Sequences and Series

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Sequences

- An arrangement of numbers in a definite order following some rule is known as a **sequence.** We also define a sequence as a function whose domain is the set of natural numbers or some subset of the type $\{1, 2, \ldots, k\}$.

- For example: 6, 12, 18, 24…; n, $n + 1$, $n + 2$, $n + 3$, $n + 4$, $n + 5$; etc.

- In general, a sequence is denoted by $\{a_n\}$ or $< a_n >$ which represents the sequence $a_1, a_2, a_3, \ldots a_n$.

- The numbers $a_1, a_2, a_3 \ldots$ and a_n occurring in a sequence are called its terms, where the subscript denotes the position of the term.
- The n^{th} term or the general term of a sequence is denoted by a_n.

- There are two types of sequences: finite and infinite.

- A sequence containing finite number of terms is called a finite sequence. For example: 5, 10, 15, 20, 25, 30, 35 is a **finite sequence**.
- A sequence containing infinite number of terms is called an **infinite sequence**. For example: sequence of prime numbers, sequence of natural numbers etc. are infinite sequences.

Note: Sometimes, a sequence is denoted by $\{T_n\}$ or $< T_n >$ which represents the sequence $T_1, T_2, T_3, \ldots T_n$.

Sum of first *n* terms of a sequence:

Let $\{a_n\}$ be the sequence such as $\{a_n\} = a_1, a_2, a_3, \ldots a_n$. Also, let S_n be the sum of its first n terms.

Then we have

$S_n = a_1 + a_2 + a_3 + \ldots + a_n$

It can be be observed that:

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

. . . .

. . . .

. . . .

$S_n = a_1 + a_2 + a_3 + \ldots + a_n$

From the above equations, we obtain

$S_1 = a_1$

$S_2 - S_1 = a_2$

$S_3 - S_2 = a_3$

. . .

. . .

. . .

$S_n - S_{n-1} = a_n$

$\Rightarrow a_n = S_n - S_{n-1}$

So, if S_n is known then any term of the sequence can be obtained.

Fibonacci sequence:

If a sequence is generated by a recurrence relation, each number being the sum of the previous two numbers, then it is called a **Fibonacci sequence**. For example: $a_1 = a_2 = 1$, $a_3 = a_1 + a_2$, $a_n = a_{n-2} + a_{n-1}$, $n > 2$.

Let's now try and solve the following puzzle to check whether we have understood the concept of Fibonacci sequence.

Series

- **If a_1, a_2, a_3… a_n is a given sequence, then the expression $a_1 + a_2 + a_3 + \ldots + a_n$ is called the series associated with the sequence.**

- For example:
  The series associated with the sequence 18, 36, 54, 72, 90 … is 18 + 36 + 54 + 72 + 90 + …
  The series associated with the sequence 2, 4, 6, 8, 10, 12 is 2 + 4 + 6 + 8 + 10 + 12
- In a compact form, the series associated with the sequence $a_1, a_2, a_3, \ldots, a_n$ can be written in sigma notation as $\sum_{k=1}^{n} a_k$, where sigma ($\sum$) denotes the sum.
- Note that series $a_1 + a_2 + a_3 + \ldots + a_n$ does not refer to the actual sum of the numbers $a_1, a_2, \ldots, a_n$. Infact, it just refers to the indicated sum, and it shows that a_1 is the first term, a_2 is the second term, …, a_n is the n^{th} term of the series.
- The series is finite or infinite depending on the given sequence.

  For example: The series associated with the sequence of numbers that are multiples of 4 is an infinite series, whereas the series associated with the sequence of numbers that are odd and less than 100 is a finite series.

Solved Examples

Example 1: The n^{th} term of a sequence is given by $a_n = \frac{(-2)^n\left(3n^2-4n+2\right)}{n+2}$. Find the ratio of the 6^{th} term of the sequence to its 4^{th} term.

Solution: The n^{th} term of a sequence is given by $a_n = \frac{(-2)^n\left(3n^2-4n+2\right)}{n+2}$.

Therefore,

$$a_6 = \frac{(-2)^6\left[3(6)^2-4(6)+2\right]}{6+2} = \frac{64\times 86}{8} = 688$$
$$a_4 = \frac{(-2)^4\left[3(4)^2-4(4)+2\right]}{4+2} = \frac{16\times 34}{6} = \frac{272}{3}$$

Thus, the ratio of the 6^{th} term of the sequence to its 4^{th} term is given by

$$\frac{a_6}{a_4} = \frac{688}{\frac{272}{3}} = \frac{688\times 3}{272} = \frac{43\times 3}{17} = \frac{129}{17}$$

Hence, the required ratio is 129:17.

Example 2: Write the first five terms of the sequence whose n^{th} term is given by $\frac{12n}{6n+8}$.

Solution:

It is given that the n^{th} term of the sequence is given by $\frac{12n}{6n+8}$.

Hence, $a_n = \frac{12n}{6n+8}$.

On putting $n = 1, 2, 3, 4, 5$ successively in a_n, we obtain

$a_1 = \frac{12(1)}{6(1)+8} = \frac{12}{14} = \frac{6}{7}$

$a_2 = \frac{12(2)}{6(2)+8} = \frac{24}{20} = \frac{6}{5}$

$a_3 = \frac{12(3)}{6(3)+8} = \frac{36}{26} = \frac{18}{13}$

$a_4 = \frac{12(4)}{6(4)+8} = \frac{48}{32} = \frac{3}{2}$

$a_5 = \frac{12(5)}{6(5)+8} = \frac{60}{38} = \frac{30}{19}$

Thus, the required first five terms of the sequence are $\frac{6}{7}, \frac{6}{5}, \frac{18}{13}, \frac{3}{2}, \frac{30}{19}$.

Example 3: Write the first six terms of the series associated with the following sequence:

$a_1 = 5$ $a_2 = -3$, $a_n = a_{n-1} + a_{n-2}$, $n > 2$.

Solution:

The sequence is given by

$a_1 = 5$ $a_2 = -3$

$a_n = a_{n-1} + a_{n-2}$, $n > 2$

On putting $n = 3, 4, 5, 6$, we obtain

$a_3 = a_2 + a_1 = -3 + 5 = 2$

$a_4 = a_3 + a_2 = 2 + (-3) = 2 - 3 = -1$

$a_5 = a_4 + a_3 = (-1) + 2 = 1$

$a_6 = a_5 + a_4 = 1 + (-1) = 0$

Thus, the first six terms of the given sequence are 5, −3, 2, −1, 1, 0.

Thus, the first six terms of the series associated with the given sequence is 5 + (−3) + 2 + (−1) + 1 + 0.

Example 4: The n^{th} term of a sequence is given by $T_n = 2n - 1$. Then show that $T_5 + T_{10} + T_{15} = 57$

Answer:
Put $n = 5$, 10 and15 successively in T_n to obtain T_5, T_{10} and T_{15} respectively.

$T_5 = 2(5) - 1 = 10 - 1 = 9$

$T_{10} = 2(10) - 1 = 20 - 1 = 19$

$T_{15} = 2(15) - 1 = 30 - 1 = 29$

$\therefore T_5 + T_{10} + T_{15} = 9 + 19 + 29 = 57$

Example 5:

The n^{th} term of a sequence is given by $T_n = 2n - 5$. If $\frac{T_{n+2}}{T_n} = \frac{13}{9}$, then prove that $n = 7$.

Answer:

$$\frac{T_{n+2}}{T_n} = \frac{2(n+2)-5}{2n-5} = \frac{13}{9}$$

$$\frac{2n-1}{2n-5} = \frac{13}{9}$$

$$26n - 65 = 18n - 9$$

$$8n = 56$$

$$n = 7$$

Therefore, $n = 7$

nth Term Of An Arithmetic Progression

We know what an arithmetic progression (A.P.) is. Also, we have learnt that there is a common difference between any two consecutive terms of an A.P..

Now, can we find the required term of a given A.P. with this information?

Let us consider the A.P. 3, 7, 11, 15,...

Here, first term (a) = 3 and common difference (d) = 4

Now, if we want to find the 5^{th} term of this A.P., then we will simply add the common difference to 4^{th} term. Thus, 5^{th} term of this A.P. will be 15 + 4 = 19.

What would we do if we are asked to find the 20^{th} term or 100^{th} term or n^{th} term?

Obviously, the process of adding common difference will be very time consuming.

For such problems, we must have a short cut or a formula to find the general term of an A.P.

Let us derive the same.

Consider the A.P. a, $a + d$, $a + 2d$, $a + 3d$, ...

For this A.P., we have

$a_1 = a$

$a_2 - a_1 = d$

$a_3 - a_2 = d$

. . .

. . .

. . .

$a_{n-1} - a_{n-2} = d$

$a_n - a_{n-1} = d$

Adding all these equations, we get

$a_1 + (a_2 - a_1) + (a_3 - a_2) + ... + (a_{n-1} - a_{n-2}) + (a_n - a_{n-1}) = a + \{d + d + ... + d\ (n - 1 \text{ times})\}$

$\Rightarrow (a_1 - a_1) + (a_2 - a_2) + (a_3 - a_3) + ... + (a_{n-1} - a_{n-1}) + a_n = a + (n - 1)d$

$\Rightarrow a_n = a + (n - 1)d$

Hence, the general term or n^{th} term i.e., a_n of an A.P. whose first term is a and common difference is d can be found by the following formula:

$\boldsymbol{a_n = a + (n - 1)d}$

Sometimes, we need to find three, four or five consecutive terms of an A.P. then it is convenient to take them as follows:

- Three consecutive terms can be taken as $a - d, a, a + d$
- Four consecutive terms can be taken as $a - 3d, a - d, a + d, a + 3d$. Here, common difference is $2d$.
- Five consecutive terms can be taken as $a - 2d, a - d, a, a + d, a + 2d$

Result: In an A.P., common difference, $d = \frac{T_p - T_q}{p - q}$, where T_p and T_q are p^{th} and q^{th} term respectively.

In particular, $d = \frac{T_n - a}{n - 1}$.

Proof:

$T_p = a + (p - 1)d, T_q = a + (q - 1)d$

$\Rightarrow T_p - T_q = a + (p - 1)d - \{a + (q - 1)d\} = (p - q)d$

$\Rightarrow d = \frac{T_p - T_q}{p - q}$

In particular, take $T_1 = a$.

Then, $d = \frac{T_n - a}{n - 1}$

Now, let us solve some examples to understand the concept better.

Example 1: Find the 20th term of the following arithmetic progression.

0.4, 1.5, 2.6, 3.7, 4.8 ...

Solution:

Here, $a = 0.4$ and $d = 1.5 - 0.4 = 1.1$

Thus, the 20th term is given by,

$a_{20} = a + (20 - 1)\, d$

$= 0.4 + (20 - 1)\, 1.1$

$= 0.4 + 19 \times 1.1$

$= 0.4 + 20.9$

$= 21.3$

Thus, the 20th term of the given A.P. is 21.3.

Example 2: If the 7th term of an A.P. is – 21 and 15th term is – 53, then find the first term and common difference.

Solution:

Let the first term and common difference of the A.P. be a and d respectively.

It is given that $a_7 = -21$ and $a_{15} = -53$

Using the formula for n^{th} term, we obtain

$a_7 = a + (7 - 1)\, d$

$-21 = a + 6d$ ... (1)

and, $a_{15} = a + (15 - 1)\, d$

$-53 = a + 14d$ … (2)

Subtracting equation (1) from (2), we obtain

$-32 = 8d$

$\Rightarrow d = -4$

Substituting the value of d in equation (1), we obtain

$-21 = a + 6(-4)$

$-21 = a - 24$

$\Rightarrow a = 3$

Thus, the first term is 3 and the common difference is –4.

Example 3: Is 102 a term of the A.P., 5, 11, 17, 23 …?

Solution:

Let 102 be the n^{th} term of the given sequence.

$\therefore a_n = 102$

Using the formula for n^{th} term, we obtain

$a_n = a + (n - 1)d$

$\therefore 102 = a + (n - 1)d$

For the given A.P., $a = 5$ and $d = 11 - 5 = 6$

$\therefore 102 = 5 + (n - 1)6$

$102 - 5 = (n - 1)6$

$97 = (n - 1)6$

$$n - 1 = \frac{97}{6}$$

$$n = \frac{97}{6} + 1$$

$$n = \frac{103}{6}$$

However, n should be a positive integer. Therefore, 102 is not a term of the given A.P.

Example 4: Find the number of three-digit numbers that are divisible by 5.

Solution:

The first three-digit number which is divisible by 5 is 100, second is 105, third is 110, and so on. The last three-digit number which is divisible by 5 is 995.

Thus, we obtain the following A.P.

100, 105 … 995

Here, we have to find the number of terms, n.

∴Last term of A.P. = 995

The number of terms in the A.P. is n, so the last term is the n^{th} term.

$a + (n - 1)d = 995$

Here, $a = 100$ and $d = 5$

$100 + (n - 1)5 = 995$

$(n - 1)5 = 995 - 100$

$5n - 5 = 895$

$5n = 895 + 5$

$5n = 900$

$$n = \frac{900}{5}$$

$n = 180$

Thus, there are 180 three-digit numbers, which are divisible by 5.

Example 5: The fare of a bus is Rs 10 for the first kilometre and Rs 5 for each additional kilometre. Find the fair after 12 kilometres.

Solution:

The fare after each kilometre forms an A.P. as follows.

Fare after one kilometre = Rs 10

Fare after two kilometres = 10 + 5 = Rs 15

Fare after three kilometres = 15 + 5 = Rs 20

Now the arithmetic progression is 10, 15, 20 …

Here, first term, a = 10 and common difference, d = 5

Now the fare after 12 kilometres is the 12th term of the A.P.

$\therefore a_{12} = a + (12 - 1)d$

$a_{12} = 10 + 11 \times 5$

$= 10 + 55$

$= 65$

Thus, the fare after 12 kilometres is Rs 65.

Example 6: Mohit borrowed a sum of money at a simple interest rate of 2% per annum. He has to pay an amount of Rs 1120 after 6 years. How much money did he borrow?

Solution:

Let the amount of money Mohit borrowed be Rs *x.* We know that the amount after *T* years is

$$A = P + \frac{P \times R \times T}{100}$$

Where, *P* and *R* denotes the principal and rate respectively

The amount after every year forms an A.P.

Amount after first year $= x + \frac{x \times 2 \times 1}{100}$

$= x + \frac{2x}{100}$

Amount after second year $= x + \frac{x \times 2 \times 2}{100}$

$= x + \frac{4x}{100}$

Thus, the A.P. is as follows.

$x + \frac{2x}{100}, x + \frac{4x}{100} \ldots$

Here, the first term is $x + \frac{2x}{100}$ and common difference is $\frac{2x}{100}$.

Now, it is given that the amount after 6 years is Rs 1120 i.e., 6^{th} term of the A.P. is 1120.

Now using the formula, $a_n = a + (n - 1)d$, we obtain

$$1120 = x + \frac{2x}{100} + (6-1)\frac{2x}{100}$$

$$1120 = x + \frac{2x}{100} + 5 \times \frac{2x}{100}$$

$$1120 = x + \frac{2x}{100} + \frac{10x}{100}$$

$$1120 = \frac{100x + 2x + 10x}{100}$$

$1120 \times 100 = 112x$

$x = 1000$

Thus, Mohit borrowed Rs 1000.

Example 7: Find four consecutive terms of an A.P. such that the difference of the middle terms is 8 and the product of the extreme terms is 217.

Solution:

Let four consecutive terms of required A.P. be $a - 3d, a - d, a + d, a + 3d$.

According to the question, we have

$a + d - (a - d) = 8$

$\Rightarrow a + d - a + d = 8$

$\Rightarrow 2d = 8$

$\Rightarrow d = 4$

Also,

$(a - 3d)(a + 3d) = 217$

$\Rightarrow a^2 - (3d)^2 = 217$

$\Rightarrow a^2 - 9d^2 = 217$

$\Rightarrow a^2 - 9(4^2) = 217$

$\Rightarrow a^2 - 144 = 217$

$\Rightarrow a^2 = 361$

$\Rightarrow a = \pm 19$

When $d = 4$ and $a = 19$, then four consecutive terms are:

$19 - 3(4), (19 - 4), (19 + 4), 19 + 3(4)$

i.e., 7, 15, 23, 31

When $d = 4$ and $a = -19$, then four consecutive terms are:

$-19 - 3(4), (-19 - 4), (-19 + 4), -19 + 3(4)$

i.e., –31, –23, –15, –7

Example 8: Find five consecutive terms of an A.P. such that the product of the extreme terms is –63 and product of second and fourth terms is –15.

Solution:

Let five consecutive terms of required A.P. be $a - 2d, a - d, a, a + d, a + 2d$.

According to the question, we have

$(a - 2d)(a + 2d) = -63$

$\Rightarrow a^2 - (2d)^2 = -63$

$\Rightarrow a^2 - 4d^2 = -63$ ...(1)

Also,

$(a - d)(a + d) = -15$

$\Rightarrow a^2 - d^2 = -15$ ...(2)

On subtracting (1) from (2), we get

$3d^2 = 48$

$\Rightarrow d^2 = 16$

$\Rightarrow d = \pm 4$

On substituting $d^2 = 16$ in (2), we get

$a^2 - 16 = -15$

$\Rightarrow a^2 = 1$

$\Rightarrow a = \pm 1$

When $a = 1$ and $d = 4$, A.P. will be –7, –3, 1, 5, 9

When $a = 1$ and $d = -4$, A.P. will be 9, 5, 1, –3, –7

When $a = -1$ and $d = 4$, A.P. will be –9, –5, –1, 3, 7

When $a = -1$ and $d = -4$, A.P. will be 7, 3, –1, –5, –9

Example 9: If the 25th and 35th terms of an arithmetic progression are 121 and 171 respectively, then find the common difference of the A.P.

Answer:

$T_{25} = 121$, $T_{35} = 171$

It is known that common difference, $d = \frac{T_p - T_q}{p - q}$.

$$\therefore d = \frac{T_{35} - T_{25}}{35 - 25} = \frac{171 - 121}{10} = 5$$

Example 10:

In an A.P, show that $d = \frac{T_{200} - T_{100}}{100}$.

Answer:

$$T_{100} = a + (100 - 1)d = a + 99d$$

$$\begin{aligned} T_{200} &= a + (200 - 1)d = a + 199d \\ &= a + 99d + 100d \\ &= T_{100} + 100d \end{aligned}$$

Therefore, $d = \frac{T_{200} - T_{100}}{100}$

Sum of n Terms of an Arithmetic Progression

We know what an arithmetic progression (A.P.) is. Sometimes, we may come across the situations when we have to find the sum of all terms involved in a series and if the series is an AP, then there is a formula which can make the process very simple.

Let us consider a similar situation.

Harry saved Rs 2000 from his salary in the first month. He increases his savings by Rs 50 every month.

Can we calculate his total savings for the first four months?

Let us try to find it.

To find the total savings for the first 4 months, we have to take the sum of the savings for the first four months.

It is given that, savings of Harry for the first month = Rs 2000

Every month, he increases his savings by Rs 50.

Thus, savings for second month = Rs (2000 + 50) = Rs 2050

Similarly, savings for the third month = Rs (2050 + 50) = Rs 2100

nd, savings for the fourth month = Rs (2100 + 50) = Rs 2150

Thus, the total savings of Harry for the first four months = Rs (2000 + 2050 + 2100 + 2150)

= Rs 8300

Now, can we calculate the total savings of Harry for 2 years?

Yes, we can find it as above but it is a very lengthy as well as time consuming process as we have to find the savings for 24 months.

We can also find the total savings of Harry for first two years using a formula. Now, let us see how we can find it.

The savings of Harry for each month forms an A.P., which is as follows.

2000, 2050, 2100, 2150 ….

The sum of savings of Harry = Rs (2000 + 2050 + 2100 + 2150 ….)

Here, we can observe that the total savings of Harry for the first month is the first term of the A.P., i.e. Rs 2000. The total savings for the first two months is the sum of first two terms of the A.P., i.e. Rs (2000 + 2050). In the same way, the total savings of Harry for first 2 years, i.e. 24 months, is the sum of first 24 terms of the A.P. We can find it by using the formula for finding the sum of n terms of an A.P.

Now, let us find the sum of first 24 terms of the above discussed A.P. which is as follows:

2000, 2050, 2100, 2150 ….

Here, first term, a = 2000

Common difference, d = 2050 − 2000 = 50

The sum of first 24 terms of the A.P. is

$$S_{24} = \frac{24}{2}\left[2\times 2000+(24-1)50\right]$$
$$= 12(4000+23\times 50)$$
$$= 12(4000+1150)$$
$$= 12\times 5150$$
$$= 61800$$

Therefore, the total savings of Harry for the first two years is Rs 61800.

This formula is used when we are given the first term and the common difference of the arithmetic progression.

We can also find the sum of n terms of an A.P., if we know the first and the last term.

The sum of n terms of an A.P. whose first term is a and last term is l is given by the formula:
Sn = n/2{a+l}

For example, consider an A.P. whose first term is 2 and 30^{th} term is 263. Then, what will be the sum of 30 terms?

Here, a = 2, l = 263 and n = 30

Therefore, sum of 30 terms $= \frac{n}{2}[a+l]$

$$= \frac{30}{2}[2+263]$$

= 15 × 265

= 3975

Thus, the sum of 30 terms is 3975.

Result: The sum of the first n natural numbers is given by $\frac{n(n+1)}{2}$.

Proof:

This can be proved by two methods.

1st method (Using concept of A.P.):

The first n natural numbers can be listed as 1, 2, 3, …, n.

Here, $a = 1$, $d = 1$.

$$\therefore S_n = \frac{n}{2}\left[2a+(n-1)d\right] = \frac{n}{2}\left[2\times 1+(n-1)1\right] = \frac{n(n+1)}{2}$$

2nd method (Without using concept of A.P.):

$S_n = 1 + 2 + 3 + \ldots + n$ … (1)

$S_n = n + (n - 1) + (n - 2) + \ldots + 3 + 2 + 1$ … (2)

Adding (1) and (2):

$2S_n = (n + 1) + (n + 1) + \ldots\ldots + (n + 1)$

Here, there are n terms in the RHS.

$$\therefore S_n = \frac{n(n+1)}{2}$$

Now, let us discuss some more examples based on sum of n terms of an A.P.

Example 1: Find the sum of first 25 terms of the following A.P.

2, 7, 12 …

Solution:

Here, $a = 2$ and $d = 7 - 2 = 5$.

Sum of the first 25 terms is given by

$$S_{25} = \frac{25}{2}\left[2a + (25-1)d\right]$$

$$= \frac{25}{2}\left[2\times 2 + (25-1)5\right]$$

$$= \frac{25}{2}\left[4 + 24\times 5\right]$$

$$= \frac{25}{2}\left[4 + 120\right]$$

$$= \frac{25}{2}\left[124\right]$$

= 25 × 62

= 1550

Thus, the sum of first 25 terms of the given A.P. is 1550.

Example 2: Find the sum of first 8 terms of the A.P whose n^{th} term is given by 6*n* − 5.

Solution:

The n^{th} term is given by

$a_n = 6n - 5$

On replacing *n* by 1, 2, 3 … respectively, we get the first, second, third … terms of the A.P.

∴ $a_1 = 6(1) - 5 = 1$

$a_2 = 6(2) - 5 = 7$

$a_3 = 6(3) - 5 = 13$ … and so on.

The A.P. so obtained is as follows.

1, 7, 13 …

Here, the first term, $a = 1$

and the common difference $d = a_2 - a_1 = 7 - 1 = 6$

Using the formula, $S_n = \frac{n}{2}\left[2a + (n-1)d\right]$, the sum of first 8 terms is given by

$$S_8 = \frac{8}{2}\left[2\times1+(8-1)6\right]$$
$$= 4\left[2+7\times6\right]$$
$$= 4\left[2+42\right]$$
$$= 4\left[44\right]$$

= 176

Thus, the sum of first 8 terms is 176.

Example 3: How many terms of the A.P. −28, −24, −20 … should be taken so that the sum will be zero?

Solution:

Let the sum of *n* terms be zero.

Here, $a = -28$ and $d = -24 - (-28) = 4$

Sum of *n* terms of an A.P. is given by

$$S_n = \frac{n}{2}\left[2a+(n-1)d\right]$$

But, it is given that the sum of *n* terms of the given A.P. is zero.

$$S_n = 0$$
$$\frac{n}{2}\left[2a+(n-1)d\right] = 0$$
$$\frac{n}{2}\left[2\times(-28)+(n-1)\times4\right] = 0$$
$$-56+4n-4 = 0$$
$$4n-60 = 0$$
$$4n = 60$$
$$n = \frac{60}{4}$$
$$n = 15$$

Thus, the sum of 15 terms of the A.P is zero.

Example 4: Sapna's father planted 4 trees in his garden, when he was 22 years old. After that, every year he planted one more tree than the number of trees he planted in the previous year. How many trees will be there in his garden when he will become 40 years old?

Solution:

We can write the given information in the form of an A.P. as follows

Number of trees he planted in the first year = 4

Number of trees he planted in the second year = 4 + 1 = 5

Number of trees he planted in the third year = 5 + 1 = 6

And so on.

Now, the A.P. is 4, 5, 6 …

He planted trees from the age of 22 years to 40 years, i.e. for 19 years.

Thus, we have to find the sum of 19 terms of this A.P.

Here, a = 4, d = 1

And, n = 19

Using the formula, $S_n = \frac{n}{2}\left[2a + (n-1)d\right]$, the sum of 19 terms of this A.P. is given by

$$\begin{aligned} S_{19} &= \tfrac{19}{2}\ [2 \times 4 + (19 - 1)\,1] \\ &= \tfrac{19}{2} \times [8 + 18] \\ &= \tfrac{19}{2} \times 26 = 19 \times 13 \\ &= 247 \end{aligned}$$

Thus, there will be 247 trees in his garden when he will become 40 years old.

Example 5: The p^{th} term of an A.P. is *q* and the q^{th} term of the A.P. is *p*. What is the sum of (*p* + *q*) terms of the A.P.?

Solution:

Let the first term and the common difference of the A.P. be a and d respectively.

It is given that the p^{th} term is q.

$\therefore a_p = q$

$\Rightarrow a + (p - 1)\, d = q$ … (1)

Similarly, the q^{th} term is p, therefore we obtain

$a + (q - 1)\, d = p$ … (2)

On subtracting equation (1) from (2), we obtain

$p - q = (q - 1)\, d - (p - 1)\, d$

$p - q = d\, [q - 1 - p + 1]$

$$d = \frac{p-q}{q-p}$$
$$d = -1$$

By putting the value of d in equation (1), we obtain

$q = a + (p - 1)\,(-1)$

$q = a + 1 - p$

$a = p + q - 1$

But we know that the sum of n terms of an A.P. is

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

Thus, the sum of $(p + q)$ terms is

$$S_{p+q} = \frac{p+q}{2}\left[2(p+q-1)+(p+q-1)(-1)\right]$$

$$S_{p+q} = \frac{p+q}{2}\left[2p+2q-2-p-q+1\right]$$

$$S_{p+q} = \frac{(p+q)}{2}\left[p+q-1\right]$$

$$S_{p+q} = \frac{(p+q)(p+q-1)}{2}$$

Thus, the sum of $(p + q)$ terms is $\frac{(p+q)(p+q-1)}{2}$.

Example 6: Find the n^{th} term of the A.P., the sum of whose n terms is $n^2 + 2n$.

Solution:

Let S_n be the sum of n terms.

It is given that the sum of n terms of the A.P. is $n^2 + 2n$.

$\therefore S_n = n^2 + 2n$ … (1)

On replacing n by $(n - 1)$ in the equation, we obtain

$$S_{n-1} = (n-1)^2 + 2(n-1)$$

Let a_n be the n^{th} term of the A.P.

Therefore, we can write

$S_n = S_{n-1} + a_n$

Thus, $a_n = S_n - S_{n-1}$

$= n^2 + 2n - [(n - 1)^2 + 2(n - 1)]$

$= n^2 + 2n - [n^2 + 1 - 2n + 2n - 2]$

$= n^2 + 2n - n^2 + 1$

$= 2n + 1$

Thus, the n^{th} term of the A.P is $(2n + 1)$.

Example 7: Find the sum of first 1000 natural numbers.
Solution:

The sum of first n natural number is given by $S_n = \frac{n(n+1)}{2}$.

$$\therefore\ S_{1000} = \frac{1000 \times (1000+1)}{2} = 500 \times 1001 = 500500$$

Hence, the sum of first 1000 natural numbers is 500500.

Example 8:
If sum of the first *n* natural numbers is 5050, find the value of *n*.
Solution:

The sum of first n natural number is given by $S_n = \frac{n(n+1)}{2}$.
Now,

$$5050 = \frac{n \times (n+1)}{2}$$
$$n^2 + n - 2 \times 5050 = 0$$
$$n^2 - 100n + 101n - 2 \times 5050 = 0$$
$$(n-100)(n+101)$$
$$n = 100$$

Hence, $n = 100$

Properties of Arithmetic Progressions and the Concept of Arithmetic Mean

Arithmetic progression is a sequence of numbers such that the difference between the consecutive terms is a constant. It exhibits some properties which are used in solving various problems.

Properties of an Arithmetic Progression

- If a constant is added to each term of an A.P., the resulting sequence will also be an A.P.
- If a constant is subtracted from each term of an A.P., the resulting sequence will also be an A.P.
- If a constant is multiplied to each term of an A.P., the resulting sequence will also be an A.P.
- If each term of an A.P. is divided by a non-zero constant, the resulting sequence is also an A.P.

Arithmetic Mean

- If we are given two numbers a and b, then we can insert a number A between these two numbers so that the sequence a, A, b becomes an A.P. Such a number i.e., A is called an **arithmetic mean (A.M.)** of the numbers a and b.
- If A is the A.M. of the numbers a and b, then A is given by $A = \frac{a+b}{2}$.

  For example, the A.M. of the two numbers 18 and 16 is $\frac{18+16}{2}=17$.
- For any two given numbers a and b, we can insert as many numbers between them as we want such that the resulting sequence becomes an A.P.

Example 1: Between −12 and 40, p numbers have been inserted in such a way that the resulting sequence is an A.P. Find the value of p if the ratio of the 4^{th} and the $(p-3)^{th}$ number is 1 : 6.

Solution:

Let $A_1, A_2, \ldots A_p$ be p numbers such that $-12, A_1, A_2, \ldots A_p, 40$ is an A.P.

Here, $a = -12$, $b = 40$, $n = p + 2$

$\therefore 40 = -12 + (p + 2 - 1)(d)$

$\Rightarrow 52 = (p + 1)\, d$

$$\Rightarrow d = \frac{52}{p+1} \qquad \ldots(1)$$

$A_1 = a + d$

$A_2 = a + 2d$

$A_3 = a + 3d \ldots$

$\therefore A_4 = a + 4d$

$A_{p-3} = a + (p-3)\, d$

According to the given information,

$$\frac{a+4d}{a+(p-3)d}=\frac{1}{6}$$

$$\Rightarrow \frac{-12+4\left(\frac{52}{p+1}\right)}{-12+(p-3)\left(\frac{52}{p+1}\right)}=\frac{1}{6} \qquad \text{[From (1)]}$$

$$\Rightarrow \frac{-12(p+1)+4(52)}{-12(p+1)+52(p-3)}=\frac{1}{6}$$

$$\Rightarrow \frac{-12p-12+208}{-12p-12+52p-156}=\frac{1}{6}$$

$$\Rightarrow \frac{-12p+196}{40p-168}=\frac{1}{6}$$

$$\Rightarrow 6(-12p+196)=40p-168$$

$$\Rightarrow -72p+1176=40p-168$$

$$\Rightarrow 40p-168+72p-1176=0$$

$$\Rightarrow 112p=1344$$

$$\Rightarrow p=\frac{1344}{112}$$

$$\Rightarrow p=12$$

Thus, the value of *p* is 12.

Example 2: Between 10 and 30, *m* numbers are inserted such that the resulting sequence is an A.P. If the sum of all the terms of the A.P. is 140, then find the value of *m*.

Solution:

It is given that between 10 and 30, *m* numbers are inserted such that the resulting sequence is an A.P.

It is also given that the sum of all the terms in the A.P. is 140.

We know that the sum of *n* terms of an A.P. is given by

$S_n=\frac{n}{2}[a+l]$, where *a* is the first term and *l* is the last term

Here, a = 10, l = 30

Therefore,

$$140 = \frac{n}{2}[10 + 30]$$
$$\Rightarrow 140 = \frac{n}{2}[40]$$
$$\Rightarrow 140 = 20n$$
$$\Rightarrow n = 7$$

Thus, the total number of terms in the A.P. is 7.

Thus, the value of m is 7 − 2 = 5.

Geometric Progressions

- A sequence $a_1, a_2, \ldots, a_n, \ldots$ is called a **geometric progression** (G.P.), if each term is non-zero and $\frac{a_{k+1}}{a_k} = r$ (constant), for $k \geq 1$.
- For example, $\frac{4}{7}, \frac{4}{21}, \frac{4}{63}, \frac{4}{189}, \ldots$ is in G.P.
- A G.P. is written in its standard form as $a, ar, ar^2, ar^3, ar^4 \ldots$
- Here, ***a*** is called the **first term** of the G.P.
- Here, ***r*** is called the **common ratio** of the G.P.
- There are two types of geometric progressions: finite and infinite.
- A **finite geometric progression** has finite number of terms. In general, a finite G.P. with n terms can be written as $a, ar, ar^2, ar^3, ar^4 \ldots ar^{n-1}$.
- An **infinite geometric progression** has infinite number of terms. In general, an infinite G.P. can be written as $a, ar, ar^2, ar^3, ar^4 \ldots ar^{n-1} \ldots$
- There are two types of geometric series: finite and infinite.
- The series $a + ar + ar^2 + ar^3 + ar^4 + \ldots + ar^{n-1}$ is a **finite geometric series**.
- The series $a + ar + ar^2 + ar^3 + ar^4 + \ldots + ar^{n-1} + \ldots$ is an **infinite geometric series**.
- The last term of a G.P. is denoted by l and the sum of n terms of a G.P. is denoted by S_n. For example, in the G.P. 8, −16, 32, −64, the last term is given by $l = -64$.

Note: n^{th} term, a_n is also denoted by T_n.

- **Example:** In the G.P. $8, 12, 18, 27, \frac{81}{2}, \frac{243}{4}$,
- First term, $a = 8$
- Common ratio, $r = \frac{12}{8} = \frac{3}{2}$
- Last term, $l = \frac{243}{4}$
- $S_n = S_6 = 8 + 12 + 18 + 27 + \frac{81}{2} + \frac{243}{4}$

Note:

- Since the division by zero is undefined, neither any term nor the common ratio of a G. P. can be 0.
- If the common ratio (r) is positive, then all the terms of the G. P. will be of same sign. Thus, all the terms of the G. P.will be either negative or positive.
- If the common ratio (r) is negative, then any two consecutive terms of the G. P. will be of different sign. Thus, the G. P. will be having negative and positive terms alternatively.

Solved Examples

Example 1: Which of the following sequences is not a geometric progression?

1. 11, 2.2, 0.44, 0.088, 0.0176 …
2. 17, 17.5, 18, 18.5, 19
3. 81, $\frac{81}{2}, \frac{243}{4}, \frac{243}{8}$
4. $16, 4, 1, \frac{1}{4}, \frac{1}{16}, \frac{1}{64} \ldots$

Solution:

1. The series 11, 2.2, 0.44, 0.088, 0.0176 … is a G.P. since
$\frac{2.2}{11} = \frac{0.44}{2.2} = \frac{0.088}{0.44} = \frac{0.0176}{0.088} = 0.2$

2. The series 17, 17.5, 18, 18.5, 19 is not a G.P. since
$\frac{17.5}{17} \neq \frac{18}{17.5} \neq \frac{18.5}{18} \neq \frac{19}{18.5}$

3. The series 81, $\frac{81}{2}, \frac{243}{4}, \frac{243}{8}$ is not a G.P. since
$\frac{\frac{81}{2}}{81} = \frac{1}{2}$ and $\frac{\frac{243}{4}}{\frac{81}{2}} = \frac{3}{2}$

4. The series 16, 4, 1, $\frac{1}{4}, \frac{1}{16}, \frac{1}{64} \ldots$ is a G.P. since
$\frac{4}{16} = \frac{1}{4} = \frac{\frac{1}{4}}{1} = \frac{\frac{1}{16}}{\frac{1}{4}} = \frac{\frac{1}{64}}{\frac{1}{16}} = \frac{1}{4}$

Example 2: Write the first term, last term and the common ratio of the following geometric progressions:

1. $3, 6\sqrt{6}, 72, 144\sqrt{6}, 1728$

2. $\frac{9}{5}, \frac{18}{25}, \frac{36}{125}, \frac{72}{625}$

Solution:

1. In the G.P. $3, 6\sqrt{6}, 72, 144\sqrt{6}, 1728$,
First term, a = 3
Last term, l = 1728
Common Ratio, r = $\frac{6\sqrt{6}}{3} = 2\sqrt{6}$

2. In the G.P., $\frac{9}{5}, \frac{18}{25}, \frac{36}{125}, \frac{72}{625}$,
First term, a = $\frac{9}{5}$

Last term, $l = \frac{72}{625}$

Common Ratio, $r = \frac{\frac{18}{25}}{\frac{9}{5}} = \frac{18}{25} \times \frac{5}{9} = \frac{2}{5}$

n^{th} Term and Sum of n-Terms of a Geometric Progression

- The n^{th} term of a geometric progression (G.P.), a, ar, ar^2, ar^3, … is given by $\boldsymbol{a_n = ar^{n-1}}$.
- For example, the 20^{th} term of a geometric progression 3, $\frac{9}{2}, \frac{27}{4}, \frac{81}{8}$ is given by

$$a_{20} = (3) \times \left(\frac{3}{2}\right)^{20-1} = (3) \times \left(\frac{3}{2}\right)^{19} = \frac{(3)^{20}}{(2)^{19}}.$$

Note: 1. To obtain the succeeding term of a given term T_n, multiply it by r.
$T_{n+1} = T_n \times r$

2. To obtain the preceding term of the given term T_n, divide it by r.

$$T_{n-1} = \frac{T_n}{r}$$

If a is the first term and r is the common ratio of a G.P., then the sum of first n terms of the G.P. is given by

- $S_n = na$, if $r = 1$
- $S_n = \frac{a(1-r^n)}{1-r} \quad or \quad \frac{a(r^n-1)}{r-1} \quad if\ r \neq 1$

Sum of infinite terms of a G.P.: $S_\infty = \frac{a}{1-r}$

Derivation:

$S_n = \frac{a(1-r^n)}{1-r}$ when $r < 1$.

Since r^n approaches to 0 as n approaches ∞, we have

$$\therefore S_\infty = \frac{a(1-0)}{1-r} = \frac{a}{1-r}$$

Note: $S_{2n} \div S_n = r^n + 1$

Derivation: $$S_{2n} \div S_n = \frac{a\left(1-r^{2n}\right)}{1-r} \div \frac{a\left(1-r^n\right)}{1-r} = \frac{1-r^{2n}}{1-r^n} = \frac{\left(1-r^n\right)\left(1+r^n\right)}{1-r^n} = 1+r^n$$

Sometimes, we need to find three or four consecutive terms of an G.P. then it is convenient to take them as follows:

- Three consecutive terms can be taken as $\frac{a}{r}$, a, ar. Here, common ratio is r.
- Four consecutive terms can be taken as $\frac{a}{r^3}$, $\frac{a}{r}$, ar, ar^3. Here, common ratio is r^2.

Example 1 A scientist kept a solution on fire. The original temperature of the solution was 24°C. He noticed that the temperature of the solution increased by 20% of the original temperature every hour. Find the temperature of the solution after the 5^{th} hour.

Solution:

It is given that the original temperature of the solution was 24°C.

It is also given that the temperature of the solution increased by 20% of the original temperature every hour. Hence, the temperature after the 1^{st} hour will be

$$T_1 = \frac{20}{100} \times 24 + 24 = 24\left(\frac{120}{100}\right)$$

Temperature after 2^{nd} hour:

$$T_2 = 24\left(\frac{120}{100}\right)\left(\frac{120}{100}\right)$$

Temperature after 3^{rd} hour:

$$T_3 = 24\left(\frac{120}{100}\right)\left(\frac{120}{100}\right)\left(\frac{120}{100}\right)$$

And so on…

Hence, the temperature of the solution noted after every hour will form a geometric progression with

First term, $a = 24$

Common ratio, $r = \left(\frac{120}{100}\right) = 1.2$

We know that the n^{th} term of a G.P. is given by $a_n = ar^{n-1}$.

Also, the temperature of the solution after the 5^{th} hour will be the 6^{th} term of the G.P., which is given by

$a_6 = 24 \times (1.2)^5 = 59.72$ (approx.)

Thus, the temperature of the solution after the 5^{th} hour will be 59.72°C approximately.

Example 2 Find the sum of the sequence 0.4, 0.44, 0.444……. up to n terms.

Solution:

The given sequence is not a G.P. However, we can relate it to a G.P. by writing the terms as

$S_n = 0.4 + 0.44 + 0.444 + \ldots$ to n terms

$$
\begin{aligned}
&= 4\left[0.1 + 0.11 + 0.111 + \ldots n \text{ terms}\right] \\
&= \frac{4}{9}\left[0.9 + 0.99 + 0.999 + \ldots n \text{ terms}\right] \\
&= \frac{4}{9}\left[(1-0.1) + \left[1-(0.1)^2\right] + \left[1-(0.1)^3\right] + \ldots n \text{ terms}\right] \\
&= \frac{4}{9}\left\{(1+1+1\ldots n \text{ terms}) - \left[0.1 + (0.1)^2 + (0.1)^3 + \ldots n \text{ terms}\right]\right\} \\
&= \frac{4}{9}\left[n - 0.1 \times \left(\frac{1-(0.1)^n}{1-0.1}\right)\right] \\
&= \frac{4}{9}\left[n - \frac{1-(0.1)^n}{9}\right]
\end{aligned}
$$

Example 3 Find the number of terms of the G.P. $-6, \frac{12}{5}, -\frac{24}{25}, \frac{48}{125}, -\frac{96}{625}$ that are required for giving the sum as $-\frac{2706}{625}$.

Solution:

Let n be the required number of terms.

It is given that

First term, $a = -6$

Common ratio, $r = -\frac{2}{5}$

We know that the sum of n terms of a G.P. is given by $S_n = \frac{a(1-r^n)}{1-r}$. Therefore,

$$S_n = \frac{(-6)\left[1-\left(\frac{-2}{5}\right)^n\right]}{1-\left(\frac{-2}{5}\right)}$$

$$\Rightarrow \frac{-2706}{625} = \frac{(-6)\left[1-\left(\frac{-2}{5}\right)^n\right]}{\frac{7}{5}}$$

$$\Rightarrow \left[1-\left(\frac{-2}{5}\right)^n\right] = \frac{-2706}{625} \times \frac{7}{5} \times \frac{1}{-6} = \frac{3157}{3125}$$

$$\Rightarrow \left(\frac{-2}{5}\right)^n = 1 - \frac{3157}{3125} = \frac{-32}{3125}$$

$$\Rightarrow \left(\frac{-2}{5}\right)^n = \left(\frac{-2}{5}\right)^5$$

$$\Rightarrow n = 5$$

Thus, the required number of terms of the G.P. is 5.

Example 4 In a G.P., the ratio of the 11^{th} term and the 14^{th} term is given by 27:125. Find the 18^{th} term of the G.P. if the 5^{th} term of the G.P. is $\frac{125}{81}$.

Solution:

Let the first term and the common ratio of the G.P. be a and r respectively.

We know that the n^{th} term of a G.P. is given by $a_n = ar^{n-1}$. Therefore,

$a_{11} = ar^{10}$

$a_{14} = ar^{13}$

It is given that the ratio of the 11th term to the 14th term is 27:125. Hence,

$$\frac{ar^{10}}{ar^{13}} = \frac{27}{125}$$
$$\Rightarrow \frac{1}{r^3} = \left(\frac{3}{5}\right)^3$$
$$\Rightarrow \frac{1}{r} = \frac{3}{5}$$
$$\Rightarrow r = \frac{5}{3}$$

It is also given that the 5th term of the G.P. is $\frac{125}{81}$. Hence,

$$a_5 = ar^4 = \frac{125}{81}$$
$$\Rightarrow a\left(\frac{5}{3}\right)^4 = \frac{125}{81}$$
$$\Rightarrow a \times \frac{625}{81} = \frac{125}{81}$$
$$\Rightarrow a = \frac{1}{5}$$

Thus, the 18th term of the G.P. is given by

$$a_{18} = ar^{17} = \left(\frac{1}{5}\right)\left(\frac{5}{3}\right)^{17} = \frac{(5)^{16}}{(3)^{17}}$$

Example 5: If S_n denotes the sum of *n* terms of a geometric series such that $\frac{S_{10}}{S_5} = \frac{33}{32}$, find the value of the common ratio.

Solution:

$$\frac{S_{10}}{S_5} = \frac{33}{32}$$
$$\frac{S_{2\times5}}{S_5} = \frac{33}{32}$$
$$r^5 + 1 = \frac{33}{32} \qquad \left[\frac{S_{2n}}{S_n} = r^n + 1\right]$$
$$r^5 + 1 = \left(\frac{1}{2}\right)^5 + 1$$
$$r = \frac{1}{2}$$

Thus, the value of the common ratio is $\frac{1}{2}$.

Example 6: For a G.P. with the first term as 2, if $S_\infty = \frac{5}{2}$, then find S_3.
Solution:

$$S_\infty = \frac{a}{1-r}$$
$$\frac{5}{2} = \frac{2}{1-r}$$
$$\frac{2}{5} = \frac{1-r}{2}$$
$$\frac{4}{5} = 1 - r$$
$$r = 1 - \frac{4}{5} = \frac{1}{5}$$
$$S_n = \frac{a(1-r^n)}{1-r} = \frac{2\left(1-\left(\frac{1}{5}\right)^3\right)}{1-\frac{1}{5}} = \frac{2\left(\frac{124}{125}\right)}{\frac{4}{5}} = 2\left(\frac{124}{125}\right) \times \frac{5}{4} = \frac{62}{25}$$

Example 7: The sum of three consecutive terms of a geometric progression is 21/2 and their product is 27. Find the terms.
Solution: Let the three consecutive terms of G.P. be a/r, a, ar. Then,

$$\frac{a}{r} + a + ar = \frac{21}{2}$$
$$\Rightarrow a\left(\frac{1}{r} + 1 + r\right) = \frac{21}{2}$$
$$\Rightarrow a\left(\frac{1+r+r^2}{r}\right) = \frac{21}{2} \qquad \ldots\ldots(i)$$

Also,

$$\frac{a}{r} \times a \times ar = 27$$
$$\Rightarrow a^3 = 27$$
$$\Rightarrow a = 3$$

Substituting the value of *a* in (*i*),

$$a\left(\frac{1+r+r^2}{r}\right) = \frac{21}{2}$$
$$\Rightarrow 3\left(\frac{1+r+r^2}{r}\right) = \frac{21}{2}$$
$$\Rightarrow \frac{1+r+r^2}{r} = \frac{7}{2}$$
$$\Rightarrow 2 + 2r + 2r^2 = 7r$$
$$\Rightarrow 2r^2 - 5r + 2 = 0$$
$$\Rightarrow 2r^2 - 4r - r + 2 = 0$$
$$\Rightarrow 2r(r-2) - 1(r-2) = 0$$
$$\Rightarrow (r-2)(2r-1) = 0$$

$$\Rightarrow r = 2 \text{ or } r = \frac{1}{2}$$

So, the terms are $\frac{3}{2},\ 3,\ 6 \text{ or } 6,\ 3,\ \frac{3}{2}$.

Geometric Mean and Its Relation with Arithmetic Mean

Like arithmetic mean, geometric mean (G. M.) is also a number which lies between two different numbers.

Arithmetic mean is the number which gives the common difference with the numbers on its left and right.

For example, if *a*, *b, c* is a sequence of numbers such that *b* is the arithmetic mean of *a* and *c* then we have *b* – *a* = *c* – *b*.

Similarly, geometric mean is the number which gives the common ratio with the numbers on its left and right.

For example, if *a*, *b, c* is a sequence of numbers such that *b* is the geometric mean of *a* and *c,* then we have

$$\frac{b}{a}=\frac{c}{b}$$
$$\Rightarrow b^2=ac$$
$$\Rightarrow b=\sqrt{ac}$$

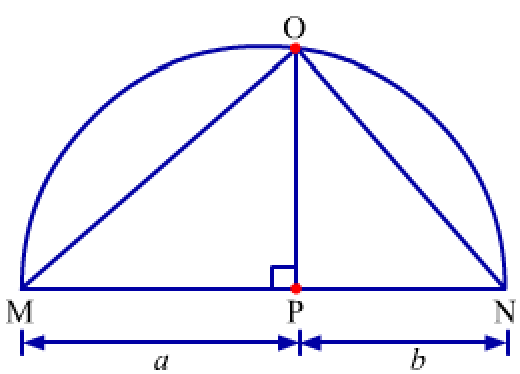
Hence the geometric mean (G.M.) of two numbers *a* and *b* is $\sqrt{ab}$ and it is denoted by *G*.

For example, the G.M. of the two numbers 36 and 25 is $\sqrt{36\times 25}$ = 30.

Note:

- *G* comes out be a real number if and only if both *a* and *b* have the same sign.
- For any two numbers *a* and *b*, there exist two values of *G* such that $\sqrt{ab}$ and $-\sqrt{ab}$.
- If *a* = *b* then $\text{GM}(G) = \pm\sqrt{aa} = \pm a$ and $\text{AM}(A) = \frac{a+a}{2} = a$.
- If we are given two numbers *a* and *b*, then we can insert a number *G* between these two numbers so that the sequence *a*, *G*, *b* becomes a G.P. Here, *G* is the **geometric mean (G.M.)** of the numbers *a* and *b*.
- For any two positive numbers, we can insert as many numbers between them as we want so that the resulting sequence becomes a G.P.

Consider a line segment MP = *a* units and PN = *b* units as shown in the following figure.



Thus, MN = MP + PN = (*a* + *b*) units

It can be seen that that a semicircle is drawn with diameter MN and OP is perpendicular to MN such that point O lies on the circle. On joining O to M and N, we get two right-angled triangles such as ΔMPO and ΔNPO.

Since ΔMON is inscribed in a semicircle, it is also a right-angled triangle such that ∠MON = 90°.

In the given figure, we have
∠MON = 90°

⇒ ∠MOP + ∠NOP = 90° ...(1)

Also,

∠PON + ∠PNO + ∠OPN = 180° (By angle sum property in ΔOPN)

⇒∠PON + ∠PNO + 90°= 180°

⇒∠PON + ∠PNO = 90° ...(2)

From (1) and (2), we get

∠MOP = ∠PNO ...(3)

Now, in ΔMPO and ΔOPN,
∠MOP = ∠PNO
and, ∠OPM = ∠OPN.
Thus, by AA similarity, these triangles are similar.

Using property of similar triangles, we get

$$\frac{MP}{OP} = \frac{OP}{PN}$$
$$\Rightarrow OP^2 = MP \times PN$$
$$\Rightarrow OP^2 = ab$$
$$\Rightarrow OP = \sqrt{ab}$$

Thus, OP is the geometric mean of MP and PN.

Hence, it can be concluded that **the length of perpendicular drawn from any point on the circumference to the diameter is the geometric mean of the lengths of the two segments obtained on diameter.**

Relation between A.M. and G.M.:

If A and G are the respective arithmetic mean and geometric mean of the numbers a and b, then we will always have the following relation between A and G:

$\boldsymbol{A \geq G}$, since $A - G = \dfrac{a+b}{2} - \sqrt{ab} = \dfrac{a+b-2\sqrt{ab}}{2} = \dfrac{\left(\sqrt{a}-\sqrt{b}\right)^2}{2} \geq 0$

Solved Examples

Example 1: Insert five numbers between $-\frac{2}{3}$ and $\frac{-2}{2187}$ so that the resulting sequence is a G.P.

Solution: Let G_1, G_2, G_3, G_4 and G_5 be the five numbers between $-\frac{2}{3}$ and $\frac{-2}{2187}$ such that $-\frac{2}{3}$, G_1, G_2, G_3, G_4, G_5, $\frac{-2}{2187}$ is a G.P.

We know that the n^{th} term of a G.P. is given by

$a_n = ar^{n-1}$

Therefore,

$$\frac{-2}{2187} = -\frac{2}{3} \times r^6$$
$$\Rightarrow r^6 = \frac{1}{729}$$
$$\Rightarrow r^6 = \left(\frac{1}{3}\right)^6$$
$$\Rightarrow r = \pm\frac{1}{3}$$

For $r = \frac{1}{3}$,

$G_1 = ar = -\frac{2}{3} \times \frac{1}{3} = -\frac{2}{9}$

$G_2 = ar^2 = -\frac{2}{3}\times\left(\frac{1}{3}\right)^2 = -\frac{2}{27}$

$G_3 = ar^3 = -\frac{2}{3}\times\left(\frac{1}{3}\right)^3 = -\frac{2}{81}$

$G_4 = ar^4 = -\frac{2}{3}\times\left(\frac{1}{3}\right)^4 = -\frac{2}{243}$

$G_5 = ar^5 = -\frac{2}{3}\times\left(\frac{1}{3}\right)^5 = -\frac{2}{729}$

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$G_5 = ar^5 = -\frac{2}{3}\times\left(-\frac{1}{3}\right)^5 = \frac{2}{729}$

Thus, we can obtain two G.P.s by inserting five numbers between the given numbers $-\frac{2}{3}$ and $\frac{-2}{2187}$. The two G.P.s are

$-\frac{2}{3}, -\frac{2}{9}, -\frac{2}{27}, -\frac{2}{81}, -\frac{2}{243}, -\frac{2}{729}, \frac{-2}{2187}$ and $-\frac{2}{3}, \frac{2}{9}, -\frac{2}{27}, \frac{2}{81}, -\frac{2}{243}, \frac{2}{729}, \frac{-2}{2187}$.

Example 2: The ratio of two numbers is 4:1. If the arithmetic mean of the two numbers is 5 more than their geometric mean, then find the two numbers.

Solution:

It is given that the ratio of the two numbers is 4:1. Hence, let the two numbers be $4x$ and x.

The arithmetic mean, A, is given by

$$\frac{4x+x}{2}=\frac{5x}{2}$$

The geometric mean, G, is given by

$$\sqrt{4x\times x}=\sqrt{4x^2}=2x$$

It is given that the arithmetic mean of the two numbers is 5 more than their geometric mean. Therefore,

$$\frac{5x}{2}=2x+5$$
$$\Rightarrow 5x=4x+10$$
$$\Rightarrow x=10$$

Thus, the two numbers are 10 and 4 × 10 = 40.

Sum of n Terms of Special Series

- The sum of the first n natural numbers is given by $1 + 2 + 3 + 4 + \ldots + n = \frac{n(n+1)}{2}$
- The sum of squares of the first n natural numbers is given by

$$1^2+2^2+3^2+4^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$$

- The sum of cubes of the first n natural numbers is given by

$$1^3+2^3+3^3+4^3+\ldots+n^3=\frac{\left[n(n+1)\right]^2}{4}$$

Example 1 Find the sum of n terms of the series: 8 + 17 + 28 + 41 + 56 +...

Solution:

The given series is 8 + 17 + 28 + 41 + 56 + ……

Let $S_n = 8 + 17 + 28 + 41 + 56 + \ldots + a_{n-1} + a_n \ldots$ (1)

$S_n = 8 + 17 + 28 + 41 + 56 + \ldots + a_{n-1} + a_n$ … (2)

On subtracting (2) from (1), we obtain

$0 = 8 + [9 + 11 + 13 + 15 + \ldots (n-1) \text{ terms}] - a_n$

$$\Rightarrow 0 = 8 + \frac{(n-1)\left[2\times 9+(n-2)\times 2\right]}{2} - a_n$$

$\Rightarrow 0 = 8 + (n-1)(n+7) - a_n$

$\Rightarrow 0 = 8 + (n-1)(n+7) - a_n$

$\Rightarrow a_n = n^2 + 6n + 1$

Hence, $S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n}\left(n^2+6n+1\right)$

$$= \sum_{k=1}^{n} n^2 + 6\sum_{k=1}^{n} n + n$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{6n(n+1)}{2} + n$$

$$= \frac{n(n+1)(2n+1)+18n(n+1)+6n}{6}$$

$$= \frac{n\left(2n^2+3n+1\right)+18\left(n^2+n\right)+6n}{6}$$

$$= \frac{2n^3+3n^2+n+18n^2+18n+6n}{6}$$

$$= \frac{2n^3+21n^2+25n}{6}$$

Example 2 Find the sum of n terms of the series whose n^{th} term is given by $a_n = n(n^2 + 1) + 3^n$.

Solution:

The n^{th} term of a series is given by $a_n = n(n^2 + 1) + 3^n = n^3 + n + 3^n$

$$\therefore S_n = \sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} k + 3^k \qquad (1)$$

Consider $\sum_{k=1}^{n} 3^k = 3^1 + 3^2 + 3^3 + \ldots$

The above series 3, 3^2, 3^3 … is a G.P. with both the first term and common ratio equal to 3.

$$\therefore \sum_{k=1}^{n} 3^k = \frac{(3)\left[(3)^n - 1\right]}{3-1} = \frac{3\left(3^n - 1\right)}{2} \qquad (2)$$

Therefore, from (1) and (2), we obtain

$$\begin{aligned} S_n &= \sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} k + \frac{3\left(3^n - 1\right)}{2} \\ &= \left\{\frac{n(n+1)}{2}\right\}^2 + \frac{n(n+1)}{2} + \frac{3\left(3^n - 1\right)}{2} \\ &= \frac{n^2(n+1)^2}{4} + \frac{n(n+1)}{2} + \frac{3\left(3^n - 1\right)}{2} \\ &= \frac{n^2(n+1)^2 + 2n(n+1) + 6\left(3^n - 1\right)}{4} \\ &= \frac{(n+1)\left\{n^2(n+1) + 2n\right\} + 6\left(3^n - 1\right)}{4} \end{aligned}$$

Example 3 Find the sum of the given series upto *n* terms.

2 × 2 + 3 × 4 + 4 × 6 + 5 × 8 + …

Solution:

The given series is 2 × 2 + 3 × 4 + 4 × 6 + 5 × 8 + …

n^{th} term, $a_n = (n + 1) \times (2n)$

$$\begin{aligned}
\therefore S_n &= \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} (k+1)(2k) \\
&= 2\sum_{k=1}^{n} k^2 + 2\sum_{k=1}^{n} k \\
&= 2\times\frac{n(n+1)(2n+1)}{6} + 2\times\frac{n(n+1)}{2} \\
&= \frac{n(n+1)(2n+1)}{3} + n(n+1) \\
&= n(n+1)\left(\frac{2n+1}{3}+1\right) \\
&= n(n+1)\left(\frac{2n+4}{3}\right) \\
&= \frac{n(n+1)(2n+4)}{3}
\end{aligned}$$