23. Scalar, or Dot, Product of Vectors

Exercise 23

1. Question

Find $\vec{a} \cdot \vec{b}$ when

i. $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$ and $\vec{b} = 3\hat{i} - 4\hat{j} - 2\hat{k}$

ii. $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b} = -2\hat{j} + 4\hat{k}$

iii. $\vec{a} = \hat{i} - \hat{j} + 5\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{k}$

Answer

i)

$\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$

$\vec{b} = 3\hat{i} - 4\hat{j} - 2\hat{k}$

$\vec{a}.\vec{b} = (\hat{i} - 2\hat{j} + \hat{k}).(3\hat{i} - 4\hat{j} - 2\hat{k})$

$\Rightarrow \vec{a}.\vec{b} = (1 \times 3) + (-2 \times -4) + (1 \times -2)$

$\Rightarrow \vec{a}.\vec{b} = 3 + 8 - 2 = 9$

Ans: $\vec{a}.\vec{b} = 9$

ii)

$\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$

$\vec{b} = 0\hat{i} - 2\hat{j} + 4\hat{k}$

$\vec{a}.\vec{b} = (\hat{i} + 2\hat{j} + 3\hat{k}).(0\hat{i} - 2\hat{j} + 4\hat{k})$

$\Rightarrow \vec{a}.\vec{b} = (1 \times 0) + (2 \times -2) + (3 \times 4)$

$\Rightarrow \vec{a}.\vec{b} = 0 - 4 + 12 = 8$

Ans: $\Rightarrow \vec{a}.\vec{b} = 8$

iii)

$\vec{a} = \hat{i} - \hat{j} + 5\hat{k}$

$\vec{b} = 3\hat{i} + 0\hat{j} - 2\hat{k}$

$\vec{a}.\vec{b} = (\hat{i} - \hat{j} + 5\hat{k}).(3\hat{i} + 0\hat{j} - 2\hat{k})$

$\Rightarrow \vec{a}.\vec{b} = (1 \times 3) + (-1 \times 0) + (5 \times -2)$

$\Rightarrow \vec{a}.\vec{b} = 3 - 0 - 10 = -7$

Ans: $\Rightarrow \vec{a}.\vec{b} = -7$

2. Question

Find the value of λ for which $\vec{a}$ and $\vec{b}$ are perpendicular, where

i. $\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$ and $\vec{b} = \left(\hat{i} - 2\hat{j} + 3\hat{k}\right)$

ii. $\vec{a} = 3\hat{i} - \hat{j} + 4\hat{k}$ and $\vec{b} = -\lambda\hat{i} + 3\hat{j} + 3\hat{k}$

iii. $\vec{a} = 2\hat{i} + 4\hat{j} - \hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + \lambda\hat{k}$

iv. $\vec{a} = 3\hat{i} + 2\hat{j} - 5\hat{k}$ and $\vec{b} = -5\hat{j} + \lambda\hat{k}$

Answer

i)

$\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$

$\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$

Since these two vectors are perpendicular,their dot product is zero.

$\Rightarrow \vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta = |\vec{a}||\vec{b}|\cos\frac{\pi}{2} = 0$

$\Rightarrow \vec{a}.\vec{b} = (2\hat{i} + \lambda\hat{j} + \hat{k}).(\hat{i} - 2\hat{j} + 3\hat{k}) = 0$

$\Rightarrow \vec{a}.\vec{b} = (2 \times 1) + (\lambda \times -2) + (1 \times 3) = 0$

$\Rightarrow \vec{a}.\vec{b} = 2 - 2\lambda + 3 = 0$

$\Rightarrow 5 = 2\lambda$

$\Rightarrow \lambda = \frac{5}{2}$

Ans: $\lambda = \frac{5}{2}$

ii)

$\vec{a} = 3\hat{i} - \hat{j} + 4\hat{k}$

$\vec{b} = -\lambda + 3\hat{j} + 3\hat{k}$

Since these two vectors are perpendicular,their dot product is zero.

$\Rightarrow \vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta = |\vec{a}||\vec{b}|\cos\frac{\pi}{2} = 0$

$\Rightarrow \vec{a}.\vec{b} = (3\hat{i} - \hat{j} + 4\hat{k}).(-\lambda + 3\hat{j} + 3\hat{k}) = 0$

$\Rightarrow \vec{a}.\vec{b} = (3 \times -\lambda) + (-1 \times 3) + (4 \times 3) = 0$

$\Rightarrow \vec{a}.\vec{b} = -3\lambda - 3 + 12 = 0$

$\Rightarrow 9 = 3\lambda$

$\Rightarrow \lambda = \frac{9}{3} = 3$

Ans: λ = 3

iii)

$\vec{a} = 2\hat{i} + 4\hat{j} - \hat{k}$

$\vec{b} = 3\hat{i} - 2\hat{j} + \lambda\hat{k}$

Since these two vectors are perpendicular,their dot product is zero.

$\Rightarrow \vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta = |\vec{a}||\vec{b}|\cos\frac{\pi}{2} = 0$

$\Rightarrow \vec{a}.\vec{b} = (2\hat{i} + 4\hat{j} - \hat{k}).(3\hat{i} - 2\hat{j} + \lambda\hat{k}) = 0$

$\Rightarrow \vec{a}.\vec{b} = (2 \times 3) + (4 \times -2) + (-1 \times \lambda) = 0$

$\Rightarrow \vec{a}.\vec{b} = -\lambda + 6 - 8 = 0$

$\Rightarrow -2 = \lambda$

$\Rightarrow \lambda = -2$

Ans: λ = - 2

iv)

$\vec{a} = 3\hat{i} + 2\hat{j} - 5\hat{k}$

$\vec{b} = -5\hat{j} + \lambda\hat{k}$

Since these two vectors are perpendicular,their dot product is zero.

$\Rightarrow \vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta = |\vec{a}||\vec{b}|\cos\frac{\pi}{2} = 0$

$\Rightarrow \vec{a}.\vec{b} = (3\hat{i} + 2\hat{j} - 5\hat{k}).(-5\hat{j} + \lambda\hat{k}) = 0$

$\Rightarrow \vec{a}.\vec{b} = (3 \times 0) + (2 \times -5) + (-5 \times \lambda) = 0$

$\Rightarrow \vec{a}.\vec{b} = -5\lambda + 0 - 10 = 0$

$\Rightarrow -10 = 5\lambda$

$\Rightarrow \lambda = \frac{-10}{5} = -2$

Ans: λ = - 2

3. Question

i. If $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{b} = 3\hat{i} - \hat{j} + 2\hat{k}$, show that $(\vec{a} + \vec{b})$ is perpendicular to $(\vec{a} - \vec{b})$.

ii. If $\vec{a} = (5\hat{i} - \hat{j} - 3\hat{k})$ and $\vec{b} = (\hat{i} + 3\hat{j} - 5\hat{k})$ then show that $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ are orthogonal.

Answer

i)

$\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$

$\vec{b} = 3\hat{i} - \hat{j} + 2\hat{k}$

$\vec{a} + \vec{b} = \hat{i} + 2\hat{j} - 3\hat{k} + 3\hat{i} - \hat{j} + 2\hat{k}$

$\Rightarrow \vec{a} + \vec{b} = 4\hat{i} + \hat{j} - \hat{k}$

$\vec{a} - \vec{b} = \hat{i} + 2\hat{j} - 3\hat{k} - (3\hat{i} - \hat{j} + 2\hat{k})$

$\Rightarrow \vec{a} - \vec{b} = -2\hat{i} + 3\hat{j} - 5\hat{k}$

Now $(\vec{a} + \vec{b}).(\vec{a} - \vec{b}) = (4\hat{i} + \hat{j} - \hat{k}).(-2\hat{i} + 3\hat{j} - 5\hat{k})$

= (4 × - 2) + (1 × 3) + (- 1 × - 5) = - 8 + 3 + 5 = 0

Since the dot product of these two vectors is 0,the vector $(\vec{a} + \vec{b})$ is perpendicular to $(\vec{a} - \vec{b})$.

Hence,proved.

ii)

$\vec{a} = 5\hat{i} - \hat{j} - 3\hat{k}$

$\vec{b} = \hat{i} + 3\hat{j} - 5\hat{k}$

$\vec{a} + \vec{b} = 5\hat{i} - \hat{j} - 3\hat{k} + \hat{i} + 3\hat{j} - 5\hat{k}$

$\Rightarrow \vec{a} + \vec{b} = 6\hat{i} + 2\hat{j} - 8\hat{k}$

$\vec{a} - \vec{b} = 5\hat{i} - \hat{j} - 3\hat{k} - (\hat{i} + 3\hat{j} - 5\hat{k})$

$\Rightarrow \vec{a} - \vec{b} = 4\hat{i} - 4\hat{j} + 2\hat{k}$

Now $(\vec{a} + \vec{b}).(\vec{a} - \vec{b}) = (6\hat{i} + 2\hat{j} - 8\hat{k}).(4\hat{i} - 4\hat{j} + 2\hat{k})$

= (6 × 4) + (2 × - 4) + (- 8 × 2) = 24 - 8 - 16 = 0

Since the dot product of these two vectors is 0,the vector $\left(\vec{a} + \vec{b}\right)$ is perpendicular to $\left(\vec{a} - \vec{b}\right)$.

Hence,proved that $\left(\vec{a} + \vec{b}\right)$ and $\left(\vec{a} - \vec{b}\right)$ are orthogonal.

4. Question

If $\vec{a} = \left(\hat{i} - \hat{j} + 7\hat{k}\right)$ and $\vec{b} = \left(5\hat{i} - \hat{j} + \lambda\hat{k}\right)$ then find the value of λ so that $\left(\vec{a} + \vec{b}\right)$ and $\left(\vec{a} - \vec{b}\right)$ are orthogonal vectors.

Answer

$\vec{a} = \hat{i} - \hat{j} + 7\hat{k}$

$\vec{b} = 5\hat{i} - \hat{j} + \lambda\hat{k}$

$(\vec{a} + \vec{b}) = \hat{i} - \hat{j} + 7\hat{k} + 5\hat{i} - \hat{j} + \lambda\hat{k}$

$\Rightarrow \vec{a} + \vec{b} = 6\hat{i} - 2\hat{j} + (7 + \lambda)\hat{k}$

$\vec{a} - \vec{b} = \hat{i} - \hat{j} + 7\hat{k} - (5\hat{i} - \hat{j} + \lambda\hat{k})$

$\Rightarrow \vec{a} - \vec{b} = -4\hat{i} + 0\hat{j} + (7 - \lambda)\hat{k}$

Now $(\vec{a} + \vec{b}).(\vec{a} - \vec{b}) = (6\hat{i} - 2\hat{j} + (7 + \lambda)\hat{k}).(-4\hat{i} + 0\hat{j} + (7 - \lambda)\hat{k})$

Since these two vectors are orthogonal,their dot product is zero.

⇒ (6 × - 4) + (- 2 × 0) + ((7 + λ) × (7 - λ)) = 0⇒ - 24 + 0 + (49 - λ^2) = 0

⇒ λ^2 = 25

⇒ λ = ±5

Ans: λ = ±5

5. Question

Show that the vectors

$\frac{1}{7}\left(2\hat{i} + 3\hat{j} + 6\hat{k}\right), \frac{1}{7}\left(3\hat{i} - 6\hat{j} + 2\hat{k}\right)$ and $\frac{1}{7}\left(6\hat{i} + 2\hat{j} - 3\hat{k}\right)$

are mutually perpendicular unit vectors.

Answer

Let,

$\vec{a} = \frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k})$

$\vec{b} = \frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k})$

$\vec{c} = \frac{1}{7}(6\hat{i} + 2\hat{j} - 3\hat{k})$

$|\vec{a}| = |\vec{b}| = |\vec{c}| = 1$

We have to show that $\vec{a}.\vec{b} = \vec{b}.\vec{c} = \vec{a}.\vec{c} = 0$

L.H.S.

$\vec{a}.\vec{b} = \frac{1}{7}\left(2\hat{i} + 3\hat{j} + 6\hat{k}\right).\frac{1}{7}\left(3\hat{i} - 6\hat{j} + 2\hat{k}\right) = \frac{1}{49}(6 - 18 + 12) = 0$

$\vec{b}.\vec{c} = \frac{1}{7}\left(3\hat{i} - 6\hat{j} + 2\hat{k}\right).\frac{1}{7}\left(6\hat{i} + 2\hat{j} - 3\hat{k}\right) = \frac{1}{49}(18 - 12 - 6) = 0$

$\vec{a}.\vec{c} = \frac{1}{7}\left(2\hat{i} + 3\hat{j} + 6\hat{k}\right).\frac{1}{7}\left(6\hat{i} + 2\hat{j} - 3\hat{k}\right) = \frac{1}{49}(12 + 6 - 18) = 0$

= R.H.S.

Hence,showed that vectors are mutually perpendicular unit vectors.

6. Question

Let $\vec{a} = 4\hat{i} + 5\hat{j} - \hat{k}, \vec{b} = \hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j} - \hat{k}.$

Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b},$ and is such that $\vec{d} \cdot \vec{c} = 21.$

Answer

$\vec{a} = (4\hat{i} + 5\hat{j} - \hat{k})$

$\vec{b} = (\hat{i} - 4\hat{j} + 5\hat{k})$

$\vec{c} = (3\hat{i} + \hat{j} - \hat{k})$

Let $\vec{d} = p\hat{i} + q\hat{j} + r\hat{k}$

the vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b},$

$\Rightarrow \vec{d}.\vec{a} = \vec{d}.\vec{b} = 0$

$\left(p\hat{i} + q\hat{j} + r\hat{k}\right).\left(4\hat{i} + 5\hat{j} - \hat{k}\right) = 0$

$\Rightarrow 4p + 5q - r = 0 \ldots (1)$

$\left(p\hat{i} + q\hat{j} + r\hat{k}\right).\left(\hat{i} - 4\hat{j} + 5\hat{k}\right) = 0$

$p - 4q + 5r = 0 \ldots (2)$

$\vec{d} \cdot \vec{c} = 21.$

$\left(p\hat{i} + q\hat{j} + r\hat{k}\right).\left(3\hat{i} + \hat{j} - \hat{k}\right) = 21$

$\Rightarrow 3p + q - r = 21 \ldots (3)$

Solving equations 1,2,3 simultaneously we get p = 7,q = - 7,r = - 7

$\therefore \vec{d} = p\hat{i} + q\hat{j} + r\hat{k} = 7\hat{i} - 7\hat{j} - 7\hat{k} = 7(\hat{i} - \hat{j} - \hat{k})$

Ans: $\vec{d} = 7(\hat{i} - \hat{j} - \hat{k})$

7. Question

Let $\vec{a} = (2\hat{i} + 3\hat{j} + 2\hat{k})$ and $\vec{b} = (\hat{i} + 2\hat{j} + \hat{k})$.

Find the projection of (i) $\vec{a}$ on $\vec{b}$ and (ii) $\vec{b}$ on $\vec{a}$.

Answer

$\vec{a} = (2\hat{i} + 3\hat{j} + 2\hat{k})$

$\vec{b} = (\hat{i} + 2\hat{j} + \hat{k})$

$|\vec{a}| = \sqrt{2^2 + 3^2 + 2^2} = \sqrt{4 + 9 + 4} = \sqrt{17}$

$|\vec{b}| = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{1 + 4 + 1} = \sqrt{6}$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{2\hat{i} + 3\hat{j} + 2\hat{k}}{\sqrt{17}}$$

$$\hat{b} = \frac{\vec{b}}{|\vec{b}|} = \frac{\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{6}}$$

Projection of $\vec{a}$ on $\vec{b}$ is $\vec{a}\hat{b} = (2\hat{i} + 3\hat{j} + 2\hat{k}).\frac{\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{6}} = \frac{2 + 6 + 2}{\sqrt{6}} = \frac{10}{\sqrt{6}} = \frac{5\sqrt{6}}{3}$

Projection of $\vec{b}$ on $\vec{a}$ is $\vec{b}\hat{a} = (\hat{i} + 2\hat{j} + \hat{k}).\frac{2\hat{i} + 3\hat{j} + 2\hat{k}}{\sqrt{17}} = \frac{2 + 6 + 2}{\sqrt{17}} = \frac{10}{\sqrt{17}} = \frac{10\sqrt{17}}{17}$

Ans: i) $\frac{5\sqrt{6}}{3}$

ii) $\frac{10\sqrt{17}}{17}$

8. Question

Find the projection of $(8\hat{i} + \hat{j})$ in the direction of $(\hat{i} + 2\hat{j} - 2\hat{k})$.

Answer

Let,

$\vec{a} = (8\hat{i} + \hat{j})$

$\vec{b} = (\hat{i} + 2\hat{j} - 2\hat{k})$

$|\vec{b}| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{1 + 4 + 4} = \sqrt{9} = 3$

$$\hat{b} = \frac{\vec{b}}{|\vec{b}|} = \frac{\hat{i} + 2\hat{j} - 2\hat{k}}{3}$$

$\therefore$ The projection of $(8\hat{i} + \hat{j})$on$(\hat{i} + 2\hat{j} - 2\hat{k})$

is:$(8\hat{i} + \hat{j}).\frac{\hat{i} + 2\hat{j} - 2\hat{k}}{3} = \frac{8 + 2 + 0}{3} = \frac{10}{3}$

Ans:10/3

9. Question

Write the projection of vector $\left(\hat{i}+\hat{j}+\hat{k}\right)$ along the vector $\hat{j}$.

Answer

Let,

$\vec{a} = (\hat{\imath} + \hat{\jmath} + \hat{k})$

$\vec{b} = (\hat{\jmath})$

$|\vec{b}| = \sqrt{0^2 + 1^2 + 0^2} = \sqrt{1} = 1$

$\hat{b} = \frac{\vec{b}}{|\vec{b}|} = \frac{(\hat{\jmath})}{1}$

∴ The projection of $(\hat{\imath} + \hat{\jmath} + \hat{k})$on $(\hat{\jmath})$

is:$(\hat{\imath} + \hat{\jmath} + \hat{k}).(\hat{\jmath}) = 1$

Ans:1

10. Question

i. Find the projection of $\vec{a}$ on $\vec{b}$ if $\vec{a}\cdot\vec{b}=8$ and $\vec{b}=\left(2\hat{i}+6\hat{j}+3\hat{k}\right)$.

ii. Write the projection of the vector $\left(\hat{i}+\hat{j}\right)$ on the vector $\left(\hat{i}-\hat{j}\right)$.

Answer

i)$\vec{b} = (2\hat{\imath} + 6\hat{\jmath} + 3\hat{k})$

$|\vec{b}| = \sqrt{2^2 + 6^2 + 3^2} = \sqrt{4 + 36 + 9} = \sqrt{49} = 7$

Projection of $\vec{a}$ on $\vec{b}$

$= \vec{a}.\frac{\vec{b}}{|\vec{b}|}$

$= \frac{8}{7}$

ANS:8/7

ii) Sol:

Let,

$\vec{a} = (\hat{\imath} + \hat{\jmath})$

$\vec{b} = (\hat{\imath} - \hat{\jmath})$

$|\vec{b}| = \sqrt{1^2 + (-1)^2} = \sqrt{1 + 1} = \sqrt{2}$

$\hat{b} = \frac{\vec{b}}{|\vec{b}|} = \frac{\hat{\imath} - \hat{\jmath}}{\sqrt{2}}$

∴ The projection of $\hat{\imath} + \hat{\jmath}$ on $(\hat{\imath} - \hat{\jmath})$

is:$(\hat{\imath} + \hat{\jmath}).\frac{\hat{\imath}-\hat{\jmath}}{\sqrt{2}} = \frac{1-1}{\sqrt{2}} = 0$

Ans: 0

11. Question

Find the angle between the vectors $\vec{a}$ and $\vec{b}$, when

i. $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$

ii. $\vec{a} = 3\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} - 2\hat{j} + 4\hat{k}$

iii. $\vec{a} = \hat{i} - \hat{j}$ and $\vec{b} = \hat{j} + \hat{k}$.

Answer

i) $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$

$\vec{a} = (\hat{\imath} - 2\hat{\jmath} + 3\hat{k})$

$\vec{b} = (3\hat{\imath} - 2\hat{\jmath} + \hat{k})$

$|\vec{a}| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$

$|\vec{b}| = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{9 + 4 + 1} = \sqrt{14}$

We know that ,

$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$\Rightarrow (\hat{\imath} - 2\hat{\jmath} + 3\hat{k})(3\hat{\imath} - 2\hat{\jmath} + \hat{k}) = \sqrt{14}\sqrt{14}\cos\theta$

$\Rightarrow (3 + 4 + 3) = 14\cos\theta$

$\Rightarrow \cos\theta = 10/14$

$\Rightarrow \cos\theta = 5/7$

$\Rightarrow \theta = \cos^{-1}(5/7)$

Ans: $\theta = \cos^{-1}(5/7)$

ii) $\vec{a} = 3\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} - 2\hat{j} + 4\hat{k}$

$\vec{a} = (3\hat{\imath} + \hat{\jmath} + 2\hat{k})$

$\vec{b} = (2\hat{\imath} - 2\hat{\jmath} + 4\hat{k})$

$|\vec{a}| = \sqrt{3^2 + (1)^2 + 2^2} = \sqrt{9 + 1 + 4} = \sqrt{14}$

$|\vec{b}| = \sqrt{2^2 + (-2)^2 + 4^2} = \sqrt{4 + 4 + 16} = \sqrt{24}$

We know that ,

$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$\Rightarrow (3\hat{\imath} + \hat{\jmath} + 2\hat{k})(2\hat{\imath} - 2\hat{\jmath} + 4\hat{k}) = \sqrt{14}\sqrt{24}\cos\theta$

$\Rightarrow (6 - 2 + 8) = \sqrt{336}\cos\theta$

$\Rightarrow \cos\theta = 12/\sqrt{336}$

$\Rightarrow \cos\theta = \sqrt{(144/336)}$

$\Rightarrow \theta = \cos^{-1}\sqrt{(3/7)}$

Ans: $\theta = \cos^{-1}\sqrt{(3/7)}$

iii. $\vec{a} = \hat{i} - \hat{j}$ and $\vec{b} = \hat{j} + \hat{k}$.

Ans:

$\vec{a} = (\hat{i} - \hat{j})$

$\vec{b} = (\hat{j} + \hat{k})$

$|\vec{a}| = \sqrt{1^2 + (-1)^2} = \sqrt{1 + 1} = \sqrt{2}$

$|\vec{b}| = \sqrt{(1)^2 + 1^2} = \sqrt{1 + 1} = \sqrt{2}$

We know that ,

$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$\Rightarrow (\hat{i} - \hat{j})(\hat{j} + \hat{k}) = \sqrt{2}\sqrt{2}\cos\theta$

$\Rightarrow (- 1) = 2 \cos\theta$

$\Rightarrow \cos\theta = - 1/2$

$\Rightarrow \theta = \cos^{-1} - 1/2$

$\Rightarrow \theta = 120^o$

Ans: $\theta = 120^o$

12. Question

If $\vec{a} = \left(\hat{i} + 2\hat{j} - 3\hat{k}\right)$ and $\vec{b} = \left(3\hat{i} - \hat{j} + 2\hat{k}\right)$ then calculate the angle between $\left(2\vec{a} + \vec{b}\right)$ and $\left(\vec{a} + 2\vec{b}\right)$.

Answer

$\vec{a} = (\hat{i} + 2\hat{j} - 3\hat{k})$

$\vec{b} = (3\hat{i} - \hat{j} + 2\hat{k})$

$\vec{a} + 2\vec{b} = \left(\hat{i} + 2\hat{j} - 3\hat{k}\right) + 2\left(3\hat{i} - \hat{j} + 2\hat{k}\right) = 7\hat{i} + \hat{k}$

$2\vec{a} + \vec{b} = 2\left(\hat{i} + 2\hat{j} - 3\hat{k}\right) + \left(3\hat{i} - \hat{j} + 2\hat{k}\right) = 5\hat{i} + 3\hat{j} - 4\hat{k}$

$|\vec{a} + 2\vec{b}| = \sqrt{7^2 + (1)^2} = \sqrt{49 + 1} = \sqrt{50}$

$|2\vec{a} + \vec{b}| = \sqrt{5^2 + (3)^2 + (-4)^2} = \sqrt{25 + 9 + 16} = \sqrt{50}$

We know that ,

$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$\Rightarrow \left(7\hat{i} + \hat{k}\right)\left(5\hat{i} + 3\hat{j} - 4\hat{k}\right) = \sqrt{50}\sqrt{50}\cos\theta$

$\Rightarrow (35 - 4) = 50 \cos\theta$

$\Rightarrow \cos\theta = 31/50$

$\Rightarrow \theta = \cos^{-1}(31/50)$

Ans: $\theta = \cos^{-1}(31/50)$

13. Question

If $\vec{a}$ is a unit vector such that $\left(\vec{x} - \vec{a}\right) \cdot \left(\vec{x} + \vec{a}\right) = 8$, find $|\vec{x}|$.

Answer

If $\vec{a}$ is a unit vector

$\Rightarrow |\vec{a}| = 1$

$\Rightarrow (\vec{x} - \vec{a}).(\vec{x} + \vec{a}) = 8$

$\Rightarrow |\vec{x}|^2 - |\vec{a}|^2 = 8$

$\Rightarrow |\vec{x}|^2 = 8 + 1 = 9$

$\Rightarrow |\vec{x}| = 3$

Ans: $|\vec{x}| = 3$

14. Question

Find the angles which the vector $\vec{a} = 3\hat{i} - 6\hat{j} + 2\hat{k}$ makes with the coordinate axes.

Answer

If we have a vector $\vec{a} = a\hat{i} + b\hat{j} + c\hat{k}$

then the angle with the x - axis $= \alpha = \cos^{-1}\frac{a}{\sqrt{a^2+b^2+c^2}}$

the angle with the y - axis $= \beta = \cos^{-1}\frac{b}{\sqrt{a^2+b^2+c^2}}$

the angle with the z - axis $= \gamma = \cos^{-1}\frac{c}{\sqrt{a^2+b^2+c^2}}$

Here, $\vec{a} = 3\hat{i} - 6\hat{j} + 2\hat{k}$

$\sqrt{a^2 + b^2 + c^2} = \sqrt{3^2 + (-6)^2 + 2^2} = \sqrt{9 + 36 + 4} = \sqrt{49} = 7$

then the angle with the x - axis $= \alpha = \cos^{-1}\frac{a}{\sqrt{a^2+b^2+c^2}} = \cos^{-1}\frac{3}{7}$

the angle with the y - axis $= \beta = \cos^{-1}\frac{b}{\sqrt{a^2+b^2+c^2}} = \cos^{-1}\frac{-6}{7}$

the angle with the z - axis $= \gamma = \cos^{-1}\frac{c}{\sqrt{a^2+b^2+c^2}} = \cos^{-1}\frac{2}{7}$

Ans:

$\cos^{-1}\frac{3}{7}, \cos^{-1}\frac{-6}{7}, \cos^{-1}\frac{2}{7}$

15. Question

Show that the vector $\vec{a} = \left(\hat{i} + \hat{j} + \hat{k}\right)$ is equally inclined to the coordinate axes.

Answer

If we have a vector $\vec{a} = a\hat{i} + b\hat{j} + c\hat{k}$

then the angle with the x - axis $= \alpha = \cos^{-1}\frac{a}{\sqrt{a^2+b^2+c^2}}$

the angle with the y - axis $= \beta = \cos^{-1}\frac{b}{\sqrt{a^2+b^2+c^2}}$

the angle with the z - axis $= \gamma = \cos^{-1}\frac{c}{\sqrt{a^2+b^2+c^2}}$

Here, $\vec{a} = \hat{i} + \hat{j} + \hat{k}$

$\sqrt{a^2 + b^2 + c^2} = \sqrt{1^2 + (1)^2 + 1^2} = \sqrt{1 + 1 + 1} = \sqrt{3}$

then the angle with the x - axis $= \alpha = \cos^{-1}\frac{a}{\sqrt{a^2+b^2+c^2}} = \cos^{-1}\frac{1}{\sqrt{3}}$

the angle with the y - axis = $\beta = \cos^{-1}\frac{b}{\sqrt{a^2+b^2+c^2}} = \cos^{-1}\frac{1}{\sqrt{3}}$

the angle with the z - axis = $\gamma = \cos^{-1}\frac{c}{\sqrt{a^2+b^2+c^2}} = \cos^{-1}\frac{1}{\sqrt{3}}$

Now since, $\alpha = \beta = \gamma$

∴ the vector $\vec{a} = \left(\hat{i}+\hat{j}+\hat{k}\right)$ is equally inclined to the coordinate axes.

Hence,proved.

16. Question

Find a vector $\vec{a}$ of magnitude $5\sqrt{2}$, making an angle π/4 with x - axis, π/2 with y - axis and an acute angle θ with z - axis.

Answer

$|\vec{a}| = 5\sqrt{2}$

$l = \cos\alpha = \cos\pi/4 = 1/\sqrt{2}$

$m = \cos\beta = \cos\pi/2 = 0$

$n = \cos\theta$

we know that

$l^2 + m^2 + n^2 = 1$

$\Rightarrow \frac{1}{\sqrt{2}}^2 + 0^2 + n^2 = 1$

$\Rightarrow n^2 = 1 - \frac{1}{2}$

$\Rightarrow n^2 = \frac{1}{2}$

$\Rightarrow n = \pm\frac{1}{\sqrt{2}}$

since the vector makes an acute angle with the z axis

$\therefore n = +\frac{1}{\sqrt{2}}$

$\therefore \vec{a} = |\vec{a}|(l\hat{\imath} + m\hat{\jmath} + n\hat{k})$

$\therefore \vec{a} = 5\sqrt{2}(1/\sqrt{2}\hat{\imath} + 1/\sqrt{2}\hat{k})$

$\therefore \vec{a} = 5(\hat{\imath} + \hat{k})$

Ans: $\vec{a} = 5(\hat{\imath} + \hat{k})$

17. Question

Find the angle between $\left(\vec{a}+\vec{b}\right)$ and $\left(\vec{a}-\vec{b}\right)$, if $\vec{a} = \left(2\hat{i}-\hat{j}+3\hat{k}\right)$ and $\vec{b} = \left(3\hat{i}+\hat{j}+2\hat{k}\right)$.

Answer

$\vec{a} = (2\hat{\imath} - \hat{\jmath} + 3\hat{k})$

$\vec{b} = (3\hat{\imath} + \hat{\jmath} + 2\hat{k})$

$\vec{a} + \vec{b} = (2\hat{\imath} - \hat{\jmath} + 3\hat{k}) + (3\hat{\imath} + \hat{\jmath} + 2\hat{k}) = 5\hat{\imath} + 5\hat{k}$

$\vec{a}-\vec{b} = (2\hat{i}-\hat{j}+3\hat{k})-(3\hat{i}+\hat{j}+2\hat{k}) = -\hat{i}-2\hat{j}+\hat{k}$

$|\vec{a}+\vec{b}| = \sqrt{5^2+(5)^2} = \sqrt{25+25} = \sqrt{50}$

$|\vec{a}-\vec{b}| = \sqrt{(-1)^2+(-2)^2+(1)^2} = \sqrt{1+4+1} = \sqrt{6}$

We know that ,

$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$\Rightarrow (5\hat{i}+5\hat{k})(-\hat{i}-2\hat{j}+\hat{k}) = \sqrt{50}\sqrt{6}\cos\theta$

$\Rightarrow (-5+5) = \sqrt{300}\cos\theta$

$\Rightarrow \cos\theta = 0$

$\Rightarrow \theta = \cos^{-1}(0) = \pi/2$

Ans: $\theta = \pi/2$

18. Question

Express the vector $\vec{a}=\left(6\hat{i}-3\hat{j}-6\hat{k}\right)$ as sum of two vectors such that one is parallel to the vector $\vec{b}=\left(\hat{i}+\hat{j}+\hat{k}\right)$ and the other is perpendicular to $\vec{b}$.

Answer

$\vec{a} = (6\hat{i}-3\hat{j}-6\hat{k})$

$\vec{b} = (\hat{i}+\hat{j}+\hat{k})$

$\Rightarrow \vec{c} \parallel \vec{b} \ \& \ \vec{d} \perp \vec{b}$

$\therefore \vec{a} = \vec{c}+\vec{d}$

$\vec{c} = \lambda\vec{b} \ \& \ \vec{b}.\vec{d} = 0$

$\Rightarrow \vec{b}.\vec{a} = \vec{b}.(\vec{c}+\vec{d})$

$\Rightarrow (\hat{i}+\hat{j}+\hat{k}).(6\hat{i}-3\hat{j}-6\hat{k}) = \vec{b}.\lambda\vec{b}+0$

$\Rightarrow \mathbf{6-3-6} = \lambda\left(|\vec{b}|^2\right) = 3\lambda$

$\Rightarrow \lambda = -1$

$\vec{c} = \lambda\vec{b} = -1(\hat{i}+\hat{j}+\hat{k}) = -(\hat{i}+\hat{j}+\hat{k})$

$\therefore \vec{a} = \vec{c}+\vec{d}$

$\Rightarrow (6\hat{i}-3\hat{j}-6\hat{k}) = -(\hat{i}+\hat{j}+\hat{k})+\vec{d}$

$\Rightarrow \vec{d} = 7\hat{i}-2\hat{j}-5\hat{k}$

$\Rightarrow \vec{a} = \vec{c}+\vec{d}$

$\Rightarrow \vec{a} = -(\hat{i}+\hat{j}+\hat{k})+(7\hat{i}-2\hat{j}-5\hat{k})$

Ans: $\vec{a} = -(\hat{i}+\hat{j}+\hat{k})+(7\hat{i}-2\hat{j}-5\hat{k})$

19. Question

Prove that $\left(\vec{a}+\vec{b}\right)\cdot\left(\vec{a}-\vec{b}\right)=\left|\vec{a}\right|^2+\left|\vec{b}\right|^2 \Leftrightarrow \vec{a}\perp\vec{b}$, where $\vec{a}\neq\vec{0}$ and $\vec{b}\neq\vec{0}$.

Answer

$(\vec{a} + \vec{b}).(\vec{a} - \vec{b}) = |\vec{a}|^2 + |\vec{b}|^2$

$\Rightarrow |\vec{a}|^2 - |\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2$

$\Rightarrow |\vec{b}| = 0$

Which is not possible hence

$(\vec{a}) \perp (\vec{b})$

20. Question

If $\vec{a} + \vec{b} + \vec{c} = \vec{0}, |\vec{a}| = 3, |\vec{b}| = 5$ and $|\vec{c}| = 7,$ find the angle between $\vec{a}$ and $\vec{b}.$

Answer

$\vec{a} + \vec{b} + \vec{c} = 0$

$\Rightarrow \vec{a} + \vec{b} = -\vec{c}$

$\Rightarrow (\vec{a} + \vec{b}).(\vec{a} + \vec{b}) = -\vec{c}.-\vec{c}$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos\theta = |\vec{c}|^2$

$\Rightarrow 3^2 + 5^2 + 2 \times 3 \times 5\cos\theta = 7^2$

$\Rightarrow 2 \times 3 \times 5\cos\theta = 49 - 9 - 25$

$\Rightarrow 30\cos\theta = 15$

$\Rightarrow \cos\theta = \frac{15}{30} = \frac{1}{2}$

$\Rightarrow \theta = \cos^{-1}\frac{1}{2} = 60^0$

Ans: $\theta = 60^0 = \frac{\pi}{3}$

21. Question

Find the angle between $\vec{a}$ and $\vec{b},$ when

i. $|\vec{a}| = 2, |\vec{b}| = 1$ and $\vec{a} \cdot \vec{b} = \sqrt{3}$

ii. $|\vec{a}| = |\vec{b}| = \sqrt{2}$ and $\vec{a} \cdot \vec{b} = -1$

Answer

i)

We know that ,

$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$\Rightarrow \sqrt{3} = 2 \times 1\cos\theta$

$\Rightarrow \sqrt{3} = 2\cos\theta$

$\Rightarrow \cos\theta = \sqrt{3}/2$

$\Rightarrow \theta = \cos^{-1}(\sqrt{3}/2) = 30^o = \frac{\pi}{6}$

Ans: θ = $\cos^{-1}(\sqrt{3}/2) = 30^o = \frac{\pi}{6}$

ii)

We know that ,

$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$\Rightarrow -1 = \sqrt{2} \times \sqrt{2}\cos\theta$

$\Rightarrow -1 = 2\cos\theta$

⇒ cosθ = - 1/2

⇒ θ = $\cos^{-1}(-1/2) = 120^o = \frac{2\pi}{3}$

Ans: θ = $\cos^{-1}(-1/2) = 120^o = \frac{2\pi}{3}$

22. Question

If $\left|\vec{a}\right| = 2, \left|\vec{b}\right| = 3$ and $\vec{a} \cdot \vec{b} = 4,$ find $\left|\vec{a} - \vec{b}\right|$.

Answer

We know that ,

$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$\Rightarrow 4 = 2 \times 3\cos\theta$

$\Rightarrow 4 = 6\cos\theta$

⇒ cosθ = 4/6

⇒ cosθ = 2/3

$\Rightarrow |\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta$

$\Rightarrow |\vec{a} - \vec{b}|^2 = 2^2 + 3^2 - (2 \times 2 \times 3) \times \frac{2}{3}$

$\Rightarrow |\vec{a} - \vec{b}|^2 = 4 + 9 - 8 = 5$

$\Rightarrow |\vec{a} - \vec{b}| = \sqrt{5}$

Ans: √5

23. Question

If $\left(\vec{a} + \vec{b}\right) \cdot \left(\vec{a} - \vec{b}\right) = 8$ and $\left|\vec{a}\right| = 8\left|\vec{b}\right|,$ find $\left|\vec{a}\right|$ and $\left|\vec{b}\right|$.

Answer

$(\vec{a} + \vec{b}).(\vec{a} - \vec{b}) = 8$

$\Rightarrow |\vec{a}|^2 - |\vec{b}|^2 = 8$

$\Rightarrow (8|\vec{b}|)^2 - |\vec{b}|^2 = 8$

$\Rightarrow 64|\vec{b}|^2 - |\vec{b}|^2 = 8$

$\Rightarrow 63|\vec{b}|^2 = 8$

$\Rightarrow |\vec{b}| = \sqrt{\frac{8}{63}}$

$\Rightarrow |\vec{a}| = 8|\vec{b}| = 8\sqrt{\frac{8}{63}}$

Ans: $|\vec{a}| = 8\sqrt{\frac{8}{63}}, |\vec{b}| = \sqrt{\frac{8}{63}}$

24. Question

If $\hat{a}$ and $\hat{b}$ are unit vectors inclined at an angle θ then prove that:

i. $\cos\frac{\theta}{2} = \frac{1}{2}|\hat{a} + \hat{b}|$

ii. $\tan\frac{\theta}{2} = \frac{|\hat{a} - \hat{b}|}{|\hat{a} + \hat{b}|}$

Answer

R.H.S:

$\left(\frac{1}{2}\right)(|\hat{a} + \hat{b}|) = \frac{1}{2}(\sqrt{|\hat{a}|^2 + |\hat{b}|^2 + 2|\hat{a}||\hat{b}|\cos\theta})$

$\Rightarrow \frac{1}{2}(\sqrt{1^2 + 1^2 + 2 \times 1 \times 1\cos\theta}$

$\Rightarrow \frac{1}{2}(\sqrt{1 + 1 + 2\cos\theta}$

$\Rightarrow \sqrt{\frac{2 + 2\cos\theta}{4}}$

$\Rightarrow \sqrt{\frac{2(1 + \cos\theta)}{4}}$

$\Rightarrow \sqrt{\frac{(1 + \cos\theta)}{2}}$

$\Rightarrow \sqrt{\cos^2\frac{\theta}{2}}$

$\Rightarrow \cos\frac{\theta}{2}$ = L.H.S

Hence, proved

ii)

R.H.S. $= \frac{(|\hat{a} - \hat{b}|)}{(|\hat{a} + \hat{b}|)}$

$\Rightarrow \frac{\sqrt{|\hat{a}|^2 + |\hat{b}|^2 - 2|\hat{a}||\hat{b}|\cos\theta}}{\sqrt{|\hat{a}|^2 + |\hat{b}|^2 + 2|\hat{a}||\hat{b}|\cos\theta}}$

$\Rightarrow \frac{\sqrt{1^2 + 1^2 - 2 \times 1 \times 1\cos\theta}}{\sqrt{1^2 + 1^2 + 2 \times 1 \times 1\cos\theta}}$

$\Rightarrow \frac{\sqrt{1 + 1 - 2\cos\theta}}{\sqrt{1 + 1 + 2\cos\theta}}$

$\Rightarrow \sqrt{\frac{1 - \cos\theta}{1 + \cos\theta}}$

$\Rightarrow \sqrt{\frac{\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}}$

$\Rightarrow \sqrt{\tan^2\frac{\theta}{2}}$

⇒ tanθ/2 = L.H.S

Hence,proved.

25. Question

The dot products of a vector with the vector $\left(\hat{i}+\hat{j}-3\hat{k}\right)$, $\left(\hat{i}+3\hat{j}-2\hat{k}\right)$ and $\left(2\hat{i}+\hat{j}+4\hat{k}\right)$ are 0, 5 and 8 respectively. Find the vector.

Answer

Let the unknown vector be: $\vec{a} = a\hat{i} + b\hat{j} + c\hat{k}$

$\therefore \left(a\hat{i} + b\hat{j} + c\hat{k}\right).\left(\hat{i} + \hat{j} - 3\hat{k}\right) = 0$

⇒ a + b - 3c = 0 ...(1)

$\therefore \left(a\hat{i} + b\hat{j} + c\hat{k}\right).\left(\hat{i} + 3\hat{j} - 2\hat{k}\right) = 5$

⇒ a + 3b - 2c = 5 ...(2)

$\therefore \left(a\hat{i} + b\hat{j} + c\hat{k}\right).\left(2\hat{i} + \hat{j} + 4\hat{k}\right) = 8$

⇒ 2a + b + 4c = 8 ...(3)

Solving equations 1 ,2,3, simultaneously we get:

a = 1,b = 2,c = 1

$\therefore \vec{a} = \hat{i} + 2\hat{j} + \hat{k}$

Ans: $\vec{a} = \hat{i} + 2\hat{j} + \hat{k}$

26. Question

If $\overrightarrow{AB} = \left(3\hat{i} - \hat{j} + 2\hat{k}\right)$ and the coordinates of A are (0, - 2, - 1), find the coordinates of B.

Answer

$\overrightarrow{AB} = \vec{B} - \vec{A} = 3\hat{i} - \hat{j} + 2\hat{k}$

$\Rightarrow \vec{B} - (0\hat{i} - 2\hat{j} - \hat{k}) = 3\hat{i} - \hat{j} + 2\hat{k}$

$\Rightarrow \vec{B} = \left(0\hat{i} - 2\hat{j} - \hat{k}\right) + 3\hat{i} - \hat{j} + 2\hat{k}$

$\Rightarrow \vec{B} = 3\hat{i} - 3\hat{j} + \hat{k}$

∴ B(3, - 3,1)

Ans: B(3, - 3,1)

27. Question

If A(2, 3, 4), B(5, 4, - 1), C(3, 6, 2) and D(1, 2, 0) be four points, show that $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{CD}$.

Answer

$\vec{A} = 2\hat{i} + 3\hat{j} + 4\hat{k}$

$\vec{B} = 5\hat{i} + 4\hat{j} - \hat{k}$

$\vec{C} = 3\hat{i} + 6\hat{j} + 2\hat{k}$

$\vec{D} = \hat{i} + 2\hat{j} + 0\hat{k}$

$\overrightarrow{AB} = \vec{B} - \vec{A} = 5\hat{i} + 4\hat{j} - \hat{k} - (2\hat{i} + 3\hat{j} + 4\hat{k}) = 3\hat{i} + \hat{j} - 5\hat{k}$

$\overrightarrow{CD} = \vec{D} - \vec{C} = \hat{i} + 2\hat{j} + 0\hat{k} - (3\hat{i} + 6\hat{j} + 2\hat{k}) = -2\hat{i} - 4\hat{j} - 2\hat{k}$

$\overrightarrow{AB}.\overrightarrow{CD} = (3\hat{i} + \hat{j} - 5\hat{k}).(-2\hat{i} - 4\hat{j} - 2\hat{k}) = -6 - 4 + 10 = 0$

Hence, $\overrightarrow{AB} \perp \overrightarrow{CD}$

28. Question

Find the value of λ for which the vectors $\left(2\hat{i}+\lambda\hat{j}+3\hat{k}\right)$ and $\left(3\hat{i}+2\hat{j}-4\hat{k}\right)$ are perpendicular to each other.

Answer

$\vec{a} = 2\hat{i} + \lambda\hat{j} + 3\hat{k}$

$\vec{b} = 3\hat{i} + 2\hat{j} - 4\hat{k}$

Since these two vectors are perpendicular, their dot product is zero.

$\Rightarrow \vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta = |\vec{a}||\vec{b}|\cos\frac{\pi}{2} = 0$

$\Rightarrow \vec{a}.\vec{b} = (2\hat{i} + \lambda\hat{j} + 3\hat{k}).(3\hat{i} + 2\hat{j} - 4\hat{k}) = 0$

$\Rightarrow \vec{a}.\vec{b} = (2 \times 3) + (\lambda \times 2) + (3 \times -4) = 0$

$\Rightarrow \vec{a}.\vec{b} = 6 + 2\lambda - 12 = 0$

$\Rightarrow 6 = 2\lambda$

$\Rightarrow \lambda = \frac{6}{2} = 3$

Ans: $\lambda = 3$

29. Question

Show that the vectors $\vec{a}=\left(3\hat{i}-2\hat{j}+\hat{k}\right)$, $\vec{b}=\left(\hat{i}-3\hat{j}+5\hat{k}\right)$ and $\vec{c}=\left(2\hat{i}+\hat{j}-4\hat{k}\right)$ form a right - angled triangle.

Answer

$\vec{a} = 3\hat{i} - 2\hat{j} + \hat{k}$

$\vec{b} = \hat{i} - 3\hat{j} + 5\hat{k}$

$\vec{c} = 2\hat{i} + \hat{j} - 4\hat{k}$

$|\vec{a}| = \sqrt{9 + 4 + 1} = \sqrt{14}$

$|\vec{c}| = \sqrt{4 + 1 + 16} = \sqrt{21}$

$\cos\theta = \frac{\vec{a}.\vec{c}}{|\vec{a}||\vec{c}|} = \frac{(3\hat{i} - 2\hat{j} + \hat{k}).(2\hat{i} + \hat{j} - 4\hat{k})}{\sqrt{14}\sqrt{21}} = \frac{6 - 2 - 4}{\sqrt{14}\sqrt{21}} = 0$

$\Rightarrow \theta = \cos^{-1}0 = \frac{\pi}{2}$

Hence, the triangle is a right angled triangle at c

30. Question

Three vertices of a triangle are A(0, - 1, - 2), B(3, 1, 4) and C(5, 7, 1). Show that it is a right - angled triangle. Also, find its other two angles.

Answer

$\vec{a} = 0\hat{\imath} - \hat{\jmath} - 2\hat{k}$

$\vec{b} = 3\hat{\imath} + \hat{\jmath} + 4\hat{k}$

$\vec{c} = 5\hat{\imath} + 7\hat{\jmath} + \hat{k}$

$|\overrightarrow{AB}| = \sqrt{9 + 4 + 36} = \sqrt{49} = 7$

$|\overrightarrow{BC}| = \sqrt{4 + 36 + 9} = \sqrt{49} = 7$

$|\overrightarrow{CA}| = \sqrt{25 + 64 + 9} = \sqrt{98} = 7\sqrt{2}$

$\overrightarrow{AB} = \vec{B} - \vec{A} = 3\hat{\imath} + \hat{\jmath} + 4\hat{k} - (0\hat{\imath} - \hat{\jmath} - 2\hat{k}) = 3\hat{\imath} + 2\hat{\jmath} + 6\hat{k}$

$\overrightarrow{BC} = \vec{C} - \vec{B} = 5\hat{\imath} + 7\hat{\jmath} + \hat{k} - (3\hat{\imath} + \hat{\jmath} + 4\hat{k}) = 2\hat{\imath} + 6\hat{\jmath} - 3\hat{k}$

$\overrightarrow{CA} = \vec{A} - \vec{C} = 0\hat{\imath} - \hat{\jmath} - 2\hat{k} - (5\hat{\imath} + 7\hat{\jmath} + \hat{k}) = -5\hat{\imath} - 8\hat{\jmath} - 3\hat{k}$

$$\cos\theta = \frac{\overrightarrow{AB}.\overrightarrow{BC}}{|\overrightarrow{AB}||\overrightarrow{BC}|} = \frac{(3\hat{\imath} + 2\hat{\jmath} + 6\hat{k}).(2\hat{\imath} + 6\hat{\jmath} - 3\hat{k})}{7 \times 7} = \frac{6 + 12 - 18}{49} = 0$$

$\therefore \theta = \frac{\pi}{2}$

$$\cos\alpha = \frac{\overrightarrow{CA}.\overrightarrow{BC}}{|\overrightarrow{CA}||\overrightarrow{BC}|} = \frac{(-5\hat{\imath} - 8\hat{\jmath} - 3\hat{k}).(2\hat{\imath} + 6\hat{\jmath} - 3\hat{k})}{7\sqrt{2} \times 7} = \frac{-10 - 48 + 9}{49\sqrt{2}}$$
$$= |\frac{-1}{\sqrt{2}}|$$

$\therefore \theta = \frac{\pi}{4} = 45°$

$$\cos\alpha = \frac{\overrightarrow{CA}.\overrightarrow{AB}}{|\overrightarrow{CA}||\overrightarrow{AB}|} = \frac{(-5\hat{\imath} - 8\hat{\jmath} - 3\hat{k}).(3\hat{\imath} + 2\hat{\jmath} + 6\hat{k})}{7\sqrt{2} \times 7} = \frac{-15 - 16 + 18}{49\sqrt{2}}$$
$$= |\frac{-1}{\sqrt{2}}|$$

$\therefore \theta = \frac{\pi}{4} = 45°$

Ans:45°,90°,45°

31. Question

If the position vectors of the verticesA, B and C of a ΔABC be (1, 2, 3), (- 1, 0, 0) and (0, 1, 2) respectively then find ∠ABC.

Answer

$\vec{a} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$

$\vec{b} = -\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$

$\vec{c} = 0\hat{\imath} + \hat{\jmath} + 2\hat{k}$

$|\overrightarrow{AB}| = \sqrt{4 + 4 + 9} = \sqrt{17}$

$|\overrightarrow{BC}| = \sqrt{1 + 1 + 4} = \sqrt{6}$

$|\overrightarrow{CA}| = \sqrt{1 + 1 + 1} = \sqrt{3}$

$\overrightarrow{AB} = \vec{B} - \vec{A} = -\hat{i} + 0\hat{j} + 0\hat{k} - (\hat{i} + 2\hat{j} + 3\hat{k}) = -2\hat{i} - 2\hat{j} - 3\hat{k}$

$\overrightarrow{BC} = \vec{C} - \vec{B} = 0\hat{i} + 1\hat{j} + 2\hat{k} - (-\hat{i} + 0\hat{j} + 0\hat{k}) = \hat{i} + \hat{j} + 2\hat{k}$

$\overrightarrow{CA} = \vec{A} - \vec{C} = \hat{i} + 2\hat{j} + 3\hat{k} - (0\hat{i} + 1\hat{j} + 2\hat{k}) = \hat{i} + \hat{j} + \hat{k}$

$\cos\theta = \frac{\overrightarrow{AB}.\overrightarrow{BC}}{|\overrightarrow{AB}||\overrightarrow{BC}|} = \frac{(-2\hat{i} - 2\hat{j} - 3\hat{k}).(\hat{i} + \hat{j} + 2\hat{k})}{\sqrt{17} \times \sqrt{6}} = \frac{-2 - 2 - 6}{\sqrt{102}} = |\frac{-10}{\sqrt{102}}|$

$\therefore \theta = \cos^{-1}\frac{10}{\sqrt{102}}$

Ans: $\theta = \cos^{-1}\frac{10}{\sqrt{102}} = \angle ABC$

32. Question

If $\vec{a}$ and $\vec{b}$ are two unit vectors such that $|\vec{a} + \vec{b}| = \sqrt{3}$, find $(2\vec{a} - 5\vec{b}) \cdot (3\vec{a} + \vec{b})$.

Answer

$|\vec{a}| = |\vec{b}| = 1$

$|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos\theta$

$\Rightarrow 3 = 1 + 1 + 2\cos\theta$

$\Rightarrow \cos\theta = 1/2$

$\therefore (2\vec{a} - 5\vec{b}).(3\vec{a} + \vec{b}) = 6|\vec{a}|^2 - 5|\vec{b}|^2 - 13\vec{a}.\vec{b}$

$\Rightarrow (2\vec{a} - 5\vec{b}).(3\vec{a} + \vec{b}) = 6 - 5 - 13|\vec{a}||\vec{b}|\cos\theta = 1 - 13 \times 1 \times 1 \times (1/2)|$

$\Rightarrow (2\vec{a} - 5\vec{b}).(3\vec{a} + \vec{b}) = 1 - \frac{13}{2} = \frac{-11}{2}$

Ans: $(2\vec{a} - 5\vec{b}).(3\vec{a} + \vec{b}) = \frac{-11}{2}$

33. Question

If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a} + \vec{b}| = |\vec{a}|$ then prove that vector $(2\vec{a} + \vec{b})$ is perpendicular to the vector $\vec{b}$.

Answer

$|\vec{a} + \vec{b}| = |\vec{a}|$

$\Rightarrow |\vec{a} + \vec{b}|^2 = |\vec{a}|^2$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos\theta = |\vec{a}|^2$

$\Rightarrow |\vec{b}| = -2|\vec{a}|\cos\theta$

NOW,

$(2\vec{a} + \vec{b}).(\vec{b}) = 2\vec{a}.\vec{b} + |\vec{b}|^2$

$\Rightarrow (2\vec{a} + \vec{b}).(\vec{b}) = 2|\vec{a}||\vec{b}|\cos\theta + ((2|\vec{a}|\cos\theta)^2)$

$\Rightarrow (2\vec{a} + \vec{b}).(\vec{b}) = 2|\vec{a}|(-2|\vec{a}|\cos\theta)\cos\theta + ((2|\vec{a}|\cos\theta)^2) = 0$

Hence, $(2\vec{a} + \vec{b}) \perp (\vec{b})$

34. Question

If $\vec{a} = (3\hat{i} - \hat{j})$ and $\vec{b} = (2\hat{i} + \hat{j} - 3\hat{k})$ then express $\vec{b}$ in the form $\vec{b} = (\vec{b}_1 + \vec{b}_2)$, where $\vec{b}_1 \parallel \vec{a}$ and $\vec{b}_2 \perp \vec{a}$.

Answer

Let b_1 = c and b_2 = d

$\vec{a} = (3\hat{i} - \hat{j})$

$\vec{b} = (2\hat{i} + \hat{j} - 3\hat{k})$

$\Rightarrow \vec{c} \parallel \vec{a} \,\&\, \vec{d} \perp \vec{a}$

$\therefore \vec{b} = \vec{c} + \vec{d}$

$\vec{c} = \lambda\vec{a} \,\&\, \vec{a}.\vec{d} = 0$

$\Rightarrow \vec{a}.\vec{b} = \vec{a}.(\vec{c} + \vec{d})$

$\Rightarrow (3\hat{i} - \hat{j}).(2\hat{i} + \hat{j} - 3\hat{k}) = \vec{a}.\lambda\vec{a} + 0$

$\Rightarrow 6 - 1 = \lambda(|\vec{a}|^2) = 10\lambda$

$\Rightarrow \lambda = 5/10 = 1/2$

$\vec{c} = \lambda\vec{a} = (1/2)(3\hat{i} - \hat{j}) = \left(\frac{3}{2}\hat{i} - \frac{1}{2}\hat{j}\right)$

$\therefore \vec{b} = \vec{c} + \vec{d}$

$\Rightarrow (2\hat{i} + \hat{j} - 3\hat{k}) = \left(\frac{3}{2}\hat{i} - \frac{1}{2}\hat{j}\right) + \vec{d}$

$\Rightarrow \vec{d} = \left(\frac{1}{2}\hat{i} + \frac{3}{2}\hat{j}\right) - 3\hat{k}$

$\Rightarrow \vec{b} = b_1 + b_2$

$\Rightarrow \vec{b} = \left(\frac{3}{2}\hat{i} - \frac{1}{2}\hat{j}\right) + (\left(\frac{1}{2}\hat{i} + \frac{3}{2}\hat{j}\right) - 3\hat{k})$

Ans: $\vec{b} = \left(\frac{3}{2}\hat{i} - \frac{1}{2}\hat{j}\right) + (\left(\frac{1}{2}\hat{i} + \frac{3}{2}\hat{j}\right) - 3\hat{k})$