SOME SPECIAL INTEGRALS (XII, R. S. AGGARWAL)

EXERCISE 14 A (Pg.No.: 719)

Evaluate:

1. $\int \frac{1}{(1-9x^2)} dx$

Sol. Let $I = \int \frac{1}{(1-9x^2)} dx \quad \Rightarrow I = \int \frac{1}{9\left(\frac{1}{9}-x^2\right)} dx \quad \Rightarrow I = \frac{1}{9}\int \frac{1}{\left(\frac{1}{3}\right)^2 - (x)^2} dx$

$\Rightarrow I = \frac{1}{9} \cdot \frac{1}{2\left(\frac{1}{3}\right)} \log \left| \frac{\frac{1}{3}+x}{\frac{1}{3}-x} \right| + c \qquad \therefore\ I = \frac{1}{6} \log \left| \frac{1+3x}{1-3x} \right| + c$

2. $\int \frac{1}{(25-4x^2)} dx$

Sol. Let $I = \int \frac{1}{(25-4x^2)} dx \quad \Rightarrow I = \int \frac{1}{4\left(\frac{25}{4}-x^2\right)} dx \quad \Rightarrow I = \frac{1}{4}\int \frac{1}{\left(\frac{5}{2}\right)^2 - (x)^2} dx$

$\Rightarrow I = \frac{1}{4} \cdot \frac{1}{2\left(\frac{5}{2}\right)} \log \left| \frac{\frac{5}{2}+x}{\frac{5}{2}-x} \right| + c \qquad \therefore\ I = \frac{1}{20} \log \left| \frac{5+2x}{5-2x} \right| + c$

3. $\int \frac{1}{(x^2+16)} dx$

Sol. Let $I = \int \frac{1}{(x^2+16)} dx \quad \Rightarrow I = \int \frac{1}{(x)^2+(4)^2} dx \qquad \therefore\ I = \frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right) + c$

4. $\int \frac{1}{(4+9x^2)} dx$

Sol. Let $I = \int \frac{1}{(4+9x^2)} dx \quad \Rightarrow I = \int \frac{1}{9\left(\frac{4}{9}+x^2\right)} dx \quad \Rightarrow I = \frac{1}{9}\int \frac{1}{\left(\frac{2}{3}\right)^2 + (x)^2} dx$

$\Rightarrow I = \frac{1}{9} \frac{1}{\left(\frac{2}{3}\right)} \tan^{-1}\left(\frac{x}{2/3}\right) + c \qquad \therefore\ I = \frac{1}{6}\tan^{-i}\left(\frac{3x}{2}\right) + c$

5. $\int \frac{1}{(50+2x^2)} dx$

Sol. Let $I = \int \frac{1}{(50+2x^2)} dx \quad \Rightarrow I = \int \frac{1}{2(25+x^2)} dx \quad \Rightarrow I = \frac{1}{2}\int \frac{1}{(5)^2+(x)^2} dx$

$$\Rightarrow I=\frac{1}{2}.\frac{1}{5}\tan^{-1}\frac{x}{5}+c \qquad \therefore I=\frac{1}{10}\tan^{-1}\frac{x}{5}+c$$

6. $\int\frac{1}{\left(16x^2-25\right)}dx$

Sol. Let $I=\int\frac{1}{\left(16x^2-25\right)}dx \Rightarrow I=\int\frac{1}{16\left(x^2-\frac{25}{16}\right)}dx \Rightarrow I=\frac{1}{16}\int\frac{1}{(x)^2-\left(\frac{5}{4}\right)^2}dx$

$$\Rightarrow I=\frac{1}{16}.\frac{1}{2\left(\frac{5}{4}\right)}\log\left|\frac{x-\frac{5}{4}}{x+\frac{5}{4}}\right|+c \quad \therefore I=\frac{1}{40}\log\left|\frac{4x-5}{4x+5}\right|+c$$

7. $\int\frac{\left(x^2-1\right)}{\left(x^2+4\right)}dx$

Sol. Let $I=\int\frac{\left(x^2-1\right)}{\left(x^2+4\right)}dx=\int\left(\frac{x^2+4-5}{x^2+4}\right)dx=\int\left(\frac{x^2+4}{x^2+4}-\frac{5}{x^2+4}\right)dx \Rightarrow I=\int\left(1-\frac{5}{x^2+4}\right)dx$

$$\Rightarrow I=\int dx-5\int\frac{1}{\left(x^2+4\right)}dx \Rightarrow I=x-5\int\frac{1}{(x)^2+(2)^2}dx \quad \therefore I=x-\frac{5}{2}\tan^{-1}\frac{x}{2}+c$$

8. $\int\frac{x^2}{\left(9+4x^2\right)}dx$

Sol. Let $I=\int\frac{x^2}{\left(9+4x^2\right)}dx \Rightarrow I=\frac{1}{4}\int\frac{4x^2}{4x^2+9}dx=\frac{1}{4}\int\frac{4x^2+9-9}{4x^2+9}dx$

$$\Rightarrow \frac{1}{4}\int dx-\frac{9}{4}.\frac{1}{4}\int\frac{1}{\left(\frac{9}{4}+x^2\right)}dx \Rightarrow I=\frac{1}{4}\int dx-\frac{9}{16}\int\frac{1}{(x)^2+\left(\frac{3}{2}\right)^2}dx$$

$$\Rightarrow I=\frac{1}{4}x-\frac{9}{3/2}\tan^{-1}\left(\frac{x}{3/2}\right)+c=\frac{1}{4}x-\frac{9}{16}\times\frac{2}{3}\tan^{-1}\left(\frac{2x}{3}\right)+c.\ \therefore I=\frac{1}{4}x-\frac{3}{8}\tan^{-1}\left(\frac{2x}{3}\right)+c$$

9. $\int\frac{e^x}{\left(e^{2x}+1\right)}dx$

Sol. Let $I=\int\frac{e^x}{\left(e^{2x}+1\right)}dx$, Put, $e^x=t \Rightarrow e^xdx=dt$

$$\Rightarrow I=\int\frac{1}{\left(t^2+1\right)}dt \Rightarrow I=\tan^{-1}t+c \quad \therefore I=\tan^{-1}\left(e^x\right)+c$$

10. $\int\frac{\sin x}{\left(1+\cos^2 x\right)}dx$

Sol. Let $I=\int\frac{\sin x}{\left(1+\cos^2 x\right)}dx$, Put $\cos x=t \Rightarrow -\sin x\,dx=dt \Rightarrow \sin x\,dx=(-dt)$

$$\Rightarrow I=\int\frac{-dt}{\left(1+t^2\right)} \Rightarrow I=-\int\frac{1}{\left(1+t^2\right)}dt \Rightarrow I=-\tan^{-1}t+c \quad \therefore I=-\tan^{-1}(\cos x)+c$$

11. $\int \frac{\cos x}{\left(1+\sin^2 x\right)}dx$

Sol. Let $I=\int \frac{\cos x}{\left(1+\sin^2 x\right)}dx$, Put, $\sin x = t \quad \Rightarrow \cos x\; dx = dt$

$\Rightarrow I=\int \frac{1}{\left(1+t^2\right)}dt \quad \Rightarrow I=\tan^{-1} t + c \quad \therefore\; I=\tan^{-1}\left(\sin x\right)+c$

12. $\int \frac{3x^5}{\left(1+x^{12}\right)}dx$

Sol. Let $I=\int \frac{3x^5}{\left(1+x^{12}\right)}dx \;\Rightarrow I=\int \frac{3x^5}{\left\{1+\left(x^6\right)^2\right\}}dx$, Put $x^6=t \;\Rightarrow 6x^5\,dx=dt \;\Rightarrow 3x^5\,dx=\frac{dt}{2}$

$\Rightarrow I=\int \frac{1}{\left(1+t^2\right)}\frac{dt}{2} \;\Rightarrow I=\frac{1}{2}\int \frac{1}{\left(1+t^2\right)}dt \;\Rightarrow I=\frac{1}{2}\tan^{-1} t+c \quad \therefore\; I=\frac{1}{2}\tan^{-1}\left(x^6\right)+c$

13. $\int \frac{2x^3}{\left(4+x^8\right)}dx$

Sol. Let $I=\int \frac{2x^3}{\left(4+x^8\right)}dx \;\Rightarrow I=\int \frac{2x^3}{\left\{4+\left(x^4\right)^2\right\}}dx$, Put $x^4=t \;\Rightarrow 4x^3dx=dt \;\Rightarrow 2x^3\;dx=\frac{dt}{2}$

$\Rightarrow I=\int \frac{1}{\left(4+t^2\right)}\frac{dt}{2} \;\Rightarrow I=\frac{1}{2}\int \frac{1}{\left(4+t^2\right)}dt \;\Rightarrow I=\frac{1}{2}\int \frac{1}{(2)^2+(t)^2}dt$

$\Rightarrow I=\frac{1}{2}.\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)+c \quad \therefore\; I=\frac{1}{4}\tan^{-1}\left(\frac{x^4}{2}\right)+c$

14. $\int \frac{dx}{\left(e^x+e^{-x}\right)}$

Sol. Let $I=\int \frac{dx}{\left(e^x+e^{-x}\right)} \;\Rightarrow I=\int \frac{1}{\left(e^x+\frac{1}{e^x}\right)}dx \;\Rightarrow I=\int \frac{e^x}{\left(e^x\right)^2+(1)^2}dx$, Put $e^x=t \;\Rightarrow e^x dx=dt$

$\Rightarrow I=\int \frac{dt}{t^2+1} \quad \therefore\; I=\tan^{-1}\left(e^x\right)+c$

15. $\int \frac{x}{\left(1-x^4\right)}dx$

Sol. Let $I=\int \frac{x}{\left(1-x^4\right)}dx \;\Rightarrow I=\int \frac{x}{\left\{1-\left(x^2\right)^2\right\}}dx$, Put $x^2=t \;\Rightarrow 2x\;dx=dt \;\Rightarrow x\;dx=\frac{dt}{2}$

$\Rightarrow I=\int \frac{1}{\left(1-t^2\right)}\frac{dt}{2} \;\Rightarrow I=\frac{1}{2}\int \frac{1}{\left(1-t^2\right)}dt \;\Rightarrow I=\frac{1}{2}\int \frac{1}{(1)^2-(t)^2}dt$

$\Rightarrow I=\frac{1}{2}.\frac{1}{2(1)}\log\left|\frac{1+t}{1-t}\right|+c \quad \therefore\; I=\frac{1}{4}\log\left|\frac{1+x^2}{1-x^2}\right|+c$

16. $\int \frac{x^2}{\left(a^6 - x^6\right)} dx$

Sol. Let $I = \int \frac{x^2}{\left(a^6 - x^6\right)} dx \Rightarrow I = \int \frac{x^2}{\left\{a^6 - \left(x^3\right)^2\right\}} dx$, Put $x^3 = t \Rightarrow 3x^2 dx = dt \Rightarrow x^2 dx = \frac{dt}{3}$

$\Rightarrow I = \int \frac{1}{\left(a^6 - t^2\right)} \cdot \frac{dt}{3} \Rightarrow I = \frac{1}{3} \int \frac{1}{\left(a^3\right)^2 - (t)^2} dt$

$\Rightarrow I = \frac{1}{3} \cdot \frac{1}{2\left(a^3\right)} \log \left| \frac{a^3 + t}{a^3 - t} \right| + c \quad \therefore I = \frac{1}{6a^3} \log \left| \frac{a^3 + x^3}{a^3 - x^3} \right| + c$

17. $\int \frac{1}{\left(x^2 + 4x + 8\right)} dx$

Sol. Let $I = \int \frac{1}{\left(x^2 + 4x + 8\right)} dx \Rightarrow I = \int \frac{1}{(x)^2 + 2x.2 + (2)^2 - (2)^2 + 8} dx$

$\Rightarrow I = \int \frac{1}{(x+2)^2 + 4} dx \Rightarrow I = \int \frac{1}{(x+2)^2 + (2)^2} dx \quad \therefore I = \frac{1}{2} \tan^{-1}\left(\frac{x+2}{2}\right) + c$

18. $\int \frac{1}{\left(4x^2 - 4x + 3\right)} dx$

Sol. Let $I = \int \frac{1}{\left(4x^2 - 4x + 3\right)} dx \Rightarrow I = \int \frac{1}{4\left(x^2 - x + 3/4\right)} dx \Rightarrow I = \frac{1}{4} \int \frac{1}{\left(x^2 - x + 3/4\right)} dx$

$\Rightarrow I = \frac{1}{4} \int \frac{1}{(x)^2 - 2x.\frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + \left(\frac{3}{4}\right)} dx \Rightarrow I = \frac{1}{4} \int \frac{1}{\left(x - \frac{1}{2}\right)^2 + \frac{1}{2}} dx$

$\Rightarrow I = \frac{1}{4} \int \frac{1}{\left(x - \frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} dx \Rightarrow I = \frac{1}{4} \cdot \frac{1}{\left(\frac{1}{\sqrt{2}}\right)} \tan^{-1}\left[\frac{\left(x - \frac{1}{2}\right)}{\left(\frac{1}{\sqrt{2}}\right)}\right] + c \quad \therefore I = \frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{2x-1}{\sqrt{2}}\right) + c$

19. $\int \frac{1}{\left(2x^2 + x + 3\right)} dx$

Sol. Let $I = \int \frac{1}{\left(2x^2 + x + 3\right)} dx \Rightarrow I = \int \frac{1}{2\left(x^2 + \frac{x}{2} + \frac{3}{2}\right)} dx$

$\Rightarrow I = \frac{1}{2} \int \frac{1}{(x)^2 + 2x.\frac{1}{4} + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2 + \frac{3}{2}} dx \Rightarrow I = \frac{1}{2} \int \frac{1}{\left(x + \frac{1}{4}\right)^2 + \left(\frac{\sqrt{23}}{4}\right)^2} dx$

$\Rightarrow I = \frac{1}{2} \cdot \frac{1}{\sqrt{23}/4} \tan^{-1}\left[\frac{\left(x + \frac{1}{4}\right)}{\left(\frac{\sqrt{23}}{4}\right)}\right] + c \quad \therefore I = \frac{2}{\sqrt{23}} \tan^{-1}\left(\frac{4x+1}{\sqrt{23}}\right) + c$

20. $\int \frac{1}{\left(2x^2-x-1\right)}dx$

Sol. Let $I=\int \frac{1}{\left(2x^2-x-1\right)}dx \Rightarrow I=\frac{1}{2}\int \frac{1}{\left(x^2-\frac{x}{2}-\frac{1}{2}\right)}dx$

$\Rightarrow I=\frac{1}{2}\int \frac{1}{(x)^2-2x.\frac{1}{4}+\left(\frac{1}{4}\right)^2-\left(\frac{1}{4}\right)^2-\frac{1}{2}}dx \qquad \Rightarrow I=\frac{1}{2}\int \frac{1}{\left(x-\frac{1}{4}\right)^2-\left(\frac{3}{4}\right)^2}dx$

$\Rightarrow I=\frac{1}{2}.\frac{1}{2\left(\frac{3}{4}\right)}\log\left[\frac{x-\frac{1}{4}-\frac{3}{4}}{x-\frac{1}{4}+\frac{3}{4}}\right]+c \quad \Rightarrow I=\frac{1}{3}\log\left(\frac{2x-2}{2x+1}\right)+c \quad \therefore\ I=\frac{1}{3}\log\left[\frac{2(x-1)}{(2x+1)}\right]+c$

21. $\int \frac{1}{\left(3-2x-x^2\right)}dx$

Sol. Let $I=\int \frac{1}{\left(3-2x-x^2\right)}dx \quad \Rightarrow I=\int \frac{1}{-\left[x^2+2x-3\right]}dx$

$\Rightarrow I=\int \frac{1}{-\left[(x)^2+2.x.1+(1)^2-(1)^2-3\right]}dx \quad \Rightarrow I=\int \frac{1}{-\left[(x+1)^2-(2)^2\right]}dx$

$\Rightarrow I=\int \frac{1}{(2)^2-(x+1)^2}dx \quad \Rightarrow I=\frac{1}{2(2)}\log\left(\frac{2+x+1}{2-x-1}\right)+c \quad \therefore\ I=\frac{1}{4}\log\left(\frac{3+x}{1-x}\right)+c$

22. $\int \frac{x}{\left(x^2+3x+2\right)}dx$

Sol. For given $I=\int \frac{x}{\left(x^2+3x+2\right)}dx$ Let $x=A.\frac{d}{dx}\left(x^2+3x+2\right)+B$

$\Rightarrow x=A(2x+3)+B \quad \Rightarrow x=2Ax+3A+B$

Equating co-effecting we get, $2A=1 \ \therefore\ A=\frac{1}{2}$

And $0=3A+B \Rightarrow B=-3A \quad \Rightarrow B=-3\left(\frac{1}{2}\right) \quad \therefore B=\frac{-3}{2}$

$\Rightarrow I=\int \frac{A(2x+3)+B}{\left(x^2+3x+2\right)}dx \quad \Rightarrow I=A\int \frac{(2x+3)}{\left(x^2+3x+2\right)}dx+B\int \frac{1}{\left(x^2+3x+2\right)}dx$

$\Rightarrow I=\frac{1}{2}.I_1-\frac{3}{2}.I_2 \qquad \ldots(1)$

$\Rightarrow I_1=\int \frac{(2x+3)}{\left(x^2+3x+2\right)}dx$, Put $x^2+3x+2=t \quad \Rightarrow (2x+3)dx=dt$

$\Rightarrow I_1=\int \frac{dt}{t} \quad \Rightarrow I_1=\log|t|+c \quad \Rightarrow I_1=\log\left|x^2+3x+2\right|+c_1$

Now, $I_2=\int \frac{1}{x^2+3x+2}dx \quad \Rightarrow I_2=\int \frac{1}{(x)^2+2x.\frac{3}{2}+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2+2}dx$

$$\Rightarrow I_2=\int\frac{1}{\left(x+\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}dx \Rightarrow I_2=\frac{1}{2\left(\frac{1}{2}\right)}\log\left[\frac{x+\frac{3}{2}-\frac{1}{2}}{x+\frac{3}{2}+\frac{1}{2}}\right]+c$$

$$\Rightarrow I_2=\log\left(\frac{2x+3-1}{2x+3+1}\right)+c \quad \Rightarrow I_2=\log\left(\frac{2x+2}{2x+4}\right)+c \quad \Rightarrow I_2=\log\left(\frac{x+1}{x+2}\right)+c$$

Putting the value of I_1 & I_2 in equation (1), $I=\frac{1}{2}\log\left|x^2+3x+2\right|-\frac{3}{2}\log\left|\frac{x+1}{x+2}\right|+c$

23. $\int\frac{(x-3)}{\left(x^2+2x-4\right)}dx$

Sol. Let $I=\int\frac{(x-3)}{\left(x^2+2x-4\right)}dx$ and suppose $x-3=A.\frac{d}{dx}\left(x^2+2x-4\right)+B$

$$\Rightarrow x-3=A(2x+2)+B \quad \Rightarrow x-3=2Ax+2A+B$$

Equating co-efficient we get, $1=2A \quad \therefore A=\frac{1}{2}$

And $-3=2A+B \quad \Rightarrow B=-3-2A \quad \Rightarrow B=-3-2\left(\frac{1}{2}\right) \quad \therefore B=-4$

$$\Rightarrow I=\int\frac{A(2x+2)+B}{\left(x^2+2x-4\right)}dx \Rightarrow I=A\int\frac{(2x+2)}{\left(x^2+2x-4\right)}dx+B\int\frac{1}{\left(x^2+2x-4\right)}dx$$

$$\Rightarrow I=A\int\frac{(2x+2)}{\left(x^2+2x-4\right)}dx+B\int\frac{1}{\left(x^2+2x-4\right)}dx \quad \Rightarrow I=\frac{1}{2}I_1-4I_2 \qquad \ldots(1)$$

$$\Rightarrow I_1=\int\frac{(2x+2)}{\left(x^2+2x-4\right)}dx,\ \text{Put, } x^2+2x-4=t \quad \Rightarrow (2x+2)dx=dt$$

$$\Rightarrow I_1=\int\frac{dt}{t} \quad \Rightarrow I=\log|t|+c_1 \quad \Rightarrow I=\log\left|x^2+2x-4\right|+c_1$$

Now, $I_2=\int\frac{1}{\left(x^2+2x-4\right)}dx \Rightarrow I_2=\int\frac{1}{(x)^2+2.x.1+(1)^2-(1)^2-4}dx$

$$\Rightarrow I_2=\int\frac{1}{(x+1)^2-\left(\sqrt{5}\right)^2}dx \Rightarrow I_2=\frac{1}{2\sqrt{5}}\log\left(\frac{x+1-\sqrt{5}}{x+1+\sqrt{5}}\right)+c$$

Putting the value of I_1 & I_2 in equation (1), $I=\frac{1}{2}\log\left|x^2+2x-4\right|-\frac{2}{\sqrt{5}}\log\left|\frac{x+1-\sqrt{5}}{x+1+\sqrt{5}}\right|+c$

24. $\int\frac{(2x-3)}{\left(x^2+3x-18\right)}dx$

Sol. Let $I=\int\frac{(2x-3)}{\left(x^2+3x-18\right)}dx$ and suppose $(2x-3)=A.\frac{d}{dx}\left(x^2+3x-18\right)+B$

$$\Rightarrow (2x-3)=A(2x+3)+B \quad \Rightarrow 2x-3=2Ax+3A+B$$

Equating co-efficient we get, $2=2A \quad \therefore A=1$

And $-3=3A+B \quad \Rightarrow B=-3-3A=-3-3(1)=-6 \quad \therefore B=-6$

$$\Rightarrow I=\int\frac{A(2x+3)+B}{(x^2+3x-18)}dx \quad \Rightarrow I=A\int\frac{(2x+3)}{(x^2+3x-18)}dx+B\int\frac{1}{(x^2+3x-18)}dx$$

$$\Rightarrow I=I_1-6\,I_2 \qquad \ldots(1)$$

$\Rightarrow I_1=\int\frac{(2x+3)}{(x^2+3x-18)}dx$, Put $x^2+3x-18=t \;\Rightarrow (2x+3)dx=dt$

$$\Rightarrow I_1=\int\frac{dt}{t} \quad \Rightarrow I_1=\log|t|+c_1 \quad \Rightarrow I_1=\log\left|x^2+3x-18\right|+c_1$$

Now, $I_2=\int\frac{1}{(x^2+3x-18)}dx \quad \Rightarrow I_2=\int\frac{1}{(x)^2+2.x.\frac{3}{2}+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2-18}dx$

$$\Rightarrow I_2=\int\frac{1}{\left(x+\frac{3}{2}\right)^2-\left(\frac{9}{2}\right)^2}dx \quad \Rightarrow I_2=\frac{1}{2\left(\frac{9}{2}\right)}\log\left[\frac{x+\frac{3}{2}-\frac{9}{2}}{x+\frac{3}{2}+\frac{9}{2}}\right]+c_2$$

$$\Rightarrow I_2=\frac{1}{9}\log\left(\frac{2x+3-9}{2x+3+9}\right)+c_2 \quad \Rightarrow I_2=\frac{1}{9}\log\left(\frac{2x-6}{2x+12}\right)+c_2 \quad \Rightarrow I_2=\frac{1}{9}\log\left|\frac{x-3}{x+6}\right|+c_2$$

Putting the value of I_1 & I_2 in equation (1), $I=\log\left|x^2+3x-18\right|-\frac{2}{3}\log\left|\frac{x-3}{x+6}\right|+c$

25. $\int\frac{x^2}{(x^2+6x-3)}dx$

Sol. Let $I=\int\frac{x^2}{(x^2+6x-3)}dx \;\Rightarrow I=\int\left(1-\frac{6x-3}{x^2+6x-3}\right)dx$

$$\Rightarrow I=\int dx-\int\frac{6x-3}{x^2+6x-3}dx \quad \Rightarrow I=x-\int\frac{6x-3}{x^2+6x-3}dx$$

Now let $6x-3=A(2x+6)+B \;\Rightarrow 6x-3=2Ax+6A+B$

Equating co-efficient, we get, $6=2A \;\therefore A=3$

And $-3=6A+B \quad \Rightarrow B=-3-6A=-3-6(3)=-3-18=-21 \quad \therefore\; B=-21$

$$\Rightarrow I=x-\int\frac{A(2x+6)+B}{(x^2+6x-3)}dx \quad \Rightarrow I=x-\left\{A\int\frac{2x+6}{x^2+6x-3}dx+B\int\frac{1}{x^2+6x-3}dx\right\}$$

$$\Rightarrow I=x-(3I_1-21I_2) \qquad \ldots(1)$$

$\Rightarrow I_1=\int\frac{2x+6}{x^2+6x-3}dx$, Put $x^2+6x-3=t \;\Rightarrow (2x+6)dx=dt$

$$\Rightarrow I_1=\int\frac{dt}{t} \quad \Rightarrow I_1=\log|t|+c_1 \quad \therefore\; I_1=\log\left|x^2+6x-3\right|+c$$

Now, $I_2=\int\frac{1}{(x^2+6x-3)}dx \quad \Rightarrow I_2=\int\frac{1}{(x)^2+2.x.3+(3)^2-(3)^2-3}dx$

$$\Rightarrow I_2=\int\frac{1}{(x+3)^2-(2\sqrt{3})^2} \quad \Rightarrow I_2=\frac{1}{2(2\sqrt{3})}\log\left|\frac{x+3-2\sqrt{3}}{x+3+2\sqrt{3}}\right|+c_2$$

$$\Rightarrow I_1 = \frac{1}{4\sqrt{3}}\log\left(\frac{x+3-2\sqrt{3}}{x+3+2\sqrt{3}}\right)+c$$

Putting the values of I_1 & I_2 in equation (i), we get,

$$I = x - \left\{3\log\left|x^2+6x-3\right| + (-21)\frac{1}{4\sqrt{3}}\log\left(\frac{x+3-2\sqrt{3}}{x+3+2\sqrt{3}}\right)\right\}+c$$

$$\therefore\quad I = x - 3\log\left|x^2+6x-3\right| + \frac{7\sqrt{3}}{4}\log\left|\frac{x+3-2\sqrt{3}}{x+3+2\sqrt{3}}\right|+c$$

26. $\displaystyle\int\frac{(2x-1)}{(2x^2+2x+1)}dx$

Sol. Let $I = \displaystyle\int\frac{(2x-1)}{(2x^2+2x+1)}dx$ and suppose $2x-1 = A.\frac{d}{dx}(2x^2+2x+1)+B$

$$\Rightarrow 2x-1 = A(4x+2)+B \quad \Rightarrow 2x-1 = 4Ax+2A+B$$

Equating co-efficient both side, we get, $2 = 4A \quad \therefore\ A = \frac{1}{2}$

And $-1 = 2A+B \quad \Rightarrow B = -1-2A = -1-2\left(\frac{1}{2}\right) = -2 \quad \therefore\ B = -2$

$$\Rightarrow I = \int\frac{A(4x+2)+B}{(2x^2+2x+1)}dt \quad \Rightarrow I = A\int\frac{(4x+2)}{(2x^2+2x+1)}dx + B\int\frac{1}{(2x^2+2x+1)}dx$$

$$\Rightarrow I = \frac{1}{2}I_1 - 2I_2 \qquad \ldots(1)$$

$$\Rightarrow I_1 = \int\frac{4x+2}{2x^2+2x+1}dx,\quad \text{Put } 2x^2+2x+1 = t \ \Rightarrow (4x+2)dx = dt$$

$$\Rightarrow I_1 = \int\frac{dt}{t} \quad \Rightarrow I_1 = \log|t|+c_1 \quad \Rightarrow I_1 = \log\left|2x^2+2x+1\right|+c_1$$

Now, $I_2 = \displaystyle\int\frac{1}{(2x^2+2x+1)}dx \quad \Rightarrow I_2 = \frac{1}{2}\int\frac{1}{\left(x^2+x+\frac{1}{2}\right)}dx$

$$\Rightarrow I_2 = \frac{1}{2}\int\frac{1}{(x)^2+2.x.\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)}dx \quad \Rightarrow I_2 = \frac{1}{2}\int\frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}dx$$

$$\Rightarrow I_2 = \frac{1}{2}.\frac{1}{1/2}\tan^{-1}\left[\frac{(x+1/2)}{1/2}\right]+c \quad \Rightarrow I_2 = \tan^{-1}(2x+1)+c$$

Putting the value of I_1 & I_2 in equation (1), we get,

$$I = \frac{1}{2}\log\left|2x^2+2x+1\right| + (-2)\tan^{-1}(2x+1)+c$$

$$\therefore\quad I = \frac{1}{2}\log\left|2x^2+2x+1\right| - 2\tan^{-1}(2x+1)+c$$

27. $\int \frac{(1-3x)}{(3x^2+4x+2)}dx$

Sol. Let $I = \int \frac{(1-3x)}{(3x^2+4x+2)}dx$ and suppose $(1-3x) = A.\frac{d}{dx}(3x^2+4x+2)+B$

$\Rightarrow 1-3x = A(6x+4)+B \quad \Rightarrow 1-3x = 6Ax+4A+B$

Equating co-efficient both side, we get, $-3 = 6A \quad \therefore A = -\frac{1}{2}$

And $1 = 4A+B \quad \Rightarrow B = 1-4A = 1-4\left(-\frac{1}{2}\right) = 3$

$\Rightarrow I = \int \frac{A(6x+4)+B}{(3x^2+4x+2)}dx + B\int \frac{1}{(3x^2+4x+2)}dx \quad \Rightarrow I = -\frac{1}{2}I_1 + 3I_2 \qquad \ldots(1)$

$\Rightarrow I_1 = \int \frac{6x+4}{3x^2+4x+2}dx$, Put $3x^2+4x+2 = t \quad \Rightarrow (6x+4)dx = dt$

$\Rightarrow I_1 = \int \frac{dt}{t} \quad \Rightarrow I_1 = \log|t| + c_1 \quad \Rightarrow I_1 = \log|3x^2+4x+2| + c_1$

Now, $I_2 = \int \frac{1}{(3x^2+4x+2)}dx \quad \Rightarrow I_2 = \frac{1}{3}\int \frac{1}{\left(x^2+\frac{4x}{3}+\frac{2}{3}\right)}dx$

$\Rightarrow I_2 = \frac{1}{3}\int \frac{1}{(x)^2+2.x.\frac{2}{3}+\left(\frac{2}{3}\right)^2-\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)}dx \quad \Rightarrow I_2 = \frac{1}{3}\int \frac{1}{\left(x+\frac{2}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2}$

$\Rightarrow I_2 = \frac{1}{3}.\frac{1}{\frac{\sqrt{2}}{3}}\tan^{-1}\left[\frac{\left(x+\frac{2}{3}\right)}{\left(\frac{\sqrt{2}}{3}\right)}\right]+c \quad \Rightarrow I_2 = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+2}{\sqrt{2}}\right)+c$

Putting the value of I_1 & I_2 in equation (1), we get,

$$I = -\frac{1}{2}\log|3x^2+4x+2| + 3\left(\frac{1}{\sqrt{2}}\right)\tan^{-1}\left(\frac{3x+2}{\sqrt{2}}\right)+c$$

$$\therefore \quad I = -\frac{1}{2}\log|3x^2+4x+2| + \frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{3x+2}{\sqrt{2}}\right)+c$$

28. $\int \frac{2x}{(2+x-x^2)}dx$

Sol. Let $I = \int \frac{2x}{(2+x-x^2)}dx$ and suppose $2x = A.\frac{d}{dx}(2+x-x^2)+B$

$\Rightarrow 2x = A(1-2x)+B \quad \Rightarrow 2x = A-2Ax+B$

Equating co-efficient both side, we get, $2 = -2A \quad \therefore A = -1$

And $0 = A+B \quad \Rightarrow B = -A = 1$

$\Rightarrow I = \int \frac{A(1-2x)+B\,dx}{(2+x-x^2)} \quad \Rightarrow I = A\int \frac{(1-2x)}{(2+x-x^2)}dx + B\int \frac{1}{(2+x-x^2)}dx$

$\Rightarrow I = -I_1 + I_2 \qquad \ldots(1)$

$\Rightarrow I_1 = \int \frac{1-2x}{2+x-x^2}dx$, Put $2+x-x^2 = t \Rightarrow (1-2x)dx = dt$

$\Rightarrow I_1 = \int \frac{dt}{t} \quad \Rightarrow I_1 = \log|t| + c_1 \quad \Rightarrow I_1 = \log\left|2+x-x^2\right| + c_1$

Now, $I_2 = \int \frac{1}{\left(2+x-x^2\right)}dx \quad \Rightarrow I_2 = \int \frac{1}{-\left[x^2-x-2\right]}dx$

$\Rightarrow I_2 = \int \frac{1}{-\left[\left(x^2\right)-2.x.\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2-2\right]}dx \quad \Rightarrow I_2 = \int \frac{1}{-\left[\left(x-\frac{1}{2}\right)^2-\left(\frac{3}{2}\right)^2\right]}dx$

$\Rightarrow I_2 = \int \frac{1}{\left(\frac{3}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}dx \quad \Rightarrow I_2 = \frac{1}{2\left(\frac{3}{2}\right)}\log\left|\frac{\frac{3}{2}+x-\frac{1}{2}}{\frac{3}{2}-x+\frac{1}{2}}\right| + c_2$

$\Rightarrow I_2 = \frac{1}{3}\log\left|\frac{3+2x-1}{3-2x+1}\right| + c_2 \quad \Rightarrow I_2 = \frac{1}{3}\log\left|\frac{2+2x}{4-2x}\right| + c_2 \quad \Rightarrow I_2 = \frac{1}{3}\log\left|\frac{1+x}{2-x}\right| + c_2$

Putting the value of I_1 & I_2 in equation (1), we get, $I = -\log\left|2+x-x^2\right| + \frac{1}{3}\log\left|\frac{1+x}{2-x}\right| + c$

29. $\int \frac{1}{\left(1+\cos^2 x\right)}dx$

Sol. Let $I = \int \frac{1}{\left(1+\cos^2 x\right)}dx$

$\Rightarrow I = \int \frac{\frac{1}{\cos^2 x}}{\frac{1+\cos^2 x}{\cos^2 x}}dx \quad \Rightarrow I = \int \frac{\sec^2 x}{\left(\sec^2 x+1\right)}dx \quad \Rightarrow I = \int \frac{\sec^2 x}{\left(1+\tan^2 x\right)+1}dx$

$\Rightarrow I = \int \frac{\sec^2 x}{\left(2+\tan^2 x\right)}dx$, Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$\Rightarrow I = \int \frac{dt}{2+t^2} \quad \Rightarrow I = \int \frac{dt}{\left(\sqrt{2}\right)^2+(t)^2} \quad \Rightarrow I = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)+c \quad \therefore I = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right)+c$

30. $\int \frac{dx}{\left(2+\sin^2 x\right)}$

Sol. Let $I = \int \frac{dx}{\left(2+\sin^2 x\right)}dx$

$\Rightarrow I = \int \frac{\frac{1}{\cos^2 x}}{\frac{2+\sin^2 x}{\cos^2 x}}dx \quad \Rightarrow I = \int \frac{\sec^2 x}{2\sec^2 x+\tan^2 x}dx \quad \Rightarrow I = \int \frac{\sec^2 x}{2\left(1+\tan^2 x\right)+\tan^2 x}dx$

$\Rightarrow I = \int \frac{\sec^2 x}{2+2\tan^2 x+\tan^2 x}dx \quad \Rightarrow I = \int \frac{\sec^2 x}{2+3\tan^2 x}dx$, Put $\tan x = t \quad \Rightarrow \sec^2 x\, dx = dt$

$$\Rightarrow I=\int\frac{dt}{\left(2+3t^2\right)} \quad \Rightarrow I=\frac{1}{3}\int\frac{1}{\left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2+(t)^2}dt$$

$$\Rightarrow I=\frac{1}{3}.\frac{1}{\left(\sqrt{2}/\sqrt{3}\right)}\tan^{-1}\left(\frac{t}{\sqrt{2}/\sqrt{3}}\right)+c \quad \therefore I=\frac{1}{\sqrt{6}}\tan^{-1}\left(\frac{\sqrt{3}\tan x}{\sqrt{2}}\right)+c$$

31. $\int\frac{1}{\left(a^2\cos^2 x+b^2\sin^2 x\right)}dx$

Sol. Let $I=\int\frac{1}{\left(a^2\cos^2 x+b^2\sin^2 x\right)}dx$,

$$\Rightarrow I=\int\frac{\frac{1}{\cos^2 x}}{\left(\frac{a^2\cos^2 x+b^2\sin^2 x}{\cos^2 x}\right)}dx \quad \Rightarrow I=\int\frac{\sec^2 x}{a^2+b^2\tan^2 x}dx \quad \Rightarrow I=\frac{1}{b^2}\int\frac{\sec^2 x}{\frac{a^2}{b^2}+\tan^2 x}dx$$

Put $\tan x=t \quad \Rightarrow \sec^2 x\,dx=dt$, we get, $I=\frac{1}{b^2}\int\frac{dt}{\left(\frac{a}{b}\right)^2+(t)^2}$

$$\Rightarrow I=\frac{1}{b}.\frac{1}{(a/b)}\tan^{-1}\left(\frac{t}{a/b}\right)+c \quad \therefore I=\frac{1}{ab}\tan^{-1}\left(\frac{b\tan x}{a}\right)+c$$

32. $\int\frac{dx}{\left(\cos^2 x-3\sin^2 x\right)}$

Sol. Let $I=\int\frac{1}{\left(\cos^2 x-3\sin^2 x\right)}dx$,

$$\Rightarrow I=\int\frac{\frac{1}{\cos^2 x}}{\frac{\cos^2 x-3\sin^2 x}{\cos^2 x}}dx \quad \Rightarrow I=\int\frac{\sec^2 x}{1-3\tan^2 x}dx$$

Put $\tan x=t \quad \Rightarrow \sec^2 x\,dx=dt$; we get, $I=\int\frac{dt}{1-3t^2} \quad \Rightarrow I=\frac{1}{3}\int\frac{1}{\frac{1}{3}-t^2}dt$

$$\Rightarrow I=\frac{1}{3}\int\frac{1}{\left(\frac{1}{\sqrt{3}}\right)^2-(t)^2}dt \quad \Rightarrow I=\frac{1}{3}\frac{1}{2.\left(\frac{1}{\sqrt{3}}\right)}\log\left|\frac{\frac{1}{\sqrt{3}}+t}{\frac{1}{\sqrt{3}}-t}\right|+c \quad \therefore I=\frac{1}{2\sqrt{3}}\log\left|\frac{1+\sqrt{3}\tan x}{1-\sqrt{3}\tan x}\right|+c$$

33. $\int\frac{1}{\left(\sin^2 x-4\cos^2 x\right)}dx$

Sol. Let $I=\int\frac{1}{\left(\sin^2 x-4\cos^2 x\right)}dx$, Dividing numerator and denominator by $\cos^2 x$.

$$\Rightarrow I=\int\frac{\frac{1}{\cos^2 x}}{\left(\frac{\sin^2 x-4\cos^2 x}{\cos^2 x}\right)}dx \quad \Rightarrow I=\int\frac{\sec^2 x}{\tan^2 x-4}dx \Rightarrow I=\int\frac{dt}{(t)^2-(2)^2}$$

Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$, we get, $I = \frac{1}{2(2)}.\log\left|\frac{t-2}{t+2}\right| + c \quad \therefore\ I = \frac{1}{4}\log\left|\frac{\tan x - 2}{\tan x + 2}\right| + c$

34. $\int \frac{1}{\left(\sin x \cos x + 2\cos^2 x\right)} dx$

Sol. Let $I = \int \frac{1}{\left(\sin x \cos x + 2\cos^2 x\right)} dx$, Dividing numerator and denominator by $\cos^2 x$.

$$\Rightarrow I = \int \frac{\frac{1}{\cos^2 x}}{\left(\frac{\sin x \cos x + 2\cos^2 x}{\cos^2 x}\right)} dx \quad \Rightarrow I = \int \frac{\sec^2 x}{\left(\frac{\sin x \cos x}{\cos^2 x} + \frac{2\cos^2 x}{\cos^2 x}\right)} dx \quad \Rightarrow I = \int \frac{\sec^2 x}{(\tan x + 2)} dx$$

Put $\tan x + 2 = t \quad \Rightarrow \sec^2 x\, dx = dt$, we get, $I = \int \frac{dt}{t} \quad \Rightarrow I = \log|t| + c \quad \therefore\ I = \log|\tan x + 2| + c$

35. $\int \frac{\sin 2x}{\sin^4 x + \cos^4 x} dx$

Sol. Let $I = \int \frac{\sin 2x}{\sin^4 x + \cos^4 x} dx$, Dividing numerator and denominator by $\cos^4 x$.

$$\Rightarrow I = \int \frac{\frac{\sin 2x}{\cos^4 x}}{\frac{\sin^4 x + \cos^4 x}{\cos^4 x}} dx = \int \frac{\frac{2\sin x.\cos x}{\cos^4 x}}{\frac{\sin^4 x + \cos^4 x}{\cos^4 x}} dx$$

$\Rightarrow I = \int \frac{2\tan x.\sec^2 x}{1 + \left(\tan^2 x\right)^2} dx$, Put $\tan^2 x = t \quad \Rightarrow 2\tan x.\sec^2 x\, dx = dt$

$$\Rightarrow I = \int \frac{dt}{1+t^2} \quad \Rightarrow I = \tan^{-1}(t) + c \quad \therefore\ I = \tan^{-1}\left(\tan^2 x\right) + c$$

36. $\int \frac{(2\sin 2\phi - \cos\phi)}{\left(6 - \cos^2\phi - 4\sin\phi\right)} d\phi$

Sol. Let $I = \int \left(\frac{2\sin 2\phi - \cos\phi}{6 - \cos^2\phi - 4\sin\phi}\right) d\phi \quad \Rightarrow I = \int \frac{2.2\sin\phi.\cos\phi - \cos\phi}{6 - \left(1 - \sin^2\phi\right) - 4\sin\phi} d\phi$

$$\Rightarrow I = \int \frac{(4\sin\phi - 1)\cos\phi}{6 - 1 + \sin^2\phi - 4\sin\phi} d\phi \quad \Rightarrow I = \int \frac{(4\sin\phi - 1)\cos\phi}{\sin^2\phi - 4\sin\phi + 5} d\phi$$

Put $\sin\phi = t \Rightarrow \cos\phi\, d\phi = dt$, we get,

$I = \int \frac{(4t-1)}{\left(t^2 - 4t + 5\right)} dt$ Now suppose $4t - 1 = A(2t - 4) + B \Rightarrow 4t - 1 = 2At - 4A + B$

Equating co-efficient both side, we get, $4 = 2A \quad \therefore\ A = 2$

And $-1 = -4A + B \quad \Rightarrow B = -1 + 4A = -1 + 4(2) = -1 + 8 = 7$

$$\Rightarrow I = \int \frac{A(2t-4) + B}{\left(t^2 - 4t + 5\right)} dt \quad \Rightarrow I = A\int \frac{(2t-4)}{\left(t^2 - 4t + 5\right)} dt + B\int \frac{1}{\left(t^2 - 4t + 5\right)} dt$$

$\Rightarrow I = 2I_1 + 7I_2 \qquad \ldots(1)$

$$\Rightarrow I_1 = \int \frac{2t-4}{t^2 - 4t - 5} dt$$

Put $t^2-4t+5=t \Rightarrow (2t-4)dt=dt$, we get,

$$I_1=\int\frac{dt}{t} \Rightarrow I_1=\log|t|+c_1 \Rightarrow I_1=\log\left|t^2-4t+5\right|+c_1$$

Now, $I_2=\int\frac{1}{(t^2-4t+5)}dt \Rightarrow I_2=\int\frac{1}{(t)^2-2.t.2+(2)^2-(2)^2+5}dt$

$$\Rightarrow I_2=\int\frac{1}{(t-2)^2+(1)^2} \Rightarrow I_2=\tan^{-1}\left(\frac{t-2}{1}\right)+c_2 \Rightarrow I_2=\tan^{-1}(\sin\phi-2)+c_2$$

Putting the value of I_1 & I_2 in equation (1), we get,

$$I=2\log\left|t^2-4t+5\right|+7\tan^{-1}(\sin\phi-2)+c$$

$$\therefore\ I=2\log\left|\sin^2\phi-4\sin\phi+5\right|+7\tan^{-1}(\sin\phi-2)+c$$

37. $\int\frac{dx}{(\sin x-2\cos x)(2\sin x+\cos x)}$

Sol. Let $I=\int\frac{1}{(\sin x-2\cos x)(2\sin x+\cos x)}dx$

$$\Rightarrow I=\int\frac{1}{2\sin^2 x+\sin x\cos x-4\sin x\cos x-2\cos^2 x}dx$$

$\Rightarrow I=\int\frac{1}{2\sin^2 x-2\cos^2 x-3\sin x\cos x}dx$, dividing numerator and denominator by $\cos^2 x$

$$\Rightarrow I=\int\frac{\sec^2 x}{2\tan^2 x-2-3\tan x}dx$$

Put $\tan x=t \Rightarrow \sec^2 x\,dx=dt$, we get,

$$I=\int\frac{dt}{2t^2-3t-2} \Rightarrow I=\int\frac{1}{2\left(t^2-\frac{3}{2}t-1\right)}dt \Rightarrow I=\frac{1}{2}\int\frac{1}{(t)^2-2.t.\frac{3}{4}+\left(\frac{3}{4}\right)^2-\left(\frac{3}{4}\right)^2-1}dt$$

$$\Rightarrow I=\frac{1}{2}\int\frac{1}{\left(t-\frac{3}{4}\right)^2-\frac{25}{16}}dt \Rightarrow I=\frac{1}{2}\int\frac{1}{\left(t-\frac{3}{4}\right)^2-\left(\frac{5}{4}\right)^2}dt$$

$$\Rightarrow I=\frac{1}{2}\cdot\frac{1}{2.\left(\frac{5}{4}\right)}\log\left|\frac{t-\frac{3}{4}-\frac{5}{4}}{t-\frac{3}{4}+\frac{5}{4}}\right|+c \Rightarrow I=\frac{1}{5}\log\left|\frac{t-2}{2t+1}\right|+c \quad \therefore\ I=\frac{1}{5}\log\left|\frac{\tan x-2}{2\tan x+1}\right|+c$$

38. $\int\frac{(1-x^2)}{(1+x^4)}dx$

Sol. Let $I=\int\frac{(1-x^2)}{(1+x^4)}dx \Rightarrow I=\int\frac{\frac{1}{x^2}-1}{x^2+\frac{1}{x^2}}dx=\int\frac{\left(1-\frac{1}{x^2}\right)}{x^2+\frac{1}{x^2}+2-2}dx$

$$\Rightarrow I=-\int\frac{\left(1-\frac{1}{x^2}\right)}{\left(x+\frac{1}{x}\right)^2-\left(\sqrt{2}\right)^2}dx,\ \text{Put } t=x+\frac{1}{x}\ \Rightarrow dt=\left(1-\frac{1}{x^2}\right)dx$$

$$I=-\int\frac{dt}{t^2-\left(\sqrt{2}\right)^2}-\int\frac{dt}{\left(\sqrt{2}\right)^2-t^2}=\frac{1}{2\sqrt{2}}\log\left(\frac{\sqrt{2}+t}{\sqrt{2}-t}\right)+c=\frac{1}{2\sqrt{2}}\log\left(\frac{\sqrt{2}+x+\frac{1}{x}}{\sqrt{2}-x-\frac{1}{x}}\right)+c$$

39. $\int\frac{\left(x^2+1\right)}{\left(x^4+x^2+1\right)}dx$

Sol. Let $I=\int\frac{\left(x^2+1\right)}{\left(x^4+x^2+1\right)}dx \quad \Rightarrow I=\int\frac{x^2\left(1+\frac{1}{x^2}\right)}{x^2\left(x^2+1+\frac{1}{x^2}\right)}dx \quad \Rightarrow I=\int\frac{\left(1+\frac{1}{x^2}\right)}{\left(x^2+\frac{1}{x^2}\right)+1}dx$

$$\Rightarrow I=\int\frac{\left(1+\frac{1}{x^2}\right)}{\left[(x)^2-2.x.\frac{1}{x}+\left(\frac{1}{x}\right)^2+2\right]+1}dx \quad \Rightarrow I=\int\frac{\left(1+\frac{1}{x^2}\right)}{\left(x-\frac{1}{x}\right)^2+\left(\sqrt{3}\right)^2}dx$$

Put $\left(x-\frac{1}{x}\right)=t \quad \Rightarrow\left(1+\frac{1}{x^2}\right)dx=dt$, we get, $I=\int\frac{\left(1+\frac{1}{x^2}\right)}{\left(x-\frac{1}{x}\right)^2+\left(\sqrt{3}\right)^2}dx$

$$\Rightarrow I=\int\frac{dt}{(t)^2+\left(\sqrt{3}\right)^2} \quad \Rightarrow I=\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right)+c$$

$$\Rightarrow I=\frac{1}{\sqrt{3}}\tan^{-1}\left[\frac{\left(x-\frac{1}{x}\right)}{\sqrt{3}}\right]+c \quad \therefore\ I=\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{3}x}\right)+c$$

40. $\int\frac{dx}{\sin^4x+\cos^4x}$

Sol. Let $I=\int\frac{1}{\sin^4x+\cos^4x}dx$, Dividing numerator and denominator by $\cos^4x$.

$$\Rightarrow I=\int\frac{\frac{1}{\cos^4x}}{\frac{\sin^4x+\cos^4x}{\cos^4x}}dx \quad \Rightarrow I=\int\frac{\sec^4x}{\tan^4x+1}dx \quad \Rightarrow I=\int\frac{\left(1+\tan^2x\right)\sec^2x}{\left(\tan^4x+1\right)}dx$$

Put $\tan x=t \quad \Rightarrow \sec^2x\,dx=dt$, we get, $I=\int\frac{t^2+1}{t^4+1}dt \quad \Rightarrow I=\int\frac{t^2\left(1+\frac{1}{t^2}\right)}{t^2\left(t^2+\frac{1}{t^2}\right)}dt$

$$\Rightarrow I=\int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}dt \quad \Rightarrow I=\int \frac{1+\frac{1}{t^2}}{(t)^2-2.t.\frac{1}{t}+\left(\frac{1}{t}\right)^2+2}dt \quad \Rightarrow I=\int \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+\left(\sqrt{2}\right)^2}dt$$

Put $t-\frac{1}{t}=y \quad \Rightarrow \left(1+\frac{1}{t^2}\right)dt=dy$, we get,

$$I=\int \frac{dy}{y^2+\left(\sqrt{2}\right)^2} \quad \Rightarrow I=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{y}{\sqrt{2}}\right)+c \quad \Rightarrow I=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t-\frac{1}{t}}{\sqrt{2}}\right)+c$$

$$\Rightarrow I=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t^2-1}{\sqrt{2}t}\right)+c \qquad \Rightarrow I=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan^2 x-1}{\sqrt{2}\tan x}\right)+c$$

EXERCISE 14B (Pg.No.: 732)

1. $\int \frac{1}{\sqrt{16-x^2}}dx$

Sol. Let $I=\int \frac{1}{\sqrt{16-x^2}}dx \quad \Rightarrow I=\int \frac{1}{\sqrt{(4)^2-(x)^2}}dx \quad \therefore\ I=\sin^{-1}\left(\frac{x}{4}\right)+c$

2. $\int \frac{1}{\sqrt{1-9\,x^2}}\,dx$

Sol. Let $I=\int \frac{1}{\sqrt{1-9x^2}}dx \quad \Rightarrow I=\frac{1}{3}\int \frac{1}{\sqrt{\frac{1}{9}-x^2}}dx \quad \Rightarrow I=\frac{1}{3}\int \frac{1}{\sqrt{\left(\frac{1}{3}\right)^2-(x)^2}}dx$

$\Rightarrow I=\frac{1}{3}\sin^{-1}\left(\frac{x}{1/3}\right)+c \quad \therefore\ I=\frac{1}{3}\sin^{-1}(3x)+c$

3. $\int \frac{1}{\sqrt{15-8x^2}}dx$

Sol. Let $I=\int \frac{1}{\sqrt{15-8x^2}}dx \quad \Rightarrow I=\frac{1}{\sqrt{8}}\int \frac{1}{\sqrt{\frac{15}{8}-x^2}}dx \quad \Rightarrow I=\frac{1}{2\sqrt{2}}\int \frac{1}{\sqrt{\left(\frac{\sqrt{15}}{2\sqrt{2}}\right)^2-(x)^2}}\,dx$

$\Rightarrow I=\frac{1}{2\sqrt{2}}\sin^{-1}\left(\frac{x}{\sqrt{\frac{15}{8}}}\right)+c \quad \therefore\ I=\frac{1}{2\sqrt{2}}\sin^{-1}\left(\sqrt{\frac{8}{15}}x\right)+c$

4. $\int \frac{dx}{\sqrt{x^2-4}}$

Sol. Let $I=\int \frac{dx}{\sqrt{x^2-4}} \quad \Rightarrow I=\int \frac{1}{\sqrt{(x)^2-(2)^2}}dx$

$\because \int \frac{1}{\sqrt{x^2-a^2}}dx=\log\left[x+\sqrt{x^2-a^2}\right]+c$

$\Rightarrow I=\log\left|x+\sqrt{x^2-(2)^2}\right|+c \quad \therefore\ I=\log\left|x+\sqrt{x^2-4}\right|+c$

5. $\int \frac{1}{\sqrt{4x^2-1}}dx$

Sol. Let $I=\int \frac{1}{\sqrt{4x^2-1}}dx \quad \Rightarrow I=\frac{1}{2}\int \frac{1}{\sqrt{(x)^2-1/4}}dx \quad \Rightarrow I=\frac{1}{2}\int \frac{1}{\sqrt{(x)^2-\left(\frac{1}{2}\right)^2}}dx$

$\Rightarrow I=\frac{1}{2}\log\left|x+\sqrt{x^2-1/4}\right|+c \quad \therefore\ I=\frac{1}{2}\log\left|2x+\sqrt{4x^2-1}\right|+c$

6. $\int \frac{1}{\sqrt{9x^2-7}}dx$

Sol. Let $I=\int\frac{1}{\sqrt{9x^2-7}}dx \quad \Rightarrow I=\frac{1}{3}\int\frac{1}{\sqrt{x^2-\frac{7}{9}}}dx \quad \Rightarrow I=\frac{1}{3}\int\frac{1}{\sqrt{(x)^2-\left(\frac{\sqrt{7}}{3}\right)^2}}dx$

$\Rightarrow I=\frac{1}{3}\log\left|x+\sqrt{(x)^2-\left(\frac{\sqrt{7}}{3}\right)^2}\right|+c \quad \Rightarrow I=\frac{1}{3}\log\left|x+\sqrt{x^2-\frac{7}{9}}\right|+c$

$\therefore I=\frac{1}{3}\log\left|3x+\sqrt{9x^2-7}\right|+c$

7. $\int\frac{1}{\sqrt{x^2+9}}dx$

Sol. Let $I=\int\frac{1}{\sqrt{x^2+9}}dx \quad \Rightarrow I=\int\frac{1}{\sqrt{(x)^2+(3)^2}}dx$

$\Rightarrow I=\log\left|x+\sqrt{x^2+(3)^2}\right|+c \quad \therefore I=\log\left|x+\sqrt{x^2+9}\right|+c$

8. $\int\frac{1}{\sqrt{1+4x^2}}dx$

Sol. Let $I=\int\frac{1}{\sqrt{1+4x^2}}dx \quad \Rightarrow I=\frac{1}{2}\int\frac{1}{\sqrt{\frac{1}{4}+x^2}}dx \quad \Rightarrow I=\frac{1}{2}\int\frac{1}{\sqrt{\left(\frac{1}{2}\right)^2+(x)^2}}dx$

$\Rightarrow I=\frac{1}{2}\log\left|x+\sqrt{x^2+\frac{1}{4}}\right|+c \quad \therefore I=\frac{1}{2}\log\left|2x+\sqrt{4x^2+1}\right|+c$

9. $\int\frac{1}{\sqrt{9+4x^2}}dx$

Sol. Let $I=\int\frac{1}{\sqrt{9+4x^2}}dx \quad \Rightarrow I=\frac{1}{2}\int\frac{1}{\sqrt{\frac{9}{4}+x^2}}dx \quad \Rightarrow I=\frac{1}{2}\int\frac{1}{\sqrt{\left(\frac{3}{2}\right)^2+(x)^2}}dx$

$\Rightarrow I=\frac{1}{2}\log\left|x+\sqrt{x^2+\frac{9}{4}}\right|+c \quad \therefore I=\frac{1}{2}\log\left|2x+\sqrt{4x^2+9}\right|+c$

10. $\int\frac{x}{\sqrt{9-x^4}}dx$

Sol. Let $I=\int\frac{x}{\sqrt{9-x^4}}dx$

Put $x^2=t \quad \Rightarrow 2x\,dx=dt \quad \Rightarrow x\,dx=\frac{dt}{2}$, we get, $I=\frac{1}{2}\int\frac{dt}{\sqrt{9-t^2}}$

$\Rightarrow I=\frac{1}{2}\int\frac{1}{\sqrt{(3)^2-(t)^2}}dt \quad \Rightarrow I=\frac{1}{2}\sin^{-1}\left(\frac{t}{3}\right)+c \quad \therefore I=\frac{1}{2}\sin^{-1}\left(\frac{x^2}{3}\right)+c$

11. $\int\frac{3x^2}{\sqrt{9-16x^6}}dx$

Sol. Let $I = \int \frac{3x^2}{\sqrt{9-16x^6}} dx \Rightarrow I = \frac{1}{4}\int \frac{3x^2}{\sqrt{\frac{9}{16}-x^6}} dx$

Put $x^3 = t \Rightarrow 3x^2\, dx = dt$, we get,

$$I = \frac{1}{4}\int \frac{dt}{\sqrt{(3/4)^2-(t)^2}} \Rightarrow I = \frac{1}{4}\sin^{-1}\left(\frac{t}{3/4}\right)+c \quad \therefore\ I = \frac{1}{4}\sin^{-1}\left(\frac{4x^3}{3}\right)+c$$

12. $\int \frac{\sec^2 x}{\sqrt{16+\tan^2 x}} dx$

Sol. Let $I = \int \frac{\sec^2 x}{\sqrt{16+\tan^2 x}} dx$

Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$, we get, $I = \int \frac{1}{\sqrt{(4)^2+(t)^2}} dt$

$\Rightarrow I = \log\left| t+\sqrt{t^2+4^2} \right|+c \quad \therefore\ I = \log\left| \tan x+\sqrt{\tan^2 x+16} \right|+c$

13. $\int \frac{\sin x}{\sqrt{4+\cos^2 x}} dx$

Sol. Let $I = \int \frac{\sin x}{\sqrt{4+\cos^2 x}} dx$, Put $\cos x = t \Rightarrow -\sin x\, dx = dt \Rightarrow \sin x\, dx = (-dt)$, we get,

$$I = \int \frac{(-dt)}{\sqrt{4+t^2}} \Rightarrow I = -\int \frac{1}{\sqrt{(2)^2+(t)^2}} dt \Rightarrow I = -\log\left| t+\sqrt{(2)^2+(t)^2} \right|+c$$

$\therefore\ I = -\log\left| \cos x+\sqrt{4+\cos^2 x} \right|+c$

14. $\int \frac{\cos x}{\sqrt{9\sin^2 x-1}} dx$

Sol. Let $I = \int \frac{\cos x}{\sqrt{9\sin^2 x-1}} dx$, Put $\sin x = t \Rightarrow \cos x\, dx = dt$, we get, $I = \int \frac{dt}{\sqrt{9t^2-1}}$

$$\Rightarrow I = \frac{1}{3}\int \frac{dt}{\sqrt{(t)^2-\left(\frac{1}{3}\right)^2}} \Rightarrow I = \frac{1}{3}\log\left| t+\sqrt{(t)^2-\left(\frac{1}{3}\right)^2} \right|+c$$

$\Rightarrow I = \frac{1}{3}\log\left| \sin x+\sqrt{\sin^2 x-1/9} \right|+c \quad \therefore\ I = \frac{1}{3}\log\left| 3\sin x+\sqrt{9\sin^2 x-1} \right|+c$

15. $\int \frac{e^x}{\sqrt{4+e^{2x}}} dx$

Sol. Let $I = \int \frac{e^x}{\sqrt{4+e^{2x}}} dx \Rightarrow I = \int \frac{e^x}{\sqrt{4+(e^x)^2}} dx$, Put $e^x = t \Rightarrow e^x\, dx = dt$, we get,

$$I = \int \frac{dt}{\sqrt{(2)^2+(t)^2}} \Rightarrow I = \log\left| t+\sqrt{(2)^2+(t)^2} \right|+c \quad \therefore\ I = \log\left| e^x+\sqrt{4+e^{2x}} \right|+c$$

16. $\int \frac{2e^x}{\sqrt{4-e^{2x}}}dx$

Sol. Let $I=\int \frac{2e^x}{\sqrt{4-e^{2x}}}dx \Rightarrow I=\int \frac{2e^x}{\sqrt{4-\left(e^x\right)^2}}dx$, Put $e^x=t \Rightarrow e^x\,dx=dt$, we get,

$$I=2\int \frac{dt}{\sqrt{(2)^2-(t)^2}} \Rightarrow I=2\sin^{-1}\left(\frac{t}{2}\right)+c \quad \therefore\ I=2\sin^{-1}\left(\frac{e^x}{2}\right)+c$$

17. $I=\int \frac{1}{\sqrt{1-e^x}}dx$

Sol. Let $I=\int \frac{1}{\sqrt{1-e^x}}dx=\int \frac{1}{\sqrt{\frac{e^x\left(1-e^x\right)}{e^x}}}dx=\int \frac{dx}{e^{x/2}\sqrt{\left(e^{-x}-1\right)}}=\int \frac{e^{-x/2}}{\sqrt{\left(e^{-x/2}\right)^2-1}}dx$

Put $e^{-x/2}=t \Rightarrow e^{-x/2}\left(-\frac{1}{2}\right)dx=dt \Rightarrow e^{-x/2}dx=-2dt$, we get,

$$\Rightarrow I=\int \frac{-2dt}{\sqrt{t^2-1}} \Rightarrow I=-2\log\left|t+\sqrt{t^2-1}\right|+c \quad \therefore\ I=-2\log\left|e^{-x/2}+\sqrt{e^{-x}-1}\right|+c$$

18. $\int \sqrt{\frac{a-x}{a+x}}dx$

Sol. Let $I=\int \sqrt{\frac{a-x}{a+x}}dx \Rightarrow I=\int \frac{\sqrt{a-x}}{\sqrt{a+x}}.\frac{\sqrt{a-x}}{\sqrt{a-x}}dx \Rightarrow I=\int \frac{a-x}{\sqrt{a^2-x^2}}dx$

$$\Rightarrow I=a\int \frac{1}{\sqrt{a^2-x^2}}dx-\int \frac{x}{\sqrt{a^2-x^2}}dx \Rightarrow I=a\sin^{-1}\left(\frac{x}{a}\right)-I_1 \qquad \ldots(1)$$

$\Rightarrow I_1=\int \frac{xdx}{\sqrt{a^2-x^2}}$, Put $a^2-x^2=t \Rightarrow -2x\,dx=dt \Rightarrow x\,dx=-\frac{dt}{2}$,

We get, $I_1=-\frac{1}{2}\int \frac{dt}{\sqrt{t}} \Rightarrow I_1=\frac{-1}{2}.\frac{t^{1/2}}{1/2}+c \quad \therefore I_1=-\sqrt{a^2-x^2}+c$

Putting the value of I_1 in equation (1), we get, $I=a\sin^{-1}\left(\frac{x}{a}\right)+\sqrt{a^2-x^2}+c$

19. $\int \frac{1}{\sqrt{x^2+6x+5}}dx$

Sol. Let $I=\int \frac{1}{\sqrt{x^2+6x+5}}dx \Rightarrow I=\int \frac{1}{\sqrt{(x)^2+2.x.3+(3)^2-(3)^2+5}}dx$

$$\Rightarrow I=\int \frac{1}{\sqrt{(x+3)^2-(2)^2}}dx \Rightarrow I=\log\left|(x+3)+\sqrt{(x+3)^2-(2)^2}\right|+c$$

$$\therefore\ I=\log\left|(x+3)+\sqrt{x^2+6x+5}\right|+c$$

20. $\int \frac{1}{\sqrt{(2-x)^2+1}}dx$

Sol. Let $I=\int \frac{1}{\sqrt{(2-x)^2+1}}dx$, Put $2-x=t \Rightarrow -dx=dt \Rightarrow dx=-dt$.

We get, $I=\int \frac{-dt}{\sqrt{t^2+1}} \quad \Rightarrow I=-\log\left|t+\sqrt{t^2+1}\right|+c$

$\Rightarrow I=-\log\left|(2-x)+\sqrt{(2-x)^2+1}\right|+c \quad \therefore I=-\log\left|(2-x)+\sqrt{x^2-4x+5}\right|+c$

21. $\int \frac{1}{\sqrt{(x-3)^2-1}}dx$

Sol. Let $I=\int \frac{1}{\sqrt{(x-3)^2-1}}dx$, Put $x-3=t \Rightarrow dx=dt$

We get, $I=\int \frac{1}{\sqrt{t^2-1}}dt \quad \Rightarrow I=\log\left|t+\sqrt{t^2-1}\right|+c$

$\Rightarrow I=\log\left|(x-3)+\sqrt{(x-3)^2-1}\right|+c \quad \therefore I=\log\left|(x-3)+\sqrt{x^2-6x+8}\right|+c$

22. $\int \frac{1}{\sqrt{x^2-6x+10}}dx$

Sol. Let $I=\int \frac{1}{\sqrt{x^2-6x+10}}dx \Rightarrow I=\int \frac{1}{\sqrt{(x)^2-2.x.3+(3)^2-(3)^2+10}}dx$

$\Rightarrow I=\int \frac{1}{\sqrt{(x-3)^2+(1)^2}}dx \quad \Rightarrow I=\log\left|(x-3)+\sqrt{(x-3)^2+(1)^2}\right|+c$

$\therefore \quad I=\log\left|(x-3)+\sqrt{x^2-6x+10}\right|+c$

23. $\int \frac{dx}{\sqrt{2+2x-x^2}}$

Sol. Let $I=\int \frac{dx}{\sqrt{2+2x-x^2}} \quad \Rightarrow I=-\int \frac{1}{\sqrt{-\left[x^2-2x-2\right]}}dx$

$\Rightarrow I=\int \frac{1}{\sqrt{-\left[(x)^2-2.x.1+(1)^2-(1)^2-2\right]}}dx \quad \Rightarrow I=\int \frac{1}{\sqrt{-\left[(x-1)^2-\left(\sqrt{3}\right)^2\right]}}dx$

$\Rightarrow I=\int \frac{1}{\sqrt{\left(\sqrt{3}\right)^2-(x-1)^2}}dx \quad \therefore I=\sin^{-1}\left(\frac{x-1}{\sqrt{3}}\right)+c$

24. $\int \frac{1}{\sqrt{8-4x-2x^2}}dx$

Sol. Let $I=\int \frac{1}{\sqrt{8-4x-2x^2}}dx \quad \Rightarrow I=\int \frac{1}{\sqrt{-2\left[x^2+2x-4\right]}}dx$

$\Rightarrow I=\frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{-\left[x^2+2x-4\right]}}dx \quad \Rightarrow I=\frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{-\left[(x)^2+2.x.1+(1)^2-(1)^2-4\right]}}dx$

$$\Rightarrow I=\frac{1}{\sqrt{2}}\int\frac{1}{\sqrt{-\left[(x+1)^2-\left(\sqrt{5}\right)^2\right]}}dx \quad \Rightarrow I=\frac{1}{\sqrt{2}}\int\frac{1}{\sqrt{\left(\sqrt{5}\right)^2-(x+1)^2}}dx$$

$$\therefore \quad I=\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{x+1}{\sqrt{5}}\right)+c$$

25. $\int\frac{1}{\sqrt{16-6x-x^2}}dx$

Sol. Let $I=\int\frac{1}{\sqrt{16-6x-x^2}}dx \quad \Rightarrow I=\int\frac{1}{\sqrt{-\left[x^2+6x-16\right]}}dx$

$$\Rightarrow I=\int\frac{1}{\sqrt{-\left[(x)^2+2.x.3+(3)^2-(3)^2-16\right]}}dx \quad \Rightarrow I=\int\frac{1}{\sqrt{-\left[(x+3)^2-(5)^2\right]}}dx$$

$$\Rightarrow I=\int\frac{1}{\sqrt{(5)^2-(x+3)^2}}dx \qquad \therefore \; I=\sin^{-1}\left(\frac{x+3}{5}\right)+c$$

26. $\int\frac{1}{\sqrt{7-6x-x^2}}dx$

Sol. Let $I=\int\frac{1}{\sqrt{7-6x-x^2}}dx \quad \Rightarrow I=\int\frac{1}{\sqrt{-\left[x^2+6x-7\right]}}dx$

$$\Rightarrow I=\int\frac{1}{\sqrt{-\left[(x)^2+2.x.3+(3)^2-(3)^2-7\right]}}dx \quad \Rightarrow I=\int\frac{1}{\sqrt{-\left[(x+3)^2-(4)^2\right]}}dx$$

$$\Rightarrow I=\int\frac{1}{\sqrt{(4)^2-(x+3)^2}}dx \qquad \therefore \; I=\sin^{-1}\left(\frac{x+3}{4}\right)+c$$

27. $I=\int\frac{dx}{\sqrt{x-x^2}}$

Sol. Let $I=\int\frac{1}{\sqrt{-\left[x^2-x\right]}}dx \quad \Rightarrow I=\int\frac{1}{\sqrt{-\left[(x)^2-2.x.\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2\right]}}$

$$\Rightarrow I=\int\frac{1}{\sqrt{-\left[\left(x-\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2\right]}} \quad \Rightarrow I=\int\frac{1}{\sqrt{\left(\frac{1}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}}dx$$

$$\Rightarrow I=\sin^{-1}\left(\frac{x-\frac{1}{2}}{\frac{1}{2}}\right)+c \qquad \therefore \; I=\sin^{-1}(2x-1)+c$$

28. $\int\frac{1}{\sqrt{8+2x-x^2}}dx$

Sol. Let $I=\int\frac{1}{\sqrt{8+2x-x^2}}dx \quad \Rightarrow I=\int\frac{1}{\sqrt{-\left[x^2-2x-8\right]}}dx$

$$\Rightarrow I=\int \frac{1}{\sqrt{-\left[(x)^2-2.x.1+(1)^2-(1)^2-8\right]}}dx \quad \Rightarrow I=\int \frac{1}{\sqrt{-\left[(x-1)^2-(3)^2\right]}}dx$$

$$\Rightarrow I=\int \frac{1}{\sqrt{(3)^2-(x-1)^2}}dx \qquad \therefore\ I=\sin^{-1}\left(\frac{x-1}{3}\right)+c$$

29. $\int \frac{1}{\sqrt{x^2-3x+2}}dx$

Sol. Let $I=\int \frac{1}{\sqrt{x^2-3x+2}}dx \quad \Rightarrow I=\int \frac{1}{\sqrt{(x)^2-2.x.\frac{3}{2}+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2+2}}dx$

$$\Rightarrow I=\int \frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx \quad \Rightarrow I=\log\left|\left(x-\frac{3}{2}\right)+\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right|+c$$

$$\therefore\ I=\log\left|\left(x-\frac{3}{2}\right)+\sqrt{x^2-3x+2}\right|+c$$

30. $\int \frac{1}{\sqrt{2x^2+3x-2}}dx$

Sol. Let $I=\int \frac{1}{\sqrt{2x^2+3x-2}}dx \ \Rightarrow I=\int \frac{1}{\sqrt{2\left(x^2+\frac{3}{2}x-1\right)}}dx$

$$\Rightarrow I=\frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{x^2+\frac{3}{2}x-1}}dx \ \Rightarrow I=\frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{(x)^2+2.x.\frac{3}{4}+\left(\frac{3}{4}\right)^2-\left(\frac{3}{4}\right)^2-1}}dx$$

$$\Rightarrow I=\frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{\left(x+\frac{3}{4}\right)^2-\left(\frac{5}{4}\right)^2}}dx \quad \Rightarrow I=\frac{1}{\sqrt{2}}\log\left|\left(x+\frac{3}{4}\right)+\sqrt{\left(x+\frac{3}{4}\right)^2-\left(\frac{5}{4}\right)^2}\right|+c$$

$$\therefore\ I=\frac{1}{\sqrt{2}}\log\left|\left(x+\frac{3}{4}\right)+\sqrt{x^2+\frac{3}{2}x-1}\right|+c$$

31. $\int \frac{1}{\sqrt{2x^2+4x+6}}dx$

Sol. Let $I=\int \frac{1}{\sqrt{2x^2+4x+6}}dx \quad \Rightarrow I=\frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{x^2+2x+3}}dx$

$$\Rightarrow I=\frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{(x)^2+2.x.1+(1)^2-(1)^2+3}}dx \quad \Rightarrow I=\frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{(x+1)^2+\left(\sqrt{2}\right)^2}}dx$$

$$\Rightarrow I=\frac{1}{\sqrt{2}}\log\left|(x+1)+\sqrt{(x+1)^2+\left(\sqrt{2}\right)^2}\right|+c \quad \therefore\ I=\frac{1}{\sqrt{2}}\log\left|(x+1)+\sqrt{x^2+2x+3}\right|+c$$

32. $\int \frac{1}{\sqrt{1+2x-3x^2}}dx$

Sol. Let $I=\int\frac{1}{\sqrt{1+2x-3x^2}}dx \quad \Rightarrow I=\int\frac{1}{\sqrt{-3\left[x^2-\frac{2}{3}x-\frac{1}{3}\right]}}dx$

$$\Rightarrow I=\frac{1}{\sqrt{3}}\int\frac{1}{\sqrt{-\left[(x)^2-2.x.\frac{1}{3}+\left(\frac{1}{3}\right)^2-\left(\frac{1}{3}\right)^2-\left(\frac{1}{3}\right)\right]}}dx$$

$$\Rightarrow I=\frac{1}{\sqrt{3}}\int\frac{1}{\sqrt{-\left[\left(x-\frac{1}{3}\right)^2-\left(\frac{2}{3}\right)^2\right]}}dx \quad \Rightarrow I=\frac{1}{\sqrt{3}}\int\frac{1}{\sqrt{\left(\frac{2}{3}\right)^2-\left(x-\frac{1}{3}\right)^2}}dx$$

$$\Rightarrow I=\frac{1}{\sqrt{3}}\sin^{-1}\left[\frac{\left(x-\frac{1}{3}\right)}{\left(\frac{2}{3}\right)}\right]+c \quad \therefore\ I=\frac{1}{\sqrt{3}}\sin^{-1}\left(\frac{3x-1}{2}\right)+c$$

33. $\int\frac{1}{\sqrt{x}\sqrt{5-x}}dx$

Sol. Let $I=\int\frac{1}{\sqrt{x}\sqrt{5-x}}dx \quad \Rightarrow I=\int\frac{1}{\sqrt{-\left[x^2-5x\right]}}dx$

$$\Rightarrow I=\int\frac{1}{\sqrt{-\left[(x)^2-2.x.\frac{5}{2}+\left(\frac{5}{2}\right)^2-\left(\frac{5}{2}\right)^2\right]}}dx \quad \Rightarrow I=\int\frac{1}{\sqrt{-\left[\left(x-\frac{5}{2}\right)^2-\left(\frac{5}{2}\right)^2\right]}}dx$$

$$\Rightarrow I=\int\frac{1}{\sqrt{\left(\frac{5}{2}\right)^2-\left(x-\frac{5}{2}\right)^2}}dx \quad \Rightarrow I=\sin^{-1}\left[\frac{\left(x-\frac{5}{2}\right)}{\left(\frac{5}{2}\right)}\right]+c \quad \therefore\ I=\sin^{-1}\left(\frac{2x-5}{5}\right)+c$$

34. $\int\frac{1}{\sqrt{3+4x-2x^2}}dx$

Sol. Let $I=\int\frac{1}{\sqrt{3+4x-2x^2}}dx \quad \Rightarrow I=\int\frac{1}{\sqrt{-\left[2x^2-4x-3\right]}}dx$

$$\Rightarrow I=\frac{1}{\sqrt{2}}\int\frac{1}{\sqrt{-\left[x^2-2x-\frac{3}{2}\right]}}dx \quad \Rightarrow I=\frac{1}{\sqrt{2}}\int\frac{1}{\sqrt{-\left[(x)^2-2.x.1+(1)^2-(1)^2-\frac{3}{2}\right]}}dx$$

$$\Rightarrow I=\frac{1}{\sqrt{2}}\int\frac{1}{\sqrt{-\left[(x-1)^2-\left(\frac{\sqrt{5}}{\sqrt{2}}\right)^2\right]}}dx \quad \Rightarrow I=\frac{1}{\sqrt{2}}\int\frac{1}{\sqrt{\left(\sqrt{\frac{5}{2}}\right)^2-(x-1)^2}}dx$$

$$\Rightarrow I=\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{x-1}{\sqrt{5}/\sqrt{2}}\right)+c \quad \therefore\ I=\frac{1}{\sqrt{2}}\sin^{-1}\left[\frac{\sqrt{2}(x-1)}{\sqrt{5}}\right]+c$$

35. $\int\frac{x^2}{\sqrt{x^6+2x^3+3}}dx$

Sol. Let $I=\int\frac{x^2}{\sqrt{x^6+2x^3+3}}dx \quad \Rightarrow I=\int\frac{x^2}{\sqrt{\left(x^3\right)^2+2x^3+3}}$

Put $x^3=t \quad \Rightarrow 3x^2dx=dt \quad \Rightarrow x^2dx=\frac{dt}{3}$

We get, $I=\frac{1}{3}\int\frac{dt}{\sqrt{t^2+2t+3}} \quad \Rightarrow I=\frac{1}{3}\int\frac{1}{\sqrt{(t)^2+2.t.1+(1)^2-(1)^2+3}}dt$

$\Rightarrow I=\frac{1}{3}\int\frac{1}{\sqrt{(t+1)^2+\left(\sqrt{2}\right)^2}}dt \quad \Rightarrow I=\frac{1}{3}\log\left|(t+1)+\sqrt{(t+1)^2+\left(\sqrt{2}\right)^2}\right|+c$

$\therefore \quad I=\frac{1}{3}\log\left|\left(x^3+1\right)+\sqrt{x^6+2x^3+3}\right|+c$

36. $\int\frac{(2x+3)}{\sqrt{x^2+x+1}}dx$

Sol. Let $I=\int\frac{(2x+3)}{\sqrt{x^2+x+1}}dx \quad \Rightarrow I-\int\frac{(2x+3)}{\sqrt{x^2+x+1}}dx$

$\Rightarrow 2x+3=A(2x+1)+B \quad \Rightarrow 2x+3=2Ax+A+B$

Equating co-efficient both side, we get, $2=2A \quad \therefore A=1$

And $3=A+B \quad \Rightarrow B=3-A=3-1=2$

$\Rightarrow I=\int\frac{A(2x+1)+B}{\sqrt{x^2+x+1}}dx \quad \Rightarrow I=A\int\frac{(2x+1)}{\sqrt{x^2+x+1}}dx+B\int\frac{1}{\sqrt{x^2+x+1}}dx$

$\Rightarrow I=I_1+2I_2 \qquad \ldots(1)$

$\Rightarrow I_1=\int\frac{2x+1}{\sqrt{x^2+x+1}}dx$, Put $x^2+x+1=t \quad \Rightarrow (2x+1)dx=dt$

$\Rightarrow I_1=\int\frac{1}{\sqrt{t}}dt \quad \Rightarrow I_1=\frac{t^{1/2}}{1/2}+c_1 \quad \Rightarrow I_1=2\sqrt{x^2+x+1}+c_1$

Now, $I_2=\int\frac{1}{\sqrt{x^2+x+1}}dx \quad \Rightarrow I_2=\int\frac{1}{\sqrt{(x)^2+2.x.\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1}}dx$

$\Rightarrow I_2=\int\frac{1}{\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}dx \quad \Rightarrow I_2=\log\left|\left(x+\frac{1}{2}\right)+\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right|+c$

$\Rightarrow I_2=\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right|+c$

Putting the value of I_1 & I_2 in equation (1), we get, $I=2\sqrt{x^2+x+1}+2\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right|+c$

37. $\int\frac{(2x+3)}{\sqrt{x^2+4x+3}}dx$

Sol. Let $I=\int\frac{(2x+3)}{x^2+4x+3}dx \quad \Rightarrow 2x+3=A(2x+4)+B \quad \Rightarrow 2x+3=2Ax+4A+B$

Equating co-efficient both side, we get, $2 = 2A \quad \therefore\ A = 1$

And $3 = 4A + B \ \Rightarrow\ B = 3 - 4A = 3 - 4(1) = -1$

$$\Rightarrow I = \int \frac{A(2x+4)+B}{\sqrt{x^2+4x+3}}dx \quad \Rightarrow I = A\int \frac{(2x+4)}{\sqrt{x^2+4x+3}}dx + B\int \frac{1}{\sqrt{x^2+4x+3}}dx$$

$$\Rightarrow I = I_1 - I_2 \qquad \ldots(1)$$

$$\Rightarrow I_1 = \int \frac{2x+4}{\sqrt{x^2+4x+3}}dx$$

Put $x^2+4x+3 = t \quad \Rightarrow (2x+4)dx = dt$, we get,

$$\Rightarrow I_1 = \int \frac{dt}{\sqrt{t}} \quad \Rightarrow I_1 = 2\sqrt{t} + c_1 \quad \Rightarrow I_1 = 2\sqrt{x^2+4x+3} + c_1$$

Now, $I_2 = \int \frac{1}{\sqrt{x^2+4x+3}}dx \quad \Rightarrow I_2 = \int \frac{1}{\sqrt{(x)^2 + 2.x.2 + (2)^2 - (2)^2 + 3}}dx$

$$\Rightarrow I_2 = \int \frac{1}{\sqrt{(x+2)^2 - (1)^2}}dx \quad \Rightarrow I_2 = \log\left|x+2+\sqrt{x^2+4x+3}\right| + c$$

Putting the values of I_1 & I_2 in equation (1), we get,

$$\Rightarrow I = 2\sqrt{x^2+4x+3} + (-1)\log\left|(x+2)+\sqrt{x^2+4x+3}\right| + c$$

$$\therefore\ I = 2\sqrt{x^2+4x+3} - \log\left|(x+2)+\sqrt{x^2+4x+3}\right| + c$$

38. $\int \frac{(4x+3)}{\sqrt{2x^2+2x-3}}dx$

Sol. Let $I = \int \frac{(4x+3)}{\sqrt{2x^2+2x-3}}dx \quad \Rightarrow 4x+3 = A(4x+2)+B \quad \Rightarrow 4x+3 = 4Ax+2A+B$

Equating co-efficient both side, we get, $4 = 4A \quad \therefore\ A = 1$ and $3 = 2A + B$

$$\Rightarrow B = 3 - 2A = 3 - 2(1) = 1 \quad \Rightarrow I = \int \frac{A(4x+2)+B}{\sqrt{2x^2+2x-3}}dx$$

$$\Rightarrow I = A\int \frac{(4x+2)}{\sqrt{2x^2+2x-3}}dx + B\int \frac{1}{\sqrt{2x^2+2x-3}}dx$$

$$\Rightarrow I = I_1 + I_2 \qquad \ldots(1)$$

$\Rightarrow I_1 = \int \frac{4x+2}{\sqrt{2x^2+2x-3}}dx$, Put $2x^2+2x-3 = t \quad \Rightarrow (4x+2)dx = dt$

$$\Rightarrow I_1 = \int \frac{dt}{\sqrt{t}} \quad \Rightarrow I_1 = 2\sqrt{t} + c_1 \quad \Rightarrow I_1 = 2\sqrt{2x^2+2x-3} + c_1$$

Now, $I_2 = \int \frac{1}{\sqrt{2x^2+2x-3}}dx \quad \Rightarrow I_2 = \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{x^2+x-3/2}}dx$

$$\Rightarrow I_2 = \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{(x)^2 + 2.x.\frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)}}dx$$

$$\Rightarrow I_2=\frac{1}{\sqrt{2}}\int\frac{1}{\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{\sqrt{7}}{4}\right)^2}}dx \quad \Rightarrow I_2=\frac{1}{\sqrt{2}}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{\sqrt{7}}{4}\right)^2}\right|+c$$

$$\Rightarrow I_2=\frac{1}{\sqrt{2}}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x-\frac{3}{2}}\right|+c$$

Putting the value of I_1 & I_2 in equation (1), we get,

$$\therefore \quad I=2\sqrt{2x^2+2x-3}+\frac{1}{\sqrt{2}}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x-\frac{3}{2}}\right|+c$$

39. $\int\frac{(3-2x)}{\sqrt{2+x-x^2}}dx$

Sol. Let $I=\int\frac{(3-2x)}{\sqrt{2+x-x^2}}dx \Rightarrow 3-2x=A(1-2x)+B \Rightarrow 3-2x=A-2Ax+B$

Equating co-efficient both side, we get, $-2=-2A \quad \therefore A=1$

And $3=A+B \quad \Rightarrow B=3-A=3-1=2$

$$\Rightarrow I=\int\frac{A(1-2x)+B}{\sqrt{2+x-x^2}}dx \quad \Rightarrow I=A\int\frac{(1-2x)}{\sqrt{2+x-x^2}}dx+B\int\frac{1}{\sqrt{2+x-x^2}}dx$$

$\Rightarrow I=AI_1+BI_2 \qquad \ldots(1)$

$\Rightarrow I_1=\int\frac{1-2x}{\sqrt{2+x-x^2}}dx$ Put $2+x-x^2=t \quad \Rightarrow (1-2x)\,dx=dt$, we get,

$$I_1=\int\frac{dt}{\sqrt{t}} \quad \Rightarrow I_1=2\sqrt{t}+c_1 \quad \Rightarrow I_1=2\sqrt{2+x-x^2}+c_1$$

Now, $I_2=\int\frac{1}{\sqrt{2+x-x^2}}dx \quad \Rightarrow I_2=\int\frac{1}{\sqrt{-\left[x^2-x-2\right]}}dx$

$$\Rightarrow I_2=\int\frac{1}{\sqrt{-\left[(x)^2-2.x.\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2-2\right]}}dx \quad \Rightarrow I_2=\int\frac{1}{\sqrt{-\left[\left(x-\frac{1}{2}\right)^2-\left(\frac{3}{2}\right)^2\right]}}$$

$$\Rightarrow I_2=\int\frac{1}{\sqrt{\left(\frac{3}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}}dx \quad \Rightarrow I_2=\sin^{-1}\left[\frac{\left(x-\frac{1}{2}\right)}{\left(\frac{3}{2}\right)}\right]+c \quad \Rightarrow I_2=\sin^{-1}\left(\frac{2x-1}{3}\right)+c$$

Putting the value of I_1 & I_2 in equation (1), we get, $I=2\sqrt{2+x-x^2}+2\sin^{-1}\left(\frac{2x-1}{3}\right)+c$

40. $\int\frac{(x+2)}{\sqrt{x^2+2x-1}}dx$

Sol. $I=\int\frac{(x+2)}{\sqrt{x^2+2x-1}}dx \quad \Rightarrow x+2=A(2x+2)+B \quad \Rightarrow x+2=2Ax+2A+B$

Equating co-efficient both side, we get, $1=2A \quad \therefore A=\frac{1}{2}$

And $2 = 2A + B \quad \Rightarrow B = 2 - 2A = 2 - 2\left(\frac{1}{2}\right) = 2 - 1 = 1$

$\Rightarrow I = \int \frac{A(2x+2)+B}{\sqrt{x^2+2x-1}}dx \Rightarrow I = A\int \frac{(2x+2)}{\sqrt{x^2+2x-1}}dx + B\int \frac{1}{\sqrt{x^2+2x-1}}dx$

$\Rightarrow I = \frac{1}{2}I_1 + I_2 \qquad \ldots(1)$

$\Rightarrow I_1 = \int \frac{2x+2}{\sqrt{x^2+2x-1}}dx$, Put $x^2+2x-1 = t \quad \Rightarrow (2x+2)dx = dt$, we get,

$\therefore \quad I_1 = \int \frac{dt}{\sqrt{t}} \quad \Rightarrow I_1 = 2\sqrt{t} + c_1 \quad \Rightarrow I_1 = 2\sqrt{x^2+2x-1} + c_1$

Now, $I_2 = \int \frac{1}{\sqrt{x^2+2x-1}}dx \quad \Rightarrow I_2 = \int \frac{1}{\sqrt{(x)^2+2.x.1+(1)^2-(1)^2-1}}dx$

$\Rightarrow I_2 = \int \frac{1}{\sqrt{(x+1)^2-(\sqrt{2})^2}}dx \quad \Rightarrow I_2 = \log\left|(x+1)+\sqrt{x^2+2x-1}\right| + c_2$

Putting the value of I_1 & I_2 in equation (1), we get,

$I = 2\left(\frac{1}{2}\right)\sqrt{x^2+2x-1} + \log\left|(x+1)+\sqrt{x^2+2x-1}\right| + c$

$\therefore \quad I = \sqrt{x^2+2x-1} + \log\left|(x+1)+\sqrt{x^2+2x-1}\right| + c$

41. $\int \frac{(3x+1)}{\sqrt{5-2x-x^2}}dx$

Sol. Let $I = \int \frac{(3x+1)}{\sqrt{5-2x-x^2}}dx \Rightarrow 3x+1 = A(-2-2x)+B \quad \Rightarrow 3x+1 = -2A-2Ax+B$

Equating co-efficient both side, we get, $3 = -2A \quad \therefore A = -\frac{3}{2}$ and $1 = -2A+B$

$\Rightarrow B = 1+2A = 1+2\left(-\frac{3}{2}\right) = 1-3 = -2 \Rightarrow I = \int \frac{A(-2-2x)+B}{\sqrt{5-2x-x^2}}dx$

$\Rightarrow I = A\int \frac{(-2-2x)}{\sqrt{5-2x-x^2}}dx + B\int \frac{1}{\sqrt{5-2x-x^2}}dx \qquad \Rightarrow I = -\frac{3}{2}I_1 - 2I_2 \qquad \ldots(1)$

$\therefore \quad I_1 = \int \frac{(-2-2x)}{\sqrt{5-2x-x^2}}dx$, Put $5-2x-x^2 = t \quad \Rightarrow (-2-2x)dx = dt$

$\Rightarrow I_1 = \int \frac{dt}{\sqrt{t}} \quad \Rightarrow I_1 = 2\sqrt{t} + c_1 \quad \Rightarrow I_1 = 2\sqrt{5-2x-x^2} + c_1$

Now, $I_2 = \int \frac{1}{\sqrt{5-2x-x^2}}dx \quad \Rightarrow I_2 = \int \frac{1}{\sqrt{-\left[x^2+2x-5\right]}}dx$

$\Rightarrow I_2 = \int \frac{1}{\sqrt{-\left[(x)^2+2.x.1+(1)^2-(1)^2-5\right]}}dx \quad \Rightarrow I_2 = \int \frac{1}{\sqrt{-\left[(x+1)^2-(\sqrt{6})^2\right]}}dx$

$$\Rightarrow I_2=\int\frac{1}{\sqrt{\left(\sqrt{6}\right)^2-(x+1)^2}}dx \quad \Rightarrow I_2=\sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right)+c$$

Putting the value of I_1 & I_2 in equation (1), we get,

$$\therefore \quad I=2\left(-\frac{3}{2}\right)\sqrt{5-2x-x^2}+(-2)\sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right)+c \quad \therefore\ I=-3\sqrt{5-2x-x^2}-2\sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right)+c$$

42. $\displaystyle\int\frac{(6x+5)}{\sqrt{6+x-2x^2}}dx$

Sol. Let $I=\displaystyle\int\frac{(6x+5)}{\sqrt{6+x-2x^2}}dx \Rightarrow 6x+5=A(1-4x)+B \quad \Rightarrow 6x+5=A-4Ax+B$

Equating co-efficient both side, we get, $6=-4A \quad \therefore\ A=-\dfrac{3}{2}$

And $5=A+B \quad \Rightarrow B=5-A=5+\dfrac{3}{2} \quad \therefore B=\dfrac{13}{2}$

$$\Rightarrow I=\int\frac{A(1-4x)+B}{\sqrt{6+x-2x^2}}dx \quad \Rightarrow I=A\int\frac{(1-4x)}{\sqrt{6+x-2x^2}}dx+B\int\frac{1}{\sqrt{6+x-2x^2}}dx$$

$$\Rightarrow I=-\frac{3}{2}I_1+\frac{13}{2}I_2 \qquad \ldots(1)$$

$\Rightarrow I_1=\displaystyle\int\frac{1-4x}{\sqrt{6+x-2x^2}}dx$, Put $6+x-2x^2=t \quad \Rightarrow (1-4x)dx=dt$, we get,

$$\therefore \quad I_1=\int\frac{dt}{\sqrt{t}} \quad \Rightarrow I_1=2\sqrt{t}+c_1 \quad \Rightarrow I_1=2\sqrt{6+x-2x^2}+c_1$$

Now, $I_2=\displaystyle\int\frac{1}{\sqrt{6+x-2x^2}}dx \quad \Rightarrow I_2=\int\frac{1}{\sqrt{-\left[2x^2-x-6\right]}}dx$

$$\Rightarrow I_2=\frac{1}{\sqrt{2}}\int\frac{1}{\sqrt{-\left[x^2-\frac{x}{2}-3\right]}}dx \quad \Rightarrow I_2=\frac{1}{\sqrt{2}}\int\frac{1}{\sqrt{-\left[(x)^2-2.x.\frac{1}{4}+\left(\frac{1}{4}\right)^2-\left(\frac{1}{4}\right)^2-3\right]}}dx$$

$$\Rightarrow I_2=\frac{1}{\sqrt{2}}\int\frac{1}{\sqrt{-\left[\left(x-\frac{1}{4}\right)^2-\left(\frac{7}{4}\right)^2\right]}}dx \quad \Rightarrow I_2=\frac{1}{\sqrt{2}}\int\frac{1}{\sqrt{\left(\frac{7}{4}\right)^2-\left(x-\frac{1}{4}\right)^2}}dx$$

$$\Rightarrow I_2=\frac{1}{\sqrt{2}}\sin^{-1}\left[\frac{\left(x-\frac{1}{4}\right)}{\left(\frac{7}{4}\right)}\right]+c \quad \Rightarrow I_2=\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{4x-1}{7}\right)+c$$

Putting the value of I_1 & I_2 in equation (1), we get,

$$I=2\left(-\frac{3}{2}\right)\sqrt{6+x-2x^2}+\frac{13}{2}\left(\frac{1}{\sqrt{2}}\right)\sin^{-1}\left(\frac{4x-1}{7}\right)+c$$

$$\therefore \quad I=-3\sqrt{6+x-2x^2}+\frac{13}{2\sqrt{2}}\sin^{-1}\left(\frac{4x-1}{7}\right)+c$$

43. $I=\int\sqrt{\frac{1+x}{x}}dx$

Sol. Let $I=\int\sqrt{\frac{1+x}{x}}dx \Rightarrow I=\int\frac{\sqrt{1+x}}{\sqrt{x}}.\frac{\sqrt{1+x}}{\sqrt{1+x}}dx \Rightarrow I=\int\frac{1+x}{\sqrt{x^2+x}}dx$

$(1+x)=A(2x+1)+B \Rightarrow 1+x=2Ax+A+B$

Equating co-efficient both side, we get, $A+B=1 \quad \ldots(1) \qquad 2A=1 \quad \therefore\ A=\frac{1}{2}$

Putting the value of A in equation (1), we get, $B=\frac{1}{2}$

$\Rightarrow I=\int\frac{A(2x+1)+B}{\sqrt{x^2+x}}dx \Rightarrow I=A\int\frac{2x+1}{\sqrt{x^2+x}}dx+B\int\frac{1}{\sqrt{x^2+x}}dx$

$\Rightarrow I=\frac{1}{2}I_1+\frac{1}{2}I_2 \qquad \ldots(2)$

$\Rightarrow I_1=\int\frac{2x+1}{\sqrt{x^2+x}}dx$, Put $x^2+x=t \Rightarrow (2x+1)dx=dt$

We get, $I_1=\int\frac{1}{\sqrt{t}}dt \Rightarrow I_1=2\sqrt{t}+c_1 \Rightarrow I_1=2\sqrt{x^2+x}+c_1$

$\Rightarrow I_2=\int\frac{1}{\sqrt{x^2+x}}dx \Rightarrow I_2=\int\frac{1}{\sqrt{(x)^2+2.x.\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}}$

$\Rightarrow I_2=\int\frac{1}{\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx \Rightarrow I_2=\log\left|x+\frac{1}{2}+\sqrt{x+x^2}\right|+c_2$

Putting the value of I_1 & I_2 in equation (1), we get, $I=\sqrt{x+x^2}+\frac{1}{2}\log\left|x+\frac{1}{2}+\sqrt{x+x^2}\right|+c$

44. $\int\frac{(x+2)}{\sqrt{x^2+5x+6}}dx$

Sol. Let $I=\int\frac{(x+2)}{\sqrt{x^2+5x+6}}dx$

$x+2=A\frac{d}{dx}\left(x^2+5x+6\right)+B$

$x+2=A(2x+5)+B \quad \ldots(i)$

$x+2=2Ax+(5A+B)$

Comparing on the side with proper co-efficient we get $x=2Ax$

$1=2A$

$\therefore\ A=\frac{1}{2}$

$2=(5A+B)$

$$2 = 5\times\frac{1}{2}+B$$

$$2 = \frac{5}{2}+B$$

$$2-\frac{5}{2} = B$$

$$\frac{4-5}{2} = B$$

$$\therefore B = \frac{1}{2}$$

Putting the value of A and B in equation (i) we get

$$x+2 = \frac{1}{2}(2x+5)+\left(-\frac{1}{2}\right)$$

$$x+2 = \frac{1}{2}(2x+5)-\frac{1}{2}$$

$$I = \int\frac{\frac{1}{2}(2x+5)-\frac{1}{2}}{\sqrt{x^2+5x+6}}dx$$

$$I = \int\frac{\frac{1}{2}(2x+5)}{\sqrt{x^2+5x+6}}dx-\int\frac{\frac{1}{2}}{\sqrt{x^2+5x+6}}dx$$

$$I = \frac{1}{2}\int\frac{2x+5}{\sqrt{x^2+5x+6}}dx-\frac{1}{2}\int\frac{1}{\sqrt{x^2+5x+6}}dx$$

$$I = I_1 - I_2$$

$$I_1 = \frac{1}{2}\int\frac{2x+5}{\sqrt{x^2+5x+6}}dx$$

Let $x^2+5x+6 = t$

Diff on the both side w.r. to x, we get $\frac{d}{dx}\left(x^2+5x+6\right) = \frac{d}{dx}(t)$

$$2x+5 = \frac{dt}{dx}$$

$$dx = \frac{dt}{2x+5}$$

$$I_1 = \frac{1}{2}\int\frac{2x+5}{\sqrt{t}}\times\frac{dt}{2x+5}$$

$$I_1 = \frac{1}{2}\int t^{-1/2}dt$$

$$I_1 = \frac{1}{2}\times\frac{t^{1/2}}{1/2}+c$$

$$I_2 = \frac{2}{2}\times t^{1/2}+c$$

$$I_1 = \sqrt{t}+c$$

$$I_1 = \sqrt{x^2+5x+6}+c$$

$$I_2 = \frac{1}{2}\int \frac{1}{\sqrt{x^2+5x+6}}dx$$

$$I_2 = \frac{1}{2}\int \frac{1}{\sqrt{x^2+2x\frac{x}{5}+\left(\frac{5}{2}\right)^2-\left(\frac{5}{2}\right)^2+6}}dx$$

$$I_2 = \frac{1}{2}\int \frac{1}{\sqrt{\left(x+\frac{5}{2}\right)^2-\frac{25}{4}+6}}dx$$

$$I_2 = \frac{1}{2}\int \frac{1}{\sqrt{\left(x+\frac{5}{2}\right)^2-\frac{25+24}{4}}}dx$$

$$I_2 = \frac{1}{2}\int \frac{1}{\sqrt{\left(x+\frac{5}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx$$

$$I_2 = \frac{1}{2}\log\left|\left(x+\frac{5}{2}\right)+\sqrt{\left(x+\frac{5}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right|+c$$

$$I_2 = \frac{1}{2}\log\left|\left(x+\frac{5}{2}\right)+\sqrt{x^2+5x+6}\right|+c$$

From equation (i)

$$I = I_1 - I_2$$

$$I = \sqrt{x^2+5x+6}-\frac{1}{2}\log\left|\left(x+\frac{5}{2}\right)+\sqrt{x^2+5x+6}\right|+c$$

EXERCISE 14C (Pg.No.: 744)

Evaluate the following integrals :

1. $\int \sqrt{4-x^2}\, dx$

Sol. Let $I = \int \sqrt{4-x^2}\,dx \quad \Rightarrow I = \int \sqrt{(2)^2-(x)^2}\,dx$

$$\Rightarrow I = \frac{x}{2}\sqrt{4-x^2} + \frac{(2)^2}{2}\sin^{-1}\left(\frac{x}{2}\right)+c \quad \therefore I = \frac{x}{2}\sqrt{4-x^2}+2\sin^{-1}\left(\frac{x}{2}\right)+c$$

2. $\int \sqrt{4-9x^2}\,dx$

Sol. Let $I = \int \sqrt{4-9x^2}\,dx \quad \Rightarrow I = \int \sqrt{9\left(\frac{4}{9}-x^2\right)}\,dx \quad \Rightarrow I = 3\int \sqrt{\left(\frac{2}{3}\right)^2-(x)^2}\,dx$

$$\Rightarrow I = 3\left[\frac{x}{2}\sqrt{\left(\frac{2}{3}\right)^2-x^2}+\frac{\left(\frac{2}{3}\right)^2}{2}\sin^{-1}\left(\frac{x}{2/3}\right)\right]+c \quad \Rightarrow I = 3\left[\frac{x}{2}\sqrt{\frac{4}{9}-x^2}+\frac{4}{9.2}\sin^{-1}\left(\frac{3x}{2}\right)\right]+c$$

$$\therefore \quad I = \frac{x}{2}\sqrt{4-9x^2}+\frac{2}{3}\sin^{-1}\left(\frac{3x}{2}\right)+c$$

3. $\int \sqrt{x^2-2}\,dx$

Sol. Let $I = \int \sqrt{x^2-2}\,dx \quad \Rightarrow I = \int \sqrt{(x)^2-\left(\sqrt{2}\right)^2}\,dx$

$$\Rightarrow I = \frac{x}{2}\sqrt{(x)^2-\left(\sqrt{2}\right)^2}-\frac{\left(\sqrt{2}\right)^2}{2}\log\left|x+\sqrt{(x)^2-\left(\sqrt{2}\right)^2}\right|+c \quad \therefore I = \frac{x}{2}\sqrt{x^2-2}-\log\left|x+\sqrt{x^2-2}\right|+c$$

4. $\int \sqrt{2x^2-3}\,dx$

Sol. Let $I = \int \sqrt{2\left(x^2-\frac{3}{2}\right)}\,dx \quad \Rightarrow I = \int \sqrt{2\left(x^2-\frac{3}{2}\right)}\,dx \quad \Rightarrow I = \sqrt{2}\int \sqrt{\left(x^2-\frac{3}{2}\right)}\,dx$

$$\Rightarrow I = \sqrt{2}\int \sqrt{(x)^2-\left(\frac{\sqrt{3}}{\sqrt{2}}\right)^2}\,dx \quad \Rightarrow I = \sqrt{2}\left[\frac{x}{2}\sqrt{(x)^2-\left(\sqrt{\frac{3}{2}}\right)^2}-\frac{\left(\sqrt{\frac{3}{2}}\right)^2}{2}\log\left|x+\sqrt{(x)^2-\left(\sqrt{\frac{3}{2}}\right)^2}\right|\right]+c$$

$$\therefore \quad I = \frac{x}{2}\sqrt{2x^2-3}-\frac{3}{2\sqrt{2}}\log\left|\sqrt{2}x+\sqrt{2x^2-3}\right|+c$$

5. $\int \sqrt{x^2+5}\,dx$

Sol. Let $I = \int \sqrt{x^2+5}\,dx \quad \Rightarrow I = \int \sqrt{(x)^2+\left(\sqrt{5}\right)^2}\,dx$

$$\Rightarrow I = \frac{x}{2}\sqrt{(x)^2+\left(\sqrt{5}\right)^2}+\frac{\left(\sqrt{5}\right)^2}{2}\log\left|x+\sqrt{(x)^2+\left(\sqrt{5}\right)^2}\right|+c$$

$$\therefore \quad I = \frac{x}{2}\sqrt{x^2+5}+\frac{5}{2}\log\left|x+\sqrt{x^2+5}\right|+c$$

6. $\int \sqrt{4x^2+9}\,dx$

Sol. Let $I=\int \sqrt{4x^2+9}\,dx \Rightarrow I=\int \sqrt{4\left(x^2+\frac{9}{4}\right)}\,dx \Rightarrow I=2\int \sqrt{(x)^2+\left(\frac{3}{2}\right)^2}\,dx$

$$\Rightarrow I=2\left[\frac{x}{2}\sqrt{(x)^2+\left(\frac{3}{2}\right)^2}+\frac{\left(\frac{3}{2}\right)^2}{2}\log\left|x+\sqrt{(x)^2+\left(\frac{3}{2}\right)^2}\right|\right]+c$$

$$\therefore\ I=\frac{x}{2}\sqrt{4x^2+9}+\frac{9}{4}\log\left|2x+\sqrt{4x^2+9}\right|+c$$

7. $\int \sqrt{3x^2+4}\,dx$

Sol. Let $I=\int \sqrt{3x^2+4}\,dx \Rightarrow I=\int \sqrt{3\left(x^2+\frac{4}{3}\right)}\,dx \Rightarrow I=\sqrt{3}\int \sqrt{(x)^2+\left(\frac{2}{\sqrt{3}}\right)^2}\,dx$

$$\Rightarrow I=\sqrt{3}\left[\frac{x}{2}\sqrt{(x)^2+\left(\frac{2}{\sqrt{3}}\right)^2}+\frac{\left(\frac{2}{\sqrt{3}}\right)^2}{2}\log\left|x+\sqrt{(x)^2+\left(\frac{2}{\sqrt{3}}\right)^2}\right|\right]+c$$

$$\therefore\ I=\frac{x}{2}\sqrt{3x^2+4}+\frac{2}{\sqrt{3}}\log\left|\sqrt{3}x+\sqrt{3x^2+4}\right|+c$$

8. $\int \cos x\sqrt{9-\sin^2 x}\,dx$

Sol. Let $I=\int \cos x\sqrt{9-\sin^2 x}\,dx$

Now, Put $\sin x=t \Rightarrow \cos x\,dx=dt$, we get, $I=\int \sqrt{9-t^2}\,dt \Rightarrow I=\int \sqrt{(3)^2-(t)^2}\,dt$

$$\Rightarrow I=\frac{t}{2}\sqrt{(3)^2-(t)^2}+\frac{(3)^2}{2}\sin^{-1}\left(\frac{t}{3}\right)+c \quad \therefore\ I=\frac{\sin x}{2}\sqrt{9-\sin^2 x}+\frac{9}{2}\sin^{-1}\left(\frac{\sin x}{3}\right)+c$$

9. $\int \sqrt{x^2-4x+2}\,dx$

Sol. Let $I=\int \sqrt{x^2-4x+2}\,dx \Rightarrow I=\int \sqrt{(x)^2-2.x.2+(2)^2-(2)^2+2}\,dx$

$$\Rightarrow I=\int \sqrt{(x-2)^2-2}\,dx \Rightarrow I=\int \sqrt{(x-2)^2-\left(\sqrt{2}\right)^2}\,dx$$

$$\Rightarrow I=\frac{x-2}{2}\sqrt{(x-2)^2-\left(\sqrt{2}\right)^2}-\frac{\left(\sqrt{2}\right)^2}{2}\log\left|x-2+\sqrt{(x-2)^2-\left(\sqrt{2}\right)^2}\right|+c$$

$$\therefore\ I=\frac{(x-2)}{2}\sqrt{x^2-4x+2}-\log\left|(x-2)+\sqrt{x^2-4x+2}\right|+c$$

10. $\int \sqrt{x^2+6x-4}\,dx$

Sol. Let $I=\int \sqrt{x^2+6x-4}\,dx \Rightarrow I=\int \sqrt{(x)^2+2.x.3+(3)^2-(3)^2-4}\,dx$

$$\Rightarrow I=\frac{x+3}{2}\sqrt{(x+3)^2-\left(\sqrt{13}\right)^2}-\frac{\left(\sqrt{13}\right)^2}{2}\log\left|(x+3)+\sqrt{(x+3)^2-\left(\sqrt{3}\right)^2}\right|+c$$

$$\therefore\ I=\frac{(x+3)}{2}\sqrt{x^2+6x-4}-\frac{13}{2}\log\left|(x+3)+\sqrt{x^2+6x-4}\right|+c$$

11. $\int\sqrt{2x-x^2}\,dx$

Sol. Let $I=\int\sqrt{2x-x^2}\,dx \quad\Rightarrow I=\int\sqrt{-\left[x^2-2x\right]}\,dx$

$$\Rightarrow I=\int\sqrt{-\left[(x)^2-2.x.1+(1)^2-(1)^2\right]}\,dx \quad\Rightarrow I=\int\sqrt{-\left[(x-1)^2-(1)^2\right]}\,dx$$

$$\Rightarrow I=\int\sqrt{(1)^2-(x-1)^2}\,dx \quad\Rightarrow I=\frac{x-1}{2}\sqrt{(1)^2-(x-1)^2}+\frac{(1)^2}{2}\sin^{-1}\left(\frac{x-1}{1}\right)+c$$

$$\therefore\ I=\frac{(x-1)}{2}\sqrt{2x-x^2}+\frac{1}{2}\sin^{-1}(x-1)+c$$

12. $\int\sqrt{1-4x-x^2}\,dx$

Sol. Let $I=\int\sqrt{1-4x-x^2}\,dx \quad\Rightarrow I=\int\sqrt{-\left[x^2+4x-1\right]}\,dx$

$$\Rightarrow I=\int\sqrt{-\left[(x)^2+2.x.2+(2)^2-(2)^2-1\right]}\,dx \quad\Rightarrow I=\int\sqrt{-\left[(x+2)^2-5\right]}\,dx$$

$$\Rightarrow I=\int\sqrt{\left(\sqrt{5}\right)^2-(x+2)^2}\,dx \quad\Rightarrow I=\frac{x+2}{2}\sqrt{\left(\sqrt{5}\right)^2-(x+2)^2}+\frac{\left(\sqrt{5}\right)^2}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right)+c$$

$$\therefore\ I=\frac{(x+2)}{2}\sqrt{1-4x-x^2}+\frac{5}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right)+c$$

13. $\int\sqrt{2ax-x^2}\,dx$

Sol. Let $I=\int\sqrt{2ax-x^2}\,dx \quad\Rightarrow I=\int\sqrt{-\left[x^2-2ax\right]}\,dx$

$$\Rightarrow I=\int\sqrt{-\left[(x)^2-2.x.a+(a)^2-(a)^2\right]}\,dx \quad\Rightarrow I=\int\sqrt{-\left[(x-a)^2-a^2\right]}\,dx$$

$$\Rightarrow I=\int\sqrt{(a)^2-(x-a)^2}\,dx \quad\Rightarrow I=\frac{x-a}{2}\sqrt{(a)^2-(x-a)^2}+\frac{(a)^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right)+c$$

$$\therefore\ I=\frac{(x-a)}{2}\sqrt{2ax-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right)+c$$

14. $\int\sqrt{2x^2+3x+4}\,dx$

Sol. Let $I=\int\sqrt{2x^2+3x+4}\,dx \quad\Rightarrow I=\int\sqrt{2\left(x^2+\frac{3}{2}x+2\right)}\,dx$

$$\Rightarrow I=\sqrt{2}\int\sqrt{x^2+\frac{3}{2}x+2}\,dx \quad\Rightarrow I=\sqrt{2}\int\sqrt{(x)^2+2.x.\frac{3}{4}+\left(\frac{3}{4}\right)^2-\left(\frac{3}{4}\right)^2+2}\,dx$$

$$\Rightarrow I=\sqrt{2}\int\sqrt{\left(x+\frac{3}{4}\right)^2-\left(\frac{\sqrt{23}}{4}\right)^2}\,dx$$

$$\Rightarrow I=\sqrt{2}\left[\frac{x+\frac{3}{4}}{2}\sqrt{\left(x+\frac{3}{4}\right)^2-\left(\frac{\sqrt{23}}{4}\right)^2}-\frac{\left(\frac{\sqrt{23}}{4}\right)^2}{2}.\log\left|\left(x+\frac{3}{4}\right)+\sqrt{\left(x+\frac{3}{4}\right)^2-\left(\frac{\sqrt{23}}{4}\right)^2}\right|\right]+c$$

$$\therefore\ I=\frac{(4x+3)}{8}\sqrt{2x^2+3x+4}+\frac{23\sqrt{2}}{32}\log\left|\left(x+\frac{3}{4}\right)+\frac{1}{\sqrt{2}}\sqrt{2x^2+3x+4}\right|+c$$

15. $\int\sqrt{x^2+x}\,dx$

Sol. Let $I=\int\sqrt{x^2+x}\,dx \Rightarrow I=\int\sqrt{(x)^2+2.x.\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\,dx \Rightarrow I=\int\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\,dx$

$$\Rightarrow I=\frac{x+\frac{1}{2}}{2}\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}-\frac{\left(\frac{1}{2}\right)^2}{2}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right|+c$$

$$\therefore\ I=\frac{(2x+1)}{4}\sqrt{x^2+x}-\frac{1}{8}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x}\right|+c$$

16. $\int\sqrt{x^2+x+1}\,dx$

Sol. Let $I=\int\sqrt{x^2+x+1}\,dx \Rightarrow I=\int\sqrt{(x)^2+2.x.\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1}\,dx \Rightarrow I=\int\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dx$

$$\Rightarrow I=\frac{x+\frac{1}{2}}{2}\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}+\frac{\left(\frac{\sqrt{3}}{2}\right)^2}{2}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right|+c$$

$$\therefore\ I=\frac{(2x+1)}{4}\sqrt{x^2+x+1}-\frac{3}{8}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right|+c$$

17. $\int(2x-5)\sqrt{x^2-4x+3}\,dx$

Sol. Let $I=\int(2x-5)\sqrt{x^2-4x+3}\,dx$

Using, $(2x-5)=A.\frac{d}{dx}\left(x^2-4x+3\right)+B$

$\Rightarrow 2x-5=A(2x-4)+B \Rightarrow 2x-5=2Ax-4A+B$

Now, Equating co-efficient both side we get, $2A=2 \Rightarrow A=1$ and $-4A+B=-5$

$\Rightarrow B=-5+4A \Rightarrow B=-5+4(1) \quad \therefore\ B=-1$

$\Rightarrow I=\int\{A(2x-4)+B\}\sqrt{x^2-4x+3}\,dx$

$\Rightarrow I=A\int(2x-4)\sqrt{x^2-4x+3}\,dx+B\int\sqrt{x^2-4x+3}\,dx$

$\Rightarrow I=I_1-I_2 \qquad \ldots(1),\qquad$ where $I_1=\int(2x-4)\sqrt{x^2-4x+3}\,dx$

Put $x^2-4x+3=t \Rightarrow (2x-4)\,dx=dt$, we get,

$$\therefore\ I_1=\int\sqrt{t}\ dt \Rightarrow I_1=\frac{t^{3/2}}{3/2}+c_1 \Rightarrow I_1=\frac{2}{3}\left(x^2-4x+3\right)^{3/2}+c_1 \Rightarrow I_2=\int\sqrt{x^2-4x+3}\ dx$$

$$\Rightarrow I_2=\int\sqrt{(x)^2-2.x.2+(2)^2-(2)^2+3}\ dx \Rightarrow I_2=\int\sqrt{(x-2)^2-(1)^2}\ dx$$

$$\Rightarrow I_2=\frac{x-2}{2}\sqrt{(x-2)^2-(1)^2}-\frac{(1)^2}{2}\log\left|(x-2)+\sqrt{(x-2)^2-(1)^2}\right|+c$$

$$\Rightarrow I_2=\frac{x-2}{2}\sqrt{x^2-4x+3}-\frac{1}{2}\log\left|(x-2)+\sqrt{x^2-4x+3}\right|+c$$

Putting the value of I_1 & I_2 in equation (1), we get,

$$\therefore I=\frac{2}{3}\left(x^2-4x+3\right)^{3/2}-\frac{(x-2)}{2}\sqrt{x^2-4x+3}+\frac{1}{2}\log\left|(x-2)+\sqrt{x^2-4x+3}\right|+c$$

18. $\int(x+2)\sqrt{x^2+x+1}\ dx$

Sol. Let $I=\int(x+2)\sqrt{x^2+x+1}\ dx$

By Using, $x+2=A.\frac{d}{dx}\left(x^2+x+1\right)+B$

$$\Rightarrow x+2=A(2x+1)+B \Rightarrow x+2=2Ax+A+B$$

Now, Equating co-efficient both side, we get, $2A=1 \quad \therefore\ A=\frac{1}{2}$

And $A+B=2 \quad \Rightarrow B=2-A \quad \Rightarrow B=2-\frac{1}{2} \quad \therefore\ B=\frac{3}{2}$

$$\Rightarrow I=\int\{A(2x+1)+B\}\sqrt{x^2+x+1}\ dx$$

$$\Rightarrow I=A\int(2x+1)\sqrt{x^2+x+1}\ dx+B\int\sqrt{x^2+x+1}\ dx \Rightarrow I=\frac{1}{2}I_1+\frac{3}{2}I_2 \qquad \ldots(1)$$

$$\Rightarrow I_1=\int(2x+1)\sqrt{x^2+x+1}\ dx$$

Put $x^2+x+1=t \quad \Rightarrow (2x+1)\ dx=dt$, we get,

$$\therefore\ I_1=\int\sqrt{t}\,dt \Rightarrow I_1=\frac{t^{3/2}}{3/2}+c_1 \Rightarrow I_1=\frac{2}{3}\left(x^2+x+1\right)^{3/2}+c_1$$

$$\Rightarrow I_2=\int\sqrt{x^2+x+1}\ dx \Rightarrow I_2=\int\sqrt{(x)^2+2.x.\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1}\ dx$$

$$\Rightarrow I_2=\int\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\ dx \Rightarrow I_2=\int\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\ dx$$

$$\Rightarrow I_2=\frac{x+\frac{1}{2}}{2}\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}+\frac{\left(\frac{\sqrt{3}}{2}\right)^2}{2}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right|+c$$

$$\therefore\ I_2=\frac{2x+1}{4}\sqrt{x^2+x+1}+\frac{3}{8}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right|+c_2$$

Putting the value of I_1 & I_2 in equation (1), we get,

$$\therefore\ I=\frac{1}{3}\left(x^2+x+1\right)^{3/2}+\frac{3}{8}(2x+1)\sqrt{x^2+x+1}+\frac{9}{16}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right|+c$$

19. $\int(x-5)\sqrt{x^2+x}\ dx$

Sol. Let $I=\int(x-5)\sqrt{x^2+x}\ dx$

Using, $(x-5)=A.\frac{d}{dx}\left(x^2+x\right)+B$

$\Rightarrow\ x-5=A(2x+1)+B\quad\Rightarrow\ x-5=2Ax+A+B$

Equating co-efficient we get, $2A=1\quad\therefore\ A=\frac{1}{2}$

And $A+B=-5\quad\Rightarrow B=-5-A\quad\Rightarrow B=-5-\frac{1}{2}\quad\therefore\ B=\frac{-11}{2}$

$\Rightarrow I=\int\left\{A(2x+1)+B\right\}\sqrt{x^2+x}\ dx\quad\Rightarrow I=A\int(2x+1)\sqrt{x^2+x}\ dx+B\int\sqrt{x^2+x}\ dx$

$\Rightarrow I=\frac{1}{2}I_1-\frac{11}{2}I_2\qquad\ldots(1)$

$I_1=\int(2x+1)\sqrt{x^2+x}\ dx$

Now, Put $x^2+x=t\quad\Rightarrow\ (2x+1)\ dx=dt$, we get,

$\therefore\ I_1=\int\sqrt{t}\ dt\ \Rightarrow I_1=\frac{t^{3/2}}{3/2}+c_1\ \Rightarrow I_1=-\frac{2}{3}\left(x^2+x\right)^{3/2}+c_1$

$\Rightarrow I_2=\int\sqrt{x^2+x}\ dx\ \Rightarrow I_2=\int\sqrt{(x)^2+2.x.\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\ dx\ \Rightarrow I_2=\int\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\ dx$

$$\Rightarrow I_2=\frac{x+\frac{1}{2}}{2}\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}-\frac{\left(\frac{1}{2}\right)^2}{2}\log\left|x+\frac{1}{2}+\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right|+c$$

$$\Rightarrow I_2=\frac{2x+1}{4}\sqrt{x^2+x}-\frac{1}{8}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x}\right|+c_2$$

Putting the value of I_1 & I_2 in equation (1), we get,

$$\therefore\ I=\frac{1}{3}\left(x^2+x\right)^{3/2}-\frac{11}{8}(2x+1)\sqrt{x^2+x}+\frac{11}{16}.\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x}\right|+c$$

20. $\int(4x+1)\sqrt{x^2-x-2}\ dx$

Sol. Let $I=\int(4x+1)\sqrt{x^2-x-2}\ dx$

By Using, $4x+1=A.\frac{d}{dx}\left(x^2-x-2\right)+B$

$\Rightarrow\ 4x+1=A(2x-1)+B\quad\Rightarrow\ 4x+1=2Ax-A+B$

Now Equating co-efficient both side we get, $2A=4\quad\therefore\ A=2$

And $-A+B=1\quad\Rightarrow\ B=1+A\quad\Rightarrow\ B=1+2\quad\therefore\ B=3$

$\Rightarrow\ I=\int\left\{A(2x-1)+B\right\}\sqrt{x^2-x-2}\ dx$

$$\Rightarrow I = A\int(2x-1)\sqrt{x^2-x-2}\ dx + B\int\sqrt{x^2-x-2}\ dx \quad \Rightarrow I = 2I_1 + 3I_2 \qquad \ldots(1)$$

$$\therefore\ I_1 = \int(2x-1)\sqrt{x^2-x-2}\ dx$$

Put $x^2-x-2=t \ \Rightarrow (2x-1)\ dx = dt$, we get,

$$\therefore\ I_1 = \int\sqrt{t}\ dt \ \Rightarrow I_1 = \frac{t^{3/2}}{3/2}+c_1 \ \Rightarrow I_1 = \frac{2}{3}\left(x^2-x-2\right)^{3/2}+c_1$$

$$\Rightarrow I_2 = \int\sqrt{x^2-x-2}\ dx$$

$$\Rightarrow I_2 = \int\sqrt{(x)^2-2.x.\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2-2}\ dx \quad \Rightarrow I_2 = \int\sqrt{\left(x-\frac{1}{2}\right)^2-\left(\frac{3}{2}\right)^2}\ dx$$

$$\Rightarrow I_2 = \frac{x-\frac{1}{2}}{2}\sqrt{\left(x-\frac{1}{2}\right)^2-\left(\frac{3}{2}\right)^2}-\frac{\left(\frac{3}{2}\right)^2}{2}\log\left|\left(x-\frac{1}{2}\right)+\sqrt{\left(x-\frac{1}{2}\right)^2-\left(\frac{3}{2}\right)^2}\right|+c_2$$

$$\Rightarrow I_2 = \frac{2x-1}{4}\sqrt{x^2-x-2}-\frac{9}{8}\log\left|\left(x-\frac{1}{2}\right)+\sqrt{x^2-x-2}\right|+c_2$$

Putting the value of I_1 & I_2 in equation (1), we get,

$$\therefore\ I = \frac{4}{3}\left(x^2-x-2\right)^{3/2}+\frac{3}{4}(2x-1)\sqrt{x^2-x-2}-\frac{27}{8}\log\left|\left(x-\frac{1}{2}\right)+\sqrt{x^2-x-2}\right|+c$$

21. $\int(2x+1)\sqrt{2x^2+3}\ dx$

Sol. Let $I = \int(2x+1)\sqrt{2x^2+3}\ dx$

Put $2x+1 = A.\frac{d}{dx}\left(2x^2+3\right)+B \ \Rightarrow 2x+1 = A(4x)+B$

Now Equating co-efficient both side we get, $4A=2 \quad \therefore\ A=\frac{1}{2}$ & $B=1$

$$I = \int\{A(4x)+B\}\sqrt{2x^2+3}\ dx \quad \Rightarrow I = A\int 4x\sqrt{2x^2+3}\ dx + B\int\sqrt{2x^2+3}\ dx$$

$$\Rightarrow I = \frac{1}{2}I_1 + I_2 \qquad \ldots(1)$$

$I_1 = \int 4x\sqrt{2x^2+3}\ dx$, Put $2x^2+3=t \ \Rightarrow 4x\ dx = dt$, we get,

$$I_1 = \int\sqrt{t}\ dt \ \Rightarrow I_1 = \frac{t^{3/2}}{3/2}+c_1 \ \Rightarrow I_1 = \frac{2}{3}\left(2x^2+3\right)^{3/2}+c_1$$

$$I_2 = \int\sqrt{2x^2+3}\ dx \ \Rightarrow I_2 = \int\sqrt{2\left(x^2+\frac{3}{2}\right)}\ dx \ \Rightarrow I_2 = \sqrt{2}\int\sqrt{(x)^2-\left(\sqrt{\frac{3}{2}}\right)^2}\ dx$$

$$\Rightarrow I_2 = \sqrt{2}\left[\frac{x}{2}\sqrt{(x)^2+\left(\sqrt{\frac{3}{2}}\right)^2}+\frac{\left(\sqrt{\frac{3}{2}}\right)^2}{2}\log\left|x+\sqrt{(x)^2+\frac{\left(\sqrt{3}\right)^2}{\sqrt{2}}}\right|\right]+c$$

$$\Rightarrow I_2 = \frac{x}{2}\sqrt{2x^2+3}+\frac{3}{2\sqrt{2}}\log\left|\sqrt{2}x+\sqrt{2x^2+3}\right|+c_2$$

Putting the value of I_1 & I_2 in equation (1), we get,

$$\therefore \quad I=\frac{1}{3}\left(2x^2+3\right)^{3/2}+\frac{x}{2}\sqrt{2x^2+3}+\frac{3}{2\sqrt{2}}\log\left|\sqrt{2}\,x+\sqrt{2x^2+3}\right|+c$$

22. $\int x\sqrt{1+x-x^2}\;dx$

Sol. Let $I=\int x\sqrt{1+x-x^2}\;dx$

Using $x=A.\frac{d}{dx}\left(1+x-x^2\right)+B$

$\Rightarrow x=A(1-2x)+B \quad \Rightarrow x=A-2Ax+B$

By Equating co-efficient both side, we get, $-2A=1 \quad \therefore\ A=-\frac{1}{2}$

And $A+B=0 \quad \Rightarrow B=-A \quad \therefore\ B=\frac{1}{2}$

$\Rightarrow I=\int\left\{A(1-2x)+B\right\}\sqrt{1+x-x^2}\;dx$

$\Rightarrow I=A\int(1-2x)\sqrt{1+x-x^2}\;dx+B\int\sqrt{1+x-x^2}\;dx \quad \Rightarrow I=-\frac{1}{2}I_1+\frac{1}{2}I_2 \quad \ldots(1)$

$\Rightarrow I_1=\int(1-2x)\sqrt{1+x-x^2}\;dx$, Put $1+x-x^2=t \quad \Rightarrow (1-2x)dx=dt$, we get.

$\therefore\ I_1=\int\sqrt{t}\;dt \quad \Rightarrow I_1=\frac{t^{3/2}}{3/2}+c_1 \quad \Rightarrow I_1=\frac{2}{3}\left(1+x-x^2\right)^{3/2}+c_1$

Now, $I_2=\int\sqrt{1+x-x^2}\,dx$

$$\Rightarrow I_2=\int\sqrt{-\left[x^2-x-1\right]}\;dx \quad \Rightarrow I_2=\int\sqrt{-\left[(x)^2-2x.\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2-1\right]}\;dx$$

$$\Rightarrow I_2=\int\sqrt{-\left[\left(x-\frac{1}{2}\right)^2-\frac{5}{4}\right]}\;dx \quad \Rightarrow I_2=\int\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}\;dx$$

$$\Rightarrow I_2=\frac{x-\frac{1}{2}}{2}\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}+\frac{\left(\frac{\sqrt{5}}{2}\right)^2}{2}\sin^{-1}\frac{\left(x-\frac{1}{2}\right)}{\frac{\sqrt{5}}{2}}+c_2$$

$$\Rightarrow I_2=\frac{2x-1}{4}\sqrt{1+x-x^2}+\frac{5}{8}\sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right)+c_2$$

Putting the value of I_1 & I_2 in equation (1), we get,

$$\therefore I=-\frac{1}{3}\left(1+x-x^2\right)^{3/2}+\frac{1}{8}(2x-1)\sqrt{1+x-x^2}+\frac{5}{16}\sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right)+c$$

23. $\int(2x-5)\sqrt{2+3x-x^2}\;dx$

Sol. Let $I=\int(2x-5)\sqrt{2+3x-x^2}\;dx$

Using $2x-5=A.\frac{d}{dx}\left(2+3x-x^2\right)+B$

$\Rightarrow\ 2x-5=A(3-2x)+B\ \ \Rightarrow\ 2x-5=3A-2Ax+B$

By Equating co-efficient both side, we get, $-2A=2\ \ \therefore\ A=-1$

And $3A+B=-5\qquad \Rightarrow B=-5-3A\quad \therefore\ B=-2$

$\Rightarrow\ I=\int\{A(3-2x)+B\}\sqrt{2+3x-x^2}\ dx$

$\Rightarrow\ I=A\int(3-2x)\sqrt{2+3x-x^2}\ dx+B\int\sqrt{2+3x-x^2}\ dx\ \ \Rightarrow\ I=-I_1-2\,I_2 \qquad \ldots(1)$

$\Rightarrow\ I_1=\int(3-2x)\sqrt{2+3x-x^2}\ dx$, Put $2+3x-x^2=t\ \Rightarrow\ (3-2x)\ dx=dt$, we get,

$\therefore\ \ I_1=\int\sqrt{t}\ dt\ \ \Rightarrow I_1=\dfrac{t^{3/2}}{3/2}+c_1\ \ \Rightarrow I_1=\dfrac{2}{3}\left(2+3x-x^2\right)^{3/2}+c_1$

Now, $I_2=\int\sqrt{2+3x-x^2}\ dx$

$\Rightarrow\ I_2=\int\sqrt{-\left[x^2-3x-2\right]}\ dx\ \ \Rightarrow\ I_2=\int\sqrt{-\left[(x)^2-2.x.\dfrac{3}{2}+\left(\dfrac{3}{2}\right)^2-\left(\dfrac{3}{2}\right)^2-2\right]}\ dx$

$\Rightarrow\ I_2=\int\sqrt{-\left[\left(x-\dfrac{3}{2}\right)^2-\dfrac{17}{4}\right]}\ dx\ \ \Rightarrow\ I_2=\int\sqrt{\left(\dfrac{\sqrt{17}}{2}\right)^2-\left(x-\dfrac{3}{2}\right)^2}\ dx$

$$\Rightarrow\ I_2=\frac{x-\frac{3}{2}}{2}\sqrt{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}+\frac{\left(\frac{\sqrt{17}}{2}\right)^2}{2}\sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{17}}{2}}\right)+c_2$$

$$\Rightarrow\ I_2=\frac{2x-3}{4}\sqrt{2+3x-x^2}+\frac{17}{8}\sin^{-1}\left(\frac{2x-3}{\sqrt{17}}\right)+c_2$$

Putting the value of I_1 & I_2 in equation (1), we get,

$$\therefore\ \ I=-\frac{2}{3}\left(2+3x-x^2\right)^{3/2}-\frac{1}{2}(2x-3)\sqrt{2+3x-x^2}-\frac{17}{4}\sin^{-1}\left(\frac{2x-3}{\sqrt{17}}\right)+c$$

24. $\int(6x+5)\left(\sqrt{6+x-2x^2}\right)dx$

Sol. Let $I=\int(6x+5)\sqrt{6+x-2x^2}\ dx$

Using, $6x+5=A.\dfrac{d}{dx}\left(6+x-2x^2\right)+5$

$\Rightarrow\ 6x+5=A(1-4x)+B\ \ \Rightarrow\ 6x+5=A-4Ax+B$

By Equating co-efficient both side, we get, $-4A=6\ \ \therefore\ A=-\dfrac{3}{2}$

And $A+B=5\quad \Rightarrow B=5-A\quad \therefore\ B=\dfrac{13}{2}$

$\Rightarrow\ I=\int\{A(1-4x)+B\}\sqrt{6+x-2x^2}\ dx$

$\Rightarrow\ I=A\int(1-4x)\sqrt{6+x-2x^2}dx+B\int\sqrt{6+x-2x^2}\ dx\ \ \Rightarrow\ I=\dfrac{-3}{2}I_1+\dfrac{13}{2}I_2 \qquad \ldots(1)$

$\Rightarrow\ I_1=\int(1-4x)\sqrt{6+x-2x^2}\ dx$, Put $6+x-2x^2=t\ \Rightarrow\ (1-4x)\ dx=dt$, we get,

$\therefore\ I_1 = \int \sqrt{t}\, dt \quad \Rightarrow I_1 = \frac{t^{3/2}}{3/2} + c_1 \Rightarrow I_1 = \frac{2}{3}\left(6+x-2x^2\right)^{3/2} + c_1$

Now, $I_2 = \int \sqrt{6+x-2x^2}\, dx$

$\Rightarrow I_2 = \int \sqrt{-2\left[x^2 - \frac{x}{2} - 3\right]}\, dx \quad \Rightarrow I_2 = \sqrt{2}\int \sqrt{-\left[(x)^2 - 2.x.\frac{1}{4} + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2 - 3\right]}\, dx$

$\Rightarrow I_2 = \sqrt{2}\int \sqrt{-\left[\left(x-\frac{1}{4}\right)^2 - \frac{49}{16}\right]}\, dx \quad \Rightarrow I_2 = \sqrt{2}\int \sqrt{\left(\frac{7}{4}\right)^2 - \left(x-\frac{1}{4}\right)^2}\, dx$

$$\Rightarrow I_2 = \sqrt{2}\left[\frac{x-\frac{1}{4}}{2}\sqrt{\left(\frac{7}{4}\right)^2 - \left(x-\frac{1}{4}\right)^2} + \frac{\left(\frac{7}{4}\right)^2}{2}\sin^{-1}\left(\frac{x-\frac{1}{4}}{\frac{7}{4}}\right)\right] + c_2$$

$\Rightarrow I_2 = \frac{4x-1}{8}\sqrt{6+x-2x^2} + \frac{49}{16\sqrt{2}}\sin^{-1}\left(\frac{4x-1}{7}\right) + c_2$

Putting the value of I_1 & I_2 in equation (1), we get,

$\therefore\ I = -\left(6+x-2x^2\right)^{3/2} + -\frac{13(4x-1)}{16}\sqrt{6+x-2x^2} + \frac{637}{32\sqrt{2}}\sin^{-1}\left(\frac{4x-1}{7}\right) + c$

25. $\int (x+1)\sqrt{1-x-x^2}\, dx$

Sol. Let $I = \int (x+1)\sqrt{1-x-x^2}\, dx$

Using, $x+1 = A.\frac{d}{dx}\left(1-x-x^2\right) + B$

$\Rightarrow x+1 = A(-1-2x) + B \quad \Rightarrow x+1 = -A - 2Ax + B \quad \Rightarrow x+1 = (-A+B) - 2Ax$

By Equating co-efficient both side, we get, $-2A = 1 \quad \therefore\ A = -\frac{1}{2}$

And $-A + B = 1 \quad \Rightarrow B = 1 + A \Rightarrow B = 1 - \frac{1}{2} \quad \therefore\ B = \frac{1}{2}$

$\Rightarrow I = \int \left\{A(-1-2x) + B\right\}\sqrt{1-x-x^2}\, dx$

$\Rightarrow I = A\int (-1-2x)\sqrt{1-x-x^2}\, dx + B\int \sqrt{1-x-x^2}\, dx \quad \Rightarrow I = \frac{-1}{2}I_1 + \frac{1}{2}I_2 \qquad \ldots(1)$

$\Rightarrow I_1 = \int (-1-2x)\sqrt{1-x-x^2}\, dx \qquad$ Put $1-x-x^2 = t \Rightarrow (-1-2x)\, dx = dt$, we get,

$\therefore\ I_1 = \int \sqrt{t}\, dt \quad \Rightarrow I_1 = \frac{t^{3/2}}{3/2} + c_1 \quad \Rightarrow I_1 = \frac{2}{3}\left(1-x-x^2\right)^{3/2} + c_1$

Now., $I_2 = \int \sqrt{1-x-x^2}\, dx$

$\Rightarrow I_2 = \int \sqrt{-\left[x^2 + x - 1\right]}\, dx \quad \Rightarrow I_2 = \int \sqrt{-\left[(x)^2 + 2.x.\frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 - 1\right]}\, dx$

$\Rightarrow I_2 = \int \sqrt{-\left[\left(x+\frac{1}{2}\right)^2 - \frac{5}{4}\right]}\, dx \quad \Rightarrow I_2 = \int \sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - \left(x+\frac{1}{2}\right)^2}\, dx$

$$\Rightarrow I_2 = \frac{x+\frac{1}{2}}{2}\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - \left(x+\frac{1}{2}\right)^2} + \frac{\left(\frac{\sqrt{5}}{2}\right)^2}{2}\sin^{-1}\frac{\left(x+\frac{1}{2}\right)}{\sqrt{5}/2} + c_2$$

$$\Rightarrow I_2 = \frac{2x+1}{4}\sqrt{1-x-x^2} + \frac{5}{8}\sin^{-1}\left(\frac{2x+1}{\sqrt{5}}\right) + c_2$$

Putting the value of I_1 & I_2 in equation (1), we get,

$$\therefore\ I = \frac{-1}{3}\left(1-x-x^2\right)^{3/2} + \frac{2x+1}{8}\left(\sqrt{1-x-x^2}\right) + \frac{5}{16}\sin^{-1}\left(\frac{2x+1}{\sqrt{5}}\right) + c$$

26. $\int (x-3)\sqrt{x^2+3x-18}\,dx$

Sol. Let $x-3 = \lambda\frac{d}{dx}\left(x^2+3x-18\right)$

$\Rightarrow x-3 = \lambda(2x+3)+\mu$

Comparing coefficient of x and constant term we get

$1 = 2\lambda,\ -3 = 3\lambda+\mu \Rightarrow \lambda = \frac{1}{2}, \mu = -3-\frac{3}{2} = \frac{-9}{2}$

$\therefore$ Given integral $= \int\left[\frac{1}{2}(2x+3)+\left(-\frac{9}{2}\right)\right]dx$

$$= \frac{1}{2}\int(2x+3)\sqrt{x^2+3x-18}\,dx + \left(-\frac{9}{2}\right)\int\sqrt{x^2+3x-18}\,dx$$

$$= \frac{1}{2}\left[\frac{\left(x^2+3x-18\right)^{3/2}}{\frac{3}{2}}\right] - \frac{9}{2}\int\sqrt{\left(x+\frac{3}{2}\right)^2 - \frac{9}{4} - 18}\,dx$$

$$= \frac{1}{3}\left(x^2+3x-18\right)^{3/2}\ \frac{9}{2}\int\sqrt{\left(x^2+\frac{3}{2}\right)^2 - \left(\frac{9}{2}\right)^2}\,dx$$

$$= \frac{1}{3}\left(x^2+3x-8\right)^{3/2} - \frac{9}{8}\left[(2x+3)\sqrt{x^2+3x-18} - \frac{81}{2}\log\left[x+\frac{3}{2}+\sqrt{x^2+3x-18}\right]\right]$$