Chapter 2

DIFFERENTIATION OF FUNCTIONS

§ 2.1. Definition of the Derivative

The *derivative* $f'(x)$ of the function $y=f(x)$ at a given point x is defined by the equality

$$f'(x)=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}.$$

If this limit is finite, then the function $f(x)$ is called *differentiable* at the point x; and it is infallibly continuous at this point.

Geometrically, the value of the derivative $f'(x)$ represents the slope of the line tangent to the graph of the function $y=f(x)$ at the point x.

The number

$$f'_{+}(x)=\lim_{\Delta x\to +0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

is called the *right-side* derivative at the point x

The number

$$f'_{-}(x)=\lim_{\Delta x\to -0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

is called the *left-side* derivative at the point x.

The necessary and sufficient condition for the existence of the derivative $f'(x)$ is the existence of the finite right- and left-side derivatives, and also of the equality $f'_{-}(x)=f'_{+}(x)$.

If $f'(x)=\infty$, the function $f(x)$ is said to have an infinite derivative at the point x. In this case the line tangent to the graph of the function $y=f(x)$ at the point x is perpendicular to the x-axis.

2.1.1. Find the increment Δy and the ratio $\frac{\Delta y}{\Delta x}$ for the following functions:

(a) $y=\sqrt{x}$ at $x=0$ and $\Delta x=0.0001$;

(b) $y=\frac{1}{x^2+x-6}$ at $x=1$ and $\Delta x=0.2$.

Solution. (a) $\Delta y = \sqrt{x+\Delta x} - \sqrt{x} = \sqrt{0.0001} = 0.01$;

$$\frac{\Delta y}{\Delta x} = \frac{0.01}{0.0001} = 100.$$

2.1.2. Using the definition of the derivative, find the derivatives of the following functions:

(a) $y = \cos ax$; (b) $y = 5x^2 - 2x$.

Solution. (a) $\Delta y = \cos a(x+\Delta x) - \cos ax =$

$$= -2\sin\left(ax + \frac{a}{2}\Delta x\right)\sin\frac{a}{2}\quad x;$$

$$\frac{\Delta y}{\Delta x} = \frac{-2\sin\left(ax+\frac{a}{2}\Delta x\right)\sin\frac{a}{2}\Delta x}{\Delta x};$$

$$y' = \lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = -2\lim_{\Delta x \to 0}\sin\left(ax + \frac{a}{2}\Delta x\right)\lim_{\Delta x \to 0}\frac{\sin\frac{a}{2}\Delta x}{\Delta x} = -a\sin ax.$$

In particular, if $a = 1$, then $y = \cos x$ and $y' = -\sin x$.

2.1.3. Show that the following functions have no finite derivatives at the indicated points:

(a) $y = \sqrt[5]{x^3}$ at the point $x = 0$;

(b) $y = \sqrt[3]{x-1}$ at the point $x = 1$;

(c) $y = 3|x| + 1$ at the point $x = 0$.

Solution. (a) $\Delta y = \sqrt[5]{(x+\Delta x)^3} - \sqrt[5]{x^3}$.

At $x = 0$ we have $\Delta y = \sqrt[5]{\Delta x^3}$, $\dfrac{\Delta y}{\Delta x} = \dfrac{\sqrt[5]{\Delta x^3}}{\Delta x} =$

$$= \frac{1}{\sqrt[5]{\Delta x^2}};\ \text{hence,}\ y'(0) = \lim_{\Delta x \to 0}\frac{1}{\sqrt[5]{\Delta x^2}} = \infty,$$

i.e. there is no finite derivative.

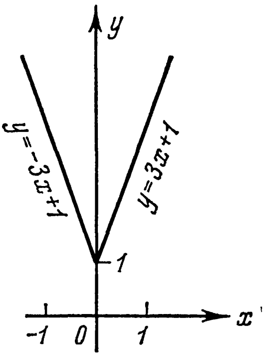

Fig. 34

(c) At $\Delta x > 0$ the increment of the function Δy at $x = 0$ will be: $\Delta y = 3(0+\Delta x) + 1 - 1 = 3\Delta x$. Therefore

$$\lim_{\Delta x \to +0}\frac{\Delta y}{\Delta x} = 3.$$

At $\Delta x < 0$ the increment of the function Δy will be

$$\Delta y = -3(0+\Delta x) + 1 - 1 = -3\Delta x,$$

hence,

$$\lim_{\Delta x \to -0}\frac{\Delta y}{\Delta x} = -3.$$

Since the one-sided limits are different, there is no derivative at the point $x = 0$ (see Fig. 34).

2.1.4. Investigate the function $y=|\ln x|$ for differentiability at the point $x=1$.

Solution. At $x=1$

$$\Delta y=|\ln(1+\Delta x)|-|\ln 1|=|\ln(1+\Delta x)|,$$

i. e.

$$\Delta y=|\ln(1+\Delta x)|=\begin{cases}\ln(1+\Delta x) \text{ at } \Delta x\geqslant 0,\\ -\ln(1+\Delta x) \text{ at } \Delta x<0.\end{cases}$$

Therefore

$$\frac{\Delta y}{\Delta x}=\begin{cases}\dfrac{\ln(1+\Delta x)}{\Delta x} \text{ at } \Delta x>0,\\ -\dfrac{\ln(1+\Delta x)}{\Delta x} \text{ at } \Delta x<0,\end{cases}$$

whence

$$\lim_{\Delta x\to+0}\frac{\Delta y}{\Delta x}=+1 \text{ and } \lim_{\Delta x\to-0}\frac{\Delta y}{\Delta x}=-1.$$

Since the one-sided limits are different, there is no derivative. Hence, the function $y=|\ln x|$ is not differentiable at the point $x=1$ (see Fig. 35).

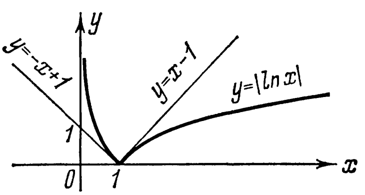

Fig. 35

2.1.5. Find the average velocity of motion specified by the formula

$$s=(t^2-5t+2) \text{ m}$$

from $t_1=5$ sec to $t_2=15$ sec.

2.1.6. Using the definition of the derivative, find the derivatives of the following functions:

(a) $y=x^3$; (b) $y=1/x^2$.

2.1.7. Investigate the function $y=|\cos x|$ for differentiability at the points $x=\pi/2+n\pi$ (n an integer).

§ 2.2. Differentiation of Explicit Functions

I. Basic Rules of Differentiation

(1) $c'=0$;
(2) $(u\pm v)'=u'\pm v'$;
(3) $(cu)'=cu'$;
(4) $(uv)'=u'v+uv'$, the product rule;
(5) $\left(\dfrac{u}{v}\right)'=\dfrac{u'v-uv'}{v^2}$ $(v\neq 0)$, the quotient rule.

Here $c=\text{const}$, and u and v are functions of x which have derivatives at a corresponding point.

(6) If the function $u=\varphi(x)$ is differentiable at the point x_0, and the function $y=f(u)$ is differentiable at the point $u_0=\varphi(x_0)$, then

the composite function $y=f(\varphi(x))$ is differentiable at the point x_0 and $y'_x(x_0)=y'_u(u_0)\,u'_x(x_0)$, the function of a function, or chain, rule.

II. Differentiation of Basic Elementary Functions

(1) $(u^n)'=nu^{n-1}u'$; (2) $(\sin u)'=\cos u\cdot u'$;
(3) $(\cos u)'=-\sin u\cdot u'$;
(4) $(\tan u)'=\dfrac{u'}{\cos^2 u}$; (5) $(\cot u)'=-\dfrac{u'}{\sin^2 u}$;
(6) $(\ln u)'=\dfrac{u'}{u}$;
(7) $(a^u)'=a^u\ln a\cdot u'$; 8) $(e^u)'=e^u u'$;
(9) $(\sinh u)'=\cosh u\cdot u'$;
(10) $(\cosh u)'=\sinh u\cdot u'$;
(11) $(\arcsin u)'=\dfrac{u'}{\sqrt{1-u^2}}=-(\arccos u)'$;
(12) $(\arctan u)'=\dfrac{u'}{1+u^2}=-(\text{arc}\cot u)'$.

2.2.1. Find y', if:
(a) $y=5x^{2/3}-3x^{5/2}+2x^{-3}$;
(b) $y=\dfrac{a}{\sqrt[3]{x^2}}-\dfrac{b}{x\sqrt[3]{x}}$ (a, b constants).

Solution. (a) $y'=5\cdot\dfrac{2}{3}x^{2/3-1}-3\cdot\dfrac{5}{2}x^{5/2-1}-2\cdot 3x^{-3-1}=\dfrac{10}{3\sqrt[3]{x}}-$
$-\dfrac{15}{2}x\sqrt{x}-\dfrac{6}{x^4}$.

2.2.2. Find y', if:
(a) $y=3\cos x+2\sin x$; (b) $y=\dfrac{\sin x+\cos x}{\sin x-\cos x}$;
(c) $y=(x^2+1)\arctan x$; (d) $y=x^3\arcsin x$.

Solution. (a) $y'=3(\cos x)'+2(\sin x)'=-3\sin x+2\cos x$;

(b) $$y'=\frac{(\sin x+\cos x)'(\sin x-\cos x)-(\sin x-\cos x)'(\sin x+\cos x)}{(\sin x-\cos x)^2}=$$
$$=\frac{(\cos x-\sin x)(\sin x-\cos x)-(\cos x+\sin x)(\sin x+\cos x)}{(\sin x-\cos x)^2}=$$
$$=-\frac{2}{(\sin x-\cos x)^2};$$

(d) $y'=(x^3)'\arcsin x+(\arcsin x)'x^3=3x^2\arcsin x+\dfrac{x^3}{\sqrt{1-x^2}}$.

2.2.3. Find the derivative of the given function and then compute the particular value of the derivative at the indicated value of the argument:

(a) $f(x)=1-\sqrt[3]{x^2}+16/x$ at $x=-8$;

(b) $f(x)=(1-\sqrt{x})^2/x$ at $x=0.01$;
(c) $f(t)=(\cos t)/(1-\sin t)$ at $t=\pi/6$.

Solution. (a) $f'(x)=-\frac{2}{3}x^{-1/3}-16x^{-2}=-\frac{2}{3\sqrt[3]{x}}-\frac{16}{x^2}$.

Putting $x=-8$, we obtain

$$f'(-8)=-\frac{2}{3\sqrt[3]{-8}}-\frac{16}{(-8)^2}=\frac{1}{12};$$

(c) $f'(t)=\frac{-\sin t\,(1-\sin t)+\cos^2 t}{(1-\sin t)^2}=\frac{1}{1-\sin t}$.

Whence $f'(\pi/6)=2$.

2.2.4. Taking advantage of the differentiation formulas, find the derivatives of the following functions:

(a) $y=2x^3+3x-5$; (b) $y=\sqrt{x}+\frac{1}{\sqrt{x}}+0.1x^{10}$;

(c) $y=\frac{2x^2+x+1}{x^2-x+1}$; (d) $y=\frac{x+\sqrt{x}}{x-2\sqrt[3]{x}}$;

(e) $y=\frac{\cos\varphi+\sin\varphi}{1-\cos\varphi}$; (f) $y=2e^x+\ln x$;

(g) $y=e^x(\cos x+\sin x)$; (h) $y=\frac{e^x+\sin x}{xe^x}$.

2.2.5. Taking advantage of the rule for differentiation of a composite function find the derivatives of the following functions:

(a) $y=\sin^3 x$; (b) $y=\ln\tan x$; (c) $y=5^{\cos x}$;
(d) $y=\ln\sin(x^3+1)$; (e) $y=\arcsin\sqrt{1-x^2}$;
(f) $y=\ln^5(\tan 3x)$; (g) $y=\sin^2\sqrt{1/(1-x)}$.

Solution. (a) Here the role of the external function is played by the power function: $\sin x$ is raised to the third power. Differentiating this power function with respect to the intermediate argument $(\sin x)$, we obtain

$$(\sin^3 x)'_{\sin x}=3\sin^2 x;$$

but the intermediate argument $\sin x$ is a function of an independent variable x; therefore we have to multiply the obtained result by the derivative of $\sin x$ with respect to the independent variable x. Thus, we obtain

$$y'_x=(\sin^3 x)'_{\sin x}(\sin x)'_x=3\sin^2 x\cos x;$$

(b) $y'_x=(\ln\tan x)'_{\tan x}(\tan x)'_x=\frac{1}{\tan x}\frac{1}{\cos^2 x}=\frac{2}{\sin 2x}$;

(c) $y'_x=(5^{\cos x})'_{\cos x}(\cos x)'_x=5^{\cos x}\ln 5\,(-\sin x)=-5^{\cos x}\sin x\ln 5$;

(d) $y'_x = [\ln \sin (x^3+1)]'_{\sin (x^3+1)} [\sin (x^3+1)]'_{x^3+1} [x^3+1]'_x =$

$$= \frac{1}{\sin (x^3+1)} \cdot \cos (x^3+1) \cdot 3x^2 = 3x^2 \cot (x^3+1);$$

(e) $y'_x = (\arcsin \sqrt{1-x^2})'_{\sqrt{1-x^2}} (\sqrt{1-x^2})'_{1-x^2} (1-x^2)'_x =$

$$= \frac{1}{\sqrt{1-(1-x^2)}} \cdot \frac{1}{2\sqrt{1-x^2}} \cdot (-2x) = -\frac{x}{|x| \sqrt{1-x^2}} \quad (x \neq 0).$$

2.2.6. Find the derivatives of the following functions:

(a) $y = (1+3x+5x^2)^4$; (b) $y = (3-\sin x)^3$;

(c) $y = \sqrt[3]{\sin^2 x + 1/\cos^2 x}$;

(d) $y = \sqrt[3]{2e^x + 2^x + 1} + \ln^5 x$;

(e) $y = \sin 3x + \cos (x/5) + \tan \sqrt{x}$;

(f) $y = \sin (x^2 - 5x + 1) + \tan (a/x)$;

(g) $y = \arccos \sqrt{x}$;

(h) $y = \arctan (\ln x) + \ln (\arctan x)$;

(i) $y = \ln^2 \arctan (x/3)$;

(j) $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$.

Solution. (a) $y' = 4(1+3x+5x^2)^3 (1+3x+5x^2)' =$

$$= 4(1+3x+5x^2)^3 (3+10x);$$

(g) $y' = -\frac{1}{\sqrt{1-(\sqrt{x})^2}} (\sqrt{x})' = -\frac{1}{\sqrt{1-x}} \frac{1}{2\sqrt{x}} = -\frac{1}{2\sqrt{x(1-x)}}$;

(j) $y' = \frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}} \left[1 + \frac{1}{2\sqrt{x+\sqrt{x}}} \left(1 + \frac{1}{2\sqrt{x}}\right)\right]$.

2.2.7. Find the derivative of the function

$$y = \arcsin \frac{2x}{1+x^2}.$$

We have

$$y' = \frac{1}{\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}} \frac{2(1+x^2)-4x^2}{(1+x^2)^2} = \frac{2(1-x^2)}{\sqrt{(1-x^2)^2}(1+x^2)} = \frac{2(1-x^2)}{|1-x^2|(1+x^2)},$$

i. e.

$$y' = \begin{cases} \frac{2}{1+x^2} & \text{at } |x| < 1, \\ -\frac{2}{1+x^2} & \text{at } |x| > 1. \end{cases}$$

At $|x| = 1$ the derivative is non-existent.

2.2.8. Find the derivatives of the following functions:

(a) $y = \sinh 5x \cosh (x/3)$;

(b) $y=\coth(\tan x)-\tanh(\cot x)$;
(c) $y=\arccos(\tanh x)+\sinh(\sin 6x)$;
(d) $y=\sinh^2 x^3+\cosh^3 x^2$;
(e) $y=\dfrac{e^{\sinh ax}}{\sinh bx-\cosh bx}$.

Solution.
(a) $y'=(\sinh 5x)'\cosh\frac{x}{3}+\sinh 5x\left(\cosh\frac{x}{3}\right)'=$

$$=5\cosh 5x\cosh\frac{x}{3}+\frac{1}{3}\sinh 5x\sinh\frac{x}{3};$$

(c) $$y'=-\frac{(\tanh x)'}{\sqrt{1-\tanh^2 x}}+\cosh(\sin 6x)(\sin 6x)'=$$

$$=-\frac{1/\cosh^2 x}{\sqrt{(\cosh^2 x-\sinh^2 x)/\cosh^2 x}}+$$

$$+6\cos 6x\cosh(\sin 6x)=-\frac{1}{\cosh x}+6\cos 6x\cosh(\sin 6x).$$

2.2.9. Find the derivatives of the following functions:
(a) $y=\sqrt[3]{\dfrac{x^3(x^2+1)}{\sqrt[5]{5-x}}}$; (b) $y=[u(x)]^{v(x)}$ $(u(x)>0)$;
(c) $y=\sqrt[3]{x^2\dfrac{1-x}{1+x^2}}\sin^3 x\cos^2 x$;
(d) $y=(\sqrt{\tan x})^{x+1}$.

Solution. (a) Apply the method of logarithmic differentiation. Consider, instead of y, the function

$$z=\ln|y|=\ln\sqrt[3]{\frac{|x^3|(x^2+1)}{\sqrt[5]{|5-x|}}}=\ln|x|+\frac{1}{3}\ln(x^2+1)-\frac{1}{15}\ln|5-x|.$$

Taking into account that $(\ln|u|)'=u'/u$, we have

$$z'=\frac{1}{x}+\frac{2x}{3(x^2+1)}+\frac{1}{15(5-x)}=\frac{-24x^3+125x^2-14x+75}{15x(x^2+1)(5-x)}.$$

But $z'=(\ln|y|)'=y'/y$, whence

$$y'=yz'=\sqrt[3]{\frac{x^3(x^2+1)}{\sqrt[5]{5-x}}}\cdot\frac{-24x^3+125x^2-14x+75}{15x(x^2+1)(5-x)}.$$

(b) Suppose the functions $u(x)$ and $v(x)$ have derivatives in the given domain of definition. Then the function

$$z=\ln y=v\ln u$$

also has a derivative in this domain, and

$$z'=(v\ln u)'=v'\ln u+v\frac{u'}{u}.$$

Hence, the function

$$y = e^{\ln y} = e^z$$

also has a derivative in the indicated domain, and

$$y' = e^z z' = yz'.$$

Thus,

$$y' = u^v \left(v' \ln u + v \frac{u'}{u} \right) = vu^{v-1}u' + u^v \ln u \cdot v'.$$

2.2.10. Show that the function $y = xe^{-x^2/2}$ satisfies the equation

$$xy' = (1 - x^2)\, y.$$

Solution.

$$y' = e^{-x^2/2} - x^2 e^{-x^2/2} = e^{-x^2/2}(1 - x^2);$$

$$xy' = xe^{-x^2/2}(1 - x^2).$$

Hence,

$$xy' = y\,(1 - x^2).$$

2.2.11. Show that the function $y = xe^{-x}$ satisfies the equation $xy' = (1 - x)\, y$.

2.2.12. Investigate the following functions for differentiability: (a) $y = \arcsin(\cos x)$; (b) $y = \sqrt{1 - \sqrt{1 - x^2}}$.

Solution. (a) $y' = \dfrac{(\cos x)'}{\sqrt{1 - \cos^2 x}} = -\dfrac{\sin x}{\sqrt{\sin^2 x}} = -\dfrac{\sin x}{|\sin x|}$.

Hence, $y' = -1$ at points where $\sin x > 0$; $y' = 1$ at points where $\sin x < 0$. At points where $\sin x = 0$, i. e. at the points $x = k\pi$ $(k = 0, \pm 1, \pm 2, \ldots)$ the function, though continuous, is not differentiable.

(b) The domain of definition of this function is the interval $-1 \leqslant x \leqslant 1$.

$$y' = \frac{1}{2\sqrt{1 - \sqrt{1 - x^2}}} \cdot \frac{-1}{2\sqrt{1 - x^2}} (-2x) \text{ at } x \neq 0 \text{ and } x \neq \pm 1.$$

As $x \to 1 - 0$ or $x \to -1 + 0$ we have $y' \to +\infty$. Let us find out whether the derivative y' exists at the point $x = 0$, i. e. whether $\lim\limits_{\Delta x \to 0} \dfrac{\sqrt{1 - \sqrt{1 - \Delta x^2}}}{\Delta x}$ exists.

Since $\sqrt{1 - \Delta x^2} - 1 \sim -\frac{1}{2} \Delta x^2$, then

$$\lim_{\Delta x \to 0} \frac{\sqrt{1 - \sqrt{1 - \Delta x^2}}}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{\frac{1}{2} \Delta x^2}}{\Delta x} = \begin{cases} \dfrac{1}{\sqrt{2}} & \text{as } \Delta x \to +0, \\ -\dfrac{1}{\sqrt{2}} & \text{as } \Delta x \to -0. \end{cases}$$

Thus, $y'_-(0) \neq y'_+(0)$, which means that the function under consideration has no derivative at the point $x=0$, though it is continuous at this point.

Note. There are cases of failure of existence of $f'(x)$ and even of $f'_+(x)$ and $f'_-(x)$ at a given point, i.e. when the graph of the function has neither a right-, nor a left-side tangent at the given point. For instance, the function

$$f(x)=\begin{cases} x\sin(1/x) & \text{at } x\neq 0, \\ 0 & \text{at } x=0 \end{cases}$$

is continuous at the point $x=0$, but does not have even one-sided derivatives, since $\frac{\Delta f(x)}{\Delta x}=\sin\frac{1}{\Delta x}$.

2.2.13. Find the derivatives of the following functions:

(a) $f(x)=\sinh(x/2)+\cosh(x/2)$;

(b) $f(x)=\ln[\cosh x]$; (c) $f(x)=2\sqrt{\cosh x-1}$;

(d) $f(x)=\arcsin[\tanh x]$;

(e) $f(x)=\sqrt{1+\sinh^2 4x}$;

(f) $f(x)=e^{ax}(\cosh bx+\sinh bx)$.

2.2.14. Applying logarithmic differentiation find the derivatives of the following functions:

(a) $y=(\cos x)^{\sin x}$; (b) $y=\sqrt[3]{\frac{\sin 3x}{1-\sin 3x}}$;

(c) $y=\frac{\sqrt{x-1}}{\sqrt[3]{(x+2)^2}\sqrt{(x+3)^3}}$.

2.2.15.

$$f(x)=\frac{\cos^2 x}{1+\sin^2 x};$$

show that

$$f(\pi/4)-3f'(\pi/4)=3.$$

2.2.16. Show that the function

$$y=\frac{x-e^{-x^2}}{2x^2}$$

satisfies the differential equation

$$xy'+2y=e^{-x^2}.$$

2.2.17. Find the derivatives of the following functions:

(a) $y=\ln\cos\sqrt{\arcsin 3^{-2x}}$ $(x>0)$;

(b) $y=\sqrt[3]{\arctan\sqrt[5]{\cos\ln^3 x}}$.

§ 2.3. Successive Differentiation of Explicit Functions. Leibniz Formula

If the derivative of the $(n-1)$th order of a function $y=f(x)$ is already found, then the derivative of the nth order is determined by the equality

$$y^{(n)}(x)=[y^{(n-1)}(x)]'.$$

In particular, $y''(x)=[y'(x)]'$, $y'''(x)=[y''(x)]'$, and so on.

If u and v are functions differentiable n times, then for their linear combination c_1u+c_2v (c_1, c_2 constants) we have the following formula:

$$(c_1u+c_2v)^{(n)}=c_1u^{(n)}+c_2v^{(n)},$$

and for their product uv the Leibniz formula (or rule)

$$(uv)^{(n)}=u^{(n)}v+nu^{(n-1)}v'+\frac{n(n-1)}{1\cdot 2}u^{(n-2)}v''+\ldots+uv^{(n)}=\sum_{k=0}^{n}C_n^k u^{(n-k)}v^{(k)},$$

where $u^{(0)}=u$, $v^{(0)}=v$ and $C_n^k=\frac{n(n-1)\ldots(n-k+1)}{1\cdot 2\cdot 3\ldots k}=\frac{n!}{k!\,(n-k)!}$ are binomial coefficients. Here are the basic formulas:

(1) $(x^m)^{(n)}=m(m-1)\ldots(m-n+1)\,x^{m-n}$.

(2) $(a^x)^{(n)}=a^x\ln^n a\ (a>0)$. In particular, $(e^x)^{(n)}=e^x$.

(3) $(\ln x)^{(n)}=(-1)^{n-1}\frac{(n-1)!}{x^n}$.

(4) $(\sin x)^{(n)}=\sin(x+n\pi/2)$.

(5) $(\cos x)^{(n)}=\cos(x+n\pi/2)$.

2.3.1. Find the derivatives of the nth order of the following functions:

(a) $y=\ln x$; (b) $y=e^{kx}$; (c) $y=\sin x$; (d) $y=\sin 5x\cos 2x$;
(e) $y=\sin x\cos x$; (f) $y=\sin 3x\cos^2 x$; (g) $y=\ln(x^2+x-2)$.

Solution.

(a) $y'=\frac{1}{x}=x^{-1}$; $y''=(-1)\,x^{-2}$; $y'''=1\cdot 2x^{-3}$;

$$y^{(4)}=-1\cdot 2\cdot 3x^{-4};\ \ldots;\ y^{(n)}=(-1)^{n-1}(n-1)!\,x^{-n}=\frac{(-1)^{n-1}(n-1)!}{x^n}.$$

(c) $y'=\cos x=\sin(x+\pi/2)$;
$y''=\cos(x+\pi/2)=\sin(x+2\pi/2)$.

In general, if we assume that for a given $n=k$

$$y^{(k)}=\sin\left(x+k\frac{\pi}{2}\right),$$

then it will turn out that

$$y^{(k+1)}=\cos\left(x+k\frac{\pi}{2}\right)=\sin\left[(k+1)\frac{\pi}{2}+x\right].$$

Whence, by virtue of mathematical induction we conclude that for any natural n

$$y^{(n)}=\sin\left(x+n\frac{\pi}{2}\right).$$

(d) $y=\sin 5x\cos 2x=\frac{1}{2}\left[\sin 7x+\sin 3x\right]$.

Therefore

$$y^{(n)}=\frac{1}{2}\left[7^n\sin\left(7x+n\frac{\pi}{2}\right)+3^n\sin\left(3x+n\frac{\pi}{2}\right)\right].$$

(g) $y'=\frac{2x+1}{x^2+x-2}$.

To simplify the computations let us transform the obtained function:

$$y'=\frac{2x+1}{x^2+x-2}=\frac{(x+2)+(x-1)}{(x-1)(x+2)}=\frac{1}{x-1}+\frac{1}{x+2}=(x-1)^{-1}+(x+2)^{-1}.$$

Whence

$$y''=-1(x-1)^{-2}-1(x+2)^{-2};$$
$$y'''=1\cdot 2(x-1)^{-3}+1\cdot 2(x+2)^{-3};$$

. . . .

$$y^{(n)}=(-1)^{n-1}(n-1)!\,[(x-1)^{-n}+(x+2)^{-n}]=$$
$$=(-1)^{n-1}(n-1)!\left[\frac{1}{(x-1)^n}+\frac{1}{(x+2)^n}\right].$$

2.3.2. $y=\frac{ax+b}{cx+d}$; find $y^{(n)}$.

Solution. Transform the given expression in the following way:

$$y=\frac{ax+b}{cx+d}=\frac{a}{c}+\frac{bc-ad}{c(cx+d)}=\frac{a}{c}+\frac{bc-ad}{c}(cx+d)^{-1}.$$

Whence

$$y'=(-1)\frac{bc-ad}{c}c(cx+d)^{-2},$$

$$y''=(-1)(-2)\frac{bc-ad}{c}c^2(cx+d)^{-3},$$

$$y'''=(-1)(-2)(-3)\frac{bc-ad}{c}c^3(cx+d)^{-4},$$

. . . .

$$y^{(n)}=(-1)^n n!\frac{bc-ad}{c}c^n(cx+d)^{-(n+1)}=$$
$$=(-1)^n\frac{n!\,c^{n-1}}{(cx+d)^{n+1}}(bc-ad).$$

2.3.3. $y=x/(x^2-1)$; find $y^{(n)}$.

Solution. Transform the given expression

$$y=\frac{x}{x^2-1}=\frac{1}{2}\left[\frac{1}{x+1}+\frac{1}{x-1}\right],$$

therefore (see Problem **2.3.2**):

$$y^{(n)}=\frac{(-1)^n n!}{2}\left[\frac{1}{(x+1)^{n+1}}+\frac{1}{(x-1)^{n+1}}\right].$$

2.3.4. Using the Leibniz formula, find the derivatives of the indicated orders for the following functions:

(a) $y=x^2\sin x$; find $y^{(25)}$;

(b) $y=e^x(x^2-1)$; find $y^{(24)}$;

(c) $y=e^{\alpha x}\sin\beta x$; find $y^{(n)}$.

Solution. (a) $y^{(25)}=(\sin x\cdot x^2)^{(25)}=(\sin x)^{(25)}x^2+25(\sin x)^{(24)}(x^2)'+$ $+\frac{25\cdot 24}{2}(\sin x)^{(23)}(x^2)''$, since the subsequent summands equal zero. Therefore

$$y^{(25)}=x^2\sin\left(x+25\frac{\pi}{2}\right)+50x\sin\left(x+24\frac{\pi}{2}\right)+600\sin\left(x+23\frac{\pi}{2}\right)=$$
$$=(x^2-600)\cos x+50x\sin x.$$

2.3.5. Compute the value of the nth derivative of the function $y=\frac{3x+2}{x^2-2x+5}$ at the point $x=0$.

Solution. By hypothesis we have $y(x)(x^2-2x+5)=3x+2$. Let us differentiate this identity n times using the Leibniz formula; then (for $n\geqslant 2$) we obtain

$$y^n(x)(x^2-2x+5)+ny^{(n-1)}(x)(2x-2)+\frac{n(n-1)}{2}y^{(n-2)}(x)\cdot 2=0.$$

Putting $x=0$, we have

$$5y^{(n)}(0)-2ny^{(n-1)}(0)+n(n-1)y^{(n-2)}(0)=0.$$

Whence

$$y^{(n)}(0)=\frac{2}{5}ny^{(n-1)}(0)-\frac{n(n-1)}{5}y^{(n-2)}(0).$$

We have obtained a recurrence relation for determining the nth derivative at the point $x=0$ $(n\geqslant 2)$. The values $y(0)$ and $y'(0)$ are found immediately: $y(0)=2/5$;

$$y'(x)=\frac{-3x^2-4x+19}{(x^2-2x+5)^2}; \qquad y'(0)=\frac{19}{25}.$$

Then, successively putting $n=2, 3, 4, \ldots$, find the values of the derivatives of higher orders with the aid of the recurrence relation.

For example,

$$y''(0)=\frac{2}{5}\cdot 2\cdot\frac{19}{25}-\frac{2\cdot 1}{5}\cdot\frac{2}{5}=\frac{56}{125},$$

$$y'''(0)=\frac{2}{5}\cdot 3\cdot\frac{56}{125}-\frac{3\cdot 2}{5}\cdot\frac{19}{25}=-\frac{234}{625}.$$

2.3.6. Find the derivatives of the second order of the following functions:

(a) $y=x\sqrt{1+x^2}$; (b) $y=\frac{\arcsin x}{\sqrt{1-x^2}}$; (c) $y=e^{-x^2}$.

2.3.7. Given the function

$$y=c_1e^{2x}+c_2xe^{2x}+e^x.$$

Show that this function satisfies the equation

$$y''-4y'+4y=e^x.$$

2.3.8. Using the Leibniz formula give the derivatives of the indicated orders for the following functions:

(a) $y=x^3\sin x$; find $y^{(20)}$;
(b) $y=e^{-x}\sin x$; find y''';
(c) $y=e^x(3x^2-4)$; find $y^{(n)}$;
(d) $y=(1-x^2)\cos x$; find $y^{(2n)}$.

2.3.9. Using the expansion into a linear combination of simpler functions find the derivatives of the 100th order of the functions:

(a) $y=\frac{1}{x^2-3x+2}$; (b) $y=\frac{1+x}{\sqrt{1-x}}$.

2.3.10. Show that the function

$$y=x^n[c_1\cos(\ln x)+c_2\sin(\ln x)]$$

(c_1, c_2, n constants) satisfies the equation

$$x^2y''+(1-2n)xy'+(1+n^2)y=0.$$

2.3.11. Prove that if $f(x)$ has a derivative of the nth order, then

$$[f(ax+b)]^{(n)}=a^nf^{(n)}(ax+b).$$

§ 2.4. Differentiation of Inverse, Implicit and Parametrically Represented Functions

1. The Derivative of an Inverse Function. If a differentiable function $y=f(x)$, $a<x<b$ has a single-valued continuous inverse function $x=g(y)$ and $y'_x \neq 0$ then there exists also

$$x'_y = \frac{1}{y'_x}.$$

For the derivative of the second order we have

$$x''_{yy} = -\frac{y''_{xx}}{(y'_x)^3}.$$

2. The Derivative of an Implicit Function. If a differentiable function $y=y(x)$ satisfies the equation $F(x, y)=0$, then we have to differentiate it with respect to x, considering y as a function of x, and solve the obtained equation $\frac{d}{dx}F(x, y)=0$ with respect to y'_x. To find y''_x, the equation should be twice differentiated with respect to x, and so on.

3. The Derivative of a Function Represented Parametrically. If the system of equations

$$x=\varphi(t), \quad y=\psi(t), \quad \alpha<t<\beta,$$

where $\varphi(t)$ and $\psi(t)$ are differentiable functions and $\varphi'(t)\neq 0$, defines y as a single-valued continuous function of x, then there exists a derivative y'_x and

$$y'_x = \frac{\psi'_t(t)}{\varphi'_t(t)} = \frac{y'_t}{x'_t}.$$

The derivatives of higher orders are computed successively:

$$y''_{xx} = \frac{(y'_x)'_t}{x'_t}, \quad y'''_{xxx} = \frac{(y''_{xx})'_t}{x'_t}, \text{ and so on.}$$

In particular, for the second derivative the following formula is true:

$$y''_{xx} = \frac{x'_t y''_{tt} - x''_{tt} y'_t}{(x'_t)^3}.$$

2.4.1. For the function

(a) $y=2x^3+3x^5+x$; find x'_y;

(b) $y=3x-(\cos x)/2$; find x''_{yy};

(c) $y= x+e^x$; find x''_{yy}.

Solution. (a) We have $y'_x = 6x^2+15x^4+1$, hence,

$$x'_y = \frac{1}{y'_x} = \frac{1}{6x^2+15x^4+1}.$$

(c) $y'_x = 1 + e^x$, $y''_{xx} = e^x$, hence,

$$x'_y = \frac{1}{1+e^x}, \quad x''_{yy} = -\frac{e^x}{(1+e^x)^3}.$$

2.4.2. Using the rule for differentiation of an inverse function, find the derivative y'_x for the following functions:

(a) $y = \sqrt[3]{x}$; (b) $y = \arcsin \sqrt{x}$; (c) $y = \ln \sqrt{1+x^2}$.

Solution. (a) The inverse function $x = y^3$ has the derivative $x'_y = 3y^2$. Hence,

$$y'_x = \frac{1}{x'_y} = \frac{1}{3y^2} = \frac{1}{3\sqrt[3]{x^2}}.$$

(c) At $x > 0$ the inverse function $x = \sqrt{e^{2y} - 1}$ has the derivative $x'_y = e^{2y}/\sqrt{e^{2y}-1}$. Hence,

$$y'_x = \frac{1}{x'_y} = \frac{\sqrt{e^{2y}-1}}{e^{2y}} = \frac{\sqrt{x^2}}{x^2+1} = \frac{x}{x^2+1}.$$

2.4.3. For each of the following functions represented parametrically find the derivative of the first order of y with respect to x:

(a) $x = a(t - \sin t)$, $\quad y = a(1 - \cos t)$;
(b) $x = k \sin t - \sin kt$, $\quad y = k \cos t + \cos kt$;
(c) $x = 2 \ln \cot t$, $\quad y = \tan t + \cot t$;
(d) $x = e^{ct}$, $\quad y = e^{-ct}$.

Solution. (a) Find the derivatives of x and y with respect to the parameter t:

$$x'_t = a(1 - \cos t); \quad y'_t = a \sin t.$$

Whence

$$\frac{dy}{dx} = \frac{a \sin t}{a(1-\cos t)} = \cot \frac{t}{2} \ (t \neq 2k\pi).$$

(c) $$\frac{dx}{dt} = \frac{-2 \operatorname{cosec}^2 t}{\cot t} = -\frac{4}{\sin 2t};$$

$$\frac{dy}{dt} = \sec^2 t - \operatorname{cosec}^2 t = -\frac{4 \cdot \cos 2t}{\sin^2 2t};$$

$$\frac{dy}{dx} = \frac{4 \cos 2t \sin 2t}{4 \sin^2 2t} = \cot 2t \ \left(t \neq \frac{k\pi}{2}\right).$$

2.4.4. The functions are defined parametrically:

(a) $\begin{cases} x = a \cos^3 t, \\ y = b \sin^3 t; \end{cases}$ (b) $\begin{cases} x = t^3 + 3t + 1, \\ y = t^3 - 3t + 1; \end{cases}$

(c) $\begin{cases} x = a(\cos t + t \sin t), \\ y = a(\sin t - t \cos t); \end{cases}$ (d) $\begin{cases} x = e^t \cos t, \\ y = e^t \sin t. \end{cases}$

Find for them the second derivative of y with respect to x.

Solution. (a) First find y'_x.

$$y'_t = 3b\sin^2 t\cos t; \quad x'_t = -3a\cos^2 t\sin t;$$

$$y'_x = -\frac{3b\sin^2 t\cos t}{3a\cos^2 t\sin t} = -\frac{b}{a}\tan t \quad \left(t \neq (2k+1)\frac{\pi}{2}\right).$$

Then we shall find y''_{xx} using the formula

$$y''_{xx} = \frac{(y'_x)'_t}{x'_t},$$

where

$$(y'_x)'_t = -\frac{b}{a\cos^2 t}.$$

Whence

$$y''_{xx} = -\frac{b}{a\cos^2 t\,(-3a\cos^2 t\sin t)} = \frac{b}{3a^2\cos^4 t\sin t}.$$

(d) $x'_t = e^t\cos t - e^t\sin t = e^t(\cos t - \sin t);$

$y'_t = e^t\sin t + e^t\cos t = e^t(\cos t + \sin t);$

$$y'_x = \frac{\cos t + \sin t}{\cos t - \sin t};$$

$$y''_{xx} = \frac{(y'_x)'_t}{x'_t} = \frac{\left(\dfrac{\cos t + \sin t}{\cos t - \sin t}\right)'_t}{e^t(\cos t - \sin t)} = \frac{2}{e^t(\cos t - \sin t)^3}.$$

2.4.5. Find y'''_{xxx}:

(a) $x = e^{-t}$; $\quad y = t^3$; $\quad$ (b) $x = \sec t$; $\quad y = \tan t$.

Solution. (a) First find

$$x'_t = -e^{-t}; \quad y'_t = 3t^2,$$

whence

$$y'_x = -3t^2/e^{-t} = -3e^t t^2.$$

Then find the second derivative

$$y''_{xx} = \frac{(y'_x)'_t}{x'_t} = \frac{-(3e^t t^2 + 6te^t)}{-e^{-t}} = 3te^{2t}(t+2).$$

And finally, find the third derivative

$$y'''_{xxx} = \frac{(y''_{xx})'_t}{x'_t} = \frac{3e^{2t}[2(t^2+2t)+2t+2]}{-e^{-t}} = -6e^{3t}(t^2+3t+1).$$

2.4.6. Find the derivative y'_x of the following implicit functions:

(a) $x^3 + x^2y + y^2 = 0$; $\quad$ (b) $\ln x + e^{-y/x} = c$;

(c) $x^2 + y^2 - 4x - 10y + 4 = 0$;

(d) $x^{2/3} + y^{2/3} = a^{2/3}$.

Solution. (a) Differentiate with respect to x, considering y as a function of x; we get:

$$3x^2 + 2xy + x^2y' + 2yy' = 0.$$

Solving this equation with respect to y' find

$$y' = -\frac{3x^2+2xy}{x^2+2y}.$$

2.4.7. Find y''_{xx} if:
(a) $\arctan y - y + x = 0$; (b) $e^x - e^y = y - x$;
(c) $x + y = e^{x-y}$.

Solution. (a) Differentiate with respect to x, considering y as a function of x and determine y':

$$\frac{y'}{1+y^2} - y' + 1 = 0, \text{ whence } y' = \frac{1+y^2}{y^2} = y^{-2} + 1.$$

Differentiate once again with respect to x:

$$y'' = -2y^{-3}y'.$$

Substituting the value of y' thus found, we finally get

$$y''_{xx} = -\frac{2(1+y^2)}{y^5}.$$

2.4.8. Find the value of y'' at the point $x = 1$ if

$$x^3 - 2x^2y^2 + 5x + y - 5 = 0 \text{ and } y|_{x=1} = 1.$$

Solution. Differentiating with respect to x, we find that

$$3x^2 - 4xy^2 - 4x^2yy' + 5 + y' = 0.$$

Putting $x = 1$ and $y = 1$, obtain the value of y' at $x = 1$:

$$3 - 4 - 4y' + 5 + y' = 0; \quad y' = 4/3.$$

Differentiate once again with respect to x:

$$6x - 4y^2 - 8xyy' - 8xyy' - 4x^2y'^2 - 4x^2yy'' + y'' = 0.$$

Putting $x = 1$; $y = 1$ and $y' = 4/3$, find the value y'' at $x = 1$:

$$6 - 4 - \frac{64}{3} - \frac{64}{9} - 3y'' = 0, \ y'' = -8\frac{22}{27}.$$

2.4.9. Find y'_x for the following implicit functions:
(a) $x + \sqrt{xy} + y = a$; (b) $\arctan(y/x) = \ln\sqrt{x^2+y^2}$;
(c) $e^x \sin y - e^{-y}\cos x = 0$;
(d) $e^y + xy = e$; find y'_x at the point (0, 1).

2.4.10. Find y''_{xx} of the following implicit functions:
(a) $y = x + \arctan y$;
(b) $x^2 + 5xy + y^2 - 2x + y - 6 = 0$; find y'' at the point (1, 1).

2.4.11. For each of the following functions represented parametrically find the indicated derivatives:

(a) $x=\frac{a\sin t}{1+b\cos t}$, $y=\frac{c\cos t}{1+b\cos t}$; find y'_x;

(b) $x=\ln(1+t^2)$, $y=t-\arctan t$; find y'_x;

(c) $x=t^2+2$, $y=t^3/3-t$; find y''_{xx};

(d) $x=e^{-t^2}$, $y=\arctan(2t+1)$; find y'_x;

(e) $x=4\tan^2(t/2)$, $y=a\sin t+b\cos t$; find y'_x;

(f) $x=\arcsin(t^2-1)$, $y=\arccos 2t$; find y'_x;

(g) $x=\arcsin t$, $y=\sqrt{1-t^2}$; find y''_{xx}.

2.4.12. Show that the function $y=f(x)$, defined by the parametric equations $x=e^t\sin t$, $y=e^t\cos t$, satisfies the relation $y''(x+y)^2=2(xy'-y)$.

§ 2.5. Applications of the Derivative

The equation of a line tangent to the curve of a differentiable function $y=y(x)$ at a point $M(x_0, y_0)$, where $y_0=y(x_0)$, has the form

$$y-y_0=y'(x_0)(x-x_0).$$

A straight line passing through the point of contact perpendicularly to the tangent line is called the *normal to the curve.* The equation of the normal at the point M will be

$$y-y_0=-\frac{1}{y'(x_0)}(x-x_0),$$

$$y'(x_0)\neq 0.$$

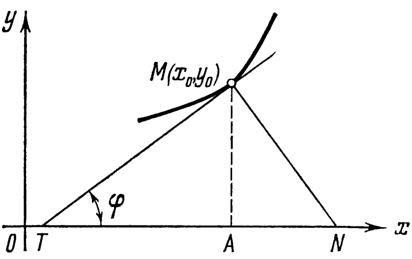

Fig. 36

The segments AT, AN are called the subtangent and the subnormal, respectively; and the lengths MT and MN are the so-called segment of the tangent and the segment of the normal, respectively (see Fig. 36). The lengths of the four indicated segments are expressed by the following formulas:

$$AT=\left|\frac{y}{y'}\right|;\quad AN=|yy'|;\quad MT=\left|\frac{y}{y'}\right|\sqrt{1+(y')^2};$$

$$MN=|y|\sqrt{1+(y')^2}.$$

2.5.1. Write the equations of the tangent line and the normal:
(a) to the curve $y=x^3-3x+2$ at the point (2, 4);

(b) to the parabola $y=2x^2-x+5$ at $x=-0.5$;

(c) to the curve $y=x^4+3x^2-16$ at the points of intersection with the parabola $y=3x^2$.

Solution. (a) Find the derivative at the point $x=2$:

$$y'=3x^2-3, \quad y'(2)=9.$$

The equation of the tangent line has the following form:

$$y-4=9(x-2) \quad \text{or} \quad 9x-y-14=0.$$

The equation of the normal is of the form:

$$y-4=-\frac{1}{9}(x-2) \text{ or } x+9y-38=0.$$

(c) Solving the system of equations

$$\begin{cases} y=x^4+3x^2-16, \\ y=3x^2, \end{cases}$$

we shall find the points of intersection of the curves

$$x_1=-2, \ x_2=2, \ y_1=y_2=12.$$

Now we find the derivatives at the points $x=-2$ and $x=2$:

$$y'=4x^3+6x, \quad y'(-2)=-44, \quad y'(2)=44.$$

Therefore, the equations of the tangent lines have the form

$$y-12=-44(x+2), \quad y-12=44(x-2).$$

The equations of the normals have the form

$$y-12=\frac{1}{44}(x+2), \quad y-12=-\frac{1}{44}(x-2).$$

2.5.2. Find the points on the curve $y=x^3-3x+5$ at which the tangent line:

(a) is parallel to the straight line $y=-2x$;

(b) is perpendicular to the straight line $y=-x/9$;

(c) forms an angle of 45° with the positive direction of the x-axis.

Solution. To find the required points we take into consideration that at the point of tangency the slope of the tangent is equal to the derivative $y'=3x^2-3$ computed at this point.

(a) By the condition of parallelism

$$3x^2-3=-2,$$

whence $x_1=-1/\sqrt{3}$, $x_2=1/\sqrt{3}$. The required points are:

$$M_1(-1/\sqrt{3},\ 5+8\sqrt{3}/9), \ M_2(1/\sqrt{3},\ 5-8\sqrt{3}/9).$$

(b) By the condition of perpendicularity

$$3x^2-3=9,$$

whence $x_1=-2$, $x_2=2$. The required points: $M_1(-2,\ 3)$, $M_2(2,\ 7)$.

2.5.3. Find the angles at which the following lines intersect:
(a) the straight line $y=4-x$ and the parabola $y=4-x^2/2$;
(b) the sinusoid $y=\sin x$ and the cosine curve $y=\cos x$.

Solution. (a) Recall that the angle between two curves at the point of their intersection is defined as the angle formed by the lines tangent to these curves and drawn at this point. Find the points of intersection of the curves by solving the system of equations

$$\begin{cases} y=4-x, \\ y=4-x^2/2. \end{cases}$$

Whence

$$M_1(0,\ 4);\quad M_2(2,\ 2).$$

Determine then the slopes of the lines tangent to the parabola at the points M_1 and M_2:

$$y'(0)=0,\quad y'(2)=-2.$$

The slope of a straight line is constant for all its points; in our case it equals -1. Finally, determine the angle between the two straight lines:

$$\tan\varphi_1=1;\quad \varphi_1=45°;$$

$$\tan\varphi_2=\frac{-1+2}{1+2}=\frac{1}{3};$$

$$\varphi_2=\operatorname{arc\,tan}\frac{1}{3}\approx 18.5°.$$

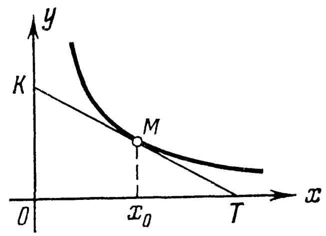

Fig. 37

2.5.4. Prove that the segment of the tangent to the hyperbola $y=c/x$ which is contained between the coordinate axes is bisected at the point of tangency.

Solution. We have $y'=-c/x^2$; hence, the value of the subtangent for the tangent at the point $M(x_0,\ y_0)$ will be

$$\left|\frac{y}{y'}\right|=|x_0|,$$

i. e. $Ox_0=x_0T$ (Fig. 37), which completes the proof.

Whence follows a simple method of constructing a tangent to the hyperbola $y=c/x$: lay off the x-intercept $OT=2x_0$. Then MT will be the desired tangent.

2.5.5. Prove that the ordinate of the catenary $y=a\cosh(x/a)$ is the geometric mean of the length of the normal and the quantity a.

Solution. Compute the length of the normal. Since

$$y'=\sinh(x/a),$$

the length of the normal will be

$$MN=|y|\sqrt{1+(y')^2}=y\sqrt{1+\sinh^2(x/a)}=y\cosh(x/a)=y^2/a,$$

whence $y^2=a\cdot MN$, and $y=\sqrt{a\cdot MN}$, which completes the proof.

2.5.6. Find the slope of the tangent to the curve

$$\begin{cases} x=t^2+3t-8, \\ y=2t^2-2t-5 \end{cases}$$

at the point $M(2, -1)$.

Solution. First determine the value of t corresponding to the given values of x and y. This value must simultaneously satisfy the two equations

$$\begin{cases} t^2+3t-8=2 \\ 2t^2-2t-5=-1. \end{cases}$$

The roots of the first equation are $t_1=2$; $t_2=-5$, the roots of the second equation $t_1=2$; $t_2=-1$. Hence, to the given point there corresponds the value $t=2$. Now determine the value of the derivative at the point M:

$$y'|_{x=2}=\left(\frac{y'_t}{x'_t}\right)_{t=2}=\left(\frac{4t-2}{2t+3}\right)_{t=2}=\frac{6}{7}.$$

And so, the slope of the tangent at the point $M(2, -1)$ is equal to 6/7.

2.5.7. Prove that the tangent to the lemniscate $\rho=a\sqrt{\cos 2\theta}$ at the point corresponding to the value $\theta_0=\pi/6$ is parallel to the x-axis.

Solution. Write in the parametric form the equation of the lemniscate:

$$x=\rho\cos\theta=a\sqrt{\cos 2\theta}\cos\theta,$$
$$y=\rho\sin\theta=a\sqrt{\cos 2\theta}\sin\theta.$$

Whence

$$x'_\theta=-\frac{a\cos\theta\sin 2\theta}{\sqrt{\cos 2\theta}}-a\sqrt{\cos 2\theta}\sin\theta,$$
$$y'_\theta=-\frac{a\sin\theta\sin 2\theta}{\sqrt{\cos 2\theta}}+a\sqrt{\cos 2\theta}\cos\theta,$$
$$x'_\theta(\pi/6)=-a\sqrt{2}, \qquad y'_\theta(\pi/6)=0.$$

Thus, the slope $k = \frac{y'_\theta(\pi/6)}{x'_\theta(\pi/6)} = 0$. Consequently, the line tangent to the lemniscate at the point with $\theta_0 = \pi/6$ and $\rho_0 = a\sqrt{\cos 2\theta_0} = a/\sqrt{2}$ is parallel to the x-axis.

2.5.8. Find the equations of the tangent and the normal to the following curves:

(a) $4x^3 - 3xy^2 + 6x^2 - 5xy - 8y^2 + 9x + 14 = 0$ at the point $(-2, 3)$;
(b) $x^5 + y^5 - 2xy = 0$ at the point $(1, 1)$.

Solution. (a) Differentiate the implicit function:

$$12x^2 - 3y^2 - 6xyy' + 12x - 5y - 5xy' - 16yy' + 9 = 0.$$

Substitute the coordinates of the point $M(-2, 3)$:

$$48 - 27 + 36y' - 24 - 15 + 10y' - 48y' + 9 = 0;$$

whence

$$y' = -9/2.$$

Thus the equation of the tangent line is

$$y - 3 = -\frac{9}{2}(x+2)$$

and the equation of the normal

$$y - 3 = \frac{2}{9}(x+2).$$

2.5.9. Through the point $(2, 0)$, which does not belong to the curve $y = x^4$, draw tangents to the latter.

Solution. Let (x_0, x_0^4) be the point of tangency; then the equation of the tangent will be of the form:

$$y - x_0^4 = y'(x_0)(x - x_0)$$

$$y - x_0^4 = 4x_0^3(x - x_0).$$

By hypothesis the desired tangent line passes through the point $(2, 0)$, hence, the coordinates of this point satisfy the equation of the tangent line:

$$-x_0^4 = 4x_0^3(2 - x_0); \quad 3x_0^4 - 8x_0^3 = 0,$$

whence $x_0 = 0$; $x_0 = 8/3$. Thus, there are two points of tangency: $M_1(0, 0)$, $M_2(8/3, 4096/81)$.

Accordingly, the equations of the tangent lines will be

$$y = 0, \quad y - \frac{4096}{81} = \frac{2048}{27}\left(x - \frac{8}{3}\right).$$

2.5.10. $f(x)=3x^5-15x^3+5x-7$. Find out at which of the points x the rate of change of the function is minimal.

Solution. The rate of change of a function at a certain point is equal to the derivative of the function at this point

$$f'(x)=15x^4-45x^2+5=15\left[(x^2-1/2)^2+1/12\right].$$

The minimum value of $f'(x)$ is attained at $x=\pm 1/\sqrt{2}$. Hence the minimum rate of change of the function $f(x)$ is at the point $x=\pm 1/\sqrt{2}$ and equals 5/4.

2.5.11. A point is in motion along a cubic parabola $12y=x^3$. Which of its coordinates changes faster?

Solution. Differentiating both members of the given equation with respect to t we get the relation between the rates of change of the coordinates:

$$12y'_t=3x^2\cdot x'_t$$

or

$$\frac{y'_t}{x'_t}=\frac{x^2}{4}.$$

Hence,

(1) at $-2<x<2$ the ratio $y'_t:x'_t$ is less than unity, i.e. the rate of change of the ordinate is less than that of the abscissa;

(2) at $x=\pm 2$ the ratio $y'_t:x'_t$ is equal to unity, i.e. at these points the rates of change of the coordinates are equal;

(3) at $x<-2$ or $x>2$ the ratio $y'_t:x'_t$ is greater than unity, i.e. the rate of change of the ordinate exceeds that of the abscissa.

2.5.12. A body of mass 6g is in rectilinear motion according to the law $s=-1+\ln(t+1)+(t+1)^3$ (s is in centimetres and t, in seconds). Find the kinetic energy ($mv^2/2$) of the body one second after it begins to move.

Solution. The velocity of motion is equal to the time derivative of the distance:

$$v(t)=s'_t=\frac{1}{t+1}+3(t+1)^2.$$

Therefore

$$v(1)=12\frac{1}{2} \text{ and } \frac{mv^2}{2}=\frac{6}{2}\left(12\frac{1}{2}\right)^2=468\frac{3}{4}\text{ (erg)}.$$

2.5.13. The velocity of rectilinear motion of a body is proportional to the square root of the distance covered (s), (as, for example, in free fall of a body). Prove that the body moves under the action of a constant force.

Solution. By hypothesis we have

$$v=s'_t=\alpha\sqrt{s}\quad(\alpha=\text{const});$$

whence

$$s''_{tt} = v'_t = \alpha \frac{1}{2\sqrt{s}} s'_t = \alpha^2/2.$$

But according to Newton's law the force

$$F = ks''_{tt} \quad (k = \text{const}).$$

Hence,

$$F = k\alpha^2/2 = \text{const}.$$

2.5.14. A raft is pulled to the bank by means of a rope which is wound on a drum, at a rate of 3 m/min. Determine the speed of the raft at the moment when it is 25 m distant from the bank if the drum is situated on the bank 4 m above water level.

Solution. Let s denote the length of the rope between the drum and the raft and x the distance from the raft to the bank. By hypothesis

$$s^2 = x^2 + 4^2.$$

Differentiating this relation with respect to t, find the relationship between their speeds:

$$2ss'_t = 2xx'_t,$$

whence

$$x'_t = \frac{s}{x} s'_t.$$

Taking into consideration that

$$s'_t = 3; \quad x = 25; \quad s = \sqrt{25^2 + 4^2} \approx 25.3,$$

we obtain

$$x'_t = \frac{\sqrt{25^2 + 4^2}}{25} \cdot 3 \approx 3.03 \text{ (m/min)}.$$

2.5.15. (a) Find the slope of the tangent to the cubic parabola $y = x^3$ at the point $x = \sqrt{3}/3$.

(b) Write the equations of the tangents to the curve $y = 1/(1 + x^2)$ at the points of its intersection with the hyperbola $y = 1/(x + 1)$.

(c) Write the equation of the normal to the parabola $y = x^2 + 4x + 1$ perpendicular to the line joining the origin of coordinates with the vertex of the parabola.

(d) At what angle does the curve $y = e^x$ intersect the y-axis?

2.5.16. The velocity of a body in rectilinear motion is determined by the formula $v = 3t + t^2$. What acceleration will the body have 4 seconds after the start?

2.5.17. The law of rectilinear motion of a body with a mass of 100 kg is $s = 2t^2 + 3t + 1$. Determine the kinetic energy ($mv^2/2$) of the body 5 seconds after the start.

2.5.18. Show that if the law of motion of a body is $s=ae^{t}+be^{-t}$, then its acceleration is numerically equal to the distance covered.

2.5.19. A body is thrown vertically with an initial velocity of a m/sec. What altitude will it reach in t seconds? Find the velocity of the body. In how many seconds and at what distance from the ground will the body reach the highest point?

2.5.20. Artificial satellites move round the Earth in elliptical orbits. The distance r of a satellite from the centre of the Earth as a function of time t can be approximately expressed by the following equation:

$$r=a\left[1-\varepsilon\cos M-\frac{\varepsilon^2}{2}(\cos 2M-1)\right]$$

where $M=\frac{2\pi}{P}(t-t_n)$

t = time parameter
a = semi-major axis of the orbit
ε = eccentricity of the orbit
P = period of orbiting
t_n = time of passing the perigee[1] ty the satellite.

Here a, ε, P and t_n are constants.

Find the rate of change in the distance r from the satellite to the centre of the Earth (i.e. find the so-called radial velocity of the satellite).

§ 2.6. The Differential of a Function. Application to Approximate Computations

If the increment Δy of the function $y=f(x)$ can be expressed as:

$$\Delta y=f(x+\Delta x)-f(x)=A(x)\Delta x+\alpha(x, \Delta x)\Delta x,$$

where

$$\lim_{\Delta x\to 0}\alpha(x, \Delta x)=0,$$

then such a function is called *differentiable* at the point x. The principal linear part of this increment $A(x)\Delta x$ is called the *differential* and is denoted $df(x)$ or dy. By definition, $dx=\Delta x$.

For the differential of the function $y=f(x)$ to exist it is necessary and sufficient that there exist a finite derivative $y'=A(x)$. The differential of a function can be written in the following way:

$$dy=y'\,dx=f'(x)\,dx.$$

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[1] The perigee of the satellite orbit is the shortest distance from the satellite to the centre of the Earth.

For a composite function $y=f(u)$, $u=\varphi(x)$ the differential is retained in the form

$$dy=f'(u)\,du$$

(the invariance of the form of the differential).

With an accuracy up to infinitesimals of a higher order than Δx the approximate formula $\Delta y \approx dy$ takes place. Only for a linear function $y=ax+b$ do we have $\Delta y=dy$.

Differentials of higher orders of the function $y=f(x)$ are successively determined in the following way:

$$d^2y=d(dy);\ d^3y=d(d^2y),\ \ldots,\ d^ny=d(d^{n-1}y).$$

If $y=f(x)$ and x is an independent variable, then

$$d^2y=y''(dx)^2;\ d^3y=y'''(dx)^3,\ \ldots,\ d^ny=y^{(n)}(dx)^n.$$

But if $y=f(u)$, where $u=\varphi(x)$, then $d^2y=f''(u)\,du^2+f'(u)\,d^2u$, and so on.

2.6.1. Find the differential of the function

$$y=\ln(1+e^{10x})+\operatorname{arc\,tan} e^{5x}.$$

Calculate dy at $x=0$; $dx=0.2$.

Solution.

$$dy=\left[\frac{(1+e^{10x})'}{1+e^{10x}}-\frac{(e^{5x})'}{1+e^{10x}}\right]dx=\frac{5e^{5x}(2e^{5x}-1)}{1+e^{10x}}dx.$$

Substituting $x=0$ and $dx=0.2$, we get

$$dy\,|_{x=0;\ dx=0.2}=\frac{5}{2}\cdot 0.2=0.5.$$

2.6.2. Find the increment and the differential of the function

$$y=3x^3+x-1$$

at the point $x=1$ at $\Delta x=0.1$.

Find the absolute and relative errors allowed when replacing the increment of the function with its differential.

Solution.

$$\Delta y=[3(x+\Delta x)^3+(x+\Delta x)-1]-(3x^3+x-1)=$$
$$=9x^2\,\Delta x+9x\,\Delta x^2+3\Delta x^3+\Delta x,$$
$$dy=(9x^2+1)\,\Delta x.$$

Whence

$$\Delta y-dy=9x\,\Delta x^2+3\Delta x^3.$$

At $x=1$ and $\Delta x=0.1$ we get

$$\Delta y-dy=0.09+0.003=0.093,$$
$$dy=1;\ \Delta y=1.093.$$

The absolute error $|\Delta y - dy| = 0.093$, the relative error $\left|\frac{\Delta y - dy}{\Delta y}\right| = \frac{0.093}{1.093} \approx 0.085$ or 8.5%.

2.6.3. Calculate approximately the increment of the function

$$y = x^3 - 7x^2 + 8$$

as x changes from 5 to 5.01.

2.6.4. Using the concept of the differential, find the approximate value of the function

$$y = \sqrt[5]{\frac{2-x}{2+x}} \text{ at } x = 0.15.$$

Solution. Notice that from $\Delta y = y(x+\Delta x) - y(x)$ we get

$$y(x+\Delta x) = y(x) + \Delta y,$$

or, putting $\Delta y \approx dy$,

$$y(x+\Delta x) \approx y(x) + dy.$$

In our problem let us put $x = 0$ and $\Delta x = 0.15$. Then

$$y' = \frac{1}{5}\sqrt[5]{\left(\frac{2+x}{2-x}\right)^4} \cdot \frac{(-4)}{(2+x)^2};$$

$$y'(0) = -\frac{1}{5}, \quad dy = -\frac{1}{5} \cdot 0.15 = -0.03.$$

Hence,

$$y(0.15) \approx y(0) + dy = 1 - 0.03 = 0.97.$$

The true value of $y(0.15) = 0.9702$ (accurate to 10^{-4}).

2.6.5. Find the approximate value of:

(a) $\cos 31°$; (b) $\log 10.21$; (c) $\sqrt[5]{33}$; (d) $\cot 45°10'$.

Solution. (a) In solving this problem we shall use the formula (*) of the preceding problem. Putting $x = \pi/6$, $\Delta x = \pi/180$, we compute:

$$y(x) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2};$$

$$y'(x) = -\sin\frac{\pi}{6} = -\frac{1}{2};$$

$$\cos 31° = \cos\left(\frac{\pi}{6} + \frac{\pi}{180}\right) \approx \frac{\sqrt{3}}{2} - \frac{1}{2}\frac{\pi}{180} = 0.851.$$

(c) Put $x = 32$; $\Delta x = 1$. By formula (*) we get

$$\sqrt[5]{33} \approx \sqrt[5]{32} + (\sqrt[5]{x})'_{x=32} \cdot 1 = 2 + \frac{1}{5\sqrt[5]{32^4}} = 2 + \frac{1}{80} = 2.0125.$$

2.6.6. All faces of a copper cube with 5-cm sides were uniformly ground down. As a result the weight of the cube was reduced by 0.96 g. Knowing the specific weight of copper (8) find the reduction in the cube size, i.e. the amount by which its side was reduced.

Solution. The volume of the cube $v = x^3$, where x is the length of the side. The volume is equal to the weight divided by the density: $v = p/d$; the change in cube's volume $\Delta v = 0.96/8 = 0.12$ (cm³). Since Δv approximately equals dv and taking into consideration that $dv = 3x^2\,dx$ we shall have $0.12 = 3 \times 5^2 \times \Delta x$, whence

$$\Delta x = \frac{0.12}{3 \cdot 25} = 0.0016 \text{ cm}.$$

Thus, the side of the cube was reduced by 0.0016 cm.

2.6.7. Find the expressions for determining the absolute errors in the following functions through the absolute errors in their arguments:

(a) $y = \ln x$;
(b) $y = \log x$;
(c) $y = \sin x \quad (0 < x < \pi/2)$;
(d) $y = \tan x \quad (0 < x < \pi/2)$;
(e) $y = \log(\sin x) \quad (0 < x < \pi/2)$;
(f) $y = \log(\tan x) \quad (0 < x < \pi/2)$.

Solution. If the function $f(x)$ is differentiable at a point x and the absolute error of the argument Δ_x is sufficiently small, then the absolute error in the function y can be expressed by the number

$$\Delta_y = | y'_x | \Delta_x.$$

(a) $\Delta_y = |(\ln x)'|_x \Delta_x = \frac{\Delta_x}{x}$, i.e. the absolute error of a natural logarithm is equal to the relative error in its argument.

(b) $\Delta_y = (\log x)' \Delta_x = \frac{M}{x} \Delta_x$, where $M = \log e = 0.43429$;

(e) $\Delta_y = | [\log(\sin x)]' | \Delta_x = M |\cot x| \Delta_x$;

(f) $\Delta_y = | [\log(\tan x)]' | \Delta_x = \frac{2M}{|\sin 2x|} \Delta_x$.

From (e) and (f) it follows that the absolute error in $\log \tan x$ is always more than that in $\log \sin x$ (for the same x and Δ_x).

2.6.8. Find the differentials dy and d^2y of the function

$$y = 4x^5 - 7x^2 + 3,$$

assuming that:

(1) x is an independent variable;
(2) x is a function of another independent variable.

Solution. By virtue of the invariance of its form the differential of the first order dy is written identically in both cases:

$$dy = y'\,dx = (20x^4 - 14x)\,dx.$$

But in the first case dx is understood as the increment of the independent variable Δx ($dx = \Delta x$), and in the second, as the differential of x as of a function (dx may not be equal to Δx).

Since differentials of higher orders do not possess the property of invariance, to find d^2y we have to consider the following two cases.

(1) Let x be an independent variable; then

$$d^2y = y'' \, dx^2 = (80x^3 - 14) \, dx^2.$$

(2) Let x be a function of some other variable. In this case

$$d^2y = (80x^3 - 14) \, dx^2 + (20x^4 - 14x) \, d^2x.$$

2.6.9. Find differentials of higher orders (x an independent variable):

(a) $y = 4^{-x^2}$; find d^2y;

(b) $y = \sqrt{\ln^2 x - 4}$; find d^2y;

(c) $y = \sin^2 x$; find d^3y.

2.6.10. $y = \ln \frac{1-x^2}{1+x^2}$; find d^2y if: (a) x is an independent variable, (b) x is a function of another variable. Consider the particular case when $x = \tan t$.

2.6.11. The volume V of a sphere of radius r is equal to $\frac{4}{3}\pi r^3$. Find the increment and differential of the volume and explain their geometrical meaning.

2.6.12. The law of the free fall of a material point is $s = gt^2/2$. Find the increment and differential of the distance at a moment t and elucidate their mechanical meaning.

§ 2.7. Additional Problems

2.7.1. Given the functions: (a) $f(x) = |x|$ and (b) $\varphi(x) = |x^3|$. Do derivatives of these functions exist at the point $x = 0$?

2.7.2. Show that the curve $y = e^{|x|}$ cannot have a tangent line at the point $x = 0$. What is the angle between the one-sided tangents to this curve at the indicated point?

2.7.3. Show that the function

$$f(x) = |x - a| \, \varphi(x),$$

where $\varphi(x)$ is a continuous function and $\varphi(a) \neq 0$, has no derivative at the point $x = a$. Find the one-sided derivatives $f'_-(a)$ and $f'_+(a)$.

2.7.4. Given the function

$$f(x)=\begin{cases} x^2\sin(1/x) & \text{at } x\neq 0,\\ 0 & \text{at } x=0.\end{cases}$$

Use this example to show that the derivative of a continuous function is not always a continuous function.

2.7.5. Let

$$f(x)=\begin{cases} x^2, & \text{if } x\leqslant x_0,\\ ax+b, & \text{if } x>x_0.\end{cases}$$

Find the coefficients a and b at which the function is continuous and has a derivative at the point x_0.

2.7.6. By differentiating the formula $\cos 3x=\cos^3 x-3\cos x\sin^2 x$ deduce the formula $\sin 3x=3\cos^2 x\sin x-\sin^3 x$.

2.7.7. From the formula for the sum of the geometric progression

$$1+x+x^2+\ldots+x^n=\frac{1-x^{n+1}}{1-x}\quad(x\neq 1)$$

deduce the formulas for the following sums:

(a) $1+2x+3x^2+\ldots+nx^{n-1}$;

(b) $1^2+2^2x+3^2x^2+\ldots+n^2x^{n-1}$.

2.7.8. Prove the identity

$$\cos x+\cos 3x+\ldots+\cos(2n-1)x=\frac{\sin 2nx}{2\sin x},\quad x\neq k\pi$$

and deduce from it the formula for the sum

$$\sin x+3\sin 3x+\ldots+(2n-1)\sin(2n-1)x.$$

2.7.9. Find y' if:

(a) $y=f(\sin^2 x)+f(\cos^2 x)$; (b) $y=f(e^x)e^{f(x)}$;

(c) $y=\log_{\varphi(x)}\psi(x)$ $(\varphi(x)>0;\ \ \psi(x)>0)$.

2.7.10. Is it reasonable to assert that the product $F(x)=f(x)g(x)$ has no derivative at the point $x=x_0$ if:

(a) the function $f(x)$ has a derivative at the point x_0, and the function $\varphi(x)$ has no derivative at this point?

(b) neither function has a derivative at the point x_0?

Consider the examples: (1) $f(x)=x$, $g(x)=|x|$;

(2) $f(x)=|x|$, $g(x)=|x|$.

Is it reasonable to assert that the sum $F(x)=f(x)+g(x)$ has no derivative at the point $x=x_0$ if:

(c) the function $f(x)$ has a derivative at the point x_0, and the function $g(x)$ has no derivative at this point?

(d) neither function has a derivative at the point x_0?

2.7.11. Prove that the derivative of a differentiable even function is an odd function, and the derivative of an odd function is an even function. Give a geometric explanation to these facts.

2.7.12. Prove that the derivative of a periodic function with period T is a periodic function with period T.

2.7.13. Find $F'(x)$ if

$$F(x)=\begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$$

2.7.14. Find the derivative of the function $y=x|x|$. Sketch the graphs of the given function and its derivative.

2.7.15. Suppose we have a composite function $y=f(u)$, where $u=\varphi(x)$. Among what points should we look for points at which the composite function may have no derivative?

Does the composite function always have no derivative at these points? Consider the function $y=u^2$, $u=|x|$.

2.7.16. Find y'' for the following functions:

(a) $y=|x^3|$; (b) $y=\begin{cases} x^2\sin(1/x), & x\neq 0, \\ 0 \text{ at } x=0. \end{cases}$

Is there $y''(0)$?

2.7.17. (a) $f(x)=x^n$; show that

$$f(1)+\frac{f'(1)}{1!}+\frac{f^{(2)}(1)}{2!}+\ldots+\frac{f^{(n)}(1)}{n!}=2^n.$$

(b) $f(x)=x^{n-1}e^{1/x}$; show that

$$[f(x)]^{(n)}=(-1)^n\frac{f(x)}{x^{2n}} \quad (n=1, 2, \ldots).$$

2.7.18. $y=x^2e^{-x/a}$; show that

$$f^{(n)}(0)=\frac{(-1)^n n(n-1)}{a^{n-2}} \quad (n\geqslant 2).$$

2.7.19. Show that the function $y=\arcsin x$ satisfies the relation $(1-x^2)y''=xy'$. Find $y^{(n)}(0)$ $(n\geqslant 2)$ by applying the Leibniz formula to both members of this identity.

2.7.20. Prove that the Chebyshev polynomials

$$T_n(x)=\frac{1}{2^{n-1}}\cos(n\arccos x) \quad (n=1, 2, \ldots)$$

satisfy the equation

$$(1-x^2)\,T_n''(x)-xT_n'(x)+n^2T_n(x)=0.$$

2.7.21. The derivative of the nth order of the function e^{-x^2} has the form

$$(e^{-x^2})^{(n)}=e^{-x^2}H_n(x),$$

where $H_n(x)$ is a polynomial of degree n called the *Chebyshev-Hermite polynomial.*

Prove that the recurrence relation

$$H_{n+1}(x)-2xH_n(x)+2nH_{n-1}(x)=0 \quad (n=1,\ 2,\ \ldots)$$

is valid.

2.7.22. Show that there exists a single-valued function $y=y(x)$ defined by the equation $y^3+3y=x$, and find its derivative y'_x.

2.7.23. Single out the single-valued continuous branches of the inverse function $x=x(y)$ and find their derivatives if $y=2x^2-x^4$.

2.7.24. $u=\frac{1}{2}\ln\frac{1+v}{1-v}$; check the relation $\frac{du}{dv}\frac{dv}{du}=1$.

2.7.25. Inverse trigonometric functions are continuous at all points of the domain of definition. Do they have a finite derivative at all points of the domain? Indicate the points at which the following functions have no finite derivative:

(a) $y=\arccos\frac{x+1}{2}$; (b) $y=\arcsin\frac{1}{x}$.

2.7.26. Show that the function $y=y(x)$, defined parametrically: $x=2t-|t|$, $y=t^2+t|t|$, is differentiable at $t=0$ but its derivative cannot be found by the usual formula.

2.7.27. Determine the parameters a, b, c in the equation of the parabola $y=ax^2+bx+c$ so that it becomes tangent to the straight line $y=x$ at the point $x=1$ and passes through the point $(-1,\ 0)$.

2.7.28. Prove that the curves $y_1=f(x)$ $(f(x)>0)$ and $y_2=f(x)\sin ax$, where $f(x)$ is a differentiable function, are tangent to each other at the common points.

2.7.29. Show that for any point $M(x_0,\ y_0)$ of the equilateral hyperbola $x^2-y^2=a^2$ the segment of the normal from the point M to the point of intersection with the abscissa is equal to the radius vector of the point M.

2.7.30. Show that for any position of the generating circle the tangent line and the normal to the cycloid $x=a(t-\sin t)$, $y=a(1-\cos t)$ pass through the highest $(at,\ 2a)$ and the lowest $(at,\ 0)$ points of the circle, respectively.

2.7.31. Show that two cardioids $\rho=a(1+\cos\varphi)$ and $\rho=a(1-\cos\varphi)$ intersect at right angles.

2.7.32. Let $y=f(u)$, where $u=\varphi(x)$. Prove the validity of the equality

$$d^3y=f'''(u)\,du^3+3f''(u)\,du\,d^2u+f'(u)\,d^3u.$$

2.7.33. Let $y=f(x)$, where $x=\varphi(t)$; the functions $f(x)$ and $\varphi(t)$ are twice differentiable and $dx\neq 0$. Prove that

$$y''_{xx}=\frac{d^2y\,dx-dy\,d^2x}{dx^3},$$

where the differentials forming the right member of the relation are differentials with respect to the variable t.

2.7.34. How will the expression

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y$$

be transformed (where y is a twice differentiable function of x) if we introduce a new independent variable t, putting $x=\cos t$?

2.7.35. In determining an electric current by means of a tangent galvanometer use is made of the formula

$$I=k\tan\varphi,$$

where I = current

k = factor of proportionality (depending on the instrument)

φ = angle of pointer deflection.

Determine the relative error of the result which depends on the inaccuracy in reading the angle φ. At what position of the pointer can one obtain the most reliable results?