8. Quadratic Equations

An equation in which the highest power of the unknown quantity is two is called **quadratic equation**.

Quadratic Equations

A equation with degree two : $ax^2 + bx + c = 0$, where $a \neq 0$
A real number α is root of a quadratic equation if $a\alpha^2 + b\alpha + c = 0$

A quadratic equation $ax^2 + bx + c = 0$, has
- two distinct roots, if $D = b^2 - 4ac > 0$
- two equal roots, i.e. coincident real roots if $D = b^2 - 4ac = 0$
- no real roots, if $D = b^2 - 4ac < 0$

where $D = b^2 - 4ac$, is called the discriminant of a quadratic equation

Quadratic formula (Shreedharacharya's rule)

The real roots α and β of a quadratic equation are given by

$$\alpha = \frac{-b + \sqrt{D}}{2a} \qquad \beta = \frac{-b - \sqrt{D}}{2a}$$

To evaluate: $x^2 - 4x + 3 = 0$
$a = 1,\ b = -4,\ c = 3$
$D = b^2 - 4ac = 16 - 4 \times 1 \times 3$
$D = 4$
$$\alpha = \frac{-b + \sqrt{D}}{2a} = \frac{4 + 2}{2} = \frac{6}{2} = 3$$
$$\beta = \frac{-b - \sqrt{D}}{2a} = \frac{4 - 2}{2} = \frac{2}{2} = 1$$

Types of quadratic equation

Quadratic equations are of two types:

Purely quadratic	Adfected quadratic
$ax^2 + c = 0$ where $a, c \in C$ and $b = 0$, $a \neq 0$	$ax^2 + bx + c = 0$ where $a, b, c \in C$ and $a \neq 0$, $b \neq 0$

Roots of a quadratic equation:

The values of variable x which satisfy the quadratic equation is called roots of quadratic equation.

Solution of quadratic equation

(1) Factorization method

Let $ax^2 + bx + c = a(x - \alpha)(x - \beta) = 0$.
Then $x = \alpha$ and $x = \beta$ will satisfy the given equation.
Hence, factorize the equation and equating each factor to zero gives roots of the equation.

Example: $3x^2 - 2x + 1 = 0 \Rightarrow (x - 1)(3x + 1) = 0$; $x = 1, -1/3$

(2) Sri Dharacharya method

By completing the perfect square as

$$ax^2 + bx + c = 0 \Rightarrow x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Adding and subtracting $\left(\frac{b}{2a}\right)^2$, $\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right] = 0$

which gives, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Hence the quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ has two roots,

given by $\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$, $\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$

Every quadratic equation has two and only two roots.

Nature of roots

In a quadratic equation $ax^2 + bx + c = 0$, let us suppose that are real and $a \neq 0$. The following is true about the nature of its roots.

1. The equation has real and distinct roots if and only if $D \equiv b^2 - 4ac > 0$.
2. The equation has real and coincident (equal) roots if and only if $D \equiv b^2 - 4ac = 0$.
3. The equation has complex roots of the form $\alpha \pm i\beta$, $\alpha \neq 0$ if and only if $D \equiv b^2 - 4ac < 0$.
4. The equation has rational roots if and only if a, b, c $\in$ Q (the set of rational numbers) and $D \equiv b^2 - 4ac$ is a perfect square (of a rational number).
5. The equation has (unequal) irrational (surd form) roots if and only if $D \equiv b^2 - 4ac > 0$ and not a perfect square even if a, b and c are rational. In this case if $p + \sqrt{q}$, p,q rational is an irrational root, then $p - \sqrt{q}$ is also a root (a, b, c being rational).
6. $\alpha + i\beta$ ($\beta \neq 0$ and $\alpha, \beta \in R$) is a root if and only if its conjugate $\alpha - i\beta$ is a root, that is complex roots occur in pairs in a quadratic equation. In case the equation is satisfied by more than two complex numbers, then it reduces to an identity.
   $0.x^2 + 0.x + 0 = 0$, i.e., $a = 0 = b = c$.

Relations between roots and coefficients

(1) Relation between roots and coefficients of quadratic equation: If α and β are the roots of quadratic equation , ($a \neq 0$) then

$$\text{Sum of roots} = S = \alpha + \beta = \frac{-b}{a} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$$

$$\text{Product of roots} = P = \alpha.\beta = \frac{c}{a} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

(2) Formation of an equation with given roots: A quadratic equation whose roots are α and β is given by $(x - \alpha)(x - \beta) = 0$.

$\therefore x^2 - (\alpha + i\beta)x + \alpha\beta = 0$

i.e. x^2 – (sum of roots)x + (product of roots) = 0

$\therefore x^2 - Sx + P = 0$

(3) Symmetric function of the roots : A function of α and β is said to be a symmetric function, if it remains unchanged when α and β are interchanged.

For example, $\alpha^2 + \beta^2 + 2\alpha\beta$ is a symmetric function of α and β whereas $\alpha^2 + \beta^2 + 2\alpha\beta$ is not a symmetric function of α and β.

In order to find the value of a symmetric function of α and β, express the given function in terms of α

+ β and αβ. The following results may be useful.

(i) $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

(ii) $\alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$

(iii) $\alpha^4 + \beta^4 = (\alpha^3+\beta^3)(\alpha+\beta) - \alpha\beta(\alpha^2+\beta^2)$

(iv) $\alpha^5 + \beta^5 = (\alpha^3+\beta^3)(\alpha^2+\beta^2) - \alpha^2\beta^2(\alpha+\beta)$

(v) $|\alpha - \beta| = \sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$

(vi) $\alpha^2 - \beta^2 = (\alpha+\beta)(\alpha-\beta)$

(vii) $\alpha^3 - \beta^3 = (\alpha-\beta)[(\alpha+\beta)^2 - \alpha\beta]$

(viii) $\alpha^4 - \beta^4 = (\alpha+\beta)(\alpha-\beta)(\alpha^2+\beta^2)$

Properties of quadratic equation

1. If f(a) and f(b) are of opposite signs then at least one or in general odd number of roots of the equation lie between a and b.
2. If then f(a) = f(b) there exists a point c between a and b such that f(c) = 0, a<c<b.
3. If α is a root of the equation f(x) = 0 then the polynomial f(x) is exactly divisible by (x – α), then (x – α) is factor of f(x).
4. If the roots of the quadratic equations $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ are in the same ratio [i.e. $\alpha_1/\beta_1 = \alpha_2/\beta_2$] then $b_1^2/b_2^2 = a_1c_1/a_2c_2$.

Nature Of The Roots Of A Quadratic Equation

The nature of the roots depends on the value of $b^2 - 4ac$. $bx^2 - 4ac$ is called the **discriminant** of the quadratic equation $ax^2 + bx + c = 0$ and is generally, denoted by D.
∴ $D = b^2 - 4ac$
If D > 0, i..e., $b^2 - 4ac > 0$, i.e., b2 – 4ac is positive; the roots are **real and unequal**. Also,
(i) If $b^2 - 4ac$ is a perfect square, the roots are **rational and unequal**.
(ii) If $b^2 - 4ac$ is positive but not perfect square, the roots are **irrational and unequal**.
If D = 0, i.e., $b^2 - 4ac = 0$; the roots are **real and equal**.
If D < 0, i.e., b2 – 4ac < 0; i.e., $b^2 - 4ac$ is negative; the roots are not real, i.e., the roots are **imaginary**.

Nature Of The Roots Of A Quadratic Equation With Examples

Example 1: Without solving, examine the nature of roots of the equation $2x^2 + 2x + 3 = 0$.

Sol. Comparing $2x^2 + 2x + 3 = 0$ with $ax^2 + bx + c = 0$
we get: a = 2, b = 2 and c = 3
$D = b^2 - 4ac = (2)^2 - 4 \times 2 \times 3 = 4 - 24$
= – 20; which is negative.
∴ The roots of the given equation are imaginary.

Example 2: Without solving, examine the nature of roots of the equation $2x^2 - 7x + 3 = 0$.

Sol. Comparing $2x^2 - 7x + 3 = 0$ with $ax^2 + bx + c = 0$;
we get: a = 2, b = – 7 and c = 3
$D = b^2 - 4ac = (-7)^2 - 4 \times 2 \times 3$
= 49 – 24 = 25, which is perfect square.
∴ The roots of the given equation are rational and unequal.

Example 3: Without solving, examine the nature of roots of the equation $x^2 - 5x - 2 = 0$.

Sol. Comparing $x^2 - 5x - 2 = 0$ with $ax^2 + bx + c = 0$;
we get: $a = 1$, $b = -5$ and $c = -2$
$D = b^2 - 4ac = (-5)^2 - 4 \times 1 \times -2$
$= 25 + 8 = 33$; which is positive but not a perfect square.
∴ The roots of the given equation are irrational and unequal.

Example 4: Without solving, examine the nature of roots of the equation $4x^2 - 4x + 1 = 0$.

Sol. Comparing $4x^2 - 4x + 1 = 0$ with $ax^2 + bx + c = 0$;
we get: $a = 4$, $b = -4$, and $c = 1$
$D = b^2 - 4ac = (-4)^2 - 4 \times 4 \times 1$
$= 16 - 16 = 0$
∴ Roots are real and equal.

Example 5: For what value of m, are the roots of the equation $(3m + 1)x^2 + (11 + m)x + 9 = 0$ equal?

Sol. Comparing the given equation with $ax^2 + bx + c = 0$;
we get : $a = 3m + 1$, $b = 11 + m$ and $c = 9$
∴ Discriminant, $D = b^2 - 4ac$
$= (11 + m)^2 - 4(3m + 1) \times 9$
$= 121 + 22m + m^2 - 108m - 36$
$= m^2 - 86m + 85$
$= m^2 - 85m - m + 85$
$= m(m - 85) - 1(m - 85)$
$= (m - 85)(m - 1)$
Since the roots are equal, $D = 0$
$\Rightarrow (m - 85)(m - 1) = 0$
$\Rightarrow m - 85 = 0$ or $m - 1 = 0$
$\Rightarrow m = 85$ or $m = 1$

Solving Quadratic Equations with Complex Roots

When the roots of a quadratic equation are imaginary, they always occur in conjugate pairs.

A root of an equation is a solution of that equation.
If a quadratic equation with real-number coefficients has a negative discriminant, then the two solutions to the equation are complex conjugates of each other.
(Remember that a negative number under a radical sign yields a complex number.)
The discriminant is the b2- 4ac part of the quadratic formula (the part under the radical sign).
If the discriminant is negative, when you solve your quadratic equation the number under the radical sign in the quadratic formula is negative — forming complex roots.

Quadratic equation: $ax^2 + bx + c = 0 \quad (a \neq 0)$

Quadratic formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Example 1:

Find the solution set of the given equation over the set of complex numbers.

$$\frac{x^2}{2} = 5x - 17$$

$a = 1$, $b = -10$, $c = 34$

	Pick out the coefficient values representing a, b, and c, and substitute into the quadratic formula, as you would do in the solution to any normal quadratic equation. Remember, when there is no number visible in front of the variable, the number **1** is there.

$$x = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(34)}}{2(1)}$$

$$x = \frac{10 \pm \sqrt{100 - 136}}{2}$$

$$x = \frac{10 \pm \sqrt{-36}}{2}$$

$$x = \frac{10 \pm 6i}{2}$$ Reduce this fraction.

$$x = 5 \pm 3i$$

$x = 5 + 3i$ or $5 - 3i$ Answer

	HINT: When the directions say: ***Express over the set of complex numbers,*** look for a negative value under the radical sign.

Example 2:

Find the solution set of the given equation over the set of complex numbers.

$$3x^2 + 10 = 4x$$

$$3x^2 - 4x + 10 = 0$$

$a = 3$, $b = -4$, $c = 10$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(3)(10)}}{2(3)}$$

$$x = \frac{4 \pm \sqrt{16 - 120}}{6}$$

$$x = \frac{4 \pm \sqrt{-104}}{6}$$

$$x = \frac{4 \pm \sqrt{-4} \bullet \sqrt{26}}{6}$$

$$x = \frac{4 \pm 2i\sqrt{26}}{6}$$ Reduce this fraction.

$$x = \frac{2 \pm i\sqrt{26}}{3}$$

$x = \frac{2 + i\sqrt{26}}{3}$ or $\frac{2 - i\sqrt{26}}{3}$ Answer

Example 3:

Find the solution set of the given equation and express its roots in a+bi form.

$$x+\frac{5}{x}=3$$

$$(x)x+(x)\frac{5}{x}=(x)3$$ Multiply each term by x to eliminate the fraction.

$$x^2+5=3x$$

$$x^2-3x+5=0$$

$$x=\frac{-(-3)\pm\sqrt{(-3)^2-4(1)(5)}}{2(1)}$$

$$x=\frac{3\pm\sqrt{9-20}}{2}$$

$$x=\frac{3\pm\sqrt{-11}}{2}$$

$$x=\frac{3\pm\sqrt{-1}\bullet\sqrt{11}}{2}$$

$$x=\frac{3\pm i\sqrt{11}}{2}$$

$$x=\frac{3}{2}+\frac{i\sqrt{11}}{2} \quad \text{or} \quad \frac{3}{2}-\frac{i\sqrt{11}}{2} \quad \text{Answer}$$

Solving A Quadratic Equation By Factoring

(i) $(x+y)^2 = x^2+2xy+y^2$

(ii) $(x-y)^2 = x^2-2xy+y^2$

(iii) $x^2-y^2=(x-y)(x+y)$

(iv) $(x+a)(x+b) = x^2+(a+b)x+ab$

(v) $(x+y+z)^2=x^2+y^2+z^2+2xy+2yz+2zx$

(vi) $(x+y)^3 = x^3+y^3+3xy(x+y)$

(vii) $(x-y)^3 = x^3-y^3-3xy(x-y)$

(viii) $x^3+y^3+z^3-3xyz = (x+y+z)(x^2+y^2+z^2-xy-yz-zx)$

$x^3+y^3+z^3=3xyz \quad if \ x+y+z=0$

Forms of a Quadratic Equation

- **Standard Form of a Quadratic Equation** $ax^2 + bx + c = 0, a \neq 0$

- **Factored Form of a Quadratic Equation** $a(x + p)(x + q) = 0, a \neq 0$

  Factoring means to write the terms in multiplication form (as a product).

- **Zero Product Property**

  If $ab = 0$ then either $a = 0$ or $b = 0$ (or both).

  The expression *must* be set equal to zero to use this property.

Zero Product Example: Quadratic in Factored Form $(x - 6)\ (x + 8) = 0$

$x - 6 = 0$ or $x + 8 = 0$

$x = 6$ or $x = 8$

Factoring Help!

Question	Strategy	Answer
$m^2+10m+16$	Both signs are positive, so both signs in answer are positive.	$(m+2)(m+8)$
$n^2-8n-48$	Two negatives, so in our answer, one will be positive (the smaller number) and one will be negative (the larger number)	$(n-12)(n+4)$
$y^2-15y+56$	Second term negative, third term positive; both signs in the answer will be negative	$(y-8)(y-7)$
p^2+p-20	Second term positive, third term negative; one will be positive (the larger number) and one will be negative (the smaller number)	$(p+5)(p-4)$

Factoring

- Before today the only way we had for solving quadratics was to factor.

Zero-factor property

$$x^2 - 2x - 15 = 0$$

$$(x + 3)(x - 5) = 0$$

$$x + 3 = 0 \quad \text{or} \quad x - 5 = 0$$

$$x = -3 \quad \text{or} \quad x = 5$$

$$x = \{-3, 5\}$$

Since, $3x^2 - 5x + 2$ is a quadratic polynomial;
$3x^2 - 5x + 2 = 0$ is a quadratic equation.
Also,
$3x^2 - 5x + 2 = 3x^2 - 3x - 2x + 2$ [Factorising]
$= 3x (x - 1) - 2(x - 1)$
$= (x - 1) (3x - 2)$
In the same way :
$3x^2 - 5x + 2 = 0 \Rightarrow 3x^2 - 3x - 2x + 2 = 0$ [Factorising L.H.S.]
$\Rightarrow (x - 1) (3x - 2) = 0$
i.e., $x - 1 = 0$ or $3x - 2 = 0$
$\Rightarrow x = 1$ or $x = 2/3$
which is the solution of given quadratic equation.

In order to solve the given Quadratic Equation:

1. Clear the fractions and brackets, if given.
2. By transfering each term to the left hand side; express the given equation as $ax^2 + bx + c = 0$ or $a + bx + cx^2 = 0$
3. Factorise left hand side of the equation obtained (the right hand side being zero).
4. By putting each factor equal to zero; solve it.

Solving A Quadratic Equation By Factoring With Examples

Example 1: Solve (i) $x^2 + 3x - 18 = 0$ (ii) $(x - 4) (5x + 2) = 0$

(iii) $2x^2 + ax - a2 = 0$; where 'a' is a real number.

Sol. **(i)** $x^2 + 3x - 18 = 0$
$\Rightarrow x^2 + 6x - 3x - 18 = 0$
$\Rightarrow x(x + 6) - 3(x + 6) = 0$
i.e., $(x + 6)(x - 3) = 0$
$\Rightarrow x + 6 = 0$ or $x - 3 = 0$
$\Rightarrow x = -6$ or $x = 3$
Roots of the given equation are -6 and 3

(ii) $(x - 4)(5x + 2) = 0$
$\Rightarrow x - 4 = 0$ or $5x + 2 = 0$
$x = 4$ or $x = -2/5$

(iii) $2x^2 + ax - a^2 = 0$
$\Rightarrow 2x^2 + 2ax - ax - a^2 = 0$
$\Rightarrow 2x(x + a) - a(x + a) = 0$
i.e., $(x + a)(2x - a) = 0$
$\Rightarrow x + a = 0$ or $2x - a = 0$
$\Rightarrow x = -a$ or $x = a/2$

Example 2: Solve the following quadratic equations
(i) $x^2 + 5x = 0$ (ii) $x^2 = 3x$ (iii) $x^2 = 4$

Sol. **(i)** $x^2 + 5x = 0 \Rightarrow x(x + 5) = 0$
$\Rightarrow x = 0$ or $x + 5 = 0$
$\Rightarrow x = 0$ or $x = -5$

(ii) $x^2 = 3x$
$\Rightarrow x^2 - 3x = 0$
$\Rightarrow x(x - 3) = 0$
$\Rightarrow x = 0$ or $x = 3$

(iii) $x^2 = 4$
$\Rightarrow x = \pm 2$

Example 3: Solve the following quadratic equations
(i) $7x^2 = 8 - 10x$ (ii) $3(x^2 - 4) = 5x$ (iii) $x(x + 1) + (x + 2)(x + 3) = 42$

Sol. **(i)** $7x^2 = 8 - 10x$
$\Rightarrow 7x^2 + 10x - 8 = 0$
$\Rightarrow 7x^2 + 14x - 4x - 8 = 0$
$\Rightarrow 7x(x + 2) - 4(x + 2) = 0$
$\Rightarrow (x + 2)(7x - 4) = 0$
$\Rightarrow x + 2 = 0$ or $7x - 4 = 0$
$\Rightarrow x = -2$ or $x = 4/7$

(ii) $3(x^2 - 4) = 5x$
$\Rightarrow 3x^2 - 5x - 12 = 0$
$\Rightarrow 3x^2 - 9x + 4x -\neg 12 = 0$
$\Rightarrow 3x(x - 3) + 4(x - 3) = 0$
$\Rightarrow (x - 3)(3x + 4) = 0$
$\Rightarrow x - 3 = 0$ or $3x + 4 = 0$
$\Rightarrow x = 3$ or $x = -4/3$

(iii) $x(x + 1) + (x + 2)(x + 3) = 42$
$\Rightarrow x^2 + x + x^2 + 3x + 2x + 6 - 42 = 0$
$\Rightarrow 2x^2 + 6x - 36 = 0$
$\Rightarrow x^2 + 3x - 18 = 0$
$\Rightarrow x^2 + 6x - 3x - 18 = 0$
$\Rightarrow x(x + 6) - 3(x + 6) = 0$

⇒ (x + 6) (x − 3) = 0
⇒ x = − 6 or x = 3

Example 4: Solve for x : $12abx^2 - (9a^2 - 8b^2)x - 6ab = 0$
Given equation is $12abx^2 - (9a^2 - 8b^2)x - 6ab = 0$
⇒ 3ax(4bx − 3a) + 2b(4bx − 3a) = 0
⇒ (4bx − 3a) (3ax + 2b) = 0
⇒ 4bx − 3a = 0 or 3ax + 2b = 0
⇒ x =3a/4b or x = − 2b/3a

Solving Factorable Quadratic Equations

A **quadratic equation** is a polynomial equation of degree two. The **standard form** is $ax^2 + bx + c = 0$.

Definition of a Quadratic Equation

A **quadratic equation** in *x* is an equation that can be written in the **standard form**

$$ax^2 + bx + c = 0$$

where *a*, *b*, and *c* are real numbers with *a not equal to* 0. A quadratic equation in *x* is also called a **second-degree polynomial equation** in *x*.

There's no magic to solving quadratic equations. Quadratic equations can be solved by **factoring** and also by **graphing.**

The factoring method of solution:

Let's do a quick review of factoring.

There are primarily three types of factoring:

*Common Monomial	$ab + ac = a(b + c)$
*Difference of Squares	$x^2 - 9 = (x + 3)(x - 3)$
*Quadratic Trinomial	$x^2 - 5x + 6 = (x - 3)(x - 2)$

If you can factor, you will be able to solve factorable quadratic equations.

Example 1:

Solve for x: $x^2 - x = 6$

Here are the steps you should follow:

1.	Move all terms to the same side of the equal sign, so the equation is set equal to 0.	$x^2 - x - 6 = 0$ This places the equation in **standard form**.
2.	Factor the algebraic expression.	$(x-3)(x+2) = 0$ $(x + 3)$ and $(x + 2)$ are called **factors.** These are factors of the **expression** $x^2 - x - 6$.
3.	Set each factor equal to 0. (This process is called the "zero product property". If the product of two factors equals 0, then either one or both of the factors must be 0.)	$x - 3 = 0; \quad x + 2 = 0$
4.	Solve each resulting equation.	$x = 3; \quad x = -2$ $x = 3$ and $x = -2$ are called **roots**. These are roots of the **equation** $x^2 - x - 6 = 0$.

Example 2:

Solve for x: $x^2 + 3x = 0$

Factor the common monomial. $\quad x(x + 3)=0$

Set each factor equal to 0 and solve for x.

$x = 0 \quad | \quad x + 3 = 0$

$\quad\quad\quad | \quad x = -3$

List all values of x. $\quad x = \{0, -3\}$

Example 3:

Solve for y: $y^2 = 16$

Get all terms on the same side. $\quad y^2 - 16 = 0$

Factor the difference of squares. $\quad (y + 4)(y - 4) = 0$

Set each factor equal to 0 and solve for y.

$y + 4 = 0 \quad | \quad y - 4 = 0$

$y = -4 \quad | \quad y = 4$

List all values of y. $\quad y = \{-4, 4\}$

Example 4:

Solve for c: $c^2 - 12 = c$

Get all terms on the same side.	$c^2 - 12 - c = 0$
Arrange the terms in standard form.	$c^2 - c - 12 = 0$
Factor the quadratic trinomial.	$(c + 3)(c - 4) = 0$
Set each factor equal to 0 and solve for c.	$c + 3 = 0$, $c = -3$ | $c - 4 = 0$, $c = 4$
List all values of c.	$c = \{-3, 4\}$

Example 5:

Solve for x: $\frac{x}{9} = \frac{144}{x}$

Employ "product of the means = product of the extremes" (cross-multiply) for this proportion.	$x \cdot x = 9 \cdot 144$ $x^2 = 1296$
Get all terms on the same side.	$x^2 - 1296 = 0$
Factor the difference of squares.	$(x + 36)(x - 36) = 0$
Set each factor equal to 0 and solve for x.	$x + 36 = 0$, $x = -36$ | $x - 36 = 0$, $x = 36$
List all values of x.	$x = \{-36, 36\}$

Example 6:

Solve for x: $\frac{x+2}{2x+1} = \frac{x-2}{3}$

Employ "product of the means = product of the extremes" (cross-multiply) for this proportion.	$3(x+2) = (2x+1)(x-2)$ $3x + 6 = 2x^2 - 3x - 2$
Get all terms on the same side.	$0 = 2x^2 - 6x - 8$
Factor the common monomial.	$0 = 2(x^2 - 3x - 4)$
Factor the quadratic trinomial.	$2(x - 4)(x + 1) = 0$
Set each factor equal to 0 and solve for x.	$2 \neq 0$ | $x - 4 = 0$, $x = 4$ | $x + 1 = 0$, $x = -1$
List all values of x.	$x = \{-1, 4\}$

Example 7:

Write a quadratic equation, in the form $ax^2 + bx + c = 0$, whose roots are 2 and 5.

The simplest answer will be an equation where the factors of the expression are $(x - 2)$ and $(x - 5)$. Create this equation.	$(x - 2)(x - 5) = 0$
Multiply.	$x^2 - 5x - 2x + 10 = 0$
Combine to get an answer equation.	$x^2 - 7x + 10 = 0$

Example 8:

The square of a number exceeds 5 times the number by 24. Find the number(s).

Translate the problem into a mathematical equation.	$x^2 = 5x + 24$
Get all terms on the same side.	$x^2 - 5x - 24 = 0$
Factor the difference of squares.	$(x - 8)(x + 3) = 0$
Set each factor equal to 0 and solve for x.	$x - 8 = 0$, $x = 8$ | $x + 3 = 0$, $x = -3$
List all values of x.	$\mathbf{x = \{8, -3\}}$

Example 9:

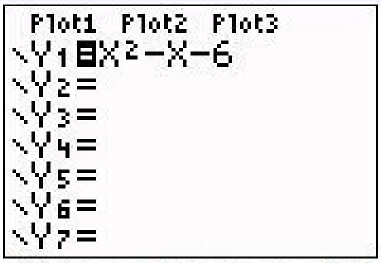
In football, the height of the football reached during a pass can be modeled by the equation $h = -16t^2 + 28t + 6$, where the height, h, is in feet and the time, t, is in seconds. How long does it take for this ball to reach a height of 12 feet?

Substitute 12 into the equation for h.	$12 = -16t^2 + 28t + 6$
Get all terms on the same side. Move terms to the left side to avoid working with a negative leading coefficient.	$16t^2 - 28t + 6 = 0$
Factor the quadratic trinomial.	$(4t - 1)(4t - 6) = 0$
Set each factor equal to 0 and solve for t.	$4t - 1 = 0$, $4t = 1$, $t = 1/4$ | $4t - 6 = 0$, $4t = 6$, $t = 6/4 = 3/2$
List all values of t that are positive. Negative time, should it appear, is not considered an answer.	$t = \{1/4, 3/2\}$. Reaches a height of 12 feet when time is 0.25 seconds (ball going up) and 1.5 seconds (ball coming down).

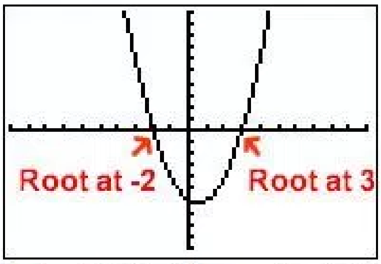
The graphing method of solution:

Quadratic equations can also be solved by graphing. Graphing can be done "by hand" or with the use of a graphing calculator. The locations where the graph intersects (crosses) the x-axis will be the solutions to the equation. These locations are called the **roots** of the equation.
(At the Algebra 1 level, the quadratic graphs will always intersect the x-axis.)

Solve for x by graphing: $x^2 - x = 6$



Put the equation into standard form
$x^2 - x - 6 = 0$.
Enter the left-hand side into Y1.



On the TI-83+/84+ use the ZERO command to find where the graph crosses the x-axis.

Answer:
$x = 3; \quad x = -2$

Solving A Quadratic Equation By Completing The Square

Every quadratic equation can be converted in the form:
$(x + a)^2 - b^2 = 0$ or $(x - a)^2 - b^2 = 0$.

Steps:

1. Bring, if required, all the term of the quadratic equation to the left hand side.
2. Express the terms containing x as $x^2 + 2xy$ or $x^2 - 2xy$.
3. Add and subtract y2 to get $x^2 + 2xy + y^2 - y^2$ or $x^2 - 2xy + y^2 - y^2$; which gives $(x + y)^2 - y^2$ or $(x - y)^2 - y^2$.

Thus,
(i) $x^2 + 8x = 0 \Rightarrow x^2 + 2x \times 4 = 0$
$\Rightarrow x^2 + 2x \times 4 + 4^2 - 4^2 = 0$
$\Rightarrow (x + 4)^2 - 16 = 0$
(ii) $x^2 - 8x = 0 \Rightarrow x^2 - 2 \times x \times 4 = 0$
$\Rightarrow x^2 - 2 \times x \times 4 + 4^2 - 4^2 = 0$
$\Rightarrow (x - 4)^2 - 16 = 0$

Solving A Quadratic Equation By Completing The Square

To solve $ax^2 + bx + c = 0$, by completing the square:

Step 1 If $a \neq 1$, divide both sides of the equation by a.
Step 2 Rewrite the equation so that the constant term is alone on one side of the equality symbol.
Step 3 Square half the coefficient of x, and add this square to both sides of the equation.
Step 4 Factor the resulting trinomial as a perfect square and combine like terms on the other side.
Step 5 Use the square root property to complete the solution.

Solving A Quadratic Equation By Completing The Square With Examples

Example 1: Find the roots of the quadratic equation $2x^2 - 7x + 3 = 0$ (if they exist) by the method of completing the square.

Sol. $2x^2 - 7x + 3 = 0$

$\Rightarrow x^2 - \frac{7}{2}x + \frac{3}{2} = 0$ [Dividing each term by 2]

$\Rightarrow x^2 - 2 \times x \times \frac{7}{4} + \frac{3}{2} = 0$

$\Rightarrow x^2 - 2 \times x \times \frac{7}{4} + \left(\frac{7}{4}\right)^2 - \left(\frac{7}{4}\right)^2 + \frac{3}{2} = 0$

$\Rightarrow \left(x - \frac{7}{4}\right)^2 - \frac{49}{16} + \frac{3}{2} = 0$

$\Rightarrow \left(x - \frac{7}{4}\right)^2 - \left(\frac{49-24}{16}\right) = 0$

$\Rightarrow \left(x - \frac{7}{4}\right)^2 - \frac{25}{16} = 0$

$i.e., \left(x - \frac{7}{4}\right)^2 = \frac{25}{16} \Rightarrow x - \frac{7}{4} = \pm\frac{5}{4}$

$i.e., x - \frac{7}{4} = \frac{5}{4}$ or $x - \frac{7}{4} = -\frac{5}{4}$

$\Rightarrow x = \frac{7}{4} + \frac{5}{4}$ or $x = \frac{7}{4} - \frac{5}{4}$

$\Rightarrow x = 3$ or $x = \frac{1}{2}$

Example 2: Find the roots of the quadratic equation $4x^2 + 4\sqrt{3}x + 3 = 0$

Sol. $4x^2 + 4\sqrt{3}x + 3 = 0$

$\Rightarrow x^2 + \sqrt{3}x + \frac{3}{4} = 0$

$i.e., \ x^2 + 2 \times x \times \frac{\sqrt{3}}{2} + \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2 + \frac{3}{4} = 0$

$\Rightarrow \left(x + \frac{\sqrt{3}}{2}\right)^2 - \frac{3}{4} + \frac{3}{4} = 0$

$i.e., \left(x + \frac{\sqrt{3}}{2}\right)^2 = 0$

$\Rightarrow x + \frac{\sqrt{3}}{2} = 0$ and $x = \frac{-\sqrt{3}}{2}$

Example 3: Find the roots of the quadratic equation $2x^2 + x + 4 = 0$

Sol. $2x^2 + x + 4 = 0$

$\Rightarrow x^2 + \frac{x}{2} + 2 = 0$

$i.e., \ x^2 + 2 \times x \times \frac{1}{4} + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2 + 2 = 0$

$\Rightarrow \left(x + \frac{1}{4}\right)^2 - \frac{1}{16} + 2 = 0$

$\Rightarrow \left(x + \frac{1}{4}\right)^2 + \frac{31}{16} = 0$

$i.e., \left(x + \frac{1}{4}\right)^2 = -\frac{31}{16}$

Solving Quadratic Equations by Completing the Square

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Ex

2. *Solve* : $x^2 - 8x + 24 = 0$ by completing the square.

$x^2 - 8x + 24 = 0$	Keep all terms containing x on one side. Move the constant to the right.
$x^2 - 8x = -24$	
$x^2 - 8x + \square = -24 + \square$	Get ready to create a perfect square on the left. Balance the equation.
$x^2 - 8x + \boxed{16} = -24 + \boxed{16}$	Take half of the x-term coefficient and square it. Add this value to both sides.
$x^2 - 8x + 16 = -8$	
$(x-4)^2 = -8$	Simplify and write the perfect square on the left.
$x - 4 = \pm\sqrt{-8}$	Take the square root of both sides. Be sure to allow for both plus and minus.
$x = 4 \pm \sqrt{-8} = 4 \pm 2i\sqrt{2}$	
$x = 4 + 2i\sqrt{2}; \quad x = 4 - 2i\sqrt{2}$	Represent the negative radical as an imaginary number and solve for x.

1. *Solve* : $x^2 - 2x - 1 = 0$ by completing the square.

$x^2 - 2x - 1 = 0$	Keep all terms containing x on one side. Move the constant to the right.
$x^2 - 2x = 1$	
$x^2 - 2x + \square = 1 + \square$	Get ready to create a perfect square on the left. Balance the equation.
$x^2 - 2x + \boxed{1} = 1 + \boxed{1}$	Take half of the x-term coefficient and square it. Add this value to both sides.
$x^2 - 2x + 1 = 2$	
$(x-1)^2 = 2$	Simplify and write the perfect square on the left.
$x - 1 = \pm\sqrt{2}$	Take the square root of both sides. Be sure to allow for both plus and minus.
$x = 1 \pm \sqrt{2}$	
$x = 1 + \sqrt{2}; \quad x = 1 - \sqrt{2}$	Solve for x.

3. *Solve* : $5x^2 - 6x = 8$ by completing the square.

$5x^2 - 6x = 8$	Keep all terms containing x on one side. This equation is all set up to start.
$x^2 - \frac{6}{5}x = \frac{8}{5}$	Divide all terms by 5 to create a leading coefficient of one.
$x^2 - \frac{6}{5}x + \square = \frac{8}{5} + \square$	Prepare to get a perfect square on the left. Balance the equation.
$x^2 - \frac{6}{5}x + \boxed{\frac{9}{25}} = \frac{8}{5} + \boxed{\frac{9}{25}}$	Take half of the x-term coefficient and square it. Add this value to both sides.
$x^2 - \frac{6}{5}x + \frac{9}{25} = \frac{40}{25} + \frac{9}{25}$	Simplify
$x^2 - \frac{6}{5}x + \frac{9}{25} = \frac{49}{25}$	Simplify
$\left(x - \frac{3}{5}\right)^2 = \frac{49}{25}$	Write the perfect square on the left.
$x - \frac{3}{5} = \pm\frac{7}{5}$	Take the square root of both sides. Be sure to allow for both plus and minus.
$x = \frac{3}{5} \pm \frac{7}{5} = \frac{3 \pm 7}{5}$	Solve for x.
$x = \frac{10}{5} = 2; \quad x = \frac{-4}{5}$	