tan y = 0$ is

(a) $\log \sin y + \sin x + C$ (b) $\log \sin x \sin y = C$

(c) $\sin y + \log \sin x + C$ (d) $\sin x \sin y = C$

4. Solution $\dfrac{dy}{dx} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$ is

(a) $y = \log\left(e^x + e^{-x}\right) + C$ (b) $y = \log\left(e^x - e^{-x}\right) + C$

(c) $y = \log\left(e^x + 1\right) + C$ (d) $y = \log\left(1 - e^{-x}\right) + C$

5. Solution of $e^{-x+y}\dfrac{dy}{dx}=1$ is

(a) $e^{y}=e^{x}+C$ (b) $e^{y}=e^{-x}+C$ (c) $e^{-y}=e^{-x}+C$ (d) $e^{-y}=e^{x}+C$

6. Solution of $\dfrac{dy}{dx}+\dfrac{1}{y}+y=0$ is

(a) $x+\frac{1}{2}\log\left(1+y\right)=C$ (b) $x+\frac{1}{2}\log\left(1+y^{2}\right)=C$

(c) $x+\log\left(1+y\right)=C$ (d) $x+\log\left(1+y^{2}\right)=C$

7. Solution of $\dfrac{dy}{dx}=\cos^{2}y$ is

(a) $x+\tan y=C$ (b) $\tan y=x+C$ (c) $\sin y+x=C$ (d) $\sin y-x=C$

8. Solution of $\dfrac{dy}{dx}=e^{y+x}+e^{y}x^{2}$ is

(a) $e^{x}+e^{y}=\dfrac{x^{3}}{3}+C$ (b) $e^{-x}+e^{y}+\dfrac{x^{3}}{3}=C$ (c) $e^{-x}+^{-y}=\dfrac{x^{3}}{3}+C$ (d) $e^{x}+e^{-y}+\dfrac{x^{3}}{3}=C$

9. By what substitution will the differential equation $\dfrac{dy}{dx}+\dfrac{y}{x}=\dfrac{y^{2}}{x^{2}}$ change in the linear equation

(a) $y=t$ (b) $y^{2}=t$ (c) $\dfrac{1}{y}=t$ (d) $\dfrac{1}{y^{2}}=t$

10. By what substitution will the differential equation $\dfrac{dy}{dx}+xy=e^{-x}y^{3}$ change in the linear equation

(a) $\dfrac{1}{y}=v$ (b) $y^{-2}=v$ (c) $y^{-3}=v$ (d) $y^{3}=v$

11. Find the general solution of differential equation $\dfrac{dy}{dx}+2x=e^{2x}$.

12. Find integrating factor of differential equation $\dfrac{dy}{dx}+y\tan x=\sin x$.

13. Find integrating factor of differential equation $\dfrac{dy}{dx}+\dfrac{1}{\sin x}y=e^{x}$.

14. Differential equation $\cos\left(x+y\right)\dfrac{dy}{dx}=1$ is of which form?

15. Differential equation $\dfrac{dy}{dx}-y\tan x=e^{x}\sec x$ is of which form?

**Find general solution of following equations.**

16. $\frac{dy}{dx} = \frac{4x+3y+1}{3x+2y+1}$

17. $\frac{dy}{dx} = \frac{y}{x}\left\{\log\left(\frac{y}{x}\right)+1\right\}$

18. $x\frac{dy}{dx} = y + 2\sqrt{y^2 - x^2}$

19. $\frac{dy}{dx} = e^{x-y}\left(e^y - e^x\right)$

20. $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$

## IMPORTANT POINTS

1. An equation involving derivatives independent variable, dependent variable and derivative of the dependent variable with respect to independent variable is known as a differential equation. Differential equations are of two types:
   (i) Ordinary differentaial equation
   (ii) Partial differential equation
2. Order of a differential equationis the order of highest order derivative occuring in the differential equation.
3. Degree of a differential equationis the degree of the highest order derivative, when differential cofficients are made free from redicals and functions.
4. **Solution of differential equation:**
   The solution to the differential equation used in the equation refers to a relationship in the independent and dependent variables which does not contain any differential coefficient and the given differential equatiion is satisifed from derivative obtained.
   The solution of a differential equation is also called its primitive because the differential equation is a relation derived from it.
   (i) **General or total solution :** In the solutioin of a differential equation if there are arbitarary constants equal tot he order of it then that solution is called general solution. This is also called total solutiion or total integral or total primitive.
   (ii) **Particular solution :** The solution of a differential equation obtained by assigning particular values to the arbitrary cosntants in the general solution is called particualr solution.
   (iii) **Singular solution :** The solutions of a differential equation where arbitrary constants are not present and fail to have a particualr solutiion of general solution.
5. **Differential methods to solve differential equation of first order and first degree:**
   **(A) Variable Sepaerable Method :** Differential equations with variable separable on wriing the equation in general form $f(x)\,dx + g(y)\,dy = 0$ and then on integrating, the required solution may be accurid.
   **(B) Varable separation by substitution :** The given differential equation may be reduced to variable, separable form by suitable substitution and by getting its solution and again substituting required solution can be obtained.

**(C) Homogeneous differential equation :** If the general form of differential equation may be written in the form of $\frac{dy}{dx} = \frac{f_1(x, y)}{f_2(x, y)} = \frac{ax+by}{cx+dy}$ where $f_1(x, y)$ and $f_2(x, y), x$ are homogeneous functions of $x$ and $y$ then to reduce in variable separable equation use substitution $y = vx$.

**(D) Equation reducible to homogeneous form**

(i) form $\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'}$, where $\frac{a}{a'} \neq \frac{b}{b'}$

to reduce into homogeneous use $x = X + h,\ y = Y + k$ cosntants $h$ and $k$ are selected such that $ah + bk + c = 0$ and $a'h + b'k + c' = 0$ on solving them the vlaues of $h$ and $k$ are found. At last put $X = x - h$ and $Y = y - k$ and get the required solutions.

(ii) when $\frac{a}{a'} = \frac{b}{b'}$ then put $ax + by = v$ and reduce the equationto variable separatio form and then get the solution.

**(E) Linear differential equation**

(i) Generalrm $\frac{dy}{dx} + Py = Q$ where $P$ nd $Q$, are constants or function of $x$

Integrating factor $(I.F.) = e^{\int Pdx}$

Soution : $y(\text{I.F.}) = \int (\text{I.F.}) \times Qdx + C$

(ii) General form $\frac{dx}{dy} + P_1x = Q_1$ where $P_1$ and $Q_1$ are constants or function of $y$

then integratign factor $(I.F.) = e^{\int P_1 dy}$

Solution $x \times \text{I.F.} = \int I.F. \times Q_1 dy + C$

6. Differential equation reducible to linear differential equation (Bernoulli's equation) $\frac{dy}{dx} + Py = Qy^n$, where $P$ and $Q$, are constants or function of $x$, to reduce it into a linear differential equation divide by $y^n$, then put $\frac{1}{y^n} = t$ and solve. At last put $t = y^{-n}$ to get required solution.

# Answers

## Exercise 12.1

1. order 1 degree 1
2. order 2 degree 1
3. order 2 degree 2
4. order 1 degree 4
5. order 2 degree 2
6. order 1 degree 1
7. order 2 degree 3
8. order 1 degree 2

## Exercise 12.2

1. $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0$
2. $x+y\frac{dy}{dx}=0$
3. $\frac{d^2y}{dx^2}-8\frac{dy}{dx}+15y=0$
4. $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y=0$
5. $\frac{d^2y}{dx^2}+y=0$

## Exercise 12.4

1. $\sin x\left(e^y+1\right)=C$
2. $y-x=C\left(1+xy\right)$
3. $\log y=2\left[x-\log\left(x+1\right)\right]+C$
4. $e^y=e^x+\frac{1}{3}x^3+C$
5. $e^y\left(\sin x+\cos x\right)=C$
6. $y=e^{3x}+C$
7. $\sin^2 x+\sin^2 y=C$
8. $y\sin y=x^2\log x+C$
9. $y=2\tan\frac{x}{2}-x+C$
10. $y=\frac{1}{3}\sin^{-1}x^3+C$

## Exercise 12.5

1. $x+y=a\tan\left(\frac{y-C}{a}\right)$
2. $x+y+2=ce^y$
3. $y=\tan\left(\frac{x+y}{2}\right)+C$
4. $x+e^{-(x+y)}=C$
5. $x-y+c=\log\left(x+y\right)$
6. $2\left(y-x\right)=\log\left(1+2x+2y\right)+C_1$
7. $x=\tan\left(x+y\right)-\sec\left(x+y\right)+C$
8. $2x+\left(x-y\right)^2=0$
9. $y=\tan\left(\frac{x+y}{2}\right)+C$
10. $2\left(x-y\right)+\log\left(x-y+2\right)=x+c$

## Exercise 12.6

1. $y=Ce^{x^3/3y^3}$
2. $\tan\frac{y}{2x}=Cx$
3. $\left(x+cy\right)=y\log x$
4. $x=Ce^{\cos(y/x)}$
5. $y+\sqrt{x^2+y^2}=Cx^2$
6. $y=C\left(x^2-y^2\right)$
7. $x+ye^{x/y}=C$
8. $x^2y^2+2x^3y=C$
9. $\tan^{-1}\left(\frac{y}{x}\right)=\log x+C$
10. $\frac{x}{y}+\log\left(xy\right)=0$

## Exercise 12.7

1. $3\left(x^2+y^2\right)+4xy-10\left(x+y-1\right)=C$
2. $x-2y+\log\left(x-y+2\right)=C$
3. $x+2y+\log\left(2x+y-1\right)=C$
4. $3x+2y+C+2\log\left(1-x-y\right)=0$

5. $3(y-1)^2+4\left(x-\frac{3}{2}\right)(y-1)-6\left(x-\frac{3}{2}\right)^2=C$

## Exercise 12.8

1. $y=2x-1+Ce^{-2x}$
2. $y=\tan x-1+Ce^{-\tan x}$
3. $y=\frac{4x^3}{3(1+x^2)}+\frac{C}{(1+x^2)}$
4. $xy^2=2y^5+C$
5. $y\sin x=\frac{1}{2}x-\frac{1}{4}\sin 2x+C$
6. $y=\sqrt{1-x^2}+C(1-x^2)$
7. $x^2y=C+(2-x^2)\cos x+2x\sin x$
8. $16x^2y=4x^4\log x-x^4+C$
9. $xe^y=\tan y+C$
10. $x=\frac{1}{2}e^{\tan^{-1}y}+Ce^{-\tan^{-1}y}$

## Exercise 12.9

1. $y^{-2}=1+x^2+Ce^{x^2}$
2. $e^y=e^x-1+Ce^{-e^x}$
3. $\frac{1}{y}-\sin x+C\cos x=0$
4. $\sin x\sin y=C+e^{\sin x}$
5. $\tan y=\frac{1}{2}(x^2-1)+Ce^{-x^2}$
6. $\frac{1}{\log y}=\frac{1}{2x}+Cx$
7. $y(1+x^2)=\tan^{-1}x-\pi/4$

## Miscellaneous Exercise 12

| | | | |
|---|---|---|---|
| 1. (b) | 2 (a) | 3. (a) | 4. (b) |
| 5. (a) | 6. (b) | 7. (b) | 8. (d) |
| 9. (c) | 10. (b) | 11. $y+x^2=\frac{1}{2}e^{2x}+C$ | 12. $\sec x$ |
| 13. $\tan x/2$ | 14. Equation reducible to variable separation | | 15. Linear equation |
| 16. $2x^2+3xy+y^2+x+y=0$ | | 17. $\log\left(\frac{y}{x}\right)=Cx$ | 18. $y+\sqrt{y^2-x^2}=Cx^3$ |
| 19. $e^y=e^x+1+Ce^{e^x}$ | 20. $e^{e^2}\tan y=\frac{1}{2}(x^2-1)e^{x/2}+C$ | | |