Sequences and Series

Case Study Based Questions

Read the following passages and answer the questions that follow:

1. A.M. and G.M. Inequality
Let a, b be two positive real numbers, then
$(\sqrt{a}-\sqrt{b})^2 \geq 0$
$\Rightarrow a+b-2\sqrt{ab}\geq 0$

$$\Rightarrow \quad \frac{a+b}{2} \geq \sqrt{ab}$$

A.M≥ G.M.
Equality holds iff a = b
i.e., A.M. = G.M. iff a = b
The A.M. and G.M. inequality for two variables
can be generalised to n variables.
If $a_1, a_2, a_3....a_n$ are n positive real numbers, then

$$\frac{a_1+a_2+a_3+...+a_n}{n} \geq (a_1.a_2.a_3....a_n)^{\frac{1}{n}}.$$

(A) If x, y, z are positive integers, then find value of the expression (x + y) (y + z) (z + x).
(B) Find the minimum value of the expression $3^x + 3^{1-x}$, x∈R and if three positive real numbers a, b, c are in A.P. and abc = 4, then find the minimum possible value of b.
(C) Find the common ratio of a GP whose sum of infinite terms is 8 and its second term is 2.

Ans. (A) A.M.> G.M.

$$\therefore \quad \frac{x+y}{2} > \sqrt{xy}$$

$$\frac{y+z}{2} > \sqrt{yz}$$

And $\frac{z+x}{2} > \sqrt{zx}$

On multiplying three inequalities, we get

$$\frac{x+y}{2}.\frac{y+z}{2}.\frac{z+x}{2} > \sqrt{(xy)(yz)(xz)}$$

$\Rightarrow \quad (x+y)(y+z)(z+x) > 8xyz$

(B) :- A.M. $\geq$ G.M.

$$\therefore \quad \frac{a_1+a_2+a_3+...+2a_n}{n} \geq (a_1.a_2.a_3...2a_n)^{\frac{1}{n}}$$

$$\Rightarrow \quad \frac{3^x+3^{1-x}}{2} \geq \sqrt{3^x.3^{1-x}}$$

$$\Rightarrow \quad 3^x+3^{1-x} \geq 2\sqrt{3}$$

So, the minimum value of $3^x + 3^{1-x}$ is $2\sqrt{3}$.

Given a, b, c are in A.P.

So, 2b = a + c

and abc = 4

We know that, A.M. $\geq$ G.M.

$$\text{So,} \quad \frac{a+b+c}{3} \geq (abc)^{1/3}$$

$$\Rightarrow \quad \frac{2b+b}{3} \geq (4)^{1/3} \quad \text{[using (i) and (ii)]}$$

$$\Rightarrow \quad \frac{3b}{3} \geq (2^2)^{1/3}$$

$$\Rightarrow \quad b \geq 2^{\frac{2}{3}}$$

So, the minimum possible value of b is $2^{\frac{2}{3}}$.

(C) Let a be the first term and r be the common ratio of the G.P.

Given, $ar = 2$ and $S_\infty = 8 = \frac{a}{1-r}$

$$\Rightarrow \quad 8 = \frac{2}{r(1-r)} \quad \left[\because a = \frac{2}{r}\right]$$

$$\Rightarrow \quad 4r(1-r) = 1$$

$$\Rightarrow \quad 4r - 4r^2 - 1 = 0$$

$$\Rightarrow \quad 4r^2 - 4r + 1 = 0$$

$$\Rightarrow \left(r-\frac{1}{2}\right)(4r-2) = 0$$

$$\Rightarrow \quad r = \frac{1}{2}$$

2. A sequence of non-zero numbers is said to be a geometric progression, if the ratio of each term, except the first one, by its preceding term is always constant. Rahul, being a plant lover, decides to open a nursery and he bought few plants and pots. He wants to place pots in such a way that the number of pots in the first row is 2, in second row is 4 and in the third row is 8 and so on.....

(A) The constant multiple by which the number of pots is increasing in every row is:

(a) 2

(b) 4

(c) 8

(d) 1

(B) The number of pots in 8th row is:

(a) 156

(b) 256

(c) 300

(d) 456

(C) The difference in number of pots placed in 7th row and 5th row is:

(a) 86

(b) 50

(c) 90

(d) 96

(D) Total number of pots upto 10th row is:

(a) 1046

(b) 2046

(c) 1023

(d) 1024

(E) If Rahul wants to place 510 pots in total, then the total number of rows

formed in this arrangement is:

(a) 7

(b) 8

(c) 9

(d) 5

Ans. (A) (a) 2

Explanation: The number of pots in each row is 2, 4, 8....

This forms a geometric progression,

Where $a = 2, r = \frac{4}{2} = 2$

Hence, the constant multiple by which the number of pots is increasing in every row is 2.

(B) (b) 256

Explanation: Number of pots in 8th row = ag

$= a_8$

$a_8 = ar^{8-1} = 2(2)^7 = 2^8 = 256$

(C) (d) 96

Explanation: Number of pots in 7th row,

$$a_7 = 2(2)^{7-1} = 2.2^6 = 2^7 = 128$$

Number of pots in 5^{th} row,

$$a_5 = 2(2)^{5-1} = 2.2^4 = 2^5 = 32$$

Required answer = 128 - 32 = 96

(D) (b) 2046

Explanation: Total number of pots upto 10th row

$$\therefore \quad S_{10} = \frac{a(r^{10}-1)}{r-1} = \frac{2(2^{10}-1)}{2-1}$$

$$= \frac{2(1024-1)}{1} = 2046$$

(E) (b) 8

Explanation: Let there be n number of rows.

$$\therefore \quad S_n = 510 = \frac{2(2^n - 1)}{2-1}$$

$$\Rightarrow \quad \frac{510}{2} = 2^n - 1$$

$$\Rightarrow \quad 255 = 2^n - 1$$

$$\Rightarrow \quad 256 = 2^n$$

$$\Rightarrow \quad 2^8 = 2^n$$

$$\Rightarrow \quad n = 8$$

3. A person was employed in a company on the condition that he will be paid rupees 2, 4, 6, 8, 10,... for the work done in the company on 1st, 2nd 3rd day.

(A) Name the progression of amount received by the person.

(a) A.P.

(b) G.P.

(c) A.P. and G.P.

(d) None

(B) The common difference of the sequence is:

(a) 1

(b) 2

(c) 3

(d) 4

(C) The first term of the sequence is:

(a) 2

(b) 4

(C)6

(d) 8

(D) The total amount received in 10 days in rupees is:

(a) 115

(b) 105

(c) 120

(d) 110

(E)

Assertion (A): If the numbers $\frac{-2}{7}$, k, $\frac{-7}{2}$ are in GP, then $k = \pm 1$.

Reason (R): If a_1, a_2, a_3 are in GP, then $\frac{a_2}{a_1} = \frac{a_3}{a_2}$.

(a) Both (A) and (R) are true and (R) is the correct explanation of (A).

(b) Both (A) and (R) are true but (R) is not the correct explanation of (A).

(c) (A) is true but (R) is false.

(d) (A) is false but (R) is true.

Ans. (A) (a) A.P.

Explanation: The given series 2, 4, 6, 8, is A.P. because the common difference is same as

(B) (b) 2

a_2-a_1 = 2

a_3-a_2 = 2

Explanation: The common difference as the A.P. 2, 4, 6, 8,...

a_2-a_1 =4-2=2

a_3-a_2=6-4=2

a_4-a3=8-6=2

(C) (a) 2

Explanation: The first term of A.P 2, 4, 6, 8. is 2.

(D) (d) 110

Explanation: Here a = 2, d= 2 and n = 10

So,

$$S_{10} = \frac{10}{2}\left[2\times2+(10-1)2\right]$$

$$= \frac{10}{2}[4+18]$$

$$= \frac{10}{2}[22]$$

$= 10 \times 11 =$ ₹ 110

(E) (a) Both (A) and (R) are true and (R) is the correct explanation of (A).

Explanation: If $-\frac{2}{7}, k, -\frac{7}{2}$ are in GP

Then, $\frac{a_2}{a_1} = \frac{a_3}{a_2}$

$$\left[\because \text{common ration } (r) = \frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = \dots\right]$$

$$\frac{k}{-\frac{2}{7}} = \frac{-\frac{7}{2}}{k}$$

$\Rightarrow$ $\frac{7}{-2}k = \frac{-7}{2}\times\frac{1}{k}$

$\Rightarrow$ $2k \times 2k = -7 \times (-2)$

$\Rightarrow$ $14k^2 = 14$

$\Rightarrow$ $k^2 = 1$

$\Rightarrow$ $k = \pm 1$

12. A group of students obtain the sum of first n *natural number is* $\frac{n(n+1)}{2}$ *named as* A_n, *sum of square of n natural number is* $\frac{n(n+1)(2n+1)}{6}$ named as B_n and sum of cubes of n natural *number is* $\left(\frac{n(n+1)}{2}\right)^3$ *named as* C_n,

(A) Find the value of A_5, and B_3..

(B) Find the value of C_2, and the sum of first n natural number is also the sum of n numbers of A.P whose first term is 1 and common difference is also 1.

(C) What is the geometric mean of 6 and 24?

Ans.

(A) $A_n = \dfrac{n(n+1)}{2}$

So, Put $n = 5$

We get

$$A_5 = \frac{5(5+1)}{2}$$

$$= \frac{5 \times 6}{2}$$

$= 3 \times 5$

$= 15$

$$B_n = \frac{n(n+1)(2n+1)}{6}$$

So put $n = 3$

$$B_3 = \frac{3(3+1)(6+1)}{6}$$

$$= \frac{3 \times 4 \times 7}{6}$$

$$= \frac{28}{2}$$

$= 14$

(B)

$$C_n = \left(\frac{n(n+1)}{2}\right)^3$$

Put $n = 2$

$$C_2 = \left(\frac{2(2+1)}{2}\right)^3$$

$= (3)^3$

$= 27$

$$A_n = \frac{n(n+1)}{2}$$

and $S_n = 1 + 2 + 3 + \ldots n$

$$= \frac{n}{2}[2a + (n-1)d]$$

$$= \frac{n}{2}[2 + (n-1)1]$$

$$= \frac{n}{2}[2 + n - 1]$$

$$= \frac{n(n+1)}{2}$$

(C) Given two data values are 6 and 24. So, n = 2.
So, to find the geometric mean of 6 and 24, we have to take the square root for the product of 6 and 24.
Geometric Mean = √(6x24)
Geometric Mean = √(144)
Thus, the square root of 144 is 12.
Therefore, the geometric mean of 6 and 24 is 12.