Mathematics & Statistics

Academic Year: 2014-2015 Marks: 80
Date & Time: 28th February 2015, 11:00 am
Duration: 3h

Section I

Question 1: [12]

Question 1: Select and write the most appropriate answer from the given alternatives in each of the following sub-questions: [6]

Question 1.1.1: [2]

if $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ then A6= ....

6A
12A
16A
32A

Solution: i. **(d)**

Given that

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

we can write

$$A = 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Here 2 is scalar multiple

Terefore ,A=2 x I, where I= $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

thus , $A^6 = \left\{ 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}^6$

$= 2^6 \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^6$

$= 2^6 \times I^6,$

$=2^6$ x I,since $I^6=I$

$=2^5$ x A [∵ A=2I]

=32A

Question 1.1.2: [2]

The principal solution of $\cos^{-1}\left(-\frac{1}{2}\right)$ is :

$\frac{\pi}{3}$

$\frac{\pi}{6}$

$\frac{2\pi}{3}$

$\frac{3\pi}{2}$

Solution:

The principal solution of $\cos^{-1}\left(-\frac{1}{2}\right) =$ An angle in [0,π], whose cosine is -1/2

$\Rightarrow \cos^{-1}\left(-\frac{1}{2}\right) = \pi - \cos^{-1}\left(\frac{1}{2}\right)$ .....[because $\cos^{-1}(-x) = \pi - \cos x$]

$= \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

Question 1.1.3: If an equation hxy + gx + fy + c = 0 represents a pair of lines, then ________ [2]

fg = ch

gh = cf

Jh = cg

hf= - eg

Solution: (a)

Consider the general equation in second degree,

$a'x^2+2h'xy + b'y^2 + 2g'x + 2f'y + c' = 0$

The above equation will represent a pair of straight lines if,

$a'f'^2 + b'g'^2 + c'h'^2 = 2f'g'h' + a'b'c'$....(1)
Here, the given equation is, hxy + gx + fy + c = 0
Thus, comparing the coefficients, we have,
a' = 0, b' = 0, c' = 0, h' =h/2 , g' =g/2 , f' =f/2 , c' = c/2
Substituting the above values in the condition (1),
we have,

$$(0)\left(\frac{f}{2}\right)^2 + (0)\left(\frac{g}{2}\right)^2 + (c)\left(\frac{h}{2}\right)^2 = 2 \times \frac{f}{2} \times \frac{g}{2} \times \frac{h}{2} + (0) \times (0) \times (0)$$

$$(c)\left(\frac{h}{2}\right)^2 = 2 \times \frac{f}{2} \times \frac{g}{2} \times \frac{h}{2}$$

$$\frac{ch^2}{4} = \frac{fgh}{4}$$

$$ch^2 = fgh$$

$$ch = fg[\because h \neq 0]$$

Question 1.2 | Attempt any THREE of the following [6]

Question 1.2.1: Write the converse and contrapositive of the statement - [2]
“If two triangles are congruent, then their areas are equal.”

Solution: The given statement -
"If two triangles are congruent, then their areas are equal."

- **Converse of the above statement :**
  If the areas of the two triangles are equal, then the triangles are congruent.

- **Contrapositive of the given statement :**
  If the areas of two triangles are not equal then the triangles are not congruent.

Question 1.2.2: Find ‘k' if the sum of slopes of lines represented by equation x^2+ kxy - $3y^2$ = 0 is twice their product. [2]

Solution: Consider the given equation of the straight lines
2 2 x +kxy −3y =0

Sum of the slopes is given by the formula=-2H/B
Comparing the given equation with the

Standard equation,
Ax^2+ 2Hxy + By^2= 0, we have,

H=k/2 ,A=1, B=- 3

Thus, sum of the slopes= $\frac{-2\left(\frac{k}{2}\right)}{-3}=\frac{k}{3}$

Product of the slopes = $\frac{A}{B}=\frac{1}{-3}$

Also given that, sum of the slopes is twice their product

$\frac{k}{3}=2\left(\frac{1}{-3}\right)$

$k=-2$

Question 1.2.3: Find the angle between the planes $r.\left(2\hat{i}+\hat{j}-\hat{k}\right)=3$ and $r.\left(\hat{i}+2\hat{j}+\hat{k}\right)=1$ [2]

Solution: Given planes are:

$r.\left(2\hat{i}+\hat{j}-\hat{k}\right)=3$ and $r.\left(\hat{i}+2\hat{j}+\hat{k}\right)=1$

The angle between two planes with direction ratios,

(a_1, b_1, c_1) and (a_2, b_2, c_2) is

$$\cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

$$\cos\theta=\frac{2\times1+1\times2-1\times1}{\sqrt{2^2+1^2+(-1)^2}\sqrt{1^2+2^2+1^2}}$$

$$\cos\theta=\frac{3}{\sqrt{6}.\sqrt{6}}$$

$$\cos\theta=\frac{3}{6}$$

$$\cos\theta=\frac{1}{2}$$

$$\cos\theta=\cos\left(\frac{\pi}{3}\right)$$

$$\theta=\frac{\pi}{3}$$

Question 1.2.4: The Cartesian equations of line are 3x -1 = 6y + 2 = 1 - z. Find the vector equation of line. [2]

Solution: Given equations of the line are:
3x -1 = 6y +2 = 1 - z
Rewriting the above equation, we have,

$$3\left(x-\frac{1}{3}\right)=6\left(y+\frac{2}{6}\right)=-(z-1)$$

$$\frac{\left(x-\frac{1}{3}\right)}{\frac{1}{3}}=\frac{\left(y+\frac{1}{3}\right)}{\frac{1}{6}}=\frac{(z-1)}{-1}....(1)$$

Now consider the general equation of the line:

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}...(2)$$

Where, l,m and n are the direction ratios of the line and the point (a,b,c) lies on the line.
Compare the equation (1), with the general equation (2),

We have l=1/3, m=1/6 and n=-1

Also, a=1/3, b=-1/3 and c=1

This shows that the given line passes through (1/3, -1/3, 1)

Therefore, the given line passes through the point having

position vector $\bar{a}=\frac{1}{3}\hat{i}-\frac{i}{3}\hat{j}+\hat{k}$ and is parallel to the vector $\bar{b}=\frac{1}{3}\hat{i}+\frac{1}{6}\hat{j}-\hat{k}$

So its vector equation is

$$r=\left(\frac{1}{3}\hat{i}-\frac{1}{3}\hat{j}+\hat{k}\right)+\lambda\left(\frac{1}{3}\hat{i}+\frac{1}{6}\hat{j}-\hat{k}\right)$$

Question 1.2.5: If $\bar{a}=\bar{i}+2\bar{j},\bar{b}=-2\bar{i}+\bar{j},\bar{c}=4\bar{i}+3\bar{j}$, find x and y such that $\bar{c}=x\bar{a}+y\bar{b}$ **[2]**

Solution:

Given that $\vec{a}=\hat{i}+2\hat{j},\ \vec{b}=-2\hat{i}+\hat{j},\ \vec{c}=4\hat{i}+3\hat{j}$

We need to find x and y such that $\vec{c}=x\vec{a}+y\vec{b}$

Substituting the values of a, b and c, in $\vec{c}=x\vec{a}+y\vec{b}$, we have,

$$4\hat{i}+3\hat{j}=x\left(\hat{i}+2\hat{j}\right)+y\left(-2\hat{i}+\hat{j}\right)$$

$$4\hat{i}+3\hat{j}=(x-2y)\hat{i}+(2x+y)\hat{j}$$

Comparing the coefficients of i and j on both the sides, we have,
x-2y= 4
and

2x + y = 3
Solving the above simultaneous equations, we have,
x = 2 and y = -1

Question 2: **[14]**

Question 2.1 | Attempt any TWO of the following **[6]**

Question 2.1.1: If A, B, C, D are (1, 1, 1), (2, I, 3), (3, 2, 2), (3, 3, 4) respectively, then find the volume of parallelepiped with AB, AC and AD as the concurrent edges. **[3]**

Solution: Given that A,B,C and D are (1, 1, 1), (2, 1, 3) , (3, 2, 2) and (3, 3, 4) respectively. We need to find the volume of the parallelepiped with AB, AC and AD as the concurrent edges.

The volume of the parallelepiped whose coterminous edges are a, b and c is

$$\left[\vec{a}\,\vec{b}\,\vec{c}\right] = \vec{a}.\left(\vec{b} \times \vec{c}\right)$$

Given that A,B,C and D are (1, 1, 1) , (2, 1, 3) , (3, 2, 2) and (3, 3, 4)

$$\vec{AB} = (2-1)\hat{i} + (1-1)\hat{j} + (3-1)\hat{k}$$

$$= \hat{i} + 2\hat{k}$$

$$\vec{AC} = (3-1)\hat{i} + (2-1)\hat{j} + (2-1)\hat{k}$$

$$= 2\hat{i} + \hat{j} + \hat{k}$$

$$\vec{AD} = (3-1)\hat{i} + (3-1)\hat{j} + (4-1)\hat{k}$$

$$= 2\hat{i} + 2\hat{j} + 3\hat{k}$$

$$\left[\vec{a}\,\vec{b}\,\vec{c}\right] = \begin{vmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 2 & 2 & 3 \end{vmatrix}$$

=1(3-2)-0+2(4-2)

=1+4

= 5 cubic units

Question 2.1.2: Discuss the statement pattern, using truth table : ~(~p ∧ ~q) ∨ q **[3]**

Solution: Consider the statement pattern: ~ (~ p ∧ ~ q) ∨ q
Thus the truth table of the given logical statement: ~(~p ∧ ~q) ∨ q

p	q	~p	~q	~p∧~q	~(~p ∧ ~q)	~(~p ∧ ~q) ∨ q

T	T	F	F	F	T	T
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	T	F	F

The above statement is **contingency**.

Question 2.1.3: If point C $(\bar{c})$ divides the segment joining the points $A(\bar{a})$ and $B(\bar{b})$ internally in the ratio m : n, then prove that $\bar{c} = \frac{m\bar{b} + n\bar{a}}{m+n}$ **[3]**

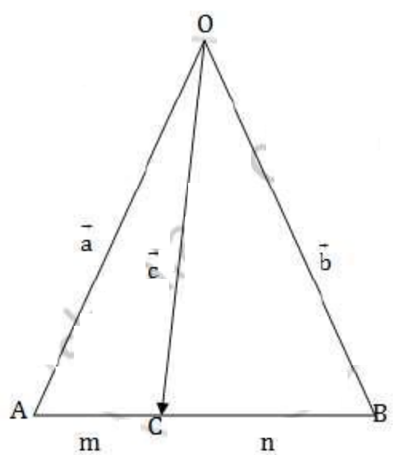
Solution:

Given that $C\left(\vec{c}\right)$ divides the segment joining the points $A\left(\vec{a}\right)$ and $B\left(\vec{b}\right)$

internally in the ratio m : n

We need to prove that $\vec{c} = \frac{m\vec{b} + n\vec{a}}{m+n}$

Consider the following figure



Since (n x length (AC)=m X length (BC)

$$n\vec{AC} = m\vec{CB}$$

$$n\left(\vec{OC} - \vec{OA}\right) = m\left(\vec{OB} - \vec{OC}\right)$$

$$n\left(\vec{c} - \vec{a}\right) = m\left(\vec{b} - \vec{c}\right)$$

$$n\vec{c} - n\vec{a} = m\vec{b} - m\vec{c}$$

$(n+m)\overrightarrow{c} = m\overrightarrow{b} + n\overrightarrow{a}$

$\overrightarrow{c} = \frac{m\overrightarrow{b} + n\overrightarrow{a}}{m+n}$

Hence proved.

Question 2.2 | Attempt any TWO of the following **[8]**

Question 2.2.1: Find the direction cosines of the line perpendicular to the lines whose direction ratios are -2, 1,-1 and -3, - 4, 1 **[4]**

Solution: Let $\bar{a}$ and $\bar{b}$ be the vectors along the lines whose direction ratios are -2, 1, -1 and -3, -4, 1 respectively.

$\therefore \bar{a} = -2\hat{i} + \hat{j} - \hat{k}$ and $\hat{b} = -3\hat{i} - 4\hat{j} + \hat{k}$

A vector perpendicular to both $\bar{a}$ and $\bar{b}$ is given by

$$\bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & -1 \\ -3 & -4 & 1 \end{vmatrix}$$

= $(1-4)\hat{i} - (-2-3)\hat{j} + (8+3)\hat{k}$

= $-3\hat{i} + 5\hat{j} + 11\hat{k}$

$\therefore$ the direction ratios of the required line are -3, 5, 11

Now, $\sqrt{9+25+12} = \sqrt{155}$

Direction cosine of the line are $-\frac{3}{\sqrt{155}}, \frac{5}{\sqrt{155}}, \frac{11}{\sqrt{155}}$.

Question 2.2.2: In any ΔABC if a^2 , b^2 , c^2 are in arithmetic progression, then prove that Cot A, Cot B, Cot C are in arithmetic progression. **[4]**

Solution: Given that a^2 ,b^2 ,c^2 are in arithmetic progression.
We need to prove that cotA, cotB and cotC are in
arithmetic progression.
a^2,b^2, c^2 are in A.P.

$-2a^2, -2b^2, -2c^2$ are in A.P

$(a^2+b^2+c^2)-2a^2, (a^2+b^2+c^2)-2b^2, (a^2+b^2+c^2)-2c^2$ are in A.P

$(b^2+c^2-a^2), (c^2+a^2-b^2), (a^2+b^2-c^2)$ are in A.P

$\frac{b^2+c^2-a^2}{2abc}, \frac{c^2+a^2-b^2}{2abc}, \frac{a^2+b^2-c^2}{2abc}$ are in A.P

$\frac{1}{a}\frac{b^2+c^2-a^2}{2bc}, \frac{1}{b}\frac{c^2+a^2-b^2}{2ac}, \frac{1}{c}\frac{a^2+b^2-c^2}{2ab}$ are in A.P

$\frac{1}{a}\cos A, \frac{1}{b}\cos B, \frac{1}{c}\cos C$ are in A.P

$\frac{k}{a}\cos A, \frac{k}{b}\cos B, \frac{k}{c}\cos C$ are in A.P

$\frac{\cos A}{\sin A}, \frac{\cos B}{\sin B}, \frac{\cos C}{\sin C}$ are in A.P

$\cot A, \cot B, \cot C$ are in A.P

Question 2.2.3: The sum of three numbers is 6. When second number is subtracted from thrice the sum of first and third number, we get number 10. Four times the sum of third number is subtracted from five times the sum of first and second number, the result is 3. Using above information, find these three numbers by matrix method. **[4]**

Solution: Given that the sum of three numbers, x, y and z is 6.
From the given statement, we have,
x + y + z = 6
3 (x + z) - y = 10
5 (x + y) - 4z = 3

Thus, the system of equations are :
x + y + z = 6 (i)
3x - y + 3z = 10 (ii)
5x + 5y - 4z = 3 (iii)

Let us write the above equations in the matrix form as:

$$\begin{bmatrix}1 & 1 & 1\\3 & -1 & 3\\5 & 5 & -4\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\10\\3\end{bmatrix}$$

AX = B

$$\mathrm{A}=\begin{bmatrix}1 & 1 & 1\\3 & -1 & 3\\5 & 5 & -4\end{bmatrix},\mathrm{X}=\begin{bmatrix}x\\y\\z\end{bmatrix},\ \mathrm{B}=\begin{bmatrix}6\\10\\3\end{bmatrix}$$

Applying $R_3 \to R_3 - 5R_1$, we have

$$\begin{bmatrix}1 & 1 & 1\\3 & -1 & 3\\0 & 0 & -9\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\10\\-27\end{bmatrix}$$

Applying $R_2 \to R_2 - 3R_1$, we get

$$\begin{bmatrix}1 & 1 & 1\\0 & -4 & 0\\0 & 0 & -9\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\-8\\-27\end{bmatrix}$$

Thus,
x + y + z = 6 [From(i)]
- 4y = - 8 (iv)
- 9z = - 27 (v)
From equation (iv), we get

y = 2
From equation (v), we get
z = 3
Putting the value of y and z in equation (i), we get x = 6 - 2 - 3 = 1
Hence, numbers are 1, 2, and 3.

Question 3: [14]

Question 3.1 | Attempt any TWO of the following [6]

Question 3.1.1: If θ is the acute angle between the lines represented by equation $ax^2 + 2hxy + by^2 = 0$ then prove that $\tan\theta = \left|\frac{2\sqrt{h^2-ab}}{a+b}\right|, a+b \neq 0$ [3]

Solution: Case (1)
Let m_1 and m_2 are the slopes of the lines represented by the equation $ax^2+2hxy+by^2=0$,
then $m_1 + m_2 = -2h/b$ and $m_1m_2 = a/b$
If θ is the acute angle between the lines,

then $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

now $(m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1m_2$

$$(m_1 - m_2)^2 = \left(\frac{-2h}{b}\right)^2 - 4\left(\frac{a}{b}\right)$$

$$(m_1 - m_2)^2 = \frac{4(h^2 - ab)}{b^2}$$

$$|m_1 - m_2| = \left|\frac{2\sqrt{h^2 - ab}}{b}\right|$$

similarly $1 + m_1m_2 = 1 + \frac{a}{b} = \frac{a+b}{b}$

substituting in $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$, we get

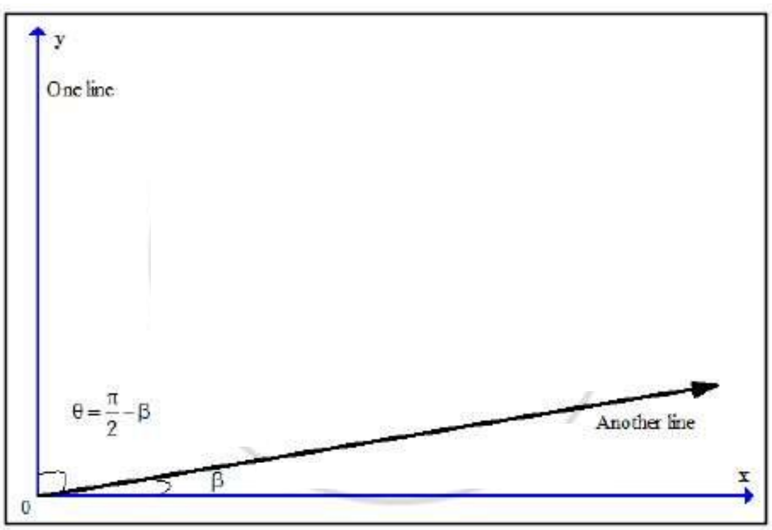
$$\tan\theta = \left|\frac{\frac{2-\sqrt{h^2-ab}}{b}}{\frac{a+b}{b}}\right|$$

$$\tan\theta = \left|\frac{2-\sqrt{h^2-ab}}{a+b}\right|, \text{if } a+b \neq 0$$

Case (2)

If one of the lines is parallel to the y-axis then one of the slopes m_1,m_2, does not exist. As the line passes through the origin so one line parallel is the y-axis, it's equation is x=0 and b=0

The other line is ax+2hy=0 whose slope $tab\beta = -\frac{a}{2h}$



$\therefore$ The acute angle between the pair of lines is $\frac{\pi}{2}-\beta$

$\therefore \tan\theta = \left|\tan\left(\frac{\pi}{2}-\beta\right)\right| = |\cot\beta| = \left|\frac{2h}{a}\right|$

put $b=0$ in $\tan\theta = \left|\frac{2\sqrt{h^2-ab}}{a+b}\right|$, we get $\tan\theta = \left|\frac{2h}{a}\right|$

Hence $\tan\theta = \left|\frac{2\sqrt{h^2-ab}}{a+b}\right|$ is valid in both the cases.

Question 3.1.2: [3]

If the lines $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}$ and $\frac{x-3}{1}=\frac{y-k}{2}=\frac{z}{1}$ intersect each other then find value of k

Solution:

Let $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}=u$ where is any constant.

So for any point on this line has co-ordinates in the form (2u+1,3u-1,4u+1)

$\frac{x-3}{1}=\frac{y-k}{2}=\frac{z}{1}=v$

So for any point on this line has co-ordinates in the form (v+3,2v+k,v).

Point of intersection of these two lines will have co-ordinates of the form

(2u +1, 3u −1,4u +1) and (v +3, 2v + k,v) .

Equating the x, y and z co-ordinates for both the forms we get three equations

2u+1=v+3

2u-v=2............(1)

3u-1=2v+k

3u-2v=k+1.......(2)

4u+1=v

4u-v=-1...........(3)

Subtracting equation (1)from equation(3) we get,

2u = -3

u=-3/2

Substitute value of u in equation (1) we get, 2(-3/2) - v=2

v=-5

Substitute value of v and in equation (2) we get,

3(-3/2) - 2(-5)=k+1

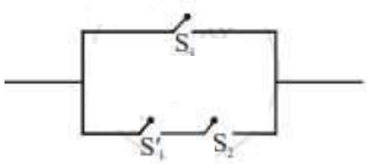
k=9/2

the value of k is 9/2

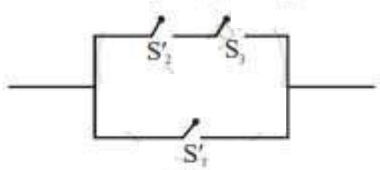
Question 3.1.3: Construct the switching circuit for the following statement : [p v (~ p ∧ q)] v [(- q ∧ r) v ~ p] **[3]**

Solution: Let, p : Switch S_1 is closed.
~ p : Switch S_1 is open.
q : Switch S_2 is closed.
~ q : Switch S_2 is open
r : Switch S_3 is closed.
~ r : Switch S_3 is open.
Now,

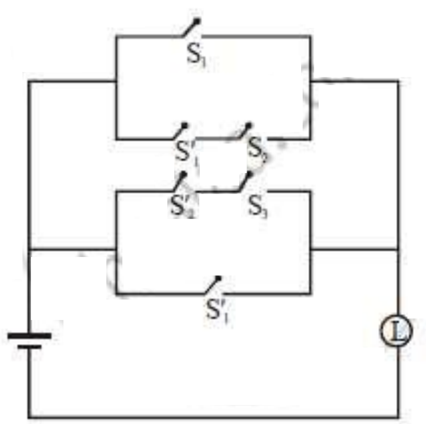
$p \vee (\sim p \wedge q)$



$(\sim q \wedge r) \vee \sim p$



$\therefore \ [p \vee (\sim p \wedge q)] \vee [(\sim q \wedge r) \vee \sim p]$



Question 3.2: Attempt any TWO of the following [8]

Question 3.2.1: Find the general solution of : cos x - sin x = 1. [3]

Solution: cos x - sin x = 1

Dividing by $\sqrt{1^2+(-1)} = \sqrt{2}$

$$\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x = \frac{1}{\sqrt{2}}$$

$$\cos\frac{\pi}{4}\cos\ x - \sin\frac{\pi}{4}\sin\ x = \frac{1}{\sqrt{2}}$$

$$\cos\left(x+\frac{\pi}{4}\right) = \cos\frac{\pi}{4} \quad ...(i)$$

The general solution of cos θ = cos α is θ = 2nπ ± $\frac{\pi}{4}; n \in z$

∴ The genera; solution of equation (i) given by

$$x + \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4}; n \in z$$

$$x = 2n\pi; x = 2n\pi - \frac{\pi}{2}; n \in z$$

Question 3.2.2: Find the equations of the planes parallel to the plane x-2y + 2z-4 = 0, which are at a unit distance from the point (1,2, 3). [3]

Solution: The equation of the planes parallel to the plane x −2y + 2z − 4 = 0
are of the form x-2y+2z+k=0

The distance of a plane ax+by+cz+ λ from a point (x_1,y_1,z_1) is given by

$$d=\left|\frac{9ax_1+by_1+cz_1+\lambda}{\sqrt{a^2+b^2+c^2}}\right|$$

It is given the plane x-2y+2z+k=0 at an unit distance from the point (1, 2, 3).

$$d=\left|\frac{1-2(2)+2(3)+k}{\sqrt{1^2+(-2)^2+(2)^2}}\right|$$

$$1=\left|\frac{k+3}{3}\right|$$

∴ |k+3|=|3|

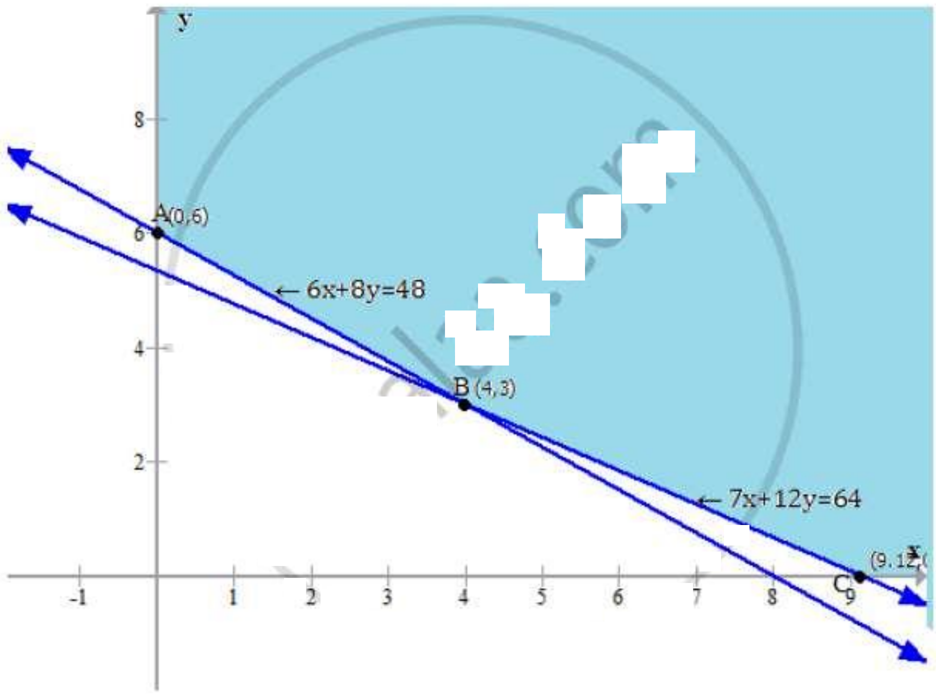
∴ k=0 or k=-6

The equation of the planes parallel to the plane x-2y+2z-4=0

are of x-2y+2z=0 and x-2y+2z=6

Question 3.2.3: A diet of a sick person must contain at least 48 units of vitamin A and 64 units of vitamin B. Two foods F_1 and F_2 are available. Food F_1 costs Rs. 6 per unit and food F_2 costs Rs. 10 per unit. One unit of food F_1 contains 6 units of vitamin A and 7 units of vitamin B. One unit of food F_2 contains 8 units of vitamin A and 12 units of vitamin B. Find the minimum cost for the diet that consists of mixture of these two foods and also meeting the minimal nutritional requirements. **[3]**

Solution: Let x and y be two different types of food.
Thus, our objective function is minimise the cost
Z = 6x + 10y, subject to the constraints,
6x + 8y ≥48
7x + 12y ≥ 64
Plotting the above lines in a graph, we have,



Thus, the region above ABC is unbounded.
Let us check the value of the function at the corner points A, B and C

Corner point	Value of Z = 6x + 10y
(0,6)	Z = 0 + 10 x 6 = 60
(4,3)	Z = 6 x 4 + 10 x 3 = 54
(64/7,0)	Z = 6 x 64/7+ 10 x 0 = 54.85

Minimum of the function is at 4, 3
Minimum cost of the optimum diet is Rs. 54

Section II

Question 4: [12]

Question 4.1 | Select and write the most appropriate answer from the given alternatives in each of the following sub-questions [6]

Question 4.1.1: A random variable X has the following probability distribution:
then E(X)=.................... [2]
0.8
0.9
0.7
1.1

Solution: (a) 0.8

X = x	-2	-1	0	1	2	3
P(x)	0.1	0.1	0.2	0.2	0.3	0.1

$$E(X) = \sum x_i P(x_i)$$

= (-2) x 0.1 + (-1) x 0.1+ 0 x 0.2 +1x 0.2+ 2 x 0.3 + 3 x 0.1

= - 0.2 - 0.1 + 0 + 0.2 + 0.6 + 0.3

= 0.8

Question 4.1.2: [2]

If $\int_0^\alpha 3x^2 dx = 8$ then the value of α is :
(a) 0
(b) -2
(c) 2
(d) ±2

Solution: (c)

$$\int_0^{\alpha} 3x^2 dx = 8$$

$$\Rightarrow \left[\frac{3x^3}{3}\right]_0^{\alpha} = 8$$

$$\Rightarrow [x^3]_0^{\alpha} = 8$$

$\Rightarrow \alpha^3 = 8$

$\therefore \alpha = 2$

Question 4.1.3: [2]

The differential equation of $y = \frac{c}{x} + c^2$ is :

(a) $x^4\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} = y$

(b) $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

(c) $x^3\left(\frac{dy}{dx}\right)^2 + x\frac{dy}{dx} = y$

(d) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0$

Solution: (a)

$y = \frac{c}{x} + c^2$.......... (i)

Differentiating w.r.t.x,

$$\frac{dy}{dx} = \frac{-c}{x^2} + 0$$

$c = -x^2 \frac{dy}{dx}$ ....(2)

Putting in equation (1)

$$y = \frac{-x^2 \frac{dy}{dx}}{x} + \left(-x^2 \frac{dy}{dx}\right)^2$$

$$y = -x\frac{dy}{dx} + x^4\left(\frac{dy}{dx}\right)^2$$

$$x^4\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} = y$$

Question 4.2 | Attempt any THREE of the following [6]

Question 4.2.1: [2]

Evaluate : $\int e^x \left[\frac{\sqrt{1-x^2}\sin^{-1}x+1}{\sqrt{1-x^2}}\right]dx$

Solution:

$$\int e^x \left[\frac{\sqrt{1-x^2}\sin^{-1}x+1}{\sqrt{1-x^2}}\right]dx$$

$$= \int e^x \left[\sin^{-1}x + \frac{1}{\sqrt{1-x^2}}\right]dx$$

We know that $\int e^x[f(x)+f\prime(x)]dx = e^x . f(x) + c$

$$= e^x . \sin^{-1}x + c$$

Question 4.2.2: [2]

If $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + ... \infty}}}$, then show that $\frac{dy}{dx} = \frac{\cos x}{2y-1}$

Solution:

$$y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + ... \infty}}}$$

Let $y = \sqrt{\sin x + y}$

$$y^2 = \sin x + y$$

Differentiat ing w.r.t.x.

$$2y. \frac{dy}{dx} = \cos x + \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{\cos x}{2y-1}$$

Question 4.2.3: [2]

Evaluate : $\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos x}dx$

Solution:

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos x} dx$$

Solving the integral without limits,

$$\int \frac{1}{1+\cos x} dx$$

$$= \int \frac{1}{2\cos^2\left(\frac{x}{2}\right)} dx$$

$$= \frac{1}{2} \int \sec^2\left(\frac{x}{2}\right) dx$$

$$= \frac{1}{2}\left[\frac{\tan\left(\frac{x}{2}\right)}{\frac{1}{2}}\right] + C$$

$$= \tan\left(\frac{x}{2}\right) + C$$

Substituting the limits,we get

$$= \left[\tan\left(\frac{x}{2}\right)\right]_0^{\frac{\pi}{2}}$$

$$= \left[\tan\left(\frac{\pi}{4}\right) - \tan 0\right]$$

= 1

Question 4.2.4: **[2]**

If y=e^{ax} ,show that $x\frac{dy}{dx} = y \log y$

Solution:

$$y = e^{ax}$$

$$y = e^{ax}..............(i)$$

$$\log y = ax............. .(ii)$$

$$\frac{dy}{dx} = ae^{ax}$$

$$\frac{dy}{dx} = ay$$

$$x\frac{dy}{dx} = axy$$

$$x\frac{dy}{dx} = y \log y$$

Question 4.2.5: A fair coin is tossed five times. Find the probability that it shows exactly three times head. [2]

Solution: Let X be the Radom variable.
let 'p' be the success and 'q' be the failure

p=1/2, q=1/2

p(Coin shows 3 heads)

$=p(x=3)=5c_3p^3q^2$

$= 10\ (1/2)^3(1/2)^2$

=10/32

=5/16

Question 5: [14]

Question 5.1: Attempt any TWO of the following [6]

Question 5.1.1: Integrate : $\sec^3 x$ w. r. t. x. [3]

Solution:

$$I = \int \sec^3 x dx$$

$$I = \int \sec x . \sec^2 x dx$$

$$I = \sec x . \int \sec^2 x dx - \int \left[\frac{d}{dx}(\sec x) . \int \sec^2 x dx \right] dx$$

$$I = \sec x . \tan x - \int \sec x . \tan x . \tan x dx$$

$$I = \sec x . \tan x - \int \sec x \left(\sec^2 x - 1 \right) dx$$

$$I = \sec x . \tan x - \int \left[\sec^3 x - \sec x \right] dx$$

$$I = \sec x . \tan x - \int \sec^3 x + \int \sec x dx$$

$$I = \sec x . \tan x - I + \log|\sec x + \tan x| + c$$

$$2I = \sec x . \tan x + \log|\sec x + \tan x| + c$$

$$\therefore I = \frac{1}{2} (\sec x . \tan x + \log|\sec x + \tan x|) + c$$

Question 5.1.2: [3]

If y = $(\tan^{-1} x)^2$, show that

$$(1+x^2)^2\frac{d^2y}{dx^2}+2x(1+x^2)\frac{dy}{dx}-2=0$$

Solution:

y = $\tan^{-1}x$

$y_1 = \frac{1}{1+x^2}$

$y_2 = \frac{d(1+x^2)^{-1}}{dx} = -\frac{1}{(1+x^2)^2}\times(2x)$

$y_2 = \frac{-2x}{(1+x^2)^2}$

$y_2\,(1+x^2) = -2x\left(\frac{1}{1+x^2}\right)$

$y_2\,(1+x^2) = -2\,xy_1$

$y_2\,(1+x^2) + 2xy_1 = 0$

Question 5.1.3: [3]

If $f(x)=\left[\tan\left(\frac{\pi}{4}+x\right)\right]^{\frac{1}{x}}$,

= k ,for x=0

is continuous at x=0 , find k.

Solution:

$$f(x)=\left[\tan\left(\frac{\pi}{4}+x\right)\right]^{\frac{1}{x}},\ \ \text{for}\ \ x\neq 0$$

$$f(0)=k$$

Since f(x) is continuos at x=0

$$\lim_{x\to 0} f(x)=f(0)$$

$$\lim_{x\to 0}\left[\tan\left(\frac{\pi}{4}+x\right)\right]^{\frac{1}{x}}=k$$

$$\lim_{x\to 0}\left[\frac{1+\tan x}{1-\tan x}\right]^{\frac{1}{x}}=k$$

$$\lim_{x\to 0}\left[1+\frac{1+\tan x}{1-\tan x}-1\right]^{\frac{1}{x}}=k$$

$$\lim_{x\to 0}\left[1+\frac{1+\tan x-1+\tan x}{1-\tan x}\right]^{\frac{1}{x}}=k$$

$$\lim_{x\to 0}\left[1+\frac{2\tan x}{1-\tan x}\right]^{\frac{1}{x}}=k$$

$$\lim_{x\to 0}\left[1+\frac{2\tan x}{1-\tan x}\right]^{\frac{1}{\frac{2\tan x}{1-\tan x}}\times\left(\frac{2\tan x}{x.(1-\tan x)}\right)}=k$$

$$e^{\lim_{x\to 0}\frac{2\tan x}{x.(1-\tan x)}}=k\left\{\because\lim_{x\to 0}\left[1+x\right]^{\frac{1}{x}}=e\right\}$$

$$e^{2\lim_{x\to 0}\frac{\tan x}{x}\times\lim_{x\to 0}\frac{1}{1-\tan x}}=k\left\{\because\lim_{x\to 0}\left[\frac{\tan x}{x}\right]=1\right\}$$

$$e^{2\times 1\times\frac{1}{1-0}}=k$$

$$k=e^2$$

Question 5.2 | Attempt any TWO of the following : [8]

Question 5.2.1: Find the co-ordinates of the points on the curve y=x-(4/x) where the tangents are parallel to the line y=2x [4]

Solution:

$y = x - \frac{4}{x} ...(1)$

Differentiat ing w.r.t. x,

$\frac{dy}{dx} = 1 + \frac{4}{x^2}$

$\left|\frac{dy}{dx}\right|_{x_1,y_1} = 1 + \frac{4}{x_1^2} = m$

Slope of the tangent , $m = 1 + \frac{4}{x_1^2}$

Also Slope of the line y= 2x, m_1 =2

m is parallel to m_1

$1 + \frac{4}{x_1^2} = 2$

$\frac{4}{x_1^2} = 1$

$x_1^2 = 4$

$\therefore x_1 = 2, x_2 = -2$

using eq (1)

$y_1 = 2 - \left(\frac{4}{2}\right) = 0$

$y_2 = -2 - \left(\frac{4}{-2}\right) = 0$

Coordinates of the point of contact are (2,0) and (-2,0).

Question 5.2.2: [4]

Prove that $\int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log\left|x + \sqrt{x^2 - a^2}\right| + c$

Solution:

Let $I=\int\sqrt{x^2-a^2}dx$

$$I=\int\sqrt{x^2-a^2}.1.\,dx$$

$$I=\sqrt{x^2-a^2}.\int dx-\int\left[\frac{d}{dx}\left(\sqrt{x^2-a^2}\right)\int dx\right]dx$$

$$I=x\sqrt{x^2-a^2}-\int\left[\frac{2x}{2\sqrt{x^2-a^2}}x\right]dx$$

$$I=x\sqrt{x^2-a^2}-\int\left[\frac{x^2}{\sqrt{x^2-a^2}}\right]dx$$

$$I=x\sqrt{x^2-a^2}-\int\left[\frac{x^2-a^2+a^2}{\sqrt{x^2-a^2}}\right]dx$$

$$I=x\sqrt{x^2-a^2}-\int\frac{x^2-a^2}{\sqrt{x^2-a^2}}dx+a^2\int\frac{dx}{\sqrt{x^2-a^2}}$$

$$I=x\sqrt{x^2-a^2}-\int\sqrt{x^2-a^2}dx+a^2\int\frac{dx}{\sqrt{x^2-a^2}}$$

$$I=x\sqrt{x^2-a^2}-I+a^2\int\frac{dx}{\sqrt{x^2-a^2}}$$

$$2I=x\sqrt{x^2-a^2}+a^2\log\left|x+\sqrt{x^2-a^2}\right|+C\prime$$

$$I=\frac{x\sqrt{x^2-a^2}}{2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+\frac{C'}{2}$$

$$I=\frac{x\sqrt{x^2-a^2}}{2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$$

Question 5.2.3: **[4]**

Evaluate : $\int_0^{\pi}\frac{x\sin x}{1+\sin x}dx$

Solution:

Let $I = \int_0^\pi \frac{x \sin x}{1 + \sin x} dx$

$= \int_0^\pi \frac{(\pi - x)\sin(\pi - x)}{1 + \sin(\pi - x)} dx \left[\because \int_0^a f(x)dx = \int_0^a f(a - x)dx\right]$

$= \int_0^\pi \frac{(\pi - x)\sin x}{1 + \sin x} dx$

$= \int_0^\pi \frac{\pi \sin x}{1 + \sin x} dx - I$

$I = \int_0^\pi \frac{\pi \sin x}{1 + \sin x} dx - I$

$2I = \int_0^\pi \frac{\pi \sin x.(1 - \sin x)}{(1 + \sin x)(1 - \sin x)} dx$

$2I = \int_0^\pi \frac{\pi \sin x.(1 - \sin x)}{1 - \sin^2 x} dx$

$\frac{2I}{\pi} = \int_0^\pi \frac{\sin x.(1 - \sin x)}{\cos^2 x} dx$

$\frac{2I}{\pi} = \int_0^\pi \frac{\sin x. - \sin^2 x}{\cos^2 x} dx$

$\frac{2I}{\pi} = \int_0^\pi \frac{\sin x}{\cos^2 x} dx - \int_0^\pi \frac{\sin^2 x}{\cos^2 x} dx$

$\frac{2I}{\pi} = \int_0^\pi \sec x.\tan x dx - \int_0^\pi \tan^2 x dx$

$\frac{2I}{\pi} = [\sec x]_0^\pi - \int_0^\pi (\sec^2 x - 1) dx$

$$\frac{2I}{\pi} = [\sec\pi - \sec 0] - \int_0^{\pi} \sec^2 x . dx + \int_0^{\pi} 1dx$$

$$\frac{2I}{\pi} = [-1 - 1] - [\tan x]_0^{\pi} - [x]_0^{\pi}$$

$$\frac{2I}{\pi} = [-2] - [\tan\pi - \tan 0] + \pi$$

$$\frac{2I}{\pi} = [-2] - 0 + \pi$$

$$\therefore I = \frac{(\pi - 2)\pi}{2}$$

Question 6: [14]

Question 6.1: Attempt any two of the following [6]

Question 6.1.1: Find a and b, so that the function f(x) defined by [3]
f(x)=-2sin x, for -π≤ x ≤ -π/2
=a sin x+b, for -π/2≤ x ≤ π/2
=cos x, for π/2≤ x ≤ π
is continuous on [- π, π]

Solution:

$$f(x) = -2\sin x, \quad \text{for} \quad -\pi \leq x \leq -\frac{\pi}{2}$$

$$= a\sin x + b, \quad \text{for} \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$= \cos x, \quad \text{for} \quad \frac{\pi}{2} \leq x < \pi$$

f(x) is continuous for x=-π/2

RHL

$$= \lim_{x \to -\frac{\pi}{2}} a\sin x + b$$

$$= a\sin\left(-\frac{\pi}{2}\right) + b$$

=-a+b

$$f\left(-\frac{\pi}{2}\right) = -2\sin\left(-\frac{\pi}{2}\right)$$

$$\therefore -a + b = 2 \ldots\ldots.(i)\left[\because f(x) \ \text{ is continuous for } \ x = -\frac{x}{2}\right]$$

$f(x)$ is continuous for $x = \frac{\pi}{2}$

LHL,

$= \lim_{x \to \frac{\pi}{2}} a \sin x + b$

$= a \sin\left(\frac{\pi}{2}\right) + b$

$\frac{2I}{\pi} = [-2] - 0 + \pi$

$\therefore I = \frac{(\pi - 2)\pi}{2}$

Question 6.1.2: [3]

If $\log_{10}\left(\frac{x^3 - y^3}{x^3 + y^3}\right) = 2 \text{then show that } \frac{dy}{dx} = \frac{-99x^2}{101y^2}$

Solution:

$\log_{10}\left(\frac{x^3 - y^3}{x^3 + y^3}\right) = 2$

Convert logarithmic form into exponential form,

$\left(\frac{x^3 - y^3}{x^3 + y^3}\right) = 10^2$

$\left(\frac{x^3 - y^3}{x^3 + y^3}\right) = 100$

$x^3 - y^3 = 100x^3 + 100y^3$

$\therefore - 99x^3 - 101y^3 = 0$

$- 101y^3 = 99x^3$

Differentiat ing w.r.t. x on both sides

$- 99(3x^2) - 101(3y^2 \frac{dy}{dx}) = 0$

$\therefore 99x^2 + 101y^2 \frac{dy}{dx} = 0$

$\therefore 101y^2 \frac{dy}{dx} = - 99x^2$

$\therefore \frac{dy}{dx} = \frac{-99x^2}{101y^2}$

Question 6.1.3: [3]

Let the p. m. f. (probability mass function) of random variable x be

$$p(x) = \binom{4}{x}\left(\frac{5}{9}\right)^{x}\left(\frac{4}{9}\right)^{4-x}, x = 0, 1, 2, 3, 4$$

=0 otherwise

find E(x) and var (x)

Solution:

p(x) = $\binom{4}{x}\left(\frac{5}{9}\right)^{x}\left(\frac{4}{9}\right)^{4-x}$, x = 0,1,2,....,4

Comparing with p(x) = $\binom{n}{x}(p)^{x}(q)^{n-x}$

$\therefore n = 4, p = \frac{5}{9}, q = \frac{4}{9}$

E(x) = np = $4 \times \frac{5}{9} = \frac{20}{9}$

V(x) = npq = $4 \times \frac{5}{9} \times \frac{4}{9} = \frac{80}{81}$

Question 6.2: Attempt any two of the following [8]

Question 6.2.1: Examine the maxima and minima of the function f(x) = $2x^3 - 21x^2 + 36x - 20$. Also, find the maximum and minimum values of f(x). [4]

Solution 1:

$f(x) = 2x^3 - 21x^2 + 36x - 20$

$f'(x) = 6x^2 - 42x + 36$

For finding critical points, we take f'(x)=0

$\therefore 6x^2 - 42x + 36 = 0$

$x^2 - 7x + 6 = 0$

(x-6)(x-1)=0

For finding the maxima and minima, find f''(x)

f'(x)=12x-42

for x=6

f''(6)=30>0

Minima

for x=1

$f''(1)=-30<0$

Maxima

Maximum values of f(x) for x=1

f(1)=-3

minimum values of f(x) for x=6

f(6)=-128

∴ the maximum values of the function is -3 and the minimum value of the function is -128.

Solution 2: $f(x) = 2x^3 - 21x^2 + 36x - 20$

$$\therefore f'(x) = 2(3x^2) - 21(2x) + 36(1) - 1$$

$$= 6x^2 - 42x + 36 = 6(x^2 - 7x + 6)$$

= 6(x - 1)(x - 6)

f has a maxima/minima if f'(x) = 0

i.e if 6(x - 1)(x-6) = 0

i.e if x - 1= 0 or x - 6 = 0

i.e if x = 0 or x = 6

Now f"(x) = 6(2x) - 42(1) = 12x - 42

∴ f"(1) = 12(1) - 42 = -30

∴ f"(1) < 0

Hence, f has a maximum at x = 1, by the second derivative test.

Also f"(6) = 12(6)- 42 = 30

∴ f"(6) > 0

Hence, f has a minimum at x = 6, by the second derivative test.

Now, the maximum value of f at 1,

$$f(1) = 2(1^3) - 21(1^2) + 36(1) - 20$$

= 2-21+36-20 = -3

and minimum value of f at x = 6

$$f(6) = 2(6^3) - 21(6^2) + 36(1) - 20$$

= 432 - 756 + 216 - 20 = -128

Question 6.2.2: Solve the differential equation $(x^2 + y^2)dx - 2xydy = 0$ **[4]**

Solution: $(x^2 + y^2)dx - 2xydy = 0$

$(x^2 + y^2)\,dx = 2xydy$

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy} \text{.........(i)}$$

The equation is a homogeneous equation

Let y= vx,

Differentiat ing w.r.t. x, we get,

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy} \quad \text{from} \quad (i)$$

$$v + x\frac{dv}{dx} = \frac{x^2 + (vx)^2}{2x.(vx)}$$

$$v + x\frac{dv}{dx} = \frac{1 + v^2}{2v}$$

$$x\frac{dv}{dx} = \frac{1 + v^2}{2v} - v$$

$$x\frac{dv}{dx} = \frac{1 + v^2 - 2v^2}{2v}$$

$$x\frac{dv}{dx} = \frac{1 - v^2}{2v}$$

$$\frac{2v}{1 - v^2}dv = \frac{1}{x}dx \text{.......(ii)}$$

Which is in variables separatable form

∴ Integrating both sides, we get

$$\int \frac{2v}{1 - v^2}dv = \int \frac{1}{x}dx + c_1$$

$$\therefore -\log\left|1 - v^2\right| = \log|x| + \log c$$

$$\therefore \log\left|x\left(1 - v^2\right)\right| = \log|c|$$

$$\therefore x\left(1 - v^2\right) = c$$

Resubstituting $v = \dfrac{y}{x}$ we get

$$x\left(1-\frac{y^2}{x^2}\right)=c$$

$$x\left(\frac{x^2-y^2}{x^2}\right)=c$$

$\therefore x^2-y^2=cx$, where c is constant

which is the required general solution

Question 6.2.3: Given the p. d. f. (probability density function) of a continuous random variable x as : **[4]**

$$f(x)=\frac{x^2}{3},-1$$

= 0 , otherwise

Determine the c. d. f. (cumulative distribution function) of x and hence find P(x < 1), P(x ≤ -2), P(x > 0), P(1 < x < 2)

Solution: c.d.f. of the continuous random variable is given by

$$F(x)=\int_{-1}^{x}\frac{y^2}{3}dx$$

$$=\left[\frac{y^3}{9}\right]_{-1}^{x}$$

$$=\frac{x^3+1}{9},x\in R$$

Consider $P(X<1)=F(1)=(1^3+1)/9=2/9$

$$P(x\leq-2)=0$$

$$P(X>0)=1-P(X\leq0)$$

$$=1-F(0)$$

$$=1-\left(\frac{0}{9}+\frac{1}{9}\right)$$

$$=\frac{8}{9}$$

$$P(1<x<2)=F(2)-F(1)$$

$$=1-\left(\frac{1}{9}+\frac{1}{9}\right)$$

$$=\frac{7}{9}$$