14

Probability

14.01 Introduction :

The concept of probability originated in 17th century in Europe. Their businessman tried to find the results of the business so that they get the maximum profit. They put up their problems to mathematicians Galileo Pascal forma cardeno etc. The mathematician developed methods to find the solutions of their problems and hence this stream of mathematics has been developed. The famous mathematicians of 18th and 19th centuary, Laplace, Gauses, Bermouli etc. have more developed this principle. In 20th centuary decision theory selection, theory etc. methods based on the principle of probability has been developed whose credit goes to R.S. Fisher and Cary Pearson. We have used even probability or even possible results to define probability. This definition is logically not proper. So another mathematician of Russia. A.N. Kolomigrove has discovered another principle of probability. That is called as axiomatic based probability principle. He decided some proved facts in his book foundation of probability published in 1993 to explain the probability.

In modern era the principle of probability is being used to get the decision related to future in various field. For example to make the budget of any state or country, insurance companies, games based on co-incidence, agriculture, economy, scientific research, soldier security terms widely in business field, field of natural science and physics and for the society and state system.

Previously we have stutied the life concept of probability based on the uncertainity of various cases. Various events happens in front of us in daily, they may have infinite and more than one definite results. A person takes the profit byexpecting the results. Find the probability of results based on the conditions and prior information of any event, the principle is called as probability.

At present we use two methods to find the probability. One of them is called as classical theory of probability in this to find the probability of any event we find the ratio of number of favourable outcomes with number of total outcomes another method is called as axiomatic approach to probability, in this method to find the probability rules or axioms are depicted.

To understand both the methods we need to understand some important words in detail. In further sections these are defined as classical theory of probability and axiomatic approach.

14.02 Definations :

(A) A Classical approach to Probality

1. Random experiment : An experiment is called random experiment if it has more than one possible outcomes or it is not possible to predict the outcome in advance. For example; when a coin is tossed it may turn up as a head, or a tail, but we are not sure which one of these results will actually happened. Such experiment are called random experiments.

2. Trial and event : An experiment is considered as trial if it surely gives result and the possible results are called events, for example :

(i) Tossing a coin is a trial and getting. head (H) of Tail (T) is an event.

(ii) Rolling a die is a trial and getting any one mumber out of 1, 2, 3, 4, 5, 6 is an event.

(iii) Selecting 2 cards from a deck of cards is a trial and from all possible ${}^{52}C_2$ results, outcomes getting both cards are king that is ${}^{4}C_2$ is an event.

3. Simple Event : If an event has only one sample point of a sample space, it is called a simple event. For example, getting an even prime number on rolling a die is a simple event.

4. Exhaustive events or total number of cases : Total number of possible outcomes (cases) of a trial is exhaustive events of the trial.

(i) Tossing a coin is an experiment and we may get head (H) or tail (T). Thus there are two exhaustive events in this experiment.

(ii) Throwing a die, 1, 2, 3, 4, 5, or 6 outcomes can orccur. Hence in this trial total cases are 6.

5. Favourable events or cases : Favourablecases of a trial is number of results of oceurence of some specific event. For example

(i) Getting even numbers 2, 4, 6 in rolling a die i.e. here the favourable cases are 3.

(ii) Selecting a card from a deck of cards and that card is of kind so favourable cases are ${}^{4}C_1$ i.e. 4

(iii) Throwing two dice and getting sum 5 has 4 favourable conditions i.e. (1,4), (2,3), (3,2), (4,1)

6. Independent and dependent events :

(i) Two events are said to be independent events if the occurence or non-occurence of one event does not affect the occurance of another event.

Example : If a coin is tossed and a die is rolled then getting a tail on coin and 4 on die are two independent events.

(ii) Dependent events : Two events are said to be dependent events if the occurance of one event affects the occurence of another event.

Example : If a card is drawn is heared without replacement followed by card of space is drawn from a well shuffeled deck of cards then both events are said to be dependent events.

7. Mutually exclusive or disjoint events : Two or more events are called mutually exclusive events if the occurence of any one of them excludes the occurence of the other event, i.e. if they can not occur simultaneously and if two events do not have any common element then they are said to be disjoint events.

(i) Tossing a coin and getting head (H) or tail (T) are equally likely events.

(ii) Selecting one card from a deck of card and having its king or queen is mutually exclusive events.

8. Equally likely events : In an experiment if there is equal possibility of events to occur then these events are called equally likely events. For example -

(i) Getting a head (H) or tail (T) in tossing a coin are equally likely events.

(ii) Selecting a card from a deck of card and it is red or black are equally likely events.

9. Compound events : If an event has more than one sample point, it is called a compound event.

For example, there are some blue and some red balls in two bags selecting a ball from a bag is a compound event because selecting one bag and then select one ball from that bag are events occuring together.

(B) Required definitions of Probability in Axiomatic View

1. Sample point and sample space : The set of all possible outcomes a random experiment is called the sample space associated with the experiment. Sample space is denoted by the symbol S. Each element of the sample space is called a sample point. In other words, each outcome of the random experiment is also called sample point.

Generally it is denoted as S for example.

(i) Sample points in throwing two coins is (H,H), (H,T), (T,H), (T,T) and S={(H,H), (H,T), (T,H), (T,T)}is sample space.

(ii) From 3 boys and 2 girls two are selected. The sample space in this trial will be (Boys, B_1, B_2, B_3, Girls G_1, G_2) :

$$S = \{B_1B_2, B_2B_3, B_3B_1, B_1G_1, B_1G_2, B_2G_1, B_2G_2, B_3G_1, B_3G_2, G_1G_2\}$$

2. Elementary events : A subset of having one element of sample space related to random experiment is called as elementary events.

Clearly, with each result of random experiment, *n* elementary event is related and conversely.

For example : Sample space of tossing a coin two times is S={HH, HT, TH, TT,} here, four elementary events in sample space $E_1 = \{HH\}$, $E_2 = \{HT\}$, $E_3 = \{TH\}$ here, $E_4 = \{TT\}$

3. Compound event : The subsets of sample space S of an experiment which are made of combination of subsets of elements of sample space S.

For example, think on throwing one die. In this sample space $S = \{1, 2, 3, 4, 5, 6\}$ elementary events $E_1 = \{1\}$, $E_2 = \{2\}, .., E_6 = \{6\}$ here, $A_1 = \{2, 4, 6\}$, $A_2 = \{1, 3, 5\}$ etc. are compound events.

4. Impossible and certain events : The empty set ϕ and the sample space S being subset of S describe events. In fact ϕ is called an impossible event and S, i.e., the whole sample space is called the sure event.

Example : Getting a number 7 in rolling a die is an impossible event.

5. Occurrence of an event : Subset A of any sample space S represents a trial if ω is the result of that random experiment and $\omega \in A$ can be said that an event has occur and if an event does not occur then it is said that $\omega \notin A$

For example, A random experiment of throwing a die, let event of getting even number is A that is $A = \{2, 4, 6\}$. If 6 is obtained in on trial and $6 \in A$ then we can say that an event has occured in this experiment if result obtained is 5 then we will say that event does not occur in experiment.

6. Algebra of events : The algebra can understood by the following table -

Event	Symbol
A Not	$\overline{A}$
A or B	$A \cup B$
A and B	$A \cap B$
A but not B	$A \cap \overline{B}$
neither A nor B	$\overline{A} \cap \overline{B} = \left(\overline{A \cup B}\right)$
any one out of A and B	$\left(A \cap \overline{B}\right) \cup \left(\overline{A} \cup B\right)$
any two out of A, B and C	$\left(A \cap B \cap \overline{C}\right) \cup \left(A \cap \overline{B} \cap C\right) \cup \left(\overline{A} \cap B \cap C\right)$
atleast one out of A, B and C	$A \cup B \cup C$
all of A, B and C	$A \cap B \cap C$

7. Mutually exclusive or disjoint event : Let S is the sample space of a random experiment A_1 and A_2 are two events then A_1 and A_2 are mutually exclusive if $A_1 \cap A_2 = \phi$ clearly, the elementary events related to random experiment are mutually exclusive. The events which are not mutually exclusive that are known as favourable events.

8. Mutually exclusive and exhaustive system of events : Let $A_1, A_2, ...A_n$, S be n number of events of sample space S then

(i) $A_i \cap A_j = \phi$, $i \neq j$ and (ii) $A_1 \cup A_2 \cup .... \cup A_n = S$

Such events are mutually exclusive and exhaustive events.

Exercise 14.1

1. 3 bulls are randomly taken form a box. Each bulb is tested and classified as defective (D) and non defective (N). Write the sample space.
2. 4 cards are drawn from a pack of cards. Find n(E) where E is the event of drawing a king, a queen, a jack and an ace.
3. A die is rolled. If getting 4 shows an event E and getting an even number shows event F then E and F mutually exclusive events ?
4. Two dice are rolled. Write the sample space of
   (i) Getting a doublet (ii) Getting a sum 8

14.03 Defination of Probability :

Classical defination of probability :

If an experiment n outcomes are equally likely, mutually exclusive and exhaustive and out of which m outcones are favourable to event A then the probability of A is defined as the ratio m/n and written as P(A)

$$\therefore \quad P(A) = \frac{\text{Favourable cases of A}}{\text{Total cases of A}} = \frac{m}{n}, \text{(numerical value)}$$

If event A is sure then $m = n$ hence

$$P(A) = \frac{n}{n} = 1,$$

If the event A is impossible then $m = 0$ and

$$P(A) = \frac{0}{n} = 0,$$

Thus for any event $0 \leq P(A) \leq 1$

thus probability of an event cannot be less then 0 and greater than 1. If an event A does not occur then it is denoted by $P(\bar{A})$

$$\therefore \quad P(\bar{A}) = \frac{\text{Unfavourable cases of A}}{\text{Total cases of A}} = \frac{n-m}{n} = 1 - \frac{m}{n}$$

$$\Rightarrow \quad P(\bar{A}) = 1 - P(A)$$

Defination of probability in axiomatic approach :

Let us consider a random experiment with sample space S and let A be the subset of S then the probability of event A is

$$P(A) = \frac{\text{No. of elements in A}}{\text{No. of elements in S}} = \frac{n(A)}{n(S)} = \frac{\text{Number of elementry events in A}}{\text{Number of elementry events in S}}$$

Clearly $P(\phi) = 0$, $P(S) = 1$ and $0 \le P(A) \le 1$.

14.04 Odds :

In an experiment if the total cases are n and favourable cases are m then for any event A then unfavourable cases will be $n - m$. The odds in favour of A will be $m : (n - m)$ and against will be $(n - m) : m$

Odds in favour of A $= \dfrac{m}{n-m} = \dfrac{\frac{m}{n}}{\frac{n-m}{n}} = \dfrac{P(A)}{P(\bar{A})}$

Odds in against of A $= \dfrac{n-m}{m} = \dfrac{\frac{n-m}{n}}{\frac{m}{n}} = \dfrac{P(\bar{A})}{P(A)}$

Theorem : In a random experiment for any event A, prove that $P(\bar{A}) = 1 - P(A)$

Proof : In an experiment if the total cases are n and for any event A the favourable cases are m then the unfavourable cases are $n - m$

Proability of non-occurance of an event A is

$$P(\bar{A}) = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - P(A)$$

Again using axiomatic approach

$$P(\bar{A}) = \frac{\text{The number of elementry events } \bar{A}}{\text{The number of elementry events } S}$$

$$= \frac{n(S) - n(A)}{n(S)} = 1 - \frac{n(A)}{n(S)} = 1 - P(A)$$

Illustrative Examples

Example 1 : Find the probability of getting an even number in rolling a dice.

Solution : Total cases = 6, Favourable number of cases = 3 i.e. 2, 4, 6

$\therefore$ Required probability $= 3/6 = 1/2$

Example 2 : Find the probability of getting a sum 7 in rolling two dice.

Solution : Total cases $6 \times 6 = 36$

Following are possibilities of getting a sum 7

(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)

$\therefore$ Favourable number of cases = 6

$\therefore$ Required probability $= 6/36 = 1/6$

Example 3 : Find the probability of getting 53 Mondays in a leap year.

Solution : We know that a leap year has 366 days. Thus having 52 complete weeks and 2 days. These two days have the following possibilities.

1. Monday & Tuesday 2. Tuesday & Wednusday 3. Wednusday & Thursday 4. Thursday & Friday 5. Friday & Saturday 6. Saturday & Sunday 7. Sunday & Monday.

Total cases : 7, Favourable number of cases : 2 (as there are two possibilities contains monday)

$\therefore$ Required probability $= 2/7$

Example 4 : Twelve tickets are marked numbers 1 to 12. A ticket is selected at random. Find the probability of it being a multiple of 2 or 3.

Solution : The number multiple of 2 or 3 are 2, 3, 4, 6, 8, 9, 10, 12

Total cases = 12 and favourable number of cases = 8

$\therefore$ Required probability $= \frac{8}{12} = \frac{2}{3}$

Example 5 : Two cards are drawn from the well shuffled deck of 52 cards. Show that the probability of getting both jacks is $\mathbf{\frac{1}{221}}$

Solution : Total cases = ${}^{52}C_2$

Favourable cases = ${}^{4}C_2$

$\therefore$ Required probability $= \frac{{}^{4}C_2}{{}^{52}C_2} = \frac{\frac{4\times3}{2\times1}}{\frac{52\times51}{2\times1}} = \frac{4\times3}{2\times1} \times \frac{2\times1}{52\times51} = \frac{1}{221}$

Example 6 : Three coins are tossed, find the probability of getting

(i) exactly two tails (ii) atleast two tails

Solution : Total cases $= 2^3 = 8$

[HHH, HHT, HTH, THH, HTT, THT, TTH, TTT]

(i) Favourable cases(Exactly two tails) = 3

$\therefore$ Required probability $= 3/8$

(ii) Favourable cases(Atleast two tails) = 4

$\therefore$ Required probability $= 4/8 = 1/2$

Example 7 : A bag contains 3 white balls and 5 black balls. Two balls are drawn at random from the bag, what is the odds favourable of drawing both black balls.

Solution : Total No.of balls in a bag $= 3+5 = 8$

Total cases 2 balls drawn from a bag containing 8 balls $= {}^{8}C_2 = 28$

Favourable cases 2 black balls drawn from a bag containing 5 black balls $= {}^{5}C_2 = 10$

$\therefore$ Unfavourable cases $= 28-10 = 18$

Ratio in favour of drawing the ball = favourable cases : unfavourable cases : $= 10 : 18 = 5 : 9$

Example 8 : 4 persons are selected randomly from a group of 4 men, 3 women and 5 children. Find the probability of selecting with exactly two children.

Solution : total person $= 4+3+5 = 12$

Total cases for selection 4 person from 12 $= {}^{12}C_4$

In such selection 2 should be children, their selection can be done in ${}^{5}C_2$ ways with 2 children remaining 2 person will be selected from 4 men + 3 women = 7 person hence their selection is ${}^{7}C_2$. Hence, total favourable conditions $= {}^{5}C_2 \times {}^{7}C_2$

$\therefore$ Required probability $= \frac{{}^{5}C_2 \times {}^{7}C_2}{{}^{12}C_4} = \frac{\frac{5\times4\times7\times6}{2\times1\times2\times1}}{\frac{12\times11\times10\times9}{4\times3\times2\times1}} = \frac{5\times7\times6}{11\times5\times9} = \frac{14}{33}$

Exercise 14.2

1. Find the probability of getting a number greater than 4 in rolling a die.
2. A coin is tossed twice. Find the probability of getting head both the times.
3. A number is choosen from a set of natural numbers 1 to 17. Find the probability that is is a prime number.
4. A coin is tossed three times. Find the probability of getting a head or tail.
5. A die is rolled, find the probability of getting a doublet or a sum 9 on the die.
6. Find the probability of getting only 52 Sundays in an ordinary year.
7. One card is drawn from a well-shuffled deck of 52 cards, find the odds in favour of getting an Ace.
8. In a class of 12 students there are 5 boys and rest are girl. A student is selected, find the odds not in favour (unfavourable) of getting a girl.
9. n people are sitting on a round table. Find the odds not in favour (unfourable) of two special people sitting together.
10. Three letters are dictated to three persons and an envelope is addressed to each of them, the letters are inserted into the envelopes at random so that each envepole contains exactly one letter. Find the probability that none of the letter is in its proper envelope.
11. A number is selected from first 200 integers. Find the probability that it is divisible by 6 or 8.
12. Three dice are thrown once. Find the probability of getting a number greater than 15.
13. The letters of the word ANGLE are arranged at random. Find the probability of geting all vowels to occur together.
14. One card is drawn from a well-shuffled deck of 52 cards.. Find the probability of it being a king or a queen or an ace.
15. A bag contains 6 white, 7 red and 5 black balls. Three balls are drawn one after other. Find the probability of drawing a white ball if the ball drawn is not replaced.

14.05 Addition theorem of probability or theorem of total probability :

When events are mutually exclusive -

Theorem 1 : Let A and B be two mutually exclusive events with respective probabilities P(A) and P(B). Then, probability of occurrence of at least one of these two events is given by the sum of the individual probabilities-

$$P(A+B) = P(A) + P(B)$$

and
$$P(A \cup B) = P(A) + P(B)$$

Proof : Let the total number of cases be n and the favourable cases for event A and B be m_1 and m_2 respectively.

$$\therefore \qquad P(A) = \frac{m_1}{n}, \; P(B) = \frac{m_2}{n}$$

Since A and B are mutually exclusive events therefore occurence of one event will be m_1+m_2

$$P(A+B) = \frac{m_1+m_2}{n} = \frac{m_1}{n} + \frac{m_2}{n} = P(A)+P(B)$$

$$P(A+B) = P(A) + P(B)$$

Proof using the set Theory :

Let S be a sample space A and B be two mutually exclusive events, then the number of elements in $(A \cup B)$

is equal to the sum of numbers of elements present in A and B individually.

So, $n(A \cup B) = n(A) + n(B)$

$$P(A \cup B) = P(A+B) = \frac{n(A \cup B)}{n(S)} = \frac{n(A) + n(B)}{n(S)}$$

$$P(A \cup B) = \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)}$$

$$\Rightarrow \quad P(A \cup B) = P(A) + P(B)$$

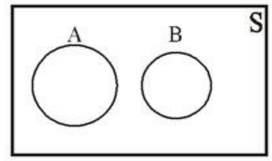

fig. 14.01

Generalisation : Let there be n mutually exclusive events, then probability of occurence of any one event is equal to the sum of individual probabilities, i.e.

$$P(A_1+A_2+A_3+...A_n) = P(A_1)+P(A_2)+P(A_3)+...+P(A_n)$$

When events are not mutually exclusive :

Theorem 2 : Let A and B are not two mutually exclusive events then probability of occurence of any one is -

$$P(A + B) = P(A) + P(B) - P(AB)$$

$$\Rightarrow \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof : Let the total number of cases be n and the favourable cases for event A and B be m_1 and m_2 respectively.

$$\therefore \quad P(A) = \frac{m_1}{n}, \quad P(B) = \frac{m_2}{n}$$

Since A and B are not mutually exclusive events therefore, let m_3 be a vourable event of A and B

$$P(AB) = \frac{m_3}{n}$$

favourable events of (A+B) are $m_1 + m_2 - m_3$

$$\therefore \quad P(A+B) = \frac{m_1 + m_2 - m_3}{n} = \frac{m_1}{n} + \frac{m_2}{n} - \frac{m_3}{n}$$

$$\Rightarrow \quad P(A+B) = P(A) + P(B) - P(AB)$$

$$\Rightarrow \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof using Set Theory:

Let denotes a sample space, set A represents the event A and set B represents the event B and these events are not mutually exclusive, thus the common events of A and B are shown as $A \cap B$

$(A \cup B) = (A - B) \cup (A \cap B) \cup (B - A)$

$A = (A - B) \cup (A \cap B)$

and $B = (B - A) \cup (A \cap B)$

$\therefore \quad P(A) = P(A - B) + P(A \cap B)$ (i)

$P(B) = P(B - A) + P(A \cap B)$ (ii)

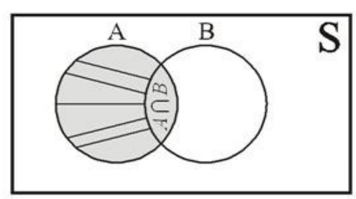

fig. 14.02

$P(A \cup B) = P[(A - B) \cup (A \cap B) \cup (B - A)]$

$= P(A - B) + P(A \cap B) + P(B - A)$

$= P(A) - P(A \cap B) + P(A \cap B) + P(B) - P(A \cap B)$ [by (i) & (ii)]

$= P(A) + P(B) - P(A \cap B)$

$\therefore \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow \quad P(A + B) = P(A) + P(B) - P(AB)$

Sub theorem : If the events are mutually exclusive then

$A \cap B = \varnothing$ and $P(A \cap B) = 0$

$\therefore \quad P(A \cup B) = P(A) + P(B)$

$\Rightarrow \quad \mathbf{P(A+B) = P(A) + P(B)}$

14.06 Multiplication theorem of probability or theorem of compound probability :

The probability of occurence of two events A and B is equal to the product of probability of event A and B conditional probability of event B (When A has alreacy occured) (or equal to the product of probability of B and conditional probability of (A) i.e.

and $P(AB) = P(A) \cdot P\left(\frac{B}{A}\right)$ or $P(A \cap B) = P(A) \cdot P\left(\frac{B}{A}\right)$

$P(AB) = P(B) \cdot P\left(\frac{A}{B}\right)$ orT $P(A \cap B) = P(B) \cdot P\left(\frac{A}{B}\right)$

Proof : Let the total number of mutually exclusive and exhaustive events are n out of which m events are favourable to event A and m_1 events are favourable to both the events A and B, then

$$P(AB) = \frac{m_1}{n} = \frac{m_1}{m} \times \frac{m}{n}$$

but $P(A) = \frac{m}{n}$

$$P\left(\frac{B}{A}\right) = \frac{\text{Events favourable to A and B}}{\text{Events favourable to A}} = \frac{m_1}{m}$$

$$P(AB) = P(A).\ P\left(\frac{B}{A}\right)$$

or $P(AB) = P\left(\frac{B}{A}\right).\ P(A)$

similarly we can prove that

$P(AB) = P(B).\ P\left(\frac{A}{B}\right)$ or $P(AB) = P\left(\frac{A}{B}\right).\ P(B)$

$\Rightarrow \quad P(AB) = P(B).\ P\left(\frac{A}{B}\right)$ or $P(AB) = P(A).\ P\left(\frac{B}{A}\right)$

Sub theorem : If the events A and B areindependent then

$$P\left(\frac{B}{A}\right) = P(B)$$

$$P(AB) = P(A).\ P(B)$$

Generalisation : If $A_1, A_2, \ldots, A_n$ are independent events

$P(A_1 A_2 A_3 \ldots . A_n) = P(A_1).\ P(A_2).\ P(A_3) \ \ldots P(A_n)$

14.07 Probability of Atleast one event :

If the probabilities of n independent events are $p_1, p_2, \ldots, p_n$ then we have to find the probability of atleast one of the event to occur.

Let $A_1, A_2, ..., A_n$ are independent events with probabilities $p_1, p_2, ..., p_n$ then $P(A_1) = p_1, P(A_2) = p_2, ... P(A_n) = p_n$ and

$$P(\bar{A}_1) = 1 - p_1, \quad P(\bar{A}_2) = 1 - p_2, \ ..., \ P(\bar{A}_n) = 1 - p_n$$

since $A_1, A_2, ..., A_n$ are independent events therefore $\bar{A}_1, \bar{A}_2, ..., \bar{A}_n$ are also independent

Therefore by multiplication theorem of probability of none of the events to occur.

$$= P(\bar{A}_1, \bar{A}_2, ...\bar{A}_n)$$

$$= P(\bar{A}_1) P(\bar{A}_2) ..... P(\bar{A}_n)$$

$$= (1-p_1)(1-p_2) ...... (1-p_n)$$

Thus probability of occurence of atleast one event

= 1− (Probability of none of the events to occur)

$$= 1-P(\bar{A}_1) P(\bar{A}_2) ..... P(\bar{A}_n)$$

$$= 1-\{(1-p_1)(1-p_2) ...... (1-p_n)\}$$

Illustrative Examples

Example 10 : If the die is tossed twice then find the probability of getting a sum of 7 or 11.

Solution : Total outcomes by throwing two dice $= 6 \times 6 = 36$

Favourable outcomes to get a sum of 7 are (6,1), (5,2), (4,3), (3,4), (2,5), (1,6) = 6

$$\therefore \quad P(7) = \frac{6}{36}$$

Favourable outcomes to get a sum of 11 are (6,5), (5,6) = 2

$$\therefore \quad P(11) = \frac{2}{36}$$

Since the events are mutually exclusive hence the probability is

$$P(7+11) = P(7) + P(11) = \frac{6}{36} + \frac{2}{36} = \frac{8}{36} = \frac{2}{9}$$

Example 10 : A bag contains 2 white, 4 black and 5 red balls. Three balls are drawn at random. Find the probability that the balls drawn are of same colour.

Solution : Total no, of balls in a bag 2+4+ 5 =11 , no. of ways of drawing 3 balls are $= {}^{11}C_3$

All the three balls can be red or black

Probability of the three balls to be red $= \frac{{}^5C_3}{{}^{11}C_3} = \frac{10}{165}$

Probability of the three balls to be black $= \frac{{}^4C_3}{{}^{11}C_3} = \frac{4}{165}$

$\because$ both the events are mutually exclusive hence the requiredprobability $= \frac{10}{165} + \frac{4}{165} = \frac{14}{165}$

Example 11 : One card is drawn from the well shuffled deck of 52 cards, find the probability that it is an ace or a card of heart.

Solution : Let event A denotes 'a card is an ace and event B denotes' a card is an heart. Here A and B are not mutually exclusive events, as the card drawn may be an ace of heart thus by addition theorem of probability.

$$P(A+B) = P(A) + P(B) - P(AB)$$

Total outcomes for event $A = {}^{52}C_1 = 52$
Drawing one ace out of 4 whose favourable condition= ${}^{4}C_1 = 4$

$$\therefore \; P(A) = \frac{4}{52} = \frac{1}{13}$$

total coutcomes for event= ${}^{52}C_1 = 52$ favourable outcomes for event= ${}^{13}C_1 = 13$ (13 cards of each suit)

$$\therefore \; P(B) = \frac{13}{52}$$

Favourable condition of event A and B to occur $= 1$

$$\therefore \; P(AB) = \frac{1}{52} \text{ (when it is an ace of heart)}$$

$$\Rightarrow \quad P(A+B) = P(A) + P(B) - P(AB)$$

$$P(A+B) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

Example 12 : A, B, C participate in different competitions. Probability of A's success is 2/5 , B's success is 1/8 and C is 5/8.

Find the Probability that

(i) All the three succeded **(ii)** Atleast one of them succeded

Solution : here $P(A) = \frac{2}{5} \quad \therefore \quad P(\bar{A}) = 1 - \frac{2}{5} = \frac{3}{5}$

$$P(B) = \frac{1}{8} \quad \therefore \quad P(\bar{B}) = 1 - \frac{1}{8} = \frac{7}{8}$$

$$P(C) = \frac{5}{8} \quad \therefore \quad P(\bar{C}) = 1 - \frac{5}{8} = \frac{3}{8}$$

(i) All the events independent event hence their probability = $P(ABC) = P(A).P(B).P(C)$

$$= \frac{2}{5} \cdot \frac{1}{8} \cdot \frac{5}{8} = \frac{1}{32}$$

(ii) Probability of atleast one of them to succeed

$$= 1 - P(\bar{A}\,\bar{B}\,\bar{C})$$

$$= 1 - P(\bar{A}) \cdot P(\bar{B}) \cdot P(\bar{C})$$

$$= 1 - \frac{3}{5} \cdot \frac{7}{8} \cdot \frac{3}{8} = 1 - \frac{63}{320} = \frac{257}{320}$$

Example 13 : Mohan speaks truth in 60% of the cases whereas Sohan speak truth 80% of the cases. Find the probability of both of them to be contradictory ot each other for some statement.

Solution : Let events A and B denotes Mohan and Sohan's speaking truth

$$P(A) = \frac{60}{100} = \frac{3}{5} \Rightarrow P(\bar{A}) = 1 - \frac{3}{5} = \frac{2}{5}$$

$$P(B) = \frac{80}{100} = \frac{4}{5} \Rightarrow P(\bar{B}) = 1 - \frac{4}{5} = \frac{1}{5}$$

(i) Mohan is speaking truth and Sohan is not speaking the truth = $A\bar{B}$

(ii) Sohan is speaking truth and Mohan is not speaking the truth = $\bar{A}B$

Since $A, \bar{B}$ and $\bar{B}, A$ are independent events

$\therefore$ $$P(\bar{A}B) = P(\bar{A}).\ P(B) = \frac{2}{5} \times \frac{4}{5} = \frac{8}{25}$$

$$P(A\bar{B}) = P(A).\ P(\bar{B}) = \frac{3}{5} \times \frac{1}{5} = \frac{3}{25}$$

Also $A\bar{B}$ and $\bar{A}B$ are mutually exclusive events

$\therefore$ Required Probability $P(\bar{A}B + A\bar{B}) = P(\bar{A}).\ P(B) + P(A).\ P(\bar{B}) = \frac{8}{25} + \frac{3}{25} = \frac{11}{25}$

Example 14 : Analysis the result of a class it is observed that 40 % students pass in maths, 25 % students pass in physics and 15 % students pass in maths and physics. A student is selected at randim then if he passes in maths then, find the probability that he passes in physics also.

Solution : Let A be the event 'a student passes in maths' and B denotes 'a student passes in Physics' then

$$P(A) = \frac{40}{100} = \frac{2}{5} \text{ and } P(B) = \frac{25}{100} = \frac{1}{4} \text{ and } P(AB) = \frac{15}{100} = \frac{3}{20}$$

Now we have to find $P\left(\frac{B}{A}\right)$ thus

$$P(AB) = P(A) \cdot P\left(\frac{B}{A}\right)$$

or $$\frac{3}{20} = \frac{2}{5} \cdot P\left(\frac{B}{A}\right)$$

$\therefore$ Required probability $$P\left(\frac{B}{A}\right) = \frac{3}{20} \times \frac{5}{2} = \frac{3}{8}$$

Example 15 : Three critics review a book odds in favour of the book are 5 : 2, 4 : 3 and 3 : 4 respectively for three critics. Find the probability that the majority are in favour of the book.

Solution : Let E_1, E_2 and E_3 denoted the events that the book will be reviewed favoured by the first, second and third critic respectively.

$$P(E_1) = \frac{5}{7}, \qquad P(E_2) = \frac{4}{7}, \qquad P(E_3) = \frac{3}{7}$$

$\therefore$ $$P(\bar{E}_1) = \frac{2}{7}, \qquad P(\bar{E}_2) = \frac{3}{7}, \qquad P(\bar{E}_3) = \frac{4}{7}$$

Different cases in favour of book by the critics are

1. $E_1E_2E_3$ 2. $\bar{E}_1E_2E_3$ 3. $E_1\bar{E}_2E_3$ 4. $E_1E_2\bar{E}_3$

Probability are

$$P(E_1E_2E_3) = P(E_1)\ P(E_2)\ P(E_3) = \frac{5}{7} \times \frac{4}{7} \times \frac{3}{7} = \frac{60}{343}$$

$$P(\bar{E}_1E_2E_3) = P(\bar{E}_1)\ P(E_2)\ P(E_3) = \frac{2}{7} \times \frac{4}{7} \times \frac{3}{7} = \frac{24}{343}$$

$$P(E_1\bar{E}_2E_3) = P(E_1)\ P(\bar{E}_2)\ P(E_3) = \frac{5}{7} \times \frac{3}{7} \times \frac{3}{7} = \frac{45}{343}$$

$$P(E_1E_2\bar{E}_3) = P(E_1)\ P(E_2)\ P(\bar{E}_3) = \frac{5}{7} \times \frac{4}{7} \times \frac{4}{7} = \frac{80}{343}$$

Above cases are mutually exclusive, hence the probability are

$$P(E_1E_2E_3) + P(\bar{E}_1E_2E_3) + P(E_1\bar{E}_2E_3) + P(E_1E_2\bar{E}_3)$$

$$= \frac{60}{343} + \frac{24}{343} + \frac{45}{343} + \frac{80}{343} = \frac{209}{343}$$

Exercise 14.3

1. If the probability of event A is 2/11 then find the probability of ' Not A'.
2. In a gram panchayat there are 4 men and 6 women. If a member is selected for a commitee randomly, find the probability that it is a women.
3. A die is rolled. Find the probability of -
   (i) Getting a prime number, (ii) Getting a number less than or equal to 1 (iii) Getting a number less than 6
4. A coin is tossed four times. Find the probability of getting head atleast three times.
5. If a coin and a die is tossed together then find the probability of getting Head on the coin and even number on the die.
6. Out of 20 people 5 are graduates. If 3 people are selected at random. What is the probability that he is a graduate ?
7. To solve a problem, the odds unfavourable to event A is 4 : 3 and odds in favour of event B is 7 : 5. What is the probability that
   (i) The problem is solved (ii) The problem is not solved (iii) It is sloved by only one.
8. An instrument will work only if its three components A, B and C work properly. Within a year the probability of a getting faulty is 0.15, of B it is 0.05 and C it is 0.10. What is the probability that the instrument gets faulty at the end of the year ?
9. Two cards are drawn in two turns at random from a pack of 52 cards. If in first turn the drawn card is not replaced then find the probability of getting two aces in first turn and two kings in second turn.
10. A and B are two events such that $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{4}$ and $P(AB) = \frac{1}{12}$ then find $P\left(\frac{B}{A}\right)$,
11. Imagine that the ratio of men and children is 1 : 2 . In a family of 5 children find the probability that (i) all are boys (ii) 3 are boys and 2 are girls.
12. A hits a target correctly 3 out of 6 and B hits 2 out of 4 correctly and C hits 1 out of 4 correctly. What is the probability that atleast two people hit target correctly ?

Miscellaneous Examples

Example 16 : Two dice A and B are rolled simultaneously. A wins if he first gets 6 before B gets 7 and B wins if he rolls and gets 7 before A gets 6. If a starts rolling then prove that the probability of A's winning is 30 / 61.

Solution : Let E_1 : getting a sum 6 on the two dice

Total outcomes for event $E_1 = 6^2 = 36$

and favourable cases = (1,5), (2,4), (3,3), (4,2) and (5,1)

Hence total favourable cases = 5

Required probability

$$\therefore\ P(E_1) = \frac{5}{36} \quad \text{and} \quad P(\bar{E}_1) = 1 - \frac{5}{36} = \frac{31}{36}$$

Again let E_2 : geting a sum 7 on the two dice

Total coutcomes for event $E_2 = 36$

and favourable cases $= (6,1), (5,2), (4,3), (3,4), (2,5)$ and $(1,6)$

Required probability

$\therefore\ P(E_2) = \frac{6}{36} = \frac{1}{6}$ and $P(\bar{E}_2) = 1 - \frac{1}{6} = \frac{5}{6}$

Probability of A's winning if he rolles first $P(E_1) = \frac{5}{36}$

(ii) Let the events $\bar{E}_1, \bar{E}_2, E_1$ represents A not getting 6 in first throw, B not geting 7 in the first throw and A getting 6 in the second throw, with probability $P(\bar{E}_1\bar{E}_2E_1) = P(\bar{E}_1).\ P(\bar{E}_2).\ P(E_1) = \frac{31}{36} \times \frac{5}{6} \times \frac{5}{36}$

(iii) Similarly probability of A's winning in the third throw

$$P(\bar{E}_1\bar{E}_2\bar{E}_1\bar{E}_2E_1) = P(\bar{E}_1).\ P(\bar{E}_2).\ P(\bar{E}_1).\ P(\bar{E}_2).\ P(E_1) = \frac{31}{36} \times \frac{5}{6} \times \frac{31}{36} \times \frac{5}{6} \times \frac{5}{36}$$

Similarly probability can be found for n number fo throws. Now probability of A's winning

$$= \frac{5}{36} + \frac{31}{36} \times \frac{5}{6} \times \frac{5}{36} + \frac{31}{36} \times \frac{5}{6} \times \frac{31}{36} \times \frac{5}{6} \times \frac{5}{36} + \ldots = \frac{5/36}{1 - \frac{31}{36} \times \frac{5}{6}} = \frac{30}{61}$$

(Sum of infinite G.P.)

Example 17 : There are 6 red and 4 white balls in a bag. Two balls are chosen two times from the bag, find the probability of getting 2 red ball first time and 2 white balls second time but taking balls first time and again.

(i) Put into the bag (ii) Don't put into the bag

Solution : **(i)** When balls are placed into the bag :

Total balls in bag $= 6 + 4 = 10$

$\therefore$ Taking two balls from by $= {}^{10}C_2$

Total way to take 2 balls from 6 red balls $= {}^{6}C_2$

$\therefore$ First time probability of getting two red balls $= \frac{{}^{6}C_2}{{}^{10}C_2}$

Ways to take out 2 balls from 4 white balls $= {}^{4}C_2$

$\therefore$ Second time probability of getting two white balls $= \frac{{}^{4}C_2}{{}^{10}C_2}$

The above events are independent, so required probability $= \frac{{}^{6}C_2}{{}^{10}C_2} \times \frac{{}^{4}C_2}{{}^{10}C_2} = \frac{1}{3} \times \frac{2}{15} = \frac{2}{45}$

(ii) When balls are not put into the bag **:**

Second time $10 - 2 = 8$ balls left in bag.

Second time, probability of getting two white balls $= \frac{{}^{4}C_2}{{}^{8}C_2}$

Henxe, required probability $= \frac{{}^{6}C_2}{{}^{10}C_2} \times \frac{{}^{4}C_2}{{}^{8}C_2} = \frac{1}{3} \times \frac{3}{14} = \frac{1}{14}$

Example 18 : If A, B C are three events associated with a random experiment, prove that

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) + P(A \cap B \cap C)$$

Solution : Consider $E = B \cup C$ so that

$$P(A\cup B\cup C)=P(A\cup E)$$
$$=P(A)+P(E)-P(A\cap E) \quad (1)$$
$$\Rightarrow \quad P(E)=P(B\cup C)=P(B)+P(C)-P(B\cap C) \quad (2)$$

(Using distribution property of intersection of sets over the union) Thus

$$A\cap E=A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$$
$$\Rightarrow \quad P(A\cap E)=P(A\cap B)+(A\cap C)-P\{(A\cap B)\cup(A\cap C)\}$$
$$=P(A\cap B)+P(A\cap C)-P(A\cap B\cap C) \quad (3)$$

Using (2) and (3) in (1), we get

$$P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(A\cap C)-P(A\cap B)-P(A\cap C)+P(A\cap B\cap C)$$

Example 19 : Find the probability that when 5 cards is drawn from a well shuffled deck of 52 cards, it contains (i) All kings (ii) Atleast 3 kings

Solution : Total number of possible events $= {}^{52}C_5$

(i) Number of events with 4 kings $= {}^{4}C_4 \times {}^{48}C_1$

$$\therefore \quad P\text{ (all kings)} = \frac{{}^{4}C_4 \times {}^{48}C_1}{{}^{52}C_5} = \frac{1}{54145}$$

(ii) P (atleast 3 kings) $=$ P (3 kings) $+$ P (4 kings)

$$= \frac{{}^{4}C_3 \times {}^{48}C_2}{{}^{52}C_5} + \frac{{}^{4}C_4 \times {}^{48}C_1}{{}^{52}C_5} = \frac{94}{54145} + \frac{1}{54145} = \frac{19}{10829}$$

Miscellaneous Exercise 14

1. A coin is tossed n times then $n(S)$ is :
   (A) $2n$ (B) 2^n (C) n^2 (D) $n/2$
2. In tossing a die twice the sample space of getting a sum 3 is :
   (A) $(1, 2)$ (B) $\{(2, 1)\}$ (C) $\{(3, 3)\}$ (D) $\{(1, 2), (2, 1)\}$
3. The number of elements in sample space if a coin is tossed and a die is rolled is :
   (A) 12 (B) 6 (C) 64 (D) 36
4. Every outcome of an experiment is :
   (A) Sample space (B)Random experiment (C) Sample point (D) Ordered pair
5. A coin is tossed three times, if the total outcomes are denoted by E then n(S) will be
   (A) 6 (B) 3 (C) 4 (D) 8
6. If $E_1 \cap E_2 = \phi$ then E_1 and E_2 events.
   (A) Exclusive (B) Independent (C) Dependent (D) Complementary
7. Favourable outcomes for getting 53 Mondays in a leap year :
   (A) 7 (B) 2 (C) 1 (D) 14
8. Aurn contains 4 white, 3 black and 2 red balls. Favourable outcomes for all the balls to be of difference colours is :
   (A) 9 (B) 24 (C) 12 (D) 7
9. If two events are mutually exclusive then $P(A\cup B)$

(A) $P(A)+P(B)$ (B) $P(A)+P(B)-P(A\cap B)$

(C) $P(A)\cdot P(B)$ (D) $P(A)\cdot P(B/A)$

10. The probability of solving a question by three students. A, B and C are 1/2, 1/3 and 11/4 then the probability that atleast one will solve the question is :

(A) 1 / 24 (B) 1 / 4 (C) 3 / 4 (D) 1 / 9

11. Two dice are rolled, the probability of getting a difference of 1 on the dice is :

(A) 5 / 18 (B) 1 / 4 (C) 2 / 9 (D) 7 / 36

12. A card is drawn from a deck of cards, the probability of it being a red or black card is :

(A) 1 / 4 (B) 1 / 2 (C) 3 / 4 (D) 26 / 51

13. Two dice are rolled, the probability of getting a sum of digit is multiple 4 is :

(A) 1 / 4 (B) 1 / 3 (C) 1 / 9 (D) 5 / 9

14. The probability of getting even numbers on both the ends of a five digit number which is formed by using the digits 1, 2, 3, 4, 5, 6 and 8 :

(A) 5 / 7 (B) 4 / 7 (C) 3 / 7 (D) 2 / 7

15. Three dice are rolled, the probability of getting same number on all the three dice is :

(A) 1 / 36 (B) 3 / 22 (C) 1 / 6 (D) 1 / 18

16. In a swimming competition the ratio in favour of A is 2 : 3 and the ratio unfavourable to B is 4 : 1 then the probability of winning the race by either of A or B is :

(A) 1 / 5 (B) 2 / 5 (C) 3 / 5 (D) 4 / 5

17. 10 students are sitting in a row. The probability of no two special students sitting together is :

(A) 1 / 5 (B) 2 / 5 (C) 3 / 5 (D) 4 / 5

18. There are 12 bulbs in a box of which 4 are defective. 3 bulbs are drawn at random without replacement, the probability of them being non-defective is :

(A) 3 / 55 (B) 13 / 55 (C) 14 / 55 (D) 17 /55

19. The probability of a sure event is :

(A) 0 (B) 1 / 2 (C) 1 (D) 2

20. In a family of three children of atleast one is a boy then find the probability of having 2 boys and 1 girl in the family :

(A) 1 / 2 (B) 1 / 3 (C) 1 / 4 (D) 3 / 4

21. The probability of taking exam in a class by a teacher is $\frac{1}{5}$ if a student remains absent 2 times, then the probability of not giving atleast one exam is :

(A) 9 / 25 (B) 11 / 25 (C) 13 / 25 (D) 23 / 25

22. Find the probability of getting 53 Sundays in a non-leap year.

23. If A and B are two mutually exclusive events and $P(A)=0.3, P(B)=K$ and $P(A\cup B)=0.5$ then find the value of K.

24. Find the probability of both 'E' occuring together in the word 'PEACE',

25. A bag contains 6 red and 8 black balls. Four balls are drawn two times. The balls drawn in the first attempt are placed again in the bag. What is the probability that the balls drawn in the first attempt are red and in the second attempt are black ?

26. A man speaks truth 3 out of 5 times. He states that in tossing 6 coins, two times head appear. What is the probability that he speaks the truth.

27. Two dice are rolled. What is the probability that neither the same digit appear nor the sum of the digits is 9 on them ?

28. Three coins are tossed, find the probability of getting.

(1) Exactly two heads (2) Atleast two heads

(3) Almost two heads (4) All the three are heads

29. In jockey race four horses A, B, C, D run. The ratio in favour of A, B, C and D is 1 : 3, 1 : 4, 1 : 5 and 1 : 6. Find the probability of winning by any one of them.

30. In the next 25 years the probability of a person to remain alive is 3 / 5 while of his wife is 2 / 3 then find the following probability :

(1) both of them alive (2) both of them are not alive

(3) atleast one of them is alive (4) only the wife remains alive

31. A and B are two independent speakers. The probability of A speaking the truth is x of B speaking the truth is y. If both A and B agrees on a statement then prove that the probability of the statement to be true is $=\dfrac{xy}{1-x-y+2xy}$

32. A,B,C tosses a coin one by one. The person wins if head appear at first. If A's turn is first, what is the probability that he wins ?

33. Sulakshna and Sunayna toss a coin simulteneously. The person wins if head appears at first. If Sulakshna's turn comes first then what is the probability of their winning.

34. One one number is selected from a group of numbers :

(1, 2, 3, 4, 5, 6, 7, 8, 9), (1, 2, 3, 4, 5, 6, 7, 8, 9)

If p_1, denotes the probability of sum of both the digits is 10 and p_2 denotes the probability of sum of both the digits is 8 then find p_1+p_2 ,

35. If $P(A)=0.4,\ P(B)=0.8,\ P(B/A)=0.6$ then find $P(A/B)$ and $P(A\cup B)$,

36. If $P(E)=0.35,\ P(F)=0.45,\ P(E\cup F)=0.65$ then find $P(F/E)$,

37. A die is thrown 5 times, find the probability of getting a number 1.

Important Points

1. An experiment in any contect may result in any one of the sevens possible outcomes. Performing an experiment is known as a trial and outcomes of the experiment are known as events.
2. Exhaustive events or total number of cases : In any experiment, the total possible outcomes is called exhaustive events or total number of cases.
3. Favourable events or cases : In any experiment, favourable events or cases are the number in which that specific events occurs.
4. Mutually exclusive or disjoint events : Two or more than two events are scid to be mutually exclusive events, if the occurrence of any one of them excludes the occurrence of the other event.
5. (i) Independent events : Two or more than two events are said to be independent events if the occurrence of one event does not affects the occurrence of another event.

   (ii) Dependent events : Two or more than two events are said to be dependent events if the occurrence of one event affects the occurrence of another event.
6. Sample point and sample space : The set of all possible outcomes of a random experiment is called

sample space associated with the experiment; sample space is denoted by the symbol 'S'. Each element of the sample space is called a sample point.

7. Elementry event : A subset only one element of sample space related to random experiment is called as elementry events.

8. Compound event : The subsets of sample space S of an experiment which are made of combination of subsets of elementary of sample space S.

9. Impossible and certain events : The empty set ϕ and the sample space S describe events being subsets of S. In fact ϕ is called impossible event and S, i.e. the whole sample space is called sure event.

10. Probability : Probability of favourable event A is

$$P(A) = \frac{\text{Favourable cases of A}}{\text{Total cases of A}} = \frac{m}{n} \quad \text{(numerical value)}$$

or $$P(A) = \frac{\text{No. of elements in A}}{\text{No. of elements in S}} = \frac{n(A)}{n(S)}$$

Probability of not A $$P(\bar{A}) = \frac{\text{Unfavourable cases of A}}{\text{Total cases of A}} = \frac{n-m}{n}$$

11. $$P(\bar{A}) = 1 - P(A)$$

or $$P(A) + P(\bar{A}) = 1$$

12. The range of probability A, $0 \leq P(A) \leq 1$

13. Odds in favour of event A $= \frac{m}{n-m} = \frac{P(A)}{P(\bar{A})}$

14. Odds against event A $= \frac{n-m}{m} = \frac{P(\bar{A})}{P(A)}$

15. **Addition theorem of Probability :**

(1) If the events are not mutually exclusive

$$P(A+B) = P(A) + P(B) \quad \text{or} \quad P(A \cup B) = P(A) + P(B)$$

$$P(A+B+C+....) = P(A) + P(B) + P(C) + ... = P(A \cup B \cup C...)$$

(2) If the events are not mutually exclusive

$$P(A+B) = P(A) + P(B) - P(AB)$$

or $$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

16. **Multiplication theorem of Probability :** Probability of happening of any two events A and B together

$$P(AB) = P(A).\ P\left(\frac{B}{A}\right) \quad \text{or} \quad P(A \cap B) = P(A).\ P\left(\frac{B}{A}\right)$$

or $$P(AB) = P(B).\ P\left(\frac{A}{B}\right) \quad \text{or} \quad P(A \cap B) = P(B).\ P\left(\frac{A}{B}\right)$$

if A , B are independent events

$$P(AB) = P(A).\ P(B) \quad \text{or} \quad P(A \cap B) = P(A).\ P(B)$$

If $A_1, A_2, \ldots, A_n$ are independent events, then

$$P(A_1 A_2 A_3, \ldots, A_n) = P(A_1).\ P(A_2).\ P(A_3), \ldots, P(A_n)$$

17. If $A_1, A_2, \ldots, A_n$ are independent events whose probabilities are respectively $p_1, p_2, \ldots, p_n$. Probability of occurence of at least one is

$$= 1 - \text{None of the events occur}$$

$$= 1 - P(\bar{A}_1)\, P(\bar{A}_2) \ldots P(\bar{A}_n)$$

$$= 1 - \{(1-p_1)\,(1-p_2)\,\ldots\,(1-p_n)\}$$

Answers

Exercise 14.1

1. {DDD, DDN, DND, NDD, DNN, NDN, NND, NNN} **2.** 256 **3.** Not

4. (i) {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)} (ii) {(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)}

Exercise 14.2

1. 1/3 **2.** 1 / 4 **3.** 7 / 17 **4.** 1 / 4 **5.** 5 / 18 **6.** 6 / 7

7. 1 : 12 **8.** 5 : 7 **9.** $\frac{n-3}{2}$ **10.** 5 / 6 **11.** 1 / 4 **12.** 5 / 108

13. 2 / 5 **14.** 3 / 13 **15.** 5 / 204

Exercise 14.3

1. 9 / 11 **2.** 3 / 5 **3.** (i) 1 / 2; (ii) 1 / 6; (iii) 5 / 6 **4.** 5 / 16 **5.** 1 / 4 **6.** 137 / 228

7. (i) 16 / 21; (ii) 5 / 21; (iii) 43 / 84 **8.** 0.27325 **9.** 6 / 270725 **10.** 1 / 4

11. (i) 1 / 32; (ii) 5 / 16 **12.** 3 / 8

Miscellaneous Exercise 14

1. (B) **2.** (D) **3.** (A) **4.** (C) **5.** (C) **6.** (D) **7.** (B)

8. (B) **9.** (A) **10.** (C) **11.** (A) **12.** (B) **13.** (A) **14.** (D)

15. (A) **16.** (C) **17.** (D) **18.** (C) **19.** (C) **20.** (B) **21.** (A)

22. $\frac{1}{7}$ **23.** 0.2 **24.** 2 / 5 **25.** (i) $\frac{150}{143143}$ **26.** $\frac{45}{143}$ **27.** $\frac{13}{18}$

28. $\frac{3}{8}, \frac{1}{2}, \frac{7}{8}, \frac{1}{8}$ **29.** $\frac{319}{420}$ **30.** $\frac{2}{5}, \frac{2}{15}, \frac{13}{15}, \frac{4}{15}$ **32.** $\frac{4}{7}, \frac{2}{7}, \frac{1}{7}$ **33.** $\frac{2}{3}, \frac{1}{3}$

34. $\frac{16}{81}$ **35.** 0.3, 0.96 **36.** $\frac{3}{7}$ **37.** $5\left(\frac{1}{6}\right)^5$

Appendix A

Logarithms

	0	1	2	3	4	5	6	7	8	9	Mean Differences 1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7

Appendix A

Logarithms

	0	1	2	3	4	5	6	7	8	9	Mean Differences								
											1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

Appendix B

Antilogarithms

	0	1	2	3	4	5	6	7	8	9	Mean Differences 1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

Appendix B

Antilogarithms

Degrees	0	1	2	3	4	5	6	7	8	9	Mean Differences 1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20