Long Answer Questions-I-A(PYQ)

[4 Marks]

$$f(x)=\begin{cases}\dfrac{1-\sin^3 x}{3\cos^2 x} & , \; if \; x<\frac{\pi}{2}\\ p & , \; if \; x=\frac{\pi}{2}\\ \dfrac{q(1-\sin x)}{(\pi-2x)^2} & , \; if \; x>\frac{\pi}{2}\end{cases}$$

Q.1.

is continuous at $x=\frac{\pi}{2}$.

Ans.

We have

$$f(x)=\begin{cases}\dfrac{1-\sin^3 x}{3\cos^2 x} & , \; if \; x<\frac{\pi}{2}\\ p & , \; if \; x=\frac{\pi}{2}\\ \dfrac{q(1-\sin x)}{(\pi-2x)^2} & , \; if \; x>\frac{\pi}{2}\end{cases} \qquad \text{is continuous at } x=\frac{\pi}{2}$$

Now, $\lim\limits_{h\to\frac{\pi^+}{2}} f(x)=\lim\limits_{h\to 0} f\left(\frac{\pi}{2}+h\right)$ $\qquad$ $\left[\text{Let } x=\frac{\pi}{2}+h,\; x\to\frac{\pi^+}{2} \Rightarrow h\to 0\right]$

$$=\lim_{h\to 0}\frac{q\left\{1-\sin\left(\frac{\pi}{2}+h\right)\right\}}{\left\{\pi-2\left(\frac{\pi}{2}+h\right)\right\}^2}=\lim_{h\to 0}\frac{q\{1-\cos h\}}{\{\pi-\pi-2h\}^2}=\lim_{h\to 0}\frac{q(1-\cos h)}{4h^2}$$

$$=\lim_{h\to 0}\frac{q.2\sin^2\frac{h}{2}}{4h^2}=\lim_{h\to 0}\frac{q.\sin^2\frac{h}{2}}{2h^2}$$

$$=q.\lim_{h\to 0}\left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^2\times\frac{1}{8}=\frac{q}{8}$$

Again $\lim\limits_{h\to\frac{\pi^-}{2}} f(x)=\lim\limits_{h\to 0} f\left(\frac{\pi}{2}-h\right)$ $\qquad$ $\left[\text{Let } x=\frac{\pi}{2}-h,\; x\to\frac{\pi^-}{2} \Rightarrow h\to 0\right]$

$$= \lim_{h \to 0} \frac{1 - \sin^3\left(\frac{\pi}{2} - h\right)}{3\cos^2\left(\frac{\pi}{2} - h\right)} = \lim_{h \to 0} \frac{1 - \cos^3 h}{3\sin^2 h}$$

$$= \lim_{h \to 0} \frac{(1 - \cos\ h)\ (1 + \cos\ h + \cos^2 h)}{3\ \sin^2 h}$$

$$= \lim_{h \to 0} \frac{2\sin^2 \frac{h}{2} . (1+1+1)}{3\sin^2 h}$$

$$= \lim_{h \to 0} \frac{2\sin^2 \frac{h}{2} . 3}{3\sin^2 h} = \lim_{h \to 0} \frac{2.\sin^2 \frac{h}{2}}{\sin^2 h}$$

$$= \lim_{h \to 0} \frac{2.\frac{\sin^2 \frac{h}{2}}{h^2}}{\frac{\sin^2 h}{h^2}} \qquad \text{[Dividing } N^r \text{ and } D^r \text{ by } h^2]$$

$$= \lim_{h \to 0} \frac{2.\frac{\sin^2 \frac{h}{2}}{\frac{h^2}{4} \times 4}}{\frac{\sin^2 h}{h^2}} = \frac{1}{2} \frac{\left(\lim_{\frac{h}{2} \to 0} \frac{\sin \frac{h}{2}}{\frac{h}{2}}\right)^2}{\left(\lim_{h \to 0} \frac{\sin h}{h}\right)^2} = \frac{1}{2}$$

Also $f\ \left(\frac{\pi}{2}\right) = p$

$\because$ $f(x)$ is continuous at $x = \frac{\pi}{2}$

$\therefore$ $\lim_{h \to \frac{\pi^+}{2}} f(x) = \lim_{h \to \frac{\pi^-}{2}} f(x) = f\ \left(\frac{\pi}{2}\right)$

$\Rightarrow$ $\frac{q}{8} = \frac{1}{2} = p$

$\Rightarrow$ $p = \frac{1}{2}$ and $q = 4$

Q.2. Show that the function *f*(*x*) = 2*x* – |*x*| is continuous but not differentiable at *x* = 0.

Ans.

Here $f(x) = 2x - |x|$

For continuity at x = 0

$$\lim_{h \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h)$$

$$= \lim_{h \to 0} \{2h - |h|\} = \lim_{h \to 0} (2h - h)$$

$$= \lim_{h \to 0} h$$

$$= 0 \qquad \ldots(i)$$

$$\lim_{h \to 0^-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$$

$$= \lim_{h \to 0} \{2(-h) - |-h|\} = \lim_{h \to 0} \{-2h - h\}$$

$$= \lim_{h \to 0} (-3h)$$

$$= 0 \qquad \ldots(ii)$$

Also, $f(0) = 2 \times 0 - |0| = 0 \qquad \ldots(iii)$

(i),(ii) and (iii) $\Rightarrow \quad \lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = f(0)$

Hence, $f(x)$ is continuous at $x = 0$

For differentiability at x = 0

$$\text{LHD} = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{(2(-h) - |-h|) - \{2 \times 0 - |0|\}}{-h} = \lim_{h \to 0} \frac{-2h - h - 0}{-h}$$

$$= \lim_{h \to 0} \frac{-3h}{-h} = \lim_{h \to 0} 3$$

$$\text{LHD} = 3 \qquad \ldots(iv)$$

Again RHD $= \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$

$$= \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{2h - |h| - 2\times 0 - |0|}{h} = \lim_{h \to 0} \frac{2h - h}{h} = \lim_{h \to 0} \frac{h}{h}$$

$$= \lim_{h \to 0} 1$$

RHD = 1 $\quad \ldots(v)$

From (iv) and (v)

LHD ≠ RHD

i.e., function $f(x) = 2x - |x|$ is not differentiable at $x = 0$

Hence, $f(x)$ is continuous but not differentiable at $x = 0$.

Q.3. Find the value of *k*, for which

$$f(x) = \begin{cases} \frac{\sqrt{1+kx} - \sqrt{1-kx}}{x}, & \text{if} -1 \leq x < 0 \\ \frac{2x+1}{x-1}, & \text{if } 0 \leq x < 1 \end{cases}$$

is continuous at *x* = 0.

Ans.

$\because$ $f(x)$ is continuous at $x = 0$

$\Rightarrow$ (LHL of $f(x)$ at $x = 0$) = (RHL of $f(x)$ at $x = 0$) = $f(0)$

$$\Rightarrow \quad \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0) \qquad \ldots(i)$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0} f(0 - h) \qquad [\text{Let } x = 0 - h, x \to 0^- \Rightarrow h \to 0]$$

$$= \lim_{x \to 0} f(-h) = \lim_{h \to 0} \frac{\sqrt{1+k(-h)} - \sqrt{1-k(-h)}}{-h}$$

$$= \lim_{h \to 0} \frac{\sqrt{1-\text{kh}} - \sqrt{1+\text{kh}}}{-h} \times \frac{\sqrt{1-\text{kh}} + \sqrt{1+\text{kh}}}{\sqrt{1-\text{kh}} + \sqrt{1+\text{kh}}}$$

$$= \lim_{h \to 0} \frac{(1-\text{kh}) - (1+\text{kh})}{-h\{\sqrt{1-\text{kh}} + \sqrt{1+\text{kh}}\}} = \lim_{h \to 0} \frac{2k}{\{\sqrt{1-\text{kh}} + \sqrt{1+\text{kh}}\}} = \frac{2k}{2}$$

$$\Rightarrow \quad \lim_{x \to 0^-} f(x) = k \qquad \ldots.(ii)$$

$$\text{Again } \lim_{x \to 0^+} f(x) = \lim_{x \to 0} f(0+h) \qquad [\text{Let } x = 0 + h, x \to 0^+ \Rightarrow h \to 0]$$

$$= \lim_{h \to 0} f(h) = \lim_{h \to 0} \frac{2h+1}{h-1} = \frac{1}{-1}$$

$$\Rightarrow \quad \lim_{x \to 0^+} f(x) = -1 \qquad \ldots(iii)$$

$$\text{Also } f(0) = \frac{2 \times 0 + 1}{0 - 1} = -1 \qquad \ldots(iv)$$

$\because$ $f(x)$ is continuous at $x = 0$

$\therefore$ (i), (ii), (iii) and (iv) $\Rightarrow$ $k = -1$.

Q.4. Find the value of '*a*' for which the function *f* defined as

$$f(x) = \begin{cases} a \sin \frac{\pi}{2}(x+1), & x \le 0 \\ \frac{\tan x - \sin x}{x^3}, & x > 0 \end{cases}$$

is continuous at *x* = 0.

Ans.

$\because \quad f(x)$ is continuous at $x = 0$.

$\Rightarrow \quad$ (LHL of $f(x)$ at $x = 0$) = (RHL of $f(x)$ at $x = 0$) = $f(0)$

$$\Rightarrow \quad \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0) \qquad \ldots(i)$$

$$\text{Now,} \quad \lim_{x\to 0^-} f(x) = \lim_{x\to 0} a \sin\tfrac{\pi}{2}(x+1) \qquad \left[\because f(x) = a \sin\tfrac{\pi}{2}(x+1), \text{ if } x \le 0\right]$$

$$= \lim_{x\to 0} a \sin\left(\tfrac{\pi}{2} + \tfrac{\pi}{2}x\right) = \lim_{x\to 0} a \cos\tfrac{\pi}{2}x = a.\cos 0 = a \qquad \ldots(ii)$$

$$\text{Again,} \quad \lim_{x\to 0^+} f(x) = \lim_{x\to 0} \frac{\tan x - \sin x}{x^3} \qquad \left[\because f(x) = \frac{\tan x - \sin x}{x^3} \text{ if } x\ 0\right]$$

$$= \lim_{x\to 0} \frac{\frac{\sin x}{\cos x} - \sin x}{x^3} = \lim_{x\to 0} \frac{\sin x - \sin x.\cos x}{\cos x.x^3} = \lim_{x\to 0} \frac{\sin x(1-\cos x)}{\cos x.x^3}$$

$$= \lim_{x\to 0} \frac{1}{\cos x}.\lim_{x\to 0} \frac{\sin x}{x}.\frac{2\sin^2\frac{x}{2}}{\frac{x^2}{4}\times 4} \qquad \left[\because 1 - \cos x = 2\sin^2\tfrac{x}{2}\right]$$

$$= \frac{1}{1}.1.\frac{1}{2}\lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2 = \frac{1}{2}.\left(\lim_{\frac{x}{2}\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right) = \frac{1}{2}\times 1 = \frac{1}{2} \qquad \ldots(iii)$$

$$\text{Also,} \quad f(0) = a\sin\tfrac{\pi}{2}(0+1) = a\sin\tfrac{\pi}{2} = a \qquad \ldots(iv)$$

$\because \quad f$ is continuous at $x = 0$

$\therefore \quad (i), (ii), (iii)$ and $(iv) \quad \Rightarrow \quad a = \frac{1}{2}$

Q.5.

$$\text{If } f(x) = \begin{cases} \dfrac{\sin (a+1)x + 2\sin x}{x}, & x < 0 \\ 2, & x = 0 \\ \dfrac{\sqrt{1+bx} - 1}{x}, & x > 0 \end{cases}$$

is continuous at *x* = 0, then find the values of *a* and *b*.

Ans.

We have

$$f(x)=\begin{cases}\frac{sin(a+1)x+2\sin x}{x} & , \ x<0\\ 2 & , \ x=0\\ \frac{\sqrt{1+bx}-1}{x} & , \ x>0\end{cases}$$ is continuous at $x = 0$

Since, $f(x)$ is continuous at $x = 0$

$$\Rightarrow \quad \lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0) \qquad \ldots (i)$$

Now, $\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h)$ [Let $x = 0 + h$, h is +ve small quantity $x \to 0^+ \Rightarrow h \to 0$]

$$= \lim_{h\to 0} f(h)$$

$$= \lim_{h\to 0} \frac{\sqrt{1+bh}-1}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{1+bh}-1}{h} \times \frac{\sqrt{1+bh}+1}{\sqrt{1+bh}+1}$$

$$= \lim_{h\to 0} \frac{1+bh-1}{h(\sqrt{1+bh}+1)}$$

$$= \lim_{h\to 0} \frac{bh}{h(\sqrt{1+bh}+1)} = \lim_{h\to 0} \frac{b}{\sqrt{1+bh}+1} = \frac{b}{2}$$

Again $\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h)$ [Let $x = 0 - h$, h is +ve small quantity $x \to 0^- \Rightarrow h \to 0$]

$$= \lim_{h\to 0} f(-h)$$

$$= \lim_{h\to 0} \frac{\sin (a+1)(-h)+2\sin (-h)}{-h}$$

$$= \lim_{h\to 0} \left[\frac{-\sin (a+1)h - 2\sin h}{-h}\right]$$

$$= \lim_{h\to 0} \left[\frac{\sin (a+1)h}{h} + \frac{2\sin h}{h}\right]$$

$$= \lim_{h\to 0} \frac{\sin (a+1)h}{h} + 2\lim_{h\to 0} \frac{\sin h}{h}$$

$$= \lim_{h\to 0} \frac{\sin (a+1)h}{(a+1)h} \times (a+1) + 2\lim_{h\to 0} \frac{\sin h}{h}$$

$$= 1 \times (a+1) + 2$$

$$= a + 3$$

Also $\quad f(0) = 2$

Now from (i) $\frac{b}{2} = a + 3 = 2$

$\Rightarrow \quad b = 4, \; a = -1$

Q.6. Find the value of *a* and *b* if the function $f(x) = \begin{cases} 3\text{ax} + b, & \text{if } x > 1 \\ 11, & \text{if } x = 1 \\ 5\text{ax} - 2b, & \text{if } x < 1 \end{cases}$ **is continuous at *x* = 1.**

Ans.

Given function $f(x) = \begin{cases} 3\text{ax} + b, & \text{if } x > 1 \\ 11, & \text{if } x = 1 \\ 5\text{ax} - 2b, & \text{if } x < 1 \end{cases}$

For continuity at $x = 1$, we have

$f(1) = 11$

LHL $= \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 5\,\text{ax} - 2b = 5a - 2b$

RHL $= \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 3\text{ax} + b = 3a + b$

For $f(x)$ to be continuous at $x = 1$, $\quad$ LHL = RHL = $f(1)$

i.e., $\quad 5a - 2b = 3a + b = 11$

On solving, $5a - 2b = 11$ and $3a + b = 11$

We get $\quad a = 3, \quad b = 2.$

Q.7. For what value of *k* is the following function continuous at *x* = 2?

$$f(x) = \begin{cases} 2x + 1, & x < 2 \\ k, & x = 2 \\ 3x - 1, & x > 2 \end{cases}$$

Ans.

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2} 2x+1 = 2\times 2+1 = 5 \qquad [\because\ f(x) = 2x+1, \text{ if } x<2]$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2} 3x-1 = 3\times 2-1 = 5 \qquad [\because\ f(x) = 3x-1, \text{ if } x>2]$$

Since, $f(x)$ is continuous at $x = 2$.

$$\therefore \qquad \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$$

$$\Rightarrow \qquad 5 = 5 = k \qquad \Rightarrow \qquad k = 5$$

Q.8. Discuss the continuity of the following function at *x* = 0:

$$f(x) = \begin{cases} \frac{x^4+2x^3+x^2}{\tan^{-1} x}, & x\neq 0 \\ 0, & x=0 \end{cases}$$

Ans.

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) \qquad [\text{Let } x = 0-h,\ \Rightarrow\ x\to 0^-\ \Rightarrow\ h\to 0]$$

$$= \lim_{h\to 0} f(-h)$$

$$= \lim_{h\to 0} \frac{(-h)^4+2(-h)^3+(-h)^2}{\tan^{-1}(-h)} \qquad [\because -h\neq 0]$$

$$= \lim_{h\to 0} \frac{h^4-2h^3+h^2}{-\tan^{-1} h} \qquad [\because \tan^{-1}(-x) = -\tan^{-1} x]$$

$$= \lim_{h\to 0} \frac{h(h^3-2h^2+h)}{-\tan^{-1}(h)} = \lim_{h\to 0} \frac{h^3-2h^2+h}{-\frac{\tan^{-1} h}{h}}$$

$$= \frac{\lim_{h\to 0}(h^3-2h^2+h)}{\lim_{h\to 0}\frac{-\tan^{-1} h}{h}} = \frac{0}{-1} = 0 \qquad \left[\because\ \lim_{h\to 0}\frac{\tan^{-1} h}{h} = 1\right]$$

$$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h)$$

$$= \lim_{h \to 0} f(h) \qquad [h \neq 0]$$

$$= \lim_{h \to 0} \frac{h^4+2h^3+h^2}{\tan^{-1} h} = \lim_{h \to 0} \frac{h(h^3+2h^2+h)}{\tan^{-1} h}$$

$$= \lim_{h \to 0} \frac{h^3+2h^2+h}{\frac{\tan^{-1} h}{h}}$$

$$= \frac{\lim_{h \to 0} (h^3+2h^2+h)}{\lim_{h \to 0} \frac{\tan^{-1} h}{h}} = \frac{0}{1} = 0 \qquad \left[\because \lim_{x \to 0} \frac{\tan^{-1} x}{x} = 1\right]$$

$$f(0) = 0 \qquad [\because f(x) = 0 \text{ for } x = 0]$$

i.e., $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0) = 0$

Hence, $f(x)$ is continuous at $x = 0$.

Q.9. Show that the function '*f*' defined by $f(x) = \begin{cases} 3x - 2, & 0 < x \leq 1 \\ 2x^2 - x, & 1 < x \leq 2 \\ 5x - 4, & x > 2 \end{cases}$ **is continuous at *x* = 2, but not differentiable.**

Ans.

For continuity:

$$\lim_{x \to 2^-} f(x) = \lim_{h \to 0} f(2-h) \qquad [\text{Let } x = 2-h, \Rightarrow x \to 2^- \Rightarrow h \to 0]$$

$$= \lim_{h \to 0} 2(2-h)^2 - (2-h) = \lim_{h \to 0} 2\{4 + h^2 - 4h\} - (2-h)$$

$$= \lim_{h \to 0} (8 + 2h^2 - 8h - 2 + h) = 6$$

$$\lim_{x \to 2^+} f(x) = \lim_{h \to 0} f(2+h) \qquad [\text{Let } x = 2+h, \Rightarrow x \to 2^+ \Rightarrow h \to 0]$$

$$= \lim_{h \to 0} 5(2+h) - 4 = 6$$

$$f(2) = 2(2)^2 - 2 = 6$$

i.e., $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$

$\Rightarrow$ $f(x)$ is continuous at $x = 2$

For Differentiability:

$$\text{LHD (at } x = 2) = \lim_{h \to 0} \frac{f(2-h) - f(2)}{-h}$$

$$= \lim_{h \to 0} \frac{2(2-h)^2 - (2-h) - \{2.2^2 - 2\}}{-h} = \lim_{h \to 0} \frac{8 + 2h^2 - 8h - 2 + h - 6}{-h}$$

$$= \lim_{h \to 0} \frac{2h^2 - 7h}{-h} = \lim_{h \to 0} \frac{2h - 7}{-1} = 7$$

$$\text{RHD (at } x = 2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{5(2+h) - 4 - \{2.2^2 - 2\}}{h}$$

$$= \lim_{h \to 0} \frac{10 + 5h - 4 - 6}{h} = \lim_{h \to 0} 5 = 5$$

$\because$ LHD $\neq$ RHD (at $x = 2$)

Hence, $f(x)$ is not differentiable at $x = 2$.

Q.10. Find the relationship between '*a*' and '*b*' so that the function *f* defined by:

$$f(x) = \begin{cases} \text{ax} + 1, & \text{if } x \le 3 \\ \text{bx} + 3, & \text{if } x > 3 \end{cases}$$ **is continuous at $x = 3$.**

Ans.

Since, $f(x)$ is continuous at $x = 3$.

$$\Rightarrow \quad \lim_{x \to 3} f(x) = \lim_{x \to 3^+} f(x) = f(3) \qquad \ldots(i)$$

Now, $\lim_{x \to 3} f(x) = \lim_{h \to 0} f(3 - h)$ $\qquad$ [Let $x = 3 - h$, $x \to 3^-$ $\Rightarrow$ $h \to 0$]

$$= \lim_{h \to 0} a(3 - h) + 1 \qquad [\because f(x) = ax + 1 \ \forall \ x \le 3]$$

$$= \lim_{h \to 0} 3a - \text{ah} + 1 = 3a + 1 \qquad \ldots(ii)$$

$\lim_{x \to 3^+} f(x) = \lim_{h \to 0} f(3 + h)$ $\qquad$ [Let $x = 3 + h$, $x \to 3^+$ $\Rightarrow$ $h \to 0$]

$$= \lim_{h \to 0} b(3 + h) + 3 = 3b + 3 \qquad [\because f(x) = bx + 3 \ \forall \ x > 3] \qquad \ldots(iii)$$

From equations (i), (ii) and (iii)

$3a + 1 = 3b + 3$

$3a - 3b = 2$ $\qquad$ or $\qquad$ $a - b = \frac{2}{3}$ which is the required relation.

Q.11. Find the value of *k* so that the function *f*, defined by

$$f(x) = \begin{cases} \text{kx} + 1, & \text{if } x \le \pi \\ \cos x, & \text{if } x > \pi \end{cases}$$ **is continuous at $x = \pi$.**

Ans.

$$\lim_{x \to \pi^-} f(x) = \lim_{h \to 0} f(\pi - h)$$ $[\text{Let } x = \pi - h,\ x \to \pi^- \Rightarrow h \to 0]$

$$= \lim_{h \to 0} k(\pi - h) + 1$$ $[\because f(x) = kx + 1 \text{ for } x \le \pi]$

$= k\pi + 1$

$[\text{Let } x = \pi + h,\ x \to \pi^+ \Rightarrow h \to 0]$

$$\lim_{h \to \pi^+} f(x) = \lim_{h \to 0} f(\pi + h)$$ $[\because f(x) = \cos x \text{ for } x > \pi]$

$$= \lim_{h \to 0} \cos(\pi + h)$$

$$= \lim_{h \to 0} - \cos h = -1$$

Also $f(\pi) = k\pi + 1$

Since, $f(x)$ is continuous at $x = \pi$.

$$\Rightarrow \quad \lim_{x \to \pi^-} f(x) = \lim_{x \to \pi^+} f(x) = f(\pi) \Rightarrow k\pi + 1 = -1 = k\pi + 1$$

$$\Rightarrow \quad k\pi = -2 \qquad \Rightarrow \quad k = -\frac{2}{\pi}$$

Q.12. Show that the function *f*(*x*) = |*x* – 3|, *x* ∈ |R, is continuous but not differentiable at *x* = 3.

Ans.

Here, $f(x) = |x - 3|$ $\Rightarrow$ $f(x) = \begin{cases} -(x-3) & ,x < 3 \\ 0 & ,x = 3 \\ (x-3) & ,x > 3 \end{cases}$

For Continuity:

Now, $\lim\limits_{x \to 3^+} f(x) = \lim\limits_{h \to 0} f(3+h)$ [Let $x = 3 + h$ and $x \to 3^+ \Rightarrow h \to 0$]

$$= \lim_{h \to 0} (3 + h - 3) = \lim_{h \to 0} h = 0$$

$$\lim_{x \to 3^+} f(x) = 0 \qquad ...(i)$$

$$\lim_{x \to 3} f(x) = \lim_{h \to 0} f(3 - h) \qquad \text{[Let } x = 3 - h \text{ and } x \to 3^- \Rightarrow h \to 0]$$

$$= \lim_{h \to 0} - (3 - h - 3) = \lim_{h \to 0} h = 0$$

$$\lim_{x \to 3^+} f(x) = 0 \qquad ...(ii)$$

Also, $f(3) = 0$ $...(iii)$

From equation (i), (ii) and (iii)

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3} f(x) = f(3)$$

Hence, $f(x)$ is continuous at $x = 3$

For Differentiability:

$$\text{RHD} = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0} \frac{(3+h-3) - 0}{h}$$

$$= \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1 \qquad ...(iv)$$

$$\text{LHD} = \lim_{h \to 0} \frac{f(3-h) - f(3)}{-h} = \lim_{h \to 0} \frac{-(3-h-3) - 0}{-h}$$

$$= \lim_{h \to 0} \frac{h}{-h} = \lim_{h \to 0} (-1) = -1 \qquad ...(v)$$

Equation (iv) and (v)

$\Rightarrow$ RHD $\neq$ LHD at $x = 3$.

Hence, $f(x)$ is not differentiable at $x = 3$.

Therefore, $f(x) = |x - 3|$, $x \in \mathbb{R}$ is continuous but not differentiable at $x = 3$.

Q.13. Discuss the continuity and differentiability of the function

***f*(*x*) = |*x*| + |*x* – 1| in the interval (–1, 2).**

Ans.

Given function is

$$f(x) = |x| + |x - 1|$$

Function is also written as

$$f(x) = \begin{cases} -x-(x-1), & if -1 < x < 0 \\ 1, & if\ 0 \leq x < 1 \\ x+(x-1), & if\ x \geq 1 \end{cases}$$

$$\Rightarrow \quad f(x) = \begin{cases} -2x+1, & \text{if } x < 0 \\ 1, & \text{if } 0 \leq x < 1 \\ 2x-1, & \text{if } x \geq 1 \end{cases}$$

Obviously, in given function we need to discuss the continuity and differentiability of the function $f(x)$ at $x = 0$ or 1 only.

For continuity at $x = 0$

$$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) \qquad [\text{Let } x = 0 + h \text{ and } x \to 0^+ \Rightarrow h \to 0]$$

$$= \lim_{h \to 0} f(h)$$

$$= \lim_{h \to 0} 1 \qquad [\because h \text{ is very small positive quantity}]$$

$$= 1 \qquad \ldots(i)$$

$$\lim_{x \to 0} f(x) = \lim_{h \to 0} f(0-h) \qquad [\text{Let } x = 0 - h \text{ and } x \to 0^- \Rightarrow h \to 0]$$

$$= \lim_{h \to 0} f(-h) = \lim_{h \to 0} \{-2(-h)+1\} = \lim_{h \to 0} (2h+1)$$

$$\lim_{x \to 0} f(x) = 1 \qquad \ldots(ii)$$

Also, $f(0) = 1 \qquad \ldots(iii)$

(i), (ii) and (iii) $\Rightarrow$ $\lim_{x \to 0^+} f(x) = \lim_{x \to 0} f(x) = f(0)$

Hence, $f(x)$ is continuous at $x = 0$

For differentiability at $x = 0$

$$\text{RHD} = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{f(h) - f(0)}{h} \qquad [\because h \text{ is very small positive quantity } \Rightarrow 0 < h < 1]$$

$$= \lim_{h \to 0} \frac{1-1}{h} = \lim_{h \to 0} \frac{0}{h} \qquad [\because |h| = h, |0| = 0]$$

$$= \lim_{h \to 0} 0$$

$$\text{RHD} = 1 \qquad \ldots(iv)$$

$$\text{LHD} = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{-2(-h)+1-1}{-h} = \lim_{h \to 0} \frac{2h}{-h}$$

$$= \lim_{h \to 0} (-2)$$

$$\text{LHD} = -2 \qquad \ldots(v)$$

(iv) and (v) $\Rightarrow$ RHD $\neq$ LHD at $x = 0$.

Hence, $f(x)$ is not differentiable at $x = 0$ but continuous at $x = 0$.

Similarly, we can prove $f(x)$ is not differentiable at $x = 1$ but continuous at $x = 1$

(Do yourself)

Q.14. Show that the function $f(x) = |x - 1| + |x + 1|$, for all $x \in R$, is not differentiable at the points $x = -1$ and $x = 1$.

Ans.

Here, given function is

$$f(x) = |x-1| + |x+1|$$

$$\Rightarrow \quad f(x)=\begin{cases} -(x-1)-(x+1), & x<-1 \\ 2, & x=-1 \\ -(x-1)+(x+1), & -1<x<1 \\ 2 & x=1 \\ (x-1)+(x+1) & x>1 \end{cases}$$

$$\Rightarrow \quad f(x)=\begin{cases} -2x, & \text{if } x<-1 \\ 2, & \text{if } x=-1 \\ 2, & \text{if} -1<x<1 \\ 2, & \text{if } x=1 \\ 2x, & \text{if } x>1 \end{cases} \qquad \Rightarrow \qquad f(x)=\begin{cases} -2x & \text{if } x<-1 \\ 2 & \text{if } -1\le x\le 1 \\ 2x & \text{if } x>1 \end{cases}$$

For $x = -1$

$$\text{RHD} = \lim_{h\to 0} \frac{f(-1+h)-f(-1)}{h}$$

$$= \lim_{h\to 0} \frac{2-2}{h} = \lim_{h\to 0} 0 = 0$$

$$\text{LHD} = \lim_{h\to 0} \frac{f(-1-h)-f(-1)}{-h}$$

$$= \lim_{h\to 0} \frac{-2(-1-h)-2}{-h} = \lim_{h\to 0} \frac{2+2h-2}{-h}$$

$$= \lim_{h\to 0} \frac{2h}{h} = \lim_{h\to 0} -2 = -2$$

i.e., RHD $\neq$ LHD.

Hence, $f(x)$ is not differentiable at $x = -1$

For $x = 1$

$$\text{RHD} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{2(1+h) - 2}{h} = \lim_{h \to 0} \frac{2+2h-2}{h}$$

$$= \lim_{h \to 0} \frac{2h}{h} = \lim_{h \to 0} 2 = 2$$

$$\text{LHD} = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h \to 0} \frac{2-2}{-h} = \lim_{h \to 0} \frac{0}{-h}$$

$$= \lim_{h \to 0} 0 = 0$$

$$\text{RHD} \neq \text{LHD}.$$

Hence, $f(x)$ not differentiable at $x = 1$.

Long Answer Questions-I-A(OIQ)

[4 Marks]

Q.1. For what value of *k*, the following function is continuous at *x* = 0?

$$f(x) = \begin{cases} \frac{1 - \cos 4x}{8x^2}, & x \neq 0 \\ k, & x = 0 \end{cases}$$

Ans.

Given function, $f(x)=\begin{cases}\frac{1-\cos 4x}{8x^2} & , \quad x \neq 0 \\ k & , \quad x=0\end{cases}$

At $x = 0$, we have $f(0) = k$

LHL $= \lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} \frac{1-\cos 4x}{8x^2}$

$= \lim\limits_{x \to 0^-} \frac{2\sin^2 2x}{8x^2} = \lim\limits_{x \to 0^-} 2 \times \left(\frac{\sin 2x}{2x}\right)^2 \times \frac{1}{2} = 1$

RHL $= \lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} \frac{1-\cos 4x}{8x^2} = \lim\limits_{x \to 0^+} 2 \times \left(\frac{\sin 2x}{2x}\right)^2 \times \frac{1}{2} = 1$

For $f(x)$ to be continuous at $x = 0$

LHL = RHL = $f(0)$

$\Rightarrow \quad 1 = 1 = k \qquad \therefore \quad k = 1$

Q.2. Examine the continuity of the following function:

$f(x)=\begin{cases}\frac{x}{2|x|}, x \neq 0 \\ \frac{1}{2}, x=0\end{cases} \qquad \text{at } x=0.$

Ans.

Given, $f(x) = \begin{cases} \frac{x}{2|x|}, & x \neq 0 \\ \frac{1}{2}, & x = 0 \end{cases}$

For continuity at $x = 0$, we have

$$f(0) = \frac{1}{2}$$

$$\text{LHL} = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{x}{2|x|} = \lim_{x \to 0^-} \frac{x}{-2x} = -\frac{1}{2}$$

$$\text{RHL} = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{x}{2|x|} = \lim_{x \to 0^+} \frac{x}{2x} = \frac{1}{2}$$

Hence, LHL $\neq$ RHL

So, $f(x)$ is discontinuous at $x = 0$.

Q.3. If the function *f*, as defined below is continuous at *x* = 0, find the values of *a*, *b* and *c*.

$$f(x) = \begin{cases} \frac{\sin\,(a+1)\,x + \sin x}{x}, & x < 0 \\ c, & x = 0 \\ \frac{\sqrt{x + bx^2} - \sqrt{x}}{bx^{3/2}}, & x > 0 \end{cases}$$

Ans.

Since $f(x)$ is continuous at $x = 0$ $\quad \therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$

Now, $\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h)$ $\quad$ [Let $x = 0 - h$, $x \to 0^- \Rightarrow h \to 0$]

$$= \lim_{h \to 0} f(-h) = \lim_{h \to 0} \frac{\sin(a+1)(-h) + \sin(-h)}{-h}$$

$$= \lim_{h \to 0} \frac{\sin(a+1)h + \sin h}{h} = \lim_{h \to 0} \frac{\sin(a+1)h}{h} + \lim_{x \to 0} \frac{\sin h}{h}$$

$$= \lim_{(a+1)h \to 0} \frac{\sin(a+1)h}{(a+1)h} \times (a+1) + 1 \qquad \left[\because \lim_{h \to 0} \frac{\sin h}{h} = 1\right]$$

$$= 1.(a+1) + 1 \qquad \left[\because \lim_{(a+1)h \to 0} \frac{\sin(a+1)h}{(a+1)h} = 1\right]$$

$$\Rightarrow \quad \lim_{x \to 0^-} f(x) = a + 2 \qquad ...(i)$$

Again, $\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h)$ $\quad$ [Let $x = 0 + h$, $x \to 0^+ \Rightarrow h \to 0$]

$$= \lim_{h \to 0} f(h) = \lim_{h \to 0} \frac{\sqrt{h + bh^2} - \sqrt{h}}{bh^{3/2}} = \lim_{h \to 0} \frac{\sqrt{h}(\sqrt{1+bh} - 1)}{bh^{3/2}}$$

$$= \lim_{h \to 0} \frac{\sqrt{1+bh} - 1}{bh} \times \frac{\sqrt{1+bh}+1}{\sqrt{1+bh}+1} = \lim_{h \to 0} \frac{1+bh-1}{bh\ (\sqrt{1+bh}+1)}$$

$$= = \lim_{h \to 0} \frac{1}{\sqrt{1+bh}+1} = \frac{1}{2}$$

$$\Rightarrow \quad \lim_{x \to 0^+} f(x) = \frac{1}{2} \qquad ...(ii)$$

Also, $\quad f(0) = c \qquad ...(iii)$

Hence, (i), (ii) and $(iii) \Rightarrow \quad a + 2 = \frac{1}{2} = c \quad \Rightarrow \quad a = -\frac{3}{2}$, $c = \frac{1}{2}$ and continuity of f does not depend on the value of b