8. Ratios and Proportions

Proportions

1. A **proportion** is a comparison of ratios.
2. A proportion is an equation that states that two ratios are equal, such as
$$\frac{4}{8}=\frac{1}{2}$$
3. Proportions always have an **EQUAL** sign!
4. A proportion can be written in two ways:
$$\frac{4}{8}=\frac{1}{2} \quad \text{or} \quad 4:8=1:2$$
Both are read "**4 is to 8 as 1 is to 2**".
5. In each proportion the first and last terms (4 and 2) are called the **extremes.** The second and third terms (8 and 1) are called the **means.**

You can tell if a simple proportion is true by just examining the fractions. If the fractions both reduce to the same value, the proportion is true.	$\frac{5}{15}=\frac{2}{6}$ This is a true proportion, since both fractions reduce to 1/3.
You can often use this same approach when solving for a missing part of a simple proportion. Remember that both fractions must represent the same value. Notice how we solve this problem by getting a common denominator for the two fractions.	$\frac{1}{3}=\frac{x}{15}$ $\frac{5}{15}=\frac{x}{15}$ To change the denominator of 3 to 15 requires multiplying by 5. The SAME must be done to the top to keep the fractions equal. Answer: x = 5

This simple approach may not be sufficient when working with more complex proportions. You need a rule:
Some people call this rule **Cross Multiply!!**
A more precise statement of the rule is:
RULE: In a true proportion, the product of the means equals the product of the extremes.
Proportions can also be solved by multiplying each side of the proportion by the common denominator for both fractions.

Example 1: Solve for x algebraically in this proportion:
$$\frac{25}{x}=\frac{5}{2}$$
Solution:

Method 1:
Appling the rule that the "in a true proportion, the product of the means equals the product of the extremes".
$$\frac{25}{x}\;\;\frac{5}{2}$$
$$5x=(25)(2)$$
$$5x=50$$
$$x=10 \text{ Answer}$$

Method 2:
Multiplying by the common denominator, 2x.
$$2x\cdot\frac{25}{x}=\frac{5}{2}\cdot 2x$$
$$2\bullet 25=5\bullet x$$
$$50=5x$$
$$10=x \text{ Answer}$$

Example 2: The length of a stadium is 100 yards and its width is 75 yards. If 1 inch represents 25 yards, what would be the dimensions of the stadium drawn on a sheet of paper?

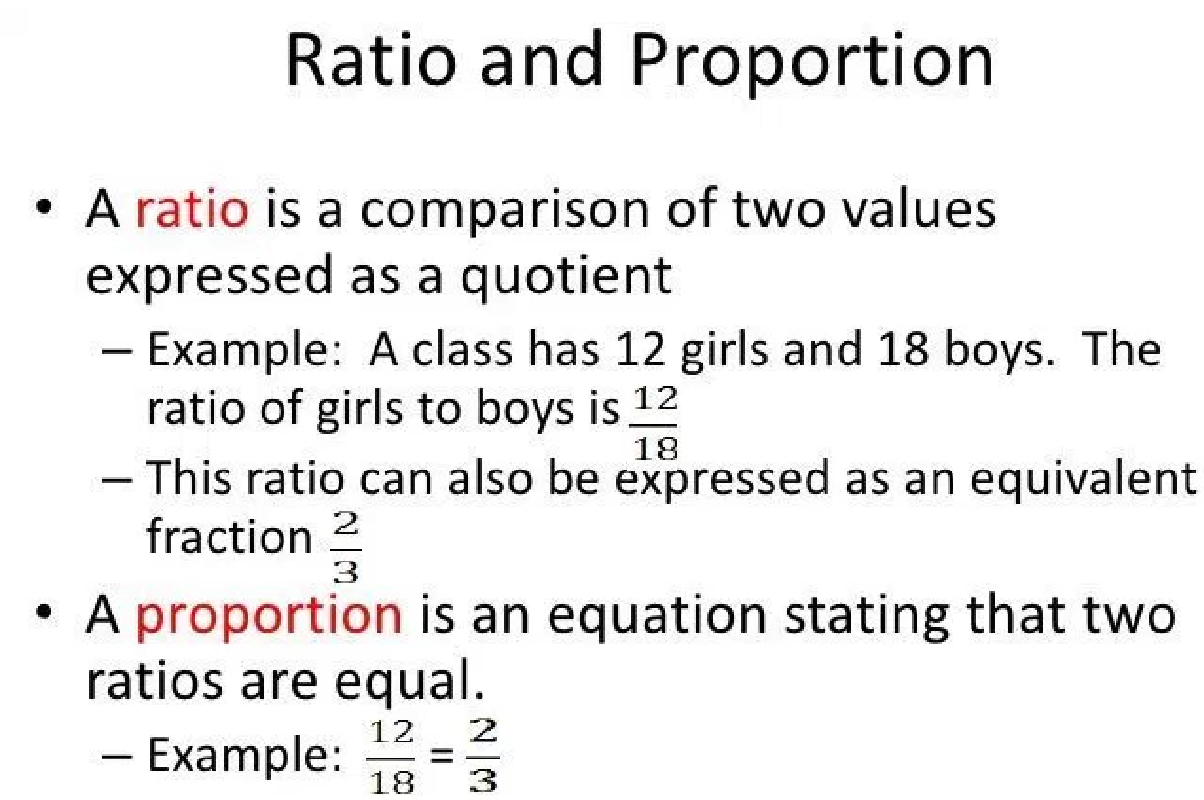
Solution: This problem can be solved by an intuitive approach, such as:

100 yards by 75 yards
100 yards = 4 inches (HINT: 100 / 25)
75 yards = 3 inches (HINT: 75 / 25)
Therefore, the dimensions would be 4 inches by 3 inches.

Solution by proportion: (Notice that the inches are all on the top and the yards are all on the bottom for this solution. Other combinations are possible.)

Length:	Width:
$\frac{1}{25} = \frac{x}{100}$	$\frac{1}{25} = \frac{y}{75}$
$25x = 100$	$25y = 75$
$x = 4$ inches	$y = 3$ inches

What is a Ratio and Proportion



1. **Ratio:**

   The ratio of two quantities a and b in the same units, is the fraction $\frac{a}{b}$ and we write it as $a : b$.

   In the ratio $a : b$, we call a as the first term or **antecedent** and b, the second term or **consequent**.

   Eg. The ratio 5 : 9 represents $\frac{5}{9}$ with antecedent = 5, consequent = 9.

   Rule: The multiplication or division of each term of a ratio by the same non-zero number does not affect the ratio.

   Eg. 4 : 5 = 8 : 10 = 12 : 15. Also, 4 : 6 = 2 : 3.

2. **Proportion:**

   The equality of two ratios is called proportion.

   If $a : b = c : d$, we write $a : b :: c : d$ and we say that a, b, c, d are in proportion.

   Here a and d are called **extremes**, while b and c are called **mean terms**.

   Product of means = Product of extremes.

   Thus, $a : b :: c : d \Leftrightarrow (b \times c) = (a \times d)$.

3. **Fourth Proportional:**

   If $a : b = c : d$, then d is called the fourth proportional to a, b, c.

   Third Proportional:

   $a : b = c : d$, then c is called the third proportion to a and b.

   Mean Proportional:

   Mean proportional between a and b is $\sqrt{ab}$.

4. **Comparison of Ratios:**

   We say that $(a : b) > (c : d) \quad \Leftrightarrow \quad \frac{a}{b} > \frac{c}{d}$.

   Compounded Ratio:

   The compounded ratio of the ratios: $(a : b)$, $(c : d)$, $(e : f)$ is $(ace : bdf)$.

5. **Duplicate Ratios:**

   Duplicate ratio of $(a : b)$ is $(a^2 : b^2)$.

   Sub-duplicate ratio of $(a : b)$ is $(\sqrt{a} : \sqrt{b})$.

   Triplicate ratio of $(a : b)$ is $(a^3 : b^3)$.

   Sub-triplicate ratio of $(a : b)$ is $(a^{1/3} : b^{1/3})$.

   If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{a-b} = \frac{c+d}{c-d}$. [componendo and dividendo]

6. **Variations:**

   We say that x is directly proportional to y, if $x = ky$ for some constant k and we write, $x \propto y$.

   We say that x is inversely proportional to y, if $xy = k$ for some constant k and we write, $x \propto \frac{1}{y}$.

RATIO

In our day-to-day life, we compare one quantity with another quantity of the same kind by using the ‘method of subtraction’ and ‘method of division’.
Example: The height of Seema is 1 m 67 cm and that of Reema is 1 m 62 cm. The difference in their heights is:
167 cm – 162 cm = 5 cm
Thus, we say Seema is 5 cm taller than Reema.
Similarly, suppose the weight of Seema is 60 kg and the weight of Reema is 50 kg. We can compare their weights by division, i.e.,

$$\frac{\text{Weight of Seema}}{\text{Weight of Reema}} = \frac{50 \text{ kg}}{60 \text{ kg}}$$

$$= \frac{6}{5}$$

So, the weight of Seema is $\frac{6}{5}$ times the weight of Reema.
When we compare two similar quantities by division, the comparison is called the **‘ratio’**. It is denoted by **‘:’** and read as **‘is to’**.

Example: $\frac{5}{8}$ = 5 : 8 (read as 5 is to 8).
As shown in the above example a ratio is like a fraction or comparison of two numbers, where a numerator and a denominator is separated by a colon (:). The first term or the quantity (5), called antecedent means 'that precedes' and the second term, called consequent means 'that follows'.

$$\frac{5}{8} = 5 : 8$$

5 ↓ Antecedent   8 ↓ Consequent

Properties of Ratio

When we compare two quantities, the following points must be taken care of:

1. A ratio is usually expressed in its simplest form.
   Example:
   $\frac{12}{36} = \frac{1}{3} = 1 : 3$
2. Both the quantities should be in the same unit. So, ratio is a number with no unit involved in it.
   Example: 200 g : 2 kg
   = 200 g : 2000 g
   $\frac{200}{2000} = \frac{1}{10} = 1 : 10$
3. The order of the quantities of a ratio is very important.
   Example: 5 : 6 is different from 6 : 5.
   They are not equal.
   5 : 6 ≠ 6 : 5

Equivalent Ratios

A ratio is similar to a fraction. So, if we divide or multiply the numerator (antecedent) and denominator (consequent) by the same number, we get an equivalent fraction (ratio).

Example: 5 : 6 = $\frac{5}{6}$

If we multiply $\frac{5}{6}$ by the same number, say 3, we get the ratio

$$\frac{5 \times 3}{6 \times 3} = \frac{15}{18} \quad \left(\begin{array}{l}\text{Multiplying by the} \\ \text{same number 3}\end{array}\right)$$

= 15 : 18 which is equivalent to 5 : 6.

Similarly,

$$\frac{12}{36} = \frac{12 \div 12}{36 \div 12} = \frac{1}{3} \quad \left(\begin{array}{l}\text{Dividing by the} \\ \text{same number 12}\end{array}\right)$$

So, 12 : 36 is equivalent to 1 : 3.

Comparison of Ratios

To compare two ratios, we have to follow these steps:
Step 1: Convert each ratio into a fraction in its simplest form.
Step 2: Find the LCM of denominators of the fractions obtained in step 1.
Step 3: Convert the denominators equal to LCM obtained in step 2 in each fraction.
Step 4: Now, compare the numerators of the fractions; the fraction with a greater numerator will be greater than the other.

Example 1: Compare the ratio 5: 6 and 7: 8.

Solution: Here, 5 : 6 = 5/6 and 7 : 8 = 7/8.

Now, compare the two fractions $\frac{5}{6}$ and $\frac{7}{8}$ making their denominators equal.

LCM of 6 and 8 = 24

$$\therefore \quad \frac{5}{6} = \frac{5 \times 4}{6 \times 4} = \frac{20}{24}$$

$$\text{and} \quad \frac{7}{8} = \frac{7 \times 3}{8 \times 3} = \frac{21}{24}$$

$$\because \quad 21 > 20,$$

$$\text{So,} \quad \frac{21}{24} > \frac{20}{24} \quad \text{or,} \quad \frac{7}{8} > \frac{5}{6}$$

Hence, 7 : 8 > 5 : 6

Example 2: Convert the ratio 125 : 275 in its simplest form.
Solution:

$$\text{Here,} \quad 125 : 275 = \frac{125}{275}$$

HCF of 125 and 275 is 25.

$$\text{So,} \quad \frac{125 \div 25}{275 \div 25} = \frac{5}{11}$$

So, 5 : 11 is the simplest form of 125 : 275.

Example 3: Write the following ratios in descending order:
4 : 3, 4 : 7, 7 : 10
Solution: We have,

$$4:3 = \frac{4}{3}, 4:7 = \frac{4}{7}, 7:10 = \frac{7}{10}$$

Now, compare these fractions making their denominators equal.

LCM of 3, 7, and 10 = 210

$$\therefore \quad \frac{4}{3} = \frac{4\times70}{3\times70} = \frac{280}{210},$$

$$\frac{4}{7} = \frac{4\times30}{7\times30} = \frac{120}{210},$$

$$\frac{7}{10} = \frac{7\times21}{10\times21} = \frac{147}{210}$$

$$\because \quad 280 > 147 > 120$$

$$\text{So,} \quad \frac{280}{210} > \frac{147}{210} > \frac{120}{210}$$

$$\text{or,} \quad \frac{4}{3} > \frac{7}{10} > \frac{4}{7}$$

Hence, $4:3 > 7:10 > 4:7$

Example 4: Mr Lai divides a sum of Rs. 1500 between his two sons in the ratio 2 : 3. How much money does each son get?
Solution: Let the first son get 2x and the second son get 3x.

$$2x + 3x = ₹\,1500$$

$$5x = ₹\,1500$$

$$x = ₹\,300$$

$$\text{First son's share} = 2 \times ₹\,300 = ₹\,600$$

$$\text{Second son's share} = 3 \times ₹\,300 = ₹\,900$$

Example 5: Two numbers are in the ratio 3 : 5 and their sum is 96. Find the numbers.

Solution: Let the first number be 3x and the second number be 5x.
Then, their sum = 3x + 5x = 96
8x = 96
x = 12
The first number = 3x = 3 × 12 = 36
The second number 5x = 5 × 12 = 60

Example 6: In a pencil box there are 100 pencils. Out of which 60 are red pencils and the restare blue pencils. Find the ratio of:
(a) blue pencils to the total number of pencils.
(b) red pencils to the total number of pencils.
(c) red pencils to blue pencils.

Solution: Total number of pencils in the pencil box = 100
Number of red pencils = 60
∴ Number of blue pencils = 100 – 60 = 40

(a) The ratio of blue pencils to the total number of pencils

$$= 40 : 100 = \frac{40}{100} = \frac{2}{5} = 2 : 5$$

(b) The ratio of red pencils to the total number of pencils

$$= 60 : 100 = \frac{60}{100} = \frac{3}{5} = 3 : 5$$

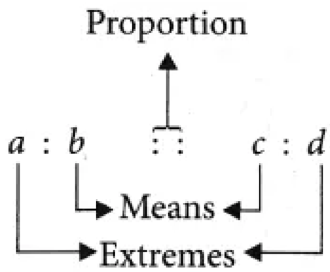
(c) The ratio of red pencils to blue pencils

$$= 60 : 40 = \frac{60}{40} = \frac{3}{2} = 3 : 2$$

PROPORTION

A proportion is an equation with a ratio on each side. It is a statement that two ratios are equal. When two ratios are equal then such type of equality of ratios is called proportion and their terms are said to be in proportion.

Example: If the cost of 3 pens is Rs. 21, and that of 6 pens is Rs. 42, then the ratio of pens is 3 : 6, and the ratio of their costs is 21 : 42. Thus, 3 : 6 = 21:42. Therefore, the terms 3,6,21, and 42 are in proportion.
Generally, the four terms, a, b, c, and d are in proportion if a : b = c : d.
Thus, a : b : : c : d means a/b = c/d or ad = ad = bc
Conversely, if ad = be, then a/b = c/d or a : b : : c : d
Here, a is the first term, b is the second term, c is the third term, and d is the fourth term. The first and the fourth terms are called extreme terms or extremes and the second and third terms are called middle terms or means.



Product of extremes	=	Product of means
ad	=	bc

Continued proportion

In a proportion, if the second and third terms are equal then the proportion is called continued proportion.
Example: If 2 : 4 : : 4 : 8, then we say that 2, 4, 8 are in continued proportion.

Mean proportion

If the terms a, b, and c are in continued proportion, then 'b' is called the mean proportion of a and c.
Example: If a, b, c are in continued proportion, then

$$a : b :: b : c$$

$$\downarrow$$

$$\frac{a}{b} = \frac{b}{c}$$

Mean proportion = b^2 = ac

Third proportion
If the terms a, b, c are in continued proportion, then c is called the third proportion.

Properties of proportions:
Convertendo: If a : b :: c : d ,then a : (a - b) :: c : (c – d).
Invertendo: If $\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{b}{a} = \frac{d}{c}$.
Alternendo: If $\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a}{c} = \frac{b}{d}$.
Componendo: If $\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a+b}{b} = \frac{c+d}{d}$.
Dividendo: $\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a-b}{b} = \frac{c-d}{d}$
Componendo and Dividendo: If $\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a+b}{a-b} = \frac{c+d}{c-d}$.

Example 1: Find x, where x : 3 : : 4 : 12.
Solution: Here, x, 3, 4, and 12 are in proportion.

$$\therefore \quad \frac{x}{3} = \frac{4}{12}$$

$$\text{or,} \quad 12 \times x = 3 \times 4$$

$$\text{or,} \quad x = \frac{3 \times 4}{12}$$

$$\text{or,} \quad x = \frac{12}{12}$$

$$\text{or,} \quad x = 1$$

Example 2: Find the third proportion of 10 and 20.
Solution: If a, b, c are in proportion, then b^2 = ac.

$$\therefore \quad (20)^2 = 10 \times c$$

$$\text{or,} \quad 10c = 400$$

$$\text{or,} \quad c = \frac{400}{10} = 40$$

So, the third proportion = 40

Example 3: Find the value of x, if 14, 42, x are in continued proportion.
Solution: Here 14, 42, and x are in proportion.

$\therefore \quad 14 : 42 :: 42 : x$

or, $\dfrac{14}{42} = \dfrac{42}{x}$

or, $14 \times x = 42 \times 42$

or, $x = \dfrac{42 \times 42}{14}$

or, $x = 126$

Example 4: The cost of 1 dozen bananas is Rs. 24. How much do 50 bananas cost?
Solution: Let the cost of 50 bananas be x.

Bananas	*Cost* (in ₹)
12	24
50	x

$12 : 24 :: 50 : x$

or, $\dfrac{12}{24} = \dfrac{50}{x}$

or, $12 \times x = 24 \times 50$

or, $x = \dfrac{24 \times 50}{12}$

or, $x = ₹\ 100$

Example 5: Rajesh drives his car at a constant speed of 12 km per 10 minutes. How long will he take to cover 48 km?
Solution: Let Rajesh take x mins, to cover 48 km.

Speed (*in km*)	*Time* (*in minutes*)
12	10
48	x

$12 : 10 :: 48 : x$

$12 \times x = 10 \times 48$

$x = \dfrac{10 \times 48}{12}$

$= 40$ minutes