HOMOGENEOUS DIFFERENTIAL EQUATIONS (XII, R. S. AGGARWAL)

EXERCISE 20 (Pg. No.: 961)

In each of the following differential equations show that it is homogenous and solve it

1. $x\,dy = (x+y)\,dx$

Sol. $x\,dy = (x+y)\,dx$

$\frac{dy}{dx} = \left(\frac{x+y}{x}\right)$

$\because (x+y)$ and x both are homogeneous function because both function having 1 degree

$\frac{dy}{dx} = \left(\frac{x+y}{x}\right)$ of a homogenous differential equation

$\frac{dy}{dx} = \left(\frac{x+y}{x}\right)$ .... (i)

Let $y = vx$

$\frac{dy}{dx} = \frac{d(vx)}{dx}$

$\frac{dy}{dx} = v\frac{dx}{dx} + x\frac{dv}{dx}$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$ ...... (ii)

From equation (i) and (ii) $\left(\frac{x+y}{x}\right) = v + x\frac{dv}{dx}$

Putting $y = vx$

$\frac{x+\sqrt{x}}{x} = v + x\frac{dv}{dx}$

$\frac{x(1+v)}{x} = v + x\frac{dv}{dx}$

$1 + v = v + x\frac{dv}{dx}$

$1 = x\frac{dv}{dx}$

$\frac{dv}{dx} = \frac{1}{x}$

$dv = \frac{1}{x}dx$

Integrating on both side we get $\int dv = \int \frac{1}{x}dx$

$v = \log|x| + C$

Putting $y = vx$

$v = \frac{y}{x}$

$\frac{y}{x} = \log|x| + c$

$y = \log|x| + cx$

2. $(x^2 - y^2)dx + 2x\,dy = 0$

Sol. $(x^2 - y^2)dx + 2xy\,dy = 0$

On separating variables we get $(x^2 - y^2)dx + 2xy\,dy = 0$

$2xy\,dy = -(x^2 - y^2)dx$

$\frac{dy}{dx} = \frac{-(x^2 - y^2)}{2xy}$

$\because -(x^2 + y^2)$ and $2xy$ both are homogenous function because both function having 2 degree

$\frac{dy}{dx} = \frac{-(x^2 - y^2)}{2xy}$ of a homogeneous diff equation

$\frac{dy}{dx} = \frac{-(x^2 - y^2)}{2xy}$ ...... (i)

Let $y = vx \Rightarrow v = \frac{y}{x}$

Diff on the both sides we get $\frac{dy}{dx} = v\frac{dx}{dx} + x\frac{dv}{dx}$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$ .... (ii)

From equation (i) and (ii)

$\frac{-(x^2 - y^2)}{2xy} = v + x\frac{dv}{dx}$

$\frac{(y^2 - x^2)}{2xy} = v + x\frac{dv}{dx}$

Putting $y = vx$

$\frac{v^2x^2 - x^2}{2x(vx)} = v + x\frac{dv}{dx}$

$\frac{x^2(v^2 - 1)}{2vx^2} = v + x\frac{dv}{dx}$

$\frac{(v^2 - 1)}{2v} = v + x\frac{dv}{dx}$

$\frac{(v^2 - 1)}{2v} = v + x\frac{dv}{dx}$

$$\frac{v^2-1}{2v}-v=x\frac{dv}{dx}$$

$$\frac{v^2-1-2v^2}{2v}=x\frac{dv}{dx}$$

$$\frac{-1-v^2}{2v}=x\frac{dv}{dx}$$

$$\frac{-\left(1+v^2\right)}{2v}=x\frac{dv}{dx}$$

$$\frac{dv}{dx}=\frac{-\left(1+x\right)}{2vx}$$

$$\frac{2v}{\left(1+v^2\right)}dv=-\frac{1}{x}dx$$

Integrating on the both side, we get

$$\int\frac{2v}{\left(1+v^2\right)}dv=\int-\frac{1}{x}dx$$

$$2\int\frac{v}{\left(1+v^2\right)}dv=-\int\frac{1}{x}dx$$

Let $\left(1+v^2\right)=t$

Diff on the both sides wrt v

$$0+2v=\frac{dt}{dv}$$

$$2v=\frac{dt}{dv}$$

$$dv=\frac{dt}{2v}$$

$$2\int\frac{v}{t}\times\frac{dt}{2v}=-\int\frac{1}{x}dx$$

$$\frac{2}{2}\int\frac{1}{t}dt=-\int\frac{1}{x}dx$$

$$\int\frac{1}{t}dt=-\int\frac{1}{x}dx$$

$$\log|t|=-\log|x|+c$$

$$\log\left|1+v^2\right|=-\log|x+c|$$

$$\log\left|1+v^2\right|+\log|x|=c$$

$$\log\left|\left(1+v^2\right)\left(x\right)\right|=\log c$$

$$\left(1+v^2\right)x=c$$

Putting $v=\frac{y}{x}$ $\left(1+\frac{y^2}{x^2}\right)x=c$

$$\left(\frac{x^2+y^2}{x^2}\right)x=c$$

$$\frac{x^2+y^2}{x}=c$$

$$x^2+y^2=cx$$

3. $x^2\,dy+y(x+y)dx=0$

Sol. $x^2\,dy+y(x+y)dx=0$

On separating variables we get

$$x^2\,dy=y(x+y)dx$$

$$\frac{dy}{dx}=\frac{-y(x+y)}{x^2}$$

$$\frac{dy}{dx}=\frac{-(xy+y^2)}{x^2}$$

$\because -(xy+y^2)$ and x^2 both are homogeneous function because both function having 2 degree

$\frac{dy}{dx}=\frac{-(xy+y^2)}{x^2}$ of a homogeneous diff equation

$$\frac{dy}{dx}=\frac{-(xy+y^2)}{x^2} \qquad \text{..... (i)}$$

Let $y=vx \Rightarrow v=\frac{y}{x}$

Diff on the both sides w.r. to x

$$\frac{dy}{dx}=\frac{vdx}{dx}+x\frac{dv}{dx}$$

$$\frac{dy}{dx}=v+x\frac{dv}{dx} \qquad \text{...... (ii)}$$

From equation (i) and (ii)

$$\frac{-(xy+y^2)}{x^2}=v+x\frac{dv}{dx}$$

Putting $y=vx$

$$\frac{-(x(vx)+v^2x^2)}{x^2}=v+x\frac{dv}{dx}$$

$$\frac{-(vx^2+v^2x^2)}{x^2}=v+x\frac{dv}{dx}$$

$$-(v+v2)=v+x\frac{d}{dx}$$

$$-v-v^2-v=x\frac{dv}{dx}$$

$$-(v^2+2v)=x\frac{dv}{dx}$$

$$-\frac{\left(v^2+2v\right)}{x}=\frac{dv}{dx}$$

$$\frac{dv}{dx}=\frac{-\left(v^2+2v\right)}{x}$$

$$\frac{dv}{\left(v^2+2v\right)}=-\frac{1}{x}dx$$

$$\frac{dv}{v(v+2)}=-\frac{1}{x}dx$$

Integrating on the both sides, we get

$$\int\frac{dv}{v(v+2)}=\int-\frac{1}{x}dx$$

$$\int\frac{dv}{v(v+2)}=-\int\frac{1}{x}dx$$

$$\because\ \frac{1}{v(v+2)}=\frac{A}{v}+\frac{B}{v+2}\quad\text{..... (iii)}$$

$$\frac{1}{v(v+2)}=\frac{A(v+2)+Bv}{v(v+2)}$$

$$1=A(v+2)+Bv$$

Putting $v=-2$

$$1=A(-2+2)+B(-2)$$

$$1=0+B(-2)$$

$$1=-2B$$

$$\therefore B=-\frac{1}{2}$$

Putting $v=0$

$$1=A(0+2)+B(0)$$

$$1=2A+0$$

$$\therefore A=\frac{1}{2}$$

Putting the value of A and B in equation (iii)

$$\frac{1}{v(v+2)}=\frac{\left(\frac{1}{2}\right)}{v}+\frac{\left(-\frac{1}{2}\right)}{v+2}\quad=\frac{1}{2}\times\frac{1}{v}-\frac{1}{2}\times\frac{1}{v+2}$$

$$\int\frac{dv}{v(v+2)}=-\int\frac{1}{x}dx$$

$$\int\frac{1}{2}\times\frac{1}{v}dv\int\frac{1}{2}\times\frac{1}{v+2}dv=-\int\frac{1}{x}dx$$

$$\frac{1}{2}\int\frac{1}{v}dv-\frac{1}{2}\int\frac{1}{v+2}dv=-\int\frac{1}{x}dx$$

$$\frac{1}{2}\log|v|-\frac{1}{2}\log|v+2|=-\log|x|+c$$

$$\frac{1}{2}\log\left|\frac{v}{v+2}\right| = -\log|x| + c$$

Putting $v = y/x$

$$\frac{1}{2}\log\left[\frac{y/x}{y/x+2}\right] = -\log|x| + \log c$$

$$\log\left[\frac{y/x}{\frac{y+2x}{x}}\right] = -2\log|x| + 2\log c$$

$$\log\left[\frac{y}{y+2x}\right] = -\log\left|x^2\right| + \log c^2$$

$$\log\left[\frac{y}{y+2x}\right] + \log\left|x^2\right| = \log c^2$$

$$\log\left|\left(\frac{y}{y+2x}\right)x^2\right| = \log c^2$$

$$\log\left|\left(\frac{vx}{vx+2x}\right)x^2\right| = \log c^2$$

Putting $v = \frac{y}{x}$

$$\left|\left[\frac{(y/x)x}{(y/x)x+2x}\right]x^2\right| = \log c^2$$

$$\log\left|\frac{yx^2}{y+2x}\right| = \log c^2$$

$$x^2 y = c^2(y+2x)$$

4. $(x-y)dy - (x+y)dx = 0$

Sol. $(x-y)dy - (x+y)dx = 0$

On separating variables we get $(x-y)dy = (x+y)dx$

$$\frac{dy}{dx} = \frac{(x+y)}{(x-y)}$$

$\because (x+y)$ and $(x-y)$ both are homogenous function because both function having 1 degree

$\frac{dy}{dx} = \frac{(x+y)}{(x-y)}$ of a homogenous differential equation $\frac{dy}{dx} = \frac{(x+y)}{(x-y)}$ ... (i)

Let $y = vx \Rightarrow v = \frac{y}{x}$

Diff on the both sides w.r. to x

$$\frac{dy}{dx} = v\frac{dx}{dx} + x\frac{dv}{dx}$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \qquad \text{... (ii)}$$

From equation (i) and (ii)

$$\frac{(x+y)}{(x-y)}=v+x\frac{dv}{dx}$$

Putting $y=vx$

$$\frac{(x+vx)}{(x-vx)}=v+x\frac{dv}{dx}$$

$$\frac{x(1+v)}{x(1-v)}=v+x\frac{dv}{dx}$$

$$\frac{(1+v)}{(1-v)}=v+x\frac{dv}{dx}$$

$$\frac{(1+v)}{(1-v)}-v=x\frac{dv}{dx}$$

$$\frac{(1+v)-v(1-v)}{(1-v)}=x\frac{dv}{dx}$$

$$\frac{1+v-v+v^2}{(1-v)}=x\frac{dv}{dx}$$

$$\frac{(1+v^2)}{(1-v)}=x\frac{dv}{dx}$$

$$\frac{(1+v^2)}{(1-v)x}=\frac{dv}{dx}$$

$$\frac{dv}{dx}=\frac{(1+v^2)}{(1-v)x}$$

$$\frac{(1-v)}{(1+v^2)}dv=\frac{1}{x}dx$$

Integrating on the both side we get

$$\int\frac{(1-v)}{(1+v^2)}dv=\int\frac{1}{x}dx$$

$$\int\frac{1}{1+v^2}dv-\int\frac{v}{1+v^2}dv=\int\frac{1}{x}dx$$

$$\int\frac{1}{1+v^2}dv-\frac{1}{2}\int\frac{2v}{1+v^2}dv=\int\frac{1}{x}dx$$

$$\frac{1}{1}\tan^{-1}\frac{v}{1}-\frac{1}{2}\int\frac{2v}{1+v^2}dv=\log|x|+c$$

$$\tan^{-1}v-\frac{1}{2}\int\frac{2v}{1+v^2}dv=\log|x|+c$$

Let $1+v^2=t$

Diff on the both side w.r. to v

$$0+2v=\frac{dt}{dv}$$

$dv = \frac{dt}{2v}$

$\tan^{-1} v - \frac{1}{2}\int \frac{2v}{t} \times \frac{dt}{2v} = \log|x| + c$

$\tan^{-1} v - \frac{1}{2}\int \frac{1}{t} dt = \log|x| + c$

$\tan^{-1} v - \frac{1}{2}\log t = \log|x| + c$

$\tan^{-1} v - \frac{1}{2}\log\left|1 + v^2\right| = \log|x| + c$

Putting $v = \frac{y}{x}$

$\tan^{-1} \frac{y}{x} - \frac{1}{2}\log\left|1 + \frac{y^2}{x^2}\right| = \log|x| + c$

$\tan^{-1} \frac{y}{x} - \frac{1}{2}\log\left|\frac{x^2 + y^2}{x^2}\right| = \log|x| + c$

$\tan^{-1} \frac{y}{x} = \frac{1}{2}\log\left|\frac{x^2 + y^2}{x^2}\right| + \log|x| + c$

$\tan^{-1} \frac{y}{x} = \log\left|\frac{x^2 + y^2}{x^2}\right|^{1/2} + \log|x| + c$

$\tan^{-1} \frac{y}{x} = \log\left|\sqrt{\frac{x^2 + y^2}{x^2}}\right| + \log|x| + c$

$\tan^{-1} \frac{y}{x} = \log\left|\frac{\sqrt{x^2 + y^2}}{x} \times x\right| + c$

$\tan^{-1} \frac{y}{x} = \log\left|\sqrt{x^2 + y^2}\right| + c$

$\tan^{-1} \frac{y}{x} = \log\left|x^2 + y^2\right|^{1/2} + c$

$\tan^{-1} \frac{y}{x} = \frac{1}{2}\log\left|x^2 + y^2\right| + c$

5. $(x + y)dy + (y - 2x)dx = 0$

Sol. $(x + y)dy + (y - 2x)dx = 0$

On separating variables we get $(x + y)dy + (y - 2x)dx = 0$

$(x + y)dy = -(y - 2x)dx$

$\frac{dy}{dx} = \frac{-(y - 2x)}{(x + y)}$

$\because -(y - 2x)$ and $(x + y)$ both are homogeneous function because both function having 1 degree

$\frac{dy}{dx} = \frac{-(y - 2x)}{(x + y)}$ of homogenous differential equation $\frac{dy}{dx} = v\frac{dx}{dx} + x\frac{dv}{dx}$ … (i)

Let $y = vx \Rightarrow v = \frac{y}{x}$

Diff on the both side w.r. to x

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \qquad \text{.... (ii)}$$

From equation (i) and (ii)

$$\frac{-(y-2x)}{(x+y)} = v + x\frac{dv}{dx}$$

Putting $y = vx$

$$\frac{-(v-2x)}{(x+vx)} = v + x\frac{dv}{dx}$$

$$\frac{(2x-vx)}{(x+vx)} = v + x\frac{dv}{dx}$$

$$\frac{x(2-v)}{x(1+v)} = v + x\frac{dv}{dx}$$

$$\frac{(2-v)}{(1+v)} = v + x\frac{dv}{dx}$$

$$\frac{(2-v)}{(1+v)} - v = x\frac{dv}{dx}$$

$$\frac{(2-v)-v(1+v)}{(1+v)} = x\frac{dv}{dx}$$

$$\frac{2-v-v-v^2}{1+v} = x\frac{dv}{dx}$$

$$\frac{2-2v-v^2}{1+v} = x\frac{dv}{dx}$$

$$\frac{-(v^2+2v-2)}{(1+v)} = x\frac{dv}{dx}$$

$$\frac{-(v^2+2v-2)}{(1+v)x} = \frac{dv}{dx}$$

$$\frac{dv}{dx} = \frac{-(v^2+2v-2)}{(v+1)x}$$

$$\frac{(v+1)dv}{v^2+2v-2} = -\frac{1}{x}dx$$

Integrating on the both sides we get

$$\int \frac{(v+1)}{v^2+2v-2}dv = \int -\frac{1}{x}dx$$

$$\frac{1}{2}\int \frac{2(v+1)}{v^2+2v-2}dv = -\int \frac{1}{x}dx$$

$$\frac{1}{2}\int \frac{2v+2}{v^2+2v-2}dv = -\int \frac{1}{x}dx$$

$$\frac{1}{2}\int \frac{2x+2}{v^2+2v-2}dv = -\log|x| + c$$

Let $v^2+2v-2=t$

Diff on the both sides w.r. to v

$$2v+2-0=\frac{dt}{dv}$$

$$(2v+2)=\frac{dt}{dv}$$

$$dv=\frac{dt}{(2v+2)}$$

$$\frac{1}{2}\int \frac{(2v+2)}{t}\times\frac{dt}{(2v+2)} = -\log|x| + c$$

$$\frac{1}{2}\int \frac{1}{t}dt = -\log|x| + c$$

$$\frac{1}{2}\log|t| + \log|x| = c$$

$$\frac{1}{2}\log|v^2+2v-2| + \log|x| = c$$

Putting $v=\frac{y}{x}$

$$\frac{1}{2}\log\left|\frac{y^2}{x^2}+2\frac{y}{x}-2\right| + \log|x| = c$$

$$\frac{1}{2}\log\left|\frac{y^2+2xy-2x^2}{x^2}\right| + \log|x| = c$$

$$\log\left|\frac{y^2+2xy-2x^2}{x^2}\right|^{1/2} + \log|x| = c$$

$$\log\left|\sqrt{\frac{y^2+2xy-2x^2}{x^2}}\right| + \log|x| = c$$

$$\log\left|\left(\frac{\sqrt{y^2+2xy-2x^2}}{x}\right)\times x\right| = \log c$$

$$(y^2+2xy-2x^2)^{1/2} = c$$

$$y^2+2xy-2x^2 = c^2$$

$$y^2+2xy-2x^2 = c$$

6. $(x^2+3xy+y^2)dx - x^2dy = 0$

Sol. $(x^2+3xy+y^2)dx = x^2dy = 0$

On separating variables we get $(x^2+3xy+y^2)dx = x^2dy$

$$x^2dy = (x^2+3xy+y^2)dx$$

$$\frac{dy}{dx}=\frac{x^2+3xy+y^2}{x^2}$$

$\because \left(x^2+3xy+y^2\right)$ and x^2 both are homogeneous function because both function having 2 degree

$$\frac{dv}{dx}=\frac{x^2+3xy+y^2}{x^2}$$ of a homogenous differential equation

$$\frac{dy}{dx}=\frac{x^2+3xy+y^2}{x^2} \quad .... \text{(i)}$$

Let $y=vx \Rightarrow v=\frac{y}{x}$

Diff on the both sides w.r. to x

$$\frac{dy}{dx}=v\frac{dx}{dx}+x\frac{dv}{dx}$$

$$\frac{dy}{dx}=v+x\frac{dv}{dx} \quad .... \text{(ii)}$$

From equation (i) and (ii)

$$\frac{x^2+3xy+y^2}{x^2}=v+x\frac{dv}{dx}$$

Putting $y=vx$

$$\frac{x^2+3x(vx)+v^2x^2}{x^2}=v+x\frac{dv}{dx}$$

$$\frac{x^2\left(1+3v+v^2\right)}{x^2}=v+x\frac{dv}{dx}$$

$$\left(1+3v+v^2\right)=v+x\frac{dv}{dx}$$

$$1+3v+v^2-v=x\frac{dv}{dx}$$

$$\left(1+2v+v^2\right)=x\frac{dv}{dx}$$

$$(1+v)^2=x\frac{dv}{dx}$$

$$x\frac{dv}{dx}=(v+1)^2$$

$$\frac{dv}{dx}=\frac{(v+1)^2}{x}$$

$$\frac{dv}{(v+1)^2}=\frac{1}{x}dx$$

Integrating on the both sides we gets

$$\int\frac{dv}{(v+1)^2}=\int\frac{1}{x}dx$$

$$\int(v+1)^{-2}\,dv=\log|x|+c$$

$$\frac{(v+1)^{-2+1}}{-2+1}=\log|x|+c$$

$$-(v+1)^{-1}=\log|x|+c$$

$$-\frac{1}{(v+1)}=\log|x|+c$$

Putting $v=\frac{y}{x}$

$$-\frac{1}{\frac{y}{x}+1}=\log|x|+c$$

$$-\frac{1}{\frac{y+x}{x}}=\log|x|+c$$

$$-\frac{x}{y+x}=\log|x|+c$$

$$-\frac{x}{y+x}-\log|x|=c$$

$$-\left(\log|x|+\frac{x}{y+x}\right)=c$$

$$\log|x|+\frac{x}{y+x}=\frac{c}{-1}$$

$$\log|x|+\frac{x}{y+x}=c$$

7. $2xy\,dx+\left(x^2+2y^2\right)dy=0$

Sol. $2xy\,dx+\left(x^2+2y^2\right)dy=0$

On separating variables we get $\left(x^2+2y^2\right)dy=-2xy\ dx$

$$\frac{dy}{dx}=\frac{-2xy}{x^2+2y^2}$$

$\because -2xy$ and x^2+2y^2 both are homogeneous function because both function having 2 degree

$\frac{dy}{dx}=\frac{-2xy}{x^2+2y^2}$ of a homogeneous differential equation $\frac{dy}{dx}=\frac{-2xy}{x^2+2y^2}$ .... (i)

Let $y=vx\Rightarrow v=\frac{y}{x}$

Diff on the both sides w.r. to x

$$\frac{dy}{dx}=v\frac{dx}{dx}+x\frac{dv}{dx}$$

$$\frac{dy}{dx}=v+x\frac{dv}{dx} \quad \text{.... (ii)}$$

From equation (i) and (ii)

$$\frac{-2xy}{x^2+2y^2}=v+x\frac{dv}{dx}$$

Putting $y = vx$

$$\frac{-2x(vx)}{x^2 + 2(vx)^2} = v + x\frac{dv}{dx}$$

$$\frac{-2vx^2}{x^2 + 2v^2x^2} = v + x\frac{dv}{dx}$$

$$\frac{-2vx^2}{x^2(1+2v^2)} = v + x\frac{dv}{dx}$$

$$\frac{-2v}{1+2v^2} = v + x\frac{dv}{dx}$$

$$\frac{-2v}{1+2v^2} - v = x\frac{dv}{dx}$$

$$\frac{-2v - v(1+2v^2)}{1+2v^2} = x\frac{dv}{dx}$$

$$\frac{-2v - v - 2v^3}{1+2v^2} = x\frac{dv}{dx}$$

$$\frac{-3v - 2v^3}{1+2v^2} = x\frac{dv}{dx}$$

$$\frac{-v(2v^2+3)}{1+2v^2} = x\frac{dv}{dx}$$

$$\frac{dv}{dx} = \frac{-v(2v^2+3)}{x(1+2v^2)}$$

$$\frac{1+2v^2}{v(2v^2+3)}dv = -\frac{1}{x}dx$$

Integrating on the both sides we get

$$\int \frac{1+2v^2}{v(2v^2+3)}dv = \int -\frac{1}{x}dx$$

Left side dividing and multiplying by 3

$$\frac{3}{3}\int \frac{1+2v^2}{v(2v^2+3)}dv = -\int \frac{1}{x}dx \quad \frac{1}{3}\int\left(\frac{3+6v^2}{2v^3+3v}\right)dv = -\int \frac{1}{x}dx$$

Let $2v^3 + 3v = t$

Diff on the both sides w.r. to v

$$6v^2 + 3 = \frac{dt}{dv}$$

$$dv = \frac{dt}{6v^2+3} = \frac{dt}{3+6v^2}$$

$$\frac{1}{3}\int\left(\frac{3+6v^2}{2v^2+3v}\right)dv = -\int \frac{1}{x}dx$$

$$\frac{1}{3}\int \frac{(3+6v^2)}{t} \times \frac{dt}{(3+6v^2)} = -\int \frac{1}{x}dx$$

$$\frac{1}{3}\int\frac{1}{t}dt = -\int\frac{1}{x}dx$$

$$\frac{1}{3}\log|t| = -\log|x| + c$$

$$\frac{1}{3}\log\left|2v^3 + 3v\right| = -\log|x| + c$$

$$\log\left|2v^3 + 3v\right| = -3\log|x| + 3\log|C|$$

Putting $v = \frac{y}{x}$

$$\log\left|2\times\frac{y^3}{x^3} + 3\times\frac{y}{x}\right| = -3\log|x| + 3\log|c|$$

$$\log\left|\frac{2y^3 + 3x^2y}{x^3}\right| = -3\log|x| + 3\log|c|$$

$$\log\left|\frac{2y^3 + 3x^2y}{x^3}\right| + 3\log|x| = 3\log|c|$$

$$\log\left|\frac{2y^3 + 3x^2y}{x^3}\right| + \log|x|^3 = 3\log|c|$$

$$\log\left|\left(\frac{2y^3 + 3x^2y}{x^3}\right)\times x^3\right| = 3\log|c|$$

$$\log\left|2y^3 + 3x^2y\right| = \log\left|c^3\right|$$

$2y^3 + 3x^2y = c$ $\left[\because c^3 = c \text{ constant}\right]$

$$2y^3 + 3x^2y = c$$

$$3x^2y + 2y^3 = c$$

8. $\frac{dy}{dx} + \frac{x-2y}{2x-y} = 0$

Sol. $\frac{dy}{dx} + \frac{x-2y}{2x-y} = 0$

On separating variables we get $\frac{dy}{dx} = -\left(\frac{x-2y}{2x-y}\right)$

$$\frac{dy}{dx} = \frac{2y-x}{2x-y}$$

$\because 2y - x$ and $2x - y$ both are homogenous function because both function having 1 degree

$\frac{dy}{dx} = \frac{2y-x}{2x-y}$ of a homogenous differential equation $\frac{dy}{dx} = \frac{2y-x}{2x-y}$ ……. (i)

Let $y = vx \Rightarrow v = \frac{y}{x}$

Differentiate on the both sides w.r. to x

$$\frac{dy}{dx} = v\frac{dx}{dx} + x\frac{dv}{dx}$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \qquad \text{..... (ii)}$$

From equation (i) and (ii)

$$\frac{2y - x}{2x - y} = v + x\frac{dv}{dx}$$

Putting $y = vx$

$$\frac{2(vx) - x}{2x - vx} = v + x\frac{dv}{dx}$$

$$\frac{x(2v-1)}{x(2-v)} = v + x\frac{dv}{dx}$$

$$\frac{2v-1}{2-v} = v + x\frac{dv}{dx}$$

$$\frac{2v-1}{2-v} - v = x\frac{dv}{dx}$$

$$\frac{2v-1-v(2-v)}{2-v} = x\frac{dv}{dx}$$

$$\frac{2v-1-2v+v^2}{2-v} = x\frac{dv}{dx}$$

$$\frac{(v^2-1)}{(2-v)} = x\frac{dv}{dx}$$

$$\frac{dv}{dx} = \frac{(v^2-1)}{x(2-v)}$$

$$\frac{(2-v)}{(v^2-1)}dv = \frac{1}{x}dx$$

Integrating on the both sides we get

$$\int\frac{2-v}{v^2-1}dv = \frac{1}{x}dx$$

$$\int\frac{2-v}{(v-1)(v+1)}dv = \frac{1}{x}dx$$

$$\therefore \quad \frac{2-v}{(v-1)(v+1)} = \frac{A}{(v-1)} + \frac{B}{(v+1)} \qquad \text{.... (iii)}$$

$$\Rightarrow \frac{2-v}{(v-1)(v+1)} = \frac{A(v+1)+B(v-1)}{(v-1)(v+1)}$$

$$\Rightarrow 2 - v = A(v+1) + B(v-1)$$

Putting $v = 1$

$$\Rightarrow 2-1 = A(1+1) + B(1-1)$$

$$\Rightarrow 1 = 2A + 0 \quad \Rightarrow 1 = 2A$$

$$\therefore A = \frac{1}{2}$$

Putting $v = -1$

$$2-(-1)=A(-1+1)+B(-1-1)$$

$$3=0+B(-2)$$

$$3=-2B$$

$$\therefore B=-\frac{3}{2}$$

Putting the value of A and B in equation (iii)

$$\frac{2-v}{(v-1)(v+1)}=\frac{\left(\frac{1}{2}\right)}{(v-1)}+\frac{\left(-\frac{3}{2}\right)}{(v+1)}=\frac{1}{2}\times\frac{1}{(v-1)}-\frac{3}{2}\times\frac{1}{(v+1)}$$

$$\int\frac{2-v}{(v-1)(v+1)}dv=\int\frac{1}{x}dx$$

$$\int\frac{1}{2}\times\frac{1}{(v-1)}dv-\int\frac{3}{2}\times\frac{1}{(v+1)}dv=\int\frac{1}{x}dx$$

$$\frac{1}{2}\int\frac{1}{(v-1)}dv-\frac{3}{2}\int\frac{1}{(v+1)}dv=\int\frac{1}{x}dx$$

$$\frac{1}{2}\log|v-1|-\frac{3}{2}\log|v+1|=\log|x|+c$$

Putting $v=\frac{y}{x}$

$$\frac{1}{2}\log\left|\frac{y}{x}-1\right|-\frac{3}{2}\log\left|\frac{y}{x}+1\right|=\log|x|+c$$

$$\frac{1}{2}\log\left|\frac{y-x}{x}\right|-\frac{3}{2}\log\left|\frac{y+x}{x}\right|=\log|x|+c$$

$$\frac{1}{2}\left[\log\left|\frac{y-x}{x}\right|-3\log\left|\frac{y+x}{x}\right|\right]=\log|x|+c$$

$$\log\left|\frac{y-x}{x}\right|-3\log\left|\frac{y+x}{x}\right|=2\log|x|+c$$

$$\log\left|\frac{y-x}{x}\right|-\log\left|\frac{(y+x)^3}{x^3}\right|=\log\left|x^2\right|+c$$

$$\log\left|\frac{\frac{y-x}{x}}{\frac{(y+x)^3}{x^3}}\right|=\log\left|x^2\right|+c$$

$$\log\left|(y-x)\times\frac{x^2}{(y+x)^3}\right|=\log\left|x^2\right|+c$$

$$\log\left|(y-x)\times\frac{x^2}{(y+x)^3}\right|=\log\left|x^2\right|+c$$

$$\log\left|\frac{(y-x)}{(y+x)^3}\times x^2\right|-\log\left|x^2\right|=c$$

$$\log\left|\frac{(y-x)x^2}{(y+x)^3}\times\frac{1}{x^2}\right|=\log|c|$$

$$\frac{(y-x)}{(y+x)^3}=c$$

$$(y-x)=c(y+x)^3$$

9. $$\frac{dy}{dx}+\frac{x^2-y^2}{3xy}=0$$

Sol. On separating variables we get $\frac{dy}{dx}=-\left(\frac{x^2-y^2}{3xy}\right)$

$$\frac{dy}{dx}=\frac{y^2-x^2}{3xy}$$

$\because y^2-x^2$ and $3xy$ both are homogenous functions because both function having 2 degree

$$\frac{dy}{dx}=\frac{y^2-x^2}{3xy} \qquad \text{.... (i)}$$

Let $y=vx \Rightarrow v=\frac{y}{x}$

Diff on the both sides w.r. to x

$$\frac{dy}{dx}=v\frac{dx}{dx}+x\frac{dv}{dx}$$

$$\frac{dy}{dx}=v+x\frac{dv}{dx} \qquad \text{..... (ii)}$$

From equation (i) and (ii)

$$\frac{y^2-x^2}{3xy}=v+x\frac{dv}{dx}$$

Putting $y=vx$

$$\frac{v^2x^2-x^2}{3x(v^x)}=v+x\frac{dv}{dx}$$

$$\frac{x^2(v^2-1)}{3vx^2}=v+x\frac{dv}{dx}$$

$$\frac{(v^2-1)}{3v}=v+x\frac{dv}{dx}$$

$$\frac{v^2-1}{3v}-v=x\frac{dv}{dx}$$

$$\frac{(v^2-1)}{3v}=v+x\frac{dv}{dx}$$

$$\frac{v^2-1}{3v}-v=x\frac{dv}{dx}$$

$$\frac{v^2-1-v(3v)}{3v}=x\frac{dv}{dx}$$

$$\frac{v^2-1-3v^2}{3v}=x\frac{dv}{dx}$$

$$\frac{-1-2v^2}{3v}=x\frac{dv}{dx}$$

$$\frac{-\left(2v^2+1\right)}{3v}=x\frac{dv}{dx}$$

$$\frac{-\left(2v^2+1\right)}{3v}=x\frac{dv}{dx}$$

$$\frac{-\left(2v^2+1\right)}{3vx}=\frac{dv}{dx}$$

$$\frac{dv}{dx}=\frac{-\left(2v^2+1\right)}{3vx}$$

$$\left(\frac{3v}{2v^2+1}\right)dv=-\frac{1}{x}dx$$

Integrating on the both sides we get

$$\int\left(\frac{3v}{2v^2+1}\right)dv=\int-\frac{1}{x}dx$$

Dividing and multiplying by 4

$$\frac{3}{4}\int\frac{4v}{2v^2+1}dv=-\int\frac{1}{x}dx$$

Let $2v^2+1=t$

Dif on the both sides w.r. to v

$$4v+0=\frac{dt}{dv}$$

$$dv=\frac{dt}{4v}$$

$$\frac{3}{4}\int\frac{4v}{t}\times\frac{dt}{4v}=-\int\frac{1}{x}dx$$

$$\frac{3}{4}\int\frac{1}{t}dt=-\int\frac{1}{x}dx$$

$$\frac{3}{4}\log|t|=-\int\frac{1}{x}$$

$$\frac{3}{4}\log\left|2v^2+1\right|=-\log|x|+c$$

Putting $v=\frac{y}{x}$

$$\frac{3}{4}\log\left|\frac{2y^2}{x^2}+1\right|=-\log|x|+c$$

$$\frac{3}{4}\log\left|\frac{2y^2+x^2}{x^2}\right|=-\log|x|+c$$

$$3\log\left|\frac{2y^2+x^2}{x^2}\right|+4\log|x|=c$$

$$\log\left|\frac{(2y^2+x^2)^3}{x^6}\right|+\log|x^4|=c$$

$$\log\left|\frac{(2y^2+x^2)^3}{x^6}\times x^4\right|=\log|c|$$

$$\frac{(2y^2+x^2)^3}{x^2}=c$$

$$(2y^2+x^2)^3=cx^2$$

10. $\frac{dy}{dx}=\frac{x^2+y^2}{2xy}$

Sol. $\frac{dy}{dx}=\frac{x^2+y^2}{2xy}$

On separating variables we get $\frac{dy}{dx}=\frac{x^2+y^2}{2xy}$

$\because x^2+y^2$ and $2xy$ both are homogeneous function because both function having 2 degrees

$\frac{dy}{dx}=\frac{x^2+y^2}{2xy}$ a homogeneous differential equation

$\frac{dy}{dx}=\frac{x^2+y^2}{2xy}$ .... (i)

Let $y=vx \Rightarrow v=\frac{y}{x}$

Diff on the both sides w.r. to x

$$\frac{dy}{dx}=v\frac{dx}{dx}+x\frac{dv}{dx}$$

$\frac{dy}{dx}=v+x\frac{dv}{dx}$ ..... (ii)

From equation (i) and (ii)

$$\frac{x^2+y^2}{2xy}=v+x\frac{dv}{dx}$$

Putting $y=vx$

$$\frac{x^2+v^2x^2}{2x(vx)}=v+x\frac{dv}{dx}$$

$$\Rightarrow\frac{x^2(1+v^2)}{2vx^2}=v+x\frac{dv}{dx}\Rightarrow\frac{1+v^2}{2v}=v+x\frac{dv}{dx}\Rightarrow\frac{1+v^2}{2v}-v=x\frac{dv}{dx}\Rightarrow\frac{1+v^2-v(2v)}{2v}=x\frac{dv}{dx}$$

$$\Rightarrow\frac{1+v^2-2v^2}{2v}=x\frac{dv}{dx}\Rightarrow\frac{1-v^2}{2v}=x\frac{dv}{dx}\Rightarrow\frac{(1-v^2)}{2vx}=\frac{dv}{dx}\Rightarrow\frac{dv}{dx}=\frac{(1-v^2)}{2vx}\Rightarrow\frac{2v}{(1-v^2)}dv=\frac{1}{x}dx$$

Integrating on the both sides we get $\int\frac{2v}{1-v^2}dv=\int\frac{1}{x}dx$

Let $1-v^2=t$

Diff on the both sides w.r. to v

$0-2v=\frac{dt}{dv}$

$dv=\frac{dt}{-2v}$

$\int\frac{2v}{t}\times\frac{dt}{-2v}=\int\frac{1}{x}dx$

$-\log|t|=\log|x|+c$

$-\log\left|1-v^2\right|=\log|x|+c$

$-\log\left|1-v^2\right|-\log|x|=c$

$-\left[\log\left|1-v^2\right|+\log|x|\right]=c$

$\log\left|1-v^2\right|+\log|x|=-c$

$\log\left|1-v^2\right|+\log|x|=c \quad [\because -c=c \text{ constant}]$

Putting $v=\frac{y}{x}$

$\log\left|1-\frac{y^2}{x^2}\right|+\log|x|=c$

$\log\left|\frac{x^2-y^2}{x^2}\right|+\log|x|=c$

$\log\left|\frac{(x^2-y^2)}{x^2}\times x\right|=\log|c|$

$\frac{x^2-y^2}{x}=c$

$x^2-y^2=cx$

11. $\frac{dy}{dx}=\frac{2xy}{x^2-y^2}$

Sol. $\frac{dy}{dx}=\frac{2xy}{x^2-y^2}$

On separating variables we get $\frac{dy}{dx}=\frac{2xy}{x^2-y^2}$

$\because 2xy$ and $\left(x^2-y^2\right)$ both are homogeneous function because both function having 2 degree

$\frac{dy}{dx}=\frac{2xy}{x^2-y^2}$ of a homogeneous differential equation $\frac{dy}{dx}=\frac{2xy}{x^2-y^2}$ …… (i)

Let $y=vx\Rightarrow v=\frac{y}{x}$

Diff on the both sides w.r. to x

$\frac{dy}{dx}=v\frac{dv}{dx}+x\frac{dv}{dx}$

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \qquad \ldots\ldots \text{(ii)}$$

From equation (i) and (ii)

$$\frac{2xy}{x^2 - y^2} = v + x\frac{dv}{dx}$$

Putting $y = vx$

$$\frac{2x(vx)}{x^2 - v^2x^2} = v + x\frac{dv}{dx}$$

$$\frac{2vx^2}{x^2(1-v^2)} = v + x\frac{dv}{dx}$$

$$\frac{2v}{1-v^2} = v + x\frac{dv}{dx}$$

$$\frac{2v - v(1-v^2)}{(1-v^2)} = x\frac{dv}{dx}$$

$$2v - v + v^3 = x\frac{dv}{dx}$$

$$\frac{(v+v^3)}{(1-v^2)} = x\frac{dv}{dx}$$

$$\frac{(v+v^3)}{(1-v^2)x} = \frac{dv}{dx}$$

$$\frac{dv}{dx} = \frac{(v+v^3)}{x(1-v^2)}$$

$$\frac{1-v^2}{v+v^3}dv = \frac{1}{x}dx$$

Integrating on the both sides we get $\int\frac{1-v^2}{v+v^3}dv = \int\frac{1}{x}dx$

$$\int\frac{(1-v^2)}{v(1+v^2)}dv = \int\frac{1}{x}dx$$

$$\because \frac{(1-v^2)}{v(1+v^2)} = \frac{A}{v} + \frac{(Bv+c)}{1+v^2} \quad \ldots \text{(ii)}$$

$$\frac{1-v^2}{v(1+v^2)} = \frac{A(1+v^2) + (Bv+c)v}{v(1+v^2)}$$

$$1 - v^2 = A + Av^2 + Bv^2 + Cv$$

$$1 + 0v - v^2 = A + Av^2 + Bv^2 + Cv$$

$$1 + 0v - v^2 = v^2(A+B) + Cv + A$$

Comparing on the both sides proper coefficient

$-v^2 = v^2(A+B)$ $\qquad$ $Cv = 0v$ $\qquad$ $A = 1$

$-1 = A + B$ $\qquad$ $c = 0$

$B=-A-1$

Putting the value of $A=1$ in equation (iv)

$B=-1-1$

$B=-2$

Putting the value of A, B and C in equation (iii)

$$\frac{1-v^2}{v\left(1+v^2\right)}=\frac{1}{v}+\frac{\left(-2v+0\right)}{1+v^2}=\frac{1}{v}+\frac{\left(-2v\right)}{1+v^2}$$

$$\int\frac{1}{v}dv-2\int\frac{v}{1+v}dv=\int\frac{1}{x}dx$$

$$\int\frac{1}{v}dv-\int\frac{2v}{1+v^2}dv=\int\frac{1}{x}dx$$

Let $1+v^2=t$

Diff on the both sides w.r. to v

$$2v=\frac{dt}{dv}$$

$$dv=\frac{dt}{2v}$$

$$\int\frac{1}{v}dv-\int\frac{2v}{t}\times\frac{dt}{2v}=\log\left|x\right|+c$$

$$\log\left|v\right|-\log\left|t\right|=\log\left|x\right|+c$$

$$\log\left|v\right|-\log\left|1+v^2\right|=\log\left|x\right|+c$$

$$\log\left|\frac{y}{x}\right|-\log\left|1+\frac{y^2}{x^2}\right|=\log\left|x\right|+c$$

$$\log\left|\frac{y}{x}\right|-\log\left|\frac{x^2+y^2}{x^2}\right|=\log\left|x\right|+c$$

$$\log\left|\frac{\frac{y}{x}}{x}\right|-\log\left|\frac{x^2+y^2}{x^2}\right|=c$$

$$\log\left|\frac{y}{x^2}\right|-\log\left|\frac{x^2+y^2}{x^2}\right|=\log\left|c\right|$$

$$\log\left|\frac{\frac{y}{x}}{\frac{x^2+y^2}{x^2}}\right|=\log\left|c\right|$$

$$\log\left|\frac{y}{x^2+y^2}\right|=\log\left|c\right|$$

$$\frac{y}{x^2+y^2}=c$$

$$y=c\left(x^2+y^2\right), \qquad y=c\left(y^2+x^2\right)$$

In each of the following differential equations show that it is homogenous and solve it

12. $\frac{x^2dy}{dx} = 2xy + y^2$

Sol. $x^2\frac{dy}{dx} = 2xy + y^2$

On separable variables we get $\frac{dy}{dx} = \frac{2xy + y^2}{x^2}$

$\because \left(2xy + y^2\right)$ and x^2 both are homogeneous function because both function having 2 degree

$\frac{dy}{dx} = \frac{2xy + y^2}{x^2}$ of a homogenous differential equation $\frac{dy}{dx} = \frac{2xy + y^2}{x^2}$ ..... (i)

Let $y = vx = v = \frac{y}{x}$

Diff on the both sides w.r.to x

$$\frac{dy}{dx} = \frac{vdx}{dx} + x\frac{dv}{dx}$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad .... (ii)$$

From equation (i) and (ii)

$$\frac{2xy + y^2}{x^2} = v + x\frac{dv}{dx}$$

Putting $y = vx$

$$\frac{2x(vx) + v^2x^2}{x^2} = v + x\frac{dv}{dx}$$

$$\frac{2vx^2 + v^2x^2}{x^2} = v + x\frac{dv}{dx}$$

$$\frac{x^2\left(2v + v^2\right)}{x^2} = v + x\frac{dv}{dx}$$

$$2v + v^2 - v = x\frac{dv}{dx}$$

$$\left(v^2 + v\right) = x\frac{dv}{dx}$$

$$\frac{dv}{dx} = \frac{v^2 + v}{x}$$

$$\frac{dv}{dx} = \frac{v(v+1)}{x}$$

$$\frac{dv}{v(v+1)} = \frac{1}{x}dx$$

Integrating on the both sides we get $\int \frac{dv}{v(v+1)} = \int \frac{1}{x}dx$

$$\because \frac{1}{v(v+1)} = \frac{A}{v} + \frac{B}{v+1} \quad .... (iii)$$

$$\frac{1}{v(v+1)}=\frac{A(v+1)+Bv}{v(v+1)}$$

$$1=A(v+1)+Bv$$

Putting $1=-1$

$$1=A(-1+1)+B(-1)$$

$$1=0+(-B)$$

$$B=-1$$

Putting $v=0$

$$1=A(0+1)+B(0)$$

$$A=1$$

Putting the value of A and B in equation (iii)

$$\frac{1}{v(v+1)}=\frac{1}{v}+\frac{(-1)}{(v+1)}$$

$$\int\frac{1}{v(v+1)}dv=\int\frac{1}{x}dx$$

$$\int\frac{1}{v}dv-\int\frac{1}{(v+1)}dv=\int\frac{1}{x}dx$$

$$\log|v|-\log|v+1|=\log|x|+c$$

Putting $v=\frac{y}{x}$

$$\log\left|\frac{y}{x}\right|-\log\left|\frac{y}{x}+1\right|=\log|x|+c$$

$$\log\left|\frac{y}{x}\right|-\log\left|\frac{y+x}{x}\right|=\log|x|+c$$

$$\log\left|\frac{\frac{y}{x}}{\frac{y+x}{x}}\right|=\log|x|+c$$

$$\log\left|\frac{y}{y+x}\right|=\log|x|+c$$

$$\log\left|\frac{y}{y+x}\right|-\log|x|=\log|c|$$

$$\log\left|\left(\frac{y}{y+x}\right)\frac{1}{x}\right|=\log|c|$$

$$\log\left|\frac{y}{x(y+x)}\right|=\log|c|$$

$$y=cx(y+x)$$

$$y=cx(x+y)$$

13. $x^2\frac{dy}{dx}=x^2+xy+y^2$

Sol. $x^2\frac{dy}{dx}=x^2+xy+y^2$

On separating variable we get

$$\frac{dy}{dx}=\frac{x^2+xy+y^2}{x^2}$$

$\because\left(x^2+xy+y^2\right)$ and x^2 both are homogenous function because both function having 2 degree

$\frac{dy}{dx}=\frac{x^2+xy+y^2}{x^2}$ of a homogenous differential equation $\frac{dy}{dx}=\frac{x^2+xy+y^2}{x^2}$ .... (i)

Let $y=vx\Rightarrow v=\frac{y}{x}$

Diff on the both sides w.r to x

$$\frac{dy}{dx}=v\frac{dx}{dx}+x\frac{dv}{dx}$$

$$\frac{dy}{dx}=v+x\frac{dv}{dx} \qquad \text{...... (ii)}$$

From equation (i) and (ii)

$$\frac{x^2+xy+y^2}{x^2}=v+x\frac{dv}{dx}$$

Putting $y=vx$

$$\frac{x^2+x(vx)+v^2x^2}{x^2}=v+x\frac{dv}{dx}$$

$$\frac{x^2\left(1+v+v^2\right)}{x^2}=v+x\frac{dv}{dx}$$

$$v^2+v+1=v+x\frac{dv}{dx}$$

$$v^2+v-v+1=x\frac{dv}{dx}$$

$$v^2+1=x\frac{dv}{dx}$$

$$\frac{v^2+1}{x}=\frac{dv}{dx}$$

$$\frac{dv}{v^2+1}=\frac{1}{x}dx$$

Integrating on the both sides we get

$$\int\frac{dv}{v^2+1}=\int\frac{1}{x}dx$$

$$\frac{1}{1}\tan^{-1}\frac{v}{1}=\log|x|+c$$

$$\tan^{-1}v=\log|x|+c$$

Putting $v=\frac{y}{x}$

$$\tan^{-1}\left(\frac{y}{x}\right)=\log|x|+c$$

$$\tan^{-1}\frac{y}{x}=\log|x|+c$$

14. $y^2+\left(x^2-xy\right)\frac{dy}{dx}=0$

Sol. $y^2+\left(x^2-xy\right)\frac{dy}{dx}=0$

On separating variables we get $\left(x^2-xy\right)\frac{dy}{dx}=-y^2$

$$\frac{dy}{dx}=\frac{-y^2}{x^2-xy}$$

$\because -y^2$ and $\left(x^2-xy\right)$ both are homogeneous function because both function having 2 degree

$\frac{dy}{dx}=\frac{-y^2}{x^2-xy}$ of a homogeneous differential equation $\frac{dy}{dx}=\frac{-y^2}{x^2-xy}$ .... (i)

Let $y=vx \Rightarrow v=\frac{y}{x}$

Diff on the both sides w.r. to x

$$\frac{dy}{dx}=v\frac{dx}{dx}+x\frac{dv}{dx}$$

$$\frac{ddxy}{}=v+x\frac{dv}{dx} \quad \text{..... (ii)}$$

From equation (i) and (ii)

$$\frac{-y^2}{x^2-xy}=v+x\frac{dv}{dx}$$

Putting $y=vx$

$$\frac{-v^2x^2}{x^2-x(vx)}=v+x\frac{dv}{dx}$$

$$\frac{-v^2x^2}{x^2(1-v)}=v+x\frac{dv}{dx}$$

$$\frac{-v^2}{(1-v)}=v+x\frac{dv}{dx}$$

$$\frac{-v^2-v}{(1-v)}=x\frac{dv}{dx}$$

$$\frac{-v^2-v(1-v)}{(1-v)}=x\frac{dv}{dx}$$

$$\frac{-v-v(1-v)}{(1-v)}=x\frac{dv}{dx}$$

$$\frac{-v^2-v+v^2}{(1-v)}=x\frac{dv}{dx}$$

$$\frac{-v}{-(v-1)} = x\frac{dv}{dx}$$

$$\frac{v}{(v-1)} = x\frac{dv}{dx}$$

$$\frac{dv}{dx} = \frac{v}{x(v-1)}$$

$$\frac{(v-1)}{v}dv = \frac{1}{x}dx$$

$$\left(\frac{v}{v} - \frac{1}{v}\right)dv = \frac{1}{x}dx$$

$$\left(1 - \frac{1}{v}\right)dv = \frac{1}{x}dx$$

Integrating on the both sides we get

$$\int\left(1 - \frac{1}{v}\right)dv = \int\frac{1}{x}dx$$

$$\int dv - \int\frac{1}{v}dv = \int\frac{1}{x}dx$$

$$v - \log|v| = \log|x| + c$$

Putting $v = \frac{y}{x}$

$$\frac{y}{x} - \log\left|\frac{y}{x}\right| = \log|x| + c$$

$$\frac{y}{x} - \log\left|\frac{y}{x}\right| - \log|x| = c$$

$$\frac{y}{x} - \log\left|\frac{y}{x} \times x\right| = c$$

$$\frac{y}{x} - \log|y| = c$$

$$\frac{y}{x} = c + \log|y|$$

$$y = x\{c + \log|y|\}$$

5. $x\frac{dy}{dx} - y = 2\sqrt{y^2 - x^2}$

Sol. $x\frac{dy}{dx} - y = 2\sqrt{y^2 - x^2}$

On separating variables we get $x\frac{dy}{dx} = 2\sqrt{y^2 - x^2} + y$

$$\frac{dy}{dx} = \frac{2\sqrt{y^2 - x^2} + y}{x}$$

$\because \left(2\sqrt{y^2 - x^2} + y\right)$ and x both are homogeneous function because

$\frac{dy}{dx}=\frac{2\sqrt{y^2-x^2}+y}{x}$ of a homogeneous differential equation

$$\frac{dy}{dx}=\frac{2\sqrt{y^2-x^2}+y}{x} \quad \text{...... (i)}$$

Let $y=vx \Rightarrow v=\frac{y}{x}$

Differentiation on the both sides w.r. to x

$$\frac{dy}{dx}=v\frac{dx}{dx}+x\frac{dv}{dx}$$

$$\frac{dy}{dx}=v+x\frac{dv}{dx} \quad \text{..... (ii)}$$

From equation (i) and (ii)

$$\frac{2\sqrt{y^2-x^2}+y}{x}=v+x\frac{dv}{dx}$$

Putting $y=vx$

$$\frac{2\sqrt{v^2x^2-x^2}+vx}{x}=v+x\frac{dv}{dx}$$

$$\frac{2x\sqrt{v^2-1}+vx}{x}=v+x\frac{dv}{dx}$$

$$\frac{x\left(v\sqrt{v^2-1}+v\right)}{x}=v+x\frac{dv}{dx}$$

$$2\sqrt{v^2-1}+v-v=x\frac{dv}{dx}$$

$$2\sqrt{v^2-1}=x\frac{dv}{dx}$$

$$\frac{dv}{dx}=\frac{2\sqrt{v^2-1}}{x}$$

$$\frac{dv}{2\sqrt{v^2-1}}=\frac{1}{x}dx$$

Integrating on the both sides we get $\int\frac{dv}{2\sqrt{v^2-1}}=\int\frac{1}{x}dx$

$$\frac{1}{2}\log\left|v+\sqrt{v^2-1}\right|=\log|x|+c$$

Putting $v=\frac{y}{x}$

$$\frac{1}{2}\log\left|\frac{y}{x}+\sqrt{\frac{y^2}{x^2}-1}\right|=\log|x|+c$$

$$\frac{1}{2}\log\left|\frac{y}{x}+\sqrt{\frac{y^2-x^2}{x^2}}\right|=\log|x|+c$$

$$\frac{1}{2}\log\left|\frac{y}{x}+\frac{\sqrt{y^2-x^2}}{x}\right|=\log|x|+c$$

$$\frac{1}{2}\log\left|\frac{y+\sqrt{y^2-x^2}}{x}\right|=\log|x|+c$$

$$\log\left|\frac{y+\sqrt{y^2-x^2}}{x}\right|=2\log|x|+c$$

$$\log\left|\frac{y+\sqrt{y^2-x^2}}{x}\right|=\log\left|x^2\right|+c$$

$$\log\left|y+\sqrt{y^2-x^2}\right|-\log|x|=\log\left|x^2\right|+c$$

$$\log\left|y+\sqrt{y^2-x^2}\right|=\log\left|x^2+x\right|+\log|c|$$

$$\log\left|y+\sqrt{y-x^2}\right|=\log\left|x^3\right|+\log|c|$$

$$\log\left|y+\sqrt{y^2-x^2}\right|=\log\left|x^3+c\right|$$

$$y+\sqrt{y^2-x^2}=cx^3$$

16. $y^2dx+\left(x^2+xy+y^2\right)dy=0$

Sol. The given equation may be written as $\frac{dy}{dx}=-\frac{y^2}{x^2+xy+y^2}$

For checking homogeneous putting tx for x and ty for y.

The given equation remains unchanged so it is a homogeneous equation.

Put $y=vx$, we get, $\frac{dy}{dx}=v+x\frac{dv}{dx}$ $\Rightarrow v+x\frac{dv}{dx}=-\frac{v^2x^2}{x^2+x.vx+v^2x^2}$ $\Rightarrow v+x\frac{dv}{dx}=-\frac{v^2}{1+v+v^2}$

$\Rightarrow x\frac{dv}{dx}=-\frac{v^2}{1+v+v^2}-\frac{v}{1}$ $\Rightarrow x\frac{dv}{dx}=-\left[\frac{v^2+v+v^2+v^3}{1+v+v^2}\right]$ $\Rightarrow \int\frac{1+v+v^2}{v^3+2v^2+v}dv=-\int\frac{dx}{x}$

$\Rightarrow \int\frac{v^2+v+1}{v\left(v^2+2v+1\right)}dv=-\log|x|+\log c$ $\Rightarrow \int\frac{v^2+v+1}{v(v+1)^2}dv=-\log|x|+\log c$

$\Rightarrow I_1=-\log|x|+\log c$ ...(1)

$I_1=\int\frac{v^2+v+1}{v(v+1)^2}\,dv$ $\Rightarrow \frac{v^2+v+1}{v(v+1)^2}=\frac{A}{v}+\frac{B}{v+1}+\frac{C}{(v+1)^2}$

$\Rightarrow \frac{v^2+v+1}{v(v+1)^2}=\frac{A(v+1)^2+Bv(v+1)+Cv}{v(v+1)^2}$ $\Rightarrow v^2+v+1=A\left(v^2+2v+1\right)+B\left(v^2+v\right)+Cv$

$\Rightarrow v^2+v+1=v^2(A+B)+V(2A+B+C)+A.$

Equating co-efficient both side we get,

$A+B=1$ ...(2)

$2A+B+C=1$ ...(3)

$A = 1 \qquad \ldots(4)$

From (3), $A = 1 \Rightarrow A + B = 1 \quad \therefore B = 0$

Putting the value of A, B in equation (3), $2.1 + 0 + C = 1 \Rightarrow C = -1 \quad \therefore A = 1, B = 0, C = -1$

$$\Rightarrow I_1 = \int\left(\frac{A}{v} + \frac{B}{v+1} + \frac{C}{(v+1)^2}\right)dv \quad \Rightarrow I = \log v$$

$$\Rightarrow I_1 = \log\left(\frac{y}{x}\right) + \frac{1}{\frac{y}{x}+1} + c \qquad \therefore I_1 = \log\left(\frac{y}{x}\right) + \frac{x}{x+y} + c$$

Putting the value of I_1 in equation (1), $\log\left(\frac{y}{x}\right) + \frac{x}{x+y} = -\log|x| + \log c \quad \therefore \log|y| + \frac{x}{x+y} = c$

17. $(x-y)\frac{dy}{dx} = x + 3y$

Sol. The given equation may be written as $\frac{dy}{dx} = \frac{x+3y}{x-y}$

For checking homogeneous putting tx for x and ty for y.

The given equation remains unchanged so it is a homogeneous equation.

Put $y = vx$, we get, $\frac{dy}{dx} = v + x\frac{dv}{dx} \quad \Rightarrow v + x\frac{dv}{dx} = \frac{x+3.vx}{x-vx} \quad \Rightarrow v + x\frac{dv}{dx} = \frac{x+3.vx}{x-vx}$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+3v}{1-v} - \frac{v}{1} \quad \Rightarrow x\frac{dv}{dx} = \frac{1+3v-v+v^2}{1-v} \quad \Rightarrow x\frac{dv}{dx} = \frac{1+2v+v^2}{1-v}$$

$$\Rightarrow \int\frac{1-v}{v^2+2v+1}dv = \int\frac{dx}{x} \quad \Rightarrow I_1 = \log|x| + \log c \qquad \ldots(1)$$

$$I_1 = \int\frac{1-v}{v^2+2v+1}dv$$

Putting $1 - v = A\frac{d}{dv}\left(v^2 + 2v + 1\right) + B$, we get,

$$1 - v = A(2v+2) + B \quad \Rightarrow 1 - v = 2Av + 2A + B \quad \Rightarrow 2A = -1 \ \& \ 2A + B = 1$$

$$A = -\frac{1}{2} \ \& \ B = 1 - 2A \quad \Rightarrow B = 1 + 1 \quad \therefore B = 2$$

$$\Rightarrow I_1 = \int\frac{A(2v+2)+B}{v^2+2v+1}dv \quad \Rightarrow I_1 = A\int\frac{2v+2}{v^2+2v+1}dv + B\int\frac{1}{v^2+2v+1}dv$$

Put $v^2 + 2v + 1 = t \quad \Rightarrow (2v+2) = \frac{dt}{dv} \quad \Rightarrow (2v+2)dv = dt$

$$\Rightarrow I_1 = -\frac{1}{2}\int\frac{1}{t}dt + 2\int\frac{1}{(v)^2 + 2.v.1 + (1)^2 - (1)^2 + 1}dv$$

$$\Rightarrow I_1 = -\frac{1}{2}\log\left|v^2+2v+1\right| + 2\int\frac{1}{(v+1)^2}dv \quad \Rightarrow I_1 = -\frac{1}{2}\log\left|v^2+2v+1\right| - 2.\frac{1}{v+1}$$

$$\Rightarrow I_1 = -\frac{1}{2}\log\left|(v+1)^2\right| - 2\frac{1}{v+1} + c \quad \Rightarrow I_1 = -\log|v+1| - \frac{2}{v+1} + c$$

$$\Rightarrow I_1 = -\log\left|\frac{y}{x}+1\right| - \frac{2}{\frac{y}{x}+1} + c \qquad \Rightarrow I_1 = -\log\left|\frac{x+y}{x}\right| - \frac{2x}{x+y} + c$$

Putting the value of I_1 in equation (1),

$$-\log\left(\frac{x+y}{x}\right) - \frac{2x}{x+y} = \log x + \log c \qquad \therefore \log\left|c(x+y)\right| + \frac{2x}{x+y} = 0$$

18. $\left(x^3+3xy^2\right)dx + \left(y^3+3x^2y\right)dy = 0$

Sol. The given equation may be written as $\dfrac{dy}{dx} = \dfrac{-\left(x^3+3xy^2\right)}{y^3+3x^2y}$

For checking homogeneous putting tx for x and ty for y.

The given equation remains unchanged so it is a homogeneous equation.

Put $y = vx$, we get, $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx} \quad \Rightarrow v + x\dfrac{dv}{dx} = \dfrac{-\left(x^3+3x.v^2x^2\right)}{v^3x^3+3x^2.vx}$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{-\left(1+3v^2\right)}{v^3+3v} \quad \Rightarrow x\frac{dv}{dx} = \frac{-1-3v^2}{v^3+3v} - \frac{v}{1} \quad \Rightarrow x\frac{dv}{dx} = \frac{-1-3v^2-v^4-3v^2}{v^3+3v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{-1-6v^2-v^4}{v^3+3v} \quad \Rightarrow \int\frac{v^3+3v}{v^4+6v^2+1}\,dv = -\int\frac{dx}{x}$$

Put $v^4+6v^2+1 = t \quad \Rightarrow 4v^3+12v = \dfrac{dt}{dv} \quad \Rightarrow \left(v^3+3v\right)dv = \dfrac{dt}{4}$

$$\Rightarrow \frac{1}{4}\int\frac{1}{t}\,dt = -\log|x| + \log(c) \quad \Rightarrow \frac{1}{4}\log|t| = 4\log xc - \log|x| + \log c$$

$$\Rightarrow \frac{1}{4}\log\left|v^4+6v^2+1\right| = -\log|x| + \log c \quad \Rightarrow \log\left(\frac{y^4}{x^4}+\frac{6y^2}{x^2}+1\right) = \log x^{-1}.c$$

$$\Rightarrow \log\frac{\left(y^4+6x^2y^2+x^4\right)}{x^4} = 4\log x^{-1}c = \log x^{-4}c \quad \Rightarrow \frac{y^4+6x^2y^2+x^4}{x^4} = x^{-4}c$$

$$\Rightarrow y^4+6x^2y^2+x^4 = c$$

19. $\left(x-\sqrt{xy}\right)dy = ydx$

Sol. The given equation may be written as $\left(x-\sqrt{xy}\right)dx = y\,dx \Rightarrow \dfrac{dy}{dx} = \dfrac{y}{x-\sqrt{xy}}$

For checking homogeneous putting tx for x and ty for y.

The given equation remains unchanged so it is a homogeneous equation.

Put $y = vx$, we get, $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx} \Rightarrow v + x\dfrac{dv}{dx} - \dfrac{vx}{x-\sqrt{x.vx}} \quad \Rightarrow v + x\dfrac{dv}{dx} = \dfrac{v}{1-\sqrt{v}}$

$$\Rightarrow x\frac{dv}{dx} = \frac{v}{1-\sqrt{v}} - \frac{v}{1} \Rightarrow x\frac{dv}{dx} = \frac{v-v+v\sqrt{v}}{1-\sqrt{v}} \quad \Rightarrow \int\frac{1-\sqrt{v}}{v\sqrt{v}}\,dv = \int\frac{dx}{x}$$

$$\Rightarrow \int\frac{1}{v^{3/2}}\,dv - \int\frac{1}{v}\,dv = \log|x| \quad \Rightarrow \frac{v^{-\frac{1}{2}}}{-1/2} - \log|v| = \log|x| + c \quad \Rightarrow -\frac{2}{\sqrt{v}} - \log|v| = \log|x| + c$$

$\Rightarrow -\frac{2\sqrt{x}}{\sqrt{y}} - \log\left|\frac{y}{x}\right| = \log|x| + c \quad \Rightarrow -2\sqrt{\frac{x}{y}} - \log y = c$

$\therefore \; 2\sqrt{\frac{x}{y}} + \log|y| = cx^2$ ***[Answer given in the book is wrong.]***

). $x^2\frac{dy}{dx} + y^2 = xy$

ol. The given equation may be written as $x\frac{dy}{dx} = y - \frac{y^2}{x} \Rightarrow \frac{dy}{dx} = \frac{xy - y^2}{x^2}$

For checking homogeneous putting tx for x and ty for y.

The given equation remains unchanged so it is a homogeneous equation.

Put $y = vx$, we get, $\frac{dy}{dx} = v + x\frac{dv}{dx} \Rightarrow v + x\frac{dv}{dx} = \frac{x.vx - v^2x^2}{x^2} \Rightarrow v + x\frac{dv}{dx} = v - v^2$

$\Rightarrow x\frac{dv}{dx} = -v^2 \quad \Rightarrow \int\frac{dv}{v^2} = -\int\frac{dx}{x} \quad \Rightarrow -\frac{1}{v} = -\log|x| + c \quad \Rightarrow \frac{1}{v} = \log|x| - c$

$\Rightarrow \frac{x}{y} = \log|x| - c \qquad \therefore \; c = \log|x| - \frac{x}{y}$

1. $x\frac{dy}{dx} = y(\log y - \log x + 1)$

ol. The given equation may be written as $\frac{dy}{dx} = \frac{y(\log y - \log x + 1)}{x}$

For checking homogeneous putting tx for x and ty for y.

The given equation remains unchanged so it is a homogeneous equation.

Put $y = vx$, we get $\frac{dy}{dx} = v + x\frac{dv}{dx} \quad \Rightarrow v + x\frac{dv}{dx} = \frac{vx(\log(vx) - \log x + 1)}{x}$

$\Rightarrow v + x\frac{dv}{dx} = v(\log v + \log x - \log x + 1) \quad \Rightarrow v + x\frac{dv}{dx} = v(\log v + 1)$

$\Rightarrow x\frac{dv}{dx} = v(\log v + 1) \quad \Rightarrow x\frac{dv}{dx} = v\log v \quad \Rightarrow \int\frac{1}{v\log v}dv = \int\frac{dx}{x}$

$\Rightarrow \log(\log v) = \log(x) + \log c \quad \Rightarrow \log v = xc \quad \Rightarrow \log\left(\frac{y}{x}\right) = xc \quad \Rightarrow \frac{y}{x} = e^{xc} \quad \therefore \; y = x\,e^{xc}$

2. $x\frac{dy}{dx} - y + x\sin\frac{y}{x} = 0$

ol. Given differential equation is $x\cdot\frac{dy}{dx} - y + x\cdot\sin\frac{y}{x} = 0$

$\Rightarrow x\cdot\frac{dy}{dx} = y - x\cdot\sin\frac{y}{x} \quad \Rightarrow \frac{dy}{dx} = \frac{y}{x} - \sin\frac{y}{x}$ …. (i)

This is of the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$

This the given differential equation is homogeneous differential equation

Let $\frac{y}{x} = v \quad \Rightarrow y = vx$

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

From (i)

$v + x\frac{dv}{dx} = v - \sin v$

$\Rightarrow x \cdot \frac{dv}{dx} = -\sin v \quad \Rightarrow \operatorname{cosec} v dv = -\frac{dx}{x}$

Integrating both sides we get $\int \operatorname{cosec} v\, dv = -\int \frac{dx}{x}$

$\Rightarrow \log\left|\tan\frac{v}{2}\right| = -\log|x| + \log C_1$

$\Rightarrow \log|x| \log\left|\tan\frac{v}{2}\right| = \log C_1$

$\Rightarrow \log\left|x \cdot \tan\frac{v}{2}\right| = \log C_1$

$\Rightarrow \left|x \cdot \tan\frac{v}{2}\right| = C_1$

$\Rightarrow x \cdot \tan\left(\frac{y}{2x}\right) = \pm C_1$

$\Rightarrow x \cdot \tan\left(\frac{y}{2x}\right) = C \quad \{\text{Let } \pm C_1 = C$

This is the required solution of given differential equation

23. $x\frac{dy}{dx} = y - x\cos^2\left(\frac{y}{x}\right)$

Sol. Given differential equation is

$x \cdot \frac{dy}{dx} = y - x \cdot \cos^2 \frac{y}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{y}{x} - \cos^2 \frac{y}{x} \qquad \text{..... (i)}$

This is of the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$

Thus the given differential equation is a hamogenous differential equation

Let $\frac{t}{x} = v$

$\Rightarrow v = vx$

$\Rightarrow \frac{dy}{dx} = v + x \cdot \frac{dv}{dx}$

Now, from (i) we have $v + x \cdot \frac{dv}{dx} = v - \cos^2 v$

$\Rightarrow x \cdot \frac{dv}{dx} = -\cos^2 v$

$\Rightarrow \sec^2 v\, dv = -\dfrac{dx}{x}$

Integrating both sides we have $\int \sec^2 v\, dv = -\int \dfrac{dx}{x}$

$\Rightarrow \tan v = -\log|x| + C$

$\Rightarrow \tan\left|\dfrac{y}{x}\right| + \log|x| = C$

This is the required solution of given differential equation

24. $\left(x\cos\dfrac{y}{x}\right)\dfrac{dy}{dx} = \left(y\cos\dfrac{y}{x}\right) + x$

Sol. Find $\dfrac{dy}{dx}$

$x\cos\left(\dfrac{y}{x}\right)\dfrac{dy}{dx} = y\cos\left(\dfrac{y}{x}\right) + x$

$\dfrac{dy}{dx} = \dfrac{y\cos\left(\dfrac{y}{x}\right) + x}{x\cos\left(\dfrac{y}{x}\right)} = \dfrac{y}{x} + \sec\left(\dfrac{y}{x}\right) = F\left(\dfrac{y}{x}\right)$

Thus the given differential equation is homogeneous differential equation

Let $\dfrac{y}{x} = v$

$\Rightarrow y = vx$

$\Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Now the differential equation is,

$v + x\dfrac{dv}{dx} = v + \sec v$

$\cos v\, dv = \dfrac{dx}{x}$

Integrating both sides

$\int \cos v\, dv = \int \dfrac{dx}{x}$

$\sin v = \log|x| + c_1$

Putting $v = \dfrac{y}{x}$ & $t\, c_1 = \log c$

$\sin\dfrac{y}{x} = \log|x| + \log|c|$

$\sin\dfrac{y}{x} = \log|cx|$

25. Find the particular solution of the differential equation $2xy + y^2 - 2x^2\dfrac{dy}{dx} = 0$, it being given that $y = 2$ when $x = 1$

Sol. Given differential equation is

$$2xy + y^2 - 2x^2 \frac{dy}{dx} = 0$$

$$\Rightarrow 2x^2 \frac{dy}{dx} = 2xy + y^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} + \frac{1}{2}\left(\frac{y}{x}\right)^2 \quad \ldots \text{(i)}$$

This is of the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$

This the given differential is homogeneous differential equation

Let $\frac{y}{x} = v$

$$\Rightarrow y = vx$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Now, form (i) we have $v + x\frac{dv}{dx} = v + \frac{v^2}{2}$

$$\Rightarrow x \cdot \frac{dv}{dx} = \frac{v^2}{2}$$

$$\Rightarrow \frac{2}{v^2} dv = \frac{dx}{x}$$

Integrating both sides we have $2\int \frac{dv}{v^2} = \int \frac{dx}{x}$

$$\Rightarrow \frac{-2}{v} = \log|x| + C$$

$$\Rightarrow -\frac{2x}{v} = \log|x| + C \quad \ldots\ldots \text{(ii)}$$

Putting $x = 1$ and $y = 2$ we have $-\frac{2 \times 1}{2} = \log|1| + C$

$$\Rightarrow C = -1$$

Putting $C = -1$ in equation (i) we have

$$-\frac{2x}{y} = \log|x| - 1$$

$$\Rightarrow -2x = y\log|x| - y$$

$$\Rightarrow -2x = y - y\log|x|$$

$$\Rightarrow y = \frac{2x}{1 - \log|x|}$$

This is the required solution of given differential equation

26. Find the particular solution of the differential equation $\left\{x\sin^2\frac{y}{x} - y\right\}dx + xdy = 0$, it being given that $y = \frac{\pi}{4}$ when $x = 1$

Sol. Given differential equation is $\left\{x \cdot \sin^2\frac{y}{x} - y\right\}dx + x\,dy = 0$

$\Rightarrow x\,dy = \left(y - x\sin^2 \frac{y}{x}\right)dx$

$\Rightarrow \frac{dy}{dx} = \frac{y}{x} - \sin^2 \frac{y}{x}$ .... (i)

This is of the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$

This the given differential equation is homogeneous differential equation

Let $\frac{y}{x} = v$

$\Rightarrow y = vx$

$\Rightarrow \frac{dy}{dx} = v + x \cdot \frac{dv}{dx}$

Now, from (ii)

$v + x \cdot \frac{dv}{dx} = v - \sin^2 v$

$\Rightarrow x \cdot \frac{dv}{dx} = -\sin^2 v$

$\Rightarrow \text{cosec}^2 v \cdot dv = -\sin^2 v$

Integrating both sides we have $\int \text{cosec}^2 v \cdot dv = -\int \frac{dx}{x}$

$\Rightarrow -\cot v = -\log|x| + C$

$\Rightarrow \cot \frac{y}{x} = \log|x| - C$

Putting $x = 1$ and $y = \frac{\pi}{4}$ we have $\cot \frac{\pi}{4} = \log|1| - C$

$\Rightarrow 1 = -C$

$\Rightarrow C = -1$

Putting $C = -1$ in (i) we have $\cot\left(\frac{y}{x}\right) = \log|x| + 1$

This is the required solution of given differential equation

27. Find the particular solution of the differential equation $\frac{dy}{dx} = \frac{y(2y-x)}{x(2y+x)}$ given that $y = 1$ when $x = 1$

Sol. For checking homogeneous putting tx for x and ty for y.

The given equation remains unchanged so it is a homogeneous equation.

Put $y = vx$, we get, $\frac{dy}{dx} = v + x\frac{dv}{dx}$ $\Rightarrow v + x\frac{dv}{dx} = \frac{vx(2vx - x)}{x(2vx + x)}$ $\Rightarrow v + x\frac{dv}{dx} = \frac{2v^2 - v}{2v+1}$

$\Rightarrow x\frac{dv}{dx} = \frac{2v^2 - v}{2v+1} - \frac{v}{1}$ $\Rightarrow x\frac{dv}{dx} = \frac{-2v}{2v+1}$ $\Rightarrow \int \frac{2v+1}{2v}dv = -\int \frac{dx}{x}$

$\Rightarrow v + \frac{1}{2}\log v = -\log|x| + c$ $\Rightarrow \frac{y}{x} + \frac{1}{2}\log\left(\frac{y}{x}\right) = -\log|x| + c$

$\Rightarrow \frac{y}{x} + \frac{1}{2}\log y - \frac{1}{2}\log x + \log|x| = c$ $\Rightarrow \frac{y}{x} + \frac{1}{2}\log y + \frac{1}{2}\log x = c$

$\Rightarrow \dfrac{y}{x}+\dfrac{1}{2}\log(x\,y)=c \qquad \ldots(1)$

Given, $x=1,\ y=1,\quad \dfrac{1}{1}+\dfrac{1}{2}\log(1.1)=c \quad \Rightarrow c=1$

Putting the value of c in equation (1), $\quad \therefore\ \dfrac{y}{x}+\dfrac{1}{2}\log(x\,y)=1$

28. Find the particular solution of the differential equation $\left(xe^{y/x}+y\right)dx=x\,dy$, given that $y(1)=0$

Sol. Given differential equation is $\left(x\cdot e^{y/x}+y\right)dx=x\,dy$

$\Rightarrow \dfrac{dy}{dx}=e^{y/x}+\dfrac{y}{x} \qquad \ldots.\ (i)$

This is of the form $\dfrac{dy}{dx}=f\left(\dfrac{y}{x}\right)$

This the given differential equation is homogeneous differential equation

Let $\dfrac{y}{x}=v$

$\Rightarrow y=vx$

$\Rightarrow \dfrac{dy}{dx}=v+x\dfrac{dv}{dx}$

Now from (ii) we have $v+x\cdot\dfrac{dv}{dx}=e^{v}+v$

$\Rightarrow x\cdot\dfrac{dv}{dx}=e^{v}$

$\Rightarrow e^{-v}dv=\dfrac{dx}{x}$

Integrating both sides we have

$\int e^{-v}dv=\int\dfrac{dx}{x}$

$\Rightarrow -e^{-v}=\log|x|+C$

$\Rightarrow -e^{-y/x}=\log|x|+C \qquad \ldots..\ (i)$

Putting $x=1\ \&\ y=0$ in (i) we have

$-e^{0}=\log(1)+C$

$\Rightarrow C=-1$

Putting $C=-1$ in (ii) we have

$-e^{y/x}=\log|x|-1$

$\Rightarrow \log|x|+e^{-y/x}=1$

This is the required solution of given differential equation

29. Find the particular solution of the differential equation $xe^{y/x}-y+x\dfrac{dy}{dx}=0$, given that $y(e)=0$

Sol. Given differential equation is

$x\cdot e^{y/x}-y+x\cdot\dfrac{dy}{dx}=0$

$\Rightarrow x\cdot\frac{dy}{dx}=y-x\cdot e^{y/x}$

$\Rightarrow \frac{dy}{dx}=\frac{y}{x}-e^{-y/x}$ ... (i)

This is of the form $\frac{dy}{dx}=f\left\{\frac{y}{x}\right\}$

This the given differential equation is homogeneous differential equation

Let $\frac{y}{x}=v$

$\Rightarrow y=vx$

$\Rightarrow \frac{dy}{dx}=v+x\cdot\frac{dv}{dx}$

Now, from (i) $v+x\cdot\frac{dv}{dx}=v-e^{v}$

$\Rightarrow x\cdot\frac{dv}{dx}=-e^{v}$

$\Rightarrow -e^{-v}dv=\frac{dx}{x}$

Integrating both sides we get $-\int -e^{-v}dv=\int\frac{dx}{x}$

$\Rightarrow e^{-v}=\log|x|+C$

$\Rightarrow e^{-y/x}=\log|x|+C$ .... (i)

$\because y(e)=0$

$\therefore e^{0}=\log e+C$

$\Rightarrow 1=1+C$

$\Rightarrow C=0$

Putting $C=0$ in (i) we have $e^{-y/x}=\log|x|$

Taking log on both sides $-\frac{y}{x}=\log(\log|x|)$

$\Rightarrow y=-x\log(\log|x|)$

This is the required solution of given differential equation

30. The slope of the tangent to a curve at any point (x,y) on it is given by $\frac{y}{x}-\left(\cot\frac{y}{x}\right)\left(\cos\frac{y}{x}\right)$, where $x>0$ and $y>0$ If the curve passes through the point $\left(1,\frac{\pi}{4}\right)$, find the equation of the curve

Sol. Since slape of the tangent to a curve at (x,y) on it is $\frac{y}{x}-\left(\cot\frac{y}{x}\right)\cdot\cos\left(\frac{y}{x}\right)$

$\therefore \frac{dy}{dx}=\frac{y}{x}-\cot\left(\frac{y}{x}\right)\cdot\cos\left(\frac{y}{x}\right)$ ..... (i)

This is of the form $\frac{dy}{dx}=f\left(\frac{y}{x}\right)$

Thus the differential equation is hangogerous differential equation

Let $\frac{y}{x} = v$

$\Rightarrow y = vx$

$\Rightarrow \frac{dy}{dx} = v + x \cdot \frac{dv}{dx}$

Now from (i) we have

$v + x \cdot \frac{dv}{dx} = v - \cot(v) \cdot \cos(v)$

$\Rightarrow x \cdot \frac{dv}{dx} = -\cot v \cdot \cos v$

$\Rightarrow \sec v \cdot \tan v \, dv = -\frac{dx}{x}$

Integrating both sides we have $\int \sec v \cdot \tan v \, dv = -\int \frac{dx}{x}$

$\Rightarrow \sec v = -\log|x| + C$

$\Rightarrow \sec\left(\frac{y}{x}\right) = -\log|x| + C \quad \ldots. \text{(ii)}$

Since the curve passes through the point $\left(1, \frac{\pi}{4}\right)$

$\therefore \sec\left(\frac{\pi}{4}\right) = -\log|1| + C$

$\Rightarrow \sqrt{2} = C$

Putting $C = \sqrt{2}$ in equation we have $\sec\left(\frac{y}{x}\right) + \log|x| = \sqrt{2}$

This is the required solution of given differential equation