CHAPTER

2

Different Products of Vectors and Their Geometrical Applications

- Dot (Scalar) Product
- Applications of Dot (Scalar) Product
- Vector (or Cross) Product of Two Vectors
- Scalar Triple Product
- Vector Triple Product
- Reciprocal System of Vectors

DOT (SCALAR) PRODUCT

The scalar product of vectors $\vec{a}$ and $\vec{b}$, written as $\vec{a}\cdot\vec{b}$, is defined to be the number $|\vec{a}||\vec{b}|\cos\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$.

i.e., $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$, where $0 \le \theta \le \pi$.

Notes:

1. *$\vec{a}\cdot\vec{b}$ is positive if θ is acute.*
2. *$\vec{a}\cdot\vec{b}$ is negative if θ is obtuse.*
3. *$\vec{a}\cdot\vec{b}$ is zero if θ is a right angle.*

Physical Interpretation of Scalar Product

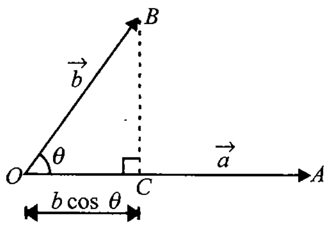


Fig. 2.1

Let $\overrightarrow{OA} = \vec{a}$ represent a force acting on a particle at O and let $\overrightarrow{OB} = \vec{b}$ represent the displacement of the particle from O to B as shown in the figure. Then the displacement in the direction of the force $= OC = b\cos\theta$. Therefore the work done by a force is a scalar quantity equal to the product of the magnitude of the force and the resolved part of the displacement in the direction of force work done by force $\vec{a}$ in moving its point of application from O to $B = |\vec{a}||\vec{b}|\cos\theta = \vec{a}\cdot\vec{b}$.

Geometrical Interpretation of Scalar Product

Let $\vec{a}$ and $\vec{b}$ be two vectors represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$, respectively.

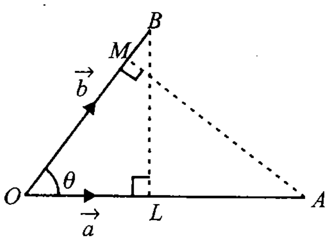


Fig. 2.2

Here OL and OM are known as projections of $\vec{b}$ on $\vec{a}$ and $\vec{a}$ on $\vec{b}$, respectively.

Now, $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$= |\vec{a}|(OB\cos\theta)$

$= |\vec{a}|(OL)$

$=$ (magnitude of $\vec{a}$) (projection of $\vec{b}$ on $\vec{a}$) (i)

Again, $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$= |\vec{b}|(|\vec{a}|\cos\theta)$

$= |\vec{b}|(OA\cos\theta)$

$= |\vec{b}|(OM)$

$=$ (magnitude of $\vec{b}$) (projection of $\vec{a}$ on $\vec{b}$) (ii)

Thus, geometrically interpreted, the scalar product of two vectors is the product of modulus of either vectors and the projection of the other in its direction.

Thus projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \vec{a}\cdot\dfrac{\vec{b}}{|\vec{b}|} = \vec{a}\cdot\hat{b}$

Projection of $\vec{b}$ on $\vec{a} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \dfrac{\vec{a}}{|\vec{a}|}\cdot\vec{b} = \hat{a}\cdot\vec{b}$

Properties of Dot (Scalar) Product

i. $\vec{a}\cdot\vec{a} = |\vec{a}||\vec{a}|\cos 0° = |\vec{a}|^2 = a^2 \Rightarrow \hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

ii. $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$ (commutative)

iii. $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$ (distributive)

Proof:

Let $\vec{OA} = \vec{a}, \vec{OB} = \vec{b}, \vec{BC} = \vec{c}$ so that

$\vec{OC} = \vec{OB} + \vec{BC} = \vec{b} + \vec{c}$

From B draw $BM \perp OA$ and from C, drawn $CN \perp OA$

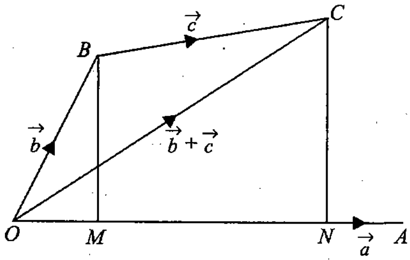


Fig. 2.3

L.H.S. $= \vec{a}\cdot(\vec{b}+\vec{c})$

$= \vec{OA}\cdot\vec{OC}$

$= (OA)(OC)\cos\theta$ (where $\theta = \angle CON$)

$= (OA)(ON)$ (as $ON = OC\cos\theta$)

$= (OA)(OM + MN)$

$= (OA)(OM) + (OA)(MN)$

$$= \vec{OA}\cdot\vec{OB} + \vec{OA}\cdot\vec{BC}$$

$$= \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c} = \text{R.H.S.}$$

iv. $(l\vec{a})\cdot(m\vec{b}) = lm(\vec{a}\cdot\vec{b})$, where l and m are scalars

v. If $\vec{a}$ and $\vec{b}$ are two non-zero vectors, then $\vec{a}\cdot\vec{b} = 0 \Leftrightarrow \vec{a}$ and $\vec{b}$ are perpendicular to each other

$\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$

vi. $(\vec{a} \pm \vec{b})^2 = (\vec{a} \pm \vec{b})\cdot(\vec{a} \pm \vec{b})$

$$= |\vec{a}|^2 + |\vec{b}|^2 \pm 2\vec{a}\cdot\vec{b}$$

$$= |\vec{a}|^2 + |\vec{b}|^2 \pm 2|\vec{a}||\vec{b}|\cos\theta$$

vii. $(\vec{a} + \vec{b})\cdot(\vec{a} - \vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$

viii. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

$(\because \hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$ and $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0)$

ix. Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. Taking dot product with $\hat{i}, \hat{j}$ and $\hat{k}$ alternatively, we have

$x = \vec{r}\cdot\hat{i},\ y = \vec{r}\cdot\hat{j}$ and $z = \vec{r}\cdot\hat{k}$

$\Rightarrow \vec{r} = (\vec{r}\cdot\hat{i})\hat{i} + (\vec{r}\cdot\hat{j})\hat{j} + (\vec{r}\cdot\hat{k})\hat{k}$

APPLICATIONS OF DOT (SCALAR) PRODUCT

Finding Angle between Two Vectors

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are non-zero vectors, then the angle between them is given by

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\ \sqrt{b_1^2 + b_2^2 + b_3^2}}$$

Also

$$\frac{(a_1b_1 + a_2b_2 + a_3b_3)^2}{(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)} = \cos^2\theta \le 1$$

$$\Rightarrow \quad (a_1b_1 + a_2b_2 + a_3b_3)^2 \le (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$$

Cosine Rule Using Dot Product

Using vector method, prove that in a triangle $a^2 = b^2 + c^2 - 2bc\cos A$ (Cosine law)

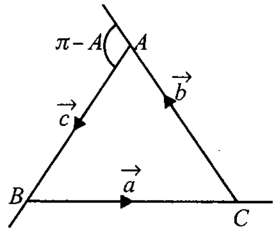


Fig. 2.4

In ΔABC,

Let $\overrightarrow{AB} = \vec{c}, \overrightarrow{BC} = \vec{a}, \overrightarrow{CA} = \vec{b}$,

Since $\vec{a} + \vec{b} + \vec{c} = 0$, we have $\vec{a} = -(\vec{b} + \vec{c})$

$\therefore |\vec{a}| = |-(\vec{b} + \vec{c})|$

$\Rightarrow |\vec{a}|^2 = |\vec{b} + \vec{c}|^2$

$\Rightarrow |\vec{a}|^2 = |\vec{b}|^2 + |\vec{c}|^2 + 2\vec{b}\cdot\vec{c}$

$\Rightarrow |\vec{a}|^2 = |\vec{b}|^2 + |\vec{c}|^2 + 2|\vec{b}||\vec{c}|\cos(\pi - A)$

(Since angle between $\vec{b}$ and $\vec{c}$ = the angle between CA produced and AB)

$\Rightarrow a^2 = b^2 + c^2 - 2bc \cos A$

Finding Components of a Vector $\vec{b}$ Along and Perpendicular to Vector $\vec{a}$ or Resolving a Given Vector in the Direction of Given Two Perpendicular Vectors

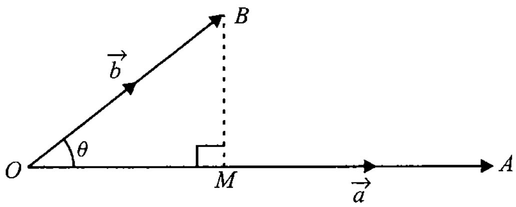


Fig. 2.5

Let $\vec{a}$ and $\vec{b}$ be two vectors represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ and let θ be the angle between $\vec{a}$ and $\vec{b}$.

$\therefore \vec{b} = \overrightarrow{OM} + \overrightarrow{MB}$

Also $\overrightarrow{OM} = (OM)\hat{a}$

$= (OB \cos\theta)\hat{a}$

$= (|\vec{b}|\cos\theta)\hat{a}$

$$= \left(|\vec{b}| \frac{(\vec{a}\cdot\vec{b})}{|\vec{a}||\vec{b}|} \right) \hat{a}$$

$$= \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} \right) \hat{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{a}|} \vec{a} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2} \right) \vec{a}$$

Also $\vec{b} = \overrightarrow{OM} + \overrightarrow{MB}$

$$\Rightarrow \overrightarrow{MB} = \vec{b} - \overrightarrow{OM} = \vec{b} - \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2} \right) \vec{a}$$

Thus, the components of $\vec{b}$ along and perpendicular to $\vec{a}$ are $\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2} \right) \vec{a}$ and $\vec{b} - \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2} \right) \vec{a}$, respectively.

Example 2.1 **If $\vec{a}, \vec{b}$ and $\vec{c}$ are non-zero vectors such that $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c}$, then find the geometrical relation between the vectors.**

Sol. $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c}$

$\Rightarrow \vec{a}\cdot\vec{b} - \vec{a}\cdot\vec{c} = \vec{0}$

$\Rightarrow \vec{a}\cdot(\vec{b} - \vec{c}) = \vec{0}$

$\Rightarrow$ Either $\vec{b} - \vec{c} = \vec{0}$ or $\vec{a} \perp (\vec{b} - \vec{c})$

$\Rightarrow \vec{b} = \vec{c}$ or $\vec{a} \perp (\vec{b} - \vec{c})$

Example 2.2 **If $\vec{r}\cdot\hat{i} = \vec{r}\cdot\hat{j} = \vec{r}\cdot\hat{k}$ and $|\vec{r}| = 3$, then find vector $\vec{r}$.**

Sol. Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. Since $\vec{r}\cdot\hat{i} = \vec{r}\cdot\hat{j} = \vec{r}\cdot\hat{k}$

$x = y = z$

Also $|\vec{r}| = \sqrt{x^2 + y^2 + z^2} = 3$

$\Rightarrow x = \pm\sqrt{3}$

Hence, the required vector $\vec{r} = \pm\sqrt{3}(\hat{i} + \hat{j} + \hat{k})$

Example 2.3 **If $\vec{a}, \vec{b}$ and $\vec{c}$ are unit vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, then find the value of $\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}$.**

Sol. Squaring $(\vec{a} + \vec{b} + \vec{c}) = \vec{0}$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2\vec{a}\cdot\vec{b} + 2\vec{b}\cdot\vec{c} + 2\vec{c}\cdot\vec{a} = 0$

$\Rightarrow 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = -3$

$\Rightarrow \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a} = -\frac{3}{2}$

Example 2.4 **If $\vec{a}, \vec{b}$ and $\vec{c}$ are mutually perpendicular vectors of equal magnitudes, then find the angle between vectors $\vec{a}$ and $\vec{a}+\vec{b}+\vec{c}$.**

Sol. Since $\vec{a}, \vec{b}$ and $\vec{c}$ are mutually perpendicular, $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}=0$

Angle between $\vec{a}$ and $\vec{a}+\vec{b}+\vec{c}$ is

$$\cos\theta = \frac{\vec{a}\cdot(\vec{a}+\vec{b}+\vec{c})}{|\vec{a}||\vec{a}+\vec{b}+\vec{c}|} \quad \text{(i)}$$

Now $|\vec{a}|=|\vec{b}|=|\vec{c}|=a$

$$|\vec{a}+\vec{b}+\vec{c}|^2 = |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2\vec{a}\cdot\vec{b}+2\vec{b}\cdot\vec{c}+2\vec{c}\cdot\vec{a}$$

$$= a^2+a^2+a^2+0+0+0$$

$$\Rightarrow |\vec{a}+\vec{b}+\vec{c}|^2 = 3a^2$$

$$\Rightarrow |\vec{a}+\vec{b}+\vec{c}| = \sqrt{3}a$$

Putting this value in (i), we get $\theta = \cos^{-1}\frac{1}{\sqrt{3}}$

Example 2.5 **If $|\vec{a}|+|\vec{b}|=|\vec{c}|$ and $\vec{a}+\vec{b}=\vec{c}$, then find the angle between $\vec{a}$ and $\vec{b}$.**

Sol. $\vec{a}+\vec{b}=\vec{c}$

$$\Rightarrow |\vec{a}|^2+|\vec{b}|^2+2\vec{a}\cdot\vec{b}=|\vec{c}|^2 \quad \text{(i)}$$

and $|\vec{a}|+|\vec{b}|=|\vec{c}|$

$$\Rightarrow |\vec{a}|^2+|\vec{b}|^2+2|\vec{a}||\vec{b}|=|\vec{c}|^2 \quad \text{(ii)}$$

$\therefore\ \vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|$ (from (i) and (ii))

$\Rightarrow \cos\theta = 1 \Rightarrow \theta = 0°$

Example 2.6 **If three unit vectors $\vec{a}, \vec{b}$ and $\vec{c}$ satisfy $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, then find the angle between $\vec{a}$ and $\vec{b}$.**

Sol. $\vec{a}+\vec{b}=-\vec{c}$

$$\Rightarrow |\vec{a}+\vec{b}|^2=|\vec{c}|^2=1$$

$$\Rightarrow |\vec{a}|^2+|\vec{b}|^2+2\vec{a}\cdot\vec{b}=1$$

$$\Rightarrow \vec{a}\cdot\vec{b}=-\frac{1}{2}$$

$$\Rightarrow |\vec{a}||\vec{b}|\cos\theta=-\frac{1}{2}$$

$$\Rightarrow \cos\theta=-\frac{1}{2}$$

$$\Rightarrow \theta=\frac{2\pi}{3}$$

Example 2.7 **If θ be the angle between the unit vectors $\vec{a}$ and $\vec{b}$, then prove that**

i. $\cos\frac{\theta}{2} = \frac{1}{2}|\vec{a}+\vec{b}|$

ii. $\sin\frac{\theta}{2} = \frac{1}{2}|\vec{a}-\vec{b}|$

Sol. **i.** $(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}\cdot\vec{b}$

$$= 1 + 1 + 2(1)(1)\cos\theta$$

$$= 2 + 2\cos\theta$$

$$\Rightarrow |\vec{a}+\vec{b}|^2 = 2\cdot 2\cos^2\frac{\theta}{2}$$

$$\Rightarrow \cos\frac{\theta}{2} = \frac{1}{2}|\vec{a}+\vec{b}|$$

ii. $(\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b}) = |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a}\cdot\vec{b}$

$$= 1 + 1 - 2(1)(1)\cos\theta$$

$$= 2 - 2\cos\theta$$

$$\Rightarrow |\vec{a}-\vec{b}|^2 = 2\cdot 2\sin^2\frac{\theta}{2}$$

$$\Rightarrow \sin\frac{\theta}{2} = \frac{1}{2}|\vec{a}+\vec{b}|$$

Example 2.8 **If the scalar projection of vector $x\hat{i}-\hat{j}+\hat{k}$ on vector $2\hat{i}-\hat{j}+5\hat{k}$ is $\frac{1}{\sqrt{30}}$, then find the value of x.**

Sol. Projection of $x\hat{i}-\hat{j}+\hat{k}$ on $2\hat{i}-\hat{j}+5\hat{k}$ $= \frac{(x\hat{i}-\hat{j}+\hat{k})\cdot(2\hat{i}-\hat{j}+5\hat{k})}{\sqrt{4+1+25}}$

$$= \frac{2x+1+5}{\sqrt{30}}$$

But, given $\frac{2x+6}{\sqrt{30}} = \frac{1}{\sqrt{30}}$ $\Rightarrow 2x+6=1 \Rightarrow x = \frac{-5}{2}$

Example 2.9 **If $\vec{a} = x\hat{i} + (x-1)\hat{j} + \hat{k}$ and $\vec{b} = (x+1)\hat{i} + \hat{j} + a\hat{k}$ make an acute angle $\forall\ x \in R$, then find the values of a.**

Sol. $\vec{a}\cdot\vec{b} = (x\hat{i}+(x-1)\hat{j}+\hat{k})\cdot((x+1)\hat{i}+\hat{j}+a\hat{k})$

$$= x(x+1)+x-1+a$$

$$= x^2+2x+a-1$$

We must have $\vec{a}\cdot\vec{b} > 0\ \forall\ x\in R$

$$\Rightarrow x^2+2x+a-1>0\ \forall\ x\in R$$
$$\Rightarrow 4-4(a-1)<0$$
$$\Rightarrow a>2$$

Example 2.10 **If $\vec{a}\cdot\hat{i}=\vec{a}\cdot(\hat{i}+\hat{j})=\vec{a}\cdot(\hat{i}+\hat{j}+\hat{k})$, then find the unit vector $\vec{a}$.**

Sol. Let $\vec{a}=x\hat{i}+y\hat{j}+z\hat{k}$

Then, $\vec{a}\cdot\hat{i}=(x\hat{i}+y\hat{j}+z\hat{k})\cdot\hat{i}=x$ and $\vec{a}\cdot(\hat{i}+\hat{j})=x+y$

and $\vec{a}\cdot(\hat{i}+\hat{j}+\hat{k})=x+y+z$ (given that $x=x+y=x+y+z$)

Now $x=x+y\Rightarrow y=0$ and $x+y=x+y+z\Rightarrow z=0$

Hence $x=1$ (Since $\vec{a}$ is a unit vector)

$\therefore\ \vec{a}=\hat{i}$

Example 2.11 **Prove by vector method that $\cos(A+B)=\cos A\cos B-\sin A\sin B$.**

Sol. Let $\hat{i}$ and $\hat{j}$ be unit vectors along OX and OY, respectively.

Let $\vec{OP},\vec{OQ}$ be two unit vectors drawn in the plane XOY such that

$\angle XOP=A,\ \angle XOQ=B$

$\therefore\quad POQ=A+B$

Now $\vec{OP}=\hat{i}\cos A+\hat{j}\sin A$

$\vec{OQ}=\hat{i}\cos B-\hat{j}\sin B$

$\therefore\quad \vec{OP}\cdot\vec{OQ}=\cos A\cos B-\sin A\sin B$

$\Rightarrow\quad (1)(1)\cos(A+B)=\cos A\cos B-\sin A\sin B$

$\Rightarrow\quad \cos(A+B)=\cos A\cos B-\sin A\sin B$

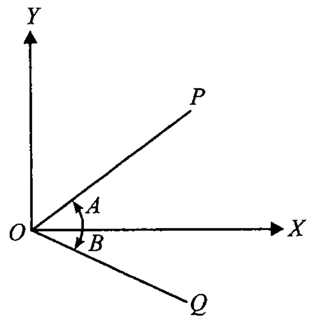


Fig. 2.6

Example 2.12 **In any triangle ABC, prove the projection formula $a=b\cos C+c\cos B$ using vector method.**

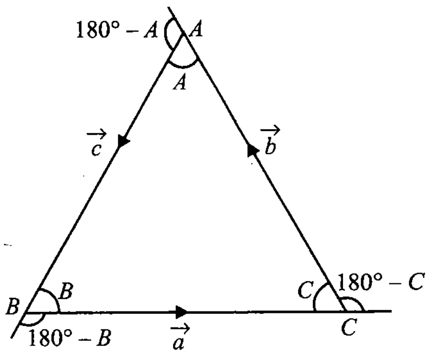


Fig. 2.7

Sol. Let $\vec{BC}=\vec{a},\ \vec{CA}=\vec{b},\ \vec{AB}=\vec{c}$, so that

$BC=a,\ CA=b,\ AB=c$

Now $\vec{a}+\vec{b}+\vec{c}=\vec{0}$

$\therefore\quad \vec{a}\cdot(\vec{a}+\vec{b}+\vec{c})=0$

$\vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}=0$

$a^2+ab\cos(180^\circ-C)+ac\cos(180^\circ-B)=0$

$a^2-ab\cos C-ac\cos B=0$

$a-b\cos C-c\cos B=0$

$a=b\cos C+c\cos B$

Example 2.13 **Prove that an angle inscribed in a semi-circle is a right angle using vector method.**

Sol. Let O be the centre of the semi-circle and BA be the diameter. Let P be any point on the circumference of the semi-circle.

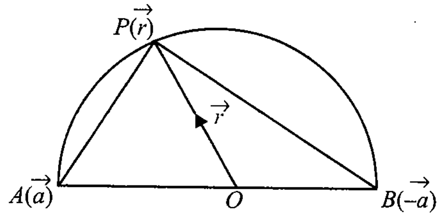


Fig. 2.8

Let $\vec{OA} = \vec{a}$, then $\vec{OB} = -\vec{a}$

Let $\vec{OP} = \vec{r}$

$\therefore \quad \vec{AP} = \vec{OP} - \vec{OA} = \vec{r} - \vec{a}$

$$\vec{BP} = \vec{OP} - \vec{OB} = \vec{r} - (-\vec{a}) = \vec{r} + \vec{a}$$

$$\vec{AP} \cdot \vec{BP} = (\vec{r} - \vec{a}) \cdot (\vec{r} + \vec{a})$$

$$= \vec{r}^2 - \vec{a}^2$$

$$= a^2 - a^2 \qquad [\because r = a \text{ as } OP = OA]$$

$\therefore \overrightarrow{AP}$ is perpendicular to $\overrightarrow{BP}$

$\Rightarrow \angle APB = 90°$

Example 2.14 **Using dot product of vectors, prove that a parallelogram, whose diagonals are equal, is a rectangle.**

Sol. Let $OACB$ be a parallelogram such that $OC = AB$

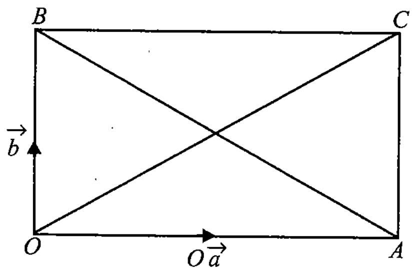


Fig. 2.9

Let $\vec{OA} = \vec{a}, \vec{OB} = \vec{b}$

Now $OC = AB$

$\Rightarrow \quad OC^2 = AB^2$

$$\Rightarrow \quad \left(\vec{OA} + \vec{AC}\right)^2 = \left(\vec{AO} + \vec{OB}\right)^2$$

$$\Rightarrow \quad \left(\vec{OA} + \vec{OB}\right)^2 = \left(-\vec{OA} + \vec{OB}\right)^2$$

$$\Rightarrow \quad (\vec{a} + \vec{b}) = (-\vec{a} + \vec{b})^2$$

$$\Rightarrow \quad \vec{a}^2 + \vec{b}^2 + 2\vec{a} \cdot \vec{b} = \vec{a}^2 + \vec{b}^2 - 2\vec{a} \cdot \vec{b}$$

$$\Rightarrow \quad 2\vec{a} \cdot \vec{b} = -2\vec{a} \cdot \vec{b}$$

$$\Rightarrow \quad 4\vec{a} \cdot \vec{b} = 0$$

$$\Rightarrow \quad \vec{a} \cdot \vec{b} = \vec{0}$$

$\Rightarrow \quad \vec{a}$ and $\vec{b}$ are perpendicular

$\Rightarrow \quad \angle AOB = 90°$

$\Rightarrow \quad OACB$ is a rectangle

Example 2.15 **If $a + 2b + 3c = 4$, then find the least value of $a^2 + b^2 + c^2$.**

Sol. Consider vectors $\vec{p} = a\hat{i} + b\hat{j} + c\hat{k}$ and $\vec{q} = \hat{i} + 2\hat{j} + 3\hat{k}$

$$\text{Now } \cos\theta = \frac{a + 2b + 3c}{\sqrt{a^2 + b^2 + c^2}\sqrt{1^2 + 2^2 + 3^2}}$$

$$\text{or} \qquad \cos^2\theta = \frac{(a + 2b + 3c)^2}{14(a^2 + b^2 + c^2)} \leq 1$$

$$\Rightarrow \qquad a^2 + b^2 + c^2 \geq \frac{8}{7}$$

$\Rightarrow$ Hence least value of $a^2 + b^2 + c^2$ is $\frac{8}{7}$

Example 2.16 **Find a unit vector $\vec{a}$ which makes an angle of $\pi/4$ with the z-axis and it is such that $(\vec{a}+\hat{i}+\hat{j})$ is a unit vector.**

Sol. Let $\vec{a} = x\hat{i}+y\hat{j}+z\hat{k}$

Given $|\vec{a}|=1$, therefore

$x^2+y^2+z^2=1$ (i)

Angle between $\vec{a}$ and z-axis is $\pi/4$, therefore

$$\cos\left(\frac{\pi}{4}\right) = \frac{\vec{a}\cdot\hat{k}}{|\vec{a}||\hat{k}|}$$

$$\Rightarrow z = \frac{1}{\sqrt{2}}$$

Now $\vec{a}+\hat{i}+\hat{j} = (x+1)\hat{i}+(y+1)\hat{j}+z\hat{k}$

Given that $\vec{a}+\hat{i}+\hat{j}$ is a unit vector. Therefore,

$|\vec{a}+\hat{i}+\hat{j}| = \sqrt{[(x+1)^2+(y+1)^2+z^2]} = 1$

$\Rightarrow x^2+y^2+z^2+2x+2y+1=0$

$\Rightarrow 1+2x+2y+1=0$, using (i)

$\Rightarrow y = -(x+1)$

From (i), we have

$x^2+(x+1)^2+(1/2)=1$

$\Rightarrow 4x^2+4x+1=0$ or $(2x+1)^2=0$

$$x = -\frac{1}{2} \Rightarrow y = -\frac{1}{2}$$

Hence $\vec{a} = -\frac{1}{2}\hat{i}-\frac{1}{2}\hat{j}+\frac{1}{\sqrt{2}}\hat{k}$

Example 2.17 **Vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are of the same length and taken pair-wise they form equal angles. If $\vec{a}=\hat{i}+\hat{j}$ and $\vec{b}=\hat{j}+\hat{k}$, then find vector $\vec{c}$.**

Sol. Let $\vec{c} = x\hat{i}+y\hat{j}+z\hat{k}$. Then $|\vec{a}|=|\vec{b}|=|\vec{c}| \Rightarrow x^2+y^2+z^2=2$

It is given that the angles between the vectors taken in pairs are equal, say θ. Therefore,

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{0+1+0}{\sqrt{2}\sqrt{2}} = \frac{1}{2}$$

$$\Rightarrow \frac{\vec{a}\cdot\vec{c}}{|\vec{a}||\vec{c}|} = \frac{1}{2} \text{ and } \frac{\vec{b}\cdot\vec{c}}{|\vec{b}||\vec{c}|} = \frac{1}{2}$$

$$\Rightarrow \frac{x+y}{\sqrt{2}\sqrt{2}} = \frac{1}{2} \text{ and } \frac{y+z}{\sqrt{2}\sqrt{2}} = \frac{1}{2}$$

$\Rightarrow x+y=1$ and $y+z=1$

$\Rightarrow y=1-x$ and $z=1-y=1-(1-x)=x$

Also $x^2+y^2+z^2=2 \Rightarrow x^2+(1-x)^2+x^2=2$

$\Rightarrow (3x+1)(x-1)=0 \Rightarrow x=1, -1/3$

Now, $y=1-x \Rightarrow y=0$ for $x=1$ and $y=4/3$ for $x=-1/3$

Hence, $\vec{c}=\hat{i}+0\hat{j}+\hat{k}$ and $\vec{c}=-\frac{1}{3}\hat{i}+\frac{4}{3}\hat{j}-\frac{1}{3}\hat{k}$

Example 2.18 **If $\vec{a}, \vec{b}$ and $\vec{c}$ are three mutually perpendicular unit vectors and $\vec{d}$ is a unit vector which makes equal angles with $\vec{a}, \vec{b}$ and $\vec{c}$, then find the value of $|\vec{a}+\vec{b}+\vec{c}+\vec{d}|^2$.**

Sol. $|\vec{a}+\vec{b}+\vec{c}+\vec{d}|^2=\Sigma|\vec{a}|^2+2\Sigma\vec{a}\cdot\vec{b}=4+2\vec{d}\cdot(\vec{a}+\vec{b}+\vec{c})$ ($\because$ $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular)

Let $\vec{d}=\lambda\vec{a}+\mu\vec{b}+\nu\vec{c}$. Then $\vec{d}\cdot\vec{a}=\vec{d}\cdot\vec{b}=\vec{d}\cdot\vec{c}=\cos\theta$. Therefore,

$\lambda=\mu=\nu=\cos\theta$

Also $\lambda^2+\mu^2+\nu^2=1 \Rightarrow 3\cos^2\theta=1 \Rightarrow \cos\theta=\pm\frac{1}{\sqrt{3}}$

$\therefore |\vec{a}+\vec{b}+\vec{c}+\vec{d}|^2=4\pm\frac{2\cdot 3}{\sqrt{3}}=4\pm 2\sqrt{3}$

Example 2.19 **A particle acted by constant forces $4\hat{i}+\hat{j}-3\hat{k}$ and $3\hat{i}+\hat{j}-\hat{k}$ is displaced from point $\hat{i}+2\hat{j}+3\hat{k}$ to point $5\hat{i}+4\hat{j}+\hat{k}$. Find the total work done by the forces in units.**

Sol. Here $\vec{F}=\vec{F_1}+\vec{F_2}=(4\hat{i}+\hat{j}-3\hat{k})+(3\hat{i}+\hat{j}-\hat{k})=7\hat{i}+2\hat{j}-4\hat{k}$

and $\vec{d}=\vec{d_2}-\vec{d_1}=(5\hat{i}+4\hat{j}+\hat{k})-(\hat{i}+2\hat{j}+3\hat{k})=4\hat{i}+2\hat{j}-2\hat{k}$

$\therefore$ Work done $=\vec{F}\cdot\vec{d}$

$=(7\hat{i}+2\hat{j}-4\hat{k})\cdot(4\hat{i}+2\hat{j}-2\hat{k})$

$=(7)(4)+(2)(2)+(-4)(-2)$

$=28+4+8=40$ units

Example 2.20 **If $\vec{a}=4\hat{i}+6\hat{j}$ and $\vec{b}=3\hat{j}+4\hat{k}$, then find the component of $\vec{a}$ along $\vec{b}$.**

Sol. The component of vector $\vec{a}$ along $\vec{b}$ is $\frac{(\vec{a}.\vec{b})\vec{b}}{|\vec{b}|^2}=\frac{18}{25}(3\hat{i}+4\hat{k})$

Example 2.21 **If $|\vec{a}|=|\vec{b}|=|\vec{a}+\vec{b}|=1$, then find the value of $|\vec{a}-\vec{b}|$.**

Sol. We have

$|\vec{a}+\vec{b}|^2+|\vec{a}-\vec{b}|^2=2(|\vec{a}|^2+|\vec{b}|^2)$

$\Rightarrow 1+|\vec{a}-\vec{b}|^2=4 \Rightarrow |\vec{a}-\vec{b}|=\sqrt{3}$

Example 2.22 **If $\vec{a}=-\hat{i}+\hat{j}+\hat{k}$ and $\vec{b}=2\hat{i}+0\hat{j}+\hat{k}$, then find vector $\vec{c}$ satisfying the following conditions: (i) that it is coplanar with $\vec{a}$ and $\vec{b}$, (ii) that it is $\perp$ to $\vec{b}$ and (iii) that $\vec{a}\cdot\vec{c}=7$.**

Sol. Let $\vec{c}=x\hat{i}+y\hat{j}+z\hat{k}$

Then from condition (i)

$$\begin{vmatrix} x & y & z \\ -1 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix} = 0 \text{ or } x + 3y - 2z = 0 \quad \text{(i)}$$

From condition (ii)

$2x + z = 0$ (ii)

From condition (iii)

$-x + y + z = 7$ (iii)

Solving (i), (ii) and (iii), we get the values of x, y and z and hence vector $\vec{c} = \frac{1}{2}(-3\hat{i} + 5\hat{j} + 6\hat{k})$

Example 2.23 **Let $\vec{a}, \vec{b}$ and $\vec{c}$ are vectors such that $|\vec{a}| = 3$, $|\vec{b}| = 4$ and $|\vec{c}| = 5$, and $(\vec{a}+\vec{b})$ is perpendicular to $\vec{c}$, $(\vec{b}+\vec{c})$ is perpendicular to $\vec{a}$ and $(\vec{c}+\vec{a})$ is perpendicular to $\vec{b}$. Then find the value of $|\vec{a}+\vec{b}+\vec{c}|$.**

Sol. Given, $(\vec{a}+\vec{b})\cdot\vec{c} = 0 \Rightarrow \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c} = 0$

$(\vec{b}+\vec{c})\cdot\vec{a} = 0 \Rightarrow \vec{a}\cdot\vec{b} + \vec{c}\cdot\vec{a} = 0$

$(\vec{c}+\vec{a})\cdot\vec{b} = 0 \Rightarrow \vec{b}\cdot\vec{c} + \vec{a}\cdot\vec{b} = 0$

$\therefore 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = 0$

Now, $|\vec{a}+\vec{b}+\vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = 50$

$\Rightarrow |\vec{a}+\vec{b}+\vec{c}| = 5\sqrt{2}$

Example 2.24 **Prove that in a tetrahedron if two pairs of opposite edges are perpendicular, then the third pair is also perpendicular.**

Sol. Let $ABCD$ be the tetrahedron and A be at the origin.

Let $\overrightarrow{AB} = \vec{b}$, $\overrightarrow{AC} = \vec{c}$ and $\overrightarrow{AD} = \vec{d}$

Let the edge AB be perpendicular to the opposite edge CD.

$\Rightarrow \overrightarrow{AB}\cdot\overrightarrow{CD} = 0$

$\Rightarrow \vec{b}\cdot(\vec{d} - \vec{c}) = 0$

$\Rightarrow \vec{b}\cdot\vec{d} = \vec{b}\cdot\vec{c}$ (i)

Also let AC be perpendicular to the opposite edge BD. Therefore,

$\overrightarrow{AC}\cdot\overrightarrow{BD} = 0$

$\Rightarrow \vec{c}\cdot(\vec{d} - \vec{b}) = 0$

$\Rightarrow \vec{c}\cdot\vec{d} = \vec{b}\cdot\vec{c}$ (ii)

Now from (i) and (ii), we have

$\Rightarrow \vec{b}\cdot\vec{d} = \vec{c}\cdot\vec{d}$

$\Rightarrow (\vec{c}-\vec{b})\cdot\vec{d} = 0$

$\Rightarrow \overrightarrow{BC}\cdot\overrightarrow{AD} = 0$

$\Rightarrow AD$ is perpendicular to opposite edge BC.

Example 2.25 **In isosceles triangle ABC, $|\overrightarrow{AB}| = |\overrightarrow{BC}| = 8$, a point E divides AB internally in the ratio $1:3$, then find the angle between $\overrightarrow{CE}$ and $\overrightarrow{CA}$ (where $|\overrightarrow{CA}| = 12$).**

Sol.

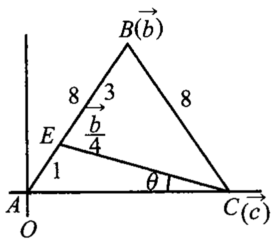


Fig. 2.10

Given $|\vec{c}| = 12$ and $|\vec{b}| = |\vec{b}-\vec{c}| = 8$

$\Rightarrow b^2 = b^2 + c^2 - 2\vec{b}\cdot\vec{c}$

$\Rightarrow \vec{b}\cdot\vec{c} = 72$

$$\cos\theta = \frac{\vec{c}\cdot\left(\vec{c}-\frac{\vec{b}}{4}\right)}{|\vec{c}|\left|\vec{c}-\frac{\vec{b}}{4}\right|} = \frac{\vec{c}\cdot\vec{c}-\frac{\vec{c}\cdot\vec{b}}{4}}{12\left|\vec{c}-\frac{\vec{b}}{4}\right|} = \frac{144-18}{12\left|\vec{c}-\frac{\vec{b}}{4}\right|}$$

$$\text{Now } \left|\vec{c}-\frac{\vec{b}}{4}\right|^2 = |\vec{c}|^2 + \frac{|\vec{b}|^2}{16} - \frac{\vec{b}\cdot\vec{c}}{2} = 144+4-36 = 112$$

$$\Rightarrow \cos\theta = \frac{21}{2\times\sqrt{112}} = \frac{21}{2\times4\sqrt{7}} = \frac{3\sqrt{7}}{8}$$

Example 2.26 **Arc AC of a circle subtends a right angle at the centre O. Point B divides the arc in the ratio $1:2$. If $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$, then calculate $\overrightarrow{OC}$ in terms of $\vec{a}$ and $\vec{b}$.**

Sol. Vector $\vec{c}$ is coplanar with vectors $\vec{a}$ and $\vec{b}$. Therefore, $\vec{c} = x\vec{a} + y\vec{b}$ (i)

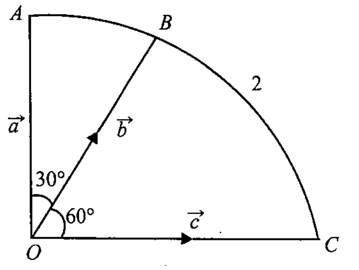


Fig. 2.11

Point B divides arc AC in the ratio 1 : 2 so that $\angle AOB = 30°$ and $\angle BOC = 60°$.

We have to find the values of x and y when we are given $|\vec{a}| = |\vec{b}| = |\vec{c}| = r$ (say).

$\vec{a}\cdot\vec{b} = r^2 \cos 30° = r^2 \dfrac{\sqrt{3}}{2}$ and $\vec{a}\cdot\vec{c} = 0$

$\vec{b}\cdot\vec{c} = r^2 \cos 60° = \dfrac{r^2}{2}$

Multiplying both sides of (i) scalarly by $\vec{c}$ and $\vec{a}$, $\vec{c}\cdot\vec{c} = x\vec{a}\cdot\vec{c} + y\vec{b}\cdot\vec{c}$

and $\vec{c}\cdot\vec{a} = x\vec{a}\cdot\vec{a} + y\vec{b}\cdot\vec{a}$

$r^2 = 0 + \dfrac{r^2}{2}y,\ y = 2$

and $0 = xr^2 + yr^2 \dfrac{\sqrt{3}}{2}$

Putting $y = 2,\ x = -\sqrt{3}$

$\vec{c} = -\sqrt{3}\,\vec{a} + 2\vec{b}$

Example 2.27 **Vector $\overrightarrow{OA} = \hat{i} + 2\hat{j} + 2\hat{k}$ turns through a right angle passing through the positive x-axis on the way. Show that the vector in its new position is $\dfrac{4\hat{i} - \hat{j} - \hat{k}}{\sqrt{2}}$.**

Sol. Let the new vector be $\overrightarrow{OB} = x\hat{i} + y\hat{j} + z\hat{k}$.

According to the given condition, we have

$|\overrightarrow{OB}| = |\overrightarrow{OA}| = 3 \Rightarrow x^2 + y^2 + z^2 = 9$ (i)

Also $\overrightarrow{OA} \perp \overrightarrow{OB} \Rightarrow x + 2y + 2z = 0$ (ii)

Since while turning $\overrightarrow{OA}$, it passes through the positive x-axis on the way,

Vectors $\overrightarrow{OA}, \overrightarrow{OB}$ and $\lambda\hat{i}$ are coplanar.

$$\Rightarrow \begin{vmatrix} x & y & z \\ 1 & 2 & 2 \\ \lambda & 0 & 0 \end{vmatrix} = 0$$

$\Rightarrow y - z = 0$ (iii)

Solving (i), (ii) and (iii) for x, y and z, we have $x = -4y = -4z$

$\Rightarrow 16y^2 + y^2 + y^2 = 9$

$\Rightarrow y = \pm\frac{1}{\sqrt{2}}, z = \pm\frac{1}{\sqrt{2}}$ and $x = \mp 4\frac{1}{\sqrt{2}}$

$\Rightarrow \overrightarrow{OB} = \pm\left(\frac{4}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j} - \frac{1}{\sqrt{2}}\hat{k}\right)$

Since angle between $\overrightarrow{OB}$ and $\hat{i}$ is acute, $\overrightarrow{OB} = \frac{4}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j} - \frac{1}{\sqrt{2}}\hat{k}$

Concept Application Exercise 2.1

1. If $|\vec{a}| = 3, |\vec{b}| = 4$ and the angle between $\vec{a}$ and $\vec{b}$ is 120°, then find the value of $|4\vec{a}+3\vec{b}|$.
2. If vectors $\hat{i}-2x\hat{j}-3y\hat{k}$ and $\hat{i}+3x\hat{j}+2y\hat{k}$ are orthogonal to each other, then find the locus of the point (x, y).
3. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be pairwise mutually perpendicular vectors, such that $|\vec{a}| = 1$, $|\vec{b}| = 2$, $|\vec{c}| = 2$. Then find the length of $\vec{a}+\vec{b}+\vec{c}$.
4. If $\vec{a} + \vec{b} + \vec{c} = 0$, $|\vec{a}| = 3$, $|\vec{b}| = 5$, $|\vec{c}| = 7$, then find the angle between $\vec{a}$ and $\vec{b}$.
5. If the angle between unit vectors $\vec{a}$ and $\vec{b}$ is 60°, then find the value of $|\vec{a} - \vec{b}|$.
6. Let $\vec{u} = \hat{i} + \hat{j}$, $\vec{v} = \hat{i} - \hat{j}$ and $\vec{w} = \hat{i}+2\hat{j}+3\hat{k}$. If $\hat{n}$ is a unit vector such that $\vec{u}\cdot\hat{n} = 0$ and $\vec{v}\cdot\hat{n} = 0$, then find the value of $|\vec{w}\cdot\hat{n}|$.
7. A, B, C, D are any four points, prove that $\overrightarrow{AB}\cdot\overrightarrow{CD} + \overrightarrow{BC}\cdot\overrightarrow{AD} + \overrightarrow{CA}\cdot\overrightarrow{BD} = 0$.
8. $P(1, 0, -1)$, $Q(2, 0, -3)$, $R(-1, 2, 0)$ and $S(3, -2, -1)$, then find the projection length of $\overrightarrow{PQ}$ on $\overrightarrow{RS}$.
9. If the vectors $3\vec{p} + \vec{q}$; $5\vec{p} - 3\vec{q}$ and $2\vec{p}+\vec{q}$; $4\vec{p}-2\vec{q}$ are pairs of mutually perpendicular vectors, then find the angle between vectors $\vec{p}$ and $\vec{q}$.
10. Let $\vec{A}$ and $\vec{B}$ be two non-parallel unit vectors in a plane. If $(\alpha\vec{A} + \vec{B})$ bisects the internal angle between $\vec{A}$ and $\vec{B}$, then find the value of α.
11. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be unit vectors, such that $\vec{a} + \vec{b} + \vec{c} = \vec{x}$, $\vec{a}\cdot\vec{x} = 1$, $\vec{b}\cdot\vec{x} = \frac{3}{2}$, $|\vec{x}| = 2$. Then find the angle between $\vec{c}$ and $\vec{x}$.
12. If $\vec{a}$ and $\vec{b}$ are unit vectors, then find the greatest value of $|\vec{a} + \vec{b}| + |\vec{a} - \vec{b}|$.
13. Constant forces $P_1 = \hat{i} - \hat{j} + \hat{k}$, $P_2 = -\hat{i} + 2\hat{j} - \hat{k}$ and $P_3 = \hat{j} - \hat{k}$ act on a particle at a point A. Determine the work done when particle is displaced from position $A(4\hat{i} - 3\hat{j} - 2\hat{k})$ to $B(6\hat{i} + \hat{j} - 3\hat{k})$

VECTOR (OR CROSS) PRODUCT OF TWO VECTORS

The cross product is just a shorthand invented for the purpose of quickly writing down the angular momentum of an object. Here's how the cross product arises naturally from angular momentum. Recall that if we have a fixed axis and an object distance r away with velocity v and mass m is moving around the axis in a circle, the magnitude of the angular momentum is $m|r||v|$, where $|r|$ is the magnitude of vector r. But what direction should the angular momentum vector point in? Well, if you follow the path of the object, it lies in a plane, an infinite two-dimensional surface. One way to represent a plane is to write down two different vectors that lie in the plane.

Another method used by mathematicians to represent a plane is to write down a single vector that is normal to the plane (normal is a synonym for perpendicular). If a plane is a flat sheet, the normal vector points straight up. Now, for any plane, there are two vectors that are normal to it, since if a vector n is normal to a plane, $-n$ will be normal as well. So how do we determine whether to use n or $-n$?

A long time ago, physicists just made an arbitrary decision known today as the right-hand rule. Given vectors $\vec{a}$ and $\vec{b}$, just curl your fingers from $\vec{a}$ to $\vec{b}$ and the thumb points in the direction of the normal used.

The vector product of two vectors $\vec{a}$ and $\vec{b}$, written as $\vec{a} \times \vec{b}$, is the vector $\vec{c} = |a||b| \sin\theta\, \hat{n}$, where θ is the angle between $\vec{a}$ and $\vec{b}$ $(0 \le \theta \le \pi)$, and $\hat{n}$ is a unit vector along the line perpendicular to both $\vec{a}$ and $\vec{b}$.

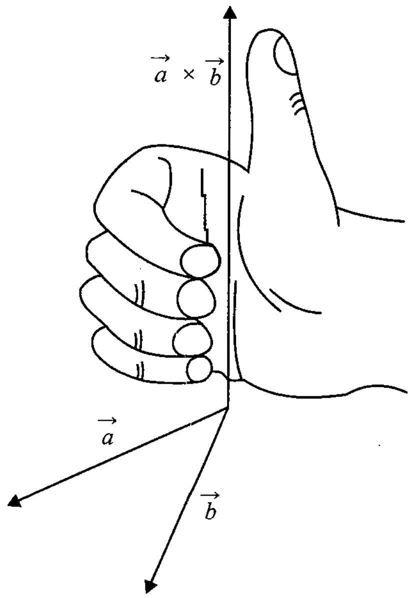


Fig. 2.12

Then direction of $\vec{c}$ is such that $\vec{a}$, $\vec{b}$ and $\vec{c}$ form a right-handed system.

We see that the direction of $\vec{b} \times \vec{a}$ is opposite to that of $\vec{a} \times \vec{b}$ as shown in Fig. 2.13.

$$\vec{b} \times \vec{a} = -\vec{a} \times \vec{b}$$

So the vector product is not commutative. In practice, this means that the order in which we do the calculation does matter.

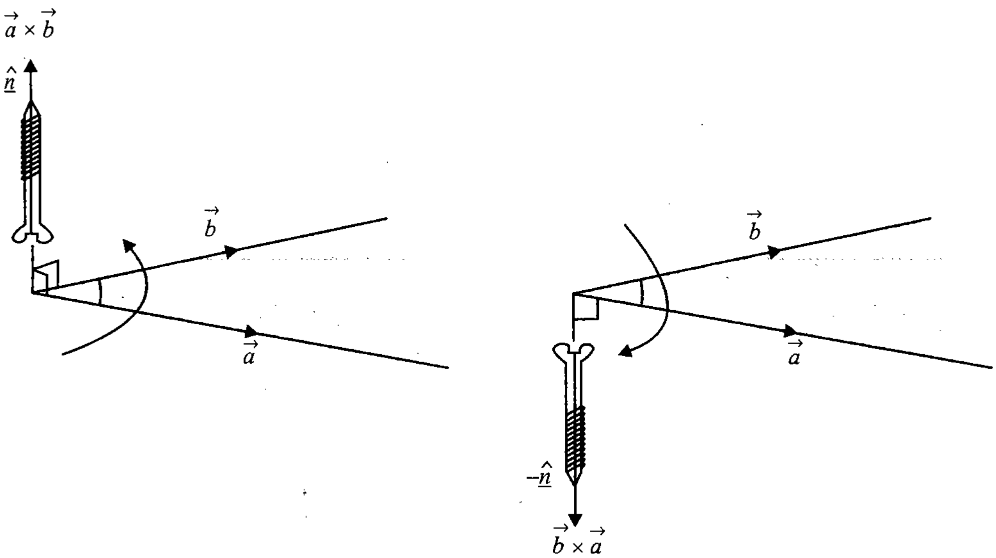


Fig. 2.13

Properties of Cross Product

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
2. $\vec{a} \times \vec{a} = 0$
3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
4. $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$ and $\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{k} = \hat{i}$, $\hat{k} \times \hat{i} = \hat{j}$
5. Two non-zero vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if $\vec{a} \times \vec{b} = \vec{0}$.
6. If $\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$ and $\vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$, then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2)\hat{i} + (a_3 b_1 - a_1 b_3)\hat{j} + (a_1 b_2 - a_2 b_1)\hat{k}$$

7. The unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ is $\frac{(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|}$, and a vector of magnitude λ perpendicular to the plane of $\vec{a}$ and $\vec{b}$ is $\pm \frac{\lambda(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|}$.

Physical Interpretation of Cross Product as a Moment of Force

Moment of force (often just *moment*) is the tendency of a force to twist or rotate an object. This is an important, basic concept in engineering and physics. A moment is valued mathematically as the product of the force and the moment arm. Moment arm is the perpendicular distance from the point of rotation to the *line of action* of the force. The moment may be thought of as a measure of the tendency of the force to cause rotation about an imaginary axis through a point.

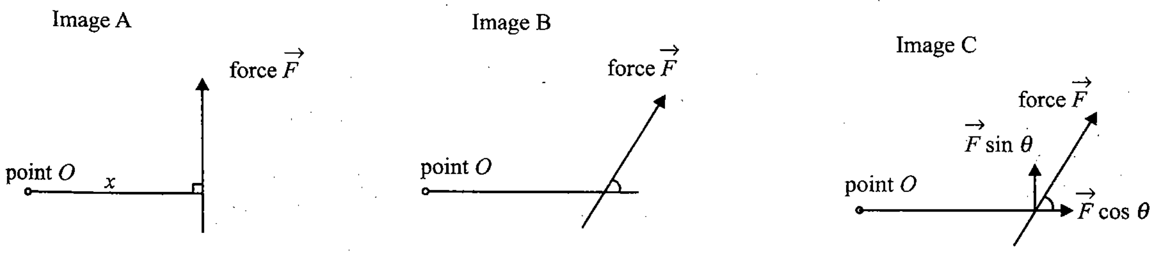


Fig. 2.14

The moment of a force can be calculated about any point and not just the points in which the line of action of the force is perpendicular.

Image A shows the components, the force F and the moment arm x when they are perpendicular to one another. When the force is not perpendicular to the point of interest, such as point O in Images B and C, the magnitude of moment $\vec{M}$ of a vector $\vec{F}$ about point $\boldsymbol{O}$ is

$\vec{M}_O = \vec{r}_{OF} \times \vec{F}$, where $\vec{r}_{OF}$ is the vector from point O to the position where quantity $\boldsymbol{F}$ is applied.

Image C represents the vector components of the force in Image B. In order to determine moment $\vec{M}$ of vector $\vec{F}$ about point $\boldsymbol{O}$, when vector $\vec{F}$ is not perpendicular to point $\boldsymbol{O}$, one must resolve the force $\vec{F}$ into its horizontal and vertical components. The sum of the moments of the two components of F about point O is

$$\vec{M}_{OF} = \vec{F} \sin\theta(x) + \vec{F} \cos\theta(0)$$

The moment arm to the vertical component of $\vec{F}$ is a distance x. The moment arm to the horizontal component of $\vec{F}$ does not exist. There is no rotational force about point O due to the horizontal component of $\vec{F}$. Thus, the moment arm distance is zero.

Thus $\vec{M}$ can be referred to as "moment $\vec{M}$ with respect to the axis that goes through point O", or simply "moment $\vec{M}$ about point O". If O is the origin, or informally, if the axis involved is clear from context, one often omits O and says simply *moment*, rather than *moment about O*. Therefore, the moment about *point O* is indeed the cross product, $\vec{M}_O = \vec{r}_{OF} \times \vec{F}$, since the cross product $= \vec{F} \sin\theta(x)$.

Geometric Interpretation of Cross Product

1.

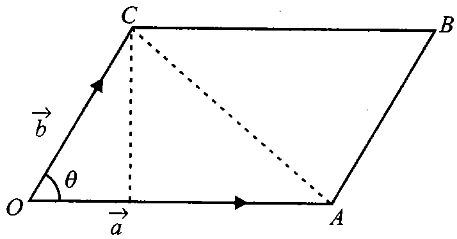


Fig. 2.15

$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta \, \hat{n}$

$\Rightarrow |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$

$= 2\left(\frac{1}{2}|\vec{a}||\vec{b}|\sin\theta\right)$

$= 2$ (Area of triangle AOC)

$=$ Area of parallelogram

Area of the triangle OAB is $\frac{1}{2}|\vec{a} \times \vec{b}|$.

$\vec{a} \times \vec{b}$ is said to be the vector area of the parallelogram with adjacent sides OA and OB.

2. If $\vec{a}, \vec{b}$ are diagonals of a parallelogram, its area $= \frac{1}{2}\ |\vec{a} \times \vec{b}|$.

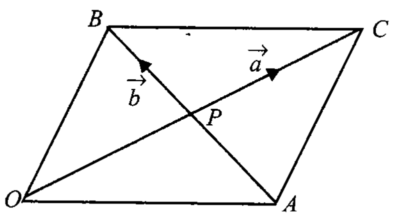


Fig. 2.16

In the above diagram $\overrightarrow{OC} = \vec{a}$ and $\overrightarrow{AB} = \vec{b}$

$\Rightarrow$ Area parallelogram $= 4 \times \frac{1}{2}|\overrightarrow{PC} \times \overrightarrow{PB}|$

$= 4 \times \frac{1}{2}\left|\frac{\vec{a}}{2} \times \frac{\vec{b}}{2}\right|$

$= \frac{1}{2}|\vec{a} \times \vec{b}|$

3. If AC and BD are the diagonals of a quadrilateral, then its vector area is $\frac{1}{2}\overrightarrow{AC}\times\overrightarrow{BD}$.

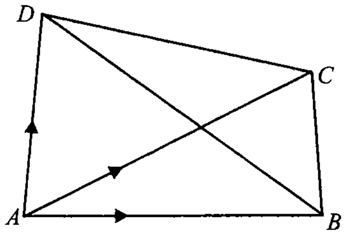


Fig. 2.17

Vector area of the quadrilateral $ABCD$ = vector area of ΔABC + vector area of ΔACD.

$$= \frac{1}{2}\overrightarrow{AB}\times\overrightarrow{AC}+\frac{1}{2}\overrightarrow{AC}\times\overrightarrow{AD}$$

$$= -\frac{1}{2}\overrightarrow{AC}\times\overrightarrow{AB}+\frac{1}{2}\overrightarrow{AC}\times\overrightarrow{AD}$$

$$= \frac{1}{2}\overrightarrow{AC}\times(\overrightarrow{AD}-\overrightarrow{AB})$$

$$= \frac{1}{2}\overrightarrow{AC}\times\overrightarrow{BD}$$

4. The area of a triangle whose vertices are $A(\vec{a})$, $B(\vec{b})$, $C(\vec{c})$ is $\frac{1}{2}|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|$

Area of triangle $= \frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$

$$= \frac{1}{2}|(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})|$$

$$= \frac{1}{2}|\vec{b}\times\vec{c}-\vec{b}\times\vec{a}-\vec{a}\times\vec{c}+\vec{a}\times\vec{a}|$$

$$= \frac{1}{2}|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|$$

Example 2.28 **If A, B and C are the vertices of a triangle ABC, prove sine rule $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$.**

Sol. Let $\overrightarrow{BC}=\vec{a}$, $\overrightarrow{CA}=\vec{b}$, $\overrightarrow{AB}=\vec{c}$, so that $\vec{a}+\vec{b}=-\vec{c}$

$\therefore\quad \vec{a}\times\vec{a}+\vec{a}\times\vec{b}=-\vec{a}\times\vec{c}$

$\vec{0}+\vec{a}\times\vec{b}=\vec{c}\times\vec{a}$

$|\vec{a}\times\vec{b}|=|\vec{c}\times\vec{a}|$

$ab\sin(180° - C) = ca\sin(180° - B)$

$ab\sin C = ca\sin B$

Dividing both sides by abc, we get

$$\frac{\sin C}{c} = \frac{\sin B}{b}$$

$$\therefore \quad \frac{b}{\sin B} = \frac{c}{\sin C} \qquad \text{(i)}$$

Similarly $\dfrac{c}{\sin C} = \dfrac{a}{\sin A}$ (ii)

From (i) and (ii), we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

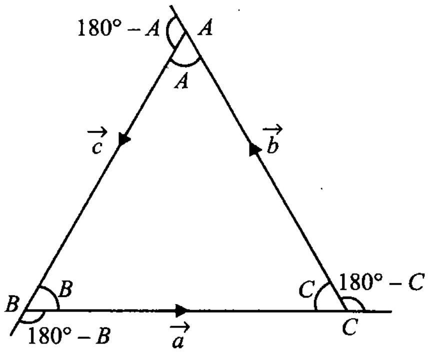


Fig. 2.18

Example 2.29 **Using cross product of vectors, prove that $\sin(A+B) = \sin A \cos B + \cos A \sin B$.**

Sol. Let OP and OQ be unit vectors making angles A and B with X-axis such that $\angle POQ = A + B$

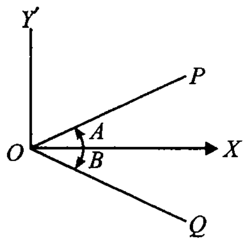


Fig. 2.19

$$\therefore \quad \vec{OP} = \hat{i}\cos A + \hat{j}\sin A$$

$$\vec{OQ} = \hat{i}\cos B - \hat{j}\sin B$$

Now $\vec{OP} \times \vec{OQ}$

$$= (1)(1)\sin(A+B)(-\hat{k})$$

$$= -\sin(A+B)\,\hat{k} \qquad \text{(i)}$$

Also $$\vec{OP} \times \vec{OQ} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos A & \sin A & 0 \\ \cos B & -\sin B & 0 \end{vmatrix}$$

$$= (-\cos A \sin B - \sin A \cos B)\,\hat{k}$$

$$\therefore \quad \vec{OP} \times \vec{OQ} = -(\sin A \cos B + \cos A \sin B)\,\hat{k} \qquad \text{(ii)}$$

From (i) and (ii), we get

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

Example 2.30 **Find a unit vector perpendicular to the plane determined by the points (1, –1, 2), (2, 0, –1) and (0, 2, 1).**

Sol. Given points are $A(1, -1, 2)$, $B(2, 0, -1)$ and $C(0, 2, 1)$

$$\Rightarrow \overline{AB} = \vec{a} = \hat{i} + \hat{j} - 3\hat{k},\ \overline{BC} = \vec{b} = -2\hat{i} + 2\hat{j} + 2\hat{k}$$

$$\therefore \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -3 \\ -2 & 2 & 2 \end{vmatrix} = 8\hat{i} + 4\hat{j} + 4\hat{k}$$

Hence unit vector $= \pm \dfrac{2\hat{i} + \hat{j} + \hat{k}}{\sqrt{6}}$

Example 2.31 **If $\vec{a}$ and $\vec{b}$ are two vectors, then prove that $(\vec{a} \times \vec{b})^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} \end{vmatrix}$**

Sol.
$$(\vec{a}\times\vec{b})^2 = (ab\sin\theta\cdot\hat{n})^2$$
$$= a^2b^2\sin^2\theta$$
$$= a^2b^2 - a^2b^2\cos^2\theta$$
$$= (\vec{a}\cdot\vec{a})\,(\vec{b}\cdot\vec{b}) - (\vec{a}\cdot\vec{b})^2$$
$$= \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} \\ \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{b} \end{vmatrix}$$

Example 2.32 **If $|\vec{a}| = 2$, then find the value of $|\vec{a}\times\hat{i}|^2 + |\vec{a}\times\hat{j}|^2 + |\vec{a}\times\hat{k}|^2$.**

Sol. $|\vec{a}\times\hat{i}|^2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ 1 & 0 & 0 \end{vmatrix}^2$ (since $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$)

$$= |a_3\hat{j} - a_2\hat{k}|^2 = a_3^2 + a_2^2$$

Similarly, $|\vec{a}\times\vec{j}|^2 = a_1^2 + a_3^2$ and $|\vec{a}\times\vec{k}|^2 = a_1^2 + a_2^2$

Hence the required result can be given as $2(a_1^2 + a_2^2 + a_3^2) = 2|\vec{a}|^2 = 8$

Example 2.33 **$\vec{r}\times\vec{a} = \vec{b}\times\vec{a}$; $\vec{r}\times\vec{b} = \vec{a}\times\vec{b}$; $\vec{a} \neq \vec{0}$; $\vec{b} \neq \vec{0}$; $\vec{a} \neq \lambda\vec{b}$, and $\vec{a}$ is not perpendicular to $\vec{b}$, then find $\vec{r}$ in terms of $\vec{a}$ and $\vec{b}$.**

Sol. $\vec{r}\times\vec{a} - \vec{b}\times\vec{a} = 0$ and $\vec{r}\times\vec{b} + \vec{b}\times\vec{a} = 0$

Adding, we get $\vec{r}\times(\vec{a} + \vec{b}) = 0$

But as we are given $\vec{a} \neq \lambda\vec{b}$, therefore

$$\vec{r} = \mu(\vec{a} + \vec{b})$$

Example 2.34 **A, B, C and D are any four points in the space, then prove that $|\overrightarrow{AB}\times\overrightarrow{CD} + \overrightarrow{BC}\times\overrightarrow{AD} + \overrightarrow{CA}\times\overrightarrow{BD}| = 4(\text{area of } \Delta ABC)$.**

Sol. Let P.V. of A, B, C and D be $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{0}$, respectively.

$\Rightarrow \overrightarrow{AB}\times\overrightarrow{CD} = (\vec{b} - \vec{a})\times(-\vec{c})$, $\overrightarrow{BC}\times\overrightarrow{AD} = (\vec{c} - \vec{b})\times(-\vec{a})$ and $\overrightarrow{CA}\times\overrightarrow{BD} = (\vec{a} - \vec{c})\times(-\vec{b})$

$$\overrightarrow{AB}\times\overrightarrow{CD} + \overrightarrow{BC}\times\overrightarrow{AD} + \overrightarrow{CA}\times\overrightarrow{BD} = \vec{c}\times\vec{b} + \vec{a}\times\vec{c} + \vec{a}\times\vec{c} + \vec{b}\times\vec{a} - \vec{a}\times\vec{b} + \vec{c}\times\vec{b}$$
$$= 2(\vec{c}\times\vec{b} + \vec{b}\times\vec{a} + \vec{a}\times\vec{c})$$
$$= 2(\vec{c}\times(\vec{b} - \vec{a}) - \vec{a}\times(\vec{b} - \vec{a}))$$

$$= 2((\vec{c}-\vec{a})\times(\vec{b}-\vec{a}))$$

$$= 2(\overrightarrow{AC}\times\overrightarrow{AB})$$

$$\Rightarrow |\overrightarrow{AB}\times\overrightarrow{CD}+\overrightarrow{BC}\times\overrightarrow{AD}+\overrightarrow{CA}\times\overrightarrow{BD}| = 4\left|\frac{1}{2}(\overrightarrow{AC}\times\overrightarrow{AB})\right| = 4\Delta ABC$$

Example 2.35 **If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are the position vectors of the vertices A, B and C, respectively, of ΔABC, prove that the perpendicular distance of the vertex A from the base BC of the triangle ABC is $\dfrac{|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|}{|\vec{c}-\vec{b}|}$.**

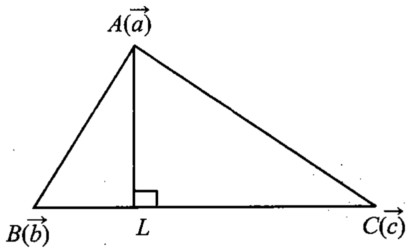


Fig. 2.20

Sol. $|\overrightarrow{BC}\times\overrightarrow{BA}| = |\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|$

$$\Rightarrow |\overrightarrow{BC}||\overrightarrow{BA}|\sin B = |\vec{a}\times\vec{b}\times\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|$$

$$\Rightarrow |\vec{c}-\vec{b}|(AB\sin B) = |\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|$$

Therefore, the length of perpendicular from A on $BC = AL = AB\sin B = \dfrac{|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|}{|\vec{b}-\vec{c}|}$.

Example 2.36 **Find the area of the triangle whose vertices are $A(1,-1,2)$, $B(2,1,-1)$ and $C(3,-1,2)$.**

Sol. Here $\overrightarrow{OA} = \hat{i}-\hat{j}+2\hat{k}$ and $\overrightarrow{OB} = 2\hat{i}+\hat{j}-\hat{k}$ and $\overrightarrow{OC} = 3\hat{i}-\hat{j}+2\hat{k}$

$$\Rightarrow \overrightarrow{AB} = \overrightarrow{OB}-\overrightarrow{OA} = \hat{i}+2\hat{j}-3\hat{k} \text{ and } \overrightarrow{AC} = \overrightarrow{OC}-\overrightarrow{OA} = 2\hat{i}$$

Hence, the required area $= \dfrac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$

Now,
$$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & -3\\ 2 & 0 & 0\end{vmatrix} = -2(3\hat{j}+2\hat{k})$$

$$\Rightarrow \text{Area of triangle} = \frac{1}{2}\times 2|3\hat{j}+2\hat{k}| = \sqrt{13}$$

Example 2.37 **Find the area of a parallelogram whose two adjacent sides are represented by vectors $3\hat{i}-\hat{k}$ and $\hat{i}+2\hat{j}$.**

Sol. The area of parallelogram is given by $= |\overrightarrow{AB}\times\overrightarrow{AD}|$

Here we are given adjacent sides. Therefore,

$$\overrightarrow{AB}\times\overrightarrow{AD} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 0 & -1\\ 1 & 2 & 0\end{vmatrix} = 2\hat{i}-\hat{j}+6\hat{k}$$

Hence the required area is $= |2\hat{i}-\hat{j}+6\hat{k}| = \sqrt{41}$

Example 2.38 **Find the area of a parallelogram whose diagonals are $\vec{a} = 3\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = \hat{i} - 3\hat{j} + 4\hat{k}$.**

Sol. $\Delta = \frac{1}{2}|\vec{a} \times \vec{b}|$

But $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix} = -2\hat{i} - 14\hat{j} - 10\hat{k}$

Hence $\Delta = \frac{1}{2}|\vec{a} \times \vec{b}| = \frac{1}{2}\sqrt{4+196+100} = 5\sqrt{3}$

Example 2.39 **Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three vectors such that $\vec{a} \neq 0$, $|\vec{a}| = |\vec{c}| = 1$, $|\vec{b}| = 4$ and $|\vec{b} \times \vec{c}| = \sqrt{15}$. If $\vec{b} - 2\vec{c} = \lambda\vec{a}$, then find the value of λ.**

Sol. Let the angle between $\vec{b}$ and $\vec{c}$ be α

$|\vec{b} \times \vec{c}| = \sqrt{15}$

$\Rightarrow |\vec{b}||\vec{c}|\sin\alpha = \sqrt{15}$

$\Rightarrow \sin\alpha = \frac{\sqrt{15}}{4}$

$\Rightarrow \cos\alpha = \frac{1}{4}$

$\Rightarrow \vec{b} - 2\vec{c} = \lambda\vec{a}$

$\Rightarrow |\vec{b} - 2\vec{c}|^2 = \lambda^2|\vec{a}|^2$

$\Rightarrow |\vec{b}|^2 + 4|\vec{c}|^2 - 4\cdot\vec{b}\cdot\vec{c} = \lambda^2|\vec{a}|^2$

$\Rightarrow 16 + 4 - 4\{|\vec{b}||\vec{c}|\cos\alpha\} = \lambda^2$

$\Rightarrow 16 + 4 - 4 \times 4 \times 1 \times \frac{1}{4} = \lambda^2$

$\Rightarrow \lambda^2 = 16 \Rightarrow \lambda = \pm 4$

Example 2.40 **Find the moment about (1, – 1, – 1) of the force $3\hat{i} + 4\hat{j} - 5\hat{k}$ acting at (1, 0, – 2).**

Sol. $\vec{F} = 3\hat{i} + 4\hat{j} - 5\hat{k}$

$\vec{PA}$ = P.V. of A – P.V. of P

$= (\hat{i} - 2\hat{j}) - (\hat{i} - \hat{j} - \hat{k})$

$= -\hat{j} + \hat{k}$

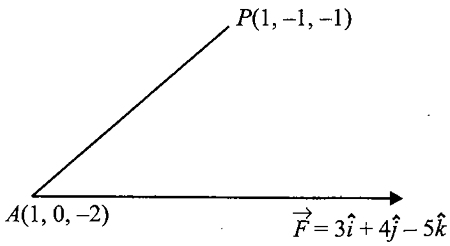


Fig. 2.21

Required vector moment $= \vec{PA} \times \vec{F}$

$= (-\hat{j} + \hat{k}) \times (3\hat{i} + 4\hat{j} - 5\hat{k})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -1 & 1 \\ 3 & 4 & -5 \end{vmatrix}$$

$= \hat{i} + 3\hat{j} + 3\hat{k}$

Example 2.41 **A rigid body is spinning about a fixed point (3, – 2, –1) with an angular velocity of 4 rad/s, the axis of rotation being in the direction of (1, 2, – 2). Find the velocity of the particle at point (4, 1, 1).**

Sol.

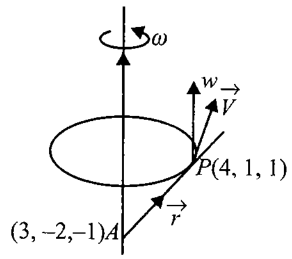


Fig. 2.22

$$\vec{\omega} = 4\left(\frac{\hat{i}+2\hat{j}-2\hat{k}}{\sqrt{1+4+4}}\right) = \frac{4}{3}(\hat{i} + 2\hat{j} - 2\hat{k})$$

$$\vec{r} = \overrightarrow{OP} - \overrightarrow{OA}$$

$$= (4\hat{i} + \hat{j} + \hat{k}) - (3\hat{i} - 2\hat{j} - \hat{k})$$

$$= \hat{i} + 3\hat{j} + 2\hat{k}$$

$$\vec{v} = \vec{\omega} \times \vec{r} = \frac{4}{3}(\hat{i} + 2\hat{j} - 2\hat{k}) \times (\hat{i} + 3\hat{j} + 2\hat{k})$$

$$= \frac{4}{3}(10\hat{i} - 4\hat{j} + \hat{k})$$

Example 2.42 **If $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ and $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$, then show that $\vec{a} - \vec{d}$ is parallel to $\vec{b} - \vec{c}$ provided $\vec{a} \neq \vec{d}$ and $\vec{b} \neq \vec{c}$.**

Sol. We have $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ and $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$ (i)

$\vec{a} - \vec{d}$ will be parallel to $\vec{b} - \vec{c}$

if $(\vec{a} - \vec{d}) \times (\vec{b} - \vec{c}) = \vec{0}$

i.e., if $\vec{a} \times \vec{b} - \vec{a} \times \vec{c} - \vec{d} \times \vec{b} + \vec{d} \times \vec{c} = \vec{0}$

i.e., if $(\vec{a} \times \vec{b} + \vec{d} \times \vec{c}) - (\vec{a} \times \vec{c} + \vec{d} \times \vec{b}) = \vec{0}$

i.e., if $(\vec{a} \times \vec{b} - \vec{c} \times \vec{d}) - (\vec{a} \times \vec{c} - \vec{b} \times \vec{d}) = \vec{0}$

i.e., if $\vec{0} - \vec{0} = \vec{0}$ [from (i)]

i.e., $\vec{0} = \vec{0}$, which is true

Hence the result

Example 2.43 **Show by a numerical example and geometrically also that $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ does not imply $\vec{b} = \vec{c}$.**

Sol. Let $\vec{a} = 3\hat{i} + 2\hat{j} + 5\hat{k}$, $\vec{b} = 6\hat{i} + 5\hat{j} + 8\hat{k}$, $\vec{c} = 3\hat{i} + 3\hat{j} + 3\hat{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 5 \\ 6 & 5 & 8 \end{vmatrix}$$

$$= (16 - 25)\hat{i} - (24 - 30)\hat{j} + (15 - 12)\hat{k}$$

$$= -9\hat{i} + 6\hat{j} + 3\hat{k}$$

$$\vec{a} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 5 \\ 3 & 3 & 3 \end{vmatrix}$$

$$= (6 - 15)\hat{i} - (9 - 15)\hat{j} + (9 - 6)\hat{k} = -9\hat{i} + 6\hat{j} + 3\hat{k}$$

$\therefore \ \vec{a} \times \vec{b} = \vec{a} \times \vec{c}$, but $\vec{b} \neq \vec{c}$.

Geometrically

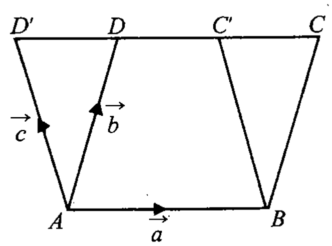


Fig. 2.23

Let $\overrightarrow{AB} = \vec{a}$, $\overrightarrow{AD} = \vec{b}$, $\overrightarrow{AD'} = \vec{c}$

Vector area of parallelogram $ABCD = \vec{a} \times \vec{b}$

Vector area of parallelogram $ABC'D' = \vec{a} \times \vec{c}$

Now vector area of parallelogram $ABCD$ = vector area of parallelogram $ABC'D'$

($\because$ both parallelograms have same base and same height)

$\therefore\ \vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ but $\vec{b} \neq \vec{c}$

Example 2.44 **If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are the position vectors of the vertices of a cyclic quadrilateral $ABCD$, prove that** $\dfrac{|\vec{a} \times \vec{b} + \vec{b} \times \vec{d} + \vec{d} \times \vec{a}|}{(\vec{b} - \vec{a})\cdot(\vec{d} - \vec{a})} + \dfrac{|\vec{b} \times \vec{c} + \vec{c} \times \vec{d} + \vec{d} \times \vec{b}|}{(\vec{b} - \vec{c})\cdot(\vec{d} - \vec{c})} = 0.$

Sol.

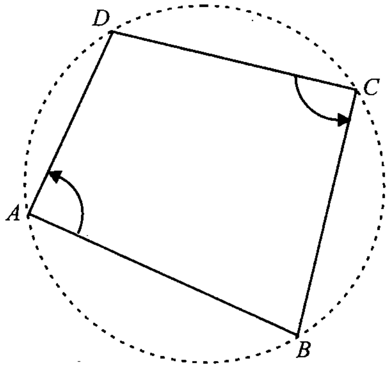


Fig. 2.24

Consider

$$\frac{|\vec{a} \times \vec{b} + \vec{b} \times \vec{d} + \vec{d} \times \vec{a}|}{(\vec{b} - \vec{a})\cdot(\vec{d} - \vec{a})} = \frac{|(\vec{a} - \vec{d}) \times (\vec{b} - \vec{a})|}{(\vec{b} - \vec{a})\cdot(\vec{d} - \vec{a})}$$

$$= \frac{|\vec{a} - \vec{d}||\vec{b} - \vec{a}|\sin A}{|\vec{b} - \vec{a}||\vec{d} - \vec{a}|\cos A}$$

$$= \tan A \qquad \text{(i)}$$

Also $$\frac{|\vec{b} \times \vec{c} + \vec{c} \times \vec{d} + \vec{d} \times \vec{b}|}{(\vec{b} - \vec{c})\cdot(\vec{d} - \vec{c})} = \frac{|(\vec{b} - \vec{c}) \times (\vec{c} - \vec{d})|}{(\vec{b} - \vec{c})\cdot(\vec{d} - \vec{c})}$$

$$= \frac{|\vec{b} - \vec{c}||\vec{c} - \vec{d}|\sin C}{|\vec{b} - \vec{c}|\cdot|\vec{d} - \vec{c}|\cos C}$$

$$= \tan C \qquad \text{(iii)}$$

As cyclic quadrilateral

$A = 180° - C$

$\Rightarrow \tan A = \tan(180° - C)$

$\Rightarrow \tan A + \tan C = 0$

$$\Rightarrow \frac{|\vec{a}\times\vec{b}+\vec{b}\times\vec{d}+\vec{d}\times\vec{a}|}{(\vec{b}-\vec{a})\cdot(\vec{d}-\vec{a})}+\frac{|\vec{b}\times\vec{c}+\vec{c}\times\vec{d}+\vec{d}\times\vec{b}|}{(\vec{b}-\vec{c})\cdot(\vec{d}-\vec{c})}=0$$

Example 2.45 **The position vectors of the vertices of a quadrilateral with A as origin are $B(\vec{b})$, $D(\vec{d})$ and $C(l\vec{b}+m\vec{d})$. Prove that the area of the quadrilateral is $\frac{1}{2}(l+m)|\vec{b}\times\vec{d}|$.**

Sol. Area of quadrilateral is $\frac{1}{2}|\overrightarrow{AC}\times\overrightarrow{BD}| = \frac{1}{2}|(l\vec{b}+m\vec{d})\times(\vec{d}-\vec{b})|$

$$= \frac{1}{2}|l\vec{b}\times\vec{d} - m\vec{d}\times\vec{b}|$$

$$= \frac{1}{2}(l+m)|\vec{b}\times\vec{d}|$$

Example 2.46 **Let $\vec{a}$ and $\vec{b}$ be unit vectors such that $|\vec{a}+\vec{b}| = \sqrt{3}$. Then find the value of $(2\vec{a}+5\vec{b})\cdot(3\vec{a}+\vec{b}+\vec{a}\times\vec{b})$.**

Sol. $(2\vec{a}+5\vec{b})\cdot(3\vec{a}+\vec{b}+\vec{a}\times\vec{b}) = 6\vec{a}\cdot\vec{a}+17\vec{a}\cdot\vec{b}+5\vec{b}\cdot\vec{b}$

($\because \vec{a}\cdot(\vec{a}\times\vec{b}) = \vec{b}\cdot(\vec{a}\times\vec{b}) = 0$, as $\vec{a}$ and $\vec{b}$ are perpendicular to $\vec{a}\times\vec{b}$)

$= 11 + 17\vec{a}\cdot\vec{b}$

Now $|\vec{a}+\vec{b}| = \sqrt{3}$

$\Rightarrow |\vec{a}+\vec{b}|^2 = 3$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}\cdot\vec{b} = 3$

$\Rightarrow \vec{a}\cdot\vec{b} = \frac{1}{2}$

$\Rightarrow (2\vec{a}+5\vec{b})\cdot(3\vec{a}+\vec{b}+\vec{a}\times\vec{b}) = 11+\frac{17}{2} = \frac{39}{2}$

Example 2.47 **$\hat{u}$ and $\hat{v}$ are two non-collinear unit vectors such that $\left|\frac{\hat{u}+\hat{v}}{2}+\hat{u}\times\hat{v}\right| = 1$. Prove that $|\hat{u}\times\hat{v}| = \left|\frac{\hat{u}-\hat{v}}{2}\right|$**

Sol. Given that $\left|\frac{\hat{u}+\hat{v}}{2}+\hat{u}\times\hat{v}\right| = 1$

$$\Rightarrow \left|\frac{\hat{u}+\hat{v}}{2}+\hat{u}\times\hat{v}\right|^2 = 1$$

$$\Rightarrow \frac{2+2\cos\theta}{4}+\sin^2\theta = 1 \quad (\because \hat{u}\cdot(\hat{u}\times\hat{v}) = \hat{v}\cdot(\hat{u}\times\hat{v}) = 0)$$

$$\Rightarrow \cos^2\frac{\theta}{2} = \cos^2\theta$$

$$\Rightarrow \theta = n\pi \pm \frac{\theta}{2},\ n \in Z$$

$$\Rightarrow \theta = \frac{2\pi}{3}$$

$$\Rightarrow |\hat{u}\times\hat{v}| = \sin\frac{2\pi}{3} = \sin\frac{\pi}{3} = \left|\frac{\hat{u}-\hat{v}}{2}\right|$$

Example 2.48 **In triangle ABC, points D, E and F are taken on the sides BC, CA and AB, respectively, such that $\frac{BD}{DC} = \frac{CE}{EA} = \frac{AF}{FB} = n$. Prove that $\Delta_{DEF} = \frac{n^2-n+1}{(n+1)^2}\Delta_{ABC}$.**

Sol. Take A as the origin and let the position vectors of points B and C be $\vec{b}$ and $\vec{c}$, respectively.

Therefore, the position vectors of D, E and F are, respectively, $\frac{n\vec{c}+\vec{b}}{n+1}$, $\frac{\vec{c}}{n+1}$ and $\frac{n\vec{b}}{n+1}$. Therefore,

$$\overrightarrow{ED} = \overrightarrow{AD} - \overrightarrow{AE} = \frac{(n-1)\vec{c}+\vec{b}}{n+1} \text{ and } \overrightarrow{EF} = \frac{n\vec{b}-\vec{c}}{n+1}$$

Now the vector area of $\Delta ABC = \frac{1}{2}(\vec{b}\times\vec{c})$

and the vector area of $\Delta DEF = \frac{1}{2}(\overrightarrow{EF}\times\overrightarrow{ED})$

$$= \frac{1}{2(n+1)^2}[(n\vec{b}-\vec{c})\times\{(n-1)\vec{c}+\vec{b}\}]$$

$$= \frac{1}{2(n+1)^2}[(n^2-n)\,\vec{b}\times\vec{c}+\vec{b}\times\vec{c}]$$

$$= \frac{1}{2(n+1)^2}[(n^2-n+1)(\vec{b}\times\vec{c})] = \frac{n^2-n+1}{(n+1)^2}\Delta_{ABC}$$

Concept Application Exercise 2.2

1. If $\vec{a} = 2\hat{i}+3\hat{j}-5\hat{k}$, $\vec{b} = m\hat{i}+n\hat{j}+12\hat{k}$ and $\vec{a}\times\vec{b} = \vec{0}$, then find (m, n).
2. If $|\vec{a}| = 2, |\vec{b}| = 5$ and $|\vec{a}\times\vec{b}| = 8$, then find the value of $\vec{a}\cdot\vec{b}$.
3. If $\vec{a}\times\vec{b} = \vec{b}\times\vec{c} \neq 0$, where $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar vectors, then for some scalar k prove that $\vec{a}+\vec{c} = k\vec{b}$.
4. If $\vec{a} = 2\vec{i}+3\vec{j}-\vec{k}$, $\vec{b} = -\vec{i}+2\vec{j}-4\vec{k}$ and $\vec{c} = \vec{i}+\vec{j}+\vec{k}$, then find the value of $(\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{c})$.

5. If the vectors $\vec{c}$, $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{b} = \hat{j}$ are such that $\vec{a}, \vec{c}$ and $\vec{b}$ form a right-handed system, then find $\vec{c}$.

6. Given that $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c}$, $\vec{a}\times\vec{b} = \vec{a}\times\vec{c}$ and $\vec{a}$ is not a zero vector. Show that $\vec{b} = \vec{c}$.

7. Show that $(\vec{a} - \vec{b})\times(\vec{a} + \vec{b}) = 2\vec{a}\times\vec{b}$ and give a geometrical interpretation of it.

8. If $\vec{x}$ and $\vec{y}$ are unit vectors and $|\vec{z}| = \frac{2}{\sqrt{7}}$ such that $\vec{z} + \vec{z}\times\vec{x} = \vec{y}$, then find the angle θ between $\vec{x}$ and $\vec{z}$.

9. Prove that $(\vec{a}\cdot\hat{i})(\vec{a}\times\hat{i}) + (\vec{a}\cdot\hat{j})(\vec{a}\times\hat{j}) + (\vec{a}\cdot\hat{k})(\vec{a}\times\hat{k}) = \vec{0}$.

10. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three non-zero vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ and $\lambda\vec{b}\times\vec{a} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a} = \vec{0}$, then find the value of λ.

11. A particle has an angular speed of 3 rad/s and the axis of rotation passes through the points (1, 1, 2) and (1, 2, –2). Find the velocity of the particle at point $P(3, 6, 4)$.

12. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be unit vectors such that $\vec{a}\cdot\vec{b} = 0 = \vec{a}\cdot\vec{c}$. If the angle between $\vec{b}$ and $\vec{c}$ is $\frac{\pi}{6}$, then find $\vec{a}$.

13. If $(\vec{a}\times\vec{b})^2 + (\vec{a}\cdot\vec{b})^2 = 144$ and $|\vec{a}| = 4$, then find the value of $|\vec{b}|$.

14. Given $|\vec{a}| = |\vec{b}| = 1$ and $|\vec{a} + \vec{b}| = \sqrt{3}$. If $\vec{c}$ be a vector such that $\vec{c} - \vec{a} - 2\vec{b} = 3(\vec{a}\times\vec{b})$, then find the value of $\vec{c}\cdot\vec{b}$.

15. Find the moment of $\vec{F}$ about point (2, –1, 3), when force $\vec{F} = 3\hat{i} + 2\hat{j} - 4\hat{k}$ is acting on point (1, –1, 2).

SCALAR TRIPLE PRODUCT

The **scalar triple product** (also called the **mixed** or **box product**) is defined as the *dot product* of one of the vectors with the *cross product* of the other two.

Thus scalar triple product of three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is defined as $\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}$

We denote it by $[\vec{a}\ \vec{b}\ \vec{c}]$

The scalar triple product can be evaluated numerically using any one of the following equivalent characterizations:

$$\left(\vec{a}\times\vec{b}\right)\cdot\vec{c} = \vec{a}\cdot\left(\vec{b}\times\vec{c}\right) = \vec{b}\cdot\left(\vec{c}\times\vec{a}\right) = \vec{c}\cdot\left(\vec{a}\times\vec{b}\right)$$

(The parentheses may be omitted without causing ambiguity, since the *dot product* cannot be evaluated first. If it were, it would leave the cross product of a scalar and a vector, which is not defined.)

i.e., $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}] = -[\vec{b}\ \vec{a}\ \vec{c}] = -[\vec{c}\ \vec{b}\ \vec{a}]$

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}_2$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$, then

$$\left[\vec{a}\,\vec{b}\,\vec{c}\right]=\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}\cdot(c_1\hat{i}+c_2\hat{j}+c_3\hat{k}).$$

$$=\begin{vmatrix}\hat{i}\cdot(c_1\hat{i}+c_2\hat{j}+c_3\hat{k}) & \hat{j}\cdot(c_1\hat{i}+c_2\hat{j}+c_3\hat{k}) & \hat{k}\cdot(c_1\hat{i}+c_2\hat{j}+c_3\hat{k})\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

$$=\begin{vmatrix}c_1 & c_2 & c_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

$$=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

Also $\left[\vec{a}\,\vec{b}\,\vec{c}\right]=\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)=(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\cdot\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$

$$=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

Geometrical Interpretation

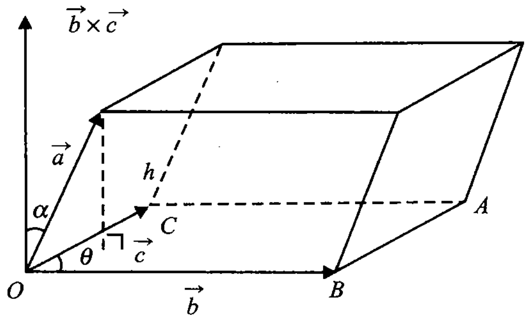


Fig. 2.25

Here $(\vec{a}\times\vec{b})\cdot\vec{c}$ represents (and is equal to) the volume of the parallelepiped whose adjacent sides are represented by the vectors $\vec{a},\vec{b}$ and $\vec{c}$.

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{a}\cdot(bc\sin\theta\,\hat{n})$$

$$= bc\sin\theta\,(\vec{a}\cdot\hat{n})$$
$$= bc\sin\theta\cdot a\cdot 1\cdot\cos\alpha$$
$$= (a\cos\alpha)\,(bc\sin\theta)$$
$$= \text{height}\times(\text{area of base})$$
$$= \text{volume of parallelepiped}$$

Also the volume of the tetrahedron $ABCD$ is equal to $\frac{1}{6}(\overrightarrow{AB}\times\overrightarrow{AC})\cdot\overrightarrow{AD}$

Properties of Scalar Triple Product

1. $(\vec{a}\times\vec{b})\cdot\vec{c} = \vec{a}\cdot(\vec{b}\times\vec{c})$, i.e., position of the dot and the cross can be interchanged without altering the product.
2. $[k\vec{a}\,\vec{b}\,\vec{c}] = k[\vec{a}\,\vec{b}\,\vec{c}]$ (where k is scalar)
3. $[\vec{a}+\vec{b}\;\vec{c}\;\vec{d}] = [\vec{a}\;\vec{c}\;\vec{d}] + [\vec{b}\;\vec{c}\;\vec{d}]$
4. $\vec{a}, \vec{b}$ and $\vec{c}$ in that order form a right-handed system if $[\vec{a}\;\vec{b}\;\vec{c}] > 0$

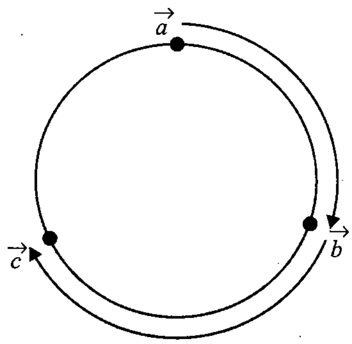


Fig. 2.26

$\vec{a}, \vec{b}$ and $\vec{c}$ in that order form a left-handed system if $[\vec{a}\,\vec{b}\,\vec{c}] < 0$.

5. The necessary and sufficient condition for three non-zero, non-collinear vectors $\vec{a}, \vec{b}$ and $\vec{c}$ to be coplanar is that $[\vec{a}\;\vec{b}\;\vec{c}] = 0$.
6. $[\vec{a}\;\vec{a}\;\vec{b}] = 0$ ($\because$ $\vec{a}$ is $\perp$ to $\vec{a}\times\vec{b}$, $\vec{a}\cdot(\vec{a}\times\vec{b}) = 0$)

Example 2.49 **If $\vec{a}, \vec{b}$ and $\vec{c}$ are three non-coplanar vectors, then find the value of** $\dfrac{\vec{a}\cdot(\vec{b}\times\vec{c})}{\vec{b}\cdot(\vec{c}\times\vec{a})} + \dfrac{\vec{b}\cdot(\vec{c}\times\vec{a})}{\vec{c}\cdot(\vec{a}\times\vec{b})} + \dfrac{\vec{c}\cdot(\vec{b}\times\vec{a})}{\vec{a}\cdot(\vec{b}\times\vec{c})}$.

Sol. Since, $[\vec{a}\;\vec{b}\;\vec{c}] \neq 0$

$$\frac{\vec{a}\cdot(\vec{b}\times\vec{c})}{\vec{b}\cdot(\vec{c}\times\vec{a})}+\frac{\vec{b}\cdot(\vec{c}\times\vec{a})}{\vec{c}\cdot(\vec{a}\times\vec{b})}+\frac{\vec{c}\cdot(\vec{b}\times\vec{a})}{\vec{a}\cdot(\vec{b}\times\vec{c})}=\frac{[\vec{a}\,\vec{b}\,\vec{c}]}{[\vec{b}\,\vec{c}\,\vec{a}]}+\frac{[\vec{b}\,\vec{c}\,\vec{a}]}{[\vec{c}\,\vec{a}\,\vec{b}]}+\frac{[\vec{c}\,\vec{b}\,\vec{a}]}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

$$=\frac{[\vec{a}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]}+\frac{[\vec{a}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]}-\frac{[\vec{a}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

$$=1+1-1=1$$

Example 2.50 **If the vectors $2\hat{i}-3\hat{j}, \hat{i}+\hat{j}-\hat{k}$ and $3\hat{i}-\hat{k}$ form three concurrent edges of a parallelopiped, then find the volume of the parallelopiped.**

Sol. Here, $\overrightarrow{OA}=2\hat{i}-3\hat{j}=\vec{a}$ (say),

$\overrightarrow{OB}=\hat{i}+\hat{j}-\hat{k}=\vec{b}$ (say),

and $\overrightarrow{OC}=3\hat{i}-\hat{k}=\vec{c}$ (say)

Hence, volume is $[\vec{a}\,\vec{b}\,\vec{c}]=\vec{a}\cdot(\vec{b}\times\vec{c})=\begin{vmatrix}2 & -3 & 0\\ 1 & 1 & -1\\ 3 & 0 & -1\end{vmatrix}=4$

Example 2.51 **Prove that $[\vec{a}+\vec{b}\;\; \vec{b}+\vec{c}\;\; \vec{c}+\vec{a}]=2\,[\vec{a}\;\;\vec{b}\;\;\vec{c}]$.**

Sol. $[\vec{a}+\vec{b}\;\;\vec{b}+\vec{c}\;\;\vec{c}+\vec{a}]=(\vec{a}+\vec{b})\cdot((\vec{b}+\vec{c})\times(\vec{c}+\vec{a}))$

$$=(\vec{a}+\vec{b})\cdot(\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a})$$

$$=[\vec{a}\,\vec{b}\,\vec{c}]+[\vec{b}\,\vec{c}\,\vec{a}]=2[\vec{a}\,\vec{b}\,\vec{c}]$$

Example 2.52 **Prove that $[\vec{l}\,\vec{m}\,\vec{n}][\vec{a}\,\vec{b}\,\vec{c}]=\begin{vmatrix}\vec{l}\cdot\vec{a} & \vec{l}\cdot\vec{b} & \vec{l}\cdot\vec{c}\\ \vec{m}\cdot\vec{a} & \vec{m}\cdot\vec{b} & \vec{m}\cdot\vec{c}\\ \vec{n}\cdot\vec{a} & \vec{n}\cdot\vec{b} & \vec{n}\cdot\vec{c}\end{vmatrix}$.**

Sol. Let $\vec{l}=l_1\hat{i}+l_2\hat{j}+l_3\hat{k}$, $\vec{m}=m_1\hat{i}+m_2\hat{j}+m_3\hat{k}$ and $\vec{n}=n_1\hat{i}+n_2\hat{j}+n_3\hat{k}$

$\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$. Therefore,

$\vec{l}\cdot\vec{a}=l_1a_1+l_2a_2+l_3a_3=\Sigma l_1a_1$

Similarly, $\vec{l}\cdot\vec{b}=\Sigma l_1b_1$, etc.

Now $[\vec{l}\,\vec{m}\,\vec{n}][\vec{a}\,\vec{b}\,\vec{c}]=\begin{vmatrix}l_1 & l_2 & l_3\\ m_1 & m_2 & m_3\\ n_1 & n_2 & n_3\end{vmatrix}\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$

$$= \begin{vmatrix} \Sigma l_1 a_1 & \Sigma l_1 b_1 & \Sigma l_1 c_1 \\ \Sigma m_1 a_1 & \Sigma m_1 b_1 & \Sigma m_1 c_1 \\ \Sigma n_1 a_1 & \Sigma n_1 b_1 & \Sigma n_1 c_1 \end{vmatrix}$$

$$= \begin{vmatrix} \vec{l}.\vec{a} & \vec{l}.\vec{b} & \vec{l}.\vec{c} \\ \vec{m}.\vec{a} & \vec{m}.\vec{b} & \vec{m}.\vec{c} \\ \vec{n}.\vec{a} & \vec{n}.\vec{b} & \vec{n}.\vec{c} \end{vmatrix}$$

Example 2.53 **Find the value of a so that the volume of the parallelopiped formed by vectors $\hat{i}+a\hat{j}+\hat{k}, \hat{j}+a\hat{k}$ and $a\hat{i}+\hat{k}$ becomes minimum.**

Sol. $V = \begin{vmatrix} 1 & a & 1 \\ 0 & 1 & a \\ a & 0 & 1 \end{vmatrix} = 1 - a + a^3$

$\Rightarrow \dfrac{dV}{da} = 3a^2 - 1$

Sign scheme for $3a^2 - 1$ is as follows

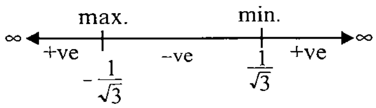


Fig. 2.27

V is minimum at $a = \dfrac{1}{\sqrt{3}}$

Example 2.54 **If $\vec{u}, \vec{v}$ and $\vec{w}$ are three non-coplanar vectors, then prove that $(\vec{u}+\vec{v}-\vec{w})\cdot(\vec{u}-\vec{v})\times(\vec{v}-\vec{w}) = \vec{u}\cdot\vec{v}\times\vec{w}$**

Sol. $(\vec{u}+\vec{v}-\vec{w})\cdot(\vec{u}-\vec{v})\times(\vec{v}-\vec{w}) = (\vec{u}+\vec{v}-\vec{w})\cdot(\vec{u}\times\vec{v}-\vec{u}\times\vec{w}-\vec{v}\times\vec{v}+\vec{v}\times\vec{w})$

$= (\vec{u}+\vec{v}-\vec{w})\cdot(\vec{u}\times\vec{v}-\vec{u}\times\vec{w}+\vec{v}\times\vec{w})$

$= 0 - 0 + \vec{u}\cdot(\vec{v}\times\vec{w}) + 0 - \vec{v}\cdot(\vec{u}\times\vec{w}) + 0 - \vec{w}\cdot(\vec{u}\times\vec{v}) + 0 - 0$

$= [\vec{u}\ \vec{v}\ \vec{w}] + [\vec{v}\ \vec{w}\ \vec{u}] - [\vec{w}\ \vec{u}\ \vec{v}] = \vec{u}\cdot(\vec{v}\times\vec{w})$

Example 2.55 **If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a}\times\vec{b}| = 2$, then find the value of $[\vec{a}\ \vec{b}\ \vec{a}\times\vec{b}]$.**

Sol. $[\vec{a}\ \vec{b}\ \vec{a}\times\vec{b}] = (\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{b})$

$= |\vec{a}\times\vec{b}|^2$

$= 4$

Example 2.56 **Find the altitude of a parallelopiped whose three coterminous edges are vectors $\vec{A}=\hat{i}+\hat{j}+\hat{k}$, $\vec{B}=2\hat{i}+4\hat{j}-\hat{k}$ and $\vec{C}=\hat{i}+\hat{j}+3\hat{k}$ with $\vec{A}$ and $\vec{B}$ as the sides of the base of the parallelopiped.**

Sol. $h=\dfrac{\text{volume of parallelopiped}}{\text{area of base}}$

$$=\frac{[\vec{A}\,\vec{B}\,\vec{C}]}{|\vec{A}\times\vec{B}|}=\frac{\begin{vmatrix}1 & 1 & 1\\ 2 & 4 & -1\\ 1 & 1 & 3\end{vmatrix}}{\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & 1\\ 2 & 4 & -1\end{vmatrix}}=\frac{4}{|-5\hat{i}+3\hat{j}+2\hat{k}|}=\frac{2\sqrt{38}}{19}$$

Example 2.57 **If $\vec{a},\vec{b}$ and $\vec{c}$ are mutually perpendicular vectors and $\vec{a}=\alpha(\vec{a}\times\vec{b})+\beta(\vec{b}\times\vec{c})+\gamma(\vec{c}\times\vec{a})$ and $[\vec{a}\,\vec{b}\,\vec{c}]=1$, then find the value of $\alpha+\beta+\gamma$.**

Sol. Taking dot product with $\vec{a},\vec{b}$ and $\vec{c}$, respectively, we get

$|\vec{a}|^2=\beta\cdot[\vec{a}\,\vec{b}\,\vec{c}]=\beta$

$0=\gamma\cdot[\vec{a}\,\vec{b}\,\vec{c}]=\gamma$

and $0=\alpha\cdot[\vec{a}\,\vec{b}\,\vec{c}]=\alpha$

$\therefore\ \alpha+\beta+\gamma=|\vec{a}|^2$

Example 2.58 **If $\vec{a},\vec{b}$ and $\vec{c}$ are non-coplanar vectors, then prove that $|(\vec{a}\cdot\vec{d})(\vec{b}\times\vec{c})+(\vec{b}\cdot\vec{d})(\vec{c}\times\vec{a})+(\vec{c}\cdot\vec{d})(\vec{a}\times\vec{b})|$ is independent of $\vec{d}$, where $\vec{d}$ is a unit vector.**

Sol. Given $[\vec{a}\,\vec{b}\,\vec{c}]\neq 0$ as $\vec{a},\vec{b},\vec{c}$ are non-coplanar. Also there does not exist any linear relation between them because if any such relation exists, then they would be coplanar.

Let $A=x(\vec{b}\times\vec{c})+y(\vec{c}\times\vec{a})+z(\vec{a}\times\vec{b})$,

where $x=\vec{a}\cdot\vec{d}$, $y=\vec{b}\cdot\vec{d}$, $z=\vec{c}\cdot\vec{d}$

We have to find the value of modulus of $\vec{A}$, i.e., $|\vec{A}|$, which is independent of $\vec{d}$.

Multiplying both sides scalarly by $\vec{a},\vec{b}$ and $\vec{c}$ and we know that scalar triple product is zero when two vectors are equal.

$\vec{A}\cdot\vec{a}=x[\vec{a}\,\vec{b}\,\vec{c}]+0$

Putting for x, we get

$(\vec{a}\cdot\vec{d})[\vec{a}\,\vec{b}\,\vec{c}]=\vec{A}\cdot\vec{a}$

Similarly, we have

$$(\vec{b}\cdot\vec{d})[\vec{a}\ \vec{b}\ \vec{c}]=\vec{A}\cdot\vec{b}$$

$$(\vec{c}\cdot\vec{d}][\vec{a}\ \vec{b}\ \vec{c}]=\vec{A}\cdot\vec{c}$$

Adding the above relations, we get

$$[(\vec{a}+\vec{b}+\vec{c})\cdot\vec{d}]\ [\vec{a}\ \vec{b}\ \vec{c}\,]=\vec{A}\cdot(\vec{a}+\vec{b}+\vec{c})$$

or $(\vec{a}+\vec{b}+\vec{c})\cdot[\vec{d}\,[\vec{a}\ \vec{b}\ \vec{c}]-\vec{A}\,]=0$

Since $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar, $\vec{a}+\vec{b}+\vec{c}\ \neq 0$ because otherwise any one is expressible as a linear combination of other two.

Hence $[\vec{a}\ \vec{b}\ \vec{c}]\,\vec{d}=\vec{A}$

$|\vec{A}|=|[\vec{a}\ \vec{b}\ \vec{c}]|$ as $\vec{d}$ is a unit vector.

It is independent of $\vec{d}$.

Example 2.59 **Prove that vectors**

$$\vec{u}=(al+a_1l_1)\hat{i}+(am+a_1m_1)\hat{j}+(an+a_1n_1)\hat{k}$$

$$\vec{v}=(bl+b_1l_1)\hat{i}+(bm+b_1m_1)\hat{j}+(bn+b_1n_1)\hat{k}$$

$$\vec{w}=(cl+c_1l_1)\hat{i}+(cm+c_1m_1)\hat{j}+(cn+c_1n_1)\hat{k}$$

are coplanar.

Sol.

$$[\vec{u}\ \vec{v}\ \vec{w}]=\begin{vmatrix} al+a_1l_1 & am+a_1m_1 & an+a_1n_1 \\ bl+b_1l_1 & bm+b_1m_1 & bn+b_1n_1 \\ cl+c_1l_1 & cm+c_1m_1 & cn+c_1n_1 \end{vmatrix}$$

$$\Rightarrow [\vec{u}\ \vec{v}\ \vec{w}]=\begin{vmatrix} a & a_1 & 0 \\ b & b_1 & 0 \\ c & c_1 & 0 \end{vmatrix}\begin{vmatrix} l & l_1 & 0 \\ m & m_1 & 0 \\ n & n_1 & 0 \end{vmatrix}=0$$

Therefore, the given vectors are coplanar.

Example 2.60 **Let G_1, G_2 and G_3 be the centroids of the trianglular faces *OBC*, *OCA* and *OAB*, respectively, of a tetrahedron *OABC*. If V_1 denotes the volume of the tetrahedron *OABC* and V_2 that of the parallelopiped with OG_1, OG_2 and OG_3 as three concurrent edges, then prove that $4V_1=9V_1$.**

Sol. Taking O as the origin, let the position vectors of A, B and C be $\vec{a}$, $\vec{b}$ and $\vec{c}$, respectively. Then the position vectors G_1, G_2 and G_3 are $\frac{\vec{b}+\vec{c}}{3}, \frac{\vec{c}+\vec{a}}{3}$ and $\frac{\vec{a}+\vec{b}}{3}$, respectively. Therefore,

$$V_1=\frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]\text{ and }V_2=[\overrightarrow{OG_1}\ \overrightarrow{OG_2}\ \overrightarrow{OG_3}]$$

Now, $V_2 = [\overrightarrow{OG_1}\ \overrightarrow{OG_2}\ \overrightarrow{OG_3}]$

$$\Rightarrow V_2 = \frac{1}{27}[\vec{b}+\vec{c}\ \ \vec{c}+\vec{a}\ \ \vec{a}+\vec{b}]$$

$$\Rightarrow V_2 = \frac{2}{27}[\vec{a}\ \vec{b}\ \vec{c}]$$

$$\Rightarrow V_2 = \frac{2}{27} \times 6V_1 \Rightarrow 9V_2 = 4V_1$$

VECTOR TRIPLE PRODUCT

The vector triple product of three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is the vector

$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$

Also $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{b}\cdot\vec{c})\vec{a}$

In general, $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$

If $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$, then the vectors $\vec{a}$ and $\vec{c}$ are collinear.

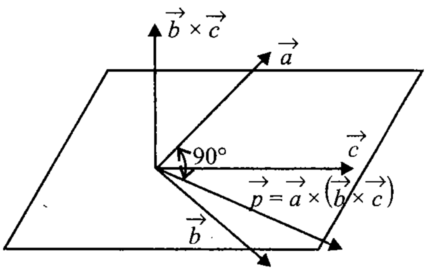


Fig. 2.28

$\vec{p} = \vec{a} \times (\vec{b} \times \vec{c})$ is a vector perpendicular to $\vec{a}$ and $\vec{b} \times \vec{c}$, but $\vec{b} \times \vec{c}$ is a vector perpendicular to the plane of $\vec{b}$ and $\vec{c}$.

$\Rightarrow$ Vector $\vec{p}$ must lie in the plane of $\vec{b}$ and $\vec{c}$.

$\Rightarrow\ \vec{p} = \vec{a} \times (\vec{b} \times \vec{c}) = x\vec{b} + y\vec{c}$ (i)

Multiplying (i) scalarly by $\vec{a}$, we have $\vec{p}\cdot\vec{a} = x(\vec{a}\cdot\vec{b}) + y(\vec{a}\cdot\vec{c})$ (ii)

But $\vec{p} \perp \vec{a} \Rightarrow \vec{p}\cdot\vec{a} = 0$. Therefore,

$x(\vec{a}\cdot\vec{b}) = -y(\vec{c}\cdot\vec{a})$, i.e., $= \dfrac{x}{\vec{c}\cdot\vec{a}} = \dfrac{-y}{\vec{a}\cdot\vec{b}} = \lambda$

$\therefore x = \lambda(\vec{c}\cdot\vec{a}),\ y = -\lambda(\vec{a}\cdot\vec{b})$ (iii)

Substituting x and y from (iii) in (i), $\vec{a} \times (\vec{b} \times \vec{c}) = \lambda[(\vec{c}\cdot\vec{a})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}]$ (iv)

The simplest way to determine λ is by taking specific vectors $\vec{a} = \hat{i}, \vec{b} = \hat{i}, \vec{c} = \hat{j}$

We have from (iv), $\hat{i}\times(\hat{i}\times\hat{j})=\lambda[(\hat{i}\cdot\hat{j})\hat{i}-(\hat{i}\cdot\hat{i})\hat{j}]$, i.e., $\hat{i}\times\hat{k}=\lambda[0\hat{i}-1\hat{j}]$, i.e., $-\hat{j}=-\lambda\hat{j}$

$\therefore \lambda=1$

Substituting λ in (iv), $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$

Lagrange's Identity

$$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})=\vec{a}\cdot[\vec{b}\times(\vec{c}\times\vec{d})]$$

$$=\vec{a}\cdot[(\vec{b}\cdot\vec{d})\vec{c}-(\vec{b}\cdot\vec{c})\vec{d}]$$

$$=(\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d})-(\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$$

$$=\begin{vmatrix}\vec{a}\cdot\vec{c} & \vec{a}\cdot\vec{d}\\ \vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{d}\end{vmatrix}$$

This is called Lagrange's identity.

Note:

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=[(\vec{a}\times\vec{b})\cdot\vec{d}]\vec{c}-[(\vec{a}\times\vec{b})\cdot\vec{c}]\vec{d}=[\vec{a}\ \vec{b}\ \vec{d}]\vec{c}-[\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$$

Thus vector $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$ *lies in the plane of* $\vec{c}$ *and* $\vec{d}$; *otherwise*

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=-(\vec{c}\times\vec{d})\times(\vec{a}\times\vec{b})=-[(\vec{c}\times\vec{d})\cdot\vec{b}]\vec{a}+[(\vec{c}\times\vec{d})\vec{a}]\vec{b}$$

which shows that the vector lies in the plane of $\vec{a}$ and $\vec{b}$. *Thus the vector lies along the common section of the plane of* $\vec{c}$ and $\vec{d}$ *and the plane of* $\vec{a}$ and $\vec{b}$.

Example 2.61 **Prove that $\hat{i}\times(\vec{a}\times\hat{i})+\hat{j}\times(\vec{a}\times\hat{j})+\hat{k}\times(\vec{a}\times\hat{k})=2\vec{a}$.**

Sol. $\hat{i}\times(\vec{a}\times\hat{i})=(\hat{i}\cdot\hat{i})\vec{a}-(\vec{a}\cdot\hat{i})\hat{i}=\vec{a}-(\vec{a}\cdot\hat{i})\hat{i}$

Similarly, $\hat{j}\times(\vec{a}\times\hat{j})=\vec{a}-(\vec{a}\cdot\hat{j})\hat{j}$ and $\hat{k}\times(\vec{a}\times\hat{k})=\vec{a}-(\vec{a}\cdot\hat{k})\hat{k}$. Therefore,

$$\hat{i}\times(\vec{a}\times\hat{i})+\hat{j}\times(\vec{a}\times\hat{j})+\hat{k}\times(\vec{a}\times\hat{k})=3\vec{a}-((\vec{a}\cdot\hat{i})\hat{i}+(\vec{a}\cdot\hat{j})\hat{j}+(\vec{a}\times\hat{k})\hat{k})=2\vec{a}$$

Example 2.62 **Let $\vec{a},\vec{b}$ and $\vec{c}$ be any three vectors, then prove that $[\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}]=[\vec{a}\ \vec{b}\ \vec{c}]^2$.**

Sol. $[\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}]=(\vec{a}\times\vec{b})\cdot((\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a}))$

$$=(\vec{a}\times\vec{b})\cdot[[\vec{b}\ \vec{c}\ \vec{a}]\vec{c}-[\vec{b}\ \vec{c}\ \vec{c}]\vec{a}]$$

$$=[\vec{a}\ \vec{b}\ \vec{c}]^2$$

Example 2.63 **For any four vectors, prove that** $(\vec{b}\times\vec{c})\cdot(\vec{a}\times\vec{d})+(\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{d})+(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})=0.$

Sol. $(\vec{b}\times\vec{c})\cdot(\vec{a}\times\vec{d}) = (\vec{b}\cdot\vec{a})(\vec{c}\cdot\vec{d})-(\vec{b}\cdot\vec{d})(\vec{c}\cdot\vec{a})$

$(\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{d}) = (\vec{c}\cdot\vec{b})(\vec{a}\cdot\vec{d})-(\vec{c}\cdot\vec{d})(\vec{a}\cdot\vec{b})$

$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d})-(\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$

$\Rightarrow (\vec{b}\times\vec{c})\cdot(\vec{a}\times\vec{d})+(\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{d})+(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})=0$

Example 2.64 **Let $\hat{a},\hat{b}$ and $\hat{c}$ be the non-coplanar unit vectors. The angle between $\hat{b}$ and $\hat{c}$ is α, between $\hat{c}$ and $\hat{a}$ is β and between $\hat{a}$ and $\hat{b}$ is γ. If $A(\hat{a}\cos\alpha), B(\hat{b}\cos\beta)$ and $C(\hat{c}\cos\gamma)$, then show that in triangle ABC,** $\dfrac{|\hat{a}\times(\hat{b}\times\hat{c})|}{\sin A}=\dfrac{|\hat{b}\times(\hat{c}\times\hat{a})|}{\sin B}=\dfrac{|\hat{c}\times(\hat{a}\times\hat{b})|}{\sin C}$

$$=\frac{\prod|\hat{a}\times(\hat{b}\times\hat{c})|}{|\sum\sin\alpha.\cos\beta.\cos\gamma\,\hat{n}_1|}, \text{ where } \hat{n}_1=\frac{\hat{b}\times\hat{c}}{|\hat{b}\times\hat{c}|},\ \hat{n}_2=\frac{\hat{c}\times\hat{a}}{|\hat{c}\times\hat{a}|} \text{ and } \hat{n}_3=\frac{\hat{a}\times\hat{b}}{|\hat{a}\times\hat{b}|}.$$

Sol. From the sine rule, we get

$$\frac{AB}{\sin C}=\frac{AC}{\sin B}=\frac{BC}{\sin A}=\frac{(AB)(BC)(CA)}{2\Delta ABC}$$

$BC=|\overrightarrow{BC}|=|\hat{c}\cos\gamma-\hat{b}\cos\beta|=|(\hat{a}\cdot\hat{b})\hat{c}-(\hat{c}\cdot\hat{a})\hat{b}|=|(\hat{a}\times(\hat{b}\times\hat{c}))|$

Similarly,

$AC=|\overrightarrow{AC}|=|\hat{b}\times(\hat{c}\times\hat{a})|$ and $AB=|\overrightarrow{AB}|=|\hat{c}\times(\hat{a}\times\hat{b})|$

Also,

$$\Delta ABC=\frac{1}{2}|\overrightarrow{BC}\times\overrightarrow{BA}|$$

$$=\frac{1}{2}|(\hat{c}\cos\gamma-\hat{b}\cos\beta)\times(\hat{a}\cos\alpha-\hat{b}\cos\beta)|$$

$$=\frac{1}{2}|(\hat{c}\times\hat{a})\cos\alpha\cos\gamma+(\hat{b}\times\hat{c})\cos\gamma\cos\beta+(\hat{a}\times\hat{b})\cos\beta\cos\alpha|$$

$\Rightarrow 2\Delta ABC=|\Sigma\hat{n}_1\sin\alpha\cos\beta\cos\gamma|$

$$\Rightarrow \frac{|\hat{a}\times(\hat{b}\times\hat{c})|}{\sin A}=\frac{|\hat{b}\times(\hat{c}\times\hat{a})|}{\sin B}=\frac{|\hat{c}\times(\hat{a}\times\hat{b})|}{\sin C}=\frac{\prod|\hat{a}\times(\hat{b}\times\hat{c})|}{|\Sigma\sin\alpha\cos\beta\cos\gamma\,\hat{n}_1|}$$

Example 2.65 **If $\vec{a},\vec{b}$ and $\vec{c}$ are three non-coplanar vectors, then prove that**

$$\vec{d}=\frac{\vec{a}\cdot\vec{d}}{[\vec{a}\ \vec{b}\ \vec{c}]}(\vec{b}\times\vec{c})+\frac{\vec{b}\cdot\vec{d}}{[\vec{a}\ \vec{b}\ \vec{c}]}(\vec{c}\times\vec{a})+\frac{\vec{c}\cdot\vec{d}}{[\vec{a}\ \vec{b}\ \vec{c}]}(\vec{a}\times\vec{b})$$

Sol. Since $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar, vectors $\vec{a}\times\vec{b}$, $\vec{b}\times\vec{c}$ and $\vec{c}\times\vec{a}$ are also non-coplanar. Let

$\vec{d} = l(\vec{b}\times\vec{c}) + m(\vec{c}\times\vec{a}) + n(\vec{a}\times\vec{b})$ (i)

Now multiplying both sides of (i) scalarly by $\vec{a}$, we have

$\vec{a}\cdot\vec{d} = l\vec{a}\cdot(\vec{b}\times\vec{c}) + m\vec{a}\cdot(\vec{c}\times\vec{a}) + n\vec{a}\cdot(\vec{a}\times\vec{b}) = l[\vec{a}\,\vec{b}\,\vec{c}]$ $\quad\because [\vec{a}\,\vec{c}\,\vec{a}] = 0 = [\vec{a}\,\vec{a}\,\vec{b}]$

$\Rightarrow l = (\vec{a}\cdot\vec{d})/[\vec{a}\,\vec{b}\,\vec{c}]$

Similarly, multiplying (i) scalarly by $\vec{b}$ and $\vec{c}$ successively, we get

$m = (\vec{b}\cdot\vec{d})/[\vec{a}\,\vec{b}\,\vec{c}]$ and $n = (\vec{c}\cdot\vec{d})/[\vec{a}\,\vec{b}\,\vec{c}]$

Putting these values of l, m and n in (i), we get the required relation.

Example 2.66 **If $\vec{b}$ is not perpendicular to $\vec{c}$, then find the vector $\vec{r}$ satisfying the equation $\vec{r}\times\vec{b} = \vec{a}\times\vec{b}$ and $\vec{r}\cdot\vec{c} = 0$.**

Sol. Given $\vec{r}\times\vec{b} = \vec{a}\times\vec{b} \Rightarrow (\vec{r}-\vec{a})\times\vec{b} = 0$

Hence $(\vec{r}-\vec{a})$ and $\vec{b}$ are parallel.

$\Rightarrow \vec{r}-\vec{a} = t\vec{b}$ (i)

Also $\vec{r}\cdot\vec{c} = 0$

$\therefore$ Taking dot product of (i) by $\vec{c}$, we get $\vec{r}\cdot\vec{c} - \vec{a}\cdot\vec{c} = t(\vec{b}\cdot\vec{c})$

$\Rightarrow 0 - \vec{a}\cdot\vec{c} = t(\vec{b}\cdot\vec{c})$ or $t = -\left(\dfrac{\vec{a}\cdot\vec{c}}{\vec{b}\cdot\vec{c}}\right)$ (ii)

From (i) and (ii), solution of $\vec{r}$ is $\vec{r} = \vec{a} - \left(\dfrac{\vec{a}\cdot\vec{c}}{\vec{b}\cdot\vec{c}}\right)\vec{b}$

Example 2.67 **If $\vec{a}$ and $\vec{b}$ are two given vectors and k is any scalar, then find the vector $\vec{r}$ satisfying $\vec{r}\times\vec{a} + k\vec{r} = \vec{b}$**

Sol. $\vec{r}\times\vec{a} + k\vec{r} = \vec{b}$ (i)

$\Rightarrow (\vec{r}\times\vec{a})\times\vec{a} + k\vec{r}\times\vec{a} = \vec{b}\times\vec{a}$

$\Rightarrow (\vec{r}\cdot\vec{a})\vec{a} - (\vec{a}\cdot\vec{a})\vec{r} + k(\vec{b} - k\vec{r}) = \vec{b}\times\vec{a}$

$\Rightarrow (\vec{r}\cdot\vec{a})\vec{a} + k\vec{b} - \vec{b}\times\vec{a} = (|\vec{a}|^2 + k^2)\vec{r}$

$$\Rightarrow \vec{r} = \frac{(\vec{r}\cdot\vec{a})\vec{a} + k\vec{b} - \vec{b}\times\vec{a}}{|\vec{a}|^2 + k^2}$$

Also in Eq. (i), taking dot product with $\vec{a}$, we have

$$(\vec{r}\times\vec{a})\cdot\vec{a}+k\,\vec{r}\cdot\vec{a}=\vec{b}\cdot\vec{a}$$

$$\Rightarrow \vec{r}\cdot\vec{a}=\frac{\vec{b}\cdot\vec{a}}{k}$$

$$\Rightarrow \vec{r}=\frac{1}{k^2+|\vec{a}|^2}\left[\frac{(\vec{a}\cdot\vec{b})\vec{a}}{k}+k\vec{b}+(\vec{a}\times\vec{b})\right]$$

Example 2.68 **If $\vec{r}\cdot\vec{a}=0$, $\vec{r}\cdot\vec{b}=1$ and $[\vec{r}\ \vec{a}\ \vec{b}]=1$, $\vec{a}\cdot\vec{b}\neq 0$, $(\vec{a}\cdot\vec{b})^2-|\vec{a}|^2|\vec{b}|^2=1$, then find $\vec{r}$ in terms of $\vec{a}$ and $\vec{b}$.**

Sol. Writing $\vec{r}$ as linear combination of $\vec{a}, \vec{b}$ and $\vec{a}\times\vec{b}$, we have

$$\vec{r}=x\vec{a}+y\vec{b}+z(\vec{a}\times\vec{b})$$

For scalars x, y and z

$0=\vec{r}\cdot\vec{a}=x|\vec{a}|^2+y\vec{a}\cdot\vec{b}$ (taking dot product with $\vec{a}$)

$1=\vec{r}\cdot\vec{b}=x\vec{a}\cdot\vec{b}+y|\vec{b}|^2$ (taking dot product with $\vec{b}$)

Solving, we get $y=\dfrac{|\vec{a}|^2}{|\vec{a}|^2|\vec{b}|^2-(\vec{a}\cdot\vec{b})^2}=|\vec{a}|^2$

and $x=\dfrac{\vec{a}\cdot\vec{b}}{(\vec{a}\cdot\vec{b})^2-|\vec{a}|^2|\vec{b}|^2}=\vec{a}\cdot\vec{b}$

Also $1=[\vec{r}\,\vec{a}\,\vec{b}]=z|\vec{a}\times\vec{b}|^2$ (taking dot product with $\vec{a}\times\vec{b}$)

$$\Rightarrow z=\frac{1}{|a\times b|^2}$$

thus $\vec{r}=((\vec{a}\cdot\vec{b})\vec{a}-|\vec{a}|^2\,\vec{b})+\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|^2}$

$$=\vec{a}\times(\vec{a}\times\vec{b})+\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|^2}$$

Example 2.69 **If vector $\vec{x}$ satisfying $\vec{x}\times\vec{a}+(\vec{x}\cdot\vec{b})\,\vec{c}=\vec{d}$ is given by $\vec{x}=\lambda\,\vec{a}+\vec{a}\times\dfrac{\vec{a}\times(\vec{d}\times\vec{c})}{(\vec{a}\cdot\vec{c})|\vec{a}|^2}$, then find the value of λ.**

Sol. $\vec{x}\times\vec{a}+(\vec{x}\cdot\vec{b})\vec{c}=\vec{d}$

$\therefore \{\vec{x}\times\vec{a}+(\vec{x}\cdot\vec{b})\,\vec{c}\}\times\vec{c}=\vec{d}\times\vec{c}$

$\Rightarrow (\vec{x}\times\vec{a})\times\vec{c}+(\vec{x}\cdot\vec{b})(\vec{c}\times\vec{c})=\vec{d}\times\vec{c}$

$\Rightarrow (\vec{x}\cdot\vec{c})\,\vec{a}-(\vec{a}\cdot\vec{c})\,\vec{x}=(\vec{d}\times\vec{c})$

$\Rightarrow \vec{a}\times\{(\vec{x}\cdot\vec{c})\vec{a}-(\vec{a}\cdot\vec{c})\,\vec{x}\}=\vec{a}\times(\vec{d}\times\vec{c})$

$\Rightarrow -(\vec{a}\cdot\vec{c})\,(\vec{a}\times\vec{x})=\vec{a}\times(\vec{d}\times\vec{c})\quad (\because \vec{a}\times\vec{a}=0)$

$$\Rightarrow \vec{x}\times\vec{a}=\frac{\vec{a}\times(\vec{d}\times\vec{c})}{\vec{a}\cdot\vec{c}}$$

$$\Rightarrow \vec{a}\times(\vec{x}\times\vec{a})=\vec{a}\times\frac{\vec{a}\times(\vec{d}\times\vec{c})}{\vec{a}\cdot\vec{c}}$$

$$\Rightarrow (\vec{a}\cdot\vec{a})\,\vec{x}-(\vec{a}\cdot\vec{x})\,\vec{a}=\vec{a}\times\frac{\vec{a}\times(\vec{d}\times\vec{c})}{\vec{a}\cdot\vec{c}}$$

$$\Rightarrow (\vec{a}\cdot\vec{a})\,\vec{x}=(\vec{a}\cdot\vec{x})\,\vec{a}+\vec{a}\times\frac{\vec{a}\times(\vec{d}\times\vec{c})}{\vec{a}\cdot\vec{c}}$$

$$\Rightarrow \vec{x}=\frac{(\vec{a}\cdot\vec{x})\,\vec{a}}{|\vec{a}|^2}+\vec{a}\times\frac{\vec{a}\times(\vec{d}\times\vec{c})}{(\vec{a}\cdot\vec{c})\,|\vec{a}|^2}\quad \text{where } \lambda=\frac{\vec{a}\cdot\vec{x}}{|\vec{a}|^2}$$

Example 2.70 **$\vec{a}$, $\vec{b}$ and $\vec{c}$ are three non-coplanar vectors and $\vec{r}$ is any arbitrary vector. Prove that $[\vec{b}\,\vec{c}\,\vec{r}]\vec{a}+[\vec{c}\,\vec{a}\,\vec{r}]\vec{b}+[\vec{a}\,\vec{b}\,\vec{r}]\vec{c}=[\vec{a}\,\vec{b}\,\vec{c}]\vec{r}$.**

Sol. Let $\vec{r}=x_1\vec{a}+x_2\vec{b}+x_3\vec{c} \Rightarrow \vec{r}\cdot(\vec{b}\times\vec{c})=x_1\vec{a}\cdot(\vec{b}\times\vec{c}) \Rightarrow x_1=\dfrac{[\vec{r}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]}$

Also, $\vec{r}\cdot(\vec{c}\times\vec{a}) = x_2\vec{b}\cdot(\vec{c}\times\vec{a}) \Rightarrow x_2=\dfrac{[\vec{r}\,\vec{c}\,\vec{a}]}{[\vec{a}\,\vec{b}\,\vec{c}]}$ and $\vec{r}\cdot(\vec{a}\times\vec{b})=x_3\vec{c}\cdot(\vec{a}\times\vec{b})$

$$\Rightarrow x_3=\frac{[\vec{r}\,\vec{a}\,\vec{b}]}{[\vec{a}\,\vec{b}\,\vec{c}]} \Rightarrow \vec{r}=\frac{[\vec{r}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]}\vec{a}+\frac{[\vec{r}\,\vec{c}\,\vec{a}]}{[\vec{a}\,\vec{b}\,\vec{c}]}\vec{b}+\frac{[\vec{r}\,\vec{a}\,\vec{b}]}{[\vec{a}\,\vec{b}\,\vec{c}]}\vec{c} \Rightarrow [\vec{b}\,\vec{c}\,\vec{r}]\vec{a}+[\vec{c}\,\vec{a}\,\vec{r}]\vec{b}+[\vec{a}\,\vec{b}\,\vec{r}]\,\vec{c}=[\vec{a}\,\vec{b}\,\vec{c}]\,\vec{r}$$

Example 2.71 **If $\vec{a},\vec{b}$ and $\vec{c}$ are non-coplanar unit vectors such that $\vec{a}\times(\vec{b}\times\vec{c})=\dfrac{\vec{b}+\vec{c}}{\sqrt{2}}$, $\vec{b}$ and $\vec{c}$ are non-parallel, then prove that the angle between $\vec{a}$ and $\vec{b}$ is $3\pi/4$.**

Sol. $\vec{a}\times(\vec{b}\times\vec{c})=\dfrac{\vec{b}+\vec{c}}{\sqrt{2}}$

$$\Rightarrow (\vec{a}\cdot\vec{c})\,\vec{b}-(\vec{a}\cdot\vec{b})\,\vec{c}=\frac{1}{\sqrt{2}}\vec{b}+\frac{1}{\sqrt{2}}\vec{c} \quad \text{(i)}$$

Since $\vec{b}$ and $\vec{c}$ are non-collinear, comparing coefficients of $\vec{c}$ on both sides of (i), we get

$$-\vec{a}.\vec{b}=\frac{1}{\sqrt{2}} \Rightarrow \vec{a}.\vec{b}=-\frac{1}{\sqrt{2}}$$

$$\Rightarrow (1)(1)\cos\theta=-\frac{1}{\sqrt{2}},$$

where θ is the angle between $\vec{a}$ and $\vec{b}$

$$\therefore \cos\theta=-\frac{1}{\sqrt{2}} \Rightarrow \cos\theta \cos 135°$$

$$\Rightarrow \theta=135°=3\pi/4$$

Example 2.72 **Prove that** $\vec{R}+\dfrac{[\vec{R}\cdot(\vec{\beta}\times(\vec{\beta}\times\vec{\alpha}))]\vec{\alpha}}{|\vec{\alpha}\times\vec{\beta}|^2}+\dfrac{[\vec{R}\cdot(\vec{\alpha}\times(\vec{\alpha}\times\vec{\beta}))]\vec{\beta}}{|\vec{\alpha}\times\vec{\beta}|^2}=\dfrac{[\vec{R}\vec{\alpha}\vec{\beta}](\vec{\alpha}\times\vec{\beta})}{|\vec{\alpha}\times\vec{\beta}|^2}$

Sol. $\vec{\alpha},\vec{\beta}$ and $\vec{\alpha}\times\vec{\beta}$ are three non-coplanar vectors. Any vector $\vec{R}$ can be represented as a linear combination of these vectors.

$$\Rightarrow \vec{R}=k_1\vec{\alpha}+k_2\vec{\beta}+k_3(\vec{\alpha}\times\vec{\beta}) \qquad \text{(i)}$$

Take dot product of (i) with $(\vec{\alpha}\times\vec{\beta})$

$$\Rightarrow \vec{R}\cdot(\vec{\alpha}\times\vec{\beta})=k_3(\vec{\alpha}\times\vec{\beta})\cdot(\vec{\alpha}\times\vec{\beta})=k_3|\vec{\alpha}\times\vec{\beta}|^2$$

$$\Rightarrow k_3=\frac{\vec{R}\cdot(\vec{\alpha}\times\vec{\beta})}{|\vec{\alpha}\times\vec{\beta}|^2}=\frac{[\vec{R}\vec{\alpha}\vec{\beta}]}{|\vec{\alpha}\times\vec{\beta}|^2}$$

Take dot product of (i) with $\vec{\alpha}\times(\vec{\alpha}\times\vec{\beta})$

$$\Rightarrow \vec{R}\cdot(\vec{\alpha}\times(\vec{\alpha}\times\vec{\beta}))=k_2(\vec{\alpha}\times(\vec{\alpha}\times\vec{\beta}))\cdot\vec{\beta}$$

$$=k_2[(\vec{\alpha}\cdot\vec{\beta})\vec{\alpha}-(\vec{\alpha}\cdot\vec{\alpha})\vec{\beta}]\cdot\vec{\beta}=k_2[(\vec{\alpha}\cdot\vec{\beta})^2-|\vec{\alpha}|^2|\vec{\beta}|^2]$$

$$=-k_2|\vec{\alpha}\times\vec{\beta}|^2$$

$$\Rightarrow k_2=\frac{-[\vec{R}\cdot(\vec{\alpha}\times(\vec{\alpha}\times\vec{\beta}))]}{|\vec{\alpha}\times\vec{\beta}|^2} \quad \text{Similarly, } k_1=-\frac{[\vec{R}\cdot(\vec{\beta}\times(\vec{\beta}\times\vec{\alpha}))]}{|\vec{\alpha}\times\vec{\beta}|^2}$$

$$\Rightarrow \vec{R}=\frac{-[\vec{R}.[\vec{\beta}\times(\vec{\beta}\times\vec{\alpha}))]\vec{\alpha}}{|\vec{\alpha}\times\vec{\beta}|^2}-\frac{[\vec{R}\cdot(\vec{\alpha}\times(\vec{\alpha}\times\vec{\beta}))]\vec{\beta}}{|\vec{\alpha}\times\vec{\beta}|^2}+\frac{[(\vec{R}\cdot(\vec{\alpha}\times\vec{\beta}))](\vec{\alpha}\times\vec{\beta})}{(\vec{\alpha}\times\vec{\beta})^2}$$

$$\Rightarrow \vec{R}+\frac{[\vec{R}\cdot(\vec{\beta}\times(\vec{\beta}\times\vec{\alpha}))]\vec{\alpha}}{|\vec{\alpha}\times\vec{\beta}|^2}+\frac{[\vec{R}\cdot(\vec{\alpha}\times(\vec{\alpha}\times\vec{\beta}))]\vec{\beta}}{|\vec{\alpha}\times\vec{\beta}|^2}=\frac{[\vec{R}\cdot(\vec{\alpha}\times\vec{\beta})](\vec{\alpha}\times\vec{\beta})}{|\vec{\alpha}\times\vec{\beta}|^2}$$

Example 2.73 **If $\vec{a}, \vec{b}$ and $\vec{c}$ are three non-coplanar non-zero vectors, then prove that $(\vec{a}\cdot\vec{a})\ \vec{b}\times\vec{c} + (\vec{a}\cdot\vec{b})\ \vec{c}\times\vec{a} + (\vec{a}\cdot\vec{c})\ \vec{a}\times\vec{b} = [\vec{b}\ \vec{c}\ \vec{a}]\vec{a}$.**

Sol. As $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar, $\vec{b}\times\vec{a}$, $\vec{c}\times\vec{a}$ and $\vec{a}\times\vec{b}$ are also non-coplanar.

So, any vector can be expressed as a linear combination of these vectors.

Let $\vec{a} = \lambda\, \vec{b}\times\vec{c} + \mu\, \vec{c}\times\vec{a} + \nu\, \vec{a}\times\vec{b}$

$$\therefore\ \vec{a}\cdot\vec{a} = \lambda[\vec{b}\ \vec{c}\ \vec{a}],\ \vec{a}\cdot\vec{b} = \mu[\vec{c}\ \vec{a}\ \vec{b}],\ \vec{a}\cdot\vec{c} = \nu\ [\vec{a}\ \vec{b}\ \vec{c}]$$

$$\therefore\ \vec{a} = \frac{(\vec{a}\cdot\vec{a})\vec{b}\times\vec{c}}{[\vec{b}\ \vec{c}\ \vec{a}]} + \frac{(\vec{a}\cdot\vec{b})\vec{c}\times\vec{a}}{[\vec{c}\ \vec{a}\ \vec{b}]} + \frac{(\vec{a}\cdot\vec{c})\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

RECIPROCAL SYSTEM OF VECTORS

Two systems of vectors are called reciprocal systems of vectors if by taking the dot product we get unity.

Thus if $\vec{a}, \vec{b}$ and $\vec{c}$ are three non-coplanar vectors, and if

$\vec{a}' = \dfrac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$, $\vec{b}' = \dfrac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}$ and $\vec{c}' = \dfrac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$, then $\vec{a}', \vec{b}', \vec{c}'$ are said to be the reciprocal systems of vectors for vectors $\vec{a}, \vec{b}$ and $\vec{c}$.

Properties

i. *If $\vec{a}, \vec{b}$ and $\vec{c}$ and $\vec{a}'$, $\vec{b}'$ and $\vec{c}'$ are reciprocal system of vectors, then* $\vec{a}\cdot\vec{a}' = \dfrac{\vec{a}\cdot(\vec{b}\times\vec{c})}{(\vec{a}\,\vec{b}\,\vec{c})} = \dfrac{[\vec{a}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]} = 1.$

Similarly, $\vec{b}\cdot\vec{b}' = \vec{c}\cdot\vec{c}' = 1\cdot$

Due to the above property, the two systems of vectors are called reciprocal systems.

ii. $\vec{a}\cdot\vec{b}' = \vec{a}\cdot\vec{c}' = \vec{b}\cdot\vec{a}' = \vec{b}\cdot\vec{c}' = \vec{c}\cdot\vec{a}' = \vec{c}\cdot\vec{b}' = 0$

iii. $[\vec{a}\,\vec{b}\,\vec{c}]\,[\vec{a}'\vec{b}'\vec{c}'] = 1$

Proof:

We have $[\vec{a}'\vec{b}'\vec{c}'] = \left[\dfrac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}\ \dfrac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}\ \dfrac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}\right] = \dfrac{1}{[\vec{a}\,\vec{b}\,\vec{c}]^3}[\vec{b}\times\vec{c}\ \vec{c}\times\vec{a}\ \vec{a}\times\vec{b}] = \dfrac{1}{[\vec{a}\,\vec{b}\,\vec{c}]^3}[\vec{a}\,\vec{b}\,\vec{c}]^2 = \dfrac{1}{[\vec{a}\,\vec{b}\,\vec{c}]}$

$\Rightarrow [\vec{a}'\vec{b}'\vec{c}']\,[\vec{a}\,\vec{b}\,\vec{c}] = 1$

iv. *The orthogonal triad of vectors $\hat{i}, \hat{j}$* and *$\hat{k}$ is self-reciprocal.*

Let $\hat{i}'$, $\hat{j}'$ and *$\hat{k}'$ be the system of vectors reciprocal to the system $\hat{i}$, $\hat{j}$* and *$\hat{k}$. Then,* *we have* $\hat{i}' = \dfrac{\hat{j}\times\hat{k}}{[\hat{i}\,\hat{j}\,\hat{k}]} = \hat{i}$. *Similarly,* $\hat{j}' = \hat{j}$ *and* $\hat{k}' = \hat{k}$.

v. $\vec{a}, \vec{b}$ and $\vec{c}$ *are non-coplanar iff* $\vec{a}', \vec{b}'$ and $\vec{c}'$ *are non-coplanar.*

As $[\vec{a}\,\vec{b}\,\vec{c}]\cdot[\vec{a}'\,\vec{b}'\,\vec{c}'] = 1$ *and* $[\vec{a}\,\vec{b}\,\vec{c}] \neq 0$ *are non-coplanar* $\Leftrightarrow \dfrac{1}{[\vec{a}\,\vec{b}\,\vec{c}]} \neq 0 \Leftrightarrow [\vec{a}'\,\vec{b}'\,\vec{c}']$ *are non-coplanar.*

Example 2.74 **Find a set of vectors reciprocal to the set** $-\hat{i}+\hat{j}+\hat{k}, \hat{i}-\hat{j}+\hat{k}, \hat{i}+\hat{j}+\hat{k}$.

Sol. Let $\vec{a} = -\hat{i}+\hat{j}+\hat{k}$, $\vec{b} = \hat{i}-\hat{j}+\hat{k}$, $\vec{c} = \hat{i}+\hat{j}+\hat{k}$

Then $\vec{b}\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -2\hat{i}+2\hat{k}$, $\vec{c}\times\vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{vmatrix} = -2\hat{j}+2\hat{k}$, $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix}$
$= 2\hat{i}+2\hat{j}$

$$[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 4$$

If a', b', c' is the reciprocal system of vectors, then

$$\vec{a}' = (\vec{b}\times\vec{c})/[\vec{a}\,\vec{b}\,\vec{c}] = \frac{1}{2}(-\hat{i}+\hat{k}),\quad \vec{b}' = (\vec{c}\times\vec{a})/[\vec{a}\,\vec{b}\,\vec{c}] = \frac{1}{2}(-\hat{j}+\hat{k}),$$

$$\vec{c}' = (\vec{a}\times\vec{b})/[\vec{a}\,\vec{b}\,\vec{c}] = \frac{1}{2}(\hat{i}+\hat{j})$$

Example 2.75 **Let** $\vec{a}, \vec{b}$ **and** $\vec{c}$ **be a set of non-coplanar vectors and** $\vec{a}', \vec{b}'$ **and** $\vec{c}'$ **be its reciprocal set.**

Prove that $\vec{a} = \dfrac{b'\times c'}{[\vec{a}'\,\vec{b}'\,\vec{c}']}$, $\vec{b} = \dfrac{c'\times a'}{[\vec{a}'\,\vec{b}'\,\vec{c}']}$ **and** $\vec{c} = \dfrac{a'\times b'}{[\vec{a}'\,\vec{b}'\,\vec{c}']}$.

Sol. We have, $\vec{b}'\times\vec{c}' = \dfrac{(\vec{c}\times\vec{a})\times(\vec{a}\times\vec{b})}{[\vec{a}\,\vec{b}\,\vec{c}]^2}$

$$= \frac{\{(\vec{c}\times\vec{a})\cdot\vec{b}\}\vec{a} - \{(\vec{c}\times\vec{a})\cdot\vec{a}\}\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]^2} = \frac{[\vec{c}\,\vec{a}\,\vec{b}]\vec{a} - [\vec{c}\,\vec{a}\,\vec{a}]\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]^2} = \frac{[\vec{a}\,\vec{b}\,\vec{c}]\vec{a} - 0}{[\vec{a}\,\vec{b}\,\vec{c}]^2} = \frac{\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

Also, $[\vec{a}'\,\vec{b}'\,\vec{c}'] = \vec{a}'\cdot(\vec{b}'\times\vec{c}') = \dfrac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}\cdot\dfrac{\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]} = \dfrac{[\vec{a}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]^2} = \dfrac{1}{[\vec{a}\,\vec{b}\,\vec{c}]}$

$$\Rightarrow \frac{\vec{b}'\times\vec{c}'}{[\vec{a}'\vec{b}'\vec{c}']} = \vec{a}$$

Similarly, $\vec{b} = \frac{\vec{c}'\times\vec{a}'}{[\vec{a}'\vec{b}'\vec{c}']}$, $\vec{c} = \frac{\vec{a}'\times\vec{b}'}{[\vec{a}'\vec{b}'\vec{c}']}$

Example 2.76 **If $\vec{a},\vec{b},\vec{c}$ and $\vec{a}',\vec{b}',\vec{c}'$ are reciprocal system of vectors, then prove that $\vec{a}'\times\vec{b}'+\vec{b}'\times\vec{c}'+\vec{c}'\times\vec{a}' = \frac{\vec{a}+\vec{b}+\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$.**

Sol. $\vec{a}'\times\vec{b}' = \frac{(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})}{[\vec{a}\,\vec{b}\,\vec{c}]^2} = \frac{\{(\vec{b}\times\vec{c})\cdot\vec{a}\}\vec{c}-\{(\vec{b}\times\vec{c})\cdot\vec{c}\}\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]^2} = \frac{[\vec{b}\,\vec{c}\,\vec{a}]\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]^2} = \frac{[\vec{a}\,\vec{b}\,\vec{c}]\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]^2} = \frac{\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$

Similarly, $\vec{b}'\times\vec{c}' = \frac{\vec{a}}{[\vec{a}\times\vec{b}\times\vec{c}]}$ and $\vec{c}'\times\vec{a}' = \frac{\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$

Adding, $\vec{a}'\times\vec{b}'+\vec{b}'\times\vec{c}'+\vec{c}'\times\vec{a}' = \frac{\vec{a}+\vec{b}+\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$

Example 2.77 **If $\vec{a},\vec{b}$ and $\vec{c}$ be three non-coplanar vectors and a', b' and c' constitute the reciprocal system of vectors, then prove that**

i. $\vec{r} = (\vec{r}\cdot\vec{a}')\vec{a} + (\vec{r}\cdot\vec{b}')\vec{b} + (\vec{r}\cdot\vec{c}')\vec{c}$

ii. $\vec{r} = (\vec{r}\cdot\vec{a})\vec{a}' + (\vec{r}\cdot\vec{b})\vec{b}' + (\vec{r}\cdot\vec{c})\vec{c}'$

Sol. i. Since a vector can be expressed as a linear combination of three non-coplanar vectors, therefore let $\vec{r} = x\vec{a}+y\vec{b}+z\vec{c}$ (i)

where x, y and z are scalars.

Multiplying both sides of (i) scalarly by $\vec{a}'$, we get

$\vec{r}\cdot\vec{a}' = x\vec{a}\cdot\vec{a}'+y\vec{b}\cdot\vec{a}'+z\vec{c}\cdot\vec{a}' = x\cdot 1 = x$

$(\because\ \vec{a}\cdot\vec{a}' = 1,\ \vec{b}.\vec{a}' = 0 = \vec{c}\cdot\vec{a}')$

Similarly multiplying both sides of (i) scalarly by $\vec{b}'$ and $\vec{c}'$, successively, we get

$y = \vec{r}\cdot\vec{b}'$ and $z = \vec{r}\cdot\vec{c}'$

Putting in (i), we get $\vec{r} = (\vec{r}\cdot\vec{a}')\vec{a}+(\vec{r}\cdot\vec{b}')\vec{b}+(\vec{r}\cdot\vec{c}')\vec{c}$

ii. Since $\vec{a}'$, $\vec{b}'$ and $\vec{c}'$ are three non-coplanar vectors, we can take $\vec{r} = x\vec{a}'+y\vec{b}'+z\vec{c}'$ (ii)

Multiplying both sides of (ii) scalarly by $\vec{a}$, we get $\vec{r}\cdot\vec{a} = x(\vec{a}'\cdot\vec{a})+y(\vec{b}'\cdot\vec{a})+z(\vec{c}'\cdot\vec{a}) = x$

$(\because\ \vec{a}'\cdot\vec{a} = 1,\ \vec{b}'\cdot\vec{a} = 0 = \vec{c}'\cdot\vec{a})$

Similarly, multiplying both sides of (i) scalarly by $\vec{b}$ and $\vec{c}$ successively, we get

$y = \vec{r}\cdot\vec{b}$ and $z = \vec{r}\cdot\vec{c}$

Putting in (ii), we get $\vec{r} = (\vec{r}\cdot\vec{a})\vec{a}'+(\vec{r}\cdot\vec{b})\vec{b}'+(\vec{r}\cdot\vec{c})\vec{c}'$.

Concept Application Exercise 2.3

1. If $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ are four non-coplanar unit vectors such that $\vec{d}$ makes equal angles with all the three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, then prove that $[\vec{d}\ \vec{a}\ \vec{b}]=[\vec{d}\ \vec{c}\ \vec{b}]=[\vec{d}\ \vec{c}\ \vec{a}]$.

2. Prove that if $[\vec{l}\ \vec{m}\ \vec{n}]$ are three non-coplanar vectors, then $[\vec{l}\ \vec{m}\ \vec{n}](\vec{a}\times\vec{b})=\begin{vmatrix}\vec{l}\cdot\vec{a} & \vec{l}\cdot\vec{b} & \vec{l}\\ \vec{m}\cdot\vec{a} & \vec{m}\cdot\vec{b} & \vec{m}\\ \vec{n}\cdot\vec{a} & \vec{n}\cdot\vec{b} & \vec{n}\end{vmatrix}$

3. If the volume of a parallelopiped whose adjacent edges are $\vec{a}=2\hat{i}+3\hat{j}+4\hat{k}$, $\vec{b}=\hat{i}+\alpha\hat{j}+2\hat{k}$, $\vec{c}=\hat{i}+2\hat{j}+\alpha\hat{k}$ is 15, then find the value of α if $(\alpha > 0)$.

4. If $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}+\hat{k}$, then find vector $\vec{c}$ such that $\vec{a}\cdot\vec{c}=2$ and $\vec{a}\times\vec{c}=\vec{b}$.

5. If $\vec{x}\cdot\vec{a}=0$, $\vec{x}\cdot\vec{b}=0$ and $\vec{x}\cdot\vec{c}=0$ for some non-zero vector $\vec{x}$, then prove that $[\vec{a}\ \vec{b}\ \vec{c}]=0$.

6. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three non-coplanar vectors, show that

$$[\vec{a}\times\vec{b}\quad \vec{b}\times\vec{c}\quad \vec{c}\times\vec{d}]=[\vec{a}\ \vec{b}\ \vec{c}]^2=\begin{vmatrix}\vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c}\\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c}\\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c}\end{vmatrix}.$$

7. If $\vec{a}, \vec{b}$ and $\vec{c}$ are three vectors such that $\vec{a}\times\vec{b}=\vec{c}, \vec{b}\times\vec{c}=\vec{a}, \vec{c}\times\vec{a}=\vec{b}$, then prove that $|\vec{a}|=|\vec{b}|=|\vec{c}|$.

8. If $\vec{a}=\vec{p}+\vec{q}, \vec{p}\times\vec{b}=\vec{0}$ and $\vec{q}\cdot\vec{b}=0$, then prove that $\dfrac{\vec{b}\times(\vec{a}\times\vec{b})}{\vec{b}\cdot\vec{b}}=\vec{q}$.

9. Prove that $(\vec{a}\cdot(\vec{b}\times\hat{i}))\hat{i}+(\vec{a}\cdot(\vec{b}\times\hat{j}))\hat{j}+(\vec{a}\cdot(\vec{b}\times\hat{k}))\hat{k}=\vec{a}\times\vec{b}$.

10. For any four vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$, prove that $\vec{d}\cdot(\vec{a}\times(\vec{b}\times(\vec{c}\times\vec{d})))=(\vec{b}\cdot\vec{d})[\vec{a}\ \vec{c}\ \vec{d}]$.

11. If $\vec{a}$ and $\vec{b}$ be two non-collinear unit vectors such that $\vec{a}\times(\vec{a}\times\vec{b})=\frac{1}{2}\vec{b}$, then find the angle between $\vec{a}$ and $\vec{b}$.

12. Show that $(\vec{a}\times\vec{b})\times\vec{c}=\vec{a}\times(\vec{b}\times\vec{c})$ if and only if $\vec{a}$ and $\vec{c}$ are collinear or $(\vec{a}\times\vec{c})\times\vec{b}=\vec{0}$.

13. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be non-zero vectors such that no two are collinear and $(\vec{a}\times\vec{b})\times\vec{c}=\frac{1}{3}|\vec{b}||\vec{c}|\vec{a}$. If θ is the acute angle between vectors $\vec{b}$ and $\vec{c}$, then find the value of $\sin\theta$.

14. If $\vec{p}, \vec{q}, \vec{r}$ denote vectors $\vec{b}\times\vec{c}, \vec{c}\times\vec{a}, \vec{a}\times\vec{b}$, respectively, show that $\vec{a}$ is parallel to $\vec{q}\times\vec{r}$, $\vec{b}$ is parallel to $\vec{r}\times\vec{p}$, $\vec{c}$ is parallel to $\vec{p}\times\vec{q}$.

Exercises

Subjective Type

Solutions on page 2.84

1. If $\begin{vmatrix} (a-x)^2 & (a-y)^2 & (a-z)^2 \\ (b-x)^2 & (b-y)^2 & (b-z)^2 \\ (c-x)^2 & (c-y)^2 & (c-a)^2 \end{vmatrix} = 0$ and vectors $\vec{A}, \vec{B}$ and $\vec{C}$, where $\vec{A} = a^2\hat{i} + a\hat{j} + \hat{k}$, etc., are non-coplanar, then prove that vectors $\vec{X}, \vec{Y}$ and $\vec{Z}$, where $\vec{X} = x^2\hat{i} + x\hat{j} + \hat{k}$, etc. may be coplanar.
2. If $OABC$ is a tetrahedron where O is the origin and A, B and C are the other three vertices with position vectors $\vec{a}, \vec{b}$ and $\vec{c}$, respectively, then prove that the centre of the sphere circumscribing the tetrahedron is given by position vector $\dfrac{a^2(\vec{b}\times\vec{c})+b^2(\vec{c}\times\vec{a})+c^2(\vec{a}\times\vec{b})}{2[\vec{a}\,\vec{b}\,\vec{c}]}$.
3. Let k be the length of any edge of a regular tetrahedron (a tetrahedron whose edges are equal in length is called a regular tetrahedron). Show that the angle between any edge and a face not containing the edge is $\cos^{-1}(1/\sqrt{3})$.
4. In ΔABC, a point P is taken on AB such that $AP/BP = 1/3$ and a point Q is taken on BC such that $CQ/BQ = 3/1$. If R is the point of intersection of the lines AQ and CP, using vector method, find the area of ΔABC if the area of ΔBRC is 1 unit.
5. Let O be an interior point of ΔABC such that $\overrightarrow{OA} + 2\overrightarrow{OB} + 3\overrightarrow{OC} = \vec{0}$. Then find the ratio of the area of ΔABC to the area of ΔAOC.
6. The lengths of two opposite edges of a tetrahedron are a and b; the shortest distance between these edges is d, and the angle between them is θ. Prove using vectors that the volume of the tetrahedron is $\dfrac{abd\sin\theta}{6}$.
7. Find the volume of a parallelopiped having three coterminus vectors of equal magnitude $|\vec{a}|$ and equal inclination θ with each other.
8. Let $\vec{p}$ and $\vec{q}$ be any two orthogonal vectors of equal magnitude 4 each. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be any three vectors of lengths 7, $\sqrt{15}$ and $2\sqrt{33}$, mutually perpendicular to each other. Then find the distance of the vector $(\vec{a}\cdot\vec{p})\,\vec{p} + (\vec{a}\cdot\vec{q})\,\vec{q} + (\vec{a}\cdot(\vec{p}\times\vec{q}))\,(\vec{p}\times\vec{q}) + (\vec{b}\cdot\vec{p})\,\vec{p} + (\vec{b}\cdot\vec{q})\,\vec{q} + (\vec{b}\cdot(\vec{p}\times\vec{q}))\,(\vec{p}\times\vec{q}) + (\vec{c}\cdot\vec{p})\,\vec{p} + (\vec{c}\cdot\vec{q})\,\vec{q} + (\vec{c}\cdot(\vec{p}\times\vec{q}))\,(\vec{p}\times\vec{q})$ from the origin.
9. Given that vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ form a triangle such that $\vec{A} = \vec{B} + \vec{C}$. Find a, b, c and d such that the area of the triangle is $5\sqrt{6}$ where
$\vec{A} = a\hat{i} + b\hat{j} + c\hat{k}$
$\vec{B} = d\hat{i} + 3\hat{j} + 4\hat{k}$
$\vec{C} = 3\hat{i} + \hat{j} - 2\hat{k}$

10. A line l is passing through the point $\vec{b}$ and is parallel to vector $\vec{c}$. Determine the distance of point $A(\vec{a})$ from the line l in the form $\left|\vec{b}-\vec{a}+\dfrac{(\vec{a}-\vec{b})\cdot\vec{c}}{|\vec{c}|^2}\vec{c}\right|$ or $\dfrac{|(\vec{b}-\vec{a})\times\vec{c}|}{|\vec{c}|}$.

11. If $\vec{e}_1, \vec{e}_2, \vec{e}_3$ and $\vec{E}_1, \vec{E}_2, \vec{E}_3$ are two sets of vectors such that $\vec{e}_i\cdot\vec{E}_j = 1$, if $i = j$ and $\vec{e}_i\cdot\vec{E}_j = 0$ and if $i \neq j$, then prove that $[\vec{e}_1\ \vec{e}_2\ \vec{e}_3][\vec{E}_1\ \vec{E}_2\ \vec{E}_3] = 1$.

Objective Type

Solutions on page 2.90

Each question has four choices *a, b, c* and *d*, out of which *only one* answer is correct. Find the correct answer.

1. Two vectors in space are equal only if they have equal component in

a. a given direction
b. two given directions
c. three given directions
d. in any arbitrary direction

2. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be the three vectors having magnitudes 1, 5 and 3, respectively, such that the angle between $\vec{a}$ and $\vec{b}$ is θ and $\vec{a}\times(\vec{a}\times\vec{b}) = \vec{c}$. Then $\tan\theta$ is equal to

a. 0
b. 2/3
c. 3/5
d. 3/4

3. $\vec{a}, \vec{b}$ and $\vec{c}$ are three vectors of equal magnitude. The angle between each pair of vectors is $\pi/3$ such that $|\vec{a}+\vec{b}+\vec{c}| = \sqrt{6}$. Then $|\vec{a}|$ is equal to

a. 2
b. −1
c. 1
d. $\sqrt{6}/3$

4. If $\vec{a}, \vec{b}$ and $\vec{c}$ are three mutually perpendicular vectors, then the vector which is equally inclined to these vectors is

a. $\vec{a}+\vec{b}+\vec{c}$
b. $\dfrac{\vec{a}}{|\vec{a}|}+\dfrac{\vec{b}}{|\vec{b}|}+\dfrac{\vec{c}}{|\vec{c}|}$
c. $\dfrac{\vec{a}}{|\vec{a}|^2}+\dfrac{\vec{b}}{|\vec{b}|^2}+\dfrac{\vec{c}}{|\vec{c}|^2}$
d. $|\vec{a}|\vec{a}-|\vec{b}|\vec{b}+|\vec{c}|\vec{c}$

5. Let $\vec{a} = \hat{i}+\hat{j}$; $\vec{b} = 2\hat{i}-\hat{k}$. Then vector $\vec{r}$ satisfying the equations $\vec{r}\times\vec{a} = \vec{b}\times\vec{a}$ and $\vec{r}\times\vec{b} = \vec{a}\times\vec{b}$ is

a. $\hat{i}-\hat{j}+\hat{k}$
b. $3\hat{i}-\hat{j}+\hat{k}$
c. $3\hat{i}+\hat{j}-\hat{k}$
d. $\hat{i}-\hat{j}-\hat{k}$

6. If $\vec{a}$ and $\vec{b}$ are two vectors, such that $\vec{a}\cdot\vec{b} < 0$ and $|\vec{a}\cdot\vec{b}| = |\vec{a}\times\vec{b}|$, then the angle between vectors $\vec{a}$ and $\vec{b}$ is

a. π
b. $7\pi/4$
c. $\pi/4$
d. $3\pi/4$

7. If $\hat{a}, \hat{b}$ and $\hat{c}$ are three unit vectors, such that $\hat{a}+\hat{b}+\hat{c}$ is also a unit vector and θ_1, θ_2 and θ_3 are angles between the vectors $\hat{a}, \hat{b}$; $\hat{b}, \hat{c}$ and $\hat{c}, \hat{a}$, respectively, then among θ_1, θ_2 and θ_3

a. all are acute angles
b. all are right angles
c. at least one is obtuse angle
d. none of these

8. If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a}\cdot\vec{b} = 0 = \vec{a}\cdot\vec{c}$ and the angle between $\vec{b}$ and $\vec{c}$ is $\pi/3$, then the value of $|\vec{a}\times\vec{b} - \vec{a}\times\vec{c}|$ is

a. 1/2 b. 1 c. 2 d. none of these

9. $P(\vec{p})$ and $Q(\vec{q})$ are the position vectors of two fixed points and $R(\vec{r})$ is the position vector of a variable point. If R moves such that $(\vec{r}-\vec{p})\times(\vec{r}-\vec{q}) = \vec{0}$, then the locus of R is

a. a plane containing the origin O and parallel to two non-collinear vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$
b. the surface of a sphere described on PQ as its diameter
c. a line passing through points P and Q
d. a set of lines parallel to line PQ

10. Two adjacent sides of a parallelogram $ABCD$ are $2\hat{i}+4\hat{j}-5\hat{k}$ and $\hat{i}+2\hat{j}+3\hat{k}$. Then the value of $|\overrightarrow{AC}\times\overrightarrow{BD}|$ is

a. $20\sqrt{5}$ b. $22\sqrt{5}$ c. $24\sqrt{5}$ d. $26\sqrt{5}$

11. If $\hat{a}, \hat{b}$ and $\hat{c}$ are three unit vectors inclined to each other at an angle θ, then the maximum value of θ is

a. $\frac{\pi}{3}$ b. $\frac{\pi}{2}$ c. $\frac{2\pi}{3}$ a. $\frac{5\pi}{6}$

12. Let the pairs $\vec{a}, \vec{b}$ and $\vec{c}, \vec{d}$ each determine a plane. Then the planes are parallel if

a. $(\vec{a}\times\vec{c})\times(\vec{b}\times\vec{d}) = \vec{0}$ b. $(\vec{a}\times\vec{c})\cdot(\vec{b}\times\vec{d}) = \vec{0}$

c. $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = \vec{0}$ d. $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = \vec{0}$

13. If $\vec{r}\cdot\vec{a} = \vec{r}\cdot\vec{b} = \vec{r}\cdot\vec{c} = 0$, where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-coplanar, then

a. $\vec{r}\perp(\vec{c}\times\vec{a})$ b. $\vec{r}\perp(\vec{a}\times\vec{b})$ c. $\vec{r}\perp(\vec{b}\times\vec{c})$ d. $\vec{r} = \vec{0}$

14. If $\vec{a}$ satisfies $\vec{a}\times(\hat{i}+2\hat{j}+\hat{k}) = \hat{i}-\hat{k}$, then $\vec{a}$ is equal to

a. $\lambda\hat{i}+(2\lambda-1)\hat{j}+\lambda\hat{k}, \lambda\in R$ b. $\lambda\hat{i}+(1-2\lambda)\hat{j}+\lambda\hat{k}, \lambda\in R$

c. $\lambda\hat{i}+(2\lambda+1)\hat{j}+\lambda\hat{k}, \lambda\in R$ d. $\lambda\hat{i}-(1+2\lambda)\hat{j}+\lambda\hat{k}, \lambda\in R$

15. Vectors $3\vec{a}-5\vec{b}$ and $2\vec{a}+\vec{b}$ are mutually perpendicular. If $\vec{a}+4\vec{b}$ and $\vec{b}-\vec{a}$ are also mutually perpendicular, then the cosine of the angle between $\vec{a}$ and $\vec{b}$ is

a. $\frac{19}{5\sqrt{43}}$ b. $\frac{19}{3\sqrt{43}}$ c. $\frac{19}{2\sqrt{45}}$ d. $\frac{19}{6\sqrt{43}}$

16. The unit vector orthogonal to vector $-\hat{i}+2\hat{j}+2\hat{k}$ and making equal angles with the x- and y-axes is

a. $\pm\frac{1}{3}(2\hat{i}+2\hat{j}-\hat{k})$ b. $\pm\frac{1}{3}(\hat{i}+\hat{j}-\hat{k})$ c. $\pm\frac{1}{3}(2\hat{i}-2\hat{j}-\hat{k})$ d. None of these

17. The value of x for which the angle between $\vec{a} = 2x^2\,\hat{i} + 4x\,\hat{j} + \hat{k}$ and $\vec{b} = 7\hat{i} - 2\hat{j} + x\hat{k}$ is obtuse and the angle between $\vec{b}$ and the z-*axis* is acute and less than $\pi/6$, is

a. $a < x < 1/2$ **b.** $1/2 < x < 15$ **c.** $x > 1/2$ or $x < 0$ **d.** none of these

18. If vectors $\vec{a}$ and $\vec{b}$ are two adjacent sides of a parallelogram, then the vector representing the altitude of the parallelogram which is perpendicular to $\vec{a}$ is

a. $\vec{b} + \dfrac{\vec{b} \times \vec{a}}{|\vec{a}|^2}$ **b.** $\dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2}$ **c.** $\vec{b} - \dfrac{\vec{b} \cdot \vec{a}}{|\vec{a}|^2}\vec{a}$ **d.** $\dfrac{\vec{a} \times (\vec{b} \times \vec{a})}{|\vec{b}|^2}$

19. A parallelogram is constructed on $3\vec{a} + \vec{b}$ and $\vec{a} - 4\vec{b}$, where $|\vec{a}| = 6$ and $|\vec{b}| = 8$, and $\vec{a}$ and $\vec{b}$ are anti-parallel. Then the length of the longer diagonal is

a. 40 **b.** 64 **c.** 32 **d.** 48

20. Let $\vec{a} \cdot \vec{b} = 0$, where $\vec{a}$ and $\vec{b}$ are unit vectors and the unit vector $\vec{c}$ is inclined at an angle θ to both $\vec{a}$ and $\vec{b}$. If $\vec{c} = m\vec{a} + n\vec{b} + p(\vec{a} \times \vec{b})$, $(m, n, p \in R)$, then

a. $-\dfrac{\pi}{4} \le \theta \le \dfrac{\pi}{4}$ **b.** $\dfrac{\pi}{4} \le \theta \le \dfrac{3\pi}{4}$ **c.** $0 \le \theta \le \dfrac{\pi}{4}$ **d.** $0 \le \theta \le \dfrac{3\pi}{4}$

21. $\vec{a}$ and $\vec{c}$ are unit vectors and $|\vec{b}| = 4$. The angle between $\vec{a}$ and $\vec{c}$ is $\cos^{-1}(1/4)$ and $\vec{b} - 2\vec{c} = \lambda\vec{a}$. The value of λ is

a. $3, -4$ **b.** $1/4, 3/4$ **c.** $-3, 4$ **d.** $-1/4, 3/4$

22. Let the position vectors of the points P and Q be $4\hat{i} + \hat{j} + \lambda\hat{k}$ and $2\hat{i} - \hat{j} + \lambda\hat{k}$, respectively. Vector $\hat{i} - \hat{j} + 6\hat{k}$ is perpendicular to the plane containing the origin and the points P and Q. Then λ equals

a. $-1/2$ **b.** $1/2$ **c.** 1 **d.** none of these

23. A vector of magnitude $\sqrt{2}$ coplanar with the vectors $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$, and perpendicular to the vector $\vec{c} = \hat{i} + \hat{j} + \hat{k}$, is

a. $-\hat{j} + \hat{k}$ **b.** $\hat{i} - \hat{k}$ **c.** $\hat{i} - \hat{j}$ **d.** $\hat{i} - \hat{j}$

24. P be a point interior to the acute triangle ABC. If $\vec{PA} + \vec{PB} + \vec{PC}$ is a null vector then w.r.t. triangle ABC, point P is its

a. centroid **b.** orthocentre **c.** incentre **d.** circumcentre

25. G is the centroid of triangle ABC and A_1 and B_1 are the midpoints of sides AB and AC, respectively. If Δ_1 be the area of quadrilateral GA_1AB_1 and Δ be the area of triangle ABC, then Δ/Δ_1 is equal to

a. $\dfrac{3}{2}$ **b.** 3 **c.** $\dfrac{1}{3}$ **d.** none of these

26. Points $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are coplanar and $(\sin\alpha)\,\vec{a} + (2\sin 2\beta)\,\vec{b} + (3\sin 3\gamma)\,\vec{c} - \vec{d} = \vec{0}$. Then the least value of $\sin^2\alpha + \sin^2 2\beta + \sin^2 3\gamma$ is

a. 1/14 **b.** 14 **c.** 6 **d.** $1/\sqrt{6}$

27. If $\vec{a}$ and $\vec{b}$ are any two vectors of magnitudes 1 and 2, respectively, and $(1-3\vec{a}\cdot\vec{b})^2 + |2\vec{a}+\vec{b}+3(\vec{a}\times\vec{b})|^2 = 47$, then the angle between $\vec{a}$ and $\vec{b}$ is

a. $\pi/3$ b. $\pi - \cos^{-1}(1/4)$ c. $\frac{2\pi}{3}$ d. $\cos^{-1}(1/4)$

28. If $\vec{a}$ and $\vec{b}$ are any two vectors of magnitudes 2 and 3, respectively, such that $|2(\vec{a}\times\vec{b})|+|3(\vec{a}\cdot\vec{b})| = k$, then the maximum value of k is

a. $\sqrt{13}$ b. $2\sqrt{13}$ c. $6\sqrt{13}$ d. $10\sqrt{13}$

29. $\vec{a}$, $\vec{b}$ and $\vec{c}$ are unit vectors such that $|\vec{a}+\vec{b}+3\vec{c}|=4$. Angle between $\vec{a}$ and $\vec{b}$ is θ_1, between $\vec{b}$ and $\vec{c}$ is θ_2 and between $\vec{a}$ and $\vec{c}$ varies $[\pi/6, 2\pi/3]$. Then the maximum value of $\cos\theta_1 + 3\cos\theta_2$ is

a. 3 b. 4 c. $2\sqrt{2}$ d. 6

30. If the vector product of a constant vector $\overrightarrow{OA}$ with a variable vector $\overrightarrow{OB}$ in a fixed plane OAB be a constant vector, then the locus of B is

a. a straight line perpendicular to $\overrightarrow{OA}$

b. a circle with centre O and radius equal to $|\overrightarrow{OA}|$

c. a straight line parallel to $\overrightarrow{OA}$

d. none of these

31. Let $\vec{u}, \vec{v}$ and $\vec{w}$ be such that $|\vec{u}|=1, |\vec{v}|=2$ and $|\vec{w}|=3$. If the projection of $\vec{v}$ along $\vec{u}$ is equal to that of $\vec{w}$ along $\vec{u}$ and vectors $\vec{v}$ and $\vec{w}$ are perpendicular to each other, then $|\vec{u}-\vec{v}+\vec{w}|$ equals

a. 2 b. $\sqrt{7}$ c. $\sqrt{14}$ d. 14

32. If the two adjacent sides of two rectangles are represented by vectors $\vec{p} = 5\vec{a} - 3\vec{b}$; $\vec{q} = -\vec{a} - 2\vec{b}$ and $\vec{r} = -4\vec{a} - \vec{b}$; $\vec{s} = -\vec{a} + \vec{b}$, respectively, then the angle between the vectors $\vec{x} = \frac{1}{3}(\vec{p}+\vec{r}+\vec{s})$ and $\vec{y} = \frac{1}{5}(\vec{r}+\vec{s})$ is

a. $-\cos^{-1}\left(\frac{19}{5\sqrt{43}}\right)$ b. $\cos^{-1}\left(\frac{19}{5\sqrt{43}}\right)$

c. $\pi\cos^{-1}\left(\frac{19}{5\sqrt{43}}\right)$ d. cannot be evaluated

33. If $\vec{\alpha} \parallel (\vec{\beta}\times\vec{\gamma})$, then $(\vec{\alpha}\times\vec{\beta})\cdot(\vec{\alpha}\times\vec{\gamma})$ equals to

a. $|\vec{\alpha}|^2(\vec{\beta}\cdot\vec{\gamma})$ b. $|\vec{\beta}|^2(\vec{\gamma}\cdot\vec{\alpha})$ c. $|\vec{\gamma}|^2(\vec{\alpha}\cdot\vec{\beta})$ d. $|\vec{\alpha}||\vec{\beta}||\vec{\gamma}|$

34. The position vectors of points A, B, and C are $\hat{i}+\hat{j}+\hat{k}$, $\hat{i}+5\hat{j}-\hat{k}$ and $2\hat{i}+3\hat{j}+5\hat{k}$, respectively. The greatest angle of triangle ABC is

a. 120° b. 90° c. $\cos^{-1}(3/4)$ d. none of these

35. Given three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, two of which are non-collinear. Further if $(\vec{a} + \vec{b})$ is collinear with $\vec{c}$, $(\vec{b} + \vec{c})$ is collinear with $\vec{a}$, $|\vec{a}| = |\vec{b}| = |\vec{c}| = \sqrt{2}$. Find the value of $\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}$.

a. 3 b. -3
c. 0 d. cannot be evaluated

36. If $\vec{a}$ and $\vec{b}$ are unit vectors such that $(\vec{a}+\vec{b})\cdot(2\vec{a}+3\vec{b})\times(3\vec{a}-2\vec{b}) = \vec{0}$, then angle between $\vec{a}$ and $\vec{b}$ is

a. 0 b. $\pi/2$ c. π d. indeterminate

37. If in a right-angled triangle ABC, the hypotenuse $AB = p$, then $\overrightarrow{AB}\cdot\overrightarrow{AC} + \overrightarrow{BC}\cdot\overrightarrow{BA} + \overrightarrow{CA}\cdot\overrightarrow{CB}$ is equal to

a. $2p^2$ b. $\dfrac{p^2}{2}$ c. p^2 d. none of these

38. Resolved part of vector $\vec{a}$ and along vector $\vec{b}$ is $\vec{a}_1$ and that perpendicular to $\vec{b}$ is $\vec{a}_2$, then $\vec{a}_1 \times \vec{a}_2$ is equal to

a. $\dfrac{(\vec{a}\times\vec{b})\cdot\vec{b}}{|\vec{b}|^2}$ b. $\dfrac{(\vec{a}\cdot\vec{b})\,\vec{a}}{|\vec{a}|^2}$ c. $\dfrac{(\vec{a}\cdot\vec{b})(\vec{b}\times\vec{a})}{|\vec{b}|^2}$ d. $\dfrac{(\vec{a}\cdot\vec{b})(\vec{b}\times\vec{a})}{|\vec{b}\times\vec{a}|}$

39. $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$, $\vec{c} = \hat{i} + \hat{j} - 2\hat{k}$. A vector coplanar with $\vec{b}$ and $\vec{c}$ whose projection on $\vec{a}$ is magnitude $\sqrt{\dfrac{2}{3}}$ is

a. $2\hat{i} + 3\hat{j} - 3\hat{k}$ b. $-2\hat{i} - \hat{j} + 5\hat{k}$ c. $2\hat{i} + 3\hat{j} + 3\hat{k}$ d. $2\hat{i} + \hat{j} + 5\hat{k}$

40. If 'P' is any arbitrary point on the circumcircle of the equilateral triangle of side length l units, then $|\overrightarrow{PA}|^2 + |\overrightarrow{PB}|^2 + |\overrightarrow{PC}|^2$ is always equal to

a. $2l^2$ b. $2\sqrt{3}\,l^2$ c. l^2 d. $3l^2$

41. If $\vec{r}$ and $\vec{s}$ are non-zero constant vectors and the scalar b is chosen such that $|\vec{r} + b\vec{s}|$ is minimum, then the the value of $|b\vec{s}|^2 + |\vec{r} + b\vec{s}|^2$ is equal to

a. $2|\vec{r}|^2$ b. $|\vec{r}|^2/2$ c. $3|\vec{r}|^2$ d. $|\vec{r}|^2$

42. $\vec{a}$ and $\vec{b}$ are two unit vectors that are mutually perpendicular. A unit vector that is equally inclined to $\vec{a}$, $\vec{b}$ and $\vec{a}\times\vec{b}$ is

a. $\dfrac{1}{\sqrt{2}}(\vec{a} + \vec{b} + \vec{a}\times\vec{b})$ b. $\dfrac{1}{2}(\vec{a}\times\vec{b} + \vec{a} + \vec{b})$

c. $\dfrac{1}{\sqrt{3}}(\vec{a} + \vec{b} + \vec{a}\times\vec{b})$ d. $\dfrac{1}{3}(\vec{a} + \vec{b} + \vec{a}\times\vec{b})$

43. Given that $\vec{a}, \vec{b}, \vec{p}, \vec{q}$ are four vectors such that $\vec{a}+\vec{b}=\mu\vec{p}, \vec{b}\cdot\vec{q}=0$ and $(\vec{b})^2=1$, where μ is a scalar. Then $|(\vec{a}\cdot\vec{q})\vec{p}-(\vec{p}\cdot\vec{q})\vec{a}|$ is equal to

a. $2|\vec{p}\cdot\vec{q}|$ b. $(1/2)|\vec{p}\cdot\vec{q}|$ c. $|\vec{p}\times\vec{q}|$ d. $|\vec{p}\cdot\vec{q}|$

44. The position vectors of the vertices A, B and C of a triangle are three unit vectors $\hat{a}, \hat{b}$ and $\hat{c}$, respectively. A vector $\vec{d}$ is such that $\vec{d}\cdot\hat{a}=\vec{d}\cdot\hat{b}=\vec{d}\cdot\hat{c}$ and $\vec{d}=\lambda(\hat{b}+\hat{c})$. Then triangle ABC is

a. acute angled b. obtuse angled c. right angled d. none of these

45. If a is a real constant and A, B and C are variable angles and $\sqrt{a^2-4}\tan A+a\tan B+\sqrt{a^2+4}\tan c=6a$, then the least value of $\tan^2 A+\tan^2 B+\tan^2 C$ is

a. 6 b. 10 c. 12 d. 3

46. The vertex A of triangle ABC is on the line $\vec{r}=\hat{i}+\hat{j}+\lambda\hat{k}$ and the vertices B and C have respective position vectors $\hat{i}$ and $\hat{j}$. Let Δ be the area of the triangle and $\Delta\in[3/2, \sqrt{33}/2]$. Then the range of values of λ corresponding to A is

a. $[-8,-4]\cup[4,8]$ b. $[-4,4]$ c. $[-2,2]$ d. $[-4,-2]\cup[2,4]$

47. A non-zero vector $\vec{a}$ is such that its projections along vectors $\frac{\hat{i}+\hat{j}}{\sqrt{2}}, \frac{-\hat{i}+\hat{j}}{\sqrt{2}}$ and $\hat{k}$ are equal, then unit vector along $\vec{a}$ is

a. $\frac{\sqrt{2}\hat{j}-\hat{k}}{\sqrt{3}}$ b. $\frac{\hat{j}-\sqrt{2}\hat{k}}{\sqrt{3}}$ c. $\frac{\sqrt{2}}{\sqrt{3}}\hat{j}+\frac{\hat{k}}{\sqrt{3}}$ d. $\frac{\hat{j}-\hat{k}}{\sqrt{2}}$

48. Position vector $\hat{k}$ is rotated about origin by angle 135° in such a way that the plane made by it bisects the angle between $\hat{i}$ and $\hat{j}$. Then its new position is

a. $\pm\frac{\hat{i}}{\sqrt{2}}\pm\frac{\hat{j}}{\sqrt{2}}$ b. $\pm\frac{\hat{i}}{2}\pm\frac{\hat{j}}{2}-\frac{\hat{k}}{\sqrt{2}}$ c. $\frac{\hat{i}}{\sqrt{2}}-\frac{\hat{k}}{\sqrt{2}}$ d. none of these

49. In a quadrilateral $ABCD$, $\overrightarrow{AC}$ is the bisector of $\overrightarrow{AB}$ and $\overrightarrow{AD}$, angle between $\overrightarrow{AB}$ and $\overrightarrow{AD}$ is $2\pi/3$, $15|\overrightarrow{AC}|=3|\overrightarrow{AB}|=5|\overrightarrow{AD}|$. Then the angle between $\overrightarrow{BA}$ and $\overrightarrow{CD}$ is

a. $\cos^{-1}\frac{\sqrt{14}}{7\sqrt{2}}$ b. $\cos^{-1}\frac{\sqrt{21}}{7\sqrt{3}}$ c. $\cos^{-1}\frac{2}{\sqrt{7}}$ d. $\cos^{-1}\frac{2\sqrt{7}}{14}$

50. In the following figure, AB, DE and GF are parallel to each other and AD, BG and EF are parallel to each other. If $CD:CE=CG:CB=2:1$, then the value of area (ΔAEG) : area (ΔABD) is equal to

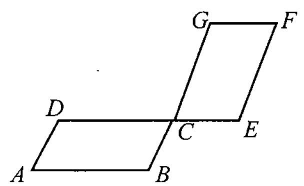


Fig. 2.29

a. 7/2 b. 3 c. 4 d. 9/2

51. Vector $\hat{a}$ in the plane of $\vec{b} = 2\hat{i} + \hat{j}$ and $\vec{c} = \hat{i} - \hat{j} + \hat{k}$ is such that it is equally inclined to $\vec{b}$ and $\vec{d}$ where $\vec{d} = \hat{j} + 2\hat{k}$. The value of $\hat{a}$ is

a. $\dfrac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$ b. $\dfrac{\hat{i}-\hat{j}+\hat{k}}{\sqrt{3}}$ c. $\dfrac{2\hat{i}+\hat{j}}{\sqrt{5}}$ d. $\dfrac{2\hat{i}+\hat{j}}{\sqrt{5}}$

52. Let $ABCD$ be a tetrahedron such that the edges AB, AC and AD are mutually perpendicular. Let the area of triangles ABC, ACD and ADB be 3, 4 and 5 sq. units, respectively. Then the area of triangle BCD is

a. $5\sqrt{2}$ b. 5 c. $\dfrac{\sqrt{5}}{2}$ d. $\dfrac{5}{2}$

53. Let $\vec{f}(t) = [t]\hat{i} + (t-[t])\hat{j} + [t+1]\hat{k}$, where [.] denotes the greatest integer function. Then the vectors $\vec{f}\left(\dfrac{5}{4}\right)$ and $\vec{f}(t), 0 < t < 1$, are

a. parallel to each other

b. perpendicular to each other

c. inclined at an angle $\cos^{-1}\dfrac{2}{\sqrt{7(1-t^2)}}$

d. inclined at $\cos^{-1}\dfrac{8+t}{9\sqrt{1+t^2}}$

54. If $\vec{a}$ is parallel to $\vec{b}\times\vec{c}$, then $(\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{c})$ is equal to

a. $|\vec{a}|^2(\vec{b}\cdot\vec{c})$ b. $|\vec{b}|^2(\vec{a}\cdot\vec{c})$ c. $|\vec{c}|^2(\vec{a}\cdot\vec{b})$ d. none of these

55. Three vectors $\hat{i}+\hat{j}$, $\hat{j}+\hat{k}$ and $\hat{k}+\hat{i}$ taken two at a time form three planes. The three unit vectors drawn perpendicular to these three planes form a parallelepiped of volume

a. 1/3 b. 4 c. $(3\sqrt{3})/4$ d. $4\sqrt{3}$

56. If $\vec{d} = \vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a}$ is a non-zero vector and $|(\vec{d}\cdot\vec{c})(\vec{a}\times\vec{b}) + (\vec{d}\cdot\vec{a})(\vec{b}\times\vec{c}) + (\vec{d}\cdot\vec{b})(\vec{c}\times\vec{a})| = 0$, then

a. $|\vec{a}| = |\vec{b}| = |\vec{c}|$

b. $|\vec{a}| + |\vec{b}| + |\vec{c}| = |\vec{d}|$

c. $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar

d. none of these

57. If $|\vec{a}| = 2$ and $|\vec{b}| = 3$ and $\vec{a}\cdot\vec{b} = 0$, then $(\vec{a}\times(\vec{a}\times(\vec{a}\times(\vec{a}\times\vec{b}))))$ is equal to

a. $48\hat{b}$ b. $-48\hat{b}$ c. $48\hat{a}$ d. $-48\hat{a}$

58. If the two diagonals of one of its faces are $6\hat{i}+6\hat{k}$ and $4\hat{j}+2\hat{k}$ and of the edges not containing the given diagonals is $\vec{c} = 4\hat{j} - 8\hat{k}$ then the volume of a parallelepiped is

a. 60 b. 80 c. 100 d. 120

59. The volume of a tetrahedron formed by the coterminus edges $\vec{a}, \vec{b}$ and $\vec{c}$ is 3. Then the volume of the parallelepiped formed by the coterminus edges $\vec{a}+\vec{b}, \vec{b}+\vec{c}$ and $\vec{c}+\vec{a}$ is

a. 6 b. 18 c. 36 d. 9

60. If $\vec{a}, \vec{b}$ and $\vec{c}$ are three mutually orthogonal unit vectors, then the triple product $[\vec{a}+\vec{b}+\vec{c}\ \ \vec{a}+\vec{b}\ \ \vec{b}+\vec{c}]$ equals

a. 0 b. 1 or −1 c. 1 d. 3

61. Vector $\vec{c}$ is perpendicular to vectors $\vec{a} = (2, -3, 1)$ and $\vec{b} = (1, -2, 3)$ and satisfies the condition $\vec{c} \cdot (\hat{i} + 2\hat{j} - 7\hat{k}) = 10$. Then vector $\vec{c}$ is equal to

a. $(7, 5, 1)$ b. $(-7, -5, -1)$ c. $(1, 1, -1)$ d. none of these

62. Given $\vec{a} = x\hat{i} + y\hat{j} + 2\hat{k}$, $\vec{b} = \hat{i} - \hat{j} + \hat{k}$, $\vec{c} = \hat{i} + 2\hat{j}$; $\vec{a} \perp \vec{b}$, $\vec{a} \cdot \vec{c} = 4$. Then

a. $[\vec{a}\,\vec{b}\,\vec{c}]^2 = |\vec{a}|$ b. $[\vec{a}\,\vec{b}\,\vec{c}] = |\vec{a}|$ c. $[\vec{a}\,\vec{b}\,\vec{c}] = 0$ d. $[\vec{a}\,\vec{b}\,\vec{c}] = |\vec{a}|^2$

63. Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ be three non-zero vectors such that $\vec{c}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$. If the angle between $\vec{a}$ and $\vec{b}$ is $\pi/6$, then the value of $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2$ is

a. 0 b. 1

c. $\frac{1}{4}(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)$ d. $\frac{3}{4}(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)$

64. Let $\vec{r}, \vec{a}, \vec{b}$ and $\vec{c}$ be four non-zero vectors such that $\vec{r} \cdot \vec{a} = 0$, $|\vec{r} \times \vec{b}| = |\vec{r}||\vec{b}|$ and $|\vec{r} \times \vec{c}| = |\vec{r}||\vec{c}|$. Then $[a\ b\ c]$ is equal to

a. $|a||b||c|$ b. $-|a||b||c|$ c. 0 d. none of these

65. If $\vec{a}, \vec{b}$ and $\vec{c}$ are such that $[\vec{a}\,\vec{b}\,\vec{c}] = 1$, $\vec{c} = \lambda\vec{a} \times \vec{b}$, angle between $\vec{a}$ and $\vec{b}$ is $2\pi/3$, $|\vec{a}| = \sqrt{2}$, $|\vec{b}| = \sqrt{3}$ and $|\vec{c}| = \frac{1}{\sqrt{3}}$, then the angle between $\vec{a}$ and $\vec{b}$ is

a. $\frac{\pi}{6}$ b. $\frac{\pi}{4}$ c. $\frac{\pi}{3}$ d. $\frac{\pi}{2}$

66. If $4\vec{a} + 5\vec{b} + 9\vec{c} = 0$, then $(\vec{a} \times \vec{b}) \times [(\vec{b} \times \vec{c}) \times (\vec{c} \times \vec{a})]$ is equal to

a. a vector perpendicular to the plane of $\vec{a}, \vec{b}$ and $\vec{c}$
b. a scalar quantity
c. $\vec{0}$
d. none of these

67. Value of $[\vec{a} \times \vec{b}\ \ \vec{a} \times \vec{c}\ \ \vec{d}]$ is always equal to

a. $(\vec{a} \cdot \vec{d})[\vec{a}\,\vec{b}\,\vec{c}]$ b. $(\vec{a} \cdot \vec{c})[\vec{a}\,\vec{b}\,\vec{d}]$ c. $(\vec{a} \cdot \vec{b})[\vec{a}\,\vec{b}\,\vec{d}]$ d. none of these

68. Let $\hat{a}$ and $\hat{b}$ be mutually perpendicular unit vectors. Then for any arbitrary $\vec{r}$,

a. $\vec{r} = (\vec{r} \cdot \hat{a})\hat{a} + (\vec{r} \cdot \hat{b})\hat{b} + (\vec{r} \cdot (\hat{a} \times \hat{b}))(\hat{a} \times \hat{b})$
b. $\vec{r} = (\vec{r} \cdot \hat{a}) - (\vec{r} \cdot \hat{b})\hat{b} - (\vec{r} \cdot (\hat{a} \times \hat{b}))(\hat{a} \times \hat{b})$
c. $\vec{r} = (\vec{r} \cdot \hat{a})\hat{a} - (\vec{r} \cdot \hat{b})\hat{b} + (\vec{r} \cdot (\hat{a} \times \hat{b}))(\hat{a} \times \hat{b})$
d. none of these

69. Let $\vec{a}$ and $\vec{b}$ be unit vectors that are perpendicular to each other. Then $[\vec{a}+(\vec{a}\times\vec{b})\quad \vec{b}+(\vec{a}\times\vec{b})\quad \vec{a}\times\vec{b}]$ will always be equal to

a. 1 **b.** 0 **c.** -1 **d.** none of these

70. $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a}|=1, |\vec{b}|=4$ and $\vec{a}\cdot\vec{b}=2$. If $\vec{c}=(2\vec{a}\times\vec{b})-3\vec{b}$, then find the angle between $\vec{b}$ and $\vec{c}$.

a. $\frac{\pi}{3}$ **b.** $\frac{\pi}{6}$ **c.** $\frac{3\pi}{4}$ **d.** $\frac{5\pi}{6}$

71. $\vec{b}$ and $\vec{c}$ are unit vectors. Then for any arbitrary vector $\vec{a}$, $(((\vec{a}\times\vec{b})+(\vec{a}\times\vec{c}))\times(\vec{b}\times\vec{c}))\cdot(\vec{b}-\vec{c})$ is always equal to

a. $|\vec{a}|$ **b.** $\frac{1}{2}|\vec{a}|$ **c.** $\frac{1}{3}|\vec{a}|$ **d.** none of these

72. If $\vec{a}.\vec{b}=\beta$ and $\vec{a}\times\vec{b}=\vec{c}$, then $\vec{b}$ is

a. $\frac{(\beta\vec{a}-\vec{a}\times\vec{c})}{|\vec{a}|^2}$ **b.** $\frac{(\beta\vec{a}+\vec{a}\times\vec{c})}{|\vec{a}|^2}$

c. $\frac{(\beta\vec{c}-\vec{a}\times\vec{c})}{|\vec{a}|^2}$ **d.** $\frac{(\beta\vec{a}+\vec{a}\times\vec{c})}{|\vec{a}|^2}$

73. If $a(\vec{\alpha}\times\vec{\beta})+b(\vec{\beta}\times\vec{\gamma})+c(\vec{\gamma}\times\vec{\alpha})=0$ and at least one of a, b and c is non-zero, then vectors $\vec{\alpha}, \vec{\beta}$ and $\vec{\gamma}$ are

a. parallel **b.** coplanar

c. mutually perpendicular **d.** none of these

74. If $(\vec{a}\times\vec{b})\times(\vec{b}\times\vec{c})=\vec{b}$, where $\vec{a}, \vec{b}$ and $\vec{c}$ are non-zero vectors, then

a. $\vec{a}, \vec{b}$ and $\vec{c}$ can be coplanar **b.** $\vec{a}, \vec{b}$ and $\vec{c}$ must be coplanar

c. $\vec{a}, \vec{b}$ and $\vec{c}$ cannot be coplanar **d.** none of these

75. If $\vec{r}.\vec{a}=\vec{r}.\vec{b}=\vec{r}.\vec{c}=\frac{1}{2}$ for some non-zero vector $\vec{r}$, then the area of the triangle whose vertices are $A(\vec{a}), B(\vec{b})$ and $C(\vec{c})$ is ($\vec{a},\vec{b},\vec{c}$ are non-coplanar)

a. $|[\vec{a}\,\vec{b}\,\vec{c}]|$ **b.** $|\vec{r}|$ **c.** $|[\vec{a}\,\vec{b}\,\vec{c}]\vec{r}|$ **d.** none of these

76. A vector of magnitude 10 along the normal to the curve $3x^2+8xy+2y^2-3=0$ at its point $P(1,0)$ can be

a. $6\hat{i}+8\hat{j}$ **b.** $-8\hat{i}+3\hat{j}$ **c.** $6\hat{i}-8\hat{j}$ **d.** $8\hat{i}+6\hat{j}$

77. If $\vec{a}$ and $\vec{b}$ are two unit vectors inclined at an angle $\pi/3$, then $\{\vec{a}\times(\vec{b}+\vec{a}\times\vec{b})\}\cdot\vec{b}$ is equal to

a. $\frac{-3}{4}$ b. $\frac{1}{4}$ c. $\frac{3}{4}$ d. $\frac{1}{2}$

78. If $\vec{a}$ and $\vec{b}$ are orthogonal unit vectors, then for a vector $\vec{r}$ non-coplanar with $\vec{a}$ and $\vec{b}$, vector $\vec{r}\times\vec{a}$ is equal to

a. $[\vec{r}\,\vec{a}\,\vec{b}]\vec{b}-(\vec{r}\cdot\vec{b})(\vec{b}\times\vec{a})$ b. $[\vec{r}\,\vec{a}\,\vec{b}](\vec{a}+\vec{b})$

c. $[\vec{r}\,\vec{a}\,\vec{b}]\vec{a}+(\vec{r}\cdot\vec{a})\vec{a}\times\vec{b}$ d. none of these

79. If $\vec{a},\vec{b},\vec{c}$ are any three non-coplanar vectors, then the equation $[\vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}\ \ \vec{a}\times\vec{b}]x^2 + [\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}]x + 1 + [\vec{b}-\vec{c}\ \ \vec{c}-\vec{a}\ \ \vec{a}-\vec{b}]=0$ has roots

a. real and distinct b. real c. equal d. imaginary

80. If $\vec{x}+\vec{c}\times\vec{y}=\vec{a}$ and $\vec{y}+\vec{c}\times\vec{x}=\vec{b}$, where $\vec{c}$ is a non-zero vector, then which of the following is not correct.

a. $\vec{x}=\frac{\vec{b}\times\vec{c}+\vec{a}+(\vec{c}\cdot\vec{a})\vec{c}}{1+\vec{c}\cdot\vec{c}}$ b. $\vec{x}=\frac{\vec{c}\times\vec{b}+\vec{b}+(\vec{c}\cdot\vec{a})\vec{c}}{1+\vec{c}\cdot\vec{c}}$

c. $\vec{y}=\frac{\vec{a}\times\vec{c}+\vec{b}+(\vec{c}\cdot\vec{b})\vec{c}}{1+\vec{c}\cdot\vec{c}}$ d. none of these

81. The condition for equations $\vec{r}\times\vec{a}=\vec{b}$ and $\vec{r}\times\vec{c}=\vec{d}$ to be consistent is

a. $\vec{b}\cdot\vec{c}=\vec{a}\cdot\vec{d}$ b. $\vec{a}\cdot\vec{b}=\vec{c}\cdot\vec{d}$ c. $\vec{b}\cdot\vec{c}+\vec{a}\cdot\vec{d}=0$ d. $\vec{a}\cdot\vec{b}+\vec{c}\cdot\vec{d}=0$

82. If $\vec{a}$ and $\vec{b}$ are non-zero non-collinear vectors, then $[\vec{a}\,\vec{b}\,\hat{i}]\hat{i}+[\vec{a}\,\vec{b}\,\hat{j}]\hat{j}+[\vec{a}\,\vec{b}\,\hat{k}]\hat{k}$ is equal to

a. $\vec{a}+\vec{b}$ b. $\vec{a}\times\vec{b}$ c. $\vec{a}-\vec{b}$ d. $\vec{b}\times\vec{a}$

83. If $\vec{a}=2\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=\hat{i}+2\hat{j}+2\hat{k}$, $\vec{c}=\hat{i}+\hat{j}+2\hat{k}$ and $(1+\alpha)\hat{i}+\beta(1+\alpha)\hat{j}+\gamma(1+\alpha)(1+\beta)\hat{k}=\vec{a}\times(\vec{b}\times\vec{c})$, then α, β and γ are

a. $-2,-4,-\frac{2}{3}$ b. $2,-4,\frac{2}{3}$ c. $-2,4,\frac{2}{3}$ d. $2,4,-\frac{2}{3}$

84. Let $\vec{a}(x)=(\sin x)\hat{i}+(\cos x)\hat{j}$ and $\vec{b}(x)=(\cos 2x)\hat{i}+(\sin 2x)\hat{j}$ be two variable vectors ($x\in R$), then $\vec{a}(x)$ and $\vec{b}(x)$ are

a. collinear for unique value of x b. perpendicular for infinite values of x

c. zero vectors for unique value of x d. none of these

85. For any two vectors $\vec{a}$ and $\vec{b}$, $(\vec{a}\times\hat{i})\cdot(\vec{b}\times\hat{i})+(\vec{a}\times\hat{j})\cdot(\vec{b}\times\hat{j})+(\vec{a}\times\hat{k})\cdot(\vec{b}\times\hat{k})$ is always equal to

a. $\vec{a}\cdot\vec{b}$ b. $2\vec{a}\cdot\vec{b}$ c. zero d. none of these

86. Let $\vec{a},\vec{b}$ and $\vec{c}$ be three non-coplanar vectors and $\vec{r}$ be any arbitrary vector. Then $(\vec{a}\times\vec{b})\times(\vec{r}\times\vec{c})+(\vec{b}\times\vec{c})\times(\vec{r}\times\vec{a})+(\vec{c}\times\vec{a})\times(\vec{r}\times\vec{b})$ is always equal to

a. $[\vec{a}\,\vec{b}\,\vec{c}]\vec{r}$ **b.** $2[\vec{a}\,\vec{b}\,\vec{c}]\vec{r}$ **c.** $3[\vec{a}\,\vec{b}\,\vec{c}]\vec{r}$ **d.** none of these

87. If $\vec{p}=\dfrac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$, $\vec{q}=\dfrac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}$ and $\vec{r}=\dfrac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$, where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three non-coplanar vectors, then the value of the expression $(\vec{a}+\vec{b}+\vec{c})\cdot(\vec{p}+\vec{q}+\vec{r})$ is

a. 3 **b.** 2 **c.** 1 **d.** 0

88. $A(\vec{a})$, $B(\vec{b})$ and $C(\vec{c})$ are the vertices of triangle ABC and $R(\vec{r})$ is any point in the plane of triangle ABC, then $\vec{r}.(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})$ is always equal to

a. zero **b.** $[\vec{a}\,\vec{b}\,\vec{c}]$ **c.** $-[\vec{a}\,\vec{b}\,\vec{c}]$ **d.** none of these

89. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-coplanar vectors and $\vec{a}\times\vec{c}$ is perpendicular to $\vec{a}\times(\vec{b}\times\vec{c})$, then the value of $[\vec{a}\times(\vec{b}\times\vec{c})]\times\vec{c}$ is equal to

a. $[\vec{a}\,\vec{b}\,\vec{c}]\,\vec{c}$ **b.** $[\vec{a}\,\vec{b}\,\vec{c}]\vec{b}$ **c.** $\vec{0}$ **d.** $[\vec{a}\,\vec{b}\,\vec{c}]\vec{a}$

90. If V be the volume of a tetrahedron and V' be the volume of another tetrahedran formed by the centroids of faces of the previous tetrahedron and $V = KV'$; then K is equal to

a. 9 **b.** 12 **c.** 27 **d.** 81

91. $[(\vec{a}\times\vec{b})\times(\vec{b}\times\vec{c})\ \ (\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})\ \ (\vec{c}\times\vec{a})\times(\vec{a}\times\vec{b})]$ is equal to (where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-zero non-coplanar vectors)

a. $[\vec{a}\,\vec{b}\,\vec{c}]^2$ **b.** $[\vec{a}\,\vec{b}\,\vec{c}]^3$ **c.** $[\vec{a}\,\vec{b}\,\vec{c}]^4$ **d.** $[\vec{a}\,\vec{b}\,\vec{c}]$

92. If $\vec{r}=x_1(\vec{a}\times\vec{b})+x_2(\vec{b}\times\vec{a})+x_3(\vec{c}\times\vec{d})$ and $4[\vec{a}\,\vec{b}\,\vec{c}]=1$, then $x_1+x_2+x_3$ is equal to

a. $\frac{1}{2}\vec{r}\cdot(\vec{a}+\vec{b}+\vec{c})$ **b.** $\frac{1}{4}\vec{r}\cdot(\vec{a}+\vec{b}+\vec{c})$ **c.** $2\vec{r}\cdot(\vec{a}+\vec{b}+\vec{c})$ **d.** $4\vec{r}\cdot(\vec{a}+\vec{b}+\vec{c})$

93. If $\vec{a}\perp\vec{b}$, then vector $\vec{v}$ in terms of $\vec{a}$ and $\vec{b}$ satisfying the equations $\vec{v}\cdot\vec{a}=0$ and $\vec{v}\cdot\vec{b}=1$ and $[\vec{v}\,\vec{a}\,\vec{b}]=1$ is

a. $\dfrac{\vec{b}}{|\vec{b}|^2}+\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|^2}$ **b.** $\dfrac{\vec{b}}{|\vec{b}|}+\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|^2}$ **c.** $\dfrac{\vec{b}}{|\vec{b}|^2}+\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$ **d.** none of these

94. If $\vec{a}'=\hat{i}+\hat{j}$, $\vec{b}'=\hat{i}-\hat{j}+2\hat{k}$ and $\vec{c}'=2\hat{i}+\hat{j}-\hat{k}$, then the altitude of the parallelepiped formed by the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ having base formed by $\vec{b}$ and $\vec{c}$ is (where $\vec{a}'$ is reciprocal vector $\vec{a}$, etc.)

a. 1 **b.** $3\sqrt{2}/2$ **c.** $1/\sqrt{6}$ **d.** $1/\sqrt{2}$

95. If $\vec{a}=\hat{i}+\hat{j}$, $\vec{b}=\hat{j}+\hat{k}$, $\vec{c}=\hat{k}+\hat{i}$, then in the reciprocal system of vectors $\vec{a},\vec{b},\vec{c}$ reciprocal $\vec{a}$ of vector $\vec{a}$ is

a. $\dfrac{\hat{i}+\hat{j}+\hat{k}}{2}$ **b.** $\dfrac{\hat{i}-\hat{j}+\hat{k}}{2}$ **c.** $\dfrac{-\hat{i}-\hat{j}+\hat{k}}{2}$ **d.** $\dfrac{\hat{i}+\hat{j}-\hat{k}}{2}$

Multiple Correct Answers Type

Solutions on page 2.114

Each question has four choices *a, b, c* and *d*, out of which *one or more* are correct.

1. If unit vectors $\vec{a}$ and $\vec{b}$ are inclined at an angle 2θ such that $|\vec{a}-\vec{b}|<1$ and $0\le\theta\le\pi$, then θ lies in the interval
   a. $[0, \pi/6)$ b. $(5\pi/6, \pi]$ c. $[\pi/6, \pi/2)$ d. $(\pi/2, 5\pi/6]$

2. $\vec{b}$ and $\vec{c}$ are non-collinear if $\vec{a}\times(\vec{b}\times\vec{c})+(\vec{a}\cdot\vec{b})\vec{b}=(4-2x-\sin y)\,\vec{b}+(x^2-1)\,\vec{c}$ and $(\vec{c}\cdot\vec{c})\vec{a}=\vec{c}$. Then
   a. $x=1$ b. $x=-1$
   c. $y=(4n+1)\dfrac{\pi}{2},\ n\in I$ d. $y=(2n+1)\dfrac{\pi}{2},\ n\in I$

3. Unit vectors $\vec{a}$ and $\vec{b}$ are perpendicular, and unit vector $\vec{c}$ is inclined at an angle θ to both $\vec{a}$ and $\vec{b}$. If $\vec{c}=\alpha\vec{a}+\beta\vec{b}+\gamma(\vec{a}\times\vec{b})$, then
   a. $a=\beta$ b. $\gamma^2=1-2\alpha^2$ c. $\gamma^2=-\cos 2\theta$ d. $\beta^2=\dfrac{1+\cos 2\theta}{2}$

4. $\vec{a}$ and $\vec{b}$ are two given vectors. With these vectors as adjacent sides, a parallelogram is constructed. The vector which is the altitude of the parallelogram and which is perpendicular to $\vec{a}$ is
   a. $\dfrac{(\vec{a}\cdot\vec{b})}{|\vec{a}|^2}\vec{a}-\vec{b}$ b. $\dfrac{1}{|\vec{a}|^2}\{|\vec{a}|^2\,\vec{b}-(\vec{a}\cdot\vec{b})\vec{a}\}$
   c. $\dfrac{\vec{a}\times(\vec{a}\times\vec{b})}{|\vec{a}|^2}$ d. $\dfrac{\vec{a}\times(\vec{b}\times\vec{a})}{|\vec{b}|^2}$

5. If $\vec{a}\times(\vec{b}\times\vec{c})$ is perpendicular to $(\vec{a}\times\vec{b})\times\vec{c}$, we may have
   a. $(\vec{a}\cdot\vec{c})|\vec{b}|^2=(\vec{a}\cdot\vec{b})(\vec{b}\cdot\vec{c})$ b. $\vec{a}\cdot\vec{b}=0$
   c. $\vec{a}\cdot\vec{c}=0$ d. $\vec{b}\cdot\vec{c}=0$

6. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be vectors forming right-hand triad. Let $\vec{p}=\dfrac{\vec{b}\times\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$, $\vec{q}=\dfrac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}$ and $\vec{r}=\dfrac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$. If $x\in R^+$, then
   a. $x[\vec{a}\ \vec{b}\ \vec{c}]+\dfrac{[\vec{p}\ \vec{q}\ \vec{r}]}{x}$ has least value 2
   b. $x^4[\vec{a}\ \vec{b}\ \vec{c}]^2+\dfrac{[\vec{p}\ \vec{q}\ \vec{r}]}{x^2}$ has least value $(3/2^{2/3})$
   c. $[\vec{p}\ \vec{q}\ \vec{r}]>0$
   d. none of these

7. $a_1, a_2, a_3 \in R - \{0\}$ and $a_1 + a_2 \cos 2x + a_3 \sin^2 x = 0$ for all $x \in R$, then

a. vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = 4\hat{i} + 2\hat{j} + \hat{k}$ are perpendicular to each other

b. vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} + 2\hat{k}$ are parallel to each other

c. if vector $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ is of length $\sqrt{6}$ units, then one of the ordered tripplet $(a_1, a_2, a_3) = (1, -1, -2)$

d. if $2a_1 + 3a_2 + 6a_3 = 26$, then $|a_1\hat{i} + a_2\hat{j} + a_3\hat{k}|$ is $2\sqrt{6}$

8. If $\vec{a}$ and $\vec{b}$ are two vectors and angle between them is θ, then

a. $|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2$

b. $|\vec{a} \times \vec{b}| = (\vec{a} \cdot \vec{b})$, if $\theta = \pi/4$

c. $\vec{a} \times \vec{b} = (\vec{a} \cdot \vec{b})\hat{n}$, ($\hat{n}$ is normal unit vector), if $\theta = \pi/4$

d. $(\vec{a} \times \vec{b}) \cdot (\vec{a} + \vec{b}) = 0$

9. Let $\vec{a}$ and $\vec{b}$ be two non-zero perpendicular vectors. A vector $\vec{r}$ satisfying the equation $\vec{r} \times \vec{b} = \vec{a}$ can be

a. $\vec{b} - \dfrac{\vec{a} \times \vec{b}}{|\vec{b}|^2}$ b. $2\vec{b} - \dfrac{\vec{a} \times \vec{b}}{|\vec{b}|^2}$ c. $|\vec{a}|\vec{b} - \dfrac{\vec{a} \times \vec{b}}{|\vec{b}|^2}$ d. $|\vec{b}|\vec{b} - \dfrac{\vec{a} \times \vec{b}}{|\vec{b}|^2}$

10. If vectors $\vec{b} = (\tan\alpha, -1, 2\sqrt{\sin\alpha/2})$ and $\vec{c} = \left(\tan\alpha, \tan\alpha, -\dfrac{3}{\sqrt{\sin\alpha/2}}\right)$ are orthogonal and vector $\vec{a} = (1, 3, \sin 2\alpha)$ makes an obtuse angle with the z-axis, then the value of α is

a. $\alpha = (4n+1)\pi + \tan^{-1}2$
b. $\alpha = (4n+1)\pi - \tan^{-1}2$
c. $\alpha = (4n+2)\pi + \tan^{-1}2$
d. $\alpha = (4n+2)\pi - \tan^{-1}2$

11. Let $\vec{r}$ be a unit vector satisfying $\vec{r} \times \vec{a} = \vec{b}$, where $|\vec{a}| = \sqrt{3}$ and $|\vec{b}| = \sqrt{2}$. Then

a. $\vec{r} = \dfrac{2}{3}(\vec{a} + \vec{a} \times \vec{b})$
b. $\vec{r} = \dfrac{1}{3}(\vec{a} + \vec{a} \times \vec{b})$
c. $\vec{r} = \dfrac{2}{3}(\vec{a} - \vec{a} \times \vec{b})$
d. $\vec{r} = \dfrac{1}{3}(-\vec{a} + \vec{a} \times \vec{b})$

12. If $\vec{a}$ and $\vec{b}$ are unequal unit vectors such that $(\vec{a} - \vec{b}) \times [(\vec{b} + \vec{a}) \times (2\vec{a} + \vec{b})] = \vec{a} + \vec{b}$, then angle θ between $\vec{a}$ and $\vec{b}$ is

a. 0 b. $\pi/2$ c. $\pi/4$ d. π

13. If $\vec{a}$ and $\vec{b}$ are two unit vectors perpendicular to each other and $\vec{c} = \lambda_1\vec{a} + \lambda_2\vec{b} + \lambda_3(\vec{a} \times \vec{b})$, then which of the following is (are) true?

a. $\lambda_1 = \vec{a} \cdot \vec{c}$
b. $\lambda_2 = |\vec{b} \times \vec{c}|$
c. $\lambda_3 = |(\vec{a} \times \vec{b}) \times \vec{c}|$
d. $\lambda_1 + \lambda_2 + \lambda_3 = (\vec{a} + \vec{b} + \vec{a} \times \vec{b}) \cdot \vec{c}$

14. If vectors $\vec{a}$ and $\vec{b}$ are non-collinear, then $\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|}$ is

a. a unit vector

b. in the plane of $\vec{a}$ and $\vec{b}$

c. equally inclined to $\vec{a}$ and $\vec{b}$

d. perpendicular to $\vec{a} \times \vec{b}$

15. If $\vec{a}$ and $\vec{b}$ are non zero vectors such that $|\vec{a} + \vec{b}| = |\vec{a} - 2\vec{b}|$, then

a. $2\vec{a}.\vec{b} = |\vec{b}|^2$

b. $\vec{a}.\vec{b} = |\vec{b}|^2$

c. least value of $\vec{a}\cdot\vec{b} + \frac{1}{|\vec{b}|^2 + 2}$ is $\sqrt{2}$

d. least value of $\vec{a}\cdot\vec{b} + \frac{1}{|\vec{b}| + 2}$ is $\sqrt{2} - 1$

16. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be non-zero vectors and $\vec{V_1} = \vec{a} \times (\vec{b} \times \vec{c})$ and $\vec{V_2} = (\vec{a} \times \vec{b}) \times \vec{c}$. Vectors $\vec{V_1}$ and $\vec{V_2}$ are equal. Then

a. $\vec{a}$ and $\vec{b}$ are orthogonal

b. $\vec{a}$ and $\vec{c}$ are collinear

c. $\vec{b}$ and $\vec{c}$ are orthogonal

d. $\vec{b} = \lambda(\vec{a} \times \vec{c})$ when λ is a scalar

17. Vectors $\vec{A}$ and $\vec{B}$ satisfying the vector equation $\vec{A} + \vec{B} = \vec{a}$, $\vec{A} \times \vec{B} = \vec{b}$ and $\vec{A}\cdot\vec{a} = 1$, where $\vec{a}$ and $\vec{b}$ are given vectors, are

a. $\vec{A} = \frac{(\vec{a} \times \vec{b}) - \vec{a}}{a^2}$

b. $\vec{B} = \frac{(\vec{b} \times \vec{a}) + \vec{a}(a^2 - 1)}{a^2}$

c. $\vec{A} = \frac{(\vec{a} \times \vec{b}) + \vec{a}}{a^2}$

d. $\vec{B} = \frac{(\vec{b} \times \vec{a}) - \vec{a}(a^2 - 1)}{a^2}$

18. A vector $\vec{d}$ is equally inclined to three vectors $\vec{a} = \hat{i} - \hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} + \hat{j}$ and $\vec{c} = 3\hat{j} - 2\hat{k}$. Let $\vec{x}, \vec{y}$ and $\vec{z}$ be three vectors in the plane of $\vec{a}, \vec{b}$; $\vec{b}, \vec{c}$; $\vec{c}, \vec{a}$, respectively. Then

a. $\vec{x}\cdot\vec{d} = -1$

b. $\vec{y}\cdot\vec{d} = 1$

c. $\vec{z}\cdot\vec{d} = 0$

d. $\vec{r}\cdot\vec{d} = 0$, where $\vec{r} = \lambda\vec{x} + \mu\vec{y} + \delta\vec{z}$

19. Vectors perpendicular to $\hat{i} - \hat{j} - \hat{k}$ and in the plane of $\hat{i} + \hat{j} + \hat{k}$ and $-\hat{i} + \hat{j} + \hat{k}$ are

a. $\hat{i} + \hat{k}$ **b.** $2\hat{i} + \hat{j} + \hat{k}$ **c.** $3\hat{i} + 2\hat{j} + \hat{k}$ **d.** $-4\hat{i} - 2\hat{j} - 2\hat{k}$

20. If side $\overrightarrow{AB}$ of an equilateral triangle ABC lying in the x–y plane is $3\hat{i}$, then side $\overrightarrow{CB}$ can be

a. $-\frac{3}{2}(\hat{i} - \sqrt{3}\hat{j})$ **b.** $\frac{3}{2}(\hat{i} - \sqrt{3}\hat{j})$ **c.** $-\frac{3}{2}(\hat{i} + \sqrt{3}\hat{j})$ **d.** $\frac{3}{2}(\hat{i} + \sqrt{3}\hat{j})$

21. The angles of a triangle, two of whose sides are represented by vectors $\sqrt{3}(\hat{a}\times\vec{b})$ and $\hat{b}-(\hat{a}\cdot\vec{b})\hat{a}$, where $\vec{b}$ is a non-zero vector and $\hat{a}$ is a unit vector in the direction of $\vec{a}$, are

a. $\tan^{-1}(\sqrt{3})$ b. $\tan^{-1}(1/\sqrt{3})$ c. $\cot^{-1}(0)$ d. $\tan^{-1}(1)$

22. $\vec{a},\vec{b}$ and $\vec{c}$ are unimodular and coplanar. A unit vector $\vec{d}$ is perpendicular to them. If $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=\frac{1}{6}\hat{i}-\frac{1}{3}\hat{j}+\frac{1}{3}\hat{k}$, and the angle between $\vec{a}$ and $\vec{b}$ is 30°, then $\vec{c}$ is

a. $(\hat{i}-2\hat{j}+2\hat{k})/3$ b. $(-\hat{i}+2\hat{j}-2\hat{k})/3$ c. $(2\hat{i}+2\hat{j}-\hat{k})/3$ d. $(-2\hat{i}-2\hat{j}+\hat{k})/3$

23. If $\vec{a}+2\vec{b}+3\vec{c}=\vec{0}$, then $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}=$

a. $2(\vec{a}\times\vec{b})$ b. $6(\vec{b}\times\vec{c})$ c. $3(\vec{c}\times\vec{a})$ d. $\vec{0}$

24. $\vec{a}$ and $\vec{b}$ are two non-collinear unit vectors, and $\vec{u}=\vec{a}-(\vec{a}\cdot\vec{b})\vec{b}$ and $\vec{v}=\vec{a}\times\vec{b}$. Then $|\vec{v}|$ is

a. $|\vec{u}|$ b. $|\vec{u}|+|\vec{u}\cdot\vec{b}|$ c. $|\vec{u}|+|\vec{u}\cdot\vec{a}|$ d. none of these

25. If $\vec{a}\times\vec{b}=\vec{c},\ \vec{b}\times\vec{c}=\vec{a}$, where $\vec{c}\neq\vec{0}$, then

a. $|\vec{a}|=|\vec{c}|$ b. $|\vec{a}|=|\vec{b}|$

c. $|\vec{b}|=1$ d. $|\vec{a}|=|\vec{b}|=|\vec{c}|=1$

26. Let $\vec{a},\vec{b}$ and $\vec{c}$ be three non-coplanar vectors and $\vec{d}$ be a non-zero vector, which is perpendicular to $(\vec{a}+\vec{b}+\vec{c})$. Now $\vec{d}=(\vec{a}\times\vec{b})\sin x+(\vec{b}\times\vec{c})\cos y+2(\vec{c}\times\vec{a})$. Then

a. $\dfrac{\vec{d}\cdot(\vec{a}+\vec{c})}{[\vec{a}\,\vec{b}\,\vec{c}]}=2$ b. $\dfrac{\vec{d}\cdot(\vec{a}+\vec{c})}{[\vec{a}\,\vec{b}\,\vec{c}]}=-2$

c. minimum value of x^2+y^2 is $\pi^2/4$ d. minimum value of x^2+y^2 is $5\pi^2/4$

27. If $\vec{a},\vec{b}$ and $\vec{c}$ are three unit vectors such that $\vec{a}\times(\vec{b}\times\vec{c})=\frac{1}{2}\vec{b}$, then ($\vec{b}$ and $\vec{c}$ being non-parallel)

a. angle between $\vec{a}$ and $\vec{b}$ is $\pi/3$ b. angle between $\vec{a}$ and $\vec{c}$ is $\pi/3$

c. angle between $\vec{a}$ and $\vec{b}$ is $\pi/2$ d. angle between $\vec{a}$ and $\vec{c}$ is $\pi/2$

28. If in triangle ABC, $\overrightarrow{AB}=\dfrac{\vec{u}}{|\vec{u}|}-\dfrac{\vec{v}}{|\vec{v}|}$ and $\overrightarrow{AC}=\dfrac{2\vec{u}}{|\vec{u}|}$, where $|\vec{u}|\neq|\vec{v}|$, then

a. $1+\cos 2A+\cos 2B+\cos 2C=0$ b. $\sin A=\cos C$

c. projection of AC on BC is equal to BC d. projection of AB on BC is equal to AB

29. $[\vec{a}\times\vec{b}\ \ \vec{c}\times\vec{d}\ \ \vec{e}\times\vec{f}]$ is equal to

a. $[\vec{a}\,\vec{b}\,\vec{d}][\vec{c}\,\vec{e}\,\vec{f}]-[\vec{a}\,\vec{b}\,\vec{c}][\vec{d}\,\vec{e}\,\vec{f}]$ b. $[\vec{a}\,\vec{b}\,\vec{e}][\vec{f}\,\vec{c}\,\vec{d}]-[\vec{a}\,\vec{b}\,\vec{f}][\vec{e}\,\vec{c}\,\vec{d}]$

c. $[\vec{c}\,\vec{d}\,\vec{a}][\vec{b}\,\vec{e}\,\vec{f}]-[\vec{a}\,\vec{d}\,\vec{b}][\vec{a}\,\vec{e}\,\vec{f}]$ d. $[\vec{a}\,\vec{c}\,\vec{e}][\vec{b}\,\vec{d}\,\vec{f}]$

30. The scalars l and m such that $l\vec{a} + m\vec{b} = \vec{c}$, where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are given vectors, are equal to

a. $l = \dfrac{(\vec{c} \times \vec{b}) \cdot (\vec{a} \times \vec{b})}{(\vec{a} \times \vec{b})^2}$

b. $l = \dfrac{(\vec{c} \times \vec{a}) \cdot (\vec{b} \times \vec{a})}{(\vec{b} \times \vec{a})}$

c. $m = \dfrac{(\vec{c} \times \vec{a}) \cdot (\vec{b} \times \vec{a})}{(\vec{b} \times \vec{a})^2}$

d. $m = \dfrac{(\vec{c} \times \vec{a}) \cdot (\vec{b} \times \vec{a})}{(\vec{b} \times \vec{a})}$

31. If $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) \cdot (\vec{a} \times \vec{d}) = 0$, then which of the following may be true?

a. $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are necessarily coplanar

b. $\vec{a}$ lies in the plane of $\vec{c}$ and $\vec{d}$

c. $\vec{b}$ lies in the plane of $\vec{a}$ and $\vec{d}$

d. $\vec{c}$ lies in the plane of $\vec{a}$ and $\vec{d}$

32. A, B, C and D are four points such that $\overrightarrow{AB} = m(2\hat{i} - 6\hat{j} + 2\hat{k})$, $\overrightarrow{BC} = (\hat{i} - 2\hat{j})$ and $\overrightarrow{CD} = n(-6\hat{i} + 15\hat{j} - 3\hat{k})$. If CD intersects AB at some point E, then

a. $m \geq 1/2$

b. $n \geq 1/3$

c. $m = n$

d. $m < n$

33. If vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-coplanar and l, m and n are distinct scalars, then $[(l\vec{a} + m\vec{b} + n\vec{c})(l\vec{b} + m\vec{c} + n\vec{a})(l\vec{c} + m\vec{a} + n\vec{b})] = 0$ implies

a. $l + m + n = 0$

b. roots of the equation $lx^2 + mx + n = 0$ are real

c. $l^2 + m^2 + n^2 = 0$

d. $l^3 + m^3 + n^3 = 3lmn$

34. Let $\vec{\alpha} = a\hat{i} + b\hat{j} + c\hat{k}$, $\vec{\beta} = b\hat{i} + c\hat{j} + a\hat{k}$ and $\vec{\gamma} = c\hat{i} + a\hat{j} + b\hat{k}$ be three coplanar vectors with $a \neq b$, and $\vec{v} = \hat{i} + \hat{j} + \hat{k}$. Then $\vec{v}$ is perpendicular to

a. $\vec{\alpha}$

b. $\vec{\beta}$

c. $\vec{\gamma}$

d. none of these

35. If vectors $\vec{A} = 2\hat{i} + 3\hat{j} + 4\hat{k}$, $\vec{B} = \hat{i} + \hat{j} + 5\hat{k}$ and $\vec{C}$ form a left-handed system, then $\vec{C}$ is

a. $11\hat{i} - 6\hat{j} - \hat{k}$

b. $-11\hat{i} + 6\hat{j} + \hat{k}$

c. $11\hat{i} - 6\hat{j} + \hat{k}$

d. $-11\hat{i} + 6\hat{j} - \hat{k}$

Reasoning Type

Solutions on page 2.126

Each question has four choices *a, b, c* and *d*, out of which *only one* is correct. Each equation contains Statement 1 and Statement 2.

a. Both the statements are true and Statement 2 is the correct explanation for Statement 1.

b. Both the statements are true but Statement 2 is not the correct explanation for Statement 1.

c. Statement 1 is true and Statement 2 is false.

d. Statement 1 is false and Statement 2 is true.

1. **Statement 1:** Vector $\vec{c} = -5\hat{i} + 7\hat{j} + 2\hat{k}$ is along the bisector of angle between $\vec{a} = \hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b} = -8\hat{i} + \hat{j} - 4\hat{k}$.

   Statement 2: $\vec{c}$ is equally inclined to $\vec{a}$ and $\vec{b}$.

2. **Statement 1:** A component of vector $\vec{b} = 4\hat{i} + 2\hat{j} + 3\hat{k}$ in the direction perpendicular to the direction of vector $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ is $\hat{i} - \hat{j}$.

   Statement 2: A component of vector in the direction of $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ is $2\hat{i} + 2\hat{j} + 2\hat{k}$.

3. **Statement 1:** Distance of point $D(1, 0, -1)$ from the plane of points $A(1, -2, 0)$, $B(3, 1, 2)$ and $C(-1, 1, -1)$ is $\frac{8}{\sqrt{229}}$.

   Statement 2: Volume of tetrahedron formed by the points A, B, C and D is $\frac{\sqrt{229}}{2}$.

4. Let $\vec{r}$ be a non-zero vector satisfying $\vec{r}\cdot\vec{a} = \vec{r}\cdot\vec{b} = \vec{r}\cdot\vec{c} = 0$ for given non-zero vectors $\vec{a}, \vec{b}$ and $\vec{c}$.

   Statement 1: $[\vec{a}-\vec{b}\ \ \vec{b}-\vec{c}\ \ \vec{c}-\vec{a}] = 0$

   Statement 2: $[\vec{a}\,\vec{b}\,\vec{c}] = 0$

5. **Statement 1:** If $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ are three mutually perpendicular unit vectors, then $a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$, $a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$ and $a_3\hat{i} + b_3\hat{j} + c_3\hat{k}$ may be mutually perpendicular unit vectors.

   Statement 2: Value of determinant and its transpose are the same.

6. **Statement 1:** If $\vec{A} = 2\hat{i} + 3\hat{j} + 6\hat{k}$, $\vec{B} = \hat{i} + \hat{j} - 2\hat{k}$ and $\vec{C} = \hat{i} + 2\hat{j} + \hat{k}$, then $|\vec{A} \times (\vec{A} \times (\vec{A} \times \vec{B}))\cdot\vec{C}| = 243$.

   Statement 2: $|\vec{A} \times (\vec{A} \times (\vec{A} \times \vec{B}))\cdot\vec{C}| = |\vec{A}|^2 |[\vec{A}\vec{B}\vec{C}]|$

7. **Statement 1:** $\vec{a}, \vec{b}$ and $\vec{c}$ are three mutually perpendicular unit vectors and $\vec{d}$ is a vector such that $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are non-coplanar. If $[\vec{d}\,\vec{b}\,\vec{c}] = [\vec{d}\,\vec{a}\,\vec{b}] = [\vec{d}\,\vec{c}\,\vec{a}] = 1$, then $\vec{d} = \vec{a} + \vec{b} + \vec{c}$.

   Statement 2: $[\vec{d}\,\vec{b}\,\vec{c}] = [\vec{d}\,\vec{a}\,\vec{b}] = [\vec{d}\,\vec{c}\,\vec{a}] \Rightarrow \vec{d}$ is equally inclined to $\vec{a}, \vec{b}$ and $\vec{c}$.

8. Consider three vectors $\vec{a}, \vec{b}$ and $\vec{c}$.

   Statement 1: $\vec{a} \times \vec{b} = ((\hat{i} \times \vec{a})\cdot\vec{b})\hat{i} + ((\hat{j} \times \vec{a})\cdot\vec{b})\hat{j} + ((\hat{k} \times \vec{a})\cdot\vec{b})\hat{k}$

   Statement 2: $\vec{c} = (\hat{i}\cdot\vec{c})\hat{i} + (\hat{j}\cdot\vec{c})\hat{j} + (\hat{k}\cdot\vec{c})\hat{k}$

Linked Comprehension Type

Solutions on page 2.128

Based on each paragraph, three multiple choice questions have to be answered. Each question has four choices *a, b, c* and *d*, out of which *only one* is correct.

For Problems 1–3

Let $\vec{u}, \vec{v}$ and $\vec{w}$ be three unit vectors such that $\vec{u} + \vec{v} + \vec{w} = \vec{a}$, $\vec{u} \times (\vec{v} \times \vec{w}) = \vec{b}$, $(\vec{u} \times \vec{v}) \times \vec{w} = \vec{c}$, $\vec{a}\cdot\vec{u} = 3/2$, $\vec{a}\cdot\vec{v} = 7/4$ and $|\vec{a}| = 2$.

1. Vector $\vec{u}$ is

a. $\vec{a}-\frac{2}{3}\vec{b}+\vec{c}$ b. $\vec{a}+\frac{4}{3}\vec{b}+\frac{8}{3}\vec{c}$ c. $2\vec{a}-\vec{b}+\frac{1}{3}\vec{c}$ d. $\frac{4}{3}\vec{a}-\vec{b}+\frac{2}{3}\vec{c}$

2. Vector $\vec{v}$ is

a. $2\vec{a}-3\vec{c}$ b. $3\vec{b}-4c$ c. $-4\vec{c}$ d. $\vec{a}+\vec{b}+2\vec{c}$

3. Vector $\vec{w}$ is

a. $\frac{2}{3}(2\vec{c}-\vec{b})$ b. $\frac{1}{3}(\vec{a}-\vec{b}-\vec{c})$ c. $\frac{1}{3}\vec{a}-\frac{2}{3}\vec{b}-2\vec{c}$ d. $\frac{4}{3}(\vec{c}-\vec{b})$

For Problems 4–6

Vectors $\vec{x},\vec{y}$ and $\vec{z}$, each of magnitude $\sqrt{2}$, make an angle of 60° with each other. $\vec{x}\times(\vec{y}\times\vec{z})=\vec{a}$, $\vec{y}\times(\vec{z}\times\vec{x})=\vec{b}$ and $\vec{x}\times\vec{y}=\vec{c}$.

4. Vector $\vec{x}$ is

a. $\frac{1}{2}[(\vec{a}-\vec{b})\times\vec{c}+(\vec{a}+\vec{b})]$ b. $\frac{1}{2}[(\vec{a}+\vec{b})\times\vec{c}+(\vec{a}-\vec{b})]$

c. $\frac{1}{2}[-(\vec{a}+\vec{b})\times\vec{c}+(\vec{a}+\vec{b})]$ d. $\frac{1}{2}[(\vec{a}+\vec{b})\times\vec{c}-(\vec{a}+\vec{b})]$

5. Vector $\vec{y}$ is

a. $\frac{1}{2}[(\vec{a}+\vec{c})\times\vec{b}-\vec{b}-\vec{a}]$ b. $\frac{1}{2}[(\vec{a}-\vec{c})\times\vec{c}+\vec{b}+\vec{a}]$

c. $\frac{1}{2}[(\vec{a}+\vec{b})\times\vec{c}+\vec{b}+\vec{a}]$ d. $\frac{1}{2}[(\vec{a}-\vec{c})\times\vec{a}+\vec{b}-\vec{a}]$

6. Vector $\vec{z}$ is

a. $\frac{1}{2}[(\vec{a}-\vec{c})\times\vec{c}-\vec{b}+\vec{a}]$ b. $\frac{1}{2}[(\vec{a}+\vec{b})\times\vec{c}+\vec{b}-\vec{a}]$

c. $\frac{1}{2}[\vec{c}\times(\vec{a}-\vec{b})+\vec{b}+\vec{a}]$ d. none of these

For Problems 7–9

If $\vec{x}\times\vec{y}=\vec{a}$, $\vec{y}\times\vec{z}=\vec{b}$, $\vec{x}\cdot\vec{b}=\gamma$, $\vec{x}\cdot\vec{y}=1$ and $\vec{y}\cdot\vec{z}=1$

7. Vector $\vec{x}$ is

a. $\frac{1}{|\vec{a}\times\vec{b}|^2}[\vec{a}\times(\vec{a}\times\vec{b})]$ b. $\frac{\gamma}{|\vec{a}\times\vec{b}|^2}[\vec{a}\times\vec{b}-\vec{a}\times(\vec{a}\times\vec{b})]$

c. $\frac{\gamma}{|\vec{a}\times\vec{b}|^2}[\vec{a}\times\vec{b}+\vec{b}\times(\vec{a}\times\vec{b})]$ d. none of these

8. Vector $\vec{y}$ is

a. $\dfrac{\vec{a}\times\vec{b}}{\gamma}$ b. $\vec{a}+\dfrac{\vec{a}\times\vec{b}}{\gamma}$ c. $\vec{a}+\vec{b}+\dfrac{\vec{a}\times\vec{b}}{\gamma}$ d. none of these

9. Vector $\vec{z}$ is

a. $\dfrac{\gamma}{|\vec{a}\times\vec{b}|^2}[\vec{a}+\vec{b}\times(\vec{a}\times\vec{b})]$ b. $\dfrac{\gamma}{|\vec{a}\times\vec{b}|^2}[\vec{a}+\vec{b}-\vec{a}\times(\vec{a}\times\vec{b})]$

c. $\dfrac{\gamma}{|\vec{a}\times\vec{b}|^2}[\vec{a}\times\vec{b}+\vec{b}\times(\vec{a}\times\vec{b})]$ d. none of these

For Problems 10–12

Given two orthogonal vectors $\vec{A}$ and $\vec{B}$ each of length unity. Let $\vec{P}$ be the vector satisfying the equation $\vec{P}\times\vec{B}=\vec{A}-\vec{P}$. Then

10. $(\vec{P}\times\vec{B})\times\vec{B}$ is equal to

a. $\vec{P}$ b. $-\vec{P}$ c. $2\vec{B}$ d. $\vec{A}$

11. $\vec{P}$ is equal to

a. $\dfrac{\vec{A}}{2}+\dfrac{\vec{A}\times\vec{B}}{2}$ b. $\dfrac{\vec{A}}{2}+\dfrac{\vec{B}\times\vec{A}}{2}$ c. $\dfrac{\vec{A}\times\vec{B}}{2}-\dfrac{\vec{A}}{2}$ d. $\vec{A}\times\vec{B}$

12. Which of the following statements is false?

a. vectors $\vec{P}$, $\vec{A}$ and $\vec{P}\times\vec{B}$ are linearly dependent.

b. vectors $\vec{P}$, $\vec{B}$ and $\vec{P}\times\vec{B}$ are linearly independent.

c. $\vec{P}$ is orthogonal to $\vec{B}$ and has length $1/\sqrt{2}$.

d. none of the above.

For Problems 13–15

Let $\vec{a}=2\hat{i}+3\hat{j}-6\hat{k}$, $\vec{b}=2\hat{i}-3\hat{j}+6\hat{k}$ and $\vec{c}=-2\hat{i}+3\hat{j}+6\hat{k}$. Let $\vec{a}_1$ be the projection of $\vec{a}$ on $\vec{b}$ and $\vec{a}_2$ be the projection of $\vec{a}_1$ on $\vec{c}$. Then

13. $\vec{a}_2$ is equal to

a. $\dfrac{943}{49}(2\hat{i}-3\hat{j}-6\hat{k})$ b. $\dfrac{943}{49^2}(2\hat{i}-3\hat{j}-6\hat{k})$

c. $\dfrac{943}{49}(-2\hat{i}+3\hat{j}+6\hat{k})$ d. $\dfrac{943}{49^2}(-2\hat{i}+3\hat{j}+6\hat{k})$

14. $\vec{a}_1 \cdot \vec{b}$ is equal to

a. -41 b. $-41/7$ c. 41 d. 287

15. Which of the following is true?

a. $\vec{a}$ and $\vec{a}_2$ are collinear
b. $\vec{a}_1$ and $\vec{c}$ are collinear
c. $\vec{a}, \vec{a}_1$ and $\vec{b}$ are coplanar
d. $\vec{a}, \vec{a}_1$ and $\vec{a}_2$ are coplanar

For Problems 16–18

Consider a triangular pyramid $ABCD$ the position vectors of whose angular points are $A(3, 0, 1)$, $B(-1, 4, 1)$, $C(5, 2, 3)$ and $D(0, -5, 4)$. Let G be the point of intersection of the medians of triangle BCD.

16. The length of vector $\overrightarrow{AG}$ is

a. $\sqrt{17}$ b. $\sqrt{51}/3$ c. $3/\sqrt{6}$ d. $\sqrt{59}/4$

17. Area of triangle ABC in sq. units is

a. 24 b. $8\sqrt{6}$ c. $4\sqrt{6}$ d. none of these

18. The length of the perpendicular from vertex D on the opposite face is

a. $14/\sqrt{6}$ b. $2/\sqrt{6}$ c. $3/\sqrt{6}$ d. none of these

For Problems 19–21

Vertices of a parallelogram taken in order are $A(2, -1, 4)$; $B(1, 0 -1)$; $C(1, 2, 3)$ and D.

19. The distance between the parallel lines AB and CD is

a. $\sqrt{6}$ b. $3\sqrt{6/5}$ c. $2\sqrt{2}$ d. 3

20. Distance of the point $P(8, 2, -12)$ from the plane of the parallelogram is

a. $\frac{4\sqrt{6}}{9}$ b. $\frac{32\sqrt{6}}{9}$ c. $\frac{16\sqrt{6}}{9}$ d. none

21. The orthogonal projections of the parallelogram on the three coordinate planes xy, yz and zx, respectively, are

a. 14, 4, 2 b. 2, 4, 14 c. 4, 2, 14 d. 2, 14, 4

For Problems 22–24

Let $\vec{r}$ be a position vector of a variable point in Cartesian OXY plane such that $\vec{r} \cdot (10\hat{j} - 8\hat{i} - \vec{r}) = 40$ and $p_1 = \max\{|\vec{r} + 2\hat{i} - 3\hat{j}|^2\}$, $p_2 = \min\{|\vec{r} + 2\hat{i} - 3\hat{j}|^2\}$. A tangent line is drawn to the curve $y = 8/x^2$ at point A with abscissa 2. The drawn line cuts the x-axis at a point B.

22. p_2 is equal to

a. 9 b. $2\sqrt{2} - 1$ c. $6\sqrt{2} + 3$ d. $9 - 4\sqrt{2}$

23. $p_1 + p_2$ is equal to

a. 2 **b.** 10 **c.** 18 **d.** 5

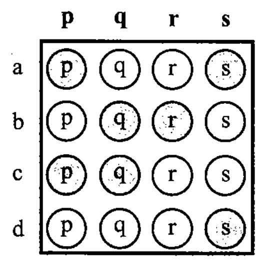
24. $\overrightarrow{AB} \cdot \overrightarrow{OB}$ is equal to

a. 1 **b.** 2 **c.** 3 **d.** 4

Matrix-Match Type

Solutions on page 2.134

Each question contains statements given in two columns which have to be matched. Statements (a, b, c, d) in Column I have to be matched with statements (p, q, r, s) in Column II. If the correct matches are a → p, s; b → q, r; c → p, q and d → s, then the correctly bubbled 4 × 4 matrix should be as follows:



1.

Column I	Column II
a. The possible value of a if $\vec{r} = (\hat{i} + \hat{j}) + \lambda(\hat{i} + 2\hat{j} - \hat{k})$ and $\vec{r} = (\hat{i} + 2\hat{j}) + \mu(-\hat{i} + \hat{j} + a\hat{k})$ are not consistent, where λ and μ are scalars, is	**p.** –4
b. The angle between vectors $\vec{a} = \lambda\hat{i} - 3\hat{j} - \hat{k}$ and $\vec{b} = 2\lambda\hat{i} + \lambda\hat{j} - \hat{k}$ is acute, whereas vector $\vec{b}$ makes an obtuse angle with the axes of coordinates. Then λ may be	**q.** –2
c. The possible value of a such that $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} + 2\hat{j} + (1+a)\hat{k}$ and $3\hat{i} + a\hat{j} + 5\hat{k}$ are coplanar is	**r.** 2
d. If $\vec{A} = 2\hat{i} + \lambda\hat{j} + 3\hat{k}$, $\vec{B} = 2\hat{i} + \lambda\hat{j} + \hat{k}$, $\vec{C} = 3\hat{i} + \hat{j}$ and $\vec{A} + \lambda\vec{B}$ is perpendicular to $\vec{C}$, then $\|2\lambda\|$ is	**s.** 3

2.

Column I	Column II
a. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three mutually perpendicular vectors where $\lvert\vec{a}\rvert = \lvert\vec{b}\rvert = 2, \lvert\vec{c}\rvert = 1$, then $[\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}]$ is	**p.** -12
b. If $\vec{a}$ and $\vec{b}$ are two unit vectors inclined at $\pi/3$, then $16[\vec{a}\ \ \vec{b}+\vec{a}\times\vec{b}\ \ \vec{b}]$ is	**q.** 0
c. If $\vec{b}$ and $\vec{c}$ are orthogonal unit vectors and $\vec{b}\times\vec{c} = \vec{a}$, then $[\vec{a}+\vec{b}+\vec{c}\ \ \vec{a}+\vec{b}\ \ \vec{b}+\vec{c}]$ is	**r.** 16
d. If $[\vec{x}\ \vec{y}\ \vec{a}] = [\vec{x}\ \vec{y}\ \vec{b}] = [\vec{a}\ \vec{b}\ \vec{c}] = 0$, each vector being a non-zero vector, then $[\vec{x}\ \vec{y}\ \vec{c}]$ is	**s.** 1

3.

Column I	Column II
a. If $\lvert\vec{a}\rvert = \lvert\vec{b}\rvert = \lvert\vec{c}\rvert$, angle between each pair of vectors is $\frac{\pi}{3}$ and $\lvert\vec{a}+\vec{b}+\vec{c}\rvert = \sqrt{6}$, then $2\lvert\vec{a}\rvert$ is equal to	**p.** 3
b. If $\vec{a}$ is perpendicular to $\vec{b}+\vec{c}$, $\vec{b}$ is perpendicular to $\vec{c}+\vec{a}$, $\vec{c}$ is perpendicular to $\vec{a}+\vec{b}$, $\lvert\vec{a}\rvert = 2$, $\lvert\vec{b}\rvert = 3$ and $\lvert\vec{c}\rvert = 6$, then $\lvert\vec{a}+\vec{b}+\vec{c}\rvert - 2$ is equal to	**q.** 2
c. $\vec{a} = 2\hat{i}+3\hat{j}-\hat{k}$, $\vec{b} = -\hat{i}+2\hat{j}-4\hat{k}$, $\vec{c} = \hat{i}+\hat{j}+\hat{k}$ and $\vec{d} = 3\hat{i}+2\hat{j}+\hat{k}$, then $\frac{1}{7}(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})$ is equal to	**r.** 4
d. If $\lvert\vec{a}\rvert = \lvert\vec{b}\rvert = \lvert\vec{c}\rvert = 2$ and $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a} = 2$, then $[\vec{a}\ \vec{b}\ \vec{c}]\cos 45^\circ$ is equal to	**s.** 5

4. Given two vectors $\vec{a} = -\hat{i}+\hat{j}+2\hat{k}$ and $\vec{b} = -\hat{i}-2\hat{j}-\hat{k}$.

Column I	Column II
a. Area of triangle formed by $\vec{a}$ and $\vec{b}$	**p.** 3
b. Area of parallelogram having sides $\vec{a}$ and $\vec{b}$	**q.** $12\sqrt{3}$
c. Area of parallelogram having diagonals $2\vec{a}$ and $4\vec{b}$	**r.** $3\sqrt{3}$
d. Volume of parallelepiped formed by $\vec{a}$, $\vec{b}$ and $\vec{c} = \hat{i}+\hat{j}+\hat{k}$	**s.** $\frac{3\sqrt{3}}{2}$

5. Given two vectors $\vec{a} = -\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b} = -2\hat{i} + \hat{j} + 2\hat{k}$.

Column I	Column II
a. A vector coplanar with $\vec{a}$ and $\vec{b}$	**p.** $-3\hat{i} + 3\hat{j} + 4\hat{k}$
b. A vector which is perpendicular to both $\vec{a}$ and $\vec{b}$	**q.** $2\hat{i} - 2\hat{j} + 3\hat{k}$
c. A vector which is equally inclined to $\vec{a}$ and $\vec{b}$	**r.** $\hat{i} + \hat{j}$
d. A vector which forms a triangle with $\vec{a}$ and $\vec{b}$	**s.** $\hat{i} - \hat{j} + 5\hat{k}$

6.

Column I	Column II
a. If $\|\vec{a} + \vec{b}\| = \|\vec{a} + 2\vec{b}\|$, then angle between $\vec{a}$ and $\vec{b}$ is	**p.** 90°
b. If $\|\vec{a} + \vec{b}\| = \|\vec{a} - 2\vec{b}\|$, then angle between $\vec{a}$ and $\vec{b}$ is	**q.** obtuse
c. If $\|\vec{a} + \vec{b}\| = \|\vec{a} - \vec{b}\|$, then angle between $\vec{a}$ and $\vec{b}$ is	**r.** 0°
d. Angle between $\vec{a} \times \vec{b}$ and a vector perpendicular to the vector $\vec{c} \times (\vec{a} \times \vec{b})$ is	**s.** acute

7. Volume of parallelepiped formed by vectors $\vec{a} \times \vec{b}$, $\vec{b} \times \vec{c}$ and $\vec{c} \times \vec{a}$ is 36 sq. units.

Column I	Column II
a. Volume of parallelepiped formed by vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is	**p.** 0 sq. units
b. Volume of tetrahedron formed by vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is	**q.** 12 sq. units
c. Volume of parallelepiped formed by vectors $\vec{a} + \vec{b}$, $\vec{b} + \vec{c}$ and $\vec{c} + \vec{a}$ is	**r.** 6 sq. units
d. Volume of parallelepiped formed by vectors $\vec{a} - \vec{b}$, $\vec{b} - \vec{c}$ and $\vec{c} - \vec{a}$ is	**s.** 1 sq. units

Integer Answer Type

Solutions on page 2.138

1. If $\vec{a}$ and $\vec{b}$ are any two unit vectors, then find the greatest positive integer in the range of $\frac{3|\vec{a}+\vec{b}|}{2} + 2\,|\vec{a}-\vec{b}|$.

2. Let $\vec{u}$ be a vector on rectangular coordinate system with sloping angle 60°. Suppose that $|\vec{u}-\hat{i}|$ is geometric mean of $|\vec{u}|$ and $|\vec{u}-2\hat{i}|$, where $\hat{i}$ is the unit vector along x-axis. Then find the value of $(\sqrt{2}+1)|\vec{u}|$.

3. Find the absolute value of parameter t for which the area of the triangle whose vertices are $A(-1, 1, 2)$; $B(1, 2, 3)$ and $C(t, 1, 1)$ is minimum.

4. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$; $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ and $[3\vec{a}+\vec{b}\ \ 3\vec{b}+\vec{c}\ \ 3\vec{c}+\vec{a}] = \lambda\begin{vmatrix} \vec{a}\cdot\hat{i} & \vec{a}\cdot\hat{j} & \vec{a}\cdot\hat{k} \\ \vec{b}\cdot\hat{i} & \vec{b}\cdot\hat{j} & \vec{b}\cdot\hat{k} \\ \vec{c}\cdot\hat{i} & \vec{c}\cdot\hat{j} & \vec{c}\cdot\hat{k} \end{vmatrix}$, then find the value of $\frac{\lambda}{4}$.

5. Let $\vec{a} = \alpha\hat{i} + 2\hat{j} - 3\hat{k}$, $\vec{b} = \hat{i} + 2\alpha\hat{j} - 2\hat{k}$ and $\vec{c} = 2\hat{i} - \alpha\hat{j} + \hat{k}$. Find the value of 6α, such that $\{(\vec{a}\times\vec{b})\times(\vec{b}\times\vec{c})\}\times(\vec{c}\times\vec{a}) = 0$.

6. If $\vec{x}, \vec{y}$ are two non-zero and non-collinear vectors satisfying $[(a-2)\alpha^2+(b-3)\alpha+c]\,\vec{x} + [(a-2)\beta^2+(b-3)\beta+c]\,\vec{y} + [(a-2)\gamma^2+(b-3)\gamma+c]\,(\vec{x}\times\vec{y}) = 0$, where α, β, γ are three distinct real numbers, then find the value of $(a^2+b^2+c^2-4)$.

7. Let $\vec{u}$ and $\vec{v}$ are unit vectors such that $\vec{u}\times\vec{v}+\vec{u}=\vec{w}$ and $\vec{w}\times\vec{u}=\vec{v}$. Find the value of $[\vec{u}\ \vec{v}\ \vec{w}]$.

8. Find the value of λ if the volume of a tetrahedron whose vertices are with position vectors $\hat{i}-6\hat{j}+10\hat{k}, -\hat{i}-3\hat{j}+7\hat{k}, 5\hat{i}-\hat{j}+\lambda\hat{k}$ and $7\hat{i}-4\hat{j}+7\hat{k}$ is 11 cubic unit.

9. Given that $\vec{u} = \hat{i}-2\hat{j}+3\hat{k}$; $\vec{v} = 2\hat{i}+\hat{j}+4\hat{k}$; $\vec{w} = \hat{i}+3\hat{j}+3\hat{k}$ and $(\vec{u}\cdot\vec{R}-15)\hat{i} + (\vec{v}\cdot\vec{R}-30)\hat{j} + (\vec{w}\cdot\vec{R}-20)\hat{k} = \vec{0}$. Then find the greatest integer less than or equal to $|\vec{R}|$.

10. Let a three-dimensional vector $\vec{V}$ satisfies the condition, $2\vec{V}+\vec{V}\times(\hat{i}+2\hat{j}) = 2\hat{i}+\hat{k}$. If $3|\vec{V}| = \sqrt{m}$, then find the value of m.

11. If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a}\cdot\vec{b} = 0 = \vec{a}\cdot\vec{c}$ and the angle between $\vec{b}$ and $\vec{c}$ is $\frac{\pi}{3}$, then find the value of $|\vec{a}\times\vec{b}-\vec{a}\times\vec{c}|$.

12. Let $\vec{OA} = \vec{a}$, $\vec{OB} = 10\vec{a}+2\vec{b}$ and $\vec{OC} = \vec{b}$, where O, A and C are non-collinear points. Let p denote the area of quadrilateral $OACB$, and let q denote the area of parallelogram with OA and OC as adjacent sides. If $p = k\,q$, then find k.

13. Find the work done by the force $F = 3\hat{i} - \hat{j} - 2\hat{k}$ acting on a particle such that the particle is displaced from point $A(-3, -4, 1)$ to point $B(-1, -1, -2)$.

Archives

Solutions on page 2.144

Subjective Type

1. From a point O inside a triangle ABC, perpendiculars OD, OE and OF are drawn to the sides BC, CA and AB, respectively. Prove that the perpendiculars from A, B and C to the sides EF, FD and DE are concurrent. **(IIT-JEE, 1978)**

2. $A_1, A_2, \ldots, A_n$ are the vertices of a regular plane polygon with n sides and O as its centre. Show that $\sum_{i=1}^{n-1} (\overrightarrow{OA_i} \times \overrightarrow{OA_{i+1}}) = (1-n)(\overrightarrow{OA_2} \times \overrightarrow{OA_1})$. **(IIT-JEE,1998)**

3. If c be a given non-zero scalar, and $\vec{A}$ and $\vec{B}$ be given non-zero vectors such that $\vec{A} \perp \vec{B}$, find the vector $\vec{X}$ which satisfies the equations $\vec{A} \cdot \vec{X} = c$ and $\vec{A} \times \vec{X} = \vec{B}$. **(IIT-JEE, 1983)**

4. If A, B, C, D are any four points in space, prove that $|\overrightarrow{AB} \times \overrightarrow{CD} + \overrightarrow{BC} \times \overrightarrow{AD} + \overrightarrow{CA} \times \overrightarrow{BD}| = 4$ (area of triangle ABC). **(IIT-JEE, 1986)**

5. If vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar, show that $\begin{vmatrix} \vec{a} & \vec{b} & \vec{c} \\ \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \end{vmatrix} = \vec{0}$. **(IIT-JEE, 1989)**

6. Let $\vec{A} = 2\vec{i} + \vec{k}, \vec{B} = \vec{i} + \vec{j} + \vec{k}$ and $\vec{C} = 4\vec{i} - 3\vec{j} + 7\vec{k}$. Determine a vector $\vec{R}$ satisfying $\vec{R} \times \vec{B} = \vec{C} \times \vec{B}$ and $\vec{R} \cdot \vec{A} = 0$. **(IIT-JEE, 1990)**

7. Determine the value of c so that for all real x, vectors $cx\hat{i} - 6\hat{j} - 3\hat{k}$ and $x\hat{i} + 2\hat{j} + 2cx\hat{k}$ make an obtuse angle with each other. **(IIT-JEE, 1991)**

8. If vectors $\vec{b}, \vec{c}$ and $\vec{d}$, are not coplanar, then prove that vector $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) + (\vec{a} \times \vec{c}) \times (\vec{d} \times \vec{b}) + (\vec{a} \times \vec{d}) \times (\vec{b} \times \vec{c})$ is parallel to $\vec{a}$. **(IIT-JEE, 1994)**

9. The position vectors of the vertices A, B and C of a tetrahedron $ABCD$ are $\hat{i} + \hat{j} + \hat{k}, \hat{i}$ and $3\hat{i}$, respectively. The altitude from vertex D to the opposite face ABC meets the median line through A of triangle ABC at a point E. If the length of the side AD is 4 and the volume of the tetrahedron is $2\sqrt{2}/3$, find the position vectors of the point E for all its possible positions. **(IIT-JEE, 1996)**

10. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be non-coplanar unit vectors, equally inclined to one another at an angle θ. If $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} = p\vec{a} + q\vec{b} + r\vec{c}$, find scalars p, q and r in terms of θ. **(IIT-JEE 1997)**

11. If $\vec{A}, \vec{B}$ and $\vec{C}$ are vectors such that $|\vec{B}| = |\vec{C}|$. Prove that $[(\vec{A} + \vec{B}) \times (\vec{A} + \vec{C})] \times (\vec{B} + \vec{C}) \cdot (\vec{B} + \vec{C}) = 0$. **(IIT-JEE, 1997)**

12. For any two vectors $\vec{u}$ and $\vec{v}$, prove that

a. $(\vec{u}.\vec{v})^2 + |\vec{u} \times \vec{v}|^2 = |\vec{u}|^2 |\vec{v}|^2$ and

b. $(\vec{1} + |\vec{u}|^2)(\vec{1} + |\vec{v}|^2) = (1 - \vec{u}.\vec{v})^2 + |\vec{u} + \vec{v} + (\vec{u} \times \vec{v})|^2$ **(IIT-JEE, 1998)**

13. Let $\vec{u}$ and $\vec{v}$ be unit vectors. If $\vec{w}$ is a vector such that $\vec{w} + (\vec{w} \times \vec{u}) = \vec{v}$, then prove that $|(\vec{u} \times \vec{v}) \cdot \vec{w}| \le 1/2$ and that the equality holds if and only if $\vec{u}$ is perpendicular to $\vec{v}$. **(IIT-JEE, 1999)**

14. Find three-dimensional vectors $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$ satisfying $\vec{v}_1 \cdot \vec{v}_1 = 4, \vec{v}_1 \cdot \vec{v}_2 = -2, \vec{v}_1 \cdot \vec{v}_3 = 6, \vec{v}_2 \cdot \vec{v}_2 = 2,$ $\vec{v}_2 \cdot \vec{v}_3 = -5, \vec{v}_3 \cdot \vec{v}_3 = 29.$ **(IIT-JEE, 2001)**

15. Let V be the volume of the parallelepiped formed by the vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$. If a_r, b_r and c_r, where $r = 1, 2, 3$, are non-negative real numbers and $\sum_{r=1}^{3}(a_r + b_r + c_r) = 3L$, show that $V \le L^3$. **(IIT-JEE, 2002)**

16. $\vec{u}, \vec{v}$ and $\vec{w}$ are three non-coplanar unit vectors and α, β and γ are the angles between $\vec{u}$ and $\vec{v}$, $\vec{v}$ and $\vec{w}$, and $\vec{w}$ and $\vec{u}$, respectively, and $\vec{x}, \vec{y}$ and $\vec{z}$ are unit vectors along the bisectors of the angles α, β and γ, respectively. Prove that $[\vec{x} \times \vec{y} \quad \vec{y} \times \vec{z} \quad \vec{z} \times \vec{x}] = \frac{1}{16}[\vec{u}\ \vec{v}\ \vec{w}]^2 \sec^2\frac{\alpha}{2} \sec^2\frac{\beta}{2} \sec^2\frac{\gamma}{2}$.

(IIT-JEE, 2003)

17. If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are distinct vectors such that $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$ and $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$, prove that $(\vec{a} - \vec{d}) \cdot (\vec{b} - \vec{c}) \ne 0$, i.e., $\vec{a} \cdot \vec{b} + \vec{d} \cdot \vec{c} \ne \vec{d} \cdot \vec{b} + \vec{a} \cdot \vec{c}$. **(IIT-JEE, 2004)**

18. P_1 and P_2 are planes passing through origin. L_1 and L_2 are two lines on P_1 and P_2, respectively, such that their intersection is the origin. Show that there exist points A, B and C, whose permutation A', B' and C', respectively, can be chosen such that (i) A is on L_1, B on P_1 but not on L_1 and C not on P_1 (ii) A' is on L_2, B' on P_2 but not on L_2 and C' not on P_2. **(IIT-JEE, 2004)**

19. If the incident ray on a surface is along the unit vector $\hat{v}$, the reflected ray is along the unit vector $\hat{w}$ and the normal is along the unit vector $\hat{a}$ outwards, express $\hat{w}$ in terms of $\hat{a}$ and $\hat{v}$.

(IIT-JEE, 2005)

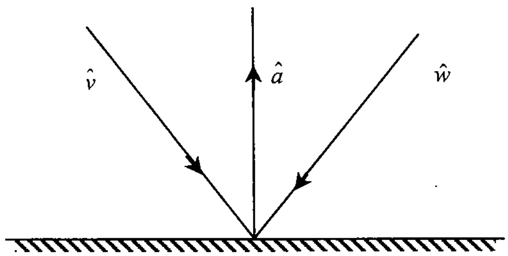


Fig. 2.30

Objective Type

Fill in the blanks

1. Let $\vec{A}, \vec{B}$ and $\vec{C}$ be vectors of length, 3, 4 and 5, respectively. Let $\vec{A}$ be perpendicular to $\vec{B}+\vec{C}$, $\vec{B}$ to $\vec{C}+\vec{A}$ and $\vec{C}$ to $\vec{A}+\vec{B}$. Then the length of vector $\vec{A}+\vec{B}+\vec{C}$ is____________.

(IIT-JEE, 1981)

2. The unit vector perpendicular to the plane determined by $P(1, -1, 2)$, $Q(2, 0, -1)$ and $R(0, 2, 1)$ is____________. **(IIT-JEE, 1983)**

3. The area of the triangle whose vertices are $A(1, -1, 2)$, $B(2, 1, -1)$, $C(3, -1, 2)$ is____________.

(IIT-JEE, 1983)

4. If $\vec{A}, \vec{B}$ and $\vec{C}$ are the three non-coplanar vectors, then $\dfrac{\vec{A}\cdot\vec{B}\times\vec{C}}{\vec{C}\times\vec{A}\cdot\vec{B}}+\dfrac{\vec{B}\cdot\vec{A}\times\vec{C}}{\vec{C}\cdot\vec{A}\times\vec{B}}=$____________.

(IIT-JEE, 1985)

5. If $\vec{A}=(1, 1, 1)$ and $\vec{C}=(0, 1, -1)$ are given vectors, then vector $\vec{B}$ satisfying the equations $\vec{A}\times\vec{B}=\vec{C}$ and $\vec{A}.\vec{B}=3$ is____________. **(IIT-JEE, 1985)**

6. Let $\vec{b}=4\hat{i}+3\hat{j}$ and $\vec{c}$ be two vectors perpendicular to each other in the xy-plane. All vectors in the same plane having projections 1 and 2 along $\vec{b}$ and $\vec{c}$, respectively, are given by____________.

(IIT-JEE, 1987)

7. The components of a vector $\vec{a}$ along and perpendicular to a non-zero vector $\vec{b}$ are____________ and____________, respectively. **(IIT-JEE, 1988)**

8. A unit vector coplanar with $\vec{i}+\vec{j}+2\vec{k}$ and $\vec{i}+2\vec{j}+\vec{k}$ and perpendicular to $\vec{i}+\vec{j}+\vec{k}$ is____________. **(IIT-JEE, 1992)**

9. A non-zero vector $\vec{a}$ is parallel to the line of intersection of the plane determined by vectors $\hat{i}$ and $\hat{i}+\hat{j}$ and the plane determined by vectors $\hat{i}-\hat{j}$ and $\hat{i}+\hat{k}$. The angle between $\vec{a}$ and vector $\hat{i}-2\hat{j}+2\hat{k}$ is____________. **(IIT-JEE, 1996)**

10. If $\vec{b}$ and $\vec{c}$ are mutually perpendicular unit vectors and $\vec{a}$ is any vector, then $(\vec{a}\cdot\vec{b})\,\vec{b}+(\vec{a}\cdot\vec{c})\,\vec{c}+\dfrac{\vec{a}\cdot(\vec{b}\times\vec{c})}{|\vec{b}\times\vec{c}|}\,(\vec{b}\times\vec{c})=$____________. **(IIT-JEE, 1996)**

11. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three vectors having magnitudes 1, 1 and 2, respectively. If $\vec{a}\times(\vec{a}\times\vec{c})+\vec{b}=\vec{0}$, then the acute angle between $\vec{a}$ and $\vec{c}$ is____________. **(IIT-JEE, 1997)**

12. A, B, C and D are four points in a plane with position vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$, respectively, such that $(\vec{a}-\vec{d})\cdot(\vec{b}-\vec{c})=(\vec{b}-\vec{d})\cdot(\vec{c}-\vec{a})=0$. Then point D is the____________of triangle ABC.

(IIT-JEE, 1984)

13. Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = 10\vec{a} + 2\vec{b}$ and $\overrightarrow{OC} = \vec{b}$, where O, A and C are non-collinear points. Let p denote the area of the quadrilateral $OABC$, and let q denote the area of the parallelogram with OA and OC as adjacent sides. If $p = kq$, then $k=$ ____________. **(IIT-JEE, 1997)**

14. If $\vec{a} = \hat{j} + \sqrt{3}\hat{k}$, $\vec{b} = -\hat{j} + \sqrt{3}\hat{k}$ and $\vec{c} = 2\sqrt{3}\hat{k}$ form a triangle, then the internal angle of the triangle between $\vec{a}$ and $\vec{b}$ is ____________. **(IIT-JEE, 2011)**

True or false

1. Let $\vec{A}, \vec{B}$ and $\vec{C}$ be unit vectors such that $\vec{A}\cdot\vec{B} = \vec{A}\cdot\vec{C} = 0$ and the angle between $\vec{B}$ and $\vec{C}$ is $\pi/3$. Then $\vec{A} = \pm 2(\vec{B}\times\vec{C})$. **(IIT-JEE, 1981)**

2. If $\vec{X}\cdot\vec{A} = 0$, $\vec{X}\cdot\vec{B} = 0$ and $\vec{X}\cdot\vec{C} = 0$ for some non-zero vector $\vec{X}$, then $[\vec{A}\,\vec{B}\,\vec{C}] = 0$. **(IIT-JEE, 1983)**

3. For any three vectors $\vec{a}, \vec{b}$ and $\vec{c}$, $(\vec{a}-\vec{b})\cdot(\vec{b}-\vec{c})\times(\vec{c}-\vec{a}) = 2\vec{a}\cdot\vec{b}\times\vec{c}$. **(IIT-JEE, 1989)**

Multiple choice questions with one correct answer

1. The scalar $\vec{A}\cdot(\vec{B}+\vec{C})\times(\vec{A}+\vec{B}+\vec{C})$ equals

   a. 0 b. $[\vec{A}\,\vec{B}\,\vec{C}] + [\vec{B}\,\vec{C}\,\vec{A}]$ c. $[\vec{A}\,\vec{B}\,\vec{C}]$ d. none of these

   (IIT-JEE, 1981)

2. For non-zero vectors $\vec{a}, \vec{b}$ and $\vec{c}$, $|(\vec{a}\times\vec{b})\cdot\vec{c}| = |\vec{a}||\vec{b}||\vec{c}|$ holds if and only if

   a. $\vec{a}\cdot\vec{b} = 0, \vec{b}\cdot\vec{c} = 0$ b. $\vec{b}.\vec{c} = 0, \vec{c}\cdot\vec{a} = 0$
   c. $\vec{c}\cdot\vec{a} = 0, \vec{a}\cdot\vec{b} = 0$ d. $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a} = 0$ **(IIT-JEE, 1982)**

3. The volume of the parallelopiped whose sides are given by $\overrightarrow{OA} = 2i - 2j$, $\overrightarrow{OB} = i + j - k$ and $\overrightarrow{OC} = 3i - k$ is

   a. 4/13 b. 4 c. 2/7 d. 2

   (IIT-JEE, 1983)

4. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three non-coplanar vectors and $\vec{p}, \vec{q}$ and $\vec{r}$ the vectors defined by the relations $\vec{p} = \dfrac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$, $\vec{q} = \dfrac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}$ and $\vec{r} = \dfrac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$. Then the value of the expression $(\vec{a}+\vec{b})\cdot\vec{p} + (\vec{b}+\vec{c})\cdot\vec{q} + (\vec{c}+\vec{a})\cdot\vec{r}$ is

   a. 0 b. 1 c. 2 d. 3

   (IIT-JEE, 1988)

5. Let $\vec{a} = \hat{i} - \hat{j}$, $\vec{b} = \hat{j} - \hat{k}$ and $\vec{c} = \hat{k} - \hat{i}$. If $\vec{d}$ is a unit vector such that $\vec{a}\cdot\vec{d} = 0 = [\vec{b}\,\vec{c}\,\vec{d}]$, then $\vec{d}$ equals

   a. $\pm\dfrac{\hat{i}+\hat{j}-2\hat{k}}{\sqrt{6}}$ b. $\pm\dfrac{\hat{i}+\hat{j}-\hat{k}}{\sqrt{3}}$ c. $\pm\dfrac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$ d. $\pm\hat{k}$

   (IIT-JEE, 1995)

6. If $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar unit vectors such that $\vec{a} \times (\vec{b} \times \vec{c}) = \frac{\vec{b} + \vec{c}}{\sqrt{2}}$, then the angle between $\vec{a}$ and $\vec{b}$ is

a. $3\pi/4$ b. $\pi/4$ c. $\pi/2$ d. π

(IIT-JEE, 1995)

7. Let $\vec{u}, \vec{v}$ and $\vec{w}$ be vectors such that $\vec{u} + \vec{v} + \vec{w} = 0$. If $|\vec{u}| = 3$, $|\vec{v}| = 4$ and $|\vec{w}| = 5$, then $\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{u}$ is

a. 47 b. -25 c. 0 d. 25

(IIT-JEE, 1995)

8. If $\vec{a}, \vec{b}$ and $\vec{c}$ are three non-coplanar vectors, then $(\vec{a} + \vec{b} + \vec{c}) \cdot [(\vec{a} + \vec{b}) \times (\vec{a} + \vec{c})]$ equals

a. 0 b. $[\vec{a}\ \vec{b}\ \vec{c}]$ c. $2[\vec{a}\ \vec{b}\ \vec{c}]$ d. $-[\vec{a}\ \vec{b}\ \vec{c}]$

(IIT-JEE, 1995)

9. $\vec{p}, \vec{q}$ and $\vec{r}$ are three mutually perpendicular vectors of the same magnitude. If vector $\vec{x}$ satisfies the equation $\vec{p} \times ((\vec{x} - \vec{q}) \times \vec{p}) + \vec{q} \times ((\vec{x} - \vec{r}) \times \vec{q}) + \vec{r} \times ((\vec{x} - \vec{p}) \times \vec{r}) = \vec{0}$, then $\vec{x}$ is given by

a. $\frac{1}{2}(\vec{p} + \vec{q} - 2\vec{r})$ b. $\frac{1}{2}(\vec{p} + \vec{q} + \vec{r})$ c. $\frac{1}{3}(\vec{p} + \vec{q} + \vec{r})$ d. $\frac{1}{3}(2\vec{p} + \vec{q} - \vec{r})$

(IIT-JEE, 1997)

10. Let $\vec{a} = 2i + j - 2k$ and $b = i + j$. If c is a vector such that $\vec{a} \cdot \vec{c} = |\vec{c}|$, $|\vec{c} - \vec{a}| = 2\sqrt{2}$ and the angle between $\vec{a} \times \vec{b}$ and $\vec{c}$ is 30°, then $|(\vec{a} \times \vec{b}) \times \vec{c}|$ is equal to

a. 2/3 b. 3/2 c. 2 d. 3

(IIT-JEE, 1999)

11. Let $\vec{a} = 2i + j + k$, $\vec{b} = i + 2j - k$ and a unit vector $\vec{c}$ be coplanar. If $\vec{c}$ is perpendicular to $\vec{a}$, then $\vec{c}$ is

a. $\frac{1}{\sqrt{2}}(-j + k)$ b. $\frac{1}{\sqrt{3}}(-i - j - k)$ c. $\frac{1}{\sqrt{5}}(i - 2j)$ d. $\frac{1}{\sqrt{3}}(i - j - k)$

12. If the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ form the sides BC, CA and AB, respectively, of triangle ABC, then

a. $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} = 0$ b. $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

c. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a}$ d. $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} = \vec{0}$

(IIT-JEE, 2000)

13. Let vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ be such that $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{0}$. Let P_1 and P_2 be planes determined by the pairs of vectors $\vec{a}, \vec{b}$ and $\vec{c}, \vec{d}$, respectively. Then the angle between P_1 and P_2 is

a. 0 b. $\pi/4$ c. $\pi/3$ d. $\pi/2$

(IIT-JEE, 2000)

14. If $\vec{a}, \vec{b}$ and $\vec{c}$ are unit coplanar vectors, then the scalar triple product $[2\vec{a} - \vec{b}\ \ 2\vec{b} - \vec{c}\ \ 2\vec{c} - \vec{a}]$ is

a. 0 b. 1 c. $-\sqrt{3}$ d. $\sqrt{3}$

(IIT-JEE, 2000)

15. If $\hat{a}, \hat{b}$ and $\hat{c}$ are unit vectors, then $|\hat{a}-\hat{b}|^2+|\hat{b}-\hat{c}|^2+|\hat{c}-\hat{a}|^2$ does not exceed

a. 4 **b.** 9 **c.** 8 **d.** 6

(IIT-JEE, 2001)

16. If $\vec{a}$ and $\vec{b}$ are two unit vectors such that $\vec{a}+2\vec{b}$ and $5\vec{a}-4\vec{b}$ are perpendicular to each other, then the angle between $\vec{a}$ and $\vec{b}$ is

a. 45° **b.** 60° **c.** $\cos^{-1}(1/3)$ **d.** $\cos^{-1}(2/7)$

(IIT-JEE, 2002)

17. Let $\vec{V}=2\hat{i}+\hat{j}-\hat{k}$ and $\vec{W}=\hat{i}+3\hat{k}$. If $\vec{U}$ is a unit vector, then the maximum value of the scalar triple product $[\vec{U}\ \vec{V}\ \vec{W}]$ is

a. -1 **b.** $\sqrt{10}+\sqrt{6}$ **c.** $\sqrt{59}$ **d.** $\sqrt{60}$

(IIT-JEE, 2002)

18. The value of a so that the volume of parallelopiped formed by $\hat{i}+a\hat{j}+\hat{k}$, $\hat{j}+a\hat{k}$ and $a\hat{i}+\hat{k}$ is minimum is

a. -3 **b.** 3 **c.** $1/\sqrt{3}$ **d.** $\sqrt{3}$

(IIT-JEE, 2003)

19. If $\vec{a}=(\hat{i}+\hat{j}+\hat{k})$, $\vec{a}\cdot\vec{b}=1$ and $\vec{a}\times\vec{b}=\hat{j}-\hat{k}$, then $\vec{b}$ is

a. $\hat{i}-\hat{j}+\hat{k}$ **b.** $2\hat{j}-\hat{k}$ **c.** $\hat{i}$ **d.** $2\hat{i}$

(IIT-JEE, 2004)

20. The unit vector which is orthogonal to the vector $5\hat{j}+2\hat{j}+6\hat{k}$ and is coplanar with vectors $2\hat{i}+\hat{j}+\hat{k}$ and $\hat{i}-\hat{j}+\hat{k}$ is

a. $\dfrac{2\hat{i}-6\hat{j}+\hat{k}}{\sqrt{41}}$ **b.** $\dfrac{2\hat{i}-3\hat{j}}{\sqrt{13}}$ **c.** $\dfrac{3\hat{i}-\hat{k}}{\sqrt{10}}$ **d.** $\dfrac{4\hat{i}+3\hat{j}-3\hat{k}}{\sqrt{34}}$

(IIT-JEE, 2004)

21. If $\vec{a}, \vec{b}$ and $\vec{c}$ are three non-zero, non-coplanar vectors and $\vec{b}_1=\vec{b}-\dfrac{\vec{b}\cdot\vec{a}}{|\vec{a}|^2}\vec{a}$, $\vec{b}_2=\vec{b}+\dfrac{\vec{b}\cdot\vec{a}}{|\vec{a}|^2}\vec{a}$,

$\vec{c}_1=\vec{c}-\dfrac{\vec{c}\cdot\vec{a}}{|\vec{a}|^2}\vec{a}+\dfrac{\vec{b}\cdot\vec{c}}{|\vec{c}|^2}\vec{b}_1$, $\vec{c}_2=\vec{c}-\dfrac{\vec{c}\cdot\vec{a}}{|\vec{a}|^2}\vec{a}-\dfrac{\vec{b}\cdot\vec{c}}{|\vec{b}_1|^2}\vec{b}_1$, $\vec{c}_3=\vec{c}-\dfrac{\vec{c}\cdot\vec{a}}{|\vec{c}|^2}\vec{a}+\dfrac{\vec{b}\cdot\vec{c}}{|\vec{c}|^2}\vec{b}_1$,

$\vec{c}_4=\vec{c}-\dfrac{\vec{c}\cdot\vec{a}}{|\vec{c}|^2}\vec{a}=\dfrac{\vec{b}\cdot\vec{c}}{|\vec{b}|^2}\vec{b}_1$, then the set of orthogonal vectors is

a. $(\vec{a},\vec{b}_1,\vec{c}_3)$ **b.** $(\vec{a},\vec{b}_1,\vec{c}_2)$ **c.** $(\vec{a},\vec{b}_1,\vec{c}_1)$ **d.** $(\vec{a},\vec{b}_2,\vec{c}_2)$

(IIT-JEE, 2005)

22. Let $\vec{a}=\hat{i}+2\hat{j}+\hat{k}$, $\vec{b}=\hat{i}-\hat{j}+\hat{k}$ and $\vec{c}=\hat{i}+\hat{j}-\hat{k}$. A vector in the plane of $\vec{a}$ and $\vec{b}$ whose projection on $\vec{c}$ is $1/\sqrt{3}$, is

a. $4\hat{i}-\hat{j}+4\hat{k}$ b. $3\hat{i}+\hat{j}-3\hat{k}$ c. $2\hat{i}+\hat{j}-2\hat{k}$ d. $4\hat{i}+\hat{j}-4\hat{k}$

(IIT-JEE, 2006)

23. Let two non-collinear unit vectors $\hat{a}$ and $\hat{b}$ form an acute angle. A point P moves so that at any time t, the position vector $\overrightarrow{OP}$ (where O is the origin) is given by $\hat{a}\cot t+\hat{b}\sin t$. When P is farthest from origin O, let M be the length of $\overrightarrow{OP}$ and $\hat{u}$ be the unit vector along $\overrightarrow{OP}$. Then

a. $\hat{u}=\dfrac{\hat{a}+\hat{b}}{|\hat{a}+\hat{b}|}$ and $M=(1+\hat{a}\cdot\hat{b})^{1/2}$

b. $\hat{u}=\dfrac{\hat{a}-\hat{b}}{|\hat{a}-\hat{b}|}$ and $M=(1+\hat{a}\cdot\hat{b})^{1/2}$

c. $\hat{u}=\dfrac{\hat{a}+\hat{b}}{|\hat{a}+\hat{b}|}$ and $M=(1+2\hat{a}\cdot\hat{b})^{1/2}$

d. $\hat{u}=\dfrac{\hat{a}-\hat{b}}{|\hat{a}-\hat{b}|}$ and $M=(1+2\hat{a}\cdot\hat{b})^{1/2}$

(IIT-JEE, 2008)

24. If $\vec{a},\vec{b},\vec{c}$ and $\vec{d}$ are unit vectors such that $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})=1$ and $\vec{a}\cdot\vec{c}=\dfrac{1}{2}$, then

a. $\vec{a},\vec{b}$ and $\vec{c}$ are non-coplanar

b. $\vec{b},\vec{c}$ and $\vec{d}$ are non-coplanar

c. $\vec{b}$ and $\vec{d}$ are non-parallel

d. $\vec{a}$ and $\vec{d}$ are parallel and $\vec{b}$ and $\vec{c}$ are parallel

(IIT-JEE, 2009)

25. Two adjacent sides of a parallelogram $ABCD$ are given by $\overrightarrow{AB}=2\hat{i}+10\hat{j}+11\hat{k}$ and $\overrightarrow{AD}=-\hat{i}+2\hat{j}+2\hat{k}$. The side AD is rotated by an acute angle α in the plane of the parallelogram so that AD becomes AD'. If AD' makes a right angle with the side AB, then the cosine of the angle α is given by

a. $\dfrac{8}{9}$ b. $\dfrac{\sqrt{17}}{9}$ c. $\dfrac{1}{9}$ d. $\dfrac{4\sqrt{5}}{9}$

(IIT-JEE, 2010)

26. Let P, Q, R and S be the points on the plane with position vectors $-2\hat{i}-\hat{j}$, $4\hat{i}$, $3\hat{i}+3\hat{j}$ and $-3\hat{j}+2\hat{j}$ respectively. The quadrilateral $PQRS$ must be a

a. parallelogram, which is neither a rhombus nor a rectangle
b. square
c. rectangle, but not a square
d. rhombus, but not a square

(IIT-JEE, 2010)

27. Let $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=\hat{i}-\hat{j}+\hat{k}$ and $\vec{c}=\hat{i}-\hat{j}-\hat{k}$ be three vectors. A vector $\vec{v}$ in the plane of $\vec{a}$ and $\vec{b}$, whose projection on $\vec{c}$ is $\frac{1}{\sqrt{3}}$, is given by

a. $\hat{i}-3\hat{j}+3\hat{k}$ **b.** $-3\hat{i}-3\hat{j}+\hat{k}$ **c.** $3\hat{i}-\hat{j}+3\hat{k}$ **d.** $\hat{i}+3\hat{j}-3\hat{k}$

(IIT-JEE, 2011)

Multiple choice questions with one or more than one correct answer

1. Let $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$ be three non-zero vectors such that $\vec{c}$ is a unit vector perpendicular to both vectors $\vec{a}$ and $\vec{b}$. If the angle between $\vec{a}$ and $\vec{b}$ is $\pi/6$, then $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2$ is equal to

a. 0
b. 1
c. $\frac{1}{4}(a_1^2+a_2^2+a_2^2)(b_1^2+b_2^2+b_3^2)$
d. $\frac{3}{4}(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)(c_1^2+c_2^2+c_3^2)$

(IIT-JEE, 1986)

2. The number of vectors of unit length perpendicular to vectors $\vec{a}=(1, 1, 0)$ and $\vec{b}=(0, 1, 1)$ is

a. one **b.** two **c.** three **d.** infinite

(IIT-JEE, 1987)

3. Let $\vec{a}=2\hat{i}-\hat{j}+\hat{k}$, $\vec{b}=\hat{i}+2\hat{j}-\hat{k}$ and $\vec{c}=\hat{i}+\hat{j}-2\hat{k}$ be three vectors. A vector in the plane of $\vec{b}$ and $\vec{c}$, whose projection on $\vec{a}$ is of magnitude $\sqrt{2/3}$, is

a. $2\hat{i}+3\hat{j}-3\hat{k}$ **b.** $2\hat{i}+3\hat{j}+3\hat{k}$ **c.** $-2\hat{i}-\hat{j}+5\hat{k}$ **d.** $2\hat{i}+\hat{j}+5\hat{k}$

(IIT-JEE, 1993)

4. For three vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ which of the following expressions is not equal to any of the remaining three?

a. $\vec{u}\cdot(\vec{v}\times\vec{w})$ **b.** $(\vec{v}\times\vec{w})\cdot\vec{u}$
c. $\vec{v}\cdot(\vec{u}\times\vec{w})$ **d.** $(\vec{u}\times\vec{v})\cdot\vec{w}$

(IIT-JEE, 1998)

5. Which of the following expressions are meaningful?

a. $\vec{u}\cdot(\vec{v}\times\vec{w})$ b. $(\vec{u}\cdot\vec{v})\cdot\vec{w}$

c. $(\vec{u}\cdot\vec{v})\,\vec{w}$ d. $\vec{u}\times(\vec{v}\cdot\vec{w})$ **(IIT-JEE, 1998)**

6. Let $\vec{a}$ and $\vec{b}$ be two non-collinear unit vectors. If $\vec{u} = \vec{a} - (\vec{a}\cdot\vec{b})\,\vec{b}$ and $\vec{v} = \vec{a}\times\vec{b}$, then $|\vec{v}|$ is

a. $|\vec{u}|$ b. $|\vec{u}|+|\vec{u}\cdot\vec{a}|$ c. $|\vec{u}|+|\vec{u}\cdot\vec{b}|$ d. $|\vec{u}|+\vec{u}\cdot(\vec{a}+\vec{b})$

(IIT-JEE, 1999)

7. Vector $\frac{1}{3}(2\hat{i}-2\hat{j}+\hat{k})$ is

a. a unit vector

b. makes an angle $\pi/3$ with vector $(2\hat{i}-4\hat{j}+3\hat{k})$

c. parallel to vector $\left(-\hat{i}+\hat{j}-\frac{1}{2}\hat{k}\right)$

d. perpendicular to vector $3\hat{i}+2\hat{j}-2\hat{k}$ **(IIT-JEE, 1994)**

8. Let $\vec{A}$ be a vector parallel to the line of intersection of planes P_1 and P_2. Plane P_1 is parallel to vectors $2\hat{j}+3\hat{k}$ and $4\hat{j}-3\hat{k}$ and P_2 is parallel to $\hat{j}-\hat{k}$ and $3\hat{i}+3\hat{j}$. Then the angle between vector $\vec{A}$ and a given vector $2\hat{i}+\hat{j}-2\hat{k}$ is

a. $\pi/2$ b. $\pi/4$ c. $\pi/6$ d. $3\pi/4$

(IIT-JEE, 2006)

9. The vector(s) which is/are coplanar with vectors $\hat{i}+\hat{j}+2\hat{k}$ and $\hat{i}+2\hat{j}+\hat{k}$, and perpendicular to vector $\hat{i}+\hat{j}+\hat{k}$ is/are

a. $\hat{j}-\hat{k}$ b. $-\hat{i}+\hat{j}$ c. $\hat{i}-\hat{j}$ d. $-\hat{j}+\hat{k}$

(IIT-JEE, 2011)

Integer Answer Type

1. If $\vec{a}$ and $\vec{b}$ are vectors in space given by $\vec{a}=\frac{\hat{i}-2\hat{j}}{\sqrt{5}}$ and $\vec{b}=\frac{2\hat{i}+\hat{j}+3\hat{k}}{\sqrt{14}}$, then find the value of $(2\vec{a}+\vec{b})\cdot[(\vec{a}\times\vec{b})\times(\vec{a}-2\vec{b})]$. **(IIT-JEE, 2010)**

2. Let $\vec{a}=-\hat{i}-\hat{k}, \vec{b}=-\hat{i}+\hat{j}$ and $\vec{c}=\hat{i}+2\hat{j}+3\hat{k}$ be three given vectors. If $\vec{r}$ is a vector such that $\vec{r}\times\vec{b}=\vec{c}\times\vec{d}$ and $\vec{r}\cdot\vec{a}=0$, then find the value of $\vec{r}\cdot\vec{b}$. **(IIT-JEE, 2011)**

ANSWERS AND SOLUTIONS

Subjective Type

1. $D = D_1 D_2$ (see determinants)

$$\approx 2\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 0$$

Since $\vec{A}$, $\vec{B}$ and $\vec{C}$ are non-coplanar, $D_1 \neq 0$,

$$D_2 = 0 \text{ or } \begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix} = 0$$

or $\vec{X}$, $\vec{Y}$ and $\vec{Z}$ are coplanar.

2.

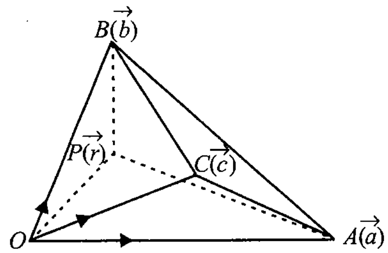


Fig. 2.31

If the centre P is with position vector $\vec{r}$, then

$\vec{a} - \vec{r} = \overrightarrow{PA}$, $\vec{b} - \vec{r} = \overrightarrow{PB}$, $\vec{c} - \vec{r} = \overrightarrow{PC}$,

where $|\overrightarrow{PA}| = |\overrightarrow{PB}| = |\overrightarrow{PC}| = |\overrightarrow{OP}| = |\vec{r}|$

Consider $|\vec{a} - \vec{r}| = |\vec{r}|$

$\Rightarrow (\vec{a} - \vec{r})\cdot(\vec{a} - \vec{r}) = \vec{r}\cdot\vec{r}$

$\Rightarrow a^2 - 2\vec{a}\cdot\vec{r} + r^2 = r^2 \Rightarrow a^2 = 2\vec{a}\cdot\vec{r}$

Similarly, $b^2 = 2\vec{b}\cdot\vec{r}$, $c^2 = 2\vec{c}\cdot\vec{r}$

Since $(\vec{b} \times \vec{c})$, $(\vec{c} \times \vec{a})$ and $(\vec{a} \times \vec{b})$ are non coplanar, then $\vec{r} = x(\vec{b} \times \vec{c}) + y(\vec{c} \times \vec{a}) + z(\vec{a} \times \vec{b})$

$\Rightarrow \vec{a}\cdot\vec{r} = x\vec{a}\cdot(\vec{b} \times \vec{c}) + y\cdot 0 + z\cdot 0 = x[\vec{a}\,\vec{b}\,\vec{c}] \Rightarrow x = \dfrac{\vec{a}\cdot\vec{r}}{[\vec{a}\,\vec{b}\,\vec{c}]} = \dfrac{a^2}{2[\vec{a}\,\vec{b}\,\vec{c}]}$

Similarly, $y = \dfrac{b^2}{2[\vec{a}\,\vec{b}\,\vec{c}]}$ and $z = \dfrac{c^2}{2[\vec{a}\,\vec{b}\,\vec{c}]}$

Hence $\vec{r} = \dfrac{a^2(\vec{b}\times\vec{c}) + b^2(\vec{c}\times\vec{a}) + c^2(\vec{a}\times\vec{b})}{2[\vec{a}\,\vec{b}\,\vec{c}]}$

3.

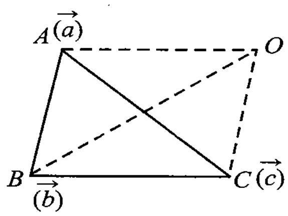


Fig. 2.32

Let O be the origin of reference and A, B, C vertices with position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, respectively. A vector normal to plane ABC is $\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a}$ and $\overrightarrow{OA} = \vec{a}$.

The angle between a line and a plane is equal to the complement of the angle between the line and the normal to the plane. Thus, if θ denotes the angle between the face and edge, then

$$\sin\theta = \frac{(\vec{b}\times\vec{c} + \vec{c}\times\vec{a} + \vec{a}\times\vec{b})\cdot\vec{a}}{|\vec{b}\times\vec{c} + \vec{c}\times\vec{a} + \vec{a}\times\vec{b}|\cdot|\vec{a}|} = \frac{[\vec{a}\,\vec{b}\,\vec{c}]}{|\vec{b}\times\vec{c} + \vec{c}\times\vec{a} + \vec{a}\times\vec{b}|\cdot|\vec{a}|}$$

Now $[\vec{a}\;\vec{b}\;\vec{c}]^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c} \end{vmatrix} = k^6 \begin{vmatrix} 1 & \cos 60° & \cos 60° \\ \cos 60° & 1 & \cos 60° \\ \cos 60° & \cos 60° & 1 \end{vmatrix}$, (where k is the length of the side of the tetrahectron)

$$= k^6\left(\frac{3}{4} - \frac{1}{8} - \frac{1}{8}\right) = \frac{1}{2}k^6$$

Also, $(\vec{b}\times\vec{c} + \vec{c}\times\vec{a} + \vec{a}\times\vec{b})$ is twice the area of triangle ABC, which is equilateral with each side k so that this is $\dfrac{\sqrt{3}}{2}k^2$.

Hence $\sin\theta = \dfrac{\dfrac{k^3}{\sqrt{2}}}{\dfrac{\sqrt{3}}{2}k^2\cdot k} = \dfrac{2}{\sqrt{6}} \Rightarrow \cos\theta = \dfrac{1}{\sqrt{3}}$.

4.

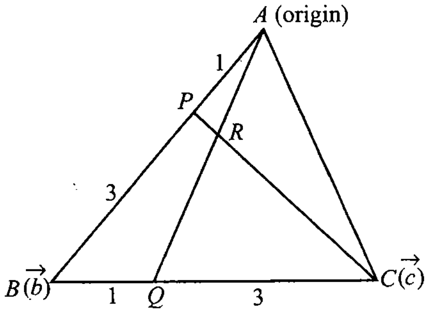


Fig. 2.33

Taking A as origin, let $\vec{b}$ and $\vec{c}$ be the position vectors of B and C, respectively.

The position vector of Q is $\dfrac{3\vec{b}+\vec{c}}{4}$ and that of P is $\dfrac{\vec{b}}{4}$.

If $\dfrac{AR}{QR}=\dfrac{\lambda}{1}$, then position vector of $R=\lambda\left(\dfrac{3\vec{b}+\vec{c}}{4}\right)$ (i)

If $\dfrac{CR}{RP}=\dfrac{\mu}{1}$, then position vector of $R=\dfrac{\mu\dfrac{\vec{b}}{4}+\vec{c}}{\mu+1}$ (ii)

Comparing (i) and (ii), we have

$$\frac{3\lambda}{4}=\frac{\mu}{4(\mu+1)} \text{ and } \frac{\lambda}{4}=\frac{1}{\mu+1}$$

Solving, $\lambda=\dfrac{4}{13}$ and $\mu=12$

Therefore, position vector R is $\dfrac{3\vec{b}+\vec{c}}{13}$.

ΔABC and ΔBRC have the same base. Therefore, areas are proportional to AQ and RQ.

$$\frac{\Delta ABC}{\Delta BRC}=\frac{\left|\dfrac{3\vec{b}+\vec{c}}{4}\right|}{\left|\dfrac{3\vec{b}+\vec{c}}{4}-\left(\dfrac{3\vec{b}+\vec{c}}{13}\right)\right|}=\frac{13}{9}$$

Area of ΔABC is 13/9 units.

5. $$\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta AOC}=\frac{\frac{1}{2}|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|}{\frac{1}{2}|\vec{a}\times\vec{c}|}$$

Now $\vec{a}+2\vec{b}+3\vec{c}=\vec{0}$

Cross multiply with $\vec{b}$, $\vec{a} \times \vec{b} + 3\vec{c} \times \vec{b} = \vec{0}$

$\Rightarrow \vec{a} \times \vec{b} = 3(\vec{b} \times \vec{c})$

Cross multiply with $\vec{a}$, $2\vec{a} \times \vec{b} + 3\vec{a} \times \vec{c} = 0$

$\Rightarrow \vec{a} \times \vec{b} = \frac{3}{2}(\vec{c} \times \vec{a})$

$\therefore \vec{a} \times \vec{b} = \frac{3}{2}(\vec{c} \times \vec{a}) = 3(\vec{b} \times \vec{c})$

Let $(\vec{c} \times \vec{a}) = \vec{p}$

$\vec{a} \times \vec{b} = \frac{3\vec{p}}{2}; \vec{b} \times \vec{c} = \frac{\vec{p}}{2}$

$$\therefore \text{Ratio} = \frac{|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|}{|\vec{c} \times \vec{a}|}$$

$$= \frac{\left|\frac{3\vec{p}}{2} + \frac{\vec{p}}{2} + \vec{p}\right|}{|\vec{p}|}$$

$$= \frac{3|\vec{p}|}{|\vec{p}|} = 3$$

6. In tetrahedron $OABC$, take O as the initial point and let the position vectors of A, B and C be $\vec{a}$, $\vec{k}$ and $\vec{c}$, respectively; then volume of the tetrahedron is equal to $\frac{1}{6}\vec{a} \cdot (\vec{k} \times \vec{c})$.

   Also $\overrightarrow{BC} = \vec{c} - \vec{k}$ so that volume of tetrahedron

   $V = \frac{1}{6}\vec{a} \cdot (\vec{k} \times (\vec{k} + \overrightarrow{BC})) = \frac{1}{6}\vec{a} \cdot (\vec{k} \times \overrightarrow{BC}) = \frac{1}{6}\vec{k} \cdot (\overrightarrow{BC} \times \vec{a})$

   $= \frac{1}{6}\vec{k} \cdot |BC|\,|a| \sin\theta\, \hat{n}$, where $\hat{n}$ is the unit vector along PN, the line perpendicular to both OA and BC.

   Also $|BC| = b$.

   Here $V = \frac{1}{6} ab \sin\theta\, \vec{k} \cdot \hat{n} = \frac{1}{6} ab \sin\theta$ (projection of OB on PN)

   $\frac{1}{6} ab \sin\theta$ = (perpendicular distance between OA and BC) $= \frac{1}{6} ab \sin\theta . d = \frac{1}{6} abd \sin\theta$

7. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three vectors of magnitude $|\vec{a}|$ and equal inclination θ with each other.

   The volume of parallelepiped $= \vec{a} \cdot (\vec{b} \times \vec{c}) = [\vec{a}\,\vec{b}\,\vec{c}]$

   $$\text{and } [\vec{a}\,\vec{b}\,\vec{c}]^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{a}\cdot\vec{c} & \vec{b}\cdot\vec{c} & \vec{c}\cdot\vec{c} \end{vmatrix}$$

$$= |\vec{a}|^6 \begin{vmatrix} 1 & \cos\theta & \cos\theta \\ \cos\theta & 1 & \cos\theta \\ \cos\theta & \cos\theta & 1 \end{vmatrix}$$

$$= |\vec{a}|^6 (2\cos^3\theta - 3\cos^2\theta + 1)$$

$$= |\vec{a}|^6 (1-\cos\theta)^2 (1+2\cos\theta)$$

$$\Rightarrow \quad [\vec{a}\,\vec{b}\,\vec{c}] = |\vec{a}|^3 \sqrt{1+2\cos\theta}(1-\cos\theta)$$

8. $\vec{p}, \vec{q}$ and $\vec{p}\times\vec{q}$ are perpendicular to each other. We have,

$$(\vec{a}\cdot\vec{p})\,\vec{p} + (\vec{a}\cdot\vec{q})\,\vec{q} + (\vec{a}\cdot(\vec{p}\times\vec{q}))\,(\vec{p}\times\vec{q}) = \vec{a}\,|\vec{p}|^2,$$

$$(\vec{b}\cdot\vec{p})\,\vec{p} + (\vec{b}\cdot\vec{q})\,\vec{q} + (\vec{b}\cdot(\vec{p}\times\vec{q}))\,(\vec{p}\times\vec{q}) = \vec{b}\,|\vec{p}|^2,$$

$$(\vec{c}\cdot\vec{p})\,\vec{p} + (\vec{c}\cdot\vec{q})\,\vec{q} + (\vec{c}\cdot(\vec{p}\times\vec{q}))\,(\vec{p}\times\vec{q}) = \vec{c}\,|\vec{p}|^2$$

Hence, the required distance is $|(\vec{a}+\vec{b}+\vec{c})|\,|\vec{p}|^2$.

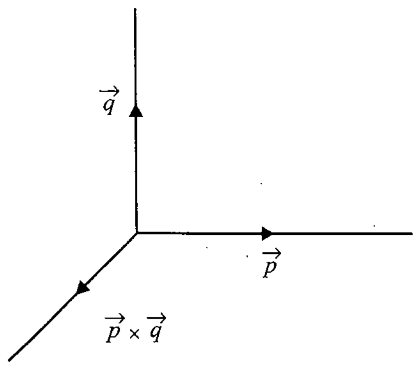


Fig. 2.34

$$= \sqrt{|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2} \times |\vec{p}|^2$$

$$= 14 \times 4^2 = 224$$

9. Here $\vec{A}$, $\vec{B}$ and $\vec{C}$ are the vectors representing the sides of triangle ABC, where $\vec{A} = a\hat{i} + b\hat{j} + c\hat{k}$, $\vec{B} = d\hat{i} + 3\hat{j} + 4\hat{k}$ and $\vec{C} = 3\hat{i} + \hat{j} - 2\hat{k}$.

Given that $\vec{A} = \vec{B} + \vec{C}$. Therefore

$$a\hat{i} + b\hat{j} + c\hat{k} = (d+3)\,\hat{i} + 4\hat{j} + 2\hat{k}$$

$$\Rightarrow \quad a = d+3, b = 4, c = 2$$

Now $$\vec{B}\times\vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ d & 3 & 4 \\ 3 & 1 & -2 \end{vmatrix}$$

$= -10\hat{i} + (2d+12)\hat{j} + (d-9)\hat{k}$

$\therefore$ Area of $\Delta ABC = \frac{1}{2} |\vec{B} \times \vec{C}|$

$= \frac{1}{2}\sqrt{[100 + (2d+12)^2 + (d-9)^2]}$

$= 5\sqrt{6}$ (Given)

$\Rightarrow \sqrt{(5d^2 + 30d + 325)} = 10\sqrt{6}$

$\Rightarrow 5d^2 + 30d - 275 = 0 \Rightarrow d^2 + 6d - 55 = 0$

$\Rightarrow (d+11)(d-5) = 0$

$\Rightarrow d = 5$ or -11

When $d = 5$, $a = 8$, $b = 4$ and $c = 2$, and when $d = -11$, $a = -8$, $b = 4$ and $c = 2$.

10.

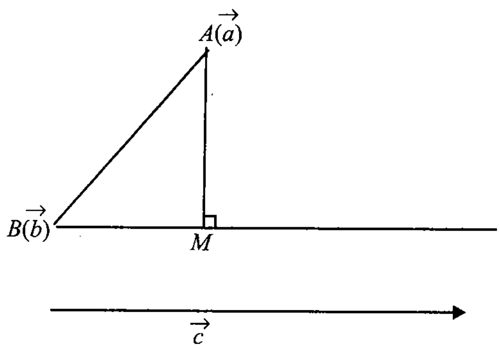


Fig. 2.35

$AM = |\overrightarrow{AB} \sin\theta|$, where θ is the angle between $\overrightarrow{AB}$ and $\vec{c}$

and $\sin\theta = \dfrac{|\overrightarrow{AB} \times \vec{c}|}{|\overrightarrow{AB}||\vec{c}|}$

$\Rightarrow AM = |\overrightarrow{AB}| \dfrac{|\overrightarrow{AB} \times \vec{c}|}{|\overrightarrow{AB}||\vec{c}|} = \dfrac{|(\vec{b} - \vec{a}) \times \vec{c}|}{|\vec{c}|}$

Also $\overrightarrow{BM} = \dfrac{(\vec{a} - \vec{b}) \cdot \vec{c}}{|\vec{c}|} \dfrac{\vec{c}}{|\vec{c}|}$

And $\overrightarrow{AM} = \overrightarrow{AB} + \overrightarrow{BM}$

$\Rightarrow |\overrightarrow{AM}| = \left| \vec{b} - \vec{a} + \dfrac{(\vec{a} - \vec{b}) \cdot \vec{c}}{|\vec{c}|^2} \vec{c} \right|$

11. We know that $[\vec{e}_1\ \vec{e}_2\ \vec{e}_3][\vec{E}_1\ \vec{E}_2\ \vec{E}_3] = \begin{vmatrix} \vec{e}_1\cdot\vec{E}_1 & \vec{e}_1\cdot\vec{E}_2 & \vec{e}_1\cdot\vec{E}_3 \\ \vec{e}_2\cdot\vec{E}_1 & \vec{e}_2\cdot\vec{E}_2 & \vec{e}_2\cdot\vec{E}_2 \\ \vec{e}_3\cdot\vec{E}_1 & \vec{e}_3\cdot\vec{E}_2 & \vec{e}_3\cdot\vec{E}_3 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 1$$

Objective Type

1. c. If $\vec{x} = \vec{y} \Rightarrow \hat{a}\cdot\vec{x} = \hat{a}\cdot\vec{y}$. This equality must hold for any arbitrary $\hat{a}$

2. d. $\vec{a}\times(\vec{a}\times\vec{b}) = \vec{c} \Rightarrow |\vec{a}||\vec{a}\times\vec{b}| = |\vec{c}|$ $(\because \vec{a}\perp(\vec{a}\times\vec{b}))$

   $1\,(1\times 5)\sin\theta = 3 \Rightarrow \sin\theta = \frac{3}{5} \Rightarrow \tan\theta = \frac{3}{4}.$

3. c. $|\vec{a}+\vec{b}+\vec{c}|^2 = 6$

   $\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = 6$

   $\Rightarrow |\vec{a}| = |\vec{b}| = |\vec{c}|$ and $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cdot\cos\frac{\pi}{3}$

   i.e., $\vec{a}\cdot\vec{b} = \frac{1}{2}|\vec{a}|^2$

   $\therefore 3|\vec{a}|^2 + 3|\vec{a}|^2 = 6$

   $\Rightarrow |\vec{a}|^2 \Rightarrow |\vec{a}| = 1$

4. b. Let $\vec{\alpha} = \frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{b}|} + \frac{\vec{c}}{|\vec{c}|}$

   Since $\vec{a}, \vec{b}$ and $\vec{c}$ are mutually perpendicular vectors, if $\vec{\alpha}$ makes angles θ, ϕ ψ with $\vec{a}, \vec{b}$ and $\vec{c}$, respectively, then

   $\vec{\alpha}\cdot\vec{a} = \frac{\vec{a}\cdot\vec{a}}{|\vec{a}|}$

   $\Rightarrow |\vec{\alpha}|\cdot|\vec{a}|\cos\theta = |\vec{a}|$

   $\Rightarrow \cos\theta = \frac{1}{|\vec{\alpha}|}$

   Similarly $\cos\phi = \frac{1}{|\vec{\alpha}|}, \cos\psi = \frac{1}{|\vec{\alpha}|}$

   $\therefore \theta = \phi = \psi$

5. **c.** $\vec{r}\times\vec{a} = \vec{b}\times\vec{a} \Rightarrow (\vec{r}-\vec{b})\times\vec{a} = 0$

$\vec{r}\times\vec{b} = \vec{a}\times\vec{b} \Rightarrow (\vec{r}-\vec{a})\times\vec{b} = 0$

If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x-2 & y & z+1 \\ 1 & 1 & 0 \end{vmatrix} = 0 \text{ and } \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x-1 & y-1 & z \\ 2 & 0 & -1 \end{vmatrix} = 0$$

$\Rightarrow z+1=0, x-y=2$ and $y-1=0, x-1+2z=0$

$\Rightarrow x=3, y=1, z=-1$

6. **d.** $|\vec{a}\cdot\vec{b}| = |\vec{a}\times\vec{b}|$

$\Rightarrow |\vec{a}||\vec{b}|\,|\cos\theta| = |\vec{a}||\vec{b}|\,|\sin\theta|$ (where θ is the angle between $\vec{a}$ and $\vec{b}$)

$\Rightarrow |\cos\theta| = |\sin\theta|$

$\Rightarrow \theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$ (as $0\le\theta\le\pi$)

But $\vec{a}\cdot\vec{b} < 0$, therefore $\theta = \frac{3\pi}{4}$

7. **c.** $|\vec{a}+\vec{b}+\vec{c}|^2 = 1$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2|\vec{a}||\vec{b}|\cos\theta_1 + 2|\vec{b}||\vec{c}|\cos\theta_2 + 2|\vec{c}||\vec{a}|\cos\theta_3 = 1$

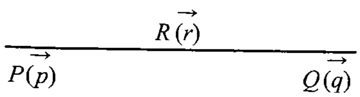
$\Rightarrow \cos\theta_1 + \cos\theta_2 + \cos\theta_3 = -1$

$\Rightarrow$ One of θ_1, θ_2 and θ_3 should be an obtuse angle.

8. **b.** $|\vec{a}\times\vec{b} - \vec{a}\times\vec{c}|^2 = |\vec{a}\times(\vec{b}-\vec{c})|^2 = |\vec{a}|^2|\vec{b}-\vec{c}|^2 - (\vec{a}\cdot(\vec{b}-\vec{c}))^2 = |\vec{b}-\vec{c}|^2$

$= |\vec{b}|^2 + |\vec{c}|^2 - 2|\vec{b}||\vec{c}|\cos\frac{\pi}{3} = 1$

9. **c.** $R(\vec{r})$ moves on PQ.



10. **b.** $|\overrightarrow{AC}\times\overrightarrow{BD}| = 2\,|\overrightarrow{AB}\times\overrightarrow{AD}|$

$$= 2\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 4 & -5 \\ 1 & 2 & 3 \end{vmatrix}$$

$= |2[\hat{i}(12+10) - \hat{j}(6+5) + \hat{k}(4-4)]|$

$= |2[22\hat{i} - 11\hat{j}]|$

$= 22|[2\hat{i} - \hat{j}]|$

$= 22 \times \sqrt{5}$

11. **c.** $(\hat{a} + \hat{b} + \hat{c})^2 \geq 0$

$3 + 2(\hat{a} \cdot \hat{b} + \vec{b} \cdot \hat{c} + \vec{c} \cdot \vec{a}) \geq 0$

$3 + 6\cos\theta \geq 0$

$\cos\theta \geq -\frac{1}{2}$

$\Rightarrow \theta = \frac{2\pi}{3}$

12. **c.** $\vec{a} \times \vec{b}$ is a vector perpendicular to the plane containing $\vec{a}$ and $\vec{b}$. Similarly, $\vec{c} \times \vec{d}$ is a vector perpendicular to the plane containing $\vec{c}$ and $\vec{d}$.

Thus, the two planes will be parallel if their normals, i.e., $\vec{a} \times \vec{b}$ and $\vec{c} \times \vec{d}$, are parallel.

$\Rightarrow (\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{0}$

13. **d.** Let $\vec{r} \neq \vec{0}$. Then $\vec{r} \cdot \vec{a} = \vec{r} \cdot \vec{b} = \vec{r} \cdot \vec{c} = 0$

$\Rightarrow \vec{a}, \vec{b}$ and $\vec{c}$ are coplanar, which is a contradiction.

Therefore, $\vec{r} = \vec{0}$

14. **c.** $\vec{a} \times (\hat{i} + 2\hat{j} + \hat{k}) = \hat{i} - \hat{k} = (\hat{j} \times (\hat{i} + 2\hat{j} + \hat{k}))$

$\Rightarrow (\vec{a} - \hat{j}) \times (\hat{i} + 2\hat{j} + \hat{k}) = \vec{0}$

$\Rightarrow \vec{a} - \hat{j} = \lambda(\hat{i} + 2\hat{j} + \hat{k})$

$\Rightarrow \vec{a} = \lambda\hat{i} + (2\lambda + 1)\hat{j} + \lambda\hat{k}, \lambda \in R$

15. **a.** $(3\vec{a} - 5\vec{b}) \cdot (2\vec{a} + \vec{b}) = 0$

$\Rightarrow 6|\vec{a}|^2 - 5|\vec{b}|^2 = 7\vec{a} \cdot \vec{b}$

Also, $(\vec{a} + 4\vec{b}) \cdot (\vec{b} - \vec{a}) = 0$

$\Rightarrow -|\vec{a}|^2 + 4|\vec{b}|^2 = 3\vec{a} \cdot \vec{b}$

$\Rightarrow \frac{6}{7}|\vec{a}|^2 - \frac{5}{7}|\vec{b}|^2 = -\frac{1}{3}|\vec{a}|^2 + \frac{4}{3}|\vec{b}|^2$

$\Rightarrow 25|\vec{a}|^2 = 43|\vec{b}|^2$

$\Rightarrow 3\vec{a} \cdot \vec{b} = -|\vec{a}|^2 + 4|\vec{b}|^2 = \frac{57}{25}|\vec{b}|^2$

$\Rightarrow 3|\vec{a}||\vec{b}|\cos\theta = \frac{57}{25}|\vec{b}|^2$

$$\Rightarrow 3\sqrt{\frac{43}{25}}|\vec{b}|^2 \cos\theta = \frac{57}{25}|\vec{b}|^2$$

$$\Rightarrow \cos\theta = \frac{19}{5\sqrt{43}}$$

16. **a.** Let l, m and n be the direction cosines of the required vector.
Then, $l = m$ (given). Therefore

Required vector $\vec{r} = l\hat{i} + m\hat{j} + n\hat{k} = l\hat{i} + l\hat{j} + n\hat{k}$

Now, $l^2 + m^2 + n^2 = 1 \Rightarrow 2l^2 + n^2 = 1$ (i)

Since, $\hat{r}$ is perpendicular to $-\hat{i} + 2\hat{j} + 2\hat{k}$,

$\vec{r}\cdot(-\hat{i} + 2\hat{j} + 2\hat{k}) = 0 \Rightarrow -l + 2l + 2n = 0 \Rightarrow l + 2n = 0$ (ii)

From (i) and (ii), we get: $n = \mp\frac{1}{3}, l = \pm\frac{2}{3}$

Hence, required vector $\vec{r} = \frac{1}{3}(\pm 2\hat{i} \pm 2\hat{j} \mp \hat{k}) = \pm\frac{1}{3}(2\hat{i} + 2\hat{j} - \hat{k})$

17. **d.** The angle between $\vec{a}$ and $\vec{b}$ is obtuse. Therefore,

$\vec{a}\cdot\vec{b} < 0$

$\Rightarrow 14x^2 - 8x + x < 0$

$\Rightarrow 7x(2x - 1) < 0$

$\Rightarrow 0 < x < 1/2$ (i)

The angle between $\vec{b}$ and the z-*axis* is acute and less than $\pi/6$. Therefore,

$\frac{\vec{b}\cdot\vec{k}}{|\vec{b}||\vec{k}|} > \cos\pi/6$ ($\because \theta < \pi/6 \Rightarrow \cos\theta > \cos\pi/6$)

$\Rightarrow \frac{x}{\sqrt{x^2 + 53}} > \frac{\sqrt{3}}{2}$

$\Rightarrow 4x^2 > 3x^2 + 159$

$\Rightarrow x^2 > 159$

$\Rightarrow x > \sqrt{159}$ or $x < -\sqrt{159}$ (ii)

Clearly, (i) and (ii) cannot hold together.

18. **c.**

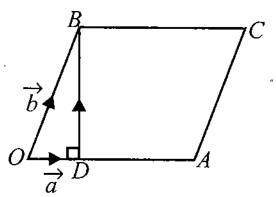


Fig. 2.36

Let $\overrightarrow{OD} = t\vec{a}$

$\therefore\ \overrightarrow{DB} = \vec{b} - t\vec{a}$

$\therefore\ (\vec{b} - t\vec{a})\cdot\vec{a} = 0\ \ (\because\ \overrightarrow{DB} \perp \overrightarrow{OA})$

$\Rightarrow t = \dfrac{\vec{b}\cdot\vec{a}}{|\vec{a}|^2}$

$\therefore\ \overrightarrow{DB} = \vec{b} - \dfrac{(\vec{b}\cdot\vec{a})\,\vec{a}}{|\vec{a}|^2}$

19. **d.** $(3\vec{a} + \vec{b})\cdot(\vec{a} - 4\vec{b})$

$= 3|\vec{a}|^2 - 11\vec{a}\cdot\vec{b} - 4|\vec{b}|^2$

$= 3 \times 36 - 11 \times 6 \times 8 \cos\pi - 4 \times 64 > 0$

Therefore, the angle between $\vec{a}$ and $\vec{b}$ is acute.

The longer diagonal is given by

$\vec{\alpha} = (3\vec{a} + \vec{b}) + (\vec{a} - 4\vec{b}) = 4\vec{a} - 3\vec{b}$

Now, $|\vec{\alpha}|^2 = |4\vec{a} - 3\vec{b}|^2 = 16|\vec{a}|^2 + 9|\vec{b}|^2 - 24\vec{a}\cdot\vec{b}$

$= 16 \times 36 + 9 \times 64 - 24 \times 6 \times 8 \cos\pi$

$= 16 \times 144$

$\Rightarrow |4\vec{a} - 3\vec{b}| = 48$

20. **b.** $\vec{c} = m\vec{a} + n\vec{b} + p(\vec{a} \times \vec{b})$

Taking dot product with $\vec{a}$ and $\vec{b}$, we have

$m = n = \cos\theta$

$\Rightarrow |\vec{c}| = |\cos\theta\,\vec{a} + \cos\theta\,\vec{b} + p\,(\vec{a} \times \vec{b})| = 1$

Squaring both sides, we get

$\cos^2\theta + \cos^2\theta + p^2 = 1$

$\Rightarrow \cos\theta = \pm\dfrac{\sqrt{1 - p^2}}{\sqrt{2}}$

Now $-\dfrac{1}{\sqrt{2}} \le \cos\theta \le \dfrac{1}{\sqrt{2}}$ (for real value of θ)

$\therefore\ \dfrac{\pi}{4} \le \cos\theta \le \dfrac{3\pi}{4}$

21. **a.** $\vec{b} - 2\vec{c} = \lambda\vec{a}$

$\Rightarrow \vec{b} = 2\vec{c} + \lambda\vec{a}$

$\Rightarrow |\vec{b}|^2 = |2\vec{c} + \lambda\vec{a}|^2$

$\Rightarrow 16 = 4\,|\vec{c}|^2 + \lambda^2\,|\vec{a}|^2 + 4\lambda\, \vec{a}\cdot\vec{c}$

$\Rightarrow 16 = 4 + \lambda^2 + 4\lambda \dfrac{1}{4}$

$\Rightarrow \lambda^2 + \lambda - 12 = 0$

$\Rightarrow \lambda = 3, -4$

22. **a.** A vector perpendicular to the plane of O, P and Q is $\overrightarrow{OP} \times \overrightarrow{OQ}$.

Now, $\overrightarrow{OP} \times \overrightarrow{OQ} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 1 & \lambda \\ 2 & -1 & \lambda \end{vmatrix} = 2\lambda\hat{i} - 2\lambda\hat{j} - 6\hat{k}$

Therefore, $\hat{i} - \hat{j} + 6\hat{k}$ is parallel to $2\lambda\hat{i} - 2\lambda\hat{j} - 6\hat{k}$

Hence $\dfrac{1}{2\lambda} = \dfrac{-1}{-2\lambda} = \dfrac{6}{-6}$

$\lambda = -\dfrac{1}{2}$

23. **a.** A vector coplanar with $\vec{a}$ and $\vec{b}$ and perpendicular to $\vec{c}$ is $\lambda\left((\vec{a}\times\vec{b})\times\vec{c}\right)$.

But $\lambda\left((\vec{a}\times\vec{b})\times\vec{c}\right) = \lambda\left[(\vec{a}\cdot\vec{c})\vec{b} - (\vec{b}\cdot\vec{c})\vec{a}\right]$

$= \lambda\left[4\vec{b} - 4\vec{a}\right]$

$= 4\lambda\left[\hat{j} - \hat{k}\right]$

Now $4\,|\lambda|\sqrt{2} = \sqrt{2}$ (Given) $\Rightarrow \lambda = \pm\dfrac{1}{4}$

Hence the required vector is $\hat{j} - \hat{k}$ or $-\hat{j} + \hat{k}$

24. **a.** $\vec{a} - \vec{p} + \vec{b} - \vec{p} + \vec{c} - \vec{p} = 0$

$\Rightarrow \vec{p} = \dfrac{\vec{a} + \vec{b} + \vec{c}}{3}$

$\Rightarrow P$ is centroid

25. **b.**

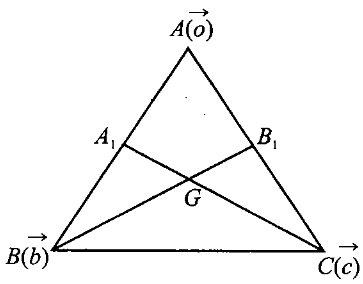


Fig. 2.37

Let P.V. of A, B and C be $\vec{0}, \vec{b}$ and $\vec{c}$, respectively. Therefore,

$$\vec{G} = \frac{\vec{b}+\vec{c}}{3}$$

$$\vec{A_1} = \frac{\vec{b}}{2}, \vec{B_1} = \frac{\vec{c}}{2}$$

$$\Delta_{AB_1G} = \frac{1}{2}|\overrightarrow{AG} \times \overrightarrow{AB_1}| = \frac{1}{2}\left|\frac{\vec{b}+\vec{c}}{3} \times \left(\frac{\vec{c}}{2}\right)\right|$$

$$= \frac{1}{12}|\vec{b} \times \vec{c}|$$

$$\Delta_{AA_1G} = \frac{1}{2}|\overrightarrow{AG} \times \overrightarrow{AA_1}| = \frac{1}{2}\left|\frac{\vec{b}+\vec{c}}{3} \times \left(\frac{\vec{b}}{2}\right)\right| = \frac{1}{12}|\vec{b} \times \vec{c}|$$

$$\Rightarrow \Delta_{GA_1AB_1} = \frac{1}{6}|\vec{b} \times \vec{c}| = \frac{1}{3}\cdot\frac{1}{2}|\vec{b} \times \vec{c}| = \frac{1}{3}\Delta_{ABC}$$

$$\Rightarrow \frac{\Delta}{\Delta_1} = 3$$

26. **a.** Points $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are coplanar. Therefore,

$\sin\alpha + 2\sin 2\beta + 3\sin 3\gamma = 1$

Now $|\sin\alpha + 2\sin 2\beta + 3\sin 3\gamma| \le \sqrt{1+4+9}.\ \sqrt{\sin^2\alpha + \sin^2 2\beta + \sin^2 3\gamma}$

$\Rightarrow \sin^2\alpha + \sin^2 2\beta + \sin^2 3\gamma \ge \frac{1}{14}$

27. **c.** $1 + 9(\vec{a}\cdot\vec{b})^2 - 6(\vec{a}\cdot\vec{b}) + 4|\vec{a}|^2 + |\vec{b}|^2 + 9|\vec{a}\times\vec{b}|^2 + 4\vec{a}\cdot\vec{b} = 47$

$\Rightarrow 1 + 4 + 4 + 36 - 4\cos\theta = 47$

$\Rightarrow \cos\theta = -\frac{1}{2}$

$\Rightarrow$ Angle between $\vec{a}$ and $\vec{b}$ is $\frac{2\pi}{3}$.

28. **c.** $k = |2(\vec{a}\times\vec{b})| + |3(\vec{a}\cdot\vec{b})|$

$= 12\sin\theta + 18\cos\theta$

$\Rightarrow$ maximum value of k is $\sqrt{12^2+18^2} = 6\sqrt{13}$

29. **b.** $|\vec{a} + \vec{b} + 3\vec{c}|^2 = 16$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + 9|\vec{c}|^2 + 2\cos\theta_1 + 6\cos\theta_2 + 6\cos\theta_3 = 16,\ \theta_3 \in [\pi/6, 2\pi/3]$

$\Rightarrow 2\cos\theta_1 + 6\cos\theta_2 = 5 - 6\cos\theta_3$

$\Rightarrow (\cos\theta_1 + 3\cos\theta_2)_{max} = 4$

30. **c.** $|\vec{a} \times \vec{r}| = |\vec{c}|$

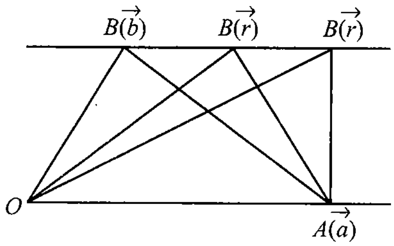


Fig. 2.38

Triangles on the same base and between the same parallel will have the same area.

31. **c.** Given $\vec{v} \cdot \vec{u} = \vec{w} \cdot \vec{u}$

and $\vec{v} \perp \vec{w} \Rightarrow \vec{v} \cdot \vec{w} = 0$

Now, $|\vec{u} - \vec{v} + \vec{w}|^2$

$= |\vec{u}|^2 + |\vec{v}|^2 + |\vec{w}|^2 - 2\vec{u} \cdot \vec{v} - 2\vec{w} \cdot \vec{v} + 2\vec{u} \cdot \vec{w}$

$= 1 + 4 + 9$

so $|\vec{u} - \vec{v} + \vec{w}| = \sqrt{14}$

32. **b.** We have

$\vec{p} \cdot \vec{q} = 0$

$\Rightarrow (5\vec{a} - 3\vec{b}) \cdot (-\vec{a} - 2\vec{b}) = 0$

$\Rightarrow 6|\vec{b}|^2 - 5|\vec{a}|^2 - 7\vec{a} \cdot \vec{b} = 0$ (i)

Also $\vec{r} \cdot \vec{s} = 0$

$\Rightarrow (-4\vec{a} - \vec{b})\ (-\vec{a} + \vec{b}) = 0$

$\Rightarrow 4|\vec{a}|^2 - |\vec{b}|^2 - 3\vec{a} \cdot \vec{b} = 0$ (ii)

Now $\vec{x} = \frac{1}{3}(\vec{p} + \vec{r} + \vec{s}) = \frac{1}{3}(5\vec{a} - 3\vec{b} - 4\vec{a} - \vec{b} - \vec{a} + \vec{b}) = -\vec{b}$

and $\vec{y} = \frac{1}{5}(\vec{r} + \vec{s}) = \frac{1}{5}(-5\vec{a}) = -\vec{a}$

Angle between $\vec{x}$ and $\vec{y}$, i.e., $\cos\theta = \dfrac{\vec{x} \cdot \vec{y}}{|\vec{x}||\vec{y}|} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$ (iii)

From (i) and (ii), $|\vec{a}| = \sqrt{\frac{25}{19}}\sqrt{\vec{a} \cdot \vec{b}}$ and $|\vec{b}| = \sqrt{\frac{43}{19}}\sqrt{\vec{a} \cdot \vec{b}}$. Therefore

$|\vec{a}||\vec{b}| = \dfrac{\sqrt{25 \times 43}}{19} \cdot \vec{a} \cdot \vec{b}$

$\theta = \cos^{-1}\left(\dfrac{19}{5\sqrt{43}}\right)$

33. **a.** $\vec{\alpha} \parallel (\vec{\beta} \times \vec{\gamma}) \Rightarrow \vec{\alpha} \perp \vec{\beta}$ and $\vec{\alpha} \perp \vec{\gamma}$

Now, $(\vec{\alpha} \times \vec{\beta}) \cdot (\vec{\alpha} \times \vec{\gamma}) = |\vec{\alpha}|^2 (\vec{\beta} \cdot \vec{\gamma}) - (\vec{\alpha} \cdot \vec{\beta})(\vec{\alpha} \cdot \vec{\gamma}) = |\vec{\alpha}|^2 \cdot (\vec{\beta} \cdot \vec{\gamma})$

34. **b.** Since, $\overrightarrow{OA} = \hat{i} + \hat{j} + \hat{k}$

$\overrightarrow{OB} = \hat{i} + 5\hat{j} - \hat{k}$

$\overrightarrow{OC} = 2\hat{i} + 3\hat{j} + 5\hat{k}$

$a = BC = |\overrightarrow{BC}| = |\overrightarrow{OC} - \overrightarrow{OB}| = |\hat{i} - 2\hat{j} + 6\hat{k}| = \sqrt{41}$

$b = CA = |\overrightarrow{CA}| = |\overrightarrow{OA} - \overrightarrow{OC}| = |-\hat{i} - 2\hat{j} - 4\hat{k}| = \sqrt{21}$

and $c = AB = |\overrightarrow{AB}| = |\overrightarrow{OB} - \overrightarrow{OA}| = |0\hat{i} + 4\hat{j} - 2\hat{k}| = \sqrt{20}$

Since $a > b > c$, A is the greatest angle. Therefore,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{21 + 20 - 41}{2 \cdot \sqrt{21} \cdot \sqrt{20}} = 0$$

$\therefore \angle A = 90°$

35. **b.** $\vec{a} + \vec{b} = \lambda \vec{c}$ (i)

and $\vec{b} + \vec{c} = \mu \vec{a}$ (ii)

$\therefore (\lambda \vec{c} - \vec{a}) + \vec{c} = \mu \vec{a}$ (putting $\vec{b} = \lambda \vec{c} - \vec{a}$)

$\Rightarrow (\lambda + 1)\vec{c} = (\mu + 1)\vec{a}$

$\Rightarrow \lambda = \mu = -1$

$\Rightarrow \vec{a} + \vec{b} + \vec{c} = 0$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = 0$

$\Rightarrow \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} = -3$

36. **d.** $\vec{0} = (\vec{a} + \vec{b}) \cdot (2\vec{a} + 3\vec{b}) \times (3\vec{a} - 2\vec{b})$

$= (\vec{a} + \vec{b}) \cdot (-4\vec{a} \times \vec{b} - 9\vec{a} \times \vec{b})$

$= -13\ (\vec{a} + \vec{b}) \cdot (\vec{a} \times \vec{b})$

which is true for all values of $\vec{a}$ and $\vec{b}$.

37. **c.** We have

$$\overrightarrow{AB}\cdot\overrightarrow{AC}+\overrightarrow{BC}\cdot\overrightarrow{BA}+\overrightarrow{CA}\cdot\overrightarrow{CB}=(AB)(AC)\cos\theta+(BC)(BA)\sin\theta+0$$

$$=AB\,(AC\cos\theta+BC\sin\theta)$$

$$=AB\left(\frac{(AC)^2}{AB}+\frac{(BC)^2}{AB}\right)$$

$$=AC^2+BC^2=AB^2=p^2$$

38. **c.** $\vec{a}_1=(\vec{a}\cdot\hat{b})\,\hat{b}=\dfrac{(\vec{a}\cdot\vec{b})\,\vec{b}}{|\vec{b}|^2}$

$$\Rightarrow \vec{a}_2=\vec{a}-\vec{a}_1=\vec{a}-\frac{(\vec{a}\cdot\vec{b})\,\vec{b}}{|\vec{b}|^2}$$

Thus, $\vec{a}_1\times\vec{a}_2=\dfrac{(\vec{a}\cdot\vec{b})\,\vec{b}}{|\vec{b}|^2}\times\left(\vec{a}-\dfrac{(\vec{a}\cdot\vec{b})\,\vec{b}}{|\vec{b}|^2}\right)=\dfrac{(\vec{a}\cdot\vec{b})(\vec{b}\times\vec{a})}{|\vec{b}|^2}$.

39. **b.** Let the required vector be $\vec{r}$. Then $\vec{r}=x_1\vec{b}+x_2\vec{c}$ and $\vec{r}\cdot\vec{a}=\sqrt{\dfrac{2}{3}}\;(|\vec{a}|)=2$

Now, $\vec{r}\cdot\vec{a}=x_1\,\vec{a}\cdot\vec{b}+x_2\,\vec{a}\cdot\vec{c}\Rightarrow 2=x_1(2-2-1)+x_2(2-1-2)\Rightarrow x_1+x_2=-2$

$\Rightarrow \vec{r}=x_1(\hat{i}+2\hat{j}-\hat{k})+x_2(\hat{i}+\hat{j}-2\hat{k})=\hat{i}\,(x_1+x_2)+\hat{j}\,(2x_1+x_2)-\hat{k}\,(2x_2+x_1)$

$$=-2\hat{i}+\hat{j}\,(x_1-2)-\hat{k}\,(-4-x_1),\ \text{where } x_1\in R$$

40. **a.** Let P.V. of P, A, B and C be $\vec{p}$, $\vec{a}$, $\vec{b}$ and $\vec{c}$, respectively, and $O(\vec{0})$ be the circumcentre of equilateral triangle ABC. Then

$$|\vec{p}|=|\vec{b}|=|\vec{a}|=|\vec{c}|=\frac{l}{\sqrt{3}}$$

Now $|\overrightarrow{PA}|^2=|\vec{a}-\vec{p}|^2=|\vec{a}|^2+|\vec{p}|^2-2\vec{p}\cdot\vec{a}$

Similarly, $|\overrightarrow{PB}|^2=|\vec{b}|^2+|\vec{p}|^2-2\vec{p}\cdot\vec{b}$

and $|\overrightarrow{PC}|^2=|\vec{c}|^2+|\vec{p}|^2-2\vec{p}\cdot\vec{c}$

$\Rightarrow \Sigma|\overrightarrow{PA}|^2=6\cdot\dfrac{l^2}{3}-2\vec{p}\cdot(\vec{a}+\vec{b}+\vec{c})=2l^2$ as $(\vec{a}+\vec{b}+\vec{c}/3=\vec{0})$

41. **d.** For minimum value $|\vec{r}+b\vec{s}|=0$.

Let $\vec{r}$ and $\vec{s}$ are anti parallel so $b\vec{s}=-\vec{r}$

so $|b\vec{s}|^2+|\vec{r}+b\vec{s}|^2=|-\vec{r}|^2+|\vec{r}-\vec{r}|^2=|\vec{r}|^2$

42. **c.** Let the required vector $\vec{r}$ be such that

$$\vec{r}=x_1\vec{a}+x_2\vec{b}+x_3\,\vec{a}\times\vec{b}$$

We must have $\vec{r}\cdot\vec{a} = \vec{r}\cdot\vec{b} = \vec{r}\cdot(\vec{a}\times\vec{b})$ (as $\vec{r}, \vec{a}, \vec{b}$ and $\vec{a}\times\vec{b}$ are unit vectors and $\vec{r}$ is equally inclined to $\vec{a}, \vec{b}$ and $\vec{a}\times\vec{b}$)

Now $\vec{r}\cdot\vec{a} = x_1$, $\vec{r}\cdot\vec{b} = x_2$, $\vec{r}\cdot(\vec{a}\times\vec{b}) = x_3$

$\Rightarrow \vec{r} = \lambda(\vec{a}+\vec{b}+(\vec{a}\times\vec{b}))$

Also, $\vec{r}\cdot\vec{r} = 1$

$\Rightarrow \lambda^2(\vec{a}+\vec{b}+\vec{a}\times\vec{b})\cdot(\vec{a}+\vec{b}+(\vec{a}\times\vec{b})) = 1$

$\Rightarrow \lambda^2(|\vec{a}|^2+|\vec{b}|^2+|\vec{a}\times\vec{b}|^2) = 1$

$\Rightarrow \lambda^2 = \dfrac{1}{3}$

$\Rightarrow \lambda = \pm\dfrac{1}{\sqrt{3}}$

$\Rightarrow \vec{r} = \pm\dfrac{1}{\sqrt{3}}(\vec{a}+\vec{b}+\vec{a}\times\vec{b})$

43. **d.** $\vec{a}+\vec{b} = \mu\vec{p}$ $\quad \vec{b}\cdot\vec{q} = 0$, $|\vec{b}|^2 = 1$

$\because \vec{a}+\vec{b} = \mu\vec{p}$

$\Rightarrow (\vec{a}+\vec{b})\times\vec{a} = \mu\vec{p}\times\vec{a}$, $\vec{b}\times\vec{a} = \mu\vec{p}\times\vec{a} \Rightarrow \vec{q}\times(\vec{b}\times\vec{a}) = \mu\vec{q}\times(\vec{p}\times\vec{a})$

$\Rightarrow (\vec{q}\cdot\vec{a})\vec{b} - (\vec{q}\cdot\vec{b})\vec{a} = \mu\vec{q}\times(\vec{p}\times\vec{a}) \Rightarrow (\vec{q}\cdot\vec{a})\vec{b} = \mu\vec{q}\times(\vec{p}\times\vec{a})$

$\because \vec{a}+\vec{b} = \mu\vec{p}$

$\Rightarrow \vec{q}\cdot(\vec{a}+\vec{b}) = \mu\vec{q}\cdot\vec{p}$

$\Rightarrow \vec{q}\cdot\vec{a}+\vec{q}\cdot\vec{b} = \mu\vec{p}\cdot\vec{q}$

$\Rightarrow \mu = \dfrac{\vec{q}\cdot\vec{a}}{\vec{p}\cdot\vec{q}}$

$\Rightarrow (\vec{q}\cdot\vec{a})\vec{b} = \dfrac{\vec{q}\cdot\vec{a}}{\vec{p}\cdot\vec{q}}[(\vec{q}\cdot\vec{a})\cdot\vec{p} - (\vec{q}\cdot\vec{p})\vec{a}]$

$\Rightarrow |(\vec{q}\cdot\vec{a})\vec{p} - (\vec{q}\cdot\vec{p})\vec{a}| = |(\vec{p}\cdot\vec{q})\vec{b}| = |(\vec{p}\cdot\vec{q})|\cdot|\vec{b}|$

$\Rightarrow |(\vec{q}\cdot\vec{a})\vec{p} - (\vec{q}\cdot\vec{p})\vec{a}| = |\vec{p}.\vec{q}|$

44. **c.** $\vec{d}\cdot\hat{a} = \vec{d}\cdot\hat{b} = \vec{d}\cdot\hat{c}$

$\Rightarrow \lambda(\hat{a}\cdot\hat{b}+\hat{a}\cdot\hat{c}) = \lambda(1+\hat{b}\cdot\hat{c}) = \lambda(1+\hat{b}\cdot\hat{c}) \Rightarrow 1+\hat{b}\cdot\hat{c} = \hat{a}\cdot\hat{b}+\hat{a}\cdot\hat{c}$

$\Rightarrow 1-\hat{a}\cdot\hat{b}+\hat{b}\cdot\hat{c}-\hat{a}\cdot\hat{c} = 0 \Rightarrow 1-\hat{a}\cdot\hat{b}+(\hat{b}-\hat{a})\cdot\hat{c} = 0 \Rightarrow \hat{a}\cdot(\hat{a}-\hat{b})+(\hat{b}-\hat{a})\cdot\hat{c} = 0$

$\Rightarrow (\hat{a}-\hat{c})\cdot(\hat{a}-\hat{b}) = 0 \Rightarrow \hat{a}-\hat{c}$ is perpendicular to $(\hat{a}-\hat{b}) \Rightarrow$ The triangle is right angled.

45. **c.** The given relation can be rewritten as the vector expression

$$(\sqrt{a^2-4}\,\hat{i}+a\hat{j}+\sqrt{a^2+4}\,\hat{k})\cdot(\tan A\hat{i}+\tan B\hat{j}+\tan C\hat{k})=6a$$

$$\Rightarrow \sqrt{a^2-4+a^2+a^2+4}\sqrt{\tan^2 A+\tan^2 B+\tan^2 C}\cdot(\cos\theta)=6a \quad (\because \vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta)$$

$$\sqrt{3}\,a\sqrt{\tan^2 A+\tan^2 B+\tan^2 C}\cdot(\cos\theta)=6a$$

$\tan^2 A+\tan^2 B+\tan^2 C=12\sec^2\theta\ge 12 \quad (\because \sec^2\theta\ge 1)$

The least value of $\tan^2 A+\tan^2 B+\tan^2 C$ is 12.

46. **d.** $\Delta=\frac{1}{2}|(\hat{j}+\lambda\hat{k})\times(\hat{i}+\lambda\hat{k})|=\frac{1}{2}|-\hat{k}+\lambda\hat{i}+\lambda\hat{j}|=\frac{1}{2}\sqrt{2\lambda^2+1}$

$$\Rightarrow \frac{9}{4}\le\frac{1}{4}(2\lambda^2+1)\le\frac{33}{4}$$

$$\Rightarrow 4\le\lambda^2\le 16$$

$$\Rightarrow 2\le|\lambda|\le 4$$

47. **c.** Let the projection be x, then $\vec{a}=\frac{x(\hat{i}+\hat{j})}{\sqrt{2}}+\frac{x(-\hat{i}+\hat{j})}{\sqrt{2}}+x\hat{k}$

$$\therefore\ \vec{a}=\frac{2x\hat{j}}{\sqrt{2}}+x\hat{k}\quad\Rightarrow\quad \hat{a}=\frac{\sqrt{2}}{\sqrt{3}}\hat{j}+\frac{\hat{k}}{\sqrt{3}}$$

48. **b.** Let $\vec{r}$ be the new position. Then $\vec{r}=\lambda\hat{k}+\mu(\hat{i}+\hat{j})$

Also $\vec{r}\cdot\hat{k}=-\frac{1}{\sqrt{2}}\Rightarrow\lambda=-\frac{1}{\sqrt{2}}$

Also, $\lambda^2+2\mu^2=1\Rightarrow 2\mu^2=\frac{1}{2}\Rightarrow\mu=\pm\frac{1}{2}$

$$\therefore\ \vec{r}=\pm\frac{1}{2}(\hat{i}+\hat{j})-\frac{\hat{k}}{\sqrt{2}}$$

49. **c.**

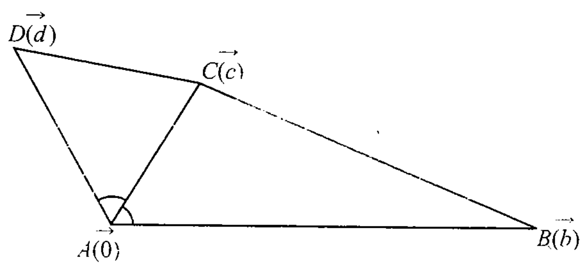


Fig. 2.39

Let $|\overrightarrow{AC}| = \lambda > 0$

Then from $15\,|\overrightarrow{AC}| = 3\,|\overrightarrow{AB}| = 5\,|\overrightarrow{AD}|$

$|\overrightarrow{AB}| = 5\lambda$

Let θ be the angle between $\overrightarrow{BA}$ and $\overrightarrow{CD}$.

$$\Rightarrow \cos\theta = \frac{\overrightarrow{BA}\cdot\overrightarrow{CD}}{|\overrightarrow{BA}||\overrightarrow{CD}|} = \frac{-\vec{b}\cdot(\vec{d}-\vec{c})}{|\vec{b}||\vec{d}-\vec{c}|} \qquad \text{(i)}$$

Now $-\vec{b}\cdot(\vec{d}-\vec{c}) = \vec{b}\cdot\vec{c} - \vec{b}\cdot\vec{d}$

$$= |\vec{b}||\vec{c}|\cos\frac{\pi}{3} - |\vec{b}||\vec{d}|\cos\frac{2\pi}{3}$$

$$= (5\lambda)(\lambda)\frac{1}{2} + (5\lambda)(3\lambda)\frac{1}{2}$$

$$= \frac{5\lambda^2 + 15\lambda^2}{2}$$

$$= 10\lambda^2$$

Denominator of (i) $= |\vec{b}||\vec{d}-\vec{c}|$

Now $|\vec{d}-\vec{c}|^2 = |\vec{d}|^2 + |\vec{c}|^2 - 2\,\vec{c}\cdot\vec{d}$

$= 9\lambda^2 + \lambda^2 - 2(\lambda)(3\lambda)(1/2)$

$= 10\lambda^2 - 3\lambda^2$

$= 7\lambda^2$

Denominator of (i) $= (5\lambda)(\sqrt{7}\,\lambda) = 5\sqrt{7}\,\lambda^2$

$$\therefore \cos\theta = \frac{10\lambda^2}{5\sqrt{7}\,\lambda^2} = \frac{2}{\sqrt{7}}$$

50. **a.** Let A be the origin. $\overrightarrow{AB} = \vec{a}$, $\overrightarrow{AD} = \vec{b}$

so, $\overrightarrow{AE} = \vec{b} + \frac{3}{2}\vec{a}$, $\overrightarrow{AG} = \vec{a} + 3\vec{b}$.

$$\text{So the required ratio} = \frac{\frac{1}{2}\left|(\vec{a}+3\vec{b})\times\left(\vec{b}+\frac{3}{2}\vec{a}\right)\right|}{\frac{1}{2}|\vec{a}\times\vec{b}|}$$

$$= \frac{7}{2}$$

51. **b.** Let $\vec{a} = \lambda\,\vec{b} + \mu\,\vec{c}$

$\vec{a}$ is equally inclined to $\vec{b}$ and $\vec{d}$ where $\vec{d} = \hat{j} + 2\hat{k}$.

$$\Rightarrow \frac{\vec{a}\cdot\vec{b}}{ab} = \frac{\vec{a}\cdot\vec{d}}{ad}$$

$$\Rightarrow \frac{(\lambda\vec{b}+\mu\vec{c})\cdot\vec{b}}{b} = \frac{(\lambda\vec{b}+\mu\vec{c})\cdot\vec{d}}{d}$$

$$\Rightarrow \frac{[\lambda(2\hat{i}+\hat{j})+\mu(\hat{i}-\hat{j}+\hat{k})]\cdot(2\hat{i}+\hat{j})}{\sqrt{5}} = \frac{[\lambda(2\hat{i}+\hat{j})+\mu(\hat{i}-\hat{j}+\hat{k})]\cdot(\hat{j}+2\hat{k})}{\sqrt{5}}$$

$$\Rightarrow \lambda(4+1)+\mu(2-1)=\lambda(1)+\mu(-1+2)$$

$$\Rightarrow 4\lambda=0, \text{ i.e., } \lambda=0$$

$$\therefore\ \hat{a} = \frac{\hat{i}-\hat{j}+\hat{k}}{\sqrt{3}}$$

52. **a.** Area of $\Delta BCD = \frac{1}{2}|\overrightarrow{BC}\times\overrightarrow{BD}| = \frac{1}{2}|(b\hat{i}-c\hat{j})\times(b\hat{i}-d\hat{k})|$

$$= \frac{1}{2}|bd\,\hat{j}+bc\,\hat{k}+dc\,\hat{i}|$$

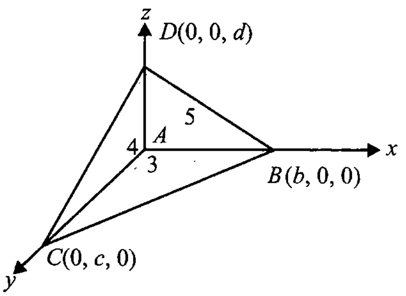


Fig. 2.40

$$= \frac{1}{2}\sqrt{b^2c^2+c^2d^2+d^2b^2} \quad \text{(i)}$$

Now $6 = bc$; $8 = cd$; $10 = bd$

$b^2c^2 + c^2d^2 + d^2b^2 = 200$

Substituting the value in (i)

$$A = \frac{1}{2}\sqrt{200} = 5\sqrt{2}$$

53. **d.** $\vec{f}\left(\frac{5}{4}\right) = \left[\frac{5}{4}\right]\hat{i} + \left(\frac{5}{4}-\left[\frac{5}{4}\right]\right)\hat{j} + \left[\frac{5}{4}+1\right]\hat{k}$

$$= \hat{i} + \left(\frac{5}{4}-1\right)\hat{j} + 2\hat{k}$$

$$= \hat{i} + \frac{1}{4}\hat{j} + 2\hat{k}$$

When $0 < t < 1$, $\vec{f}(t) = 0\vec{i} + \{t-0\}\vec{j} + \vec{k} = t\vec{j} + \vec{k}$

$\vec{f}\left(\frac{5}{4}\right) \cdot \vec{f}(t) = 2 + \frac{t}{4}$

So $\cos\theta = \dfrac{2+\frac{t}{4}}{\left|\vec{i} + \frac{1}{4}\vec{j} + 2\vec{k}\right| |t\vec{j} + \vec{k}|} = \dfrac{2+\frac{t}{4}}{\sqrt{1+\frac{1}{16}+4}\sqrt{1+t^2}}$

$$= \frac{8+t}{9\sqrt{1+t^2}}$$

54. **a.** $(\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{u}$, where $\vec{u} = \vec{a}\times\vec{c}$

$\Rightarrow \vec{a}\cdot(\vec{b}\times\vec{u}) = \vec{a}\cdot[\vec{b}\times(\vec{a}\times\vec{c})]$

$= \vec{a}\cdot[(\vec{b}\cdot\vec{c})\vec{a} - (\vec{a}\cdot\vec{b})\vec{c}]$

$= \vec{a}\cdot(\vec{b}\cdot\vec{c})\vec{a}$ $(\because \vec{a}\cdot\vec{b} = 0)$

$= |\vec{a}|^2(\vec{b}\cdot\vec{c})$

55. **d.** $(\hat{i}+\hat{j})\times(\hat{j}+\hat{k}) = \hat{i}-\hat{j}+\hat{k}$ so that unit vector perpendicular to the plane of $\hat{i}+\hat{j}$ and $\hat{j}+\hat{k}$ is $\frac{1}{\sqrt{3}}(\hat{i}-\hat{j}+\hat{k})$.

Similarly, the other two unit vectors are $\frac{1}{\sqrt{3}}(\hat{i}+\hat{j}-\hat{k})$ and $\frac{1}{\sqrt{3}}(-\hat{i}+\hat{j}+\hat{k})$.

The required volume $= \dfrac{3}{\sqrt{3}}\begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{vmatrix} = 4\sqrt{3}$

56. **c.** $\vec{d}\cdot\vec{c} = \vec{d}\cdot\vec{b} = \vec{d}\cdot\vec{c} = [\vec{a}\,\vec{b}\,\vec{c}]$

Then $|(\vec{d}\cdot\vec{c})(\vec{a}\times\vec{b}) + (\vec{d}\cdot\vec{a})(\vec{b}\times\vec{c}) + (\vec{d}\cdot\vec{b})(\vec{c}\times\vec{a})| = 0$

$\Rightarrow [\vec{a}\,\vec{b}\,\vec{c}]\,|\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a}| = 0$

$\Rightarrow [\vec{a}\,\vec{b}\,\vec{c}] = 0$ $(\because \vec{d}$ is non-zero)

$\Rightarrow \vec{a}, \vec{b}, \vec{c}$ are coplanar.

57. **a.** $(\vec{a}\times(\vec{a}\times(\vec{a}\times(\vec{a}\times\vec{b})))) = (\vec{a}\times(\vec{a}\times((\vec{a}\cdot\vec{b})\vec{a} - (\vec{a}\cdot\vec{b})\vec{b})))$

$= (\vec{a}\times(\vec{a}\times(-4\vec{b})))$

$= -4(\vec{a} \times (\vec{a} \times \vec{b}))$

$= -4((\vec{a} \cdot \vec{b})\vec{a} - (\vec{a} \cdot \vec{a})\vec{b})$

$= -4(-4\vec{b}) = 16\vec{b} = 48\hat{b}$

58. **d.** Let $\vec{a} = 6\hat{i} + 6\hat{k}, \vec{b} = 4\hat{j} + 2\hat{k}, \vec{c} = 4\hat{j} - 8\hat{k}$

then $\vec{a} \times \vec{b} = -24\hat{i} - 12\hat{j} + 24\hat{k}$

$= 12(-2\hat{i} - \hat{j} + 2\hat{k})$

$\therefore$ Area of the base of the parallelepiped $= \frac{1}{2}|\vec{a} \times \vec{b}|$

$= \frac{1}{2}(12 \times 3)$

$= 18$

Height of the parallelepiped = length of projection of $\vec{c}$ on $\vec{a} \times \vec{b}$

$= \frac{|\vec{c} \cdot \vec{a} \times \vec{b}|}{|\vec{a} \times \vec{b}|}$

$= \frac{|12(-4-16)|}{36}$

$= \frac{20}{3}$

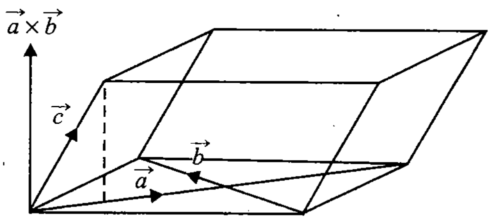


Fig. 2.41

$\therefore$ Volume of the parallelepiped $= 18 \times \frac{20}{3} = 120$

59. **c.** $3 = \frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]$

$\Rightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 18$

Volume of the required parallelepiped

$= [\vec{a} + \vec{b}\ \ \vec{b} + \vec{c}\ \ \vec{c} + \vec{a}]$

$= 2[\vec{a}\ \vec{b}\ \vec{c}] = 36$

60. **b.** Here $[\vec{a}\ \vec{b}\ \vec{c}] = \pm 1$

$[\vec{a} + \vec{b} + \vec{c}\ \ \vec{a} + \vec{b}\ \ \vec{b} + \vec{c}] = (\vec{a} + \vec{b} + \vec{c}) \times (\vec{a} + \vec{b}) \cdot (\vec{b} + \vec{c})$

$= \vec{c} \times (\vec{a} + \vec{b}) \cdot (\vec{b} + \vec{c})$

$= (\vec{c} \times \vec{a} + \vec{c} \times \vec{b}) \cdot (\vec{b} + \vec{c})$

$= \vec{c} \times \vec{a} \cdot \vec{b} = [\vec{a}\ \vec{b}\ \vec{c}] = \pm 1$

61. **a.** Let $\vec{c} = \lambda(\vec{a} \times \vec{b})$.

Hence $\lambda(\vec{a} \times \vec{b}) \cdot (\hat{i} + 2\hat{j} - 7\hat{k}) = 10$

$$\Rightarrow \lambda \begin{vmatrix} 2 & -3 & 1 \\ 1 & -2 & 3 \\ 1 & 2 & -7 \end{vmatrix} = 10$$

$\Rightarrow \lambda = -1$

$\Rightarrow \vec{c} = -(\vec{a} \times \vec{b})$

62. **d.** $\vec{a} \perp \vec{b} \Rightarrow x - y + 2 = 0$

$\vec{a} \cdot \vec{c} = 4 \Rightarrow x + 2y = 4$

Solving we get $x = 0;\ y = 2$

$\Rightarrow \vec{a} = 2\hat{j} + 2\hat{k}$

$$\Rightarrow [\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 0 & 2 & 2 \\ 1 & -1 & 1 \\ 1 & 2 & 0 \end{vmatrix} = 8 = |\vec{a}|^2$$

63. **c.** $(\vec{a} \times \vec{b} \cdot \vec{c})^2 = |\vec{a}|^2 |\vec{b}|^2 |\vec{c}|^2 \sin^2\theta \cos^2\phi$ (θ is the angle between $\vec{a}$ and $\vec{b}$, $\phi = 0$)

$= \frac{1}{4}(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$

64. **c.** $\vec{r} \cdot \vec{a} = 0, |\vec{r} \times \vec{b}| = |\vec{r}||\vec{b}|$ and $|\vec{r} \times \vec{c}| = |\vec{r}||\vec{c}|$

$\Rightarrow \vec{r} \perp \vec{a}, \vec{b}, \vec{c}$

$\therefore [\vec{a}\ \vec{b}\ \vec{c}] = 0$

65. **b.** $\vec{c} = \lambda(\vec{a} \times \vec{b})$

$\Rightarrow \vec{c} \cdot \vec{c} = \lambda(\vec{a} \times \vec{b}) \cdot \vec{c}$

$\Rightarrow \frac{1}{3} = \lambda$

Also $|\vec{c}|^2 = \lambda^2 |\vec{a} \times \vec{b}|^2$

$\Rightarrow \frac{1}{3} = \frac{1}{9}(a^2b^2\sin^2\theta) = \frac{1}{9} \times 2 \times 3\sin^2\theta$

$\Rightarrow \sin^2\theta = \frac{1}{2}$

$\Rightarrow \theta = \frac{\pi}{4}$

66. **c.** $4\vec{a} + 5\vec{b} + 9\vec{c} = 0 \Rightarrow$ Vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar.

$\Rightarrow \vec{b} \times \vec{c}$ and $\vec{c} \times \vec{a}$ are collinear $\Rightarrow (\vec{b} \times \vec{c}) \times (\vec{c} \times \vec{a}) = \vec{0}$.

67. **a.** $[\vec{a}\times\vec{b}\ \ \vec{a}\times\vec{c}\ \ \vec{d}]$

$=(\vec{a}\times\vec{b})\cdot((\vec{a}\times\vec{c})\times\vec{d})$

$=(\vec{a}\times\vec{b})\cdot((\vec{a}\cdot\vec{d})\vec{c}-(\vec{c}\cdot\vec{d})\vec{a})$

$=(\vec{a}\cdot\vec{d})[\vec{a}\,\vec{b}\,\vec{c}]$

68. **a.** Let $\vec{r}=x_1\hat{a}+x_2\hat{b}+x_3(\hat{a}\times\hat{b})$

$\Rightarrow \vec{r}\cdot\hat{a}=x_1+x_2\hat{a}\cdot\hat{b}+x_3\hat{a}\cdot(\hat{a}\times\hat{b})=x_1$

Also, $\vec{r}\cdot\hat{b}=x_1\hat{a}\cdot\hat{b}+x_2+x_3\hat{b}\cdot(\hat{a}+\hat{b})=x_2$

and $\vec{r}\cdot(\hat{a}\times\hat{b})=x_1\hat{a}\cdot(\hat{a}\times\hat{b})+x_2\hat{b}\cdot(\hat{a}\times\hat{b})+x_3(\hat{a}\times\hat{b})\cdot(\hat{a}\times\hat{b})=x_3$

$\Rightarrow \vec{r}=(\vec{r}\cdot\hat{a})\hat{a}+(\vec{r}\cdot\hat{b})\hat{b}+(\vec{r}\cdot(\hat{a}\times\hat{b}))(\hat{a}\times\hat{b})$

69. **a.** $[\vec{a}+(\vec{a}\times\vec{b})\ \ \vec{b}+(\vec{a}\times\vec{b})\ \ \vec{a}\times\vec{b}]$

$=(\vec{a}+(\vec{a}\times\vec{b}))\cdot((\vec{b}+(\vec{a}\times\vec{b}))\times(\vec{a}\times\vec{b}))$

$=(\vec{a}+(\vec{a}\times\vec{b}))\cdot(\vec{b}\times(\vec{a}\times\vec{b}))$

$=(\vec{a}+(\vec{a}\times\vec{b}))\cdot(\vec{a}-(\vec{a}\cdot\vec{b})\vec{b})$

$=\vec{a}\cdot\vec{a}=1$ (as $\vec{a}\cdot\vec{b}=0,\ \vec{a}\cdot(\vec{a}\times\vec{b})=0$)

70. **d.** $|\vec{a}|=1, |\vec{b}|=4, \vec{a}\cdot\vec{b}=2$

$\vec{c}=(2\vec{a}\times\vec{b})-3\vec{b}$

$\Rightarrow \vec{c}+3\vec{b}=2\vec{a}\times\vec{b}$

$\because\ \vec{a}\cdot\vec{b}=2$

$\Rightarrow |\vec{a}|\cdot|\vec{b}|\cos\theta=2$

$\Rightarrow \cos\theta=\dfrac{2}{|\vec{a}|\cdot|\vec{b}|}=\dfrac{2}{4}$

$\Rightarrow \cos\theta=\dfrac{1}{2}$

$\because\ \theta=\dfrac{\pi}{3}$

$\Rightarrow |\vec{c}+3\vec{b}|^2=|2\vec{a}\times\vec{b}|^2$

$\Rightarrow |\vec{c}|^2+9|\vec{b}|^2+2\vec{c}\cdot3\vec{b}=4|\vec{a}|^2|\vec{b}|^2\sin^2\theta$

$\Rightarrow |\vec{c}|^2+144+6\vec{b}\cdot\vec{c}=48$

$\Rightarrow |\vec{c}|^2+96+6(\vec{b}\cdot\vec{c})=0$ (i)

$\because\quad \vec{c}=2\vec{a}\times\vec{b}-3\vec{b}$

$\Rightarrow \quad \vec{b}\cdot\vec{c} = 0 - 3\times 16$

$\therefore \quad \vec{b}\cdot\vec{c} = -48$

Putting value of $\vec{b}\cdot\vec{c}$ in Eq. (i)

$|\vec{c}|^2 + 96 - 6\times 48 = 0$

$\Rightarrow \quad |\vec{c}|^2 = 48\times 4$

$\Rightarrow \quad |\vec{c}|^2 = 192$

Again, putting the value of $|\vec{c}|$ in Eq. (i),

$192 + 96 + 6|\vec{b}|\cdot|\vec{c}| \cos\alpha = 0$

$\Rightarrow \quad 6\times 4\times 8\sqrt{3} \cos\alpha = -288$

$\Rightarrow \quad \cos\alpha = -\dfrac{288}{6\times 4\times 8\sqrt{3}} = -\dfrac{3}{2\sqrt{3}} \Rightarrow \cos\alpha = -\dfrac{\sqrt{3}}{2}$

$\therefore \quad \alpha = \dfrac{5\pi}{6}$

71. **d.** $((\vec{a}\times\vec{b}) + (\vec{a}\times\vec{c}))\times(\vec{b}\times\vec{c})$

$= (\vec{a}\times\vec{b})\times(\vec{b}\times\vec{c}) + (\vec{a}\times\vec{c})\times(\vec{b}\times\vec{c})$

$= ((\vec{a}\times\vec{b})\cdot\vec{c})\vec{b} - ((\vec{a}\times\vec{b})\cdot\vec{b})\vec{c} + ((\vec{a}\times\vec{c})\cdot\vec{c})\vec{b} - ((\vec{a}\times\vec{c})\cdot\vec{b})\vec{c}$

$= [\vec{a}\,\vec{b}\,\vec{c}](\vec{b}+\vec{c})$

$\Rightarrow ((\vec{a}\times\vec{b}) + (\vec{a}\times\vec{c}))\times(\vec{b}\times\vec{c}))\cdot(\vec{b}-\vec{c})$

$= [\vec{a}\,\vec{b}\,\vec{c}](\vec{b}+\vec{c}).(\vec{b}-\vec{c})$

$= [\vec{a}\,\vec{b}\,\vec{c}](|\vec{b}|^2 - |c|^2) = 0$

72. **a.** $\vec{a}\times\vec{b} = \vec{c}$

$\Rightarrow \vec{a}\times(\vec{a}\times\vec{b}) = \vec{a}\times\vec{c}$

$\Rightarrow (\vec{a}\cdot\vec{b})\vec{a} - |\vec{a}|^2\vec{b} = \vec{a}\times\vec{c}$

$\Rightarrow \vec{b} = \dfrac{\beta\vec{a} - \vec{a}\times\vec{c}}{|\vec{a}|^2} \qquad (\because\ \vec{a}\cdot\vec{b} = \beta)$

73. **b.** Taking dot product of $a(\vec{\alpha}\times\vec{\beta}) + b(\vec{\beta}\times\vec{\gamma}) + c(\vec{\gamma}\times\vec{\alpha}) = 0$ with $\vec{\gamma}, \vec{\alpha}$ and $\vec{\beta}$, respectively, we have

$a[\vec{\alpha}\vec{\beta}\vec{\gamma}] = 0$

$b[\vec{\alpha}\vec{\beta}\vec{\gamma}] = 0$

$c[\vec{\alpha}\vec{\beta}\vec{\gamma}] = 0$

$\because$ At least one of a, b and $c \neq 0$

$\therefore [\vec{\alpha}\vec{\beta}\vec{\gamma}]=0$

Hence $\vec{\alpha}, \vec{\beta}$ and $\vec{\gamma}$ are coplanar.

74. **c.** $(\vec{a} \times \vec{b}) \times (\vec{b} \times \vec{c}) = \vec{b}$

$\Rightarrow [\vec{a}\vec{b}\vec{c}]\vec{b} = \vec{b}$

$\Rightarrow [\vec{a}\vec{b}\vec{c}] = 1$

$\therefore \vec{a}, \vec{b}$ and $\vec{c}$ cannot be coplanar.

75. **c.** Any vector $\vec{r}$ can be represented in terms of three non-coplanar vectors $\vec{a}, \vec{b}$ and $\vec{c}$ as

$$\vec{r} = x(\vec{a} \times \vec{b}) + y(\vec{b} \times \vec{c}) + z(\vec{c} \times \vec{a}) \quad \text{(i)}$$

Taking dot product with $\vec{a}, \vec{b}$ and $\vec{c}$, respectively, we have,

$$x = \frac{\vec{r}\cdot\vec{c}}{[\vec{a}\vec{b}\vec{c}]},\ y = \frac{\vec{r}\cdot\vec{a}}{[\vec{a}\vec{b}\vec{c}]} \text{ and } z = \frac{\vec{r}\cdot\vec{b}}{[\vec{a}\vec{b}\vec{c}]}$$

From (i)

$$[\vec{a}\vec{b}\vec{c}]\vec{r} = \frac{1}{2}(\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a})$$

$\therefore$ Area of ΔABC

$$= \frac{1}{2}|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|$$

$$= |[\vec{a}\vec{b}\vec{c}]\vec{r}|$$

76. **a.** Differentiate the curve

$6x + 8(xy_1 + y) + 4yy_1 = 0$

m_T at $(1, 0)$ is $6 + 8(y_1(0)) = 0$

$y_1(0) = -\frac{3}{4}$

$m_N = \frac{4}{3}$

Unit vector $= \pm \frac{(3\hat{i} + 4\hat{j})}{5}$

Again normal vector of magnitude $10 = \pm(6\hat{i} + 8\hat{j})$

77 **a.** $\{\vec{a} \times (\vec{b} + \vec{a} \times \vec{b})\} \cdot \vec{b}$

$= \{\vec{a} \times \vec{b} + \vec{a} \times (\vec{a} \times \vec{b})\} \cdot \vec{b}$

$= [\vec{a}\vec{b}\vec{b}] + \{(\vec{a}\cdot\vec{b})\vec{a} - (\vec{a}\cdot\vec{a})\vec{b}\} \cdot \vec{b}$

$= 0 + (\vec{a}\cdot\vec{b})^2 - (\vec{a}\cdot\vec{a})(\vec{b}\cdot\vec{b})$

$= \cos^2\frac{\pi}{3} - 1 = -\frac{3}{4}$

78. **a.** $\vec{r} \times \vec{a} = \lambda \vec{a} + \mu \vec{b} + \gamma \vec{a} \times \vec{b}$

$\therefore [\vec{r}\,\vec{a}\,\vec{a}] = \lambda \vec{a} \cdot \vec{a} + \mu \vec{b} \cdot \vec{a} + \gamma [\vec{a}\,\vec{b}\,\vec{a}]$

$0 = \lambda |\vec{a}|^2 + 0 + 0$

$\lambda = 0$

Also $[\vec{r}\,\vec{a}\,\vec{b}] = \lambda \vec{a} \cdot \vec{b} + \mu \vec{b} \cdot \vec{b} + \gamma [\vec{a}\,\vec{b}\,\vec{b}] = \mu$

Also $(\vec{r} \times \vec{a}) \times \vec{b} = \gamma (\vec{a} \times \vec{b}) \times \vec{b}$

$\Rightarrow (\vec{r} \cdot \vec{b})\vec{a} - (\vec{a} \cdot \vec{b})\vec{r} = \gamma \{(\vec{a} \cdot \vec{b})\vec{b} - (\vec{b} \cdot \vec{b})\vec{a}\}$

$\Rightarrow (\vec{r} \cdot \vec{b})\vec{a} = -\gamma \vec{a}, \quad \gamma = -(\vec{r} \cdot \vec{b})$

79. **c.** The given equation reduces to $[\vec{a}\ \vec{b}\ \vec{c}]^2 x^2 + 2[\vec{a}\ \vec{b}\ \vec{c}]x + 1 = 0 \Rightarrow D = 0$

80. **b.** $\vec{x} + \vec{c} \times \vec{y} = \vec{a}$ (i)

$\vec{y} + \vec{c} \times \vec{x} = \vec{b}$ (ii)

Taking cross with $\vec{c}$

$\vec{c} \times \vec{y} + \vec{c} \times (\vec{c} \times \vec{x}) = \vec{c} \times \vec{b}$

$\Rightarrow (\vec{a} - \vec{x}) + (\vec{c} \cdot \vec{x})\vec{c} - (\vec{c} \cdot \vec{c})\vec{x} = \vec{c} \times \vec{b}$

Also $\vec{x} + \vec{c} \times \vec{y} = \vec{a}$

$\Rightarrow \quad \vec{c} \cdot \vec{x} + \vec{c} \cdot (\vec{c} \times \vec{y}) = \vec{c} \cdot \vec{a}$

$\Rightarrow \quad \vec{c} \cdot \vec{x} + 0 = \vec{c} \cdot \vec{a}$

$\therefore \quad \vec{c} \cdot \vec{x} = \vec{c} \cdot \vec{a}$

$\Rightarrow \vec{a} - \vec{x} + (\vec{c} \cdot \vec{a})\vec{c} - (\vec{c} \cdot \vec{c})\vec{x} = \vec{c} \times \vec{b}$

$\Rightarrow \vec{x}(1 + (\vec{c} \cdot \vec{c})) = \vec{b} \times \vec{c} + \vec{a} + (\vec{c} \cdot \vec{a}) \cdot \vec{c}$

$\therefore \quad \vec{x} = \dfrac{\vec{b} \times \vec{c} + \vec{a} + (\vec{c} \cdot \vec{a})\vec{c}}{1 + \vec{c} \cdot \vec{c}}$

Similarly on taking cross product of Eq. (i), we find

$\vec{y} = \dfrac{\vec{a} \times \vec{c} + \vec{b} + (\vec{c} \cdot \vec{b})\vec{c}}{1 + \vec{c} \cdot \vec{c}}$

81. **c.** $\vec{r} \times \vec{a} = \vec{b}$

$\Rightarrow \vec{d} \times (\vec{r} \times \vec{a}) = \vec{d} \times \vec{b}$

$\Rightarrow (\vec{a} \cdot \vec{d})\vec{r} - (\vec{d} \cdot \vec{r})\vec{a} = \vec{d} \times \vec{b}$ (i)

$\vec{r} \times \vec{c} = \vec{d}$

$\Rightarrow \vec{b} \times (\vec{r} \times \vec{c}) = \vec{b} \times \vec{d}$

$\Rightarrow (\vec{b}\cdot\vec{c})\vec{r} - (\vec{b}\cdot\vec{r})\vec{c} = \vec{b}\times\vec{d}$ (ii)

Adding (i) and (ii) we get,

$(\vec{a}\cdot\vec{d} + \vec{b}\cdot\vec{c})\vec{r} - (\vec{d}\cdot\vec{r})\vec{a} - (\vec{b}\cdot\vec{r})\vec{c} = \vec{0}$

Now $\vec{r}\cdot\vec{d} = 0$ and $\vec{b}\cdot\vec{r} = 0$ as $\vec{d}$ and $\vec{r}$ as well as $\vec{b}$ and $\vec{r}$ are mutually perpendicular.

Hence, $(\vec{b}\cdot\vec{c} + \vec{a}\cdot\vec{d})\vec{r} = \vec{0}$

82. **b.** Let $\vec{a}\times\vec{b} = x\hat{i} + y\hat{j} + z\hat{k}$. Therefore,

$[\vec{a}\,\vec{b}\,\hat{i}] = (\vec{a}\times\vec{b})\cdot\hat{i} = x$

$[\vec{a}\,\vec{b}\,\hat{j}] = (\vec{a}\times\vec{b})\cdot\hat{j} = y$

$[\vec{a}\,\vec{b}\,\vec{k}] = (\vec{a}\times\vec{b})\cdot\hat{k} = z$

Hence, $[\vec{a}\,\vec{b}\,\hat{i}]\hat{i} + [\vec{a}\,\vec{b}\,\hat{j}]\hat{j} = [\vec{a}\,\vec{b}\,\hat{k}]\hat{k} = x\hat{i} + y\hat{j} + z\hat{k} = \vec{a}\times\vec{b}$

83. **a.** $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} = 5(\hat{i} + 2\hat{j} + 2\hat{k}) - 6(\hat{i} + \hat{j} + 2\hat{k})$

$\Rightarrow (1+\alpha)\hat{i} + \beta(1+\alpha)\hat{j} + \gamma(1+\alpha)(1+\beta)\hat{k} = -\hat{i} + 4\hat{j} - 2\hat{k}$

$\Rightarrow 1+\alpha = -1, \beta = -4$ and $\gamma(-1)(-3) = -2$

$\Rightarrow \gamma = -\frac{2}{3}$

84. **b.** If $\vec{a}(x)$ and $\vec{b}(x)$ are $\perp$, then $\vec{a}\cdot\vec{b} = 0$

$\Rightarrow \sin x\cos 2x + \cos x\sin 2x = 0$

$\sin(3x) = 0 = \sin 0$

$3x = n\pi \quad \Rightarrow x = \frac{n\pi}{3}$

Therefore, the two vectors are $\perp$ for infinite values of 'x'.

85. **b.** $(\vec{a}\times\hat{i})\cdot(\vec{b}\times\hat{i}) = \begin{vmatrix} \vec{a}\cdot\vec{b} & \vec{a}\cdot\hat{i} \\ \vec{b}\cdot\hat{i} & \hat{i}\cdot\hat{i} \end{vmatrix} = (\vec{a}\cdot\vec{b}) - (\vec{a}\cdot\hat{i})(\vec{b}\cdot\hat{i})$

Similarly, $(\vec{a}\times\hat{j})\cdot(\vec{b}\times\hat{j}) = (\vec{a}\cdot\vec{b}) - (\vec{a}\cdot\hat{j})(\vec{b}\cdot\hat{j})$

and $(\vec{a}\times\hat{k})\cdot(\vec{b}\times\hat{k}) = \vec{a}\cdot\vec{b} - (\vec{a}\cdot\hat{k})(\vec{b}\cdot\hat{k})$

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$. Therefore,

$(\vec{a}\cdot\hat{i}) = a_1$, $\vec{a}\cdot\hat{j} = a_2$, $\vec{a}\cdot\hat{k} = a_3$, $\vec{b}\cdot\hat{i} = b_1$, $\vec{b}\cdot\hat{j} = b_2$, $(\vec{b}\cdot\hat{k}) = b_3$

$\Rightarrow (\vec{a}\times\hat{i})\cdot(\vec{b}\times\hat{i}) + (\vec{a}\times\hat{j})\cdot(\vec{b}\times\hat{j}) + (\vec{a}\times\hat{k})\cdot(\vec{b}\times\hat{k})$

$= 3\vec{a}\cdot\vec{b} - (a_1b_1 + a_2b_2 + a_3b_3)$

$= 3\vec{a}\cdot\vec{b} - \vec{a}\cdot\vec{b} = 2\vec{a}\cdot\vec{b}$

86. **b.** $(\vec{a}\times\vec{b})\times(\vec{r}\times\vec{c})=((\vec{a}\times\vec{b})\cdot\vec{c})\vec{r}-((\vec{a}\times\vec{b})\cdot\vec{r})\cdot\vec{c}=[\vec{a}\,\vec{b}\,\vec{c}]\vec{r}-[\vec{a}\,\vec{b}\,\vec{r}]\vec{c}$

Similarly, $(\vec{b}\times\vec{c})\times(\vec{r}\times\vec{a})=[\vec{b}\,\vec{c}\,\vec{a}]\vec{r}-[\vec{b}\,\vec{c}\,\vec{r}]\vec{a}$

and, $(\vec{c}\times\vec{a})\times(\vec{r}\times\vec{b})=[\vec{c}\,\vec{a}\,\vec{b}]\vec{r}-[\vec{c}\,\vec{a}\,\vec{r}]\vec{b}$

$\Rightarrow (\vec{a}\times\vec{b})\times(\vec{r}\times\vec{c})+(\vec{b}\times\vec{c})\times(\vec{r}\times\vec{a})+(\vec{c}\times\vec{a})\times(\vec{r}\times\vec{b})$

$=3[\vec{a}\,\vec{b}\,\vec{c}]\vec{r}-([\vec{b}\,\vec{c}\,\vec{r}]\vec{a}+[\vec{c}\,\vec{a}\,\vec{r}]\vec{b}+[\vec{a}\,\vec{b}\,\vec{r}]\vec{c})$

$=3[\vec{a}\,\vec{b}\,\vec{c}]\vec{r}-[\vec{a}\,\vec{b}\,\vec{c}]\vec{r}=2[\vec{a}\,\vec{b}\,\vec{c}]\vec{r}$

87. **a.** We have,

$$\vec{a}\cdot\vec{p}=\vec{a}\cdot\frac{(\vec{b}\times\vec{c})}{[\vec{a}\,\vec{b}\,\vec{c}]}=\frac{\vec{a}\cdot(\vec{b}\times\vec{c})}{[\vec{a}\,\vec{b}\,\vec{c}]}=\frac{[\vec{a}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]}=1$$

$$\vec{a}.\vec{q}=\vec{a}\cdot\frac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}=\frac{[\vec{a}\,\vec{c}\,\vec{a}]}{[\vec{a}\,\vec{b}\,\vec{c}]}=0$$

Similarly, $\vec{a}\cdot\vec{r}=0,\ \vec{b}\cdot\vec{p}=0, \vec{b}\cdot\vec{q}=1, \vec{b}\cdot\vec{r}=0, \vec{c}\cdot\vec{p}=0,\ \vec{c}\cdot\vec{q}=0$ and $\vec{c}\cdot\vec{r}=1$

$\therefore\ (\vec{a}+\vec{b}+\vec{c})\cdot(\vec{p}+\vec{q}+\vec{r})=\vec{a}\cdot\vec{p}+\vec{a}\cdot\vec{q}+\vec{a}\cdot\vec{r}+\vec{b}\cdot\vec{p}+\vec{b}\cdot\vec{q}+\vec{b}\cdot\vec{r}+\vec{c}\cdot\vec{p}+\vec{c}\cdot\vec{q}+\vec{c}\cdot\vec{r}$

$=1+1+1=3$

88. **b.** A vector perpendicular to the plane of $A(\vec{a})$, $B(\vec{b})$ and $C(\vec{c})$ is

$(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})=\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}.$

Now for any point $R(\vec{r})$ in the plane of A, B and C is

$(\vec{r}-\vec{a})\cdot(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})=0.$

$\vec{r}\cdot(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})-\vec{a}\cdot(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})=0$

$\vec{r}\cdot(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})=0+\vec{a}\cdot\vec{b}\times\vec{c}+\vec{0}$

$\vec{r}\cdot(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})=[\vec{a}\,\vec{b}\,\vec{c}]$

89. **c.** Given that $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-coplanar.

$\Rightarrow [\vec{a}\,\vec{b}\,\vec{c}]\neq 0$ (i)

Again $\vec{a}\times(\vec{b}\times\vec{c})\cdot(\vec{a}\times\vec{c})=0$

$\Rightarrow [(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}]\cdot(\vec{a}\times\vec{c})=0$

$\Rightarrow (\vec{a}\cdot\vec{c})[\vec{b}\,\vec{a}\,\vec{c}]=0$

$\Rightarrow (\vec{a}\cdot\vec{c})=0$

$\Rightarrow \vec{a}$ and $\vec{c}$ are perpendicular. (ii)

$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} \Rightarrow [\vec{a}\times(\vec{b}\times\vec{c})]\times\vec{c} = \vec{0}$

90. **c.** Consider a tetrahedron with vertices $O(0, 0, 0), A(a, 0, 0), B(0, b, 0)$ and $C(0, 0, c)$.

Its volume $V = \frac{1}{6}[\vec{a}\,\vec{b}\,\vec{c}]$

Now centroids of the faces OAB, OAC, OBC and ABC are $G_1(a/3, b/3, 0)$, $G_2(a/3, 0, c/3)$, $G_3(0, b/3, c/3)$ and $G_4(a/3, b/3, c/3)$, respectively.

$G_4G_1 = \vec{c}/3,\ \overrightarrow{G_4G_2} = \vec{b}/3,\ \overrightarrow{G_4G_3} = \vec{a}/3$.

Volume of tetrahedron by centroids $V' = \frac{1}{6}\begin{bmatrix}\frac{\vec{a}}{3} & \frac{\vec{b}}{3} & \frac{\vec{c}}{3}\end{bmatrix} = \frac{1}{27}V$

$\Rightarrow K = 27$

91. **c.** $[(\vec{a}\times\vec{b})\times(\vec{b}\times\vec{c})\quad (\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})\quad (\vec{c}\times\vec{a})\times(\vec{a}\times\vec{b})]$

$= [[\vec{a}\,\vec{b}\,\vec{c}]\,\vec{b}\quad [\vec{a}\,\vec{b}\,\vec{c}]\,\vec{c}\quad [\vec{a}\,\vec{b}\,\vec{c}]\cdot\vec{a}] = [\vec{a}\,\vec{b}\,\vec{c}]^3\,[\vec{b}\,\vec{c}\,\vec{a}] = [\vec{a}\,\vec{b}\,\vec{c}]^4$

92. **d.** $\vec{r} = x_1(\vec{a}\times\vec{b}) + x_2(\vec{b}\times\vec{c}) + x_3(\vec{c}\times\vec{a})$

$\Rightarrow \vec{r}\cdot\vec{a} = x_2[\vec{a}\,\vec{b}\,\vec{c}],\ \vec{r}\cdot\vec{b} = x_3[\vec{b}\,\vec{c}\,\vec{a}]$

and $\vec{r}\cdot\vec{c} = x_1[\vec{c}\,\vec{a}\,\vec{b}] = x_1[\vec{a}\,\vec{b}\,\vec{c}]$

$\Rightarrow x_1 + x_2 + x_3 = 4\vec{r}.(\vec{a} + \vec{b} + \vec{c})$

93. **a.** Let $\vec{v} = x\vec{a} + y\vec{b} + z\vec{a}\times\vec{b}$

Given: $\vec{a}\cdot\vec{b} = 0,\ \vec{v}\cdot\vec{a} = 0,\ \vec{v}\cdot\vec{b} = 1,\ [\vec{v}\ \vec{a}\ \vec{b}] = 1$

$\Rightarrow \vec{v}\cdot\vec{a} = x\,\vec{a}\cdot\vec{a} = x|\vec{a}|^2 \quad (\because \vec{a}\cdot\vec{b} = 0,\ \vec{a}\cdot\vec{a}\times\vec{b} = 0)$

$\Rightarrow x = 0$

Again, $\vec{v}\cdot\vec{b} = y|\vec{b}|^2 \Rightarrow 1 = yb^2$

$\therefore y = \frac{1}{b^2}$ (ii)

Again, $\vec{v}\cdot(\vec{a}\times\vec{b}) = z(\vec{a}\times\vec{b})^2$

$\Rightarrow 1 = z(\vec{a}\times\vec{b})^2 \Rightarrow z = \frac{1}{|\vec{a}\times\vec{b}|^2}$

Hence, $\vec{v} = \frac{1}{|\vec{b}|^2}\vec{b} + \frac{1}{|\vec{a}\times\vec{b}|^2}\vec{a}\times\vec{b}$

94. **d.** Volume of the parallelepiped formed by $\vec{a}',\ \vec{b}'$ and $\vec{c}'$ is 4.

Therefore, the volume of the parallelepiped formed by $\vec{a},\ \vec{b}$ and $\vec{c}$ is $\frac{1}{4}$.

$\vec{b} \times \vec{c} = [\vec{a}\ \vec{b}\ \vec{c}]\ \vec{a}' = \frac{1}{4}\vec{a}'$

$|\vec{b} \times \vec{c}| = \frac{\sqrt{2}}{4} = \frac{1}{2\sqrt{2}}$

Length of altitude $= \frac{1}{4} \times 2\sqrt{2} = \frac{1}{\sqrt{2}}$

95. **d.** $\vec{a}' = \frac{\vec{b} \times \vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]} = \frac{\hat{i} + \hat{j} - \hat{k}}{2}$

Multiple Correct Answers Type

1. **a., b.** We have, $|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2(\vec{a}\cdot\vec{b})$

$\Rightarrow |\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos 2\theta$

$\Rightarrow |\vec{a} - \vec{b}|^2 = 2 - 2\cos 2\theta \quad (\because\ |\vec{a}| = |\vec{b}| = 1)$

$\Rightarrow |\vec{a} - \vec{b}|^2 = 4\sin^2\theta$

$\Rightarrow |\vec{a} - \vec{b}| = 2|\sin\theta|$

Now, $|\vec{a} - \vec{b}| < 1$

$\Rightarrow 2\,|\sin\theta| < 1$

$\Rightarrow |\sin\theta| < \frac{1}{2}$

$\Rightarrow \theta \in [0, \pi/6)$ or $\theta \in (5\pi/6, \pi]$

2. **a., c.** $\vec{a} \times (\vec{b} \times \vec{c}) + (\vec{a}\cdot\vec{b})\vec{b} = (4 - 2x - \sin y)\vec{b} + (x^2 - 1)\vec{c}$

$\Rightarrow (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} + (\vec{a}\cdot\vec{b})\vec{b} = (4 - 2x - \sin y)\vec{b} + (x^2 - 1)\vec{c}$

Now, $(\vec{c}\cdot\vec{c})\vec{a} = \vec{c}$. Therefore,

$(\vec{c}\cdot\vec{c})(\vec{a}\cdot\vec{c}) = (\vec{c}\cdot\vec{c}) \Rightarrow \vec{a}\cdot\vec{c} = 1$

$\Rightarrow 1 + \vec{a}\cdot\vec{b} = 4 - 2x - \sin y,\ \ x^2 - 1 = -(\vec{a}\cdot\vec{b})$

$\Rightarrow 1 = 4 - 2x - \sin y + x^2 - 1$

$\Rightarrow \sin y = x^2 - 2x + 2 = (x - 1)^2 + 1$

But $\sin y \le 1 \Rightarrow x = 1, \sin y = 1$

$\Rightarrow y = (4n + 1)\frac{\pi}{2}, \quad n \in I$

3. **a., b., c., d.** Since $\vec{a}, \vec{b}$ and $\vec{c}$ are unit vectors inclined at an angle θ.

$|\vec{a}| = |\vec{b}| = 1$ and $\cos\theta = \vec{a}\cdot\vec{c} = \vec{b}\cdot\vec{c}$

Now, $\vec{c} = \alpha\vec{a} + \beta\vec{b} + \gamma(\vec{a}\times\vec{b})$ (i)

$\Rightarrow \vec{a}\cdot\vec{c} = \alpha(\vec{a}\cdot\vec{a}) + \beta(\vec{a}\cdot\vec{b}) + \gamma\{\vec{a}\cdot(\vec{a}\times\vec{b})\}$

$\Rightarrow \cos\theta = \alpha|\vec{a}|^2 \quad (\because \vec{a}\cdot\vec{b} = 0, \vec{a}\cdot(\vec{a}\times\vec{b}) = 0)$

$\Rightarrow \cos\theta = \alpha$

Similarly, by taking dot product on both sides of (i) by $\vec{b}$, we get $\beta = \cos\theta$

$\therefore \alpha = \beta$

Again, $\vec{c} = \alpha\vec{a} + \beta\vec{b} + \gamma(\vec{a}\times\vec{b})$

$\Rightarrow |\vec{c}|^2 = |\alpha\vec{a} + \beta\vec{b} + \gamma(\vec{a}\times\vec{b})|^2$

$= \alpha^2|\vec{a}|^2 + \beta^2|\vec{b}|^2 + \gamma^2|\vec{a}\times\vec{b}|^2 + 2\alpha\beta(\vec{a}\cdot\vec{b}) + 2\alpha\gamma\{\vec{a}\cdot(\vec{a}\times\vec{b})\} + 2\beta\gamma(\vec{b}\cdot\{\vec{a}\times\vec{b}\})$

$\Rightarrow 1 = \alpha^2 + \beta^2 + \gamma^2|\vec{a}\times\vec{b}|^2$

$\Rightarrow 1 = 2\alpha^2 + \gamma^2\{|\vec{a}|^2|\vec{b}|^2\sin^2\pi/2\}$

$\Rightarrow 1 = 2\alpha^2 + \gamma^2 \Rightarrow \alpha^2 = \dfrac{1-\gamma^2}{2}$

But $\alpha = \beta = \cos\theta$.

$1 = 2\alpha^2 + \gamma^2 \Rightarrow \gamma^2 = 1 - 2\cos^2\theta = -\cos 2\theta$

$\therefore \beta^2 = \dfrac{1-\gamma^2}{2} = \dfrac{1+\cos 2\theta}{2}$

4. **a., b., c.** We have,

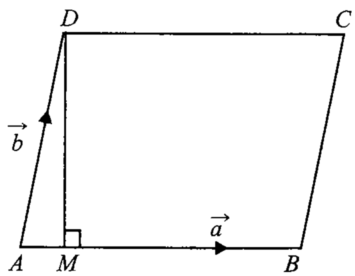


Fig. 2.42

AM = projection of $\vec{b}$ on $\vec{a} = \dfrac{\vec{a}.\vec{b}}{|\vec{a}|}$

$\therefore \overrightarrow{AM} = \left(\dfrac{\vec{a}.\vec{b}}{|\vec{a}|^2}\right)\vec{a}$

Now, in ΔADM

$\overrightarrow{AD} = \overrightarrow{AM} + \overrightarrow{MD} \Rightarrow \overrightarrow{DM} = \overrightarrow{AM} - \overrightarrow{AD}$

$\Rightarrow \overrightarrow{DM} = \dfrac{(\vec{a}\cdot\vec{b})\vec{a}}{|\vec{a}|^2} - \vec{b}$

Also, $\overrightarrow{DM} = \dfrac{1}{|\vec{a}|^2}[(\vec{a}\cdot\vec{b})\vec{a} - |\vec{a}|^2\vec{b}]$

$\Rightarrow \overrightarrow{MD} = \frac{1}{|\vec{a}|^2}[|\vec{a}|^2 \vec{b} - (\vec{a}\cdot\vec{b})\vec{a}]$

Now, $\frac{\vec{a}\times(\vec{a}\times\vec{b})}{|\vec{a}|^2} = \frac{1}{|\vec{a}|^2}[(\vec{a}\cdot\vec{b})\vec{a} - (\vec{a}\cdot\vec{a})\vec{b}] = \overrightarrow{DM}$

5. **a., c.** $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$ and $(\vec{a}\times\vec{b})\times\vec{c} = -(\vec{c}\cdot\vec{b})\vec{a} + (\vec{a}\cdot\vec{c})\vec{b}$

We have been given $(\vec{a}\times(\vec{b}\times\vec{c}))\cdot((\vec{a}\times\vec{b})\times\vec{c}) = 0$. Therefore,

$((\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c})\cdot((\vec{a}\cdot\vec{c})\vec{b} - (\vec{c}\cdot\vec{b})\vec{a}) = 0$

$\Rightarrow (\vec{a}\cdot\vec{c})^2|\vec{b}|^2 - (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{c})(\vec{a}\cdot\vec{b}) - (\vec{a}\cdot\vec{b})(\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{c}) + (\vec{a}\cdot\vec{b})(\vec{b}\cdot\vec{c})(\vec{c}\cdot\vec{a}) = 0$

$\Rightarrow (\vec{a}\cdot\vec{c})^2|\vec{b}|^2 = (\vec{a}\cdot\vec{c})(\vec{a}\cdot\vec{b})(\vec{b}\cdot\vec{c})$

$\Rightarrow (\vec{a}\cdot\vec{c})((\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{b}) - (\vec{a}\cdot\vec{b})(\vec{b}\cdot\vec{c})) = 0$

$\vec{a}\cdot\vec{c} = 0$ or $(\vec{a}\cdot\vec{c})|\vec{b}|^2 = (\vec{a}\cdot\vec{b})(\vec{b}\cdot\vec{c})$

6. **a., c.** We have $[\vec{p}\ \vec{q}\ \vec{r}] = \frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]}$. Therefore,

$[\vec{p}\ \vec{q}\ \vec{r}] > 0$

a. $x > 0, x[\vec{a}\ \vec{b}\ \vec{c}] + \frac{[\vec{p}\ \vec{q}\ \vec{r}]}{x} \geq 2$ (using A.M. ≥ G.M.)

b. Similarly, use A.M. ≥ G.M.

7. **a., b., c., d.** $a_1 + a_2 \cos 2x + a_3 \sin^2 x = 0\ \forall\ x \in R$

$\Rightarrow (a_1 + a_2) + \sin^2 x\,(a_3 - 2a_2) = 0$

$\Rightarrow a_1 + a_2 = 0$ and $a_3 - 2a_2 = 0$

$\frac{a_1}{-1} = \frac{a_2}{1} = \frac{a_3}{2} = \lambda\ (\neq 0)$

$\Rightarrow a_1 = -\lambda, a_2 = \lambda, a_3 = 2\lambda$

8. **a., b., c., d.** $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$

$\Rightarrow |\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$

$\Rightarrow \sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$ (i)

$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$\Rightarrow \cos\theta = \frac{|\vec{a}\cdot\vec{b}|}{|\vec{a}||\vec{b}|}$ (ii)

From (i) and (ii),

$\sin^2\theta + \cos^2\theta = 1$

$\Rightarrow |\vec{a}\times\vec{b}|^2 + (\vec{a}\cdot\vec{b})^2 = |\vec{a}|^2|\vec{b}|^2$

If $\theta = \pi/4$, then $\sin\theta = \cos\theta = 1/\sqrt{2}$. Therefore,

$|\vec{a}\times\vec{b}| = \dfrac{|\vec{a}||\vec{b}|}{\sqrt{2}}$ and $\vec{a}\cdot\vec{b} = \dfrac{|\vec{a}||\vec{b}|}{\sqrt{2}}$

$|\vec{a}\times\vec{b}| = \vec{a}\cdot\vec{b}$

$\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\hat{n} = \dfrac{|\vec{a}||\vec{b}|}{\sqrt{2}}\hat{n}$

$\vec{a}\times\vec{b} = (\vec{a}\cdot\vec{b})\hat{n}$

9. **a., b., c., d.** Since $\vec{a}, \vec{b}$ and $\vec{a}\times\vec{b}$ are non-coplanar,

$\vec{r} = x\vec{a} + y\vec{b} + z(\vec{a}\times\vec{b})$

$\therefore\ \vec{r}\times\vec{b} = \vec{a} \Rightarrow x\vec{a}\times\vec{b} + z\{(\vec{a}\cdot\vec{b})\vec{b} - (\vec{b}\cdot\vec{b})\vec{a}\} = \vec{a}$

$\Rightarrow -(1 + z|\vec{b}|^2)\vec{a} + x\vec{a}\times\vec{b} = 0$ (since $\vec{a}\cdot\vec{b} = 0$)

$\therefore x = 0$ and $z = -\dfrac{1}{|\vec{b}|^2}$

Thus, $\vec{r} = y\vec{b} - \dfrac{\vec{a}\times\vec{b}}{|\vec{b}|^2}$, where y is the parameter.

10. **b., d.** Since $\vec{a} = (1, 3, \sin 2\alpha)$ makes an obtuse angle with the z-axis, its z-component is negative.

$\Rightarrow -1 \le \sin 2\alpha < 0$ (i)

But $\vec{b}\cdot\vec{c} = 0$ ($\because$ orthogonal)

$\tan^2\alpha - \tan\alpha - 6 = 0$

$\therefore (\tan\alpha - 3)(\tan\alpha + 2) = 0$

$\Rightarrow \tan\alpha = 3, -2$

Now, $\tan\alpha = 3$. Therefore,

$\sin 2\alpha = \dfrac{2\tan\alpha}{1+\tan^2\alpha} = \dfrac{6}{1+9} = \dfrac{3}{5}$ (not possible as $\sin 2\alpha < 0$)

Now, if $\tan\alpha = -2$,

$\Rightarrow \sin 2\alpha = \dfrac{2\tan\alpha}{1+\tan^2\alpha} = \dfrac{-4}{1+4} = \dfrac{-4}{5}$

$\Rightarrow \tan 2\alpha > 0$

$\Rightarrow 2\alpha$ is the third quadrant. Also, $\sqrt{\sin\alpha/2}$ is meaningful. If $0 < \sin\alpha/2 < 1$, then $\alpha = (4n+1)\pi - \tan^{-1}2$ and $\alpha = (4n+2)\pi - \tan^{-1}2$.

11. **b., d.** $\vec{a} \times (\vec{r} \times \vec{a}) = \vec{a} \times \vec{b}$

$3\vec{r} - (\vec{a} \cdot \vec{r})\,\vec{a} = \vec{a} \times \vec{b}$

Also $|\vec{r} \times \vec{a}| = |\vec{b}|$

$\Rightarrow \sin^2\theta = \dfrac{2}{3}$

$\Rightarrow (1 - \cos^2\theta) = \dfrac{2}{3}$

$\Rightarrow \dfrac{1}{3} = \cos^2\theta$

$\Rightarrow \vec{a}.\vec{r} = \pm 1$

$\Rightarrow 3\vec{r} \pm \vec{a} = \vec{a} \times \vec{b}$

$\Rightarrow \vec{r} = \dfrac{1}{3}(\vec{a} \times \vec{b} \pm \vec{a})$

12. **b., d.** $(\vec{a} - \vec{b}) \times [(\vec{b} + \vec{a}) \times (2\vec{a} + \vec{b})] = \vec{b} + \vec{a}$

$\Rightarrow \{(\vec{a} - \vec{b}) \cdot (2\vec{a} + \vec{b})\}(\vec{b} + \vec{a}) - \{(\vec{a} - \vec{b}) \cdot (\vec{b} + \vec{a})\}(2\vec{a} + \vec{b}) = \vec{b} + \vec{a}$

$\Rightarrow (2 - \vec{a}.\vec{b} - 1)(\vec{b} + \vec{a}) = \vec{b} + \vec{a}$

$\Rightarrow$ either $\vec{b} + \vec{a} = \vec{0}$ or $1 - \vec{a} \cdot \vec{b} = 1$

$\Rightarrow$ either $\vec{b} = -\vec{a}$ or $\vec{a} \cdot \vec{b} = 0$

$\Rightarrow$ either $\theta = \pi$ or $\theta = \pi/2$

13. **a., d.** Given $\vec{c} = \lambda_1 \vec{a} + \lambda_2 \vec{b} + \lambda_3 (\vec{a} \times \vec{b})$ (i)

and $\vec{a} \cdot \vec{b} = 0$, $|\vec{a}| = 1$, $|\vec{b}| = 1$

From (i), $\vec{a} \cdot \vec{c} = \lambda_1$, $\vec{c} \cdot \vec{b} = \lambda_2$

and $\vec{c} \cdot (\vec{a} \times \vec{b}) = |\vec{a} \times \vec{b}|^2 \lambda_3$

$= (1.1 \sin 90°)^2 \lambda_3 = \lambda_3$

Hence $\lambda_1 + \lambda_2 + \lambda_3 = (\vec{a} + \vec{b} + \vec{a} \times \vec{b}) \cdot \vec{c}$

14. **b., c., d.** Obviously, $\dfrac{\vec{a}}{|\vec{a}|} + \dfrac{\vec{b}}{|\vec{b}|}$ is a vector in the plane of $\vec{a}$ and $\vec{b}$ and hence perpendicular to $\vec{a} \times \vec{b}$. It is also equally inclined to $\vec{a}$ and $\vec{b}$ as it is along the angle bisector.

15. **a., d.** $|\vec{a}+\vec{b}|=|\vec{a}-2\vec{b}|$

$\Rightarrow \vec{a}\cdot\vec{b}=\dfrac{|\vec{b}|^2}{2}$

Also $\vec{a}\cdot\vec{b}+\dfrac{1}{|\vec{b}|^2+2}$

$=\dfrac{|\vec{b}|^2+2}{2}+\dfrac{1}{|\vec{b}|^2+2}-1$

$\geq\sqrt{2}-1$ (using A.M. $\geq$ G.M.)

16. **b., d.** $\vec{V_1}=\vec{V_2}$

$\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\times\vec{b})\times\vec{c}$

$\Rightarrow (\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{b}\cdot\vec{c})\vec{a}$

$\Rightarrow (\vec{a}\cdot\vec{b})\vec{c}=(\vec{b}\cdot\vec{c})\vec{a}$

$\Rightarrow$ either $\vec{c}$ and $\vec{a}$ are collinear or $\vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{c}\Rightarrow\vec{b}=\lambda(\vec{a}\times\vec{c})$

17. **b., c.** We have $\vec{A}+\vec{B}=\vec{a}$

$\Rightarrow \vec{A}\cdot\vec{a}+\vec{B}\cdot\vec{a}=\vec{a}\cdot\vec{a}$

$\Rightarrow 1+\vec{B}\cdot\vec{a}=a^2$ (given $\vec{A}\cdot\vec{a}=1$)

$\Rightarrow \vec{B}\cdot\vec{a}=a^2-1$ (i)

Also $\vec{A}\times\vec{B}=\vec{b}$

$\Rightarrow \vec{a}\times(\vec{A}\times\vec{B})=\vec{a}\times\vec{b}$

$\Rightarrow (\vec{a}\cdot\vec{B})\vec{A}-(\vec{a}\cdot\vec{A})\vec{B}=\vec{a}\times\vec{b}$

$\Rightarrow (a^2-1)\vec{A}-\vec{B}=\vec{a}\times\vec{b}$ (using (i) and $\vec{a}\cdot\vec{A}=1$) (ii)

and $\vec{A}+\vec{B}=\vec{a}$ (iii)

From (ii) and (iii)

$\vec{A}=\dfrac{(\vec{a}\times\vec{b})+\vec{a}}{a^2}$

$$\vec{B} = \vec{a} - \left\{ \frac{(\vec{a} \times \vec{b}) + \vec{a}}{a^2} \right\}$$

$$\text{or } \vec{B} = \frac{(\vec{b} \times \vec{a}) + \vec{a}(a^2 - 1)}{a^2}$$

$$\text{Thus } \vec{A} = \frac{(\vec{a} \times \vec{b}) + \vec{a}}{a^2} \text{ and } \vec{B} = \frac{(\vec{b} \times \vec{a}) + \vec{a}(a^2 - 1)}{a^2}$$

18. **c., d.** Since $[\vec{a}\,\vec{b}\,\vec{c}\,] = 0$, $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar vectors.

Further, since $\vec{d}$ is equally inclined to $\vec{a}$, $\vec{b}$ and $\vec{c}$,

$$\vec{d} \cdot \vec{a} = \vec{d} \cdot \vec{b} = \vec{d} \cdot \vec{c} = 0$$

$$\vec{d} \cdot \vec{x} = \vec{d} \cdot \vec{y} = \vec{d} \cdot \vec{z} = 0$$

$$\vec{d} \cdot \vec{r} = 0$$

19. **b., d.** Let $\vec{\alpha} = \hat{i} - \hat{j} - \hat{k}$, $\vec{\beta} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{\gamma} = -\hat{i} + \hat{j} + \hat{k}$.

Let required vector $\vec{a} = x\hat{i} + y\hat{j} + z\hat{j}$.

$\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ are coplanar

$$\Rightarrow \begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{vmatrix} = 0 \Rightarrow y = z$$

Also, $\vec{a}$ and $\vec{\alpha}$ are perpendicular

$\Rightarrow \quad x - y - z = 0$

$\Rightarrow \quad x = zy$

$\Rightarrow$ Options *b* and *d* are correct.

20. **b., d.**

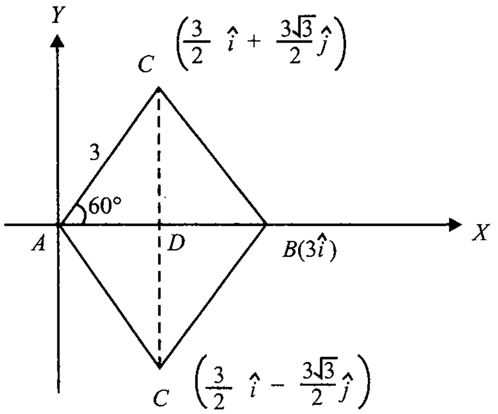


Fig. 2.43

21. **a., b., c.** Consider $\vec{V}_1 \cdot \vec{V}_2 = 0 \Rightarrow A = 90°$

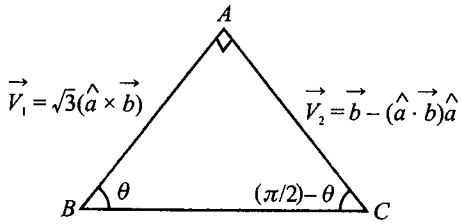


Fig. 2.44

Using the sine law, $\left|\dfrac{\vec{b} - (\hat{a} \cdot \vec{b})\hat{a}}{\sin\theta}\right| = \dfrac{\sqrt{3}\,|\hat{a} \times \vec{b}|}{\cos\theta}$

$$\Rightarrow \tan\theta = \frac{1}{\sqrt{3}} \frac{|\vec{b} - (\hat{a} \cdot \vec{b})\hat{a}|}{|\hat{a} \times \vec{b}|}$$

$$= \frac{1}{\sqrt{3}} \frac{|(\hat{a} \times \vec{b}) \times \hat{a}|}{|\hat{a} \times \vec{b}|}$$

$$= \frac{1}{\sqrt{3}} \frac{|\hat{a} \times \vec{b}||\hat{a}|\sin 90°}{|\hat{a} \times \vec{b}|} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \theta = \frac{\pi}{6}$$

22. **a., b.** Given, $\frac{1}{6}\hat{i} - \frac{1}{3}\hat{j} + \frac{1}{3}\hat{k} = (\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$

$$= [\vec{a}\,\vec{b}\,\vec{d}]\vec{c} - [\vec{a}\,\vec{b}\,\vec{c}]\vec{d}$$

$$= [\vec{a}\,\vec{b}\,\vec{d}]\vec{c} \qquad \text{(i)}$$

[$\because$ $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar]

$$[\vec{a}\,\vec{b}\,\vec{d}] = (\vec{a} \times \vec{b}) \cdot \vec{d}$$

$$= |\vec{a} \times \vec{b}||\vec{d}|\cos\theta \quad (\because \vec{d} \perp \vec{a}, \vec{d} \perp \vec{b}, \therefore \vec{d} \parallel \vec{a} \times \vec{b})$$

$$= ab \sin 30° \cdot 1 \cdot (\pm 1) \quad (\because \theta = 0 \text{ or } \pi)$$

$$= 1 \cdot 1 \cdot \frac{1}{2} . 1(\pm 1) = \pm\frac{1}{2}$$

From (i),

$$\vec{c} = \pm\left(\frac{1}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}\right) = \pm\frac{\hat{i} - 2\hat{j} + 2\hat{k}}{3}$$

23. **a., b., c.** We know that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, then $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$

Given $\vec{a}+2\vec{b}+3\vec{c}=\vec{0} \quad \Rightarrow 2\vec{a}\times\vec{b}=6\vec{b}\times\vec{c}=3\vec{c}\times\vec{a}$

Hence $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}=2(\vec{a}\times\vec{b})$ or $6(\vec{b}\times\vec{c})$ or $3(\vec{c}\times\vec{a})$

24. **a., b.** $\vec{u}=\vec{a}-(\vec{a}\cdot\vec{b})\vec{b}$

$$=\vec{a}(\vec{b}\cdot\vec{b})-(\vec{a}\cdot\vec{b})\vec{b}$$

$$=\vec{b}\times(\vec{a}\times\vec{b})$$

$$\Rightarrow |\vec{u}|=|\vec{b}\times(\vec{a}\times\vec{b})|$$

$$=|\vec{b}||\vec{a}\times\vec{b}|\sin 90^\circ$$

$$=|\vec{b}||\vec{a}\times\vec{b}|$$

$$=|\vec{v}|$$

Also $\vec{u}\cdot\vec{b}=\vec{b}\cdot\vec{b}\times(\vec{a}\times\vec{b})$

$$=[\vec{b}\ \vec{b}\ \vec{a}\times\vec{b}]$$

$$=0$$

$$\Rightarrow |\vec{v}|=|\vec{u}|+|\vec{u}\cdot\vec{b}|$$

25. **a., c.** $\vec{a}\times\vec{b}=\vec{c},\ \vec{b}\times\vec{c}=\vec{a}$

Taking cross with $\vec{b}$ in the first equation, we get $\vec{b}\times(\vec{a}\times\vec{b})=\vec{b}\times\vec{c}=\vec{a}$

$\Rightarrow |\vec{b}|^2\vec{a}-(\vec{a}\cdot\vec{b})\vec{b}=\vec{a} \Rightarrow |\vec{b}|=1$ and $\vec{a}\cdot\vec{b}=0$

Also $|\vec{a}\times\vec{b}|=|\vec{c}| \Rightarrow |\vec{a}||\vec{b}|\sin\frac{\pi}{2}=|\vec{c}| \Rightarrow |\vec{a}|=|\vec{c}|$

26. **b., d.** $\vec{d}\cdot\vec{a}=[\vec{a}\ \vec{b}\ \vec{c}]\cos y=-\vec{d}\cdot(\vec{b}+\vec{c})$

$$\Rightarrow \cos y=-\frac{\vec{d}\cdot(\vec{b}+\vec{c})}{[\vec{a}\cdot\vec{b}\cdot\vec{c}]}$$

Similarly, $\sin x = -\dfrac{\vec{d}\cdot(\vec{a}+\vec{b})}{[\vec{a}\ \vec{b}\ \vec{c}]}$ and $\dfrac{\vec{d}\cdot(\vec{a}+\vec{c})}{[\vec{a}\ \vec{b}\ \vec{c}]} = -2$

$\therefore\ \sin x + \cos y + 2 = 0$

$\Rightarrow\ \sin x + \cos y = -2$

$\Rightarrow \sin x = -1, \cos y = -1$

Since we want the minimum value of $x^2 + y^2$, $x = -\pi/2,\ y = \pi$

$\therefore$ The minimum value of $x^2 + y^2$ is $5\pi^2/4$

27. **b., c.** $\vec{a} \times (\vec{b} \times \vec{c}) = \frac{1}{2}\vec{b}$

$\Rightarrow (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} = \frac{1}{2}\vec{b}$

$\Rightarrow \vec{a}\cdot\vec{c} = \frac{1}{2}$ and $\vec{a}\cdot\vec{b} = 0$

$\Rightarrow 1\cdot 1\cos\alpha = \frac{1}{2}$ and $\vec{a} \perp \vec{b}$

$\Rightarrow \alpha = \frac{\pi}{3}$ and $\vec{a} \perp \vec{b}$

28. **a., b., c.** $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

$\overrightarrow{BC} = \dfrac{2\vec{u}}{|\vec{u}|} - \dfrac{\vec{u}}{|\vec{u}|} + \dfrac{\vec{v}}{|\vec{v}|} = \dfrac{\vec{u}}{|\vec{u}|} + \dfrac{\vec{v}}{|\vec{v}|}$

$\overrightarrow{AB}\cdot\overrightarrow{BC} = \left(\dfrac{\vec{u}}{|\vec{u}|} - \dfrac{\vec{v}}{|\vec{v}|}\right)\left(\dfrac{\vec{u}}{|\vec{u}|} + \dfrac{\vec{v}}{|\vec{v}|}\right) = (\hat{u} - \hat{v})\cdot(\hat{u} + \hat{v}) = 1 - 1 = 0$

$\Rightarrow \angle B = 90°$

$\Rightarrow 1 + \cos 2A + \cos 2B + \cos 2C = 0$

29. **a., b., c.** Let $\vec{A} = \vec{a}\times\vec{b},\ \vec{B} = \vec{c}\times\vec{d}$ and $\vec{C} = \vec{e}\times\vec{f}$

We know that $\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$

$= (\vec{a}\times\vec{b})\cdot[(\vec{c}\times\vec{d})\times(\vec{e}\times\vec{f})]$

$= (\vec{a}\times\vec{b})\cdot[\{(\vec{c}\times\vec{d})\cdot\vec{f}\}\vec{e} - \{(\vec{c}\times\vec{d})\cdot\vec{e}\}\vec{f}\}$

$= [\vec{c}\ \vec{d}\ \vec{f}][\vec{a}\ \vec{b}\ \vec{e}] - [\vec{c}\ \vec{d}\ \vec{e}][\vec{a}\ \vec{b}\ \vec{f}]$

Similarly, other parts can be obtained.

30. **a., c.** Here $(l\vec{a} + m\vec{b})\times\vec{b} = \vec{c}\times\vec{b} \Rightarrow l\vec{a}\times\vec{b} = \vec{c}\times\vec{b}$

$$\Rightarrow l(\vec{a}\times\vec{b})^2 = (\vec{c}\times\vec{b})\cdot(\vec{a}\times\vec{b}) \Rightarrow l = \frac{(\vec{c}\times\vec{b})\cdot(\vec{a}\times\vec{b})}{(\vec{a}\times\vec{b})^2}$$

Similarly, $m = \dfrac{(\vec{c}\times\vec{a})\cdot(\vec{b}\times\vec{a})}{(\vec{b}\times\vec{a})^2}$

31. **b., c., d.** $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})\cdot(\vec{a}\times\vec{d}) = 0$

$\Rightarrow ([\vec{a}\,\vec{c}\,\vec{d}]\vec{b} - [\vec{b}\,\vec{c}\,\vec{d}]\vec{a})\cdot(\vec{a}\times\vec{d}) = 0$

$\Rightarrow [\vec{a}\,\vec{c}\,\vec{d}][\vec{b}\,\vec{a}\,\vec{d}] = 0$

$\Rightarrow$ Either $\vec{c}$ or $\vec{b}$ must lie in the plane of $\vec{a}$ and $\vec{d}$.

32. **a., b.** Let $\overrightarrow{EB} = p$, $\overrightarrow{AB}$ and $\overrightarrow{CE} = q\overrightarrow{CD}$.

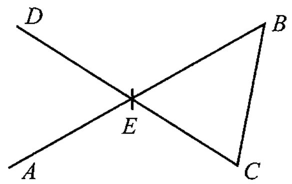


Fig. 2.45

Then $0 < p$ and $q \le 1$

Since $\overrightarrow{EB} + \overrightarrow{BC} + \overrightarrow{CE} = \vec{0}$

$pm(2\hat{i} - 6\hat{j} + 2\hat{k}) + (\hat{i} - 2\hat{j}) + qn(-6\hat{i} + 15\hat{j} - 3\hat{k}) = \vec{0}$

$\Rightarrow (2pm + 1 - 6qn)\hat{i} + (-6pm - 2 + 15qn)\hat{j} + (2pm - 6qn)\hat{k} = \vec{0}$

$\Rightarrow 2pm - 6qn + 1 = \vec{0}, -6pm - 2 + 15qn = \vec{0}, 2pm - 6qn = \vec{0}$

Solving these, we get

$p = 1/(2m)$ and $q = 1/(3n)$

$\therefore 0 < 1/(2m) \le 1$ and $0 < 1/(3n) \le 1$

$\Rightarrow m \ge 1/2$ and $n \ge 1/3$

33. **a., b., d.** $\left.\begin{aligned} \vec{V}_1 &= l\vec{a} + m\vec{b} + n\vec{c} \\ \vec{V}_2 &= n\vec{a} + l\vec{b} + m\vec{c} \\ \vec{V}_3 &= m\vec{a} + n\vec{b} + l\vec{c} \end{aligned}\right\}$ when $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-coplanar.

Therefore,

$$[\vec{V}_1\,\vec{V}_2\,\vec{V}_3] = \begin{vmatrix} l & m & n \\ n & l & m \\ m & n & l \end{vmatrix} = 0$$

$\Rightarrow (l+m+n)\,[(l-m)^2+(m-n)^2+(n-l)^2]=0$

$\Rightarrow l+m+n=0$ (i)

Obviously, $lx^2+mx+n=0$ is satisfied by $x=1$ due to (i).

$l^3+m^3+n^3=3lmn$

$\Rightarrow (l+m+n)(l^2+m^2+n^2-lm-mn-ln)=0$, which is true

34. **a., b., c.** It is given that $\vec{\alpha}, \vec{b}$ and $\vec{\gamma}$ are coplanar vectors. Therefore,

$[\vec{\alpha}\,\vec{\beta}\,\vec{\gamma}]=0$

$$\Rightarrow \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

$\Rightarrow 3abc-a^3-b^3-c^3=0$

$\Rightarrow a^3+b^3+c^3-3abc=0$

$\Rightarrow (a+b+c)(a^2+b^2+c^2-ab-bc-ca)=0$

$\Rightarrow a+b+c=0 \quad [\because a^2+b^2+c^2-ab-bc-ca \neq 0]$

$\Rightarrow \vec{v}\cdot\vec{\alpha} = \vec{v}\cdot\vec{\beta} = \vec{v}\cdot\vec{\gamma} = 0$

$\Rightarrow \vec{v}$ is perpendicular to $\vec{\alpha}$, $\vec{\beta}$ and $\vec{\gamma}$

35. **b., d.** For $\vec{A}$, $\vec{B}$ and $\vec{C}$ to form a left-handed system

$[\vec{A}\vec{B}\vec{C}]<0$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 1 & 1 & 5 \end{vmatrix} = 11\hat{i}-6\hat{j}-\hat{k}$$ (i)

(i) is satisfied by options **(b)** and **(d).**

Reasoning Type

1. **b.** A vector along the bisector is $\dfrac{\vec{a}}{|\vec{a}|}+\dfrac{\vec{b}}{|\vec{b}|}=\dfrac{-5\hat{i}+7\hat{j}+2\hat{k}}{9}$

Hence $\vec{c}=-5\hat{i}+7\hat{j}+2\hat{k}$ is along the bisector. It is obvious that $\vec{c}$ makes an equal angle with $\vec{a}$ and $\vec{b}$. However, Statement 2 does not explain Statement 1, as a vector equally inclined to given two vectors is not necessarily coplanar.

2. **c.** Component of vector $\vec{b}=4\hat{i}+2\hat{j}+3\hat{k}$ in the direction of $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ is $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\dfrac{\vec{a}}{|\vec{a}|}$ or $3\hat{i}+3\hat{j}+3\hat{k}$.

Then component in the direction perpendicular to the direction of $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ is

$\vec{b}-3\hat{i}+3\hat{j}+3\hat{k}=\hat{i}-\hat{j}$

3. **d.** $\overrightarrow{AD}=2\hat{j}-\hat{k}$, $\overrightarrow{BD}=-2\hat{i}-\hat{j}-3\hat{k}$ and $\overrightarrow{CD}=2\hat{i}-\hat{j}$

Volume of tetrahedron is $\dfrac{1}{6}[\overrightarrow{AD}\ \overrightarrow{BD}\ \overrightarrow{CD}]=\dfrac{1}{6}\left\|\begin{matrix}0 & 2 & -1\\ -2 & -1 & -3\\ 2 & -1 & 0\end{matrix}\right\|=\dfrac{8}{3}.$

Also, the area of the triangle ABC is $\dfrac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|=\dfrac{1}{2}\left\|\begin{matrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & 2\\ -2 & 3 & -1\end{matrix}\right\|$

$$=\frac{1}{2}|-9\hat{i}-2\hat{j}+12\hat{k}|$$

$$=\frac{\sqrt{229}}{2}$$

Then $\dfrac{8}{3}=\dfrac{1}{3}\times$ (distance of D from base ABC) $\times$ (area of triangle ABC)

Distance of D from base $ABC = 16/\sqrt{229}$

4. **b.** $\vec{r}\cdot\vec{a}=\vec{r}\cdot\vec{b}=\vec{r}\cdot\vec{c}=0$ only if $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar.

$\Rightarrow [\vec{a}\,\vec{b}\,\vec{c}]=0$

Hence, Statement 2 is true.

Also, $[\vec{a}-\vec{b}\ \ \vec{b}-\vec{c}\ \ \vec{c}-\vec{a}]=0$ even if $[\vec{a}\ \vec{b}\ \vec{c}]\neq 0$.

Hence, Statement 2 is not the correct explanation for Statement 1.

5. **a.** Let the three given unit vectors be $\hat{a}, \hat{b}$ and $\hat{c}$. Since they are mutually perpendicular, $\hat{a}\cdot(\hat{b}\times\hat{c})=1$. Therefore,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 1$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 1$$

Hence, $a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$, $a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$ and $a_3\hat{i} + b_3\hat{j} + c_3\hat{k}$ may be mutually perpendicular.

6. **d.** $\vec{A}\times((\vec{A}\cdot\vec{B})\vec{A}-(\vec{A}\cdot\vec{A})\vec{B})\cdot\vec{C}$

$$= \left(\underbrace{\vec{A}\times(\vec{A}\cdot\vec{B})\vec{A}}_{\text{zero}}-(\vec{A}\cdot\vec{A})\vec{A}\times\vec{B}\right)\cdot\vec{C} = -|\vec{A}|^2\,[\vec{A}\;\vec{B}\;\vec{C}]$$

Now, $|\vec{A}|^2 = 4+9+36 = 49$

$$[\vec{A}\vec{B}\vec{C}] = \begin{vmatrix} 2 & 3 & 6 \\ 1 & 1 & -2 \\ 1 & 2 & 1 \end{vmatrix} = 2(1+4) - 1(3-12) + 1(-6-6)$$

$$= 10 + 9 - 12 = 7$$

$\therefore |-|\vec{A}|^2\,[\vec{A}\vec{B}\vec{C}]| = 49\times 7 = 343$

7. **b.** Let $\vec{d} = \lambda_1\vec{a} + \lambda_2\vec{b} + \lambda_3\vec{c}$

$\Rightarrow [\vec{d}\,\vec{a}\,\vec{b}] = \lambda_3[\vec{c}\,\vec{a}\,\vec{b}] \Rightarrow \lambda_3 = 1$

$[\vec{c}\,\vec{a}\,\vec{b}] = 1$ (because $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three mutually perpendicular unit vectors)

Similarly, $\lambda_1 = \lambda_2 = 1$

$\Rightarrow \vec{d} = \vec{a} + \vec{b} + \vec{c}$

Hence Statement 1 and Statement 2 are correct, but Statement 2 does not explain Statement 1 as it does not give the value of dot products.

8. **a.** Statement 2 is true (see properties of dot product)

Also, $(\hat{i}\times\vec{a})\cdot\vec{b} = \hat{i}\cdot(\vec{a}\times\vec{b})$

$\Rightarrow \vec{a}\times\vec{b} = (\hat{i}\cdot(\vec{a}\times\vec{b}))\hat{i} + (\hat{j}\cdot(\vec{a}\times\vec{b}))\hat{j} + (\hat{k}\cdot(\vec{a}\times\vec{b}))\hat{k}$

Linked Comprehension Type

For Problems 1–3

1. b., 2. c., 3. d.

Sol.

Taking dot product of $\vec{u}+\vec{v}+\vec{w}=\vec{a}$ with $\vec{u}$, we have

$$1+\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}=\vec{a}\cdot\vec{u}=\frac{3}{2}\Rightarrow\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}=\frac{1}{2} \qquad \text{(i)}$$

Similarly, taking dot product with $\vec{v}$, we have

$$\vec{u}\cdot\vec{v}+\vec{w}\cdot\vec{v}=\frac{3}{4} \qquad \text{(ii)}$$

Also, $\vec{a}\cdot\vec{u}+\vec{a}\cdot\vec{v}+\vec{a}\cdot\vec{w}=\vec{a}\cdot\vec{a}=4$

$$\Rightarrow \vec{a}\cdot\vec{w}=4-\left(\frac{3}{2}+\frac{7}{4}\right)=\frac{3}{4}$$

Again, taking dot product with $\vec{w}$, we have

$$\vec{u}\cdot\vec{w}+\vec{v}\cdot\vec{w}=\frac{3}{4}-1=-\frac{1}{4} \qquad \text{(iii)}$$

Adding (i), (ii) and (iii), we have

$$2(\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}+\vec{v}\cdot\vec{w})=1$$

$$\Rightarrow \vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}+\vec{v}\cdot\vec{w}=\frac{1}{2} \qquad \text{(iv)}$$

Subtracting (i), (ii) and (iii) from (iv), we have

$$\vec{v}\cdot\vec{w}=0,\ \vec{u}\cdot\vec{w}=-\frac{1}{4}\ \text{and}\ \vec{u}\cdot\vec{v}=\frac{3}{4}$$

Now, the equations $\vec{u}\times(\vec{v}\times\vec{w})=\vec{b}$ and $(\vec{u}\times\vec{v})\times\vec{w}=\vec{c}$ can be written as $(\vec{u}\cdot\vec{w})\vec{v}-(\vec{u}\cdot\vec{v})\vec{w}=\vec{b}$

and $(\vec{u}\cdot\vec{w})\vec{v}-(\vec{v}\cdot\vec{w})\vec{u}=\vec{c}\Rightarrow -\frac{1}{4}\vec{v}-\frac{3}{4}\vec{w}=\vec{b},\ -\frac{1}{4}\vec{v}=\vec{c}$, i.e., $\vec{v}=-4\vec{c}$

$$\Rightarrow \vec{c}-\frac{3}{4}\vec{w}=\vec{b}\Rightarrow\vec{w}=\frac{4}{3}(\vec{c}-\vec{b})\ \text{and}\ \vec{u}=\vec{a}-\vec{v}-\vec{w}=\vec{a}+4\vec{c}-\frac{4}{3}\vec{c}+\frac{4}{3}\vec{b}=\vec{a}+\frac{4}{3}\vec{b}+\frac{8}{3}\vec{c}$$

For Problems 4–6

4. d., 5. c., 6. b.

Sol.

Given that $|\vec{x}|=|\vec{y}|=|\vec{z}|=\sqrt{2}$ and they are inclined at an angle of 60° with each other.

$$\therefore\ \vec{x}\cdot\vec{y}=\vec{y}\cdot\vec{z}=\vec{z}\cdot\vec{x}=\sqrt{2}\cdot\sqrt{2}\cos 60^\circ=1$$

$$\vec{x}\times(\vec{y}\times\vec{z})=\vec{a}\Rightarrow(\vec{x}\cdot\vec{z})\vec{y}-(\vec{x}\cdot\vec{y})\vec{z}=\vec{a}\Rightarrow\vec{y}-\vec{z}=\vec{a} \qquad \text{(i)}$$

Similarly, $\vec{y}\times(\vec{z}\times\vec{x})=\vec{b}\Rightarrow \vec{z}-\vec{x}=\vec{b}$ (ii)

$\vec{y}=\vec{a}+\vec{z},\ \vec{x}=\vec{z}-\vec{b}$ (from (i) and (ii)) (iii)

Now, $\vec{x}\times\vec{y}=\vec{c}$

$\Rightarrow (\vec{z}-\vec{b})\times(\vec{z}+\vec{a})=\vec{c}$

$\Rightarrow \vec{z}\times\vec{a}-\vec{b}\times\vec{z}-\vec{b}\times\vec{a}=\vec{c}$

$\Rightarrow \vec{z}\times(\vec{a}+\vec{b})=\vec{c}+(\vec{b}\times\vec{a})$ (iv)

$\Rightarrow (\vec{a}+\vec{b})\times\{\vec{z}\times(\vec{a}+\vec{b})\}=(\vec{a}+\vec{b})\times\vec{c}+(\vec{a}+\vec{b})\times(\vec{b}\times\vec{a})$

$\Rightarrow (\vec{a}+\vec{b})^2\vec{z}-\{(\vec{a}+\vec{b})\cdot\vec{z}\}(\vec{a}+\vec{b})=(\vec{a}+\vec{b})\times\vec{c}+|\vec{a}|^2\vec{b}-|\vec{b}|^2\vec{a}+(\vec{a}\cdot\vec{b})(\vec{b}-\vec{a})$ (v)

Now, (i) $\Rightarrow |\vec{a}|^2=|\vec{y}-\vec{z}|^2=2+2-2=2$

Similarly, (ii) $\Rightarrow |\vec{b}|^2=2$

Also (i) and (ii) $\Rightarrow \vec{a}+\vec{b}=\vec{y}-\vec{x}\Rightarrow|\vec{a}+\vec{b}|^2=2$ (vi)

Also $(\vec{a}+\vec{b})\cdot\vec{z}=(\vec{y}-\vec{x})\cdot\vec{z}=\vec{y}\cdot\vec{z}-\vec{x}\cdot\vec{z}=1-1=0$

and $\vec{a}\cdot\vec{b}=(\vec{y}-\vec{z})\cdot(\vec{z}-\vec{x})$

$=\vec{y}\cdot\vec{z}-\vec{x}\cdot\vec{y}-|\vec{z}|^2+\vec{x}\cdot\vec{z}=-1$

Thus from (v), we have $2\vec{z}=(\vec{a}+\vec{b})\times\vec{c}+2(\vec{b}-\vec{a})-(\vec{b}-\vec{a})$ or $\vec{z}=(1/2)[(\vec{a}+\vec{b})\times\vec{c}+\vec{b}-\vec{a}]$

$\therefore\ \vec{y}=\vec{a}+\vec{z}=(1/2)[(\vec{a}+\vec{b})\times\vec{c}+\vec{b}+\vec{a}]$ and $\vec{x}=\vec{z}-\vec{b}=(1/2)[(\vec{a}+\vec{b})\times\vec{c}-(\vec{a}+\vec{b})]$

For Problems 7–9

7. b., 8. a., 9. c.

Sol.

Given

$\vec{x}\times\vec{y}=\vec{a}$ (i)

$\vec{y}\times\vec{z}=\vec{b}$ (ii)

$\vec{x}\cdot\vec{b}=\gamma$ (iii)

$\vec{x}\cdot\vec{y}=1$ (iv)

$\vec{y}\cdot\vec{z}=1$ (v)

From (ii), $\vec{x}\cdot(\vec{y}\times\vec{z})=\vec{x}\cdot\vec{b}=\gamma\Rightarrow[\vec{x}\ \vec{y}\ \vec{z}]=\gamma$

From (i) and (ii), $(\vec{x}\times\vec{y})\times(\vec{y}\times\vec{z})=\vec{a}\times\vec{b}$

$\therefore\ [\vec{x}\ \vec{y}\ \vec{z}]\vec{y}-[\vec{y}\ \vec{y}\ \vec{z}]\vec{x}=\vec{a}\times\vec{b}\Rightarrow\vec{y}=\dfrac{\vec{a}\times\vec{b}}{\gamma}$ (vi)

Also from (i), we get $(\vec{x}\times\vec{y})\times\vec{y}=\vec{a}\times\vec{y}$

$\Rightarrow (\vec{x}\cdot\vec{y})\vec{y}-(\vec{y}\cdot\vec{y})\vec{x}=\vec{a}\times\vec{y}\Rightarrow\vec{x}=(1/|\vec{y}|^2)(\vec{y}-\vec{a}\times\vec{y})=\dfrac{\gamma^2}{|\vec{a}\times\vec{b}|^2}\left[\dfrac{\vec{a}\times\vec{b}}{\gamma}-\dfrac{\vec{a}\times(\vec{a}\times\vec{b})}{\gamma}\right]$

$$\Rightarrow \vec{x} = \frac{\gamma}{|\vec{a} \times \vec{b}|^2}[\vec{a} \times \vec{b} - \vec{a} \times (\vec{a} \times \vec{b})]$$

Also from (ii), $(\vec{y} \times \vec{z}) \times \vec{y} = \vec{b} \times \vec{y} \Rightarrow |\vec{y}|^2 \vec{z} - (\vec{z} \cdot \vec{y})\vec{y} = \vec{b} \times \vec{y}$

$$\Rightarrow \vec{z} = \frac{1}{|\vec{y}|^2}[\vec{y} + \vec{b} \times \vec{y}] = \frac{\gamma}{|\vec{a} \times \vec{b}|^2}[\vec{a} \times \vec{b} + \vec{b} \times (\vec{a} \times \vec{b})]$$

For Problems 10–12

10. b., 11. b., 12. d.

Sol.

$\vec{P} \times \vec{B} = \vec{A} - \vec{P}$ and $|\vec{A}| = |\vec{B}| = 1$ and $\vec{A} \cdot \vec{B} = 0$ is given

Now $\vec{P} \times \vec{B} = \vec{A} - \vec{P}$ (i)

$(\vec{P} \times \vec{B}) \times \vec{B} = (\vec{A} - \vec{P}) \times \vec{B}$ (taking cross product with $\vec{B}$ on both sides)

$\Rightarrow (\vec{P} \cdot \vec{B})\vec{B} - (\vec{B} \cdot \vec{B})\vec{P} = \vec{A} \times \vec{B} - \vec{P} \times \vec{B}$

$\Rightarrow (\vec{P} \cdot \vec{B})\vec{B} - \vec{P} = \vec{A} \times \vec{B} - \vec{A} + \vec{P}$

$\Rightarrow 2\vec{P} = \vec{A} - \vec{A} \times \vec{B} - (\vec{P} \cdot \vec{B})\vec{B}$

$\Rightarrow \vec{P} = \dfrac{\vec{A} - \vec{A} \times \vec{B} - (\vec{P} \cdot \vec{B})\vec{B}}{2}$ (ii)

Taking dot product with $\vec{B}$ on both sides of (i), we get

$\vec{P} \cdot \vec{B} = \vec{A} \cdot \vec{B} - \vec{P} \cdot \vec{B}$

$\Rightarrow \vec{P} \cdot \vec{B} = 0$ (iii)

$\Rightarrow \vec{P} = \dfrac{\vec{A} + \vec{B} \times \vec{A}}{2}$

Now

$(\vec{P} \times \vec{B}) \times \vec{B} = (\vec{P} \cdot \vec{B})\vec{B} - (\vec{B} \cdot \vec{B})\vec{P} = -\vec{P}$

$\vec{P}, \vec{A}, \vec{P} \times \vec{B} \, (= \vec{A} - \vec{P})$ are dependent

Also $\vec{P} \cdot \vec{B} = 0$

and $|\vec{P}|^2 = \left|\dfrac{\vec{A} - \vec{A} \times \vec{B}}{2}\right|^2$

$= \dfrac{|\vec{A}|^2 + |\vec{A} \times \vec{B}|^2}{4}$

$= \dfrac{1+1}{4} = \dfrac{1}{2} \Rightarrow |\vec{P}| = \dfrac{1}{\sqrt{2}}$

For Problems 13–15

13. b., 14. a., 15. c.

Sol.

13. b. $$\vec{a}_1 = \left[(2\hat{i}+3\hat{j}-6\hat{k})\cdot\frac{(2\hat{i}-3\hat{j}+6\hat{k})}{7}\right]\frac{2\hat{i}-3\hat{j}+6\hat{k}}{7} = \frac{-41}{49}(2\hat{i}-3\hat{j}+6\hat{k})$$

$$\vec{a}_2 = \frac{-41}{49}\left((2\hat{i}-3\hat{j}+6\hat{k})\cdot\frac{(-2\hat{i}+3\hat{j}+6\hat{k})}{7}\right)\frac{(-2\hat{i}+3\hat{j}+6\hat{k})}{7}$$

$$= \frac{-41}{(49)^2}(-4-9+36)(-2\hat{i}+3\hat{j}+6\hat{k}) = \frac{943}{49^2}(2\hat{i}-3\hat{j}-6\hat{k})$$

14. **a.** $\vec{a}_1\cdot\vec{b} = \frac{-41}{49}(2\hat{i}-3\hat{j}+6\hat{k})\cdot(2\hat{i}-3\hat{j}+6\hat{k}) = -41$

15. **c.** $\vec{a}$, $\vec{a}_1$ and $\vec{b}$ are coplanar because $\vec{a}_1$ and $\vec{b}$ are collinear.

For Problems 16–18

16. b., 17. c., 18. a.

Sol.

Point G is $\left(\frac{4}{3},\frac{1}{3},\frac{8}{3}\right)$. Therefore,

$$\left|\overrightarrow{AG}\right|^2 = \left(\frac{5}{3}\right)^2 + \frac{1}{9} + \left(\frac{5}{3}\right)^2 = \frac{51}{9}$$

$$\Rightarrow \left|\overrightarrow{AG}\right| = \frac{\sqrt{51}}{3}$$

$$\overrightarrow{AB} = -4\hat{i}+4\hat{j}+0\hat{k}$$

$$\overrightarrow{AC} = 2\hat{i}+2\hat{j}+2\hat{k}$$

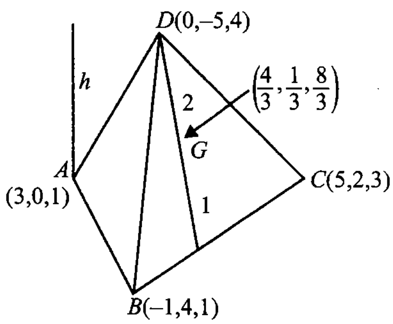


Fig. 2.46

$$\therefore\ \overrightarrow{AB}\times\overrightarrow{AC} = -8\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 0\\ 1 & 1 & 1\end{vmatrix}$$

$$= 8(\hat{i}+\hat{j}-2\hat{k})$$

Area of $\Delta ABC = \frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}| = 4\sqrt{6}$

$\overrightarrow{AD} = -3\hat{i}-5\hat{j}+3\hat{k}$

The length of the perpendicular from the vertex D on the opposite face

$= |$ projection of $\overrightarrow{AD}$ on $\overrightarrow{AB}\times\overrightarrow{AC}|$

$$= \left|\frac{(-3\hat{i}-5\hat{j}+3\hat{k})(\hat{i}+\hat{j}-2\hat{k})}{\sqrt{6}}\right|$$

$$= \left|\frac{-3-5-6}{\sqrt{6}}\right| = \frac{14}{\sqrt{6}}$$

For Problems 19–21

19. c., 20. b., 21. d.

Sol.

19. **c.** Let point D be (a_1, a_2, a_3)

$a_1 + 1 = 3 \Rightarrow a_1 = 2$

$a_2 + 0 = 1 \Rightarrow a_2 = 1$

$a_3 - 1 = 7 \Rightarrow a_3 = 8$

$\therefore D(2, 1, 8)$

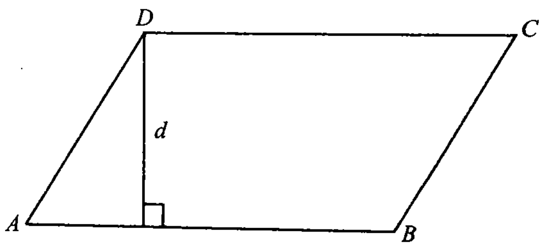


Fig. 2.47

$$\vec{d} = \left|\frac{(\overrightarrow{AB})\times(\overrightarrow{AD})}{|\overrightarrow{AB}|}\right|$$

$\overrightarrow{AB} = -\hat{i}+\hat{j}-5\hat{k}$

$\overrightarrow{AD} = 0\hat{i}+2\hat{j}+4\hat{k}$

$$\overrightarrow{AB}\times\overrightarrow{AD} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & 1 & -5\\ 0 & 2 & 4\end{vmatrix}$$

$$= 14\hat{i}+4\hat{j}-2\hat{k}$$

$$= 2(7\hat{i}+2\hat{j}-\hat{k})$$

$\Rightarrow d = 2\sqrt{2}$

20. **b.**

$\vec{n} = 7\hat{i} + 2\hat{j} - \hat{k}$ is normal to the plane $P \equiv (8, 2, -12)$.

$|\overrightarrow{AP}| = 6\hat{i} + 3\hat{j} - 16\hat{k}$

$\therefore$ distance $d = \left|\dfrac{\overrightarrow{AP}\cdot\vec{n}}{|\vec{n}|}\right|$

$= \left|\dfrac{42+6+16}{\sqrt{49+4+1}}\right|$

$= \dfrac{64}{\sqrt{54}}$

$= \dfrac{64}{3\sqrt{6}} = \dfrac{64\sqrt{6}}{18} = \dfrac{32\sqrt{6}}{9}$

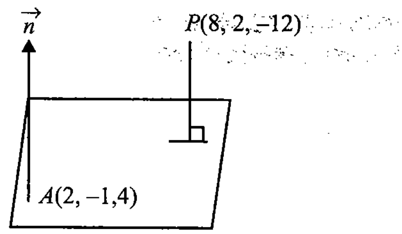


Fig. 2.48

21. **d.** Vector normal to the plane

$\overrightarrow{AD} \times \overrightarrow{AB} = +2(7\hat{i} + 2\hat{j} - \hat{k})$

Projection on $xy = 2$

Projection on $yz = 14$

Projection on $zx = 4$

For Problems 22–24

22. d., 23. c., 24. c.

Sol.

Let $\vec{r} = x\hat{i} + y\hat{j}$

$x^2 + y^2 + 8x - 10y + 40 = 0$, which is a circle

centre $C(-4, 5)$, radius $r = 1$

$p_1 = \max\{(x+2)^2 + (y-3)^2\}$

$p_2 = \min\{(x+2)^2 + (y-3)^2\}$

Let P be $(-2, 3)$. Then

$CP = 2\sqrt{2}, r = 1$

$p_2 = (2\sqrt{2} - 1)^2$

$p_1 = (2\sqrt{2} + 1)^2$

$p_1 + p_2 = 18$

Slope $= AB = \left(\dfrac{dy}{dx}\right)_{(2,2)} = -2$

Equation of AB, $2x + y = 6$

$\overrightarrow{OA} = 2\hat{i} + 2\hat{j},\ \overrightarrow{OB} = 3\hat{i}$

$\overrightarrow{AB} = \hat{i} - 2\hat{j}$

$\overrightarrow{AB}\cdot\overrightarrow{OB} = (\hat{i} - 2\hat{j})(3\hat{i}) = 3$

Matrix-Match Type

1. a → p, q, r, s; b → p, q; c → p, r; d → r

a. Given equations are consistent if

$(\hat{i} + \hat{j}) + \lambda(\hat{i} + 2\hat{j} - \hat{k}) = (\hat{i} + 2\hat{j}) + \mu(-\hat{i} + \hat{j} + a\hat{k})$

$\Rightarrow 1 + \lambda = 1 - \mu, 1 + 2\lambda = 2 + \mu, -\lambda = a\mu$

$\Rightarrow \lambda = 1/3$ and $\mu = -1/3$

$\Rightarrow a = 1$

b. $\vec{a} = \lambda\hat{i} - 3\hat{j} - \hat{k}$

$\vec{b} = 2\lambda\hat{i} + \lambda\hat{j} - \hat{k}$

Angle between $\vec{a}$ and $\vec{b}$ is acute. Therefore,

$\vec{a}\cdot\vec{b} > 0$

$\Rightarrow 2\lambda^2 - 3\lambda + 1 > 0$

$\Rightarrow (2\lambda - 1)(\lambda - 1) > 0$

$\Rightarrow \lambda \in \left(-\infty, \frac{1}{2}\right) \cup (1, \infty)$

Also $\vec{b}$ makes an obtuse angle with the axes. Therefore,

$\vec{b}\cdot\hat{i} < 0 \Rightarrow \lambda < 0$

$\vec{b}\cdot\hat{j} < 0 \Rightarrow \lambda < 0$ (ii)

Combining these two, we get $\lambda = -4, -2$

c. If vectors $2\hat{i} - \hat{j} + \hat{k},\ \hat{i} + 2\hat{j} + (1 + a)\hat{k}$ and $3\hat{i} + a\hat{j} + 5\hat{k}$ are coplanar, then

$$\begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & 1+a \\ 3 & a & 5 \end{vmatrix} = 0$$

$\Rightarrow a^2 + 2a - 8 = 0$

$\Rightarrow (a + 4)(a - 2) = 0$

$\Rightarrow a = -4, 2$

d. $\vec{A} = 2\hat{i} + \lambda\hat{j} + 3\hat{k}$

$B = 2\hat{i} + \lambda\hat{j} + \hat{k}$

$C = 3\hat{i} + \hat{j} + 0.\hat{k}$

$\therefore \vec{A} + \lambda\vec{B} = 2(1 + \lambda)\hat{i} + (\lambda + \lambda^2)\hat{j} + (3 + \lambda)\hat{k}$

Now $(\vec{A}+\lambda\vec{B})\perp\vec{C}$. Therefore,

$$(\vec{A}+\lambda\vec{B})\cdot\vec{C}=0$$
$$\Rightarrow 6(1+\lambda)+(\lambda+\lambda^2)+0=0$$
$$\Rightarrow \lambda^2+7\lambda+6=0$$
$$\Rightarrow (\lambda+6)(\lambda+1)=0$$
$$\Rightarrow \lambda=-6,=-1$$
$$\Rightarrow |2\lambda|=12, 2$$

2. a → r; b → p; c → s; d → q

a. If $\vec{a}, \vec{b}$ and $\vec{c}$ are mutually perpendicular, then

$$[\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}]=[\vec{a}\ \vec{b}\ \vec{c}]^2=(|\vec{a}||\vec{b}||\vec{c}|)^2=16$$

b. Given $\vec{a}$ and $\vec{b}$ are two unit vectors, i.e., $|\vec{a}|=|\vec{b}|=1$ and angle between them is $\pi/3$.

$$\sin\theta=\frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}\Rightarrow \sin\frac{\pi}{3}=|\vec{a}\times\vec{b}|$$

$$\frac{\sqrt{3}}{2}=|\vec{a}\times\vec{b}|$$

Now

$$[\vec{a}\ \ \vec{b}+\vec{a}\times\vec{b}\ \ \vec{b}]=[\vec{a}\ \vec{b}\ \vec{b}]+[\vec{a}\ \ \vec{a}\times\vec{b}\ \ \vec{b}]$$
$$=0+[\vec{a}\ \ \vec{a}\times\vec{b}\ \ \vec{b}]$$
$$=(\vec{a}\times\vec{b})\cdot(\vec{b}\times\vec{a})$$
$$=-(\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{b})$$
$$=-|\vec{a}\times\vec{b}|^2$$
$$=-\frac{3}{4}$$

c. If $\vec{b}$ and $\vec{c}$ are orthogonal, $\vec{b}\cdot\vec{c}=0$.

Also, it is given that $\vec{b}\times\vec{c}=\vec{a}$. Now

$$[\vec{a}+\vec{b}+\vec{c}\ \ \vec{a}+\vec{b}\ \ \vec{b}+\vec{c}]=[\vec{a}\ \ \vec{a}+\vec{b}\ \ \vec{b}+\vec{c}]+[\vec{b}+\vec{c}\ \ \vec{a}+\vec{b}\ \ \vec{b}+\vec{c}]$$
$$=[\vec{a}\ \vec{b}\ \vec{c}]$$
$$=\vec{a}\cdot(\vec{b}\times\vec{c})$$
$$=\vec{a}\cdot\vec{a}=|\vec{a}|^2=1 \quad \text{(because } \vec{a} \text{ is a unit vector)}$$

d. $[\vec{x}\ \vec{y}\ \vec{a}]=0$

Therefore, $\vec{x}$, $\vec{y}$ and $\vec{a}$ are coplanar. (i)

$[\vec{x}\ \vec{y}\ \vec{b}]=0$

Therefore, $\vec{x}$, $\vec{y}$ and $\vec{b}$ are coplanar. (ii)

Also, $[\vec{a}\ \vec{b}\ \vec{c}]=0$

Therefore, $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar (iii)

From (i), (ii) and (iii),

$\vec{x}$, $\vec{y}$ and $\vec{c}$ are coplanar. Therefore,

$[\vec{x}\ \vec{y}\ \vec{c}]=0$

3. **a → q; b → s; c → p; d → r**

a. $|\vec{a}+\vec{b}+\vec{c}| = \sqrt{6} \Rightarrow \vec{a}^2+\vec{b}^2+\vec{c}^2+2\vec{a}\cdot\vec{b}+2\vec{b}\cdot\vec{c}+2\vec{c}\cdot\vec{a}=6$

$\therefore |\vec{a}| = 1$

b. $\vec{a}$ is perpendicular to $\vec{b}+\vec{c} \Rightarrow \vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}=0$ (i)

$\vec{b}$ is perpendicular to $\vec{a}+\vec{c} \Rightarrow \vec{b}\cdot\vec{c}+\vec{b}\cdot\vec{c}=0$ (ii)

$\vec{c}$ is perpendicular to $\vec{a}+\vec{b} \Rightarrow \vec{c}\cdot\vec{a}+\vec{a}\cdot\vec{c}\cdot\vec{b}=0$ (iii)

From (i), (ii) and (iii), we get

$\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}=0$

$\therefore |\vec{a}+\vec{b}+\vec{c}|=7$

c. $(\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d})-(\vec{b}\cdot\vec{c})(\vec{a}\cdot\vec{d})=21$

d. We know that $[\vec{a}\times\vec{b}\quad \vec{b}\times\vec{c}\quad \vec{c}\times\vec{a}] = [\vec{a}\ \vec{b}\ \vec{c}]^2$

$$\text{and } [\vec{a}\ \vec{b}\ \vec{c}]^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c} \end{vmatrix}$$

$$= \begin{vmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{vmatrix}$$

$$=32$$

$\therefore \quad [\vec{a}\ \vec{b}\ \vec{c}]=4\sqrt{2}$

4. **a → s; b → r; c → q; d → p**

a. $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & 1 & 2\\ -1 & -2 & -1\end{vmatrix}=3\hat{i}-3\hat{j}+3\hat{k}$

Hence, the area of the triangle is $\frac{3\sqrt{3}}{2}$.

b. The area of the parallelogram is $3\sqrt{3}$.

c. The area of a parallelogram whose diagonals are $2\vec{a}$ and $4\vec{b}$ is $\frac{1}{2}|2\vec{a}\times 4\vec{b}|=12\sqrt{3}$.

d. The volume of the parallelepiped $=|(\vec{a}\times\vec{b})\cdot\vec{c}|=\sqrt{9+36+9}=3\sqrt{6}$

5. **a → p, r; b → q; c → s; d → p**

a. Vectors $-3\hat{i}+3\hat{j}+4\hat{k}$ and $\hat{i}+\hat{j}$ are coplanar with $\vec{a}$ and $\vec{b}$.

b. $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & 2 & 2\\ -2 & 1 & 2\end{vmatrix}$

$= 2\hat{i}-2\hat{j}+3\hat{k}$

c. If $\vec{c}$ is equally inclined to $\vec{a}$ and $\vec{b}$, then we must have $\vec{a}\cdot\vec{c}=\vec{b}\cdot\vec{c}$, which is true for $\vec{c}=\hat{i}-\hat{j}+5\hat{k}$.

d. Vector is forming a triangle with $\vec{a}$ and $\vec{b}$. Then $\vec{c}=\vec{a}+\vec{b}=-3\hat{i}+3\hat{j}+4\hat{k}$

6. **a → q; b → s; c → p; d → r**

a. $|\vec{a}+\vec{b}|=|\vec{a}+2\vec{b}|$

$a^2+b^2+2\vec{a}\cdot\vec{b}=a^2+4b^2+4\vec{a}\cdot\vec{b}$

$\Rightarrow 2\vec{a}\cdot\vec{b}=-3b^2<0$

Hence, angle between $\vec{a}$ and $\vec{b}$ is obtuse.

b. $|\vec{a}+\vec{b}|=|\vec{a}-2\vec{b}|$

$\Rightarrow a^2+b^2+2\vec{a}\cdot\vec{b}=a^2+4b^2-4\vec{a}\cdot\vec{b}$

$\Rightarrow 6\vec{a}\cdot\vec{b}=3b^2$

Hence, angle between $\vec{a}$ and $\vec{b}$ is acute.

c. $|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|$

$\Rightarrow \vec{a}\cdot\vec{b}$

$\Rightarrow$ $\vec{a}$ is perpendicular to $\vec{b}$.

d. $\vec{c}\times(\vec{a}\times\vec{b})$ lies in the plane of vectors $\vec{a}$ and $\vec{b}$.

A vector perpendicular to this plane is parallel to $\vec{a}\times\vec{b}$

Hence angle is 0°.

7. **a → r; b → s; c → q; d → p**

$[\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}]=36$

$\Rightarrow [\vec{a}\ \vec{b}\ \vec{c}]=6$

$\Rightarrow$ Volume of tetrahedron formed by vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is $\frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]=1$.

$[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}]=2[\vec{a}\ \vec{b}\ \vec{c}]=12$

$\vec{a}-\vec{b}$, $\vec{b}-\vec{c}$ and $\vec{c}-\vec{a}$ are coplanar $\Rightarrow [\vec{a}-\vec{b}\ \ \vec{b}-\vec{c}\ \ \vec{c}-\vec{a}]=0$

Integer Answer Type

1. (5) Let angle between $\vec{a}$ and $\vec{b}$ be θ.

We have $|\vec{a}|=|\vec{b}|=1$

Now $|\vec{a}+\vec{b}|=2\cos\frac{\theta}{2}$ and $|\vec{a}-\vec{b}|=2\sin\frac{\theta}{2}$

Consider $F(\theta)=\frac{3}{2}\left(2\cos\frac{\theta}{2}\right)+2\left(2\sin\frac{\theta}{2}\right)$

$\therefore F(\theta)=3\cos\frac{\theta}{2}+4\sin\frac{\theta}{2}$, $\theta\in[0,\pi]$

2. (1) Since angle between $\vec{u}$ and $\hat{i}$ is 60°,

$\vec{u}\cdot i=|\vec{u}||\hat{i}|\cos 60°=\frac{|\vec{u}|}{2}$

Given that $|\vec{u}-\hat{i}|, |\vec{u}|, |\vec{u}-2\hat{i}|$ are in G.P., so $|\vec{u}-\hat{i}|^2=|\vec{u}||\vec{u}-2\hat{i}|$

Squaring both sides, $[|\vec{u}|^2+|\hat{i}|^2-2\vec{u}\cdot\hat{i}]^2=|\vec{u}|^2[|\vec{u}|^2+4|\hat{i}|^2-4\vec{u}\cdot\hat{i}]$

$[|\vec{u}|^2+1-\frac{2|\vec{u}|}{2}]^2=|\vec{u}|^2[|\vec{u}|^2+4-4\frac{|\vec{u}|}{2}] \Rightarrow |\vec{u}|^2+2|\vec{u}|-1=0 \Rightarrow |\vec{u}|=-\frac{2\pm2\sqrt{2}}{2} \Rightarrow |\vec{u}|=\sqrt{2}-1$

3. **(2)** $\overline{AB} = 2\hat{i} + \hat{j} + \hat{k}$, $\overline{AC} = (t+1)\hat{i} + 0\hat{j} - \hat{k}$

$$\overline{AB} \times \overline{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ t+1 & 0 & -1 \end{vmatrix} = -\hat{i} + (t+3)\hat{j} - (t+1)\hat{k}$$

$$|\overline{AB} \times \overline{AC}| = \sqrt{1 + (t+3)^2 + (t+1)^2} = \sqrt{2t^2 + 8t + 11}$$

$$\text{Area of } \Delta ABC = \frac{1}{2}|\overline{AB} \times \overline{AC}| \Rightarrow \Delta = \frac{1}{2}\sqrt{2t^2 + 8t + 1}$$

$$\text{Let } f(t) = \Delta^2 = \frac{1}{4}(2t^2 + 8t + 1)$$

$f'(t) = 0 \Rightarrow t = -2$

At $t = -2$, $f''(t) > 0$

So Δ is minimum at $t = -2$

4. **(7)** $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

L.H.S. $= [3\vec{a} + \vec{b} \;\; 3\vec{b} + \vec{c} \;\; 3\vec{c} + \vec{a}]$

$= [3\vec{a}\; 3\vec{b}\; 3\vec{c}] + [\vec{b}\; \vec{c}\; \vec{a}]$

$= 3^3[\vec{a}\; \vec{b}\; \vec{c}] + [\vec{a}\; \vec{b}\; \vec{c}]$

$= 28\,[\vec{a}\; \vec{b}\; \vec{c}]$

5. **(4)** $\vec{a} = \alpha\hat{i} + 2\hat{j} - 3\hat{k}$, $\vec{b} = \hat{i} + 2\alpha\hat{j} - 2\hat{k}$, $\vec{c} = 2\hat{i} - \alpha\hat{j} + \hat{k}$

$\{(\vec{a} \times \vec{b}) \times (\vec{b} \times \vec{c})\} \times (\vec{c} \times \vec{a}) = \vec{0}$

$\Rightarrow \{[\vec{a}\; \vec{b}\; \vec{c}]\vec{b} - [\vec{a}\; \vec{b}\; \vec{b}]\vec{c}\} \times (\vec{c} \times \vec{a}) = \vec{0}$

$\Rightarrow [\vec{a}\; \vec{b}\; \vec{c}]\vec{b} \times (\vec{c} \times \vec{a}) = \vec{0}$

$\Rightarrow [\vec{a}\; \vec{b}\; \vec{c}] \quad ((\vec{a} \cdot \vec{b})\vec{c} - (\vec{b} \cdot \vec{c})\vec{a}) = \vec{0}$

$\Rightarrow [\vec{a}\; \vec{b}\; \vec{c}] = 0 \quad (\because \;\; \vec{a}$ and $\vec{c}$ are not collinear)

$$\Rightarrow \begin{vmatrix} \alpha & 2 & -3 \\ 1 & 2\alpha & -2 \\ 2 & -\alpha & 1 \end{vmatrix}$$

$\Rightarrow \quad \alpha(2\alpha-2\alpha)-2(1+4)-3(-\alpha-4\alpha)=0$

$\Rightarrow \quad 10-15\alpha=0$

$\therefore \alpha = 2/3$

6. **(9)** Since $\vec{x}$ and $\vec{y}$ are non-collinear vectors, therefore $\vec{x}$, $\vec{y}$ and $\vec{x}\times\vec{y}$ are non-coplanar vectors.

$[(a-2)\alpha^2+(b-3)\alpha+c]+[(a-2)\beta^2+(b-3)\,\beta\beta+c]\,\vec{y}\,+[(a-2)\gamma^2+(b-3)\,\gamma+c]\,(\vec{x}\times\vec{y})=0$

Coefficient of each vector $\vec{x}$, $\vec{y}$ and $\vec{x}\times\vec{y}$ is zero.

$(a-2)\alpha^2+(b-3)\alpha+c=0$

$(a-2)\beta^2+(b-3)\beta+c=0$

$(a-2)\,\gamma^2+(b-3)\,\beta+c=0$

The above three equations will satisfy if the coefficients of α, β and γ are zero because α, β and γ are three distinct real numbers

$a-2=0 \Rightarrow a=2,$

$b-3=0 \Rightarrow b=3$ and $c=0$

$\therefore a^2+b^2+c^2=2^2+3^2+0^2=4+9=13$

7. **(1)** Given, $\vec{u}\times\vec{v}+\vec{u}=\vec{w}$ and $\vec{w}\times\vec{u}=\vec{v}$

$\Rightarrow \quad (\vec{u}\times\vec{v}+\vec{u})\times\vec{u}=\vec{v} \Rightarrow (\vec{u}\times\vec{v})\times\vec{u}=\vec{v} \Rightarrow \vec{v}-(\vec{u}\cdot\vec{v})=\vec{v} \Rightarrow (\vec{u}\cdot\vec{v})\vec{u}=0 \Rightarrow (\vec{u}\,\vec{v})=0$

Now, $[\vec{u}\,\vec{v}\,\vec{w}]=\vec{u}\cdot(\vec{v}\times\vec{w})$

$=\vec{u}\cdot(\vec{v}\times(\vec{u}\times\vec{v}+\vec{u})) = \vec{u}\cdot(\vec{v}\times(\vec{u}\times\vec{v})+\vec{v}\times\vec{u}) = \vec{u}(v^2\vec{u}-(\vec{u}\cdot\vec{v})\vec{v}+\vec{v}\times\vec{u}) = \vec{v}^2\vec{u}^2=1$

8. **(7)** Let the vertices are A, B, C, D and O is the origin.

$\therefore \quad \vec{OA}=\hat{i}-6\hat{j}+10\hat{k},\ \vec{OB}=\hat{i}-3\hat{j}+7\hat{k},\ \vec{OC}=-5\hat{i}-\hat{j}+\lambda\hat{k},\ \vec{OD}=7\hat{i}-4\hat{j}+7\hat{k}$

$\therefore \quad \vec{AB}=\vec{OB}-\vec{OA}=-2\hat{i}+3\hat{j}-3\hat{k}$

$\vec{AC}=\vec{OC}-\vec{OA}=4\hat{i}+5\hat{j}+(\lambda-10)\hat{k}$

$\vec{AD}=\vec{OD}-\vec{OA}=6\hat{i}+2\hat{j}-3\hat{k}$

Volume of tetrahedron $= \frac{1}{6}[\vec{AB}\ \vec{AC}\ \vec{AD}]$

$$= \frac{1}{6}\begin{vmatrix} -2 & 3 & -3 \\ 4 & 5 & \lambda-10 \\ 6 & 2 & -3 \end{vmatrix}$$

$$= \frac{1}{6}\{-2(-15-2\lambda+20)-3(-12-6\lambda+60)-3(8-30)\}$$

$$= \frac{1}{6}\{4\lambda-10-144+18\lambda+66\}$$

$$= \frac{1}{6}(22\lambda-88)=11 \quad \text{(given)}$$

$\Rightarrow\ 2\lambda-8=6$

$\therefore\ \lambda=7$

9. **(6)** Let $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{u} = \hat{i} - 2\hat{j} + 3\hat{k};\ \vec{v} = 2\hat{i} + \hat{j} + 4\hat{k};\ \vec{w} = \hat{i} + 3\hat{j} + 3\hat{k}$

$(\vec{u}\cdot\vec{R} - 15)\hat{i} + (\vec{v}\cdot\vec{R} - 30)\hat{j} + (\vec{w}\cdot\vec{R} - 25)\hat{k} = \vec{0}$ (given)

So $\vec{u}\cdot\vec{R} = 15 \Rightarrow x - 2y + 3z = 15$ (i)

$\vec{v}\cdot\vec{R} = 30 \Rightarrow 2x + y + 4z = 30$ (ii)

$\vec{w}\cdot\vec{R} = 25 \Rightarrow x + 3y + 3z = 25$ (iii)

Solving, we get

$x=4$

$y=2$

$z=5$

10. **(6)** $2\vec{V} + \vec{V}\times(\hat{i} + 2\hat{j}) = (2\hat{i} + \hat{k})$ (i)

$\Rightarrow 2\vec{V}\cdot(\hat{i} + 2\hat{j}) + 0 = (2\hat{i} + \hat{k})\cdot(\hat{i} + 2\hat{j})$

$\Rightarrow 2\vec{V}\cdot(\hat{i} + 2\hat{j}) = 2$

$\Rightarrow |\vec{V}\cdot(\hat{i} + 2\hat{j})|^2 = 1$

$\Rightarrow |\vec{V}|^2\cdot|\hat{i} + 2\hat{j}|^2\cos^2\theta = 1$ (θ is the angle between $\vec{V}$ and $\hat{i} + 2\hat{j}$)

$\Rightarrow |\vec{V}|^2\ 5(1 - \sin^2\theta) = 1$

$\Rightarrow |\vec{V}|^2 5\sin^2\theta = 5|\vec{V}|^2 - 1$ (ii)

From Eq. (i)

$\Rightarrow |2\vec{V} + \vec{V} \times (\hat{i} + 2\hat{j})|^2 = |2\hat{i} + \hat{k}|^2$

$\Rightarrow 4|\vec{V}|^2 + |\vec{V} \times (\hat{i} + 2\hat{j})|^2 = 5$

$\Rightarrow 4|\vec{V}|^2 + |\vec{V}|^2 \cdot |\hat{i} + 2\hat{j}|^2 \sin^2\theta = 5$

$\Rightarrow 4|\vec{V}|^2 + 5|\vec{V}|^2 \sin^2\theta = 5$

$\Rightarrow 4|\vec{V}|^2 + 5|\vec{V}|^2 - 1 = 5$

$\Rightarrow 9|\vec{V}|^2 = 6$

$\Rightarrow 3|\vec{V}| = \sqrt{6}$

$\Rightarrow 3|\vec{V}| = \sqrt{6} = \sqrt{m}$

$\therefore m = 6$

11. **(1)** $\vec{a} \cdot \vec{b} = 0 \Rightarrow \vec{a} \perp \vec{b}$

$\vec{a} \cdot \vec{c} = 0 \Rightarrow \vec{a} \perp \vec{c}$

$\Rightarrow \vec{a} \perp \vec{b} - \vec{c}$

$|\vec{a} \times \vec{b} - \vec{a} \times \vec{c}| = |\vec{a} \times (\vec{b} - \vec{c})| = |\vec{a}||\vec{b} - \vec{c}| = |\vec{b} - \vec{c}|$

Now $|\vec{b} - \vec{c}|^2 = |\vec{b}|^2 + |\vec{c}|^2 - 2|\vec{b}||\vec{c}|\cos\frac{\pi}{3} = 2 - 2x \times \frac{1}{2} = 1$

$|\vec{b} - \vec{c}| = 1$

12. **(6)** Here $\vec{OA} = \vec{a}, \vec{OB} = 10\vec{a} + 2\vec{b}, \vec{OC} = \vec{b}$

q = Area of parallelogram with OA and OC as adjacent sides.

$\therefore \quad q = |\vec{a} \times \vec{b}|$ (i)

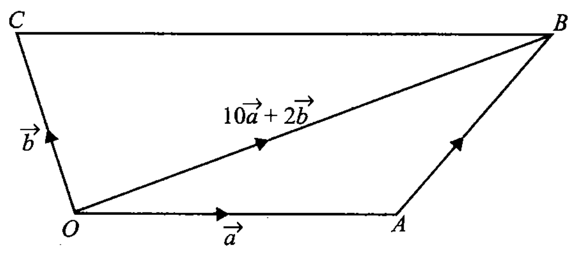


Fig. 2.49

p = Area of quadrilateral $OABC$

= Area of ΔOAB + are of ΔOBC

$= \frac{1}{2} |\vec{a} \times (10\vec{a} + 2\vec{b})| + \frac{1}{2} |(10\vec{a} + 2\vec{b}) \times \vec{b}|$

$= |\vec{a} \times \vec{b}| + 5 |\vec{a} \times \vec{b}|$

$\therefore \quad p = 6 |\vec{a} \times \vec{b}|$

$\Rightarrow \quad p = 6q$ [From Eq. (i)]

$\therefore \quad k = 6$

13. **(9)** Here $\vec{F} = 3\hat{i} - \hat{j} - 2\hat{k}$

$\vec{AB}$ = P.V. of B – P.V. of A

$\therefore \quad \vec{AB} = (-\hat{i} - \hat{j} - 2\hat{k}) - (-3\hat{i} - 4\hat{j} + \hat{k})$

$= 2\hat{i} + 3\hat{j} - 3\hat{k}$

Let $\vec{s} = \vec{AB}$ be the displacement vector

Now work done $= \vec{F} \cdot \vec{s}$

$= (3\hat{i} - \hat{j} - 2\hat{k}) \cdot (2\hat{i} + 3\hat{j} - 3\hat{k})$

$= 6 - 3 + 6 = 9$

Archives

Subjective Type

1. Let with respect to O, position vectors of points A, B, C, D, E and F be $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$, $\vec{e}$ and $\vec{f}$. Let perpendiculars from A to EF and from B to DF meet each other at H. Let position vectors of H be $\vec{r}$. We join CH. In order to prove the statement given in the question, it is sufficient to prove that CH is perpendicular to DE.

Now, as $OD \perp BC \Rightarrow \vec{d} \cdot (\vec{b} - \vec{c}) = 0$

$\Rightarrow \vec{d} \cdot \vec{b} = \vec{d} \cdot \vec{c}$ (i)

as $OE \perp AC \Rightarrow \vec{e} \cdot (\vec{c} - \vec{a}) = 0 \Rightarrow \vec{e} \cdot \vec{c} = \vec{e} \cdot \vec{a}$ (ii)

as $OF \perp AB \Rightarrow \vec{f} \cdot (\vec{a} - \vec{b}) = 0 \Rightarrow \vec{f} \cdot \vec{a} = \vec{f} \cdot \vec{b}$ (iii)

Also $AH \perp EF \Rightarrow (\vec{r} - \vec{a}) \cdot (\vec{e} - \vec{f}) = 0$

$\Rightarrow \vec{r} \cdot \vec{e} - \vec{r} \cdot \vec{f} - \vec{a} \cdot \vec{e} + \vec{a} \cdot \vec{f} = 0$ (iv)

and $BH \perp FD \Rightarrow (\vec{r} - \vec{b}) \cdot (\vec{f} - \vec{d}) = 0$

$\Rightarrow \vec{r} \cdot \vec{f} - \vec{r} \cdot \vec{d} - \vec{b} \cdot \vec{f} + \vec{b} \cdot \vec{d} = 0$ (v)

Adding (iv) and (v), we get

$$\vec{r}\cdot\vec{e}-\vec{a}\cdot\vec{e}+\vec{a}\cdot\vec{f}-\vec{r}\cdot\vec{d}-\vec{b}\cdot\vec{f}+\vec{b}\cdot\vec{d}=0$$

$$\Rightarrow \vec{r}\cdot(\vec{e}-\vec{d})-\vec{e}\cdot\vec{c}+\vec{d}\cdot\vec{c}=0 \quad \text{(using (i), (ii) and (iii))}$$

$$\Rightarrow (\vec{r}-\vec{c})\cdot(\vec{e}-\vec{d})=0$$

$$\Rightarrow \overrightarrow{CH}\cdot\overrightarrow{ED}=0 \Rightarrow CH\perp ED$$

2. $\overrightarrow{OA_1}\ \overrightarrow{OA_2},\ldots,\overrightarrow{OA_n}$. All vectors are of same magnitude, say a, and angle between any two consecutive vectors is the same, that is, $2\pi/n$. Let $\hat{p}$ be the unit vector parallel to the plane of the polygon.

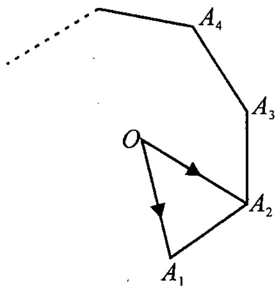


Fig. 2.50

$$\therefore \text{ Let } \overrightarrow{OA_1}\times\overrightarrow{OA_2}=a^2\sin\frac{2\pi}{n}\hat{p} \quad \text{(i)}$$

$$\text{Now, } \sum_{i=1}^{n-1}\overrightarrow{OA_i}\times\overrightarrow{OA_{i+1}}=\sum_{i=1}^{n-1}a^2\sin\frac{2\pi}{n}\hat{p}$$

$$=(n-1)\,a^2\sin\frac{2\pi}{n}\hat{p}$$

$$=(n-1)\,[-\overrightarrow{OA_2}\times\overrightarrow{OA_1}] \quad \text{(Using (i))}$$

$$=(1-n)\,[\overrightarrow{OA_2}\times\overrightarrow{OA_1}]=\text{R.H.S.}$$

3. $\vec{A}\times\vec{X}=\vec{B}$

$$\Rightarrow (\vec{A}\times\vec{X})\times\vec{A}=\vec{B}\times\vec{A}$$

$$\Rightarrow (\vec{A}\cdot\vec{A})\vec{X}-(\vec{X}\cdot\vec{A})\vec{A}=\vec{B}\times\vec{A}$$

$$\Rightarrow (\vec{A}\cdot\vec{A})\,\vec{X}-c\vec{A}=\vec{B}\times\vec{A}$$

$$\Rightarrow \vec{X}=\frac{\vec{B}\times\vec{A}+c\vec{A}}{(\vec{A}\cdot\vec{A})}$$

4. Let the position vectors of points A, B, C, D be $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$, respectively, with respect to some origin.

$|\overrightarrow{AB}\times\overrightarrow{CD}+\overrightarrow{BC}\times\overrightarrow{AD}+\overrightarrow{CA}\times\overrightarrow{BD}|$

$=[|(\vec{b}-\vec{a})\times(\vec{d}-\vec{c})+(\vec{c}-\vec{b})\times(\vec{d}-\vec{a})+(\vec{a}-\vec{c})\times(\vec{d}-\vec{b})|]$

$=2|\vec{b}\times\vec{a}+\vec{c}\times\vec{b}+\vec{a}\times\vec{c}|$ (i)

$=2(2\times(\text{area of }\Delta ABC))$
$=4\times(\text{area of }\Delta ABC)$

5. Given that $\vec{a}, \vec{b}$ and $\vec{c}$ are three coplanar vectors. Therefore, there exist scalars x, y and z, not all zero, such that

$x\vec{a}+y\vec{b}+z\vec{c}=\vec{0}$ (i)

Taking dot product of $\vec{a}$ and (i), we get

$x\vec{a}\cdot\vec{a}+y\vec{a}\cdot\vec{b}+z\vec{a}\cdot\vec{c}=0$ (ii)

Again taking dot product of $\vec{b}$ and (i), we get

$x\vec{b}\cdot\vec{a}+y\vec{b}\cdot\vec{b}+z\vec{b}\cdot\vec{c}=0$ (iii)

Now Eqs. (i), (ii) and (iii) form a homogeneous system of equations, where x, y and z are not all zero, Therefore the system must have a non-trivial solution, and for this, the determinant of coefficient matrix should be zero, i.e.,

$$\begin{vmatrix}\vec{a} & \vec{b} & \vec{c}\\ \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c}\\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c}\end{vmatrix}=0$$

6. We are given that $\vec{A}=2\hat{i}+\hat{k}$, $\vec{B}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{C}=4\hat{i}-3\hat{j}+7\hat{k}$ and to determine a vector $\vec{R}$ such that $\vec{R}\times\vec{B}=\vec{C}\times\vec{B}$ and $\vec{R}\cdot\vec{A}=0$, let $\vec{R}=x\hat{i}+y\hat{j}+z\hat{k}$

Then $\vec{R}\times\vec{B}=\vec{C}\times\vec{B}$

$$\Rightarrow\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ x & y & z\\ 1 & 1 & 1\end{vmatrix}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & -3 & 7\\ 1 & 1 & 1\end{vmatrix}$$

$\Rightarrow(y-z)\hat{i}-(x-z)\hat{j}+(x-y)\hat{k}=-10\hat{i}+(x-z)\hat{j}+7\hat{k}$

$y - z = -10$ (i)
$x - z = -3$ (ii)
$x - y = 7$ (iii)
Also $\vec{R} \cdot \vec{A} = 0$

$\Rightarrow 2x + z = 0$ (iv)

Substituting $y = x - 7$ and $z = -2x$ from (iii) and (iv), respectively in (i), we get
$x - 7 + 2x = -10$
$\Rightarrow 3x = -3$
$\Rightarrow x = -1, y = -8$ and $z = 2$

7. We have, $\vec{a} = cx\ \hat{i} - 6\hat{j} - 3\hat{k}$

$\vec{b} = x\hat{i} + 2\hat{j} + 2cx\,\hat{k}$

Now we know that $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$

As the angle between $\vec{a}$ and $\vec{b}$ is obtuse, $\cos\theta < 0 \Rightarrow \vec{a} \cdot \vec{b} < 0$
$\Rightarrow cx^2 - 12 + 6cx < 0$
$\Rightarrow -cx^2 - 6cx + 12 > 0,\ x \in R$
$\Rightarrow -c > 0$ and $D < 0$
$\Rightarrow c < 0$ and $36c^2 + 48c < 0$
$\Rightarrow c < 0$ and $(3c + 4) > 0$
$\Rightarrow c < 0$ and $c > -4/3$
$\Rightarrow -4/3 < c < 0$

8. $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) + (\vec{a} \times \vec{c}) \times (\vec{d} \times \vec{b}) + (\vec{a} \times \vec{d}) \times (\vec{b} \times \vec{c})$

Here $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = -(\vec{c} \times \vec{d} \cdot \vec{b})\,\vec{a} + (\vec{c} \times \vec{d} \cdot \vec{a})\,\vec{b}$

$= [\vec{a}\ \vec{c}\ \vec{d}]\,\vec{b} - [\vec{b}\ \vec{c}\ \vec{d}]\,\vec{a}$ (i)

$(\vec{a} \times \vec{c}) \times (\vec{d} \times \vec{b}) = -(\vec{d} \times \vec{b} \cdot \vec{c})\,\vec{a} + (\vec{d} \times \vec{b} \cdot \vec{a})\,\vec{c}$

$= [\vec{a}\ \vec{d}\ \vec{b}]\,\vec{c} - [\vec{c}\ \vec{d}\ \vec{b}]\,\vec{a}$ (ii)

$(\vec{a} \times \vec{d}) \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{d} \cdot \vec{c})\,\vec{b} - (\vec{a} \times \vec{d} \cdot \vec{b})\,\vec{c}$

$= -[\vec{a}\ \vec{c}\ \vec{d}]\,\vec{b} - [\vec{a}\ \vec{d}\ \vec{b}]\,\vec{c}$ (iii)

(Note : Here we have tried to write the given expression in such a way that we can get terms involving $\vec{a}$ and other similar terms which can get cancelled)
Adding (i), (ii) and (iii), we get
Given vector $= -2\,[\vec{b}\ \vec{c}\ \vec{d}]\ \vec{a} = k\vec{a}$

$\Rightarrow$ Given vector is parallel to $\vec{a}$.

9.

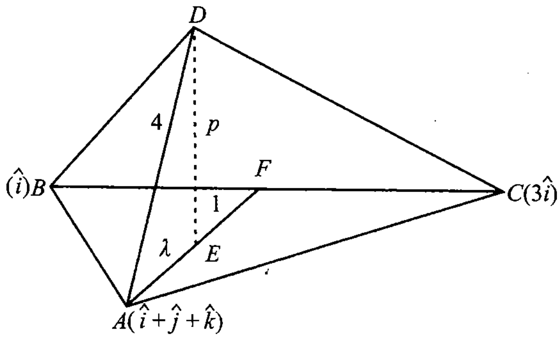


Fig. 2.51

We are given $AD = 4$

Volume of tetrahedron $= \dfrac{2\sqrt{2}}{3}$

$\Rightarrow \dfrac{1}{3}$ (Area of ΔABC) $p = \dfrac{2\sqrt{2}}{3}$

$\therefore \ \dfrac{1}{2} | \overrightarrow{BA} \times \overrightarrow{BC} | \ p = 2\sqrt{2}$

$\dfrac{1}{2} \ | (\hat{j} + \hat{k}) \times 2\hat{i} \ | \ p = 2\sqrt{2}$

$\Rightarrow | \hat{j} - \hat{k} | \ p = 2\sqrt{2}$

$\Rightarrow \sqrt{2}\, p = 2\sqrt{2}, \ p = 2$

We have to find the P.V. of point E. Let it divide median AF in the ratio $\lambda : 1$.

P.V. of E is $\dfrac{\lambda \cdot 2\hat{i} + (\hat{i} + \hat{j} + \hat{k})}{\lambda + 1}$. Therefore, (i)

$\overrightarrow{AE}$ = P.V. or E − P.V. of $A = \dfrac{\lambda (\hat{i} - \hat{j} - \hat{k})}{\lambda + 1}$

$| \overrightarrow{AE} |^2 = 3 \left(\dfrac{\lambda}{\lambda + 1} \right)^2$ (ii)

Now, $4 + 3 \left(\dfrac{\lambda}{\lambda + 1} \right)^2 = 16$

$\left(\dfrac{\lambda}{\lambda + 1} \right) = \pm 2$

$\lambda = -2$ or $-2/3$

Putting the value of λ in (i), we get the P.V. of possible positions of E as $-\hat{i} + 3\hat{j} + 3\hat{k}$ or $3\hat{i} - \hat{j} - \hat{k}$.

10. Given that $\vec{a}, \vec{b}$ and $\vec{c}$ are three unit vectors inclined at an angle θ with each other.

Also $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar. Therefore, $[\vec{a}\ \vec{b}\ \vec{c}] \neq 0$.

Also given that $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} = p\vec{a} + q\vec{b} + r\vec{c}$.

Taking dot product on both sides with $\vec{a}$, we get

$$p + q\cos\theta + r\cos\theta = [\vec{a}\ \vec{b}\ \vec{c}] \qquad \text{(i)}$$

Similarly, taking dot product on both sides with $\vec{b}$ and $\vec{c}$, we get, respectively,

$$p\cos\theta + q + r\cos\theta = 0 \qquad \text{(ii)}$$

and $p\cos\theta + q\cos\theta + r = [\vec{a}\ \vec{b}\ \vec{c}]$ (iii)

Adding (i), (ii) and (iii), we get

$$p + q + r = \frac{2[\vec{a}\ \vec{b}\ \vec{c}]}{2\cos\theta + 1} \qquad \text{(iv)}$$

Multiplying (iv) by $\cos\theta$ and subtracting (i) from it, we get

$$p(\cos\theta - 1) = \frac{2[\vec{a}\ \vec{b}\ \vec{c}]\cos\theta}{2\cos\theta + 1} - [\vec{a}\ \vec{b}\ \vec{c}]$$

or $p(\cos\theta - 1) = \dfrac{-[\vec{a}\ \vec{b}\ \vec{c}]}{2\cos\theta + 1}$

$$\Rightarrow p = \frac{[\vec{a}\ \vec{b}\ \vec{c}]}{(1 - \cos\theta)(1 + 2\cos\theta)}$$

Similarly, (iv) $\times \cos\theta -$ (ii) gives

$$q = \frac{-2[\vec{a}\ \vec{b}\ \vec{c}]\cos\theta}{(1 + 2\cos\theta)(1 - \cos\theta)}$$

and (iv) $\times \cos\theta -$ (iii) gives

$$r(\cos\theta - 1) = \frac{2[\vec{a}\ \vec{b}\ \vec{c}]\cos\theta}{2\cos\theta + 1} - [\vec{a}\ \vec{b}\ \vec{c}]$$

$$\Rightarrow r = \frac{-[\vec{a}\ \vec{b}\ \vec{c}]}{(2\cos\theta + 1)(\cos\theta - 1)}$$

But we have to find p, q and r in terms of θ only.

So let us find the value of $[\vec{a}\ \vec{b}\ \vec{c}]$

We know that

$$[\vec{a}\,\vec{b}\,\vec{c}]^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c} \end{vmatrix}.$$

$$= \begin{vmatrix} 1 & \cos\theta & \cos\theta \\ \cos\theta & 1 & \cos\theta \\ \cos\theta & \cos\theta & 1 \end{vmatrix}$$

On operating $C_1 \to C_1 + C_2 + C_3$, we get

$$\begin{vmatrix} 1+2\cos\theta & \cos\theta & \cos\theta \\ 1+2\cos\theta & 1 & \cos\theta \\ 1+2\cos\theta & \cos\theta & 1 \end{vmatrix}$$

$$= (1+2\cos\theta)\begin{vmatrix} 1 & \cos\theta & \cos\theta \\ 1 & 1 & \cos\theta \\ 1 & \cos\theta & 1 \end{vmatrix}$$

Operating $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$, we get

$$= (1+2\cos\theta)\begin{vmatrix} 0 & \cos\theta - 1 & 0 \\ 0 & 1-\cos\theta & \cos\theta - 1 \\ 1 & \cos\theta & 1 \end{vmatrix}$$

Expanding along C_1

$= (1+2\cos\theta)\,(1-\cos\theta)^2$

$\therefore [\vec{a}\,\vec{b}\,\vec{c}] = (1-\cos\theta)\sqrt{1+2\cos\theta}$

Thus, we get

$$p = \frac{1}{\sqrt{1+2\cos\theta}},\ q = \frac{-2\cos\theta}{\sqrt{1+2\cos\theta}},\ r = \frac{1}{\sqrt{1+2\cos\theta}}$$

11. We have, $(\vec{A}+\vec{B}) \times (\vec{A}+\vec{C})$

$= \vec{A}\times\vec{A} + \vec{B}\times\vec{A} + \vec{A}\times\vec{C} + \vec{B}\times\vec{C}$

$= \vec{B}\times\vec{A} + \vec{A}\times\vec{C} + \vec{B}\times\vec{C} \quad (\because \vec{A}\times\vec{A} = \vec{0})$

Thus $[(\vec{A}+\vec{B})\times(\vec{A}+\vec{C})]\times(\vec{B}\times\vec{C})$

$= [\vec{B}\times\vec{A} + \vec{A}\times\vec{C} + \vec{B}\times\vec{C}]\times(\vec{B}\times\vec{C})$

$= (\vec{B}\times\vec{A})\times(\vec{B}\times\vec{C}) + (\vec{A}\times\vec{C})\times(\vec{B}\times\vec{C}) \quad (\because x\times x = 0)$

$= \{(\vec{B}\times\vec{A})\cdot\vec{C}\}\vec{B} - \{(\vec{B}\times\vec{A})\cdot\vec{B}\}\vec{C} + \{(\vec{A}\times\vec{C})\cdot\vec{C}\}\vec{B} - \{(\vec{A}\times\vec{C})\cdot\vec{B}\}\vec{C}$

$= [\vec{B}\,\vec{A}\,\vec{C}]\vec{B} - [\vec{A}\,\vec{C}\,\vec{B}]\vec{C}$

$= [\vec{A}\,\vec{C}\,\vec{B}]\{\vec{B} - \vec{C}\}$

Thus, L.H.S. of the given expression

$= [\vec{A}\,\vec{C}\,\vec{B}]\,(\vec{B}-\vec{C})\cdot(\vec{B}+\vec{C})$

$= [\vec{A}\,\vec{C}\,\vec{B}]\,\{(\vec{B}-\vec{C})\cdot(\vec{B}+\vec{C})\}$

$= [\vec{A}\,\vec{C}\,\vec{B}]\,\{|\vec{B}|^2-|\vec{C}|^2\}=0 \quad (\because |B|=|C|)$

Alternative method:

Since $[(\vec{A}+\vec{B})\times(\vec{A}+\vec{C})]\times(\vec{B}+\vec{C})\cdot(\vec{B}+\vec{C})$ is scalar triple product of $(\vec{A}+\vec{B})\times(\vec{A}+\vec{C}), \vec{B}+\vec{C}$ and $\vec{B}+\vec{C}$, its value is 0.

12. **a.** We have $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$

and $\vec{u}\times\vec{v} = |\vec{u}||\vec{v}|\sin\theta\,\hat{n}$

(where θ is the angle between $\vec{u}$ and $\vec{v}$ and $\hat{n}$ is a unit vector perpendicular to both $\vec{u}$ and $\vec{v}$)

$\Rightarrow (\vec{u}\cdot\vec{v})^2+|\vec{u}\times\vec{v}|^2 = |\vec{u}|^2|\vec{v}|^2(\cos^2\theta+\sin^2\theta) = |\vec{u}|^2|\vec{v}|^2$

b. $(1-\vec{u}\cdot\vec{v})^2+|\vec{u}+\vec{v}+(\vec{u}\times\vec{v})|^2$

$= 1-2\vec{u}\cdot\vec{v}+(\vec{u}\cdot\vec{v})^2+|\vec{u}|^2+|\vec{v}|^2+|\vec{u}\times\vec{v}|^2+2\vec{u}\cdot\vec{v}$

$(\because \vec{u}\cdot(\vec{u}\times\vec{v}) = \vec{v}\cdot(\vec{u}\times\vec{v}) = 0)$

$= 1+|\vec{u}|^2+|\vec{v}|^2+(\vec{u}\cdot\vec{v})^2+|\vec{u}\times\vec{v}|^2$

$= 1+|\vec{u}|^2+|\vec{v}|^2+|\vec{u}|^2|\vec{v}|^2$

$= (\vec{1}+|\vec{u}|^2)(\vec{1}+|\vec{v}|^2)$

13. $[\vec{u}\,\vec{v}\,\vec{w}] = (\vec{u}\times\vec{v})\cdot(\vec{v}-\vec{w}\times\vec{u}) = (\vec{u}\times\vec{v})\cdot(\vec{u}\times\vec{w})$

$$= \begin{vmatrix} \vec{u}\cdot\vec{u} & \vec{u}\cdot\vec{w} \\ \vec{v}\cdot\vec{u} & \vec{v}\cdot\vec{w} \end{vmatrix}$$

Now, $\vec{u}\cdot\vec{u} = 1$

$\vec{u}\cdot\vec{w} = \vec{u}\cdot(\vec{v}-\vec{w}\times\vec{u}) = \vec{u}\cdot\vec{v}-[\vec{u}\,\vec{w}\,\vec{u}] = \vec{u}\cdot\vec{v}$

$\vec{v}\cdot\vec{w} = \vec{v}\cdot(\vec{v}-\vec{w}\times\vec{u}) = 1-[\vec{v}\,\vec{w}\,\vec{u}] = 1-[\vec{u}\,\vec{v}\,\vec{w}]$

$\therefore [\vec{u}\,\vec{v}\,\vec{w}] = \begin{vmatrix} 1 & \cos\theta \\ \cos\theta & 1-[\vec{u}\,\vec{v}\,\vec{w}] \end{vmatrix}$ (θ is the angle between $\vec{u}$ and $\vec{v}$)

$= 1-[\vec{u}\,\vec{v}\,\vec{w}]-\cos^2\theta$

$\therefore [\vec{u}\,\vec{v}\,\vec{w}] = \frac{1}{2}\sin^2\theta \le \frac{1}{2}$

Equality holds when $\sin^2\theta = 1$, i.e., $\theta = \pi/2$, i.e., $\vec{u}\perp\vec{v}$.

14. Given data are insufficient to uniquely determine the three vectors as there are only six equations involving nine variables.

Therefore, we can obtain infinite number of sets of three vectors, $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$, satisfying these conditions.

From the given data, we get

$\vec{v}_1 \cdot \vec{v}_1 = 4 \Rightarrow |\vec{v}_1| = 2$

$\vec{v}_2 \cdot \vec{v}_2 = 2 \Rightarrow |\vec{v}_2| = \sqrt{2}$

$\vec{v}_3 \cdot \vec{v}_3 = 29 \Rightarrow |\vec{v}_3| = \sqrt{29}$

Also $\vec{v}_1 \cdot \vec{v}_2 = -2$

$\Rightarrow |v_1||v_2| \cos\theta = -2$ (where θ is the angle between $\vec{v}_1$ and $\vec{v}_2$)

$\Rightarrow \cos\theta = \dfrac{-1}{\sqrt{2}}$

$\Rightarrow \theta = 135°$

Since any two vectors are always coplanar, let us suppose that $\vec{v}_1$ and $\vec{v}_2$ are in the x–y plane. Let $\vec{v}_1$ be along the positive direction of the x-axis. Then $\vec{v}_1 = 2\hat{i}$. $(\because |\vec{v}_1| = 2)$

As $\vec{v}_2$ makes an angle 135° with $\vec{v}_1$ and lies in the x–y plane, also keeping in mind $|\vec{v}_2| = \sqrt{2}$, we obtain $\vec{v}_2 = -\hat{i} \pm \hat{j}$

Again let $\vec{v}_3 = \alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$

$\because \vec{v}_3 \cdot \vec{v}_1 = 6 \Rightarrow 2\alpha = 6 \Rightarrow \alpha = 3$

and $\vec{v}_3 \cdot \vec{v}_2 = -5 \Rightarrow -\alpha \pm \beta = -5 \Rightarrow \beta = \pm 2$

Also $|\vec{v}_3| = \sqrt{29} \Rightarrow \alpha^2 + \beta^2 + \gamma^2 = 29$

$\Rightarrow \gamma = \pm 4$

Hence $\vec{v}_3 = 3\hat{i} \pm 2\hat{j} \pm 4\hat{k}$

15. Given that $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_2\hat{k}$ where a_r, b_r, c_r $(r = 1, 2, 3)$ are all non-negative real numbers

Also $\sum_{r=1}^{3}(a_r + b_r + c_r) = 3L$

To prove $V \le L^3$, where V is the volume of the parallolepiped formed by the vectors $\vec{a}, \vec{b}$ and $\vec{c}$, we have

$$V = [\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$\Rightarrow V = (a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2) - (a_1 b_3 c_2 + a_2 b_1 c_3 + a_3 b_2 c_1)$ (i)

Now we know that A.M. ≥ G.M., therefore

$$\frac{(a_1+b_1+c_1)+(a_2+b_2+c_2)+(a_3+b_3+c_3)}{3} \geq [(a_1+b_1+c_1)(a_2+b_2+c_2)(a_3+b_3+c_3)]^{1/3}$$

$$\Rightarrow \frac{3L}{3} \geq [(a_1+b_1+c_1)(a_2+b_2+c_2)(a_3+b_3+c_3)]^{1/3}$$

$$\Rightarrow L^3 \geq (a_1+b_1+c_1)(a_2+b_2+c_2)(a_3+b_3+c_3)$$

$$= a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 + 24 \text{ more such terms}$$

$$\geq a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 \quad (\because a_r, b_r, c_r \geq \text{or } r = 1, 2, 3)$$

$$\geq (a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2) - (a_1b_3c_2 + a_2b_1c_2 + a_3b_2c_1) \quad \text{(same reason)}$$

$$= V \text{ (from (i))}$$

Thus, $L^3 \geq V$

16. We know that $[\vec{x}\times\vec{y}\ \vec{y}\times\vec{z}\ \vec{z}\times\vec{x}] = [\vec{x}\ \vec{y}\ \vec{z}]^2$

Also a vector along the bisector of given two unit vectors $\vec{u}, \vec{v}$ is $\vec{u}+\vec{v}$.

A unit vector along the bisector is $\dfrac{\vec{u}+\vec{v}}{|\vec{u}+\vec{v}|}$

$$|\vec{u}+\vec{v}|^2 = 1+1+2\vec{u}\cdot\vec{v} = 2+2\cos\alpha = 4\cos^2\frac{\alpha}{2}$$

$$\Rightarrow \vec{x} = \frac{\vec{u}+\vec{v}}{2\cos\frac{\alpha}{2}}$$

Similarly, $\vec{y} = \dfrac{\vec{v}+\vec{w}}{2\cos\beta/2}$ and $\vec{z} = \dfrac{\vec{u}+\vec{w}}{2\cos\gamma/2}$

$$\Rightarrow [\vec{x}\ \vec{y}\ \vec{z}] = \frac{1}{8}[\vec{u}+\vec{v}\ \ \vec{v}+\vec{w}\ \ \vec{u}+\vec{w}] \sec\frac{\alpha}{2}\sec\frac{\beta}{2}\sec\frac{\gamma}{2}$$

$$= \frac{1}{8}2[\vec{u}\ \vec{v}\ \vec{w}]\sec\frac{\alpha}{2}\sec\frac{\beta}{2}\sec\frac{\gamma}{2}$$

$$= \frac{1}{4}[\vec{u}\ \vec{v}\ \vec{w}]\sec\frac{\alpha}{2}\sec\frac{\beta}{2}\sec\frac{\gamma}{2}$$

$$\Rightarrow [\vec{x}\times\vec{y}\ \vec{y}\times\vec{z}\ \vec{z}\times\vec{x}] = [\vec{x}\ \ \vec{y}\ \ \vec{z}]^2$$

$$= \frac{1}{16}[\vec{u}\ \vec{v}\ \vec{w}]^2\sec^2\frac{\alpha}{2}\sec^2\frac{\beta}{2}\sec^2\frac{\gamma}{2}$$

17. Given that $\vec{a}\times\vec{c} = \vec{b}\times\vec{d}$ (i)

and $\vec{a}\times\vec{b} = \vec{c}\times\vec{d}$ (ii)

Subtracting (ii) from (i), we get

$$\vec{a}\times(\vec{c}-\vec{b}) = (\vec{b}-\vec{c})\times\vec{d}$$

$\Rightarrow \vec{a}\times(\vec{c}-\vec{b}) = \vec{d}\times(\vec{c}-\vec{b})$

$\Rightarrow \vec{a}\times(\vec{c}-\vec{b}) - \vec{d}\times(\vec{c}-\vec{b}) = 0$

$\Rightarrow (\vec{a}-\vec{d})\times(\vec{c}-\vec{d}) = 0$

$\Rightarrow (\vec{a}-\vec{d}) \parallel (\vec{c}-\vec{d}) \quad (\because \vec{a}-\vec{d}\neq 0, \vec{c}-\vec{b}\neq 0)$

$\Rightarrow$ Angle between $\vec{a}-\vec{d}$ and $\vec{c}-\vec{b}$ is either 0 or 180°.

$\Rightarrow (\vec{a}-\vec{d})\cdot(\vec{c}-\vec{b}) = |\vec{a}-\vec{d}||\vec{c}-\vec{b}|\cos 0 \neq 0$ as $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ all are different.

18. The following figure shows the possible situation for planes P_1 and P_2 and the lines L_1 and L_2:

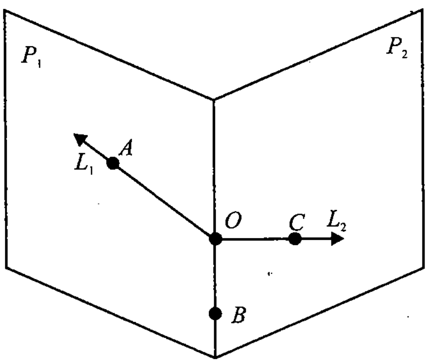


Fig. 2.52

Now if we choose points A, B and C as A on L_1, B on the line of intersection of P_1 and P_2 but other than the origin and C on L_2 again other than the origin, then we can consider

A corresponds to one of A', B', C'

B corresponds to one of the remaining of A', B' and C'

C corresponds to third of A', B' and C', e.g., $A' \equiv C$; $B' \equiv B$; $C' \equiv A$

Hence one permutation of [A B C] is [CBA]. Hence proved.

19. Given that the incident ray is along $\hat{v}$, the reflected ray is along $\hat{w}$ and the normal is along $\hat{a}$, outwards. The given figure can be redrawn as shown.

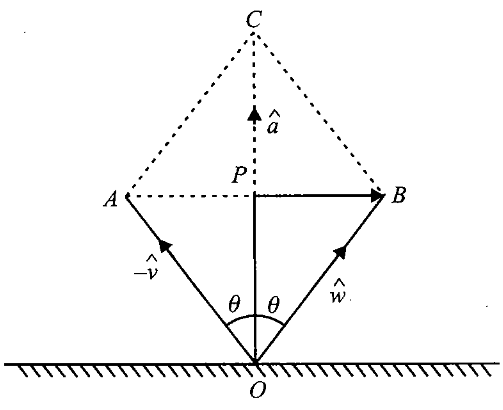


Fig. 2.53

We know that the incident ray, the reflected ray, and the normal lie in a plane, and the angle of incidence = angle of reflection.

Therefore, $\hat{a}$ will be along the angle bisector of $\hat{w}$ and $-\hat{v}$, i.e.,

$$\hat{a} = \frac{\hat{w} + (-\hat{v})}{|\hat{w} - \hat{v}|} \qquad \text{(i)}$$

But $\hat{a}$ is a unit vector

where $|\hat{w} - \hat{v}| = OC = 2OP$

$= 2|\hat{w}|\cos\theta = 2\cos\theta$

Substituting this value in (i),

$$\hat{a} = \frac{\hat{w} - \hat{v}}{2\cos\theta}$$

$\Rightarrow \hat{w} = \hat{v} + (2\cos\theta)\hat{a}$

$\Rightarrow \hat{a} = \hat{v} - 2(\hat{a}\cdot\hat{v})\hat{a} \quad (\hat{a}\cdot\hat{v} = -\cos\theta)$

Objective Type

Fill in the blanks

1. Given that $|\vec{A}| = 3$; $|\vec{B}| = 4$; $|\vec{C}| = 5$

$\vec{A} \perp (\vec{B} + \vec{C}) \Rightarrow \vec{A}\cdot(\vec{B} + \vec{C}) = 0 \Rightarrow \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C} = 0$ (i)

$\vec{B} \perp (\vec{C} + \vec{A}) \Rightarrow \vec{B}\cdot(\vec{C} + \vec{A}) = 0 \Rightarrow \vec{B}\cdot\vec{C} + \vec{B}\cdot\vec{A} = 0$ (ii)

$\vec{C} \perp (\vec{A} + \vec{B}) \Rightarrow \vec{C}\cdot(\vec{A} + \vec{B}) = 0 \Rightarrow \vec{C}\cdot\vec{A} + \vec{C}\cdot\vec{B} = 0$ (iii)

Adding (i), (ii) and (iii), we get

$2(\vec{A}\cdot\vec{B} + \vec{B}\cdot\vec{C} + \vec{C}\cdot\vec{A}) = 0$ (iv)

Now, $|\vec{A} + \vec{B} + \vec{C}|^2$

$= (\vec{A} + \vec{B} + \vec{C})\cdot(\vec{A} + \vec{B} + \vec{C})$

$= |\vec{A}|^2 + |\vec{B}|^2 + |\vec{C}|^2 + 2(\vec{A}\cdot\vec{B} + \vec{B}\cdot\vec{C} + \vec{C}\cdot\vec{A})$

$= 9 + 16 + 25 + 0$

$= 50$

$\therefore |\vec{A} + \vec{B} + \vec{C}| = 5\sqrt{2}$

2. Required unit vector

$$\hat{a} = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{|\overrightarrow{PQ} \times \overrightarrow{PR}|}$$

$\overrightarrow{PQ} = \hat{i} + \hat{j} - 3\hat{k}$; $\overrightarrow{PR} = -\hat{i} + 3\hat{j} - \hat{k}$

$$\therefore \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -3 \\ -1 & 3 & -1 \end{vmatrix}$$

$$= 8\hat{i} + 4\hat{j} + 4\hat{k}$$

$$\therefore |\overrightarrow{PQ} \times \overrightarrow{PR}| = \sqrt{64+16+16} = \sqrt{96} = 4\sqrt{6}$$

$$\therefore \hat{n} = \frac{8\hat{i} + 4\hat{j} + 4\hat{k}}{4\sqrt{6}} = \frac{2\hat{i} + \hat{j} + \hat{k}}{\sqrt{6}}$$

3. Area of $\Delta ABC = \frac{1}{2} |\overrightarrow{BA} \times \overrightarrow{BC}|$

$\overrightarrow{BA} = -\hat{i} - 2\hat{j} + 3\hat{k}$

$\overrightarrow{BC} = \hat{i} - 2\hat{j} + 3\hat{k}$

$$\therefore \text{Area} = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -2 & 3 \\ 1 & -2 & 3 \end{vmatrix} = \frac{1}{2} |6\hat{j} + 4\hat{k}|$$

$$= |3\hat{j} + 2\hat{k}|$$

$$= \sqrt{9+4} = \sqrt{13}$$

4. $$\frac{\vec{A}\cdot\vec{B}\times\vec{C}}{\vec{C}\times\vec{A}\cdot\vec{B}} + \frac{\vec{B}\cdot\vec{A}\times\vec{C}}{\vec{C}\cdot\vec{A}\times\vec{B}}$$

$$= \frac{[\vec{A}\,\vec{B}\,\vec{C}]}{[\vec{A}\,\vec{B}\,\vec{C}]} + \frac{-[\vec{A}\,\vec{B}\,\vec{C}]}{[\vec{A}\,\vec{B}\,\vec{C}]} = 0$$

5. Given $\vec{A} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{C} = \hat{j} - \hat{k}$

Let $\vec{B} = x\hat{i} + y\hat{j} + z\hat{k}$

Given that $\vec{A} \times \vec{B} = \vec{C} \Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix} = \hat{j} - \hat{k}$

$\Rightarrow (z-y)\,i+(x-z)\,\hat{j}+(y-x)\,\hat{k} = \hat{j}-\hat{k}$

$\Rightarrow z-y=0, x-z=1$ and $y-x=-1$ (i)

Also, $\vec{A}\cdot\vec{B} = 3$

$\Rightarrow x+y+z=3$ (ii)

Using (i) and (ii), we get

$y=2/3, x=5/3, z=2/3$

$\therefore\ \vec{B} = \frac{5}{3}\hat{i}+\frac{2}{3}\hat{j}+\frac{2}{3}\hat{k}$

6. Let $\vec{c} = \alpha\,\hat{i}+\beta\,\hat{j}$

Given that $\vec{b}\perp\vec{c}$

$\therefore\ \vec{b}\cdot\vec{c} = 0.$

$\Rightarrow (4\hat{i}+3\hat{j})\cdot(\alpha\hat{i}+\beta\,\hat{j}) = 0$

$\Rightarrow 4\,\alpha+3\beta=0$

$\Rightarrow \frac{\alpha}{3} = \frac{\beta}{-4} = \lambda$

$\Rightarrow \alpha=3\,\lambda, \beta=-4\,\lambda$ (i)

Now let $\vec{a} = x\hat{i}+y\,\hat{j}$ be the required vectors.

Given that projection of $\vec{a}$ along $\vec{b}$

$= \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

$= \frac{4x+3y}{\sqrt{4^2+3^2}} = 1$

$\Rightarrow 4x+3y=5$ (ii)

Also projection of $\vec{a}$ along $\vec{c}$

$\Rightarrow \frac{\vec{a}\cdot\vec{c}}{|\vec{c}|} = 2$

$\Rightarrow \frac{\alpha x+\beta y}{\sqrt{\alpha^2+\beta^2}} = 2$

$\Rightarrow 3\lambda x-4\lambda y = 10\,\lambda$

$\Rightarrow 3x-4y=10$ (iii)

Solving (ii) and (iii), we get $x=2, y=-1$

$\therefore$ The required vector is $2\hat{i}-\hat{j}$.

7.

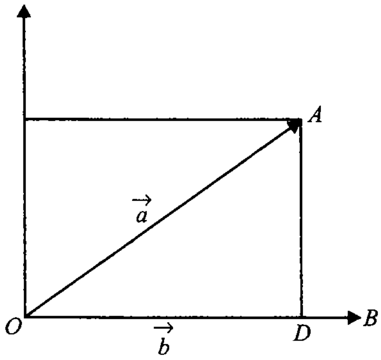


Fig. 2.54

Component of $\vec{a}$ along $\vec{b}$

$$\overrightarrow{OD} = OA\cos\theta\cdot\frac{\vec{b}}{|\vec{b}|}$$

$$= \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right)\frac{\vec{b}}{|\vec{b}|} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}$$

Component of $\vec{a}$ perpendicular to $\vec{b}$

$$\overrightarrow{DA} = \vec{a} - \overrightarrow{OD}$$

$$= \vec{a} - \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}$$

8. Let $x\hat{i} + y\hat{j} + z\hat{k}$ be a unit vector coplanar with $\hat{i} + \hat{j} + 2\hat{k}$ and $\hat{i} + 2\hat{j} + \hat{k}$ and also perpendicular to $\hat{i} + \hat{j} + \hat{k}$

Then, $\begin{vmatrix} x & y & z \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix} = 0$

$\Rightarrow -3x + y + z = 0$ (i)

and $x + y + z = 0$ (ii)

Solving the above by cross-product method, we get $\frac{x}{0} = \frac{y}{4} = \frac{z}{-4}$ or $\frac{x}{0} = \frac{y}{1} = \frac{z}{-1} = \lambda$ (say)

$\Rightarrow x = 0, y = \lambda, z = -\lambda$

As $x\hat{i} + y\hat{j} + z\hat{k}$ is a unit vector,

$\Rightarrow 0 + \lambda^2 + \lambda^2 = 1$

$\Rightarrow \lambda^2 = \frac{1}{2} \Rightarrow \lambda = \pm \frac{1}{\sqrt{2}}$

$\therefore$ The required vector is $\frac{\hat{j}-\hat{k}}{\sqrt{2}}$ or $\frac{-\hat{j}+\hat{k}}{\sqrt{2}}$.

9. A vector normal to the plane containing vectors $\hat{i}$ and $\hat{i}+\hat{j}$ is

$$\vec{p} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix} = \hat{k}$$

A vector normal to the plane containing vectors $\hat{i}-\hat{j}, \hat{i}+\hat{k}$ is

$$\vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = -\hat{i}-\hat{j}+\hat{k}.$$

Vector $\vec{a}$ is parallel to vector $\vec{p}\times\vec{q}$.

$$\vec{p}\times\vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ -1 & -1 & 1 \end{vmatrix} = \hat{i}-\hat{j}$$

$\therefore$ A vector in direction of $\vec{a}$ is $\hat{i}-\hat{j}$

Now if θ is the angle between $\vec{a}$ and $\hat{i} \;\; -2\hat{j}+2\hat{k}$, then

$$\cos\theta = \pm \frac{1\cdot 1+(-1)\cdot(-2)}{\sqrt{1+1}\;\sqrt{1+4+4}} = \pm \frac{3}{\sqrt{2}\cdot 3}$$

$$\Rightarrow \cos\theta = \pm \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4} \text{ or } \frac{3\pi}{4}$$

10. Let $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ be any three mutually perpendicular non-coplanar unit vectors and $\vec{a}$ be any vector, then

$$\vec{a} = (\vec{a}\cdot\vec{\alpha})\vec{\alpha} + (\vec{a}\cdot\vec{\beta})\vec{\beta} + (\vec{a}\cdot\vec{\gamma})\vec{\gamma}$$

Here $\vec{b}, \vec{c}$ are two mutually perpendicular vectors, therefore $\vec{b}, \vec{c}$ and $\frac{\vec{b}\times\vec{c}}{|\vec{b}\times\vec{c}|}$ are three mutually perpendicular non-coplanar unit vectors.

Hence $\vec{a} = \left(\vec{a}.\vec{b}\right)\vec{b} + \left(\vec{a}.\vec{c}\right)\vec{c} + \left(\vec{a}.\frac{\vec{b}\times\vec{c}}{|\vec{b}\times\vec{c}|}\right)\frac{\vec{b}\times\vec{c}}{|\vec{b}\times\vec{c}|}$

$$= \left(\vec{a}.\vec{b}\right)\vec{b} + \left(\vec{a}.\vec{c}\right)\vec{c} + \frac{\vec{a}.\left(\vec{b}\times\vec{c}\right)}{|\vec{b}\times\vec{c}|^2}\left(\vec{b}\times\vec{c}\right)$$

11. $\vec{a}\times(\vec{a}\times\vec{c})+\vec{b}=\vec{0}$

$\Rightarrow (\vec{a}\cdot\vec{c})\,\vec{a}-(\vec{a}\cdot\vec{a})\,\vec{c}+\vec{b}=\vec{0}$

$\Rightarrow 2\cos\theta\cdot\vec{a}-\vec{c}+\vec{b}=\vec{0}$ (using $|\vec{a}|=1, |\vec{b}|=1, |\vec{c}|=2$)

$\Rightarrow (2\cos\theta\,\vec{a}-\vec{c})^2=(-\vec{b})^2$

$\Rightarrow 4\cos^2\theta\cdot|\vec{a}|^2+|\vec{c}|^2-2\cdot 2\cos\theta\cdot\vec{a}\cdot\vec{c}=|\vec{b}|^2$

$\Rightarrow 4\cos^2\theta+4-8\cos\theta\cdot\cos\theta=1$

$\Rightarrow 4\cos^2\theta-8\cos^2\theta+4=1$

$\Rightarrow 4\cos^2\theta=3$

$\Rightarrow \cos\theta=\pm\sqrt{3}/2$

For θ to be acute, $\cos\theta=\dfrac{\sqrt{3}}{2}\Rightarrow\theta=\dfrac{\pi}{6}$

12. Given that $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are position vectors of points A, B, C and D, respectively, such that

$(\vec{a}-\vec{d})\cdot(\vec{b}-\vec{c})=(\vec{b}-\vec{d})\cdot(\vec{c}-\vec{a})=0$

$\Rightarrow \overrightarrow{DA}\cdot\overrightarrow{CB}=\overrightarrow{DB}\cdot\overrightarrow{AC}=0$

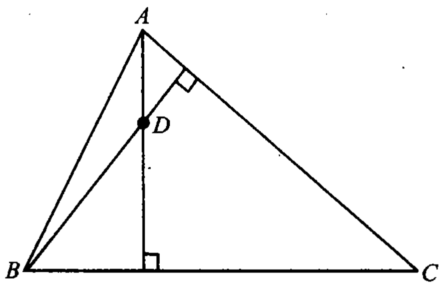


Fig. 2.55

$\Rightarrow \overrightarrow{DA}\perp\overrightarrow{CB}$ and $\overrightarrow{DB}\perp\overrightarrow{AC}$

Clearly, D is the orthocentre of ΔABC.

13. q = area of parallelogram with $\overrightarrow{OA}$ and $\overrightarrow{OC}$ as adjacent sides

$=|\overrightarrow{OA}\times\overrightarrow{OC}|$

$=|\vec{a}\times\vec{b}|$

p = area of quadrilateral $OABC$

$=\dfrac{1}{2}|\overrightarrow{OA}\times\overrightarrow{OB}|+\dfrac{1}{2}|\overrightarrow{OB}\times\overrightarrow{OC}|=\dfrac{1}{2}[|\vec{a}\times(10\vec{a}+2\vec{b})|+|(10\vec{a}+2\vec{b})\times\vec{b}|]$

$=\dfrac{1}{2}|(12\,\vec{a}\times\vec{b})|=6|\vec{a}\times\vec{b}|\Rightarrow k=6$

14. $\vec{a}\cdot\vec{b} = -1 + 3 = 2$

$|\vec{a}| = 2, |\vec{b}| = 2$

$\cos\theta = \dfrac{2}{2\times 2} = \dfrac{1}{2}$

$\theta = \dfrac{\pi}{3}$ but its value is $\dfrac{2\pi}{3}$ as it is opposite to the side of maximum length.

True or false

1. $\vec{A}, \vec{B}$ and $\vec{C}$ are three unit vectors such that $\vec{A}\cdot\vec{B} = \vec{A}\cdot\vec{C} = 0$ (i) and the angle between $\vec{B}$ and $\vec{C}$ is $\pi/3$. Now Eq. (i) shows that $\vec{A}$ is perpendicular to both $\vec{B}$ and $\vec{C}$.

$\Rightarrow \vec{B}\times\vec{C} = \lambda\vec{A}$, where λ is any scalar.

$\Rightarrow |\vec{B}\times\vec{C}| = |\lambda\vec{A}|$

$\Rightarrow \sin\pi/3 = \pm\lambda$ (as $\pi/3$ is the angle between $\vec{B}$ and $\vec{C}$)

$\Rightarrow \lambda = \pm\sqrt{3}/2$

$\Rightarrow \vec{B}\times\vec{C} = \pm\dfrac{\sqrt{3}}{2}\vec{A}$

$\Rightarrow \vec{A} = \pm\dfrac{2}{\sqrt{3}}(\vec{B}\times\vec{C})$

Therefore, the given statement is false.

2. $\vec{X}\cdot\vec{A} = 0 \Rightarrow$ either $\vec{A} = 0$ or $\vec{X}\perp\vec{A}$

$\vec{X}\cdot\vec{B} = 0 \Rightarrow$ either $\vec{B} = 0$ or $\vec{X}\perp\vec{B}$

$\vec{X}\cdot\vec{C} = 0 \Rightarrow$ either $\vec{C} = 0$ $\vec{X}\perp\vec{C}$

In any of the three cases, $\vec{A}, \vec{B}, \vec{C} = 0 \Rightarrow [\vec{A}\,\vec{B}\,\vec{C}] = 0$

Otherwise if $\vec{X}\perp\vec{A}$, $\vec{X}\perp\vec{B}$ and $\vec{X}\perp\vec{C}$, then $\vec{A}$, $\vec{B}$ and $\vec{C}$ are coplanar.

$\Rightarrow [\vec{A}\,\vec{B}\,\vec{C}] = 0$

Therefore, the statement is true.

3. Clearly vectors $\vec{a}-\vec{b}, \vec{b}-\vec{c}, \vec{c}-\vec{a}$ are coplanar

$\Rightarrow [\vec{a}-\vec{b}\;\; \vec{b}-\vec{c}\;\; \vec{c}-\vec{a}] = 0$

Therefore, the given statement is false.

Multiple choice questions with one correct answer

1. **a.** $\vec{A}\cdot(\vec{B}+\vec{C})\times(\vec{A}+\vec{B}+\vec{C}) = \vec{A}\cdot[\vec{B}\times\vec{A}+\vec{B}\times\vec{B}+\vec{B}\times\vec{C}+\vec{C}\times\vec{A}+\vec{C}\times\vec{B}+\vec{C}\times\vec{C}]$

$= \vec{A}\cdot\vec{B}\times\vec{A}+\vec{A}\cdot\vec{B}\times\vec{C}+\vec{A}\cdot\vec{C}\times\vec{A}+\vec{A}\cdot\vec{C}\times\vec{B}$ (using $\vec{a}\times\vec{a}=0$)

$=0+[\vec{A}\,\vec{B}\,\vec{C}]+0+[\vec{A}\,\vec{C}\,\vec{B}]$

$=[\vec{A}\,\vec{B}\,\vec{C}]-[\vec{A}\,\vec{B}\,\vec{C}]$

$=0$

2. **d.** $|(\vec{a}\times\vec{b})\cdot\vec{c}| = |\vec{a}||\vec{b}||\vec{c}|$

$\Rightarrow ||\vec{a}||\vec{b}|\sin\theta\,\hat{n}\cdot\vec{c}| = |\vec{a}||\vec{b}||\vec{c}|$

$\Rightarrow |\vec{a}||\vec{b}||\vec{c}||\sin\theta\cos\alpha| = |\vec{a}||\vec{b}||\vec{c}|$

$\Rightarrow |\sin\theta||\cos\alpha|=1$

$\Rightarrow \theta=\pi/2$ and $\alpha=0$

$\Rightarrow \vec{a}\perp\vec{b}$ and $\vec{c}\,||\,\hat{n}$ or perpendicular to both $\vec{a}$ and $\vec{b}$

$\Rightarrow \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a} = 0$

3. **d.** Volume of parallelopiped $= [\vec{a}\,\vec{b}\,\vec{c}]$

$$=\begin{vmatrix}2 & -2 & 0\\ 1 & 1 & -1\\ 3 & 0 & -1\end{vmatrix} = 2(-1)+2(-1+3)=2$$

4. **d.** Given that $\vec{a},\vec{b},\vec{c}$ are non-coplanar. Therefore,

$[\vec{a}\,\vec{b}\,\vec{c}]\neq 0$

Also $\vec{p}=\dfrac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]},\ \vec{q}=\dfrac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]},\ \vec{r}=\dfrac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$ (i)

Now, $(\vec{a}+\vec{b})\cdot\vec{p}+(\vec{b}+\vec{c})\cdot\vec{q}+(\vec{c}+\vec{a})\cdot\vec{r}$

$= (\vec{a}+\vec{b})\cdot\dfrac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}+(\vec{b}+\vec{c})\cdot\dfrac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}+(\vec{c}+\vec{a})\cdot\dfrac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$

$= \dfrac{\vec{a}\cdot\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}+\dfrac{\vec{b}\cdot\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}+\dfrac{\vec{c}\cdot\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$ $[\because \vec{b}\cdot\vec{b}\times\vec{c} = \vec{c}\cdot\vec{c}\times\vec{a} = \vec{a}\cdot\vec{a}\times\vec{b} = 0]$

$= \dfrac{[\vec{a}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]}+\dfrac{[\vec{a}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]}+\dfrac{[\vec{a}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]}$

$=1+1+1$

$=3$

5. **a.** Let $\vec{d} = x\hat{i} + y\hat{j} + z\hat{k}$

where $x^2 + y^2 + z^2 = 1$ (i)

($\vec{d}$ being a unit vector)

$\therefore\ \vec{a}.\vec{d} = 0$

$\Rightarrow x - y = 0 \Rightarrow x = y$ (ii)

$[\vec{b}\ \vec{c}\ \vec{d}] = 0$

$$\Rightarrow \begin{vmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ x & y & z \end{vmatrix} = 0$$

$\Rightarrow x + y + z = 0$

$\Rightarrow 2x + z = 0$ (using (ii))

$\Rightarrow z = -2x$ (iii)

From (i), (ii) and (iii)

$x^2 + x^2 + 4x^2 = 1$

$x = \pm \frac{1}{\sqrt{6}}$

$$\therefore\ \vec{d} = \pm\left(\frac{1}{\sqrt{6}}\hat{i} + \frac{1}{\sqrt{6}}\hat{j} - \frac{2}{\sqrt{6}}\vec{k}\right) = \pm\left(\frac{\hat{i} + \hat{j} - 2\hat{k}}{\sqrt{6}}\right)$$

6. **a.** Since $\vec{a} \times (\vec{b} \times \vec{c}) = \frac{\vec{b} + \vec{c}}{\sqrt{2}}$

$$\therefore\ (\vec{a}\cdot\vec{c})\,\vec{b} - (\vec{a}\cdot\vec{b})\,\vec{c} = \frac{1}{\sqrt{2}}\vec{b} + \frac{1}{\sqrt{2}}\vec{c}$$

Since $\vec{b}$ and $\vec{c}$ are non-coplanar

$\Rightarrow \vec{a}\cdot\vec{c} = \frac{1}{\sqrt{2}}$ and $\vec{a}\cdot\vec{b} = -\frac{1}{\sqrt{2}}$

$\Rightarrow \cos\theta = -\frac{1}{\sqrt{2}}$ (because $\vec{a}$ and $\vec{b}$ are unit vectors)

$\Rightarrow \theta = \frac{3\pi}{4}$

7. **b.** Since $\vec{u} + \vec{v} + \vec{w} = 0$,

$|\vec{u} + \vec{v} + \vec{w}|^2 = 0$

$\Rightarrow |\vec{u}|^2 + |\vec{v}|^2 + |\vec{w}|^2 + 2(\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{w} + \vec{w}\cdot\vec{u}) = 0$

$\Rightarrow 9 + 16 + 25 + 2\,(\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{w} + \vec{w}\cdot\vec{u}) = 0$

$\Rightarrow \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{w} + \vec{w}\cdot\vec{u} = -25$

8. **d.** $(\vec{a}+\vec{b}+\vec{c})\cdot[(\vec{a}+\vec{b})\times(\vec{a}+\vec{c})]$

$= (\vec{a}+\vec{b}+\vec{c})\cdot[\vec{a}\times\vec{a}+\vec{a}\times\vec{c}+\vec{b}\times\vec{a}+\vec{b}\times\vec{c}]$

$= (\vec{a}+\vec{b}+\vec{c})\cdot[\vec{a}\times\vec{c}+\vec{b}\times\vec{a}+\vec{b}\times\vec{c}]$

$= \vec{a}\cdot\vec{b}\times\vec{c}+\vec{b}\cdot\vec{a}\times\vec{c}+\vec{c}\cdot\vec{b}\times\vec{a}$

$= [\vec{a}\,\vec{b}\,\vec{c}]-[\vec{a}\,\vec{b}\,\vec{c}]-[\vec{a}\,\vec{b}\,\vec{c}]$

$= -[\vec{a}\,\vec{b}\,\vec{c}]$

9. **b.** As $\vec{p}, \vec{q}$ and $\vec{r}$ are three mutually perpendicular vectors of same magnitude, so let us consider

$\vec{p}=a\hat{i}, \vec{q}=a\hat{j}, \vec{r}=a\hat{k}$

Also let $\vec{x}=x_1\hat{i}+y_1\hat{j}+z_1\hat{k}$

Given that $\vec{x}$ satisfies the equation

$$\vec{p}\times[(\vec{x}-\vec{q})\times\vec{p}]+\vec{q}\times[(\vec{x}-\vec{r})\times\vec{q}]+\vec{r}\times[(\vec{x}-\vec{p})\times\vec{r}]=0 \quad \text{(i)}$$

Now $\vec{p}\times[(\vec{x}-\vec{q})\times\vec{p}] = \vec{p}\times[\vec{x}\times\vec{p}-\vec{q}\times\vec{p}]$

$= \vec{p}\times(\vec{x}\times\vec{p})-\vec{p}\times(\vec{q}\times\vec{p})$

$= (\vec{p}\cdot\vec{p})\vec{x}-(\vec{p}\cdot\vec{x})\vec{p}-(\vec{p}\cdot\vec{p})\vec{q}+(\vec{p}\cdot\vec{q})\vec{p}$

$= a^2\vec{x}-a^2x_1\hat{i}-a^3\hat{j}+0$

Similarly,

$\vec{q}\times[(\vec{x}-\vec{r})\times\vec{q}] = a^2\vec{x}-a^2y_1\hat{j}-a^3\hat{k}$

and $\vec{r}\times[(\vec{x}-\vec{p})\times\vec{r}]=a^2\vec{x}-a^2z_1\hat{k}-a^3\hat{i}$

Substituting these values in the equation, we get

$3a^2\vec{x}-a^2(x_1\hat{i}+y_1\hat{j}+z_1\hat{k})-a^2(a\hat{i}+a\hat{j}+a\hat{k})=0$

$\Rightarrow 3a^2\vec{x}-a^2\vec{x}-a^2(\vec{p}+\vec{q}+\vec{r})=\vec{0}$

$\Rightarrow 2a^2\vec{x}=(\vec{p}+\vec{q}+\vec{r})\,a^2$

$\Rightarrow \vec{x}=\frac{1}{2}(\vec{p}+\vec{q}+\vec{r})$

10. **b.** $|(\vec{a}\times\vec{b})\times\vec{c}| = |\vec{a}\times\vec{b}||\vec{c}|\sin 30°$

$$= \frac{1}{2}|\vec{a}\times\vec{b}||\vec{c}| \quad \text{(i)}$$

We have, $\vec{a} = 2\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = \hat{i} + \hat{j}$

$\Rightarrow \vec{a} \times \vec{b} = 2\hat{i} - 2\hat{j} + \hat{k}$

$\Rightarrow |\vec{a} \times \vec{b}| = \sqrt{9} = 3$

Also given $|\vec{c} - \vec{a}| = 2\sqrt{2}$

$\Rightarrow |\vec{c} - \vec{a}|^2 = 8$

$\Rightarrow |\vec{c}|^2 + |\vec{a}|^2 - 2\,\vec{a}\cdot\vec{c} = 8$

Given $|\vec{a}| = 3$ and $\vec{a}\cdot\vec{c} = |\vec{c}|$, using these we get

$|\vec{c}|^2 - 2|\vec{c}| + 1 = 0$

$\Rightarrow (|\vec{c}| - 1)^2 = 0$

$\Rightarrow |\vec{c}| = 1$

Substituting values of $|\vec{a} \times \vec{b}|$ and $|\vec{c}|$ in (i), we get

$$|(\vec{a} \times \vec{b}) \times \vec{c}| = \frac{1}{2} \times 3 \times 1 = \frac{3}{2}$$

11. **a.** As $\vec{c}$ is coplanar with $\vec{a}$ and $\vec{b}$, we take $\vec{c} = \alpha\,\vec{a} + \beta\,\vec{b}$ (i)

where α and β are scalars.

As $\vec{c}$ is perpendicular to $\vec{a}$, using (i), we get,

$0 = \alpha\,\vec{a}\cdot\vec{a} + \beta\,\vec{b}\cdot\vec{a}$

$\Rightarrow 0 = \alpha(6) + \beta(2 + 2 - 1) = 3\,(2\alpha + \beta)$

$\Rightarrow \beta = -2\alpha$

Thus, $\vec{c} = \alpha\,(\vec{a} - 2\vec{b}) = \alpha\,(-3j + 3k) = 3\alpha\,(-j + k)$

$\Rightarrow |\vec{c}|^2 = 18\alpha^2$

$\Rightarrow 1 = 18\alpha^2$

$\Rightarrow \alpha = \pm\dfrac{1}{3\sqrt{2}}$

$\therefore\ \vec{c} = \pm\dfrac{1}{\sqrt{2}}\,(-j + k)$

12. **b.** Given $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ (by triangle law). Therefore,

$\vec{a} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{a} \times \vec{0}$

$\vec{a} \times \vec{a} + \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = \vec{0}$

$\vec{a} \times \vec{b} = \vec{c} \times \vec{a}$

Similarly by taking cross product with $\vec{b}$, we get $\vec{a} \times \vec{b} = \vec{b} \times \vec{c}$

$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

13. **a.** Given that $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are vectors such that $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{0}$ (i)

P_1 is the plane determined by vectors $\vec{a}$ and $\vec{b}$. Therefore, normal vectors $\vec{n_1}$ to P_1 will be given by

$\vec{n_1} = \vec{a} \times \vec{b}$

Similarly, P_2 is the plane determined by vectors $\vec{c}$ and $\vec{d}$. Therefore, normal vectors $\vec{n_2}$ to P_2 will be given by

$\vec{n_2} = \vec{c} \times \vec{d}$

Substituting the values of $\vec{n_1}$ and $\vec{n_2}$ in (i), we get

$\vec{n_1} \times \vec{n_2} = \vec{0}$

Hence, $\vec{n_1} \,||\, \vec{n_2}$

Hence, the planes will also be parallel to each other.
Thus angle between the planes = 0.

14. **a.** $\vec{a}, \vec{b}$ and $\vec{c}$ are unit coplanar vectors, $2\vec{a} - \vec{b}, 2\vec{b} - 2\vec{c}$ and $2\vec{c} - \vec{a}$ are also coplanar vectors, being linear combination of $\vec{a}, \vec{b}$ and $\vec{c}$.

Thus, $[2\vec{a} - \vec{b} \;\; 2\vec{b} - \vec{c} \;\; 2\vec{c} - \vec{a}] = 0$

15. **b.** $\hat{a}, \hat{b}$ and $\hat{c}$ are unit vectors.

Now $x = |\hat{a} - \hat{b}|^2 + |\hat{b} - \hat{c}|^2 + |\hat{c} - \hat{a}|^2$

$= \frac{1}{2}(\hat{a}\cdot\hat{a} + \hat{b}\cdot\hat{b} + \hat{c}\cdot\hat{c}) - 2\hat{a}\cdot\hat{b} - 2\hat{b}\cdot\hat{c} - 2\hat{c}\cdot\hat{a}$

$\Rightarrow 6 - 2(\hat{a}\cdot\hat{b} + \hat{b}\cdot\hat{c} + \hat{c}\cdot\hat{a})$ (i)

Also, $|\hat{a} + \hat{b} + \hat{c}| \geq 0$

$\Rightarrow \hat{a}\cdot\hat{a} + \hat{b}\cdot\hat{b} + \hat{c}\cdot\hat{c} + 2(\hat{a}\cdot\hat{b} + \hat{b}\cdot\hat{c} + \hat{c}\cdot\hat{a}) \geq 0$

$\Rightarrow 3 + 2(\hat{a}\cdot\hat{b} + \hat{b}\cdot\hat{c} + \hat{c}\cdot\hat{a}) \geq 0$

$\Rightarrow 2(\hat{a}\cdot\hat{b} + \hat{b}\cdot\hat{c} + \hat{c}\cdot\hat{a}) \geq -3$

$\Rightarrow -2(\hat{a}\cdot\hat{b} + \hat{b}\cdot\hat{c} + \hat{c}\cdot\hat{a}) \leq 3$

$\Rightarrow 6 - 2(\hat{a}\cdot\hat{b} + \hat{b}\cdot\hat{c} + \hat{c}\cdot\hat{a}) \leq 9$ (ii)

From (i) and (ii), $x \leq 9$

Therefore, x does not exceed 9.

16. **b.** Given that $\vec{a}$ and $\vec{b}$ are two unit vectors.

$\therefore |\vec{a}| = 1$ and $|\vec{b}| = 1$

Also given that $(\vec{a} + 2\vec{b})\cdot(5\vec{a} - 4\vec{b}) = 0$

$\Rightarrow 5|\vec{a}|^2 - 8|\vec{b}|^2 - 4\vec{a}\cdot\vec{b} + 10\vec{b}\cdot\vec{a} = 0$

$\Rightarrow 5 - 8 + 6\vec{a}\cdot\vec{b} = 0$

$\Rightarrow 6|\vec{a}||\vec{b}|\cos\theta = 3$ (where θ is the angle between $\vec{a}$ and $\vec{b}$)

$\Rightarrow \cos\theta = 1/2$

$\Rightarrow \theta = 60°$

17. **c.** Given that $\vec{V} = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{W} = \hat{i} + 3\hat{k}$ and $\vec{U}$ is a unit vector

$|\vec{U}| = 1$

Now, $[\vec{U}\,\vec{V}\,\vec{W}] = \vec{U}\cdot(\vec{V}\times\vec{W})$

$= \vec{U}\cdot(2\hat{i} + \hat{j} - \hat{k})\times(\hat{i} + 3\hat{k})$

$= \vec{U}\cdot(3\hat{i} - 7\hat{j} - \hat{k})$

$= \sqrt{3^2 + 7^2 + 1^2}\,\cos\theta$ which is maximum when $\cos\theta = 1$

Therefore, maximum value of $[\vec{U}\,\vec{V}\,\vec{W}] = \sqrt{59}$

18. **c.** Volume of parallelopiped formed by $\vec{u} = \hat{i} + a\hat{j} + \hat{k}$, $\vec{v} = \hat{j} + a\hat{k}$, $\vec{w} = a\hat{i} + \hat{k}$ is

$$V = [\vec{u}\,\vec{v}\,\vec{w}] = \begin{vmatrix} 1 & a & 1 \\ 0 & 1 & a \\ a & 0 & 1 \end{vmatrix}$$

$= 1(1-0) - a(0-a^2) + 1(0-a)$

$= 1 + a^3 - a$

For V to be minimum, $\dfrac{dV}{da} = 0$

$\Rightarrow 3a^2 - 1 = 0$

$\Rightarrow a = \pm\dfrac{1}{\sqrt{3}}$

But $a > 0 \Rightarrow a = \dfrac{1}{\sqrt{3}}$

19. **c.** $(\vec{a}\times\vec{b})\times\vec{a} = (\vec{a}\cdot\vec{a})\vec{b} - (\vec{a}\cdot\vec{b})\vec{a}$

$(\hat{j} - \hat{k})\times(\hat{i} + \hat{j} + \hat{k}) = (\sqrt{3})^2\vec{b} - (\hat{i} + \hat{j} + \hat{k})$

$\Rightarrow 3\vec{b} = 3\hat{i} \Rightarrow \vec{b} = \hat{i}$

20. **c.** Any vector coplanar to $\vec{a}$ and $\vec{b}$ can be written as $\vec{r} = \mu\vec{a} + \lambda\vec{b}$

$\vec{r} = (\mu + 2\lambda)\,\hat{i} + (-\mu + \lambda)\,\hat{j} + (\mu + \lambda)\,\hat{k}$ since $\vec{r}$ is orthogonal to $5\hat{j} + 2\hat{j} + 6\hat{k}$

$\Rightarrow 5(\mu + 2\lambda) + 2(-\mu + \lambda) + 6(\mu + \lambda) = 0$

$\Rightarrow 9\mu + 18\lambda = 0$

$\Rightarrow \lambda = -\dfrac{1}{2}\mu$

$\therefore\ \vec{r} = \lambda(3\hat{j} - \hat{k})$

Since $\hat{r}$ is a unit vector, $\hat{r} = \dfrac{3\hat{i} - \hat{k}}{\sqrt{10}}$

21. **c.** We observe that $\vec{a}\cdot\vec{b}_1 = \vec{a}\cdot\vec{b} - \left(\dfrac{\vec{b}\cdot\vec{a}}{|\vec{a}|^2}\right)\vec{a}\cdot\vec{a} = \vec{a}\cdot\vec{b} - \vec{a}\,\vec{b} = 0$

$$\vec{a}\cdot\vec{c}_2 = \vec{a}\left(\vec{c} - \frac{\vec{c}\cdot\vec{a}}{|\vec{a}|^2}\vec{a} - \frac{\vec{c}\cdot\vec{b}_1}{|\vec{b}_1|^2}\vec{b}_1\right)$$

$$= \vec{a}\cdot\vec{c} - \frac{\vec{a}\cdot\vec{c}}{|\vec{a}|^2}|\vec{a}|^2 - \frac{\vec{c}\cdot\vec{b}_1}{|\vec{b}_1|^2}(\vec{a}\cdot\vec{b}_1)$$

$$= \vec{a}\cdot\vec{c} - \vec{a}\cdot\vec{c} - 0 \quad (\because\ \vec{a}\cdot\vec{b}_1 = 0)$$

And $\vec{b}_1\cdot\vec{c}_2 = \vec{b}_1\cdot\left(\vec{c} - \dfrac{\vec{c}\cdot\vec{a}}{|\vec{a}|^2}\vec{a} - \dfrac{\vec{c}\cdot\vec{b}_1}{|\vec{b}_1|^2}\vec{b}_1\right)$

$$= \vec{b}_1\cdot\vec{c} - \frac{(\vec{c}\cdot\vec{a})(\vec{b}_1\cdot\vec{a})}{|\vec{a}|^2} - \frac{\vec{c}\cdot\vec{b}_1}{|\vec{b}_1|^2}\vec{b}_1\cdot\vec{b}_1$$

$$= \vec{b}_1\cdot\vec{c} - 0 - b_1\cdot c \quad (\text{using } \vec{b}_1\cdot\vec{a} = 0)$$

$= 0$

22. **a.** A vector in the plane of $\vec{a}$ and $\vec{b}$ is $\vec{u} = \mu\vec{a} + \lambda\vec{b} = (\mu + \lambda)\,\hat{i} + (2\mu - \lambda)\,\hat{j} + (\mu + \lambda)\,\hat{k}$

Projection of $\vec{u}$ on $\vec{c} = \dfrac{1}{\sqrt{3}}$

$\Rightarrow \dfrac{\vec{u}\cdot\vec{c}}{|\vec{c}|} = \dfrac{1}{\sqrt{3}}$

$\Rightarrow \vec{u}\cdot\vec{c} = 1$

$\Rightarrow |\mu + \lambda + 2\mu - \lambda - \mu - \lambda| = 1$

$\Rightarrow |2\mu - \lambda| = 1$

$\Rightarrow \lambda = 2\mu \pm 1$

$\Rightarrow \vec{u} = 2\hat{i} + \hat{j} + 2\hat{k}$ or $4\hat{i} - \hat{j} + 4\hat{k}$

23. **a.** $|\overrightarrow{OP}| = |\hat{a}\cos t + \hat{b}\sin t|$

$= (\cos^2 t + \sin^2 t + 2\cos t \sin t\ \hat{a}\cdot\hat{b})^{1/2}$

$= (1 + 2\cos t \sin t\ \hat{a}\cdot\hat{b})^{1/2}$

$= (1 + \sin 2t\ \hat{a}\cdot\hat{b})^{1/2}$

$\therefore |\overrightarrow{OP}|_{max} = (1 + \hat{a}\cdot\hat{b})^{1/2}$ when $t = \pi/4$

$$\hat{u} = \frac{\hat{a}+\hat{b}}{\sqrt{2}\dfrac{|\hat{a}+\hat{b}|}{\sqrt{2}}}$$

$$= \frac{\hat{a}+\hat{b}}{|\hat{a}+\hat{b}|}$$

24. **c.** $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = 1$ is possible only when $|\vec{a}\times\vec{b}| = |\vec{c}\times\vec{d}| = 1$ and $(\vec{a}\times\vec{b}) \parallel (\vec{c}\times\vec{d})$.

Since $\vec{a}\cdot\vec{c} = \frac{1}{2}$ and if $\vec{b} \parallel \vec{d}$, then $|\vec{c}\times\vec{d}| \neq 1$

25. **b.** Angle between vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$ is given by

$$\cos\theta = \frac{\overrightarrow{AB}\cdot\overrightarrow{AD}}{|\overrightarrow{AB}|\cdot|\overrightarrow{AD}|} = \frac{-2+20+22}{\sqrt{4+100+121}\sqrt{1+4+4}} = \frac{8}{9}$$

$$\Rightarrow \cos\alpha = \cos(90^\circ - \theta) = \sin\theta = \frac{\sqrt{17}}{9}$$

26. **a.**

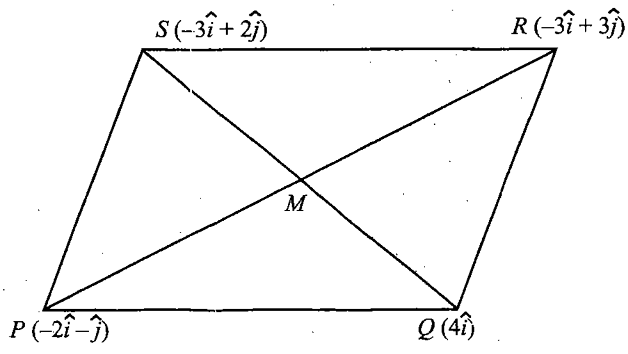


Fig. 2.56

Evaluating midpoint of PR and QS which gives $M \equiv \left[\frac{\hat{i}}{2} + \hat{j}\right]$, same for both.

$\overrightarrow{PQ} = \overrightarrow{SR} = 6\hat{i} + \hat{j}$

$\overrightarrow{PS} = \overrightarrow{QR} = -\hat{i} + 3\hat{j}$

$\Rightarrow \overrightarrow{PQ} \cdot \overrightarrow{PS} \neq 0$

$\overrightarrow{PQ} \parallel \overrightarrow{SR}, \overrightarrow{PS} \parallel \overrightarrow{QR}$ and $|\overrightarrow{PQ}| = |\overrightarrow{SR}|, |\overrightarrow{PS}| = |\overrightarrow{QR}|$

Hence, $PQRS$ is a parallelogram but not rhombus or rectangle.

27. **c.** $\vec{v} = \lambda\vec{a} + \mu\vec{b}$

$= \lambda(\hat{i} + \hat{j} + \hat{k}) + \mu(\hat{i} - \hat{j} + \hat{k})$

Projection of $\vec{v}$ on $\vec{c}$

$\frac{\vec{v} \cdot \vec{c}}{|\vec{c}|} = \frac{1}{\sqrt{3}}$

$\Rightarrow \frac{[(\lambda + \mu)\hat{i} + (\lambda - \mu)\hat{j} + (\lambda + \mu)\hat{k}] \cdot (\hat{i} - \hat{j} - \hat{k})}{\sqrt{3}} = \frac{1}{\sqrt{3}}$

$\Rightarrow \lambda + \mu - \lambda + \mu - \lambda - \mu = 1$

$\Rightarrow \mu - \lambda = 1$

$\Rightarrow \lambda = \mu - 1$

$\Rightarrow \vec{v} = (\mu - 1)(\hat{i} + \hat{j} + \hat{k}) + \mu(\hat{i} - \hat{j} + \hat{k})$

$\Rightarrow \vec{v} = (2\mu - 1)\hat{i} - \hat{j} + (2\mu - 1)\hat{k}$

At $\mu = 2$, $\vec{v} = \mathbf{3\hat{i} - \hat{j} + 3\hat{k}}$

Multiple choice questions with one or more than one correct answer

1. **c.** We are given that $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

Then $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2 = [\vec{a}\,\vec{b}\,\vec{c}]^2$

$= (\vec{a} \times \vec{b} \cdot \vec{c})^2$

$= (|\vec{a} \times \vec{b}| \cdot 1 \cos 0°)^2$ (since $\vec{c}$ is $\perp$ to $\vec{a}$ and $\vec{b}$, $\vec{c}$ is $\perp$ to $\vec{a} \times \vec{b}$)

$= (|\vec{a} \times \vec{b}|)^2$

$= \left(|\vec{a}||\vec{b}| \cdot \sin\frac{\pi}{6}\right)^2$

$= \left(\frac{1}{2}\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}\right)^2$

$= \frac{1}{4}\left(a_1^2 + a_2^2 + a_3^2\right)\left(b_1^2 + b_2^2 + b_3^2\right)$

2. **b.** We know that if $\hat{n}$ is perpendicular to $\vec{a}$ as well as $\vec{b}$, then

$$\hat{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} \text{ or } \frac{\vec{b} \times \vec{a}}{|\vec{b} \times \vec{a}|}$$

As $\vec{a} \times \vec{b}$ and $\vec{b} \times \vec{a}$ represent two vectors in opposite directions, we have two possible values of $\hat{n}$

3. **a., c.** We have $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$, $\vec{c} = \hat{i} + \hat{j} - 2\hat{k}$

Any vector in the plane of $\vec{b}$ and $\vec{c}$ is

$\vec{u} = \mu\vec{b} + \lambda\vec{c}$

$= \mu(\hat{i} + 2\hat{j} - \hat{k}) + \lambda(\hat{i} + \hat{j} - 2\hat{k})$

$= (\mu + \lambda)\hat{i} + (2\mu + \lambda)\hat{j} - (\mu + 2\lambda)\hat{k}$

Given that the magnitude of projection of $\vec{u}$ on $\vec{a}$ is $\sqrt{2/3}$

$\Rightarrow \sqrt{\frac{2}{3}} = \left|\frac{\vec{u} \cdot \vec{a}}{|\vec{a}|}\right|$

$\Rightarrow \sqrt{\frac{2}{3}} = \left|\frac{2(\mu + \lambda) - (2\mu + \lambda) - (\mu + 2\lambda)}{\sqrt{6}}\right|$

$\Rightarrow |-\lambda - \mu| = 2$

$\Rightarrow \lambda + \mu = 2$ or $\lambda + \mu = -2$

Therefore, the required vector is either $2\hat{i} + 3\hat{j} - 3\hat{k}$ or $-2\hat{i} - \hat{j} + 5\hat{k}$.

4. **c.** $[\vec{u}\ \vec{v}\ \vec{w}] = [\vec{v}\ \vec{w}\ \vec{u}] = [\vec{w}\ \vec{u}\ \vec{v}]$

but $[\vec{v}\ \vec{u}\ \vec{w}] = -[\vec{u}\ \vec{v}\ \vec{w}]$

5. **a., c.** Dot product of two vectors gives a scalar quantity.

6. **a., c.** We have $\vec{v} = \vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\, \hat{n} = \sin\theta\, \hat{n}$, where $\vec{a}$ and $\vec{b}$ are unit vectors. Therefore, $|\vec{v}| = \sin\theta$

Now, $\vec{u} = \vec{a} - (\vec{a}\cdot\vec{b})\vec{b}$

$= \vec{a} - \vec{b}\cos\theta$ (where $\vec{a}\cdot\vec{b} = \cos\theta$)

$\therefore |\vec{u}|^2 = |\vec{a} - \vec{b}\cos\theta|^2$

$= 1 + \cos^2\theta - 2\cos\theta\cdot\cos\theta$

$= 1 - \cos^2\theta = \sin^2\theta = |v|^2$

$\Rightarrow |\vec{u}| = |\vec{v}|$

Also, $\vec{u}\cdot\vec{b} = \vec{a}\cdot\vec{b} - (\vec{a}\cdot\vec{b})(\vec{b}\cdot\vec{b})$

$= \vec{a}\cdot\vec{b} - \vec{a}\cdot\vec{b}$

$= 0$

$\therefore |\vec{u}\cdot\vec{b}| = 0$

$\therefore |\vec{v}| = |\vec{u}| + |\vec{u}\cdot\vec{b}|$ is also correct.

7. **a., c., d.**

$\vec{a} = \frac{1}{3}(2\hat{i} - 2\hat{j} + \hat{k})$

$|\vec{a}|^2 = \frac{1}{9}(4+4+1) = 1 \Rightarrow |\vec{a}| = 1$

Let $\vec{b} = 2\hat{i} - 4\hat{j} + 3\hat{k}$. Then

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{5}{\sqrt{29}} \Rightarrow \theta \neq \frac{\pi}{3}$$

Let $\vec{c} = -\hat{i} + \hat{j} - \frac{1}{2}\hat{k} = \frac{-3}{2}\hat{a} \Rightarrow \vec{c} \,||\, \vec{a}$

Let $\vec{d} = 3\hat{i} + 2\hat{j} + 2\hat{k}$. Then $\vec{a}\cdot\vec{d} = 0 \Rightarrow \vec{a} \perp \vec{d}$

8. **b., d.** Normal to plane P_1 is

$\vec{n}_1 = (2\hat{j} + 3\hat{k}) \times (4\hat{j} - 3\hat{k}) = -18\hat{i}$

Normal to plane P_2 is

$\vec{n}_2 = (\hat{j} - \hat{k}) \times (3\hat{i} + 3\hat{j}) = 3\hat{i} - 3\hat{j} - 3\hat{k}$

$\therefore \vec{A}$ is parallel to $\pm(\vec{n}_1 \times \vec{n}_2) = \pm(-54\hat{j} + 54\hat{k})$

Now, the angle between $\vec{A}$ and $2\hat{i}+\hat{j}-2\hat{k}$ is given by

$$\cos\theta = \pm\frac{(-54\hat{j}+54\hat{k})\cdot(2\hat{i}+\hat{j}-2\hat{k})}{54\sqrt{2}\cdot 3} = \pm\frac{1}{\sqrt{2}}$$

$\theta = \pi/4$ or $3\pi/4$

9. **a., d.** Any vector in the plane of $\vec{a}=\hat{i}+\hat{j}+2\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+\hat{k}$ is

$$\vec{r} = \lambda(\hat{i}+\hat{j}+2\hat{k}) + \mu(\hat{i}+2\hat{j}+\hat{k})$$

$$= (\lambda+\mu)\hat{i} + (\lambda+2\mu)\hat{j} + (2\lambda+\mu)\hat{k}$$

Also $\vec{r}$ is perpendicular to the vector $\hat{i}+\hat{j}+\hat{k}$

$\Rightarrow \vec{r}\cdot\vec{c} = 0$

$\Rightarrow \lambda+\mu = 0$

Possible vectors are $\hat{j}-\hat{k}$ or $-\hat{j}+\hat{k}$

Integer Answer Type

1. **(5)** $E = (2\vec{a}+\vec{b})\cdot[|\vec{a}|^2\vec{b} - (\vec{a}\cdot\vec{b})\vec{a} - 2(\vec{a}\cdot\vec{b})\vec{b} + 2|\vec{b}|^2\vec{a}]$

$$\vec{a}\cdot\vec{b} = \frac{2-2}{\sqrt{70}} = 0$$

$|\vec{a}| = 1$

$|b| = 1$

$$\Rightarrow \quad E = (2\vec{a}+\vec{b})\cdot[2|\vec{b}|^2\vec{a} + |\vec{a}|^2\vec{b}]$$

$$= 4|\vec{a}|^2|\vec{b}|^2 + |\vec{a}|^2(\vec{a}\cdot\vec{b}) + 2|\vec{b}|^2(\vec{b}\cdot\vec{a}) + |\vec{a}|^2|\vec{b}|^2$$

$$= 5|\vec{a}|^2|\vec{b}|^2 = 5$$

2. **(9)** $\vec{r}\times\vec{b} = \vec{c}\times\vec{b}$

taking cross product with $\vec{a}$

$\vec{a}\times(\vec{r}\times\vec{b}) = \vec{a}\times(\vec{c}\times\vec{b})$

$\Rightarrow (\vec{a}\cdot\vec{b})\vec{r} - (\vec{a}\cdot\vec{r})\vec{b} = \vec{a}\times(\vec{c}\times\vec{b})$

$\Rightarrow \vec{r} = -3\hat{i}+6\hat{j}+3\hat{k}$

$\Rightarrow \vec{r}\cdot\vec{b} = 3+6 = 9$