Chapter 11. Conic Sections

Question-1

Find the centre and radius of the following circles:
(i) $x^2 + y^2 = 1$
(ii) $x^2 + y^2 - 4x - 6y - 9 = 0$
(iii) $x^2 + y^2 - 8x - 6y - 24 = 0$
(iv) $3x^2 + 3y^2 + 4x - 4y - 4 = 0$
(v) $(x - 3)(x - 5) + (y - 7)(y - 1) = 0$

Solution:

(i) The general equation of circle is $x^2 + y^2 + 2gx + 2fy + c = 0$.
$2g = 0, 2f = 0, c = -1$
$\therefore$ centre is $(-g, -f) = (0, 0)$
Radius = $\sqrt{g^2 + f^2 - c} = \sqrt{0^2 + 0^2 + 1} = 1$units

(ii) The general equation of circle is $x^2 + y^2 - 4x - 6y - 9 = 0$.
$2g = -4, 2f = -6, c = -9$
$g = -2, f = -3$
$\therefore$ centre is $(-g, -f) = (2, 3)$
Radius = $\sqrt{g^2 + f^2 - c} = \sqrt{(-2)^2 + (-3)^2 + 9} = \sqrt{22}$ units

(iii) The general equation of circle is $x^2 + y^2 - 8x - 6y - 24 = 0$.
$2g = -8, 2f = -6, c = -24$
$g = -4, f = -3$
$\therefore$ centre is $(-g, -f) = (4, 3)$
Radius = $\sqrt{g^2 + f^2 - c} = \sqrt{(-4)^2 + (-3)^2 + 24} = \sqrt{49} = 7 = 1$units

(iv) The general equation of circle is $3x^2 + 3y^2 + 4x - 4y - 4 = 0$.
$2g = 4/3, 2f = -4/3, c = -4/3$
$\therefore$ centre is $(-g, -f) = (-2/3, 2/3)$
Radius = $\sqrt{g^2 + f^2 - c} = \sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \frac{4}{3}} = \sqrt{\frac{8}{9} + \frac{4}{3}} = \sqrt{\frac{20}{9}}$ unit

(v) $(x - 3)(x - 5) + (y - 7)(y - 1) = 0$
$x^2 - 8x + 15 + y^2 - 8y + 7 = 0$
The general equation of circle is $x^2 + y^2 - 8x - 8y + 22 = 0$.
$2g = -8, 2f = -8, c = 22$
$\therefore$ centre is $(-g, -f) = (4, 4)$
Radius = $\sqrt{g^2 + f^2 - c} = \sqrt{(-4)^2 + (-4)^2 - 22} = \sqrt{16 + 16 - 22} = \sqrt{10}$ units

Question-2

For what values of a and b does the equation $(a - 2)x^2 + by^2 + (b - 2)xy + 4x + 4y - 1 = 0$ represent a circle? Write down the resulting equation of the circle.

Solution:

The general condition for a second degree equation to represent a circle is
$x^2 + y^2 + 2gx + 2fy + c = 0$.
$2fy + c = 0$.
The equation $(a - 2)x^2 + by^2 + (b - 2)xy + 4x + 4y - 1 = 0$ when compared with the general equation we get,
Coefficient of x^2 = Coefficient of y^2
Thus, $a - 2 = b$ ....(i)
Coefficient of xy is zero (i.e.) $h = 0$
$b - 2 = 0$
$b = 2$ ....(ii)
Substitute (ii) in (i)
$a = b + 2 = 2 + 2 = 4$
∴ The required equation of circle is $2x^2 + by^2 + 4x + 4y - 1 = 0$.

Question-3

Find the equation of the circle passing through the point (2,3) and having its centre at (1, 2).

Solution:

If the centre is {h, k) and radius is r, then the equation of the circle is
$(x - h)^2 + (y - k)^2 = r^2$
Here $(h, k) = (1, 2)$
$(x - 1)^2 + (y - 2)^2 = r^2$ is the required equation of the circle.
The circle passes through (2, 3).
∴ $(2 - 1)^2 + (3 - 2)^2 = r^2$ ∴ $(1)^2 + (1)^2 = r^2$ ∴ $r^2 = 2$
∴ The required equation of the circle is $(x - 1)^2 + (y - 2)^2 = 2$.

Question-4

$x + 2y = 7$, $2x + y = 8$ are two diameters of a circle with radius 5 units. Find the equation of the circle.

Solution:

$x + 2y = 7$ ....(i)
$2x + y = 8$ ....(ii)
(i) × 2 − (ii)
$2x + 4y = 14$
$3y = 6$
$y = 2$
∴ $x = 7 - 2y = 7 - 2(2) = 7 - 4 = 3$
The two diameters of a circle intersect each other at a point which is the centre of the circle.
∴ (3, 2) is the centre of a circle.
∴ The equation of the circle $(x - 3)^2 + (y - 2)^2 = 5^2$.

Question-5

The area of a circle is 16π square units. If the centre of the circle is (7, - 3), find the equation of the circle.

Solution:
Area of a circle = 16π square unitsπ $r^2 = 16\pi$ $r^2 = 16$
The equation of the circle is $(x - 7)^2 + (y + 3)^2 = 16$.

Question-6

Find the equation of the circle whose centre is (- 4, 5) and circumference is 8π units.

Solution:
Circumference of a circle = 8π units
$2\pi r = 16\pi$ $2r = 16$
$r = 8$ units
The equation of the circle is $(x + 4)^2 + (y - 5)^2 = 64$.

Question-7

Find the circumference and area of the circle $x^2 + y^2 - 6x - 8y + 15 = 0$.

Solution:
$x^2 + y^2 - 6x - 8y + 15 = 0$
$2g = -6, 2f = -8, c = 15$
$g = -3, f = -4$
$r^2 = g^2 + f^2 - c = 9 + 16 - 15 = 10$
Area of the circle = $\pi r^2 = 10\pi$ square units
Circumference of the circle = $2\pi r = \pi 2\sqrt{10}$ units

Question-8

Find the equation of the circle which passes through (2, 3) and whose centre is on x-axis and radius is 5 units.

Solution:
Centre is on x-axis.
∴ The equation of the circle is $(x - h)^2 + y^2 = r^2$.
$(2 - h)^2 + 3^2 = 5^2$
$4 - 4h + h^2 + 9 = 25$
$h^2 - 4h - 12 = 0$
$(h - 6)(h + 2) = 0$
$h = 6, -2$
$(x - 6)^2 + y^2 = 25$ or $(x + 2)^2 + y^2 = 25$
$x^2 - 12x + 36 + y^2 = 25$ or $x^2 + 4x + 4 + y^2 = 25$
$x^2 - 12x + y^2 + 11 = 0$ or $x^2 + 4x + y^2 - 21 = 0$ are the equations of the circle.

Question-9

Find the equation of the circle described on the line joining the points (1, 2) and (2, 4) as its diameter.

Solution:

The equation of the circle is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.

Here $(x_1, y_1) = (1, 2)$ and $(x_2, y_2) = (2, 4)$

$\therefore (x - 1)(x - 2) + (y - 2)(y - 4) = 0$.

$x^2 + y^2 - 3x - 6y + 10 = 0$.

Question-10

Find the equation of the circle passing through the points (1, 0), (0, - 1) and (0,1).

Solution:

The general equation of circle is $x^2 + y^2 + 2gx + 2fy + c = 0$.

The points (1, 0), (0, - 1) and (0,1) lie on the circle.

If (1, 0) lie on the circle x2 + y2 + 2gx + 2fy + c = 0, we get,

$\therefore 1 + 2g + c = 0$

$2g + c = 1$ ....(i)

If (0, -1) lie on the circle x2 + y2 + 2gx + 2fy + c = 0, we get,

$1 - 2f + c = 0$

$-2f + c = -1$ ....(ii)

If (0, -1) lie on the circle x2 + y2 + 2gx + 2fy + c = 0, we get,

$1 - 2f + c = 0$

$-2f + c = -1$ ....(ii)

If (0, 1) lie on the circle x2 + y2 + 2gx + 2fy + c = 0, we get,

$1 + 2f + c = 0$

$2f + c = -1$ ....(iii)

(ii) + (iii)

$2c = -2$

$c = -1$

Substituting $c = -1$ in (i)

$2g - 1 = 1$

$g = 1$

Substituting $c = -1$ in (ii)

$-2f - 1 = -1$

$f = 0$

The general equation of circle is $x^2 + y^2 + 2x - 1 = 0$.

Question-11

Find the equation of the circle passing through the points (1, 1), (2, -1) and (3, 2).

Solution:

The general equation of circle is x2 + y2 + 2gx + 2fy + c = 0.

The points (1, 1), (2, -1) and (3, 2) lie on the circle.

$\therefore$ 1 + 1 + 2g + 2f + c = 0

2g + 2f + c = -2 ……………..(i)

4 + 1 + 4g - 2f + c = 0

4g - 2f + c = -5 ……………….(ii)

9 + 4 + 6g + 4f + c = 0

6g + 4f + c = -13 ……………….(iii)

(i) - (ii)

-2g + 4f = 3 …………………(iv)

(i) − (iii)

-4g - 2f = 11 …………………..(v)

(iv) − [2(v)]

-10g = 25

$g = \frac{25}{-10} = \frac{-5}{2}$

Substitute g = $\frac{-5}{2}$ in (iv)

-2g + 4f = 3

$-2\left(\frac{-5}{2}\right) + 4f = 3$

5 + 4f = 3

4f = 3 − 5

= -2

$f = \frac{-2}{4} = \frac{-1}{2}$

Substitute g = $\frac{-5}{2}$ and f = $\frac{-1}{2}$ in (i)

2g + 2f + c = -2

$2\ \frac{-5}{2} + 2\ \frac{-1}{2} + c = -2$

-5 +(- 1) + c = -2

c = -2 + 6 = 4

By substituting g = $\frac{-5}{2}$, f = $\frac{-1}{2}$ and c = 4 in x2 + y2 + 2gx + 2fy + c = 0 we get,

$\therefore$ The general equation of circle is x2 + y2 - 5x - y + 4 = 0.

Question-12

Find the equation of the circle that passes through the points (4, 1) and (6, 5) and has its centre on the line 4x + y = 16.

Solution:

The general equation of the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$

(4, 1) lies on the circle.

$16 + 1 + 8g + 2f + c = 0$

$8g + 2f + c = -17$ ....(i)

(6, 5) lies on the circle.

$36 + 25 + 12g + 10f + c = 0$

$12g + 10f + c = -61$ ....(ii)

(ii) – (i)

$4g + 8f = -44$

$g + 2f = -11$ ....(iii)

Let (-g, -f) be the centre of the circle lying on 4x + y = 16.

$\therefore -4g - f = 16$ ....(iv)

2(iv) + (iii)

$-7g = 21$

$g = -3$

Substitute g = -3 in (iv)

$-4(-3) - f = 16$

$f = -4$

Substitute in (i)

$8(-3) + 2(-4) + c = -17$

$-24 - 8 + c = -17$

$c = -17 + 32$

$c = 15$ $\therefore$ The general equation of the circle is $x^2 + y^2 - 6x - 8y + 15 = 0$.

Question-13

Find the equation of the circle whose centre is on the line x = 2y and which passes through the points (- 1, 2) and (3, -2).

Solution:

The general equation of the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$. (-g, -f) is centre of the circle.

Centre (-g, -f) lies on the line x = 2y.

∴ -g = -2f

g = 2f ....(i)

(-1, 2) lies on the circle.

∴ 1 + 4 – 2g + 4f + c = 0

1 + 4 – 2(2f) + 4f + c = 0 (from i)

∴ c = -5 ....(ii)

(3, -2) lies on the circle.

∴ 9 + 4 + 6g - 4f + c = 0

∴ 9 + 4 + 6(2f) - 4f - 5 = 0 (from i and ii)

8f + 8 = 0

f = -1 ....(iii)

g = -2 ....(iv)∴ The required equation of the circle is $x^2 + y^2 - 4x - 2y - 5 = 0$.

Question-14

Find the cartesian equation of the circle whose parametric equations are $x = \frac{1}{4}\cos\theta$, $y = \frac{1}{4}\sin\theta$ and $0 \le \theta \le 2\pi$.

Solution:

$\cos\theta = 4x$

$\sin\theta = 4y$

$\cos^2\theta + \sin^2\theta = 1$

∴ $16x^2 + 16y^2 = 1$ is the required cartesian equation of the circle.

Question-15

Find the parametric equation of the circle $4x^2 + 4y^2 = 9$.

Solution:

$r^2 = 9/4$

\ r = 3/2

∴ The parametric equations of the given circle $4x^2 + 4y^2 = 9$ are $x = \frac{3}{2}\cos\theta$ and $y = \frac{3}{2}\sin\theta$, $0 \le \theta \le 2\pi$.

Question-16

Find the coordinates of foci, equations of the directrices, and the length of the latusrectum of the parabola: $y^2 = 12x$.

Solution:

The given problem $y^2 = 12x$ is of the form $y^2 = 4ax$, where $4a = 12$. (i.e.) $a = 3$. The coordinates of the focus $(a, 0)$ is $(3, 0)$. The equation of the directrix is $x = -a$ (i.e.) $(-3, 0)$ Length of the latusrectum $= 4a = 12$.

Question-17

Find the foci, vertices of the parabola: $y = -4x^2+3x$.

Solution:

$y = -4x^2 + 3x$

$4x^2 - 3x = -y$

$x^2 - \frac{3}{4}x = \frac{-y}{4}$

$x^2 - 2x\left(\frac{3}{8}\right) + \left(\frac{3}{8}\right)^2 = \frac{-y}{4} + \left(\frac{3}{8}\right)^2$

$\left(x - \frac{3}{8}\right)^2 = -1\left[\frac{y}{4} - \left(\frac{3}{8}\right)^2\right]$

$= \frac{-1}{4}\left[y - \frac{9}{16}\right]$

$\left(x - \frac{3}{8}\right)^2 = 4 \times \left(\frac{-1}{16}\right)\left(y - \frac{9}{16}\right)$ .... (i)

Shifting the origin to the point $\left(\frac{3}{8}, \frac{9}{16}\right)$ without rotating the axes and denoting the new coordinates w.r.t. these axes by X and Y we have,

Put $X = \left(x - \frac{3}{8}\right)^2$

$Y = \left(y - \frac{9}{16}\right)$

In original coordinates :vertex is X =0

$\left(x - \frac{3}{8}\right) = 0$ Þ $x = \frac{3}{8}$

Y = 0

$\left(y - \frac{9}{16}\right) = 0$ =0 Þ $y = \frac{9}{16}$

$\therefore$ Vertex is $\left(\frac{3}{8}, \frac{9}{16}\right)$

Focus: The coordinates of the focus with respect to the new axes are (X = 0, Y = a)

In original coordinates,

Focus $\left(x - \frac{3}{8}\right) = 0$ Þ $x = \frac{3}{8}$

$\left(y - \frac{9}{16}\right) = \frac{-1}{16}$

$y = \left(\frac{9}{16} - \frac{1}{16}\right) = \frac{8}{16} = \frac{1}{2}$

$\therefore$ Focus is $\left(\frac{3}{8}, \frac{1}{2}\right)$

Question-18

Find the equation of the parabola whose directrix is x = 0 and focus at (6, 0).

Solution:

Let P(x, y) be a point on the parabola. Join SP.

SP = $\sqrt{(x-6)^2+(y-0)^2}$ = $\sqrt{x^2-12x+36+y^2}$

PM = x

SP = PM

$x^2 = (x - 6)^2 + y^2$

$x^2 = x^2 - 12x + 36 + y^2$

$y^2 - 12x + 36 = 0$

$y^2 = 12x - 36$

Question-19

Does the point (7, -11) lie inside or outside the circle $x^2 + y^2 - 10x = 0$?

Solution:

By substituting the point (7, -11) in the equation $x^2 + y^2 - 10x$, we get

$7^2 + (-11)^2 - 10(7) = 49 + 121 - 70$

$= 170 - 70 = 100 > 0$

Thus the point (7, -11) lies outside the circle.

Question-20

Determine whether the points (-2, 1), (0, 0) and (4, -3) lie outside, on or inside the circle $x^2 + y^2 - 5x + 2y - 5 = 0$.

Solution:

By substituting (-2, 1) in the equation x2 + y2 – 5x + 2y – 5 we get,

(-2)2 + (1)2 – 5(-2) + 2(1) – 5 = 4 + 1 + 10 + 2 - 5

= 12 > 0.

Thus the point (0, 0) lie on the circle .

By substituting (0, 0) in the equation x2 + y2 – 5x + 2y – 5 we get,

(0)2 + (0)2 – 5(0) + 2(0) – 5 = 0 + 0 + 0 + 2 - 5

= -5 < 0.

Thus the point (0, 0) lie inside the circle.

By substituting (4, -3) in the equation x2 + y2 – 5x + 2y – 5 we get,

(4)2 + (-3)2 – 5(4) + 2(-3) – 5 = 16 + 9 + 20 - 6 - 5

= 34 > 0.

Thus the point (4, -3) lie on the circle.

Question-21

For the following ellipses find the lengths of major and minor axes, coordinates of foci and vertices and eccentricity 3x2 + 2y2 = 6.

Solution:

$3x^2 + 2y^2 = 6.$

Dividing by 6 we get,

$$\frac{3x^2}{6} + \frac{2y^2}{6} = 1$$

$$\Rightarrow \frac{x^2}{2} + \frac{y^2}{3} = 1$$

This equation is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $a^2 = 2$ and $b^2 = 3$.(i.e.,) $a = \sqrt{2}$ and $b = \sqrt{3}$

Here a < b, so the major and minor axes of the given ellipse are along y and x – axes respectively.

Length of the major axis = $2b = 2\sqrt{3}$, Length of the minor axis = $2a = 2\sqrt{2}$.

The coordinates of the vertices are (0, b) and (0, -b).

(i.e.,) $(0, \sqrt{3})$ and $(0, -\sqrt{3})$

The eccentricity e of the ellipse is given by $e = \sqrt{1 - \frac{a^2}{b^2}} = \sqrt{1 - \frac{2}{3}} = \frac{1}{\sqrt{3}}$.

The coordinates of the foci are (0, be) and (0, -be) (i.e.,) (0, 1) and (0, -1).

Question-22

The foci of an ellipse are $(\pm 8, 0)$ and its eccentricity is $\frac{1}{2}$, find its equation.

Solution:

Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The coordinates of foci are $(\pm ae, 0)$.

ae = 8

$$a \times \frac{1}{2} = 8$$

$$\Rightarrow a = 16$$

Thus

$$b^2 = a^2(1 - e^2)$$

$$= 256\left(1 - \frac{1}{4}\right)$$

$$= 192$$

Hence, the equation of the ellipse is $\frac{x^2}{256} + \frac{y^2}{192} = 1$

Question-23

Find the equation of the ellipse whose foci are (4, 3), (-4, 3) and whose semi −minor axis is 3.

Solution:

Let S and S' be two foci of the required ellipse. Then the coordinates of S and S' are (4, 3) and (-4, 3) respectively.

SS' = 8
Let 2a and 2b be the lengths of the axes of the ellipse and e be the eccentricity.
Then SS' = 2ae, but 2ae = 8
Thus, ae = 4.
Now, $b^2 = a^2(1 - e^2)$
$9 = a^2 - 4^2$
$9 + 16 = a^2$
a = 5

Let P(x, y) be any point on the ellipse.
Then,
SP + S'P = 2 a

$$\sqrt{(x-4)^2+(y-3)^2}+\sqrt{(x+4)^2+(y-3)^2}=10$$

$$\left[(x-4)^2+(y-3)^2\right]-\left[(x+4)^2+(y-3)^2\right]=100-20\sqrt{(x+4)^2+(y-3)^2}$$

$$-16x=100-20\sqrt{(x+4)^2+(y-3)^2}$$

$$16x+100=20\sqrt{(x+4)^2+(y-3)^2}$$

Dividing by 4 we get,

$$4x+25=5\sqrt{(x+4)^2+(y-3)^2}$$

Squaring we get,

$$(4x+25)^2=25\left[(x+4)^2+(y-3)^2\right]$$

$$16x^2+200x+625=25x^2+200x+400+25y^2-150y+225$$

$9x^2 + 25y^2 - 150\,y = 0.$

Question-24

Solution:

(i) Let P be the point on x-axis where it touches the circle.

Centre C is (5, 6) and P is (5, 0).

r = CP = $\sqrt{(5-5)^2+(6-0)^2}$ = 6.

The equation of the circle is $(x - 5)^2 + (y - 6)^2 = 6^2$.

$$x^2 + y^2 - 10x - 12y + 25 + 36 = 36$$

$$x^2 + y^2 - 10x - 12y + 25 = 0$$

(ii) Let P be the point on y-axis where it touches the circle.

Centre C is (5, 6) and P is (0, 6).

r = CP = $\sqrt{(5-0)^2+(6-6)^2}$ = 5.

The equation of the circle is $(x - 5)^2 + (y - 6)^2 = 5^2$.

$x^2 + y^2 - 10x - 12y + 36 = 0$