EXERCISE : 8.1

BINOMIAL THEOREM for any positive integer n

$$(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 \ldots {}^nC_r a^{n-r}b^r \ldots\ldots {}^nC_{n-1} ab^{n-1} + {}^nC_n b^n.$$

Expand each of the expressions in Exercises 1 to 5

Q No. 1 $(1-2x)^5$

$$(1-2x)^5 = [1+(-2x)]^5 = {}^5C_0 + {}^5C_1(-2x) + {}^5C_2(-2x)^2 + {}^5C_3(-2x)^3 + {}^5C_4(-2x)^4 + {}^5C_5(-2x)^5$$

$$= 1 + \frac{5}{1}(-2x) + \frac{5\times4}{2\times1}\times4x^2 + \frac{5\times4}{2\times1}(-8x^3) + \frac{5}{1}16x^4 + 1\times(-32x^5)$$

$$= 1 - 10x + 40x^2 - 8x^3 + 80x^4 - 32x^5$$

Q No 2 $\left(\frac{2}{x} - \frac{x}{2}\right)^5$

$$\left(\frac{2}{x} - \frac{x}{2}\right)^5 = \left[\frac{2}{x} + \left(-\frac{x}{2}\right)\right]^5 = {}^5C_0\left(\frac{2}{x}\right)^5 + {}^5C_1\left(\frac{2}{x}\right)^4\left(-\frac{x}{2}\right) + {}^5C_2\left(\frac{2}{x}\right)^3\left(-\frac{x}{2}\right)^2$$
$$+ {}^5C_3\left(\frac{2}{x}\right)^2\left(-\frac{x}{2}\right)^3 + {}^5C_4\left(\frac{2}{x}\right)\left(-\frac{x}{2}\right)^4 + {}^5C_5\left(\frac{2}{x}\right)\left(-\frac{x}{2}\right)^5$$

$$= 1\times\frac{32}{x^5} + \frac{5}{1}\times\frac{16}{x^4}\times\left(-\frac{x}{2}\right) + \frac{5\times4}{1\times2}\times\frac{8}{x^3}\times\frac{x^2}{4} + \frac{5\times4}{1\times2}\times\frac{4}{x^2}\times\left(-\frac{x^3}{8}\right)$$
$$+ \frac{5}{1}\times\frac{2}{x}\times\frac{x^4}{16} + 1\times\left(-\frac{x^5}{32}\right)$$

$$= \frac{32}{x^5} - \frac{40}{x^3} + \frac{20}{x} - 5x + \frac{5}{8}x^3 - \frac{x^5}{32}$$

Q No 3 $(2x-3)^6$

$$(2x-3)^6 = [2x+(-3)]^6 = {}^6C_0(2x)^6 + {}^6C_1(2x)^5(-3) + {}^6C_2(2x)^4(-3)^2 + {}^6C_3(2x)^3(-3)^3$$
$$+ {}^6C_4(2x)^2(-3)^4 + {}^6C_5(2x)(-3)^5 + {}^6C_6(2x)^0(-3)^6$$

$$= 1\times64x^6 + \frac{6}{1}\times32x^5\times(-3) + \frac{6\times5}{1\times2}(16x^4)(9) + \frac{6\times5\times4}{1\times2\times3}(8x^3)(-27)$$
$$+ \frac{6\times5}{1\times2}(4x^2)(81) + \frac{6}{1}\times(2x)\times(-243) + 1\times729$$

$$= 64x^6 - 576x^5 + 2160x^4 - 4320x^3 + 4860x^2 - 2916x + 729$$

QNO. 4 $\left(\frac{x}{3}+\frac{1}{x}\right)^5$

$$\left(\frac{x}{3}+\frac{1}{x}\right)^5 = {}^5C_0\left(\frac{x}{3}\right)^5 + {}^5C_1\left(\frac{x}{3}\right)^4\left(\frac{1}{x}\right)^1 + {}^5C_2\left(\frac{x}{3}\right)^3\left(\frac{1}{x}\right)^2 + {}^5C_3\left(\frac{x}{3}\right)^2\left(\frac{1}{x}\right)^3$$

$$+ {}^5C_4\left(\frac{x}{3}\right)^1\left(\frac{1}{x}\right)^4 + {}^5C_5\left(\frac{1}{x}\right)^5$$

$$= (1)\left(\frac{x^5}{243}\right) + \left(\frac{5}{1}\right)\left(\frac{x^4}{81}\right)\left(\frac{1}{x}\right) + \frac{5\times4}{1\times2}\left(\frac{x^3}{27}\right)\left(\frac{1}{x^2}\right) + \frac{5\times4}{1\times2}\left(\frac{x^2}{9}\right)\left(\frac{1}{x^3}\right)$$

$$+ \left(\frac{5}{1}\right)\left(\frac{x}{3}\right)\left(\frac{1}{x^4}\right) + 1\times\frac{1}{x^5}$$

$$= \frac{x^5}{243} + \frac{5x^3}{81} + \frac{10}{27}x + \frac{10}{9x} + \frac{5}{3x^3} + \frac{1}{x^5}$$

QNo5 $\left(x+\frac{1}{x}\right)^6$

$$\left(x+\frac{1}{x}\right)^6 = {}^6C_0\, x^6 + {}^6C_1\, x^5\left(\frac{1}{x}\right) + {}^6C_2\, x^4\left(\frac{1}{x}\right)^2 + {}^6C_3(x)^3\left(\frac{1}{x}\right)^3$$

$$+ {}^6C_4(x)^2\left(\frac{1}{x}\right)^4 + {}^6C_5(x)\left(\frac{1}{x}\right)^5 + {}^6C_6\left(\frac{1}{x}\right)^6$$

$$= (1)(x^6) + \left(\frac{6}{1}\right)(x^5)\left(\frac{1}{x}\right) + \left(\frac{6\times5}{1\times2}\right)(x^4)\left(\frac{1}{x^2}\right) + \left(\frac{6\times5\times4}{1\times2\times3}\right)(x^3)\left(\frac{1}{x^3}\right)$$

$$+ \left(\frac{6\times5}{1\times2}\right)(x^2)\left(\frac{1}{x^4}\right) + \left(\frac{6}{1}\right)(x)\left(\frac{1}{x^5}\right) + (1)\left(\frac{1}{x^6}\right)$$

$$= x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}$$

Using Binomial Evaluate the following:

QNo 6 $(96)^3$

$$(96)^3 = (100-4)^3 = {}^3C_0(100)^3 + {}^3C_1(100)^2(-4) + {}^3C_2(100)(-4)^2 + {}^3C_3(-4)^3$$

$$= 1\times(1000000) + \left(\frac{3}{1}\right)(10000)(-4) + \left(\frac{3}{1}\right)(100)(16) + 1(-64)$$

$$= 1000000 - 120000 + 4800 - 64 = 884736$$

QNo7 $(102)^5$

$$(102)^5 = (100+2)^5 = {}^5C_0(100)^5 + {}^5C_1(100)^4(2) + {}^5C_2(100)^3(2)^2 + {}^5C_3(100)^2(2)^3$$

$$+ {}^5C_4(100)(2)^4 + {}^5C_5(2)^5$$

$$= (1)(10000000000) + \left(\frac{5}{1}\right)(1000000000)(2) + \left(\frac{5\times4}{1\times2}\right)(1000000)(4)$$
$$+ \left(\frac{5\times4}{1\times2}\right)(10000)(8) + \left(\frac{5}{1}\right)(100)(16) + (1)(32)$$

$$= 10000000000 + 1000000000 + 400000000 + 800000 + 8000 + 32$$

$$= 11040808032.$$

QNo 8 $(101)^4$

$$(101)^4 = (100+1)^4 = {}^4C_0(100)^4 + {}^4C_1(100)^3(1) + {}^4C_2(100)^2(1)^2 + {}^4C_3(100)^1(1)^3$$
$$+ {}^4C_4(100)^0(1)^4$$

$$= 1\times100000000 + \frac{4}{1}\times1000000 + \frac{4\times3}{1\times2}\times10000 + \frac{4}{1}\times100 + 1$$

$$= 100000000 + 4000000 + 60000 + 400 + 1 = 104060401$$

QNo 9 $(99)^5$

$$(99)^5 = (100-1)^5 = [100+(-1)]^5$$

$$= {}^5C_0(100)^5(-1)^0 + {}^5C_1(100)^4(-1)^1 + {}^5C_2(100)^3(-1)^2 + {}^5C_3(100)^2(-1)^3$$
$$+ {}^5C_4(100)(-1)^4 + {}^5C_5(100)^0(-1)^5$$

$$= (1)(10000000000) + \left(\frac{5}{1}\right)(100000000)(-1) + \left(\frac{5\times4}{1\times2}\right)(1000000)$$
$$+ \left(\frac{5\times4}{1\times2}\right)(10000) + (5)(100) - 1$$

$$= 10000000000 - 500000000 + 10000000 - 100000 + 500 - 1$$

$$= 9509900499$$

QNo 10. Using Binomial Theorem, Indicate which number is larger $(1.1)^{10000}$ or 1000.

Sol: By Binomial Theorem $(1.1)^{10000} = (1+0.1)^{10000}$

$$= {}^{10000}C_0(1)^{10000}(0.1)^0 + {}^{10000}C_1(1)^{9999}(0.1) + \text{other positive terms}$$

$$= 1 + 10000\times\frac{1}{10} + \text{other positive terms}$$

$$= 1 + 1000 + \text{other positive terms}$$

$$> 1000 \qquad \therefore (1.1)^{10000} > 1000$$

QNo 11: Find $(a+b)^4-(a-b)^4$. Hence Evaluate $(\sqrt{3}+\sqrt{2})^4-(\sqrt{3}-\sqrt{2})^4$

Sol: $(a+b)^4 = {}^4c_0 a^4 + {}^4c_1 a^3 b + {}^4c_2 a^2 b^2 + {}^4c_3 ab^3 + {}^4c_4 b^4$.

ie $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$. ---(1)

Illy $(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$ [By changing b to $-b$ in (1)]

$\therefore (a+b)^4 - (a-b)^4 = 8a^3b + 8ab^3$ ----(2)

$\therefore$ For $(\sqrt{3}+\sqrt{2})^4 - (\sqrt{3}-\sqrt{2})^4$ put $a=\sqrt{3}$ and $b=\sqrt{2}$ in (2)

$\therefore (\sqrt{3}+\sqrt{2})^4 - (\sqrt{3}-\sqrt{2})^4 = 8(\sqrt{3})^3(\sqrt{2}) + 8(\sqrt{3})(\sqrt{2})^3$

$= 8\times 3\times\sqrt{3}\times\sqrt{2} + 8\times\sqrt{3}\times 2\times\sqrt{2}$

$= 24\sqrt{6} + 16\sqrt{6} = 40\sqrt{6}$.

QNo 12: Find $(x+1)^6+(x-1)^6$. Hence or otherwise Evaluate $(\sqrt{2}+1)^6+(\sqrt{2}-1)^6$

Sol. $(x+1)^6 = {}^6c_0 x^6 + {}^6c_1 x^5 + {}^6c_2 x^4 + {}^6c_3 x^3 + {}^6c_4 x^2 + {}^6c_5 x + {}^6c_6 x^0$

$= 1\times x^6 + \frac{6}{1}\times x^5 + \frac{6\times 5}{1\times 2}x^4 + \frac{6\times5\times4}{1\times2\times3}x^3 + \frac{6\times5\times4\times3}{1\times2\times3\times4}x^2 + \frac{6\times5\times4\times3\times2}{1\times2\times3\times4\times5}x + 1$

$\therefore (x+1)^6 = x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1$ ----(1)

Similarly $(x-1)^6 = x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$ ---(2)

Adding (1) and (2)

$(x+1)^6 + (x-1)^6 = 2[x^6 + 15x^4 + 15x^2 + 1]$

Put $x = \sqrt{2}$

$(\sqrt{2}+1)^6 + (\sqrt{2}-1)^6 = 2[(\sqrt{2})^6 + 15(\sqrt{2})^4 + 15(\sqrt{2})^2 + 1]$

$= 2[8 + 15\times 4 + 15\times 2 + 1]$

$= 2[8 + 60 + 30 + 1] = 2\times 99 = 198$.

Q.No.13: Show that $9^{n+1} - 8n - 9$ is divisible by 64, whenever n is +ve integer.

Sol. $9^{n+1} = (1+8)^{n+1}$

$= {}^{n+1}C_0 + {}^{n+1}C_1(8) + {}^{n+1}C_2(8)^2 + {}^{n+1}C_3(8)^3 \ldots {}^{n+1}C_{n+1}(8)^{n+1}$

$= 1 + (n+1)8 + {}^{n+1}C_2(64) + {}^{n+1}C_3(512) \ldots 8^{n+1} \quad \ldots (1)$

Now $9^{n+1} - 8n - 9 = 9^{n+1} - 8n - 1 - 8 = 9^{n+1} - 8(n+1) - 1$

$= {}^{n+1}C_2(64) + {}^{n+1}C_3(512) \ldots 8^{n+1}$ [using (1)]

$= 64\left[{}^{n+1}C_2 + 8 \cdot {}^{n+1}C_3 + \ldots 8^{n-1}\right]$

$= 64 \times$ an integer

$\therefore$ $9^{n+1} - 8n - 9$ is divisible by 64.

Q.No.14 Prove that $\sum_{r=0}^{n} 3^r \, {}^nC_r = 4^n$.

Sol. LHS $= \sum_{r=0}^{n} 3^r \, {}^nC_r = 3^0 \, {}^nC_0 + 3^1 \, {}^nC_1 + 3^2 \, {}^nC_2 + \ldots 3^n \, {}^nC_n$

$= 1 \times {}^nC_0 + 3^1 \, {}^nC_1 + 3^2 \, {}^nC_2 \ldots 3^n \, {}^nC_n$

$= {}^nC_0 + {}^nC_1(3) + {}^nC_2(3)^2 + \ldots {}^nC_n(3^n)$

$= (1+3)^n = 4^n =$ RHS.

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