CHAPTER XIII.

BINOMIAL THEOREM. POSITIVE INTEGRAL INDEX.

161. IT may be shewn by actual multiplication that

$$(x+a)(x+b)(x+c)(x+d)$$
$$= x^4 + (a+b+c+d)x^3 + (ab+ac+ad+bc+bd+cd)x^2$$
$$+ (abc+abd+acd+bcd)x + abcd \quad \ldots\ldots\ldots\ldots (1).$$

We may, however, write down this result by inspection; for the complete product consists of the sum of a number of partial products each of which is formed by multiplying together four letters, *one* being taken from *each* of the four factors. If we examine the way in which the various partial products are formed, we see that

(1) the term x^4 is formed by taking the letter x out of *each* of the factors.

(2) the terms involving x^3 are formed by taking the letter x out of *any three* factors, in every way possible, and *one* of the letters a, b, c, d out of the remaining factor.

(3) the terms involving x^2 are formed by taking the letter x out of *any two* factors, in every way possible, and *two* of the letters a, b, c, d out of the remaining factors.

(4) the terms involving x are formed by taking the letter x out of *any one* factor, and *three* of the letters a, b, c, d out of the remaining factors.

(5) the term independent of x is the product of all the letters a, b, c, d.

Example 1. $(x-2)(x+3)(x-5)(x+9)$

$$= x^4 + (-2+3-5+9)x^3 + (-6+10-18-15+27-45)x^2$$
$$+ (30-54+90-135)x + 270$$
$$= x^4 + 5x^3 - 47x^2 - 69x + 270.$$

Example 2. Find the coefficient of x^3 in the product

$$(x-3)(x+5)(x-1)(x+2)(x-8).$$

The terms involving x^3 are formed by multiplying together the x in *any three* of the factors, and *two* of the numerical quantities out of the two remaining factors; hence the coefficient is equal to the sum of the products of the quantities -3, 5, -1, 2, -8 taken two at a time.

Thus the required coefficient

$$= -15+3-6+24-5+10-40-2+8-16$$
$$= -39.$$

162. If in equation (1) of the preceding article we suppose $b=c=d=a$, we obtain

$$(x+a)^4 = x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4.$$

The method here exemplified of deducing a particular case from a more general result is one of frequent occurrence in Mathematics; for it often happens that it is more easy to prove a general proposition than it is to prove a particular case of it.

We shall in the next article employ the same method to prove a formula known as the **Binomial Theorem**, by which any binomial of the form $x+a$ can be raised to any assigned positive integral power.

163. *To find the expansion of* $(x+a)^n$ *when* n *is a positive integer.*

Consider the expression

$$(x+a)(x+b)(x+c) \ldots\ldots (x+k),$$

the number of factors being n.

The expansion of this expression is the continued product of the n factors, $x+a$, $x+b$, $x+c$, ... $x+k$, and every term in the expansion is of n dimensions, being a product formed by multiplying together n letters, *one* taken from each of these n factors.

The highest power of x is x^n, and is formed by taking the letter x from *each* of the n factors.

The terms involving x^{n-1} are formed by taking the letter x from *any* $n-1$ of the factors, and *one* of the letters a, b, c, ... k from the remaining factor; thus the coefficient of x^{n-1} in the final product is the sum of the letters a, b, c, ... k; denote it by S_1.

The terms involving x^{n-2} are formed by taking the letter x from *any* $n-2$ of the factors, and *two* of the letters a, b, c, ... k from the two remaining factors; thus the coefficient of x^{n-2} in the final product is the sum of the products of the letters a, b, c, ... k taken two at a time; denote it by S_2.

And, generally, the terms involving x^{n-r} are formed by taking the letter x from *any* $n-r$ of the factors, and r of the letters $a, b, c, \dots k$ from the r remaining factors; thus the coefficient of x^{n-r} in the final product is the sum of the products of the letters $a, b, c, \dots k$ taken r at a time; denote it by S_r.

The last term in the product is $abc \dots k$; denote it by S_n.

Hence $\quad (x+a)(x+b)(x+c) \dots\dots (x+k)$

$$= x^n + S_1 x^{n-1} + S_2 x^{n-2} + \dots + S_r x^{n-r} + \dots + S_{n-1} x + S_n.$$

In S_1 the *number of terms* is n; in S_2 *the number of terms* is the same as the number of combinations of n things 2 at a time; that is, ${}^nC_2$; in S_3 *the number of terms* is ${}^nC_3$; and so on.

Now suppose $b, c, \dots k$, each equal to a; then S_1 becomes ${}^nC_1 a$; S_2 becomes ${}^nC_2 a^2$; S_3 becomes ${}^nC_3 a^3$; and so on; thus

$$(x+a)^n = x^n + {}^nC_1 a x^{n-1} + {}^nC_2 a^2 x^{n-2} + {}^nC_3 a^3 x^{n-3} + \dots + {}^nC_n a^n;$$

substituting for ${}^nC_1$, ${}^nC_2$, ... we obtain

$$(x+a)^n = x^n + nax^{n-1} + \frac{n(n-1)}{1 \cdot 2} a^2 x^{n-2} + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a^3 x^{n-3} + \dots + a^n,$$

the series containing $n+1$ terms.

This is the *Binomial Theorem*, and the expression on the right is said to be **the expansion** of $(x+a)^n$.

164. The Binomial Theorem may also be proved as follows:

By induction we can find the product of the n factors $x+a$, $x+b$, $x+c$, ... $x+k$ as explained in Art. 158, Ex. 2; we can then deduce the expansion of $(x+a)^n$ as in Art. 163.

165. The coefficients in the expansion of $(x+a)^n$ are very conveniently expressed by the symbols ${}^nC_1$, ${}^nC_2$, ${}^nC_3$, ... ${}^nC_n$. We shall, however, sometimes further abbreviate them by omitting n, and writing C_1, C_2, C_3, ... C_n. With this notation we have

$$(x+a)^n = x^n + C_1 a x^{n-1} + C_2 a^2 x^{n-2} + C_3 a^3 x^{n-3} + \dots + C_n a^n.$$

If we write $-a$ in the place of a, we obtain

$$(x-a)^n = x^n + C_1(-a)x^{n-1} + C_2(-a)^2 x^{n-2} + C_3(-a)^3 x^{n-3} + \dots + C_n(-a)^n$$

$$= x^n - C_1 a x^{n-1} + C_2 a^2 x^{n-2} - C_3 a^3 x^{n-3} + \dots + (-1)^n C_n a^n.$$

Thus the terms in the expansion of $(x+a)^n$ and $(x-a)^n$ are *numerically* the same, but in $(x-a)^n$ they are alternately positive and negative, and the last term is positive or negative according as n is even or odd.

Example 1. Find the expansion of $(x+y)^6$.

By the formula,

$$(x+y)^6 = x^6 + {}^6C_1 x^5 y + {}^6C_2 x^4 y^2 + {}^6C_3 x^3 y^3 + {}^6C_4 x^2 y^4 + {}^6C_5 x y^5 + {}^6C_6 y^6$$
$$= x^6 + 6x^5 y + 15x^4 y^2 + 20x^3 y^3 + 15x^2 y^4 + 6xy^5 + y^6,$$

on calculating the values of ${}^6C_1$, ${}^6C_2$, ${}^6C_3$, ...

Example 2. Find the expansion of $(a-2x)^7$.

$$(a-2x)^7 = a^7 - {}^7C_1 a^6 (2x) + {}^7C_2 a^5 (2x)^2 - {}^7C_3 a^4 (2x)^3 + \ldots\ldots \text{ to 8 terms.}$$

Now remembering that ${}^nC_r = {}^nC_{n-r}$, after calculating the coefficients up to ${}^7C_3$, the rest may be written down at once; for ${}^7C_4 = {}^7C_3$; ${}^7C_5 = {}^7C_2$; and so on. Hence

$$(a-2x)^7 = a^7 - 7a^6(2x) + \frac{7 \cdot 6}{1 \cdot 2} a^5 (2x)^2 - \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} a^4 (2x)^3 + \ldots\ldots$$
$$= a^7 - 7a^6(2x) + 21a^5(2x)^2 - 35a^4(2x)^3 + 35a^3(2x)^4 - 21a^2(2x)^5 + 7a(2x)^6 - (2x)^7$$
$$= a^7 - 14a^6 x + 84a^5 x^2 - 280a^4 x^3 + 560a^3 x^4 - 672a^2 x^5 + 448ax^6 - 128x^7.$$

Example 3. Find the value of

$$(a + \sqrt{a^2-1})^7 + (a - \sqrt{a^2-1})^7.$$

We have here the sum of two expansions whose terms are numerically the same; but in the second expansion the second, fourth, sixth, and eighth terms are negative, and therefore destroy the corresponding terms of the first expansion. Hence the value

$$= 2\{a^7 + 21a^5(a^2-1) + 35a^3(a^2-1)^2 + 7a(a^2-1)^3\}$$
$$= 2a(64a^6 - 112a^4 + 56a^2 - 7).$$

166. In the expansion of $(x+a)^n$, the coefficient of the second term is ${}^nC_1$; of the third term is ${}^nC_2$; of the fourth term is ${}^nC_3$; and so on; the suffix in each term being one less than the number of the term to which it applies; hence ${}^nC_r$ is the coefficient of the $(r+1)^{\text{th}}$ term. This is called the **general term,** because by giving to r different numerical values any of the coefficients may be found from ${}^nC_r$; and by giving to x and a their appropriate indices any assigned term may be obtained. Thus the $(r+1)^{\text{th}}$ term may be written

$${}^nC_r x^{n-r} a^r, \text{ or } \frac{n(n-1)(n-2)\ldots(n-r+1)}{\lfloor r} x^{n-r} a^r.$$

In applying this formula to any particular case, it should be observed that *the index of* a *is the same as the suffix of* C, *and that the sum of the indices of* x *and* a *is* n.

Example 1. Find the fifth term of $(a+2x^3)^{17}$.

The required term
$$={}^{17}C_4\, a^{13}\,(2x^3)^4$$
$$=\frac{17\,.\,16\,.\,15\,.\,14}{1\,.\,2\,.\,3\,.\,4}\times 16a^{13}x^{12}$$
$$=38080a^{13}x^{12}.$$

Example 2. Find the fourteenth term of $(3-a)^{15}$.

The required term
$$={}^{15}C_{13}\,(3)^2\,(-a)^{13}$$
$$={}^{15}C_2\times(-9a^{13}) \qquad \text{[Art. 145.]}$$
$$=-945a^{13}.$$

167. The simplest form of the binomial theorem is the expansion of $(1+x)^n$. This is obtained from the general formula of Art. 163, by writing 1 in the place of x, and x in the place of a. Thus

$$(1+x)^n = 1 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_rx^r + \ldots + {}^nC_nx^n$$
$$= 1 + nx + \frac{n(n-1)}{1\,.\,2}x^2 + \ldots\ldots + x^n;$$

the general term being

$$\frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{\underline{|r}}x^r.$$

The expansion of a binomial may always be made to depend upon the case in which the first term is unity; thus

$$(x+y)^n = \left\{x\left(1+\frac{y}{x}\right)\right\}^n$$
$$= x^n(1+z)^n, \text{ where } z=\frac{y}{x}.$$

Example 1. Find the coefficient of x^{16} in the expansion of $(x^2-2x)^{10}$.

We have $(x^2-2x)^{10}=x^{20}\left(1-\dfrac{2}{x}\right)^{10}$;

and, since x^{20} multiplies every term in the expansion of $\left(1-\dfrac{2}{x}\right)^{10}$, we have in this expansion to seek the coefficient of the term which contains $\dfrac{1}{x^4}$.

Hence the required coefficient $={}^{10}C_4\,(-2)^4$
$$=\frac{10\,.\,9\,.\,8\,.\,7}{1\,.\,2\,.\,3\,.\,4}\times 16$$
$$=3360.$$

In some cases the following method is simpler.

Example 2. Find the coefficient of x^r in the expansion of $\left(x^2+\frac{1}{x^3}\right)^n$. Suppose that x^r occurs in the $(p+1)^{\text{th}}$ term.

The $(p+1)^{\text{th}}$ term $= {}^nC_p\,(x^2)^{n-p}\left(\frac{1}{x^3}\right)^p$

$= {}^nC_p\,x^{2n-5p}$.

But this term contains x^r, and therefore $2n-5p=r$, or $p=\frac{2n-r}{5}$.

Thus the required coefficient $= {}^nC_p = {}^nC_{\frac{2n-r}{5}}$

$$=\frac{|\underline{n}}{\left|\underline{\frac{1}{5}(2n-r)}\right.\left|\underline{\frac{1}{5}(3n+r)}\right.}.$$

Unless $\frac{2n-r}{5}$ is a positive integer there will be no term containing x^r in the expansion.

168. In Art. 163 we deduced the expansion of $(x+a)^n$ from the product of n factors $(x+a)(x+b)\ldots(x+k)$, and the method of proof there given is valuable in consequence of the wide generality of the results obtained. But the following shorter proof of the Binomial Theorem should be noticed.

It will be seen in Chap. XV. that a similar method is used to obtain the general term of the expansion of

$$(a+b+c+\ldots\ldots)^n.$$

169. *To prove the Binomial Theorem.*

The expansion of $(x+a)^n$ is the product of n factors, each equal to $x+a$, and every term in the expansion is of n dimensions, being a product formed by multiplying together n letters, *one* taken from each of the n factors. Thus each term involving $x^{n-r}a^r$ is obtained by taking a out of *any* r of the factors, and x out of the remaining $n-r$ factors. Therefore the number of terms which involve $x^{n-r}a^r$ must be equal to the number of ways in which r things can be selected out of n; that is, the coefficient of $x^{n-r}a^r$ is ${}^nC_r$, and by giving to r the values 0, 1, 2, 3, ... n in succession we obtain the coefficients of all the terms. Hence

$$(x+a)^n = x^n + {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 + \ldots + {}^nC_rx^{n-r}a^r + \ldots + a^n,$$

since ${}^nC_0$ and ${}^nC_n$ are each equal to unity.

EXAMPLES. XIII. a.

Expand the following binomials:

1. $(x-3)^5$. 2. $(3x+2y)^4$. 3. $(2x-y)^5$.

4. $(1-3a^2)^6$. 5. $(x^2+x)^5$. 6. $(1-xy)^7$.

7. $\left(2-\frac{3x^2}{2}\right)^4$. 8. $\left(3a-\frac{2}{3}\right)^6$. 9. $\left(1+\frac{x}{2}\right)^7$.

10. $\left(\frac{2}{3}x-\frac{3}{2x}\right)^6$. 11. $\left(\frac{1}{2}+a\right)^8$. 12. $\left(1-\frac{1}{x}\right)^{10}$.

Write down and simplify:

13. The 4^{th} term of $(x-5)^{13}$. 14. The 10^{th} term of $(1-2x)^{12}$.

15. The 12^{th} term of $(2x-1)^{13}$. 16. The 28^{th} term of $(5x+8y)^{30}$.

17. The 4^{th} term of $\left(\frac{a}{3}+9b\right)^{10}$.

18. The 5^{th} term of $\left(2a-\frac{b}{3}\right)^8$.

19. The 7^{th} term of $\left(\frac{4x}{5}-\frac{5}{2x}\right)^9$.

20. The 5^{th} term of $\left(\frac{x^{\frac{3}{2}}}{a^{\frac{1}{2}}}-\frac{y^{\frac{5}{2}}}{b^{\frac{3}{2}}}\right)^8$.

Find the value of

21. $(x+\sqrt{2})^4+(x-\sqrt{2})^4$. 22. $(\sqrt{x^2-a^2}+x)^5-(\sqrt{x^2-a^2}-x)^5$.

23. $(\sqrt{2}+1)^6-(\sqrt{2}-1)^6$. 24. $(2-\sqrt{1-x})^6+(2+\sqrt{1-x})^6$.

25. Find the middle term of $\left(\frac{a}{x}+\frac{x}{a}\right)^{10}$.

26. Find the middle term of $\left(1-\frac{x^2}{2}\right)^{14}$.

27. Find the coefficient of x^{18} in $\left(x^2+\frac{3a}{x}\right)^{15}$.

28. Find the coefficient of x^{18} in $(ax^4-bx)^9$.

29. Find the coefficients of x^{32} and x^{-17} in $\left(x^4-\frac{1}{x^3}\right)^{15}$.

30. Find the two middle terms of $\left(3a-\frac{a^3}{6}\right)^9$.

31. Find the term independent of x in $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$.

32. Find the 13^{th} term of $\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}$.

33. If x^r occurs in the expansion of $\left(x + \frac{1}{x}\right)^n$, find its coefficient.

34. Find the term independent of x in $\left(x - \frac{1}{x^2}\right)^{3n}$.

35. If x^p occurs in the expansion of $\left(x^2 + \frac{1}{x}\right)^{2n}$, prove that its coefficient is $\dfrac{|\underline{2n}}{\left|\underline{\tfrac{1}{3}(4n-p)}\right.\left|\underline{\tfrac{1}{3}(2n+p)}\right.}$.

170. *In the expansion of* $(1+x)^n$ *the coefficients of terms equidistant from the beginning and end are equal.*

The coefficient of the $(r+1)^{th}$ term from the beginning is ${}^nC_r$.

The $(r+1)^{th}$ term from the end has $n+1-(r+1)$, or $n-r$ terms before it; therefore counting from the beginning it is the $(n-r+1)^{th}$ term, and its coefficient is ${}^nC_{n-r}$, which has been shewn to be equal to ${}^nC_r$. [Art. 145.] Hence the proposition follows.

171. *To find the greatest coefficient in the expansion of* $(1+x)^n$.

The coefficient of the general term of $(1+x)^n$ is ${}^nC_r$; and we have only to find for what value of r this is greatest.

By Art. 154, when n is even, the greatest coefficient is ${}^nC_{\frac{n}{2}}$; and when n is odd, it is ${}^nC_{\frac{n-1}{2}}$, or ${}^nC_{\frac{n+1}{2}}$; these two coefficients being equal.

172. *To find the greatest term in the expansion of* $(x+a)^n$.

We have $$(x+a)^n = x^n\left(1 + \frac{a}{x}\right)^n;$$

therefore, since x^n multiplies every term in $\left(1 + \frac{a}{x}\right)^n$, it will be sufficient to find the greatest term in this latter expansion.

Let the r^{th} and $(r+1)^{th}$ be any two consecutive terms. The $(r+1)^{th}$ term is obtained by multiplying the r^{th} term by $\frac{n-r+1}{r} \cdot \frac{a}{x}$; that is, by $\left(\frac{n+1}{r} - 1\right)\frac{a}{x}$. [Art. 166.]

The factor $\frac{n+1}{r} - 1$ decreases as r increases; hence the $(r+1)^{th}$ term is not always greater than the r^{th} term, but only until $\left(\frac{n+1}{r} - 1\right)\frac{a}{x}$ becomes equal to 1, or less than 1.

Now $$\left(\frac{n+1}{r} - 1\right)\frac{a}{x} > 1,$$

so long as $$\frac{n+1}{r} - 1 > \frac{x}{a};$$

that is, $$\frac{n+1}{r} > \frac{x}{a} + 1,$$

or $$\frac{n+1}{\frac{x}{a}+1} > r \quad \text{.............................} (1).$$

If $\dfrac{n+1}{\frac{x}{a}+1}$ be an integer, denote it by p; then if $r = p$ the multiplying factor becomes 1, and the $(p+1)^{th}$ term is equal to the p^{th}; and these are greater than any other term.

If $\dfrac{n+1}{\frac{x}{a}+1}$ be not an integer, denote its integral part by q; then the greatest value of r consistent with (1) is q; hence the $(q+1)^{th}$ term is the greatest.

Since we are only concerned with the *numerically greatest term*, the investigation will be the same for $(x-a)^n$; therefore in any numerical example it is unnecessary to consider the sign of the second term of the binomial. Also it will be found best to work each example independently of the general formula.

Example 1. If $x=\frac{1}{3}$, find the greatest term in the expansion of $(1+4x)^8$.

Denote the r^{th} and $(r+1)^{\text{th}}$ terms by T_r and T_{r+1} respectively; then

$$T_{r+1}=\frac{8-r+1}{r}\cdot 4x\times T_r$$

$$=\frac{9-r}{r}\times\frac{4}{3}\times T_r;$$

hence $$T_{r+1}>T_r,$$

so long as $$\frac{9-r}{r}\times\frac{4}{3}>1;$$

that is $$36-4r>3r,$$

or $$36>7r.$$

The greatest value of r consistent with this is 5; hence the greatest term is the sixth, and its value

$$={}^8C_5\times\left(\frac{4}{3}\right)^5={}^8C_3\times\left(\frac{4}{3}\right)^5=\frac{57344}{243}.$$

Example 2. Find the greatest term in the expansion of $(3-2x)^9$ when $x=1$.

$$(3-2x)^9=3^9\left(1-\frac{2x}{3}\right)^9;$$

thus it will be sufficient to consider the expansion of $\left(1-\frac{2x}{3}\right)^9$.

Here $$T_{r+1}=\frac{9-r+1}{r}\cdot\frac{2x}{3}\times T_r, \text{ numerically},$$

$$=\frac{10-r}{r}\times\frac{2}{3}\times T_r;$$

hence $$T_{r+1}>T_r,$$

so long as $$\frac{10-r}{r}\times\frac{2}{3}>1;$$

that is, $$20>5r.$$

Hence for all values of r up to 3, we have $T_{r+1}>T_r$; but if $r=4$, then $T_{r+1}=T_r$, and these are the greatest terms. Thus the 4^{th} and 5^{th} terms are numerically equal and greater than any other term, and their value

$$=3^9\times{}^9C_3\times\left(\frac{2}{3}\right)^3=3^6\times84\times8=489888.$$

173. *To find the sum of the coefficients in the expansion of* $(1+x)^n$.

In the identity $(1+x)^n = 1 + C_1x + C_2x^2 + C_3x^3 + \dots + C_nx^n$, put $x=1$; thus

$$2^n = 1 + C_1 + C_2 + C_3 + \dots + C_n$$
$$= \text{sum of the coefficients.}$$

COR. $\quad C_1 + C_2 + C_3 + \dots + C_n = 2^n - 1$;

that is "the total number of combinations of n things" is $2^n - 1$. [Art. 153.]

174. *To prove that in the expansion of* $(1+x)^n$, *the sum of the coefficients of the odd terms is equal to the sum of the coefficients of the even terms.*

In the identity $(1+x)^n = 1 + C_1x + C_2x^2 + C_3x^3 + \dots + C_nx^n$, put $x = -1$; thus

$$0 = 1 - C_1 + C_2 - C_3 + C_4 - C_5 + \dots\dots;$$
$$\therefore\ 1 + C_2 + C_4 + \dots\dots = C_1 + C_3 + C_5 + \dots\dots$$
$$= \frac{1}{2}\ (\text{sum of all the coefficients})$$
$$= 2^{n-1}.$$

175. The Binomial Theorem may also be applied to expand expressions which contain more than two terms.

Example. Find the expansion of $(x^2+2x-1)^3$.

Regarding $2x-1$ as a single term, the expansion

$$=(x^2)^3 + 3(x^2)^2(2x-1) + 3x^2(2x-1)^2 + (2x-1)^3$$
$$= x^6 + 6x^5 + 9x^4 - 4x^3 - 9x^2 + 6x - 1, \text{ on reduction.}$$

176. The following example is instructive.

Example. If $\quad (1+x)^n = c_0 + c_1x + c_2x^2 + \dots\dots + c_nx^n$,

find the value of $\quad c_0 + 2c_1 + 3c_2 + 4c_3 + \dots\dots + (n+1)\,c_n \dots\dots\dots\dots(1)$,

and $\quad c_1^2 + 2c_2^2 + 3c_3^2 + \dots\dots + nc_n^2 \dots\dots\dots\dots(2)$.

The series $(1) = (c_0 + c_1 + c_2 + \dots\dots + c_n) + (c_1 + 2c_2 + 3c_3 + \dots\dots + nc_n)$

$$= 2^n + n\left\{1 + (n-1) + \frac{(n-1)(n-2)}{1\,.\,2} + \dots\dots + 1\right\}$$
$$= 2^n + n(1+1)^{n-1}$$
$$= 2^n + n\,.\,2^{n-1}.$$

To find the value of the series (2), we proceed thus:

$$c_1 x + 2c_2 x^2 + 3c_3 x^3 + \ldots\ldots + nc_n x^n$$

$$= nx\left\{1 + (n-1)x + \frac{(n-1)(n-2)}{1 \cdot 2}x^2 + \ldots\ldots + x^{n-1}\right\}$$

$$= nx(1+x)^{n-1};$$

hence, by changing x into $\frac{1}{x}$, we have

$$\frac{c_1}{x} + \frac{2c_2}{x^2} + \frac{3c_3}{x^3} + \ldots\ldots + \frac{nc_n}{x^n} = \frac{n}{x}\left(1 + \frac{1}{x}\right)^{n-1} \ldots\ldots\ldots\ldots (3).$$

Also $$c_0 + c_1 x + c_2 x^2 + \ldots\ldots + c_n x^n = (1+x)^n \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

If we multiply together the two series on the left-hand sides of (3) and (4), we see that in the product the term independent of x is the series (2); hence

the series (2) $=$ term independent of x in $\frac{n}{x}(1+x)^n\left(1+\frac{1}{x}\right)^{n-1}$

$=$ term independent of x in $\frac{n}{x^n}(1+x)^{2n-1}$

$=$ coefficient of x^n in $n(1+x)^{2n-1}$

$= n \times {}^{2n-1}C_n$

$$= \frac{|\underline{2n-1}}{|\underline{n-1}\;|\underline{n-1}}.$$

EXAMPLES. XIII. b.

In the following expansions find which is the greatest term:

1. $(x-y)^{30}$ when $x=11$, $y=4$.
2. $(2x-3y)^{28}$ when $x=9$, $y=4$.
3. $(2a+b)^{14}$ when $a=4$, $b=5$.
4. $(3+2x)^{15}$ when $x=\frac{5}{2}$.

In the following expansions find the value of the greatest term:

5. $(1+x)^n$ when $x=\frac{2}{3}$, $n=6$.
6. $(a+x)^n$ when $a=\frac{1}{2}$, $x=\frac{1}{3}$, $n=9$.

7. Shew that the coefficient of the middle term of $(1+x)^{2n}$ is equal to the sum of the coefficients of the two middle terms of $(1+x)^{2n-1}$.

8. If A be the sum of the odd terms and B the sum of the even terms in the expansion of $(x+a)^n$, prove that $A^2-B^2=(x^2-a^2)^n$.

9. The 2nd, 3rd, 4th terms in the expansion of $(x+y)^n$ are 240, 720, 1080 respectively; find x, y, n.

10. Find the expansion of $(1+2x-x^2)^4$.

11. Find the expansion of $(3x^2-2ax+3a^2)^3$.

12. Find the r^{th} term from the end in $(x+a)^n$.

13. Find the $(p+2)^{\text{th}}$ term from the end in $\left(x-\frac{1}{x}\right)^{2n+1}$.

14. In the expansion of $(1+x)^{43}$ the coefficients of the $(2r+1)^{\text{th}}$ and the $(r+2)^{\text{th}}$ terms are equal; find r.

15. Find the relation between r and n in order that the coefficients of the $3r^{\text{th}}$ and $(r+2)^{\text{th}}$ terms of $(1+x)^{2n}$ may be equal.

16. Shew that the middle term in the expansion of $(1+x)^{2n}$ is

$$\frac{1.3.5\ldots(2n-1)}{\lfloor n}2^n x^n.$$

If $c_0, c_1, c_2, \ldots c_n$ denote the coefficients in the expansion of $(1+x)^n$, prove that

17. $c_1+2c_2+3c_3+\ldots\ldots+nc_n=n.2^{n-1}$.

18. $c_0+\frac{c_1}{2}+\frac{c_2}{3}+\ldots\ldots+\frac{c_n}{n+1}=\frac{2^{n+1}-1}{n+1}$.

19. $\frac{c_1}{c_0}+\frac{2c_2}{c_1}+\frac{3c_3}{c_2}+\ldots\ldots+\frac{nc_n}{c_{n-1}}=\frac{n(n+1)}{2}$.

20. $(c_0+c_1)(c_1+c_2)\ldots\ldots(c_{n-1}+c_n)=\frac{c_1c_2\ldots\ldots c_n(n+1)^n}{\lfloor n}$.

21. $2c_0+\frac{2^2c_1}{2}+\frac{2^3c_2}{3}+\frac{2^4c_3}{4}+\ldots\ldots+\frac{2^{n+1}c_n}{n+1}=\frac{3^{n+1}-1}{n+1}$.

22. $c_0^2+c_1^2+c_2^2+\ldots\ldots+c_n^2=\frac{\lfloor 2n}{\lfloor n\,\lfloor n}$.

23. $c_0c_r+c_1c_{r+1}+c_2c_{r+2}+\ldots\ldots+c_{n-r}c_n=\frac{\lfloor 2n}{\lfloor n-r\,\lfloor n+r}$.