07

Factorisation

Polynomial

$f(x) = a_0x^n + a_1x^{n-1} + ... + a_n$ $(a_0 \neq 0)$ is called a polynomial in variable x, where $a_0, a_1, ..., a_n$ are real numbers and n is a non-negative integer, is called degree of polynomial. e.g. $(x - a)$ is a degree of 1, $x^2 - 7x + 12$ is a degree of 2.

Factor

A polynomial $g(x)$ is called a **factor** of polynomial $p(x)$, if $g(x)$ divides $p(x)$ exactly.

Factorisation

To express polynomial as the product of polynomials of degree less than that of the given polynomial is called as factorisation.

Methods of Factorisation

Some methods of factorisation are as follows

(i) Factorisation By Common Factors

A factor (s), which occurs in each term, is called common factor. In which we have to find the common factor between terms.

Example 1 Factorise $6ab + 12bc$

Sol. We have, terms $6ab = 2 \times 3 \times a \times b$ and

$$12bc = 2 \times 2 \times 3 \times b \times c$$

Here, $6ab + 12bc = 2 \times 3 \times a \times b + 2 \times 2 \times 3 \times b \times c$

$$= 2 \times 3 \times b(a + 2c) = 6a(a + 2c)$$

Thus, by common factors 6a, $a + 2c$ are factors.

(ii) Factorisation by Splitting Middle Term

Let factors of the quadratic polynomial $ax^2 + bx + c$ be $(px + q)$ and $(rx + s)$. Then,

$$ax^2 + bx + c = (px + q)(rx + s)$$
$$= prx^2 + (ps + qr)x + qs$$

On comparing the coefficients of x^2, x and constant terms from both sides, we get

$$a = pr, \; b = ps + qr \text{ and } c = qs$$

Here, b is the sum of two numbers ps and qr, whose product is $(ps)(qr) = (pr)(qs) = ac$.

Thus, to factorise $ax^2 + bx + c$, write b as the sum of two numbers, whose product is ac.

- To factorise $ax^2 + bx - c$ and $ax^2 - bx - c$, write b as the difference of two numbers whose product is $(-ac)$.

Example 2 Factors of $x^2 - 6x + 8$ are

(a) $(x-4)(x-2)$ (b) $(x+4)(x-2)$
(c) $(x-4)(x+2)$ (d) $(x+4)(x+2)$

Sol. (a) We have, $x^2 - 6x + 8$

On comparing with $ax^2 + bx + c$, we get

$$a = 1, b = -6, c = 8$$

Now, $ac = 8$

So, all possible pairs of factors of 8 are 2, 2, 2 and 4, 2.

Clearly, $4 + 2 = 6 = b$

$\therefore x^2 - 6x + 8 = x^2 - (4+2)x + 8$

$$= x^2 - 4x - 2x + 8$$
$$= x(x-4) - 2(x-4) = (x-2)(x-4)$$

Factoristion by Algebraic Indentities

To solve these types of question we have to use some algebraic identities.

Now, we consider the following identities.

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

$$(a+b)(a-b) = a^2 - b^2$$

Factorise $x^2 - (2y)^2$.

$$x^2 - (2y)^2 = (x+2y)(x-2y)$$

Here, we use $a^2 - b^2 = (a+b)(a-b)$ identity.

Example 3 Factors of $x^2 + 12x + 36$ are

(a) $(x+3)(x-3)$
(b) $(x+6)(x+6)$
(c) $(x-6)(x-3)$
(d) $(x-6)(x+6)$

Sol. (*b*) We have, $x^2 + 12x + 36$

$= x^2 + 2 \cdot 6 \cdot x + 6^2$

$= (x+6)^2$

$= (x+6)(x+6)$ $[\because a^2 + 2ab + b^2 = (a+b)^2]$

Example 4 Factors of $8a^3 - 2a$ are

(a) $2a(2a+1)(2a-1)$ (b) $2a(2a+1)(2a-3)$
(c) $2a(2a-1)(a-2)$ (d) $2a(1-2a)(1+2a)$

Sol. (*a*) We have, $8a^3 - 2a$

$= 2a(4a^2 - 1)$

$= 2a[(2a)^2 - 1]$

$= 2a(2a+1)(2a-1)$

$[\because a^2 - b^2 = (a+b)(a-b)]$

Important Theorems

Two main theorems are discussed below

(i) Remainder Theorem

Let p(x) be a polynomial in x of degree not less than one and α be a real number.

If p(x) is divided by $(x - \alpha)$, then remainder is $f(\alpha)$.

Remainder can be evaluated by substituting $x = \alpha$ in $p(x)$.

Example 5 If $p(x) = x^3 - 5x^2 + x - 5$, then find the remainder, when $p(x)$ is divided by $(x+1)$.

(a) 2 (b) 9 (c) 8 (d) – 12

Sol. (*a*) Given that,

$p(x) = x^3 - 5x^2 + x - 5$

$\therefore \quad p(-1) = (-1)^3 - 5(-1)^2 + (-1) - 5$

$= -1 - 5 - 1 - 5 = -12$

(ii) Factor Theorem

Let p(x) be a polynomial in x of degree not less than one and α be a real number.

If $p(\alpha) = 0$, then $(x - \alpha)$ is a factor of p(x).

and If $(x - \alpha)$ is a factor of p(x), then $p(\alpha) = 0$.

Example 6 For what value of k, $(x-3)$ is the factor of the polynomial $x^3 - 3x^2 + kx - 6$?

(a) 2 (b) 9 (c) 8 (d) 12

Sol. (*a*) Let $f(x) = x^3 - 3x^2 + kx - 6$

Since, $f(x)$ is divisible by $x - 3$.

$\therefore \quad f(3) = 0$

$\Rightarrow \quad (3)^3 - 3(3)^2 + k(3) - 6 = 0$

$\Rightarrow \quad 27 - 27 + 3k - 6 = 0$

$\Rightarrow \quad k = 2$

Practice Exercise

1. The factorised form of $3x-24$ is

(a) $3x \times 24$ (b) $3(x-8)$

(c) $24(x-3)$ (d) $3(x-12)$

2. Common factors of $11pq^2, 121p^2q^3, 1331p^2q$ is

(a) $121pq^2$ (b) $11pq^2$

(c) $11pq$ (d) $121p^2q^2$

3. Common factor of $17abc, 34ab^2, 51a^2b$ is

(a) $17abc$ (b) $17ab$ (c) $17ac$ (d) $17a^2b^2c$

4. The factor form of $5x^2-20xy$ is

(a) $5x(x-4y)$ (b) $10x(x-2y)$

(c) $5(x^2-2y)$ (d) None of these

5. Some of the factors of $\frac{n^2}{2}+\frac{n}{2}$ are $\frac{1}{2}n$ and $(n+1)$.

(a) True (b) False

(c) Only $\frac{n}{2}$ (d) Only $n+1$

6. Factors of $-3a^2+3ab+3ac$ are $3a$ and $(-a-b-c)$.

(a) True (b) False

(c) $3a$ only (d) $(-a-b-c)$ only

7. The factors of x^2-4 are

(a) $(x-2), (x-2)$ (b) $(x+2), (x-2)$

(c) $(x+2), (x+2)$ (d) $(x-4), (x-4)$

8. Factorisation of $xy+yz+xa+za$

(a) $(x+z)(y+a)$ (b) $(x+y)(z+a)$

(c) $(x+a)(y+z)$ (d) $(x+z)(z-y)$

9. The factor form of $5x(y+z)-7y(y+z)$ is

(a) $(5x-7y)(y-z)$ (b) $(5x-7y)(y+z)$

(c) $(5x+7y)(y+z)$ (d) $(5x+7y)(y-z)$

10. Factorise $p^2x^2+c^2x^2-ac^2-ap^2$

(a) $(p^2+c^2)(x^2-a)$ (b) $(p^2-c^2)(x^2+a)$

(c) $(p^2+c^2)(x^2+a)$ (d) $(p^2-c^2)(x^2-a)$

11. The factor form of $8-4x-2x^3+x^4$ is

(a) $(2-x)(4-x^3)$ (b) $(2+x)(4-x^3)$

(c) $(2+x)(4+x^3)$ (d) $(2-x)(4+x^3)$

12. Factors of $2ax^2+4axy+3bx^2+2ay^2+6bxy+3by^2$

(a) $(2a-3b)(x-y)^2$ (b) $(3a-2b)(x+y)^2$

(c) $(2a+3b)(x+y)^2$ (d) $(3a-2b)(x-y)^2$

13. Factorised form of $q^2-10q+21$ is

(a) $(q+7)(q-3)$ (b) $(q-7)(q+3)$

(c) $(q-7)(q-3)$ (d) $(q+7)(q+3)$

14. Factorised form of x^2+2x+1 is

(a) $(x+1)^2$ (b) $(x-1)^2$

(c) $(x+1)(x-1)$ (d) None of these

15. The factors of $(6\sqrt{3}x^2-47x+5\sqrt{3})$ are

(a) $(2x-5\sqrt{3})(3\sqrt{3}x-1)$

(b) $(5x-4\sqrt{3})(3\sqrt{4}x-1)$

(c) $(3x-5\sqrt{3})(3\sqrt{3}x+1)$

(d) $(5x-3\sqrt{3})(4\sqrt{3}x+1)$

16. The factor form of $x^2-2\sqrt{3}x+3$ is

(a) $(x+\sqrt{3})^2$ (b) $(x-\sqrt{3})^2$

(c) $(x+\sqrt{3})(x-\sqrt{3})$ (d) $(x+2)(x+\sqrt{3})$

17. Factorised form $x^2+9x+14$ is

(a) $(x+2)(x-7)$ (b) $(x-2)(x-7)$

(c) $(x+2)(x+7)$ (d) $(x-2)(x+7)$

18. Factorised form is $x^2-7x+12$

(a) $(x-3)(x-4)$ (b) $(x+3)(x-4)$

(c) $(x+3)(x+4)$ (d) $(x-3)(x+4)$

19. Factorise p^4-81

(a) $(p^2+9)(p+3)(p-3)$ (b) $(p^2-3)(p+3)(p-3)$

(c) $(p^2-3)(p-3)(2p+3)$ (d) $(p^2-9)(p-3)$

20. $(l+m)^2-(l-m)^2$ has factors

(a) $4lm$ (b) $2lm$ (c) $-4lm$ (d) $-2lm$

21. The factor form of $(a^4b^4-16c^4)$ is

(a) $4(a^2b^2+c^2)(ab-2c)(ab+2c)$

(b) $(a^2b^2-4c^2)(ab+2c)^2$

(c) $(a^2b^2+4c^2)(ab+2c)(ab-2c)$

(d) $(a^2b^2-4c^2)^2(ab+2c)(ab+4c)$

22. Factors of $x^6 - y^6$ are

(a) $(x^2 - y^2)(x^4 + y^4)$

(b) $(x^2 + y^2)(x^4 - x^2y^2 + x^4)$

(c) $(x + y)(x - y)(x^2 + xy + y^2)$

(d) $(x + y)(x - y)(x^2 - xy + y^2)(x^2 + xy + y^2)$

23. What are the factors of $32x^4 - 500x$?

(a) $16x(2x - 25)(3x^2 + 10x - 25)$

(b) $8x(4x - 65)(2x^2 + 4x + 6)$

(c) $8x(4x^2 - 5)(x^2 + 6x + 7)$

(d) $4x(2x - 5)(4x^2 + 10x + 25)$

24. Factors of $x^3 + 27y^3 + 8z^3 - 18xyz$ are

(a) $(x + 3y + 2z)$
$(x^2 + 9y^2 + 4z^2 - 3xy - 6yz - 2xz)$

(b) $(x - 3y - 2z)$
$(x^2 - 9y^2 + 4z^2 + 3xy + 6yz + 2xz)$

(c) $(x + 3y - 3z)$
$(x^2 - 9y^2 + 4z^2 - 3xy - 6yz - 2xz)$

(d) None of the above

25. What is the remainder, when $(4x^3 - 3x^2 + 2x - 1)$ is divided by $(x + 2)$?

(a) -49 (b) 55 (c) -30 (d) 37

26. The remainder, when $4a^3 - 12a^2 + 14a - 3$ is divided by $2a - 1$, is

(a) $\frac{2}{3}$ (b) $\frac{5}{3}$

(c) $\frac{6}{7}$ (d) $\frac{3}{2}$

27. Polynomial $p(x) = x^4 - 2x^3 + 3x^3 - ax + 3a - 7$, when divided by $(x + 1)$, leaves remainder 19, the value of a is

(a) 5 (b) 0 (c) 1 (d) 2

28. If $x^4 - 3x^3 + 2x^2 + x - 1$ is divided by $(x - 2)$, then remainder will be

(a) 0 (b) 2

(c) 1 (d) 3

29. The value of p, if $(2x - 1)$ is a factor of $2x^3 + px^2 + 11x + p + 3$ is

(a) -7 (b) 7 (c) -6 (d) 5

30. Which of the following is factor of polynomial $3x^2 - x - 4$?

(a) $(x + 1)$ (b) $(x - 2)$

(c) $(x - 4)$ (d) $(x - 1)$

Answers

1	(b)	2	(c)	3	(b)	4	(a)	5	(a)	6	(b)	7	(b)	8	(a)	9	(b)	10	(a)
11	(a)	12	(c)	13	(c)	14	(a)	15	(a)	16	(b)	17	(c)	18	(a)	19	(a)	20	(a)
21	(c)	22	(d)	23	(d)	24	(a)	25	(a)	26	(d)	27	(a)	28	(c)	29	(a)	30	(a)

Hints & Solutions

1. (b) We have,

$$3x - 24 = 3 \times x - 3 \times 8 = 3(x - 8)$$

[taking 3 as common]

2. (c)

We have,

$$11pq^2 = 11 \times p \times q \times q$$

$$121p^2q^3 = 11 \times 11 \times p \times p \times q \times q \times q$$

$$1331p^2q = 11 \times 11 \times 11 \times p \times p \times q$$

$\therefore$ Common factor $= 11 \times p \times q$

$= 11pq$

3. (b) Given, $17abc = 17 \times a \times b \times c$

$$34ab^2 = 2 \times 17 \times a \times b \times b$$

$$51a^2b = 3 \times 17 \times a \times a \times b$$

Now, collecting the common factors, we get

$$17 \times a \times b = 17ab$$

4. (a) $5x^2 - 20xy = 5x(x - 4y)$

5. (a) True

We have, $\frac{n^2}{2} + \frac{n}{2} = \frac{n^2 + n}{2} = \frac{1}{2} n(n + 1)$

$\therefore$ The factors are $1/2$ n and $(n + 1)$.

6. (b) False

We have, $-3a^2 + 3ab + 3ac = 3a(-a + b + c)$

7. (b) We have,

$x^2 - 4 = x^2 - 2^2 = (x + 2)(x - 2)$

$[\because a^2 - b^2 = (a + b)(a - b)]$

Hence, $(x + 2)$, $(x - 2)$ are factors of $x^2 - 4$.

8. (a) We have,

$xy + yz + xa + za = y(x + z) + a(x + z)$

$= (x + z)(y + a)$

9. (b) $5x(y + z) - 7y(y + z) = (y + z)(5x - 7y)$

10. (a) $p^2x^2 + c^2x^2 - ac^2 - ap^2$

$= p^2x^2 - ap^2 + c^2x^2 - ac^2$

$= p^2(x^2 - a) + c^2(x^2 - a) = (p^2 + c^2)(x^2 - a)$

11. (a) $8 - 4x - 2x^3 + x^4$

$= 4(2 - x) - x^3(2 - x) = (2 - x)(4 - x^3)$

12. (c) We have,

$2ax^2 + 4axy + 3bx^2 + 2ay^2 + 6bxy + 3by^2$

$= (2ax^2 + 2ay^2 + 4axy)$

$+ (3bx^2 + 3by^2 + 6bxy)$

$= 2a(x^2 + y^2 + 2xy) + 3b(x^2 + y^2 + 2xy)$

$= (2a + 3b)(x + y)^2$

13. (c) $q^2 - 10q + 21$

Here, $ab = 21$ and $a + b = -10$

Possible values of a and b are 7, 3 or -7, -3.

But $7 + 3 = 10 \neq -10$ [not possible]

$\therefore a = -7, b = -3$

Now, $q^2 - 10q + 21 = q^2 + (-7 - 3)q + 21$

$= q^2 - 7q - 3q + 21$

$= q(q - 7) - 3(q - 7) = (q - 7)(q - 3)$

Hence, required factors of given expression are $(q - 7)$ and $(q - 3)$.

14. (a) We have, $x^2 + 2x + 1$

$= x^2 + 2 \times 1 \times x + (1)^2$

$= (x + 1)^2$ $[\therefore a^2 + 2ab + b^2 = (a + b)^2]$

15. (a) We have, $6\sqrt{3}x^2 - 47x + 5\sqrt{3}$

$= 6\sqrt{3}x^2 - 45x - 2x + 5\sqrt{3}$

$= 3\sqrt{3}x(2x - 5\sqrt{3}) - 1(2x - 5\sqrt{3})$

$= (3\sqrt{3}x - 1)(2x - 5\sqrt{3})$

16. (b) $x^2 - 2\sqrt{3}x + 3$

$= x^2 - \sqrt{3}x - \sqrt{3}x + 3$

$= x(x - \sqrt{3}) - \sqrt{3}(x - \sqrt{3})$

$= (x - \sqrt{3})(x - \sqrt{3}) = (x - \sqrt{3})^2$

17. (c) $x^2 + 9x + 14$

$= x^2 + 7x + 2x + 14$

$= x(x + 7) + 2(x + 7)$

$= (x + 2)(x + 7)$

18. (a) $x^2 - 7x + 12$

$= x^2 - 3x - 4x + 12$

$= x(x - 3) - 4(x - 3) = (x - 3)(x - 4)$

19. (a) $p^4 - 81 = (p^2)^2 - (9)^2$

On comparing with $a^2 - b^2$, we get $a = p^2$ and $b = 9$

$\therefore$ $p^4 - 81 = (p^2 + 9)(p^2 - 9)$

$[\because (a^2 - b^2) = (a + b)(a - b)]$

$= (p^2 + 9)[(p)^2 - (3)^2]$

$= (p^2 + 9)(p + 3)(p - 3)$

$[\because a^2 - b^2 = (a + b)(a - b)$ and $(p^2 + 9)$ cannot be factorised further]

20. (a) We have, $(l + m)^2 - (l - m)^2$

$= (l + m + l - m)(l + m - l + m)$

$[\therefore a^2 - b^2 = (a + b)(a - b)]$

$= 2l \times 2m = 4lm$

21. (c) $(a^4b^4 - 16c^4)$

$= [(a^2b^2)^2 - (4c^2)^2]$

$= (a^2b^2 + 4c^2)(a^2b^2 - 4c^2)$

$= (a^2b^2 + 4c^2)[(ab)^2 - (2c)^2]$

$= (a^2b^2 + 4c^2)(ab + 2c)(ab - 2c)$

22. (d) We have, $x^6 - y^6$

$= (x^3)^2 - (y^3)^2 = (x^3 + y^3)(x^3 - y^3)$

$[\because a^2 - b^2 = (a + b)(a - b)]$

$= (x + y)(x^2 - xy + y^2)(x - y)(x^2 + xy + y^2)$

$= (x + y)(x - y)(x^2 - xy + y^2)(x^2 + xy + y^2)$

23. (d) We have, $32x^4 - 500x$

$= 4x\,[8x^3 - 125] = 4x\,[(2x)^3 - 5^3]$

$= 4x(2x - 5)(4x^2 + 10x + 25)$

$[\because a^3 - b^3 = (a - b)(a^2 + ab + b^2)]$

24. (a) We have, $x^3 + 27y^3 + 8z^3 - 18xyz$

$= x^3 + (3y)^3 + (2z)^3 - 3(x)(3y)(2z)$

$= (x + 3y + 2z)$

$[x^2 + 9y^2 + 4z^2 - x(3y) - 3y(2z) - x(2z)]$

$= (x + 3y + 2x)$

$(x^2 + 9y^2 + 4z^2 - 3xy - 6yz - 2xz)$

25. (a) Let $p(x) = 4x^3 - 3x^2 + 2x - 1$

$\therefore (x + 2)$ divides $p(x)$, then

$\therefore p(-2)$

$= 4(-2)^3 - 3(-2)^2 + 2\times(-2) - 1$

$= -32 - 12 - 4 - 1 = -49$

26. (d) Put $(2a - 1) = 0$

$\Rightarrow \quad a = \frac{1}{2}$

$\therefore \quad p(a) = 4a^3 - 12a^2 + 14a - 3$

$= 4\left(\frac{1}{2}\right)^3 - 12\times\left(\frac{1}{2}\right)^2 + 14\left(\frac{1}{2}\right) - 3$

$= \frac{4}{8} - \frac{12}{4} + 7 - 3$

$= \frac{1}{2} - 3 + 7 - 3$

$= \frac{1}{2} + 7 - 6$

$= \frac{1 + 14 - 12}{2} = \frac{3}{2}$

27. (a) If $(x + 1)$ divides $p(x)$, then $p(x)$ leaves remainder 19.

i.e. $\quad (x + 1) = 0$

$\Rightarrow \quad x = -1$

$\therefore \quad p(-1) = 19$

$\Rightarrow (-1)^4 - 2\times(-1)^3 + 3\times(-1)^2$

$- a\times(-1) + 3(a) - 7 = 19$

$\Rightarrow \quad 1 + 2 + 3 + a + 3a - 7 = 19$

$\Rightarrow \quad 6 + 4a - 7 = 19$

$\Rightarrow \quad 4a - 1 = 19$

$\Rightarrow \quad 4a = 20$

$\Rightarrow \quad a = \frac{20}{4} = 5$

28. (c) Let $p(x) = x^4 - 3x^3 + 2x^2 + x - 1$ ...(i)

and $g(x) = x - 2$

We have to divide $p(x)$ by $g(x)$

For this, we have to put

$g(x) = 0$

i.e. $\quad x - 2 = 0 \Rightarrow x = 2$

Put $x = 2$ in Eq. (i), we get

$p(2) = 2^4 - 3\times(2)^3 + 2\times(2)^2 + 2 - 1$

$= 16 - 24 + 8 + 1 = 1$

Hence, the value of $p(2) = 1$, which is the required remainder obtained on dividing $x^4 - 3x^3 + 2x^2 + x - 1$ by $x - 2$.

29. (a) Let $q(x) = 2x^3 + px^2 + 11x + p + 3$

If $q(x)$ is divisible by $2x - 1$, then $(2x - 1)$ is a factor of $q(x)$.

$\therefore \quad 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$

On putting $x = \frac{1}{2}$ in $q(x)$, we have

$q\left(\frac{1}{2}\right) = 2\times\left(\frac{1}{2}\right)^3 + p\left(\frac{1}{2}\right)^2 + 11\left(\frac{1}{2}\right) p + 3 = 0$

$\Rightarrow \quad 2\times\frac{1}{8} + p\times\frac{1}{4} + \frac{11}{2} + p + 3 = 0$

$\Rightarrow \quad \frac{1}{4} + \frac{p}{4} + \frac{11}{2} + p + 3 = 0$

$\Rightarrow \quad \frac{1 + p + 22 + 4p + 12}{4} = 0$

$\Rightarrow \quad 5p + 35 = 0$

$\Rightarrow \quad 5p = -35$

$\therefore \quad p = -7$

30. (a) Let $p(x) = 3x^2 - x - 4$

To check the factors

We have to check $p(x) = 0$

$(x + 1) = 0\,,\ x = -1$

$\therefore \quad p(-1) = 3\times(-1)^2 - (-1) - 4$

$= 3 + 1 - 4$

$= 4 - 4 = 0$

$\therefore$ $(x + 1)$ is the factor of $p(x)$.