6 Numerical Method

6.1 Introduction

Mathematical methods used to solve equations or evaluate integrals or solve differential equations can be classified broadly into two types.

1. Analytical Methods
2. Numerical Methods

6.1.1 Analytical Methods

Analytical methods are those which by an analysis of the equation obtain a solution directly as a readymade formulae in terms of say, the coefficients present in the equations.

Example 1.

Solve $ax^2 + bx + c$ analytically

Analytical solution: $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 2.

Evaluate $\int x^2 \, dx$ analytically

Analytical solution: $$\int_1^2 x^2 \, dx = \left[\frac{x^3}{3}\right]_1^2 = \frac{2^3 - 1^3}{3} = \frac{7}{3}$$

Example 3.

Solve the differential equation

$\frac{dy}{dx} - 2y = 0$ with initial condition $y(0) = 3$.

Analytical solution: $$\int \frac{dy}{y} = \int 2 \, dx$$

$\Rightarrow$ $$\log y = 2x$$

$$y = ce^{2x}$$

$$y(0) = 3$$

$\Rightarrow$ $$c = 3$$

$\therefore$ $y = 3e^{2x}$ is the required analytical solution.

6.1.2 Numerical Methods

Those same problems could also be solved numerically as we shall see in this chapter.

In numerical solution, instead of directly writing the answer in terms of some formulae, we perform stepwise calculations using some algorithms or numerical procedures (usually on a computer) and arrive at the same results.

The advantage of numerical methods is that usually these procedures work on a much wider range of problems as compared to analytical solutions which work only on a limited class of problems.

For example, there are no analytical solutions available for polynomials of degree 4 or more. Whereas numerical methods can be used to solve polynomial equations of any degree.

Also numerical solutions can be used on linear as well as nonlinear equations, whereas analytical solutions usually fail for nonlinear equations.

With the advent of computers and huge computational (number crunching) power, numerical methods have largely replaced analytical methods of solution and have extended the power of mathematical methods to solving a much wider class of practical problems which occur in simulation and modeling, than it was possible before using analytical methods only.

Although Numerical Methods exist to solve so many types of commonly occurring mathematical problems, we shall focus on four problems in particular in this book, where numerical methods are successfully applied.

1. Solution of system of linear equations
2. Solution of algebraic and transcendental equations in single variable
3. Evaluation of definite integrals
4. Solution of ordinary differential equations

The advantage of numerical methods is its applicability to a wider class of mathematical problems, a disadvantage of numerical methods is that these methods introduce errors in varying degrees into the solution, thereby making them approximate. These errors however, can be controlled and contained within some ordinary tolerance local.

1.1.3 Errors in Numerical Methods

1. **Round-off Error:** It occurs due to limited storage space available inside computer for storing mantissa part of a floating point number due to which these numbers are either chopped off or rounded after so many significant digits.
2. **Truncation Error:** It occurs due to usage of fixed or limited number of terms of an infinite series to approximate certain functions.

Example:

Taylor's and Mclaurin's Series expansions of functions like e^x, $\sin x$, $\cos x$ etc., with limited number of terms of the infinite series.

Although errors are introduced in Numerical Methods, they can be controlled and hence either reduced to arbitrarily low values or managed to be within tolerable limits.

For example, round-off errors can be controlled by allocating larger storage space for mantissa by using double float, instead of float for example.

Truncation errors can be controlled by developing methods in which more terms of the Taylor's series are used.

For example, truncation error in Simpson's rule of numerical integration is much less than trapezoidal rule for same problem, owing to the fact that Simpson's rule is developed by taking more terms of Taylor's Series. The order of a Numerical Method is a way of quantifying the extent of error, the higher the order, lesser the error. Some numerical methods involve starting the procedure by assuming trial guess values for the solution and then refining the answer successively to greater and greater accuracy in each iteration. These types of numerical methods are called trial and error methods or iterative methods..

For example, the Gauss-Seidel method for solving system of linear equations is a trial and error (iterative) method. So is the bisection, regula-falsi, secant and Newton-Raphson methods used for root finding (solving algebraic and transcendental equations of the form $f(x) = 0$).

Quantifying Errors in Numerical Methods: There are several measures to quantity the error which occurs in numerical methods.

$$\text{Error} = \text{Exact Value} - \text{Approximate Value}$$

$$\text{Absolute Error} = |\text{Exact Value} - \text{Approximate Value}|$$

$$\text{Relative Error} = \left|\frac{\text{Exact} - \text{Approximate}}{\text{Exact}}\right|$$

$$\text{Relative Error \%} = \left|\frac{\text{Exact} - \text{Approximate}}{\text{Exact}}\right| \times 100$$

6.2 Numerical Solution of System of Linear Equations

Consider the following m first degree equations consisting of n unknowns $x_1, x_2 ... x_n$.

$$a_{11} x_1 + a_{12} x_2 + \ldots\ldots\ldots\ldots + a_{1j} x_j + \ldots\ldots\ldots\ldots + a_{1n} x_n = b_1$$

$$a_{21} x_1 + a_{22} x_2 + \ldots\ldots\ldots\ldots + a_{2j} x_j + \ldots\ldots\ldots\ldots + a_{2n} x_n = b_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{ml} x_1 + a_{m2} x_2 + \ldots\ldots\ldots\ldots + a_{mj} x_j + \ldots\ldots\ldots\ldots + a_{mn} x_{mn} = b_m$$

or in matrix notation, we have

$$\begin{bmatrix} a_{11} a_{12} \ldots\ldots a_{1n} \\ a_{21} a_{22} \ldots\ldots a_{2n} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1} a_{m2} \ldots\ldots a_{mn} \end{bmatrix}_{m \times n} \begin{bmatrix} x_1 \\ x_2 \\ \ldots\ldots \\ x_n \end{bmatrix}_{n \times 1} = \begin{bmatrix} b_1 \\ b_2 \\ \ldots\ldots \\ b_m \end{bmatrix}_{m \times 1}$$

$$\Rightarrow \qquad AX = B$$

By finding a solution of the above system of equation we mean to obtain the values of $x_1, x_2 \ldots x_n$ such that they satisfy all the given equations simultaneously. The system of equations, given above is said to be homogenous if all b_i $(i = 1 \ldots. m)$ vanish, otherwise it is called as non homogenous system. There are number of methods to solve the above **System of Linear Equations**.

These are as follows:

1. Matrix Inversion Method
2. Cramer's Rule
3. Crout's and Dolittle's Method (Triangularisation Methods)
4. Gauss-Elimination Method
5. Gauss-Jordan's Method
6. Gauss-Seidel Iterative Method
7. Jacobi Iterative Method

In this book, we shall focus on Triangularisation, Gauss-Elimination and Gauss-Seidel Methods only.

6.2.1 Method of Factorisation or Triangularisation Method (Dolittle's Triangularisation Method)

This method is based on the fact that a square matrix A can be factorised into the form LU where L is unit lower triangular and U is a upper triangular, if all the principal minors of A are non singular i.e., it is a standard result of linear algebra that such a factorisation, when it exists, is unique.

We consider, for definiteness, the linear system

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Which can be written in the form

$$AX = B \qquad \text{... (i)}$$

Let $$A = LU \qquad \text{... (ii)}$$

where $$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \qquad \text{... (iii)}$$

and $$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \qquad \text{... (iv)}$$

(i) becomes $$LUX = B \qquad \text{... (v)}$$

If we set $$UX = Y \qquad \text{... (vi)}$$

then (v) way be written as $$LY = B \qquad \text{... (vii)}$$

which is equivalent to the system $y_1 = b_1$

$$\ell_{21}y_1 + y_2 = b_2$$
$$\ell_{31}y_1 + \ell_{32}y_2 + y_3 = b_3$$

and can be solved for y_1, y_2, y_3 by the forward substitution. Once, Y is known, the system (vi) become

$$u_{11}x_1 + u_{12}x_2 + u_{13}x_3 = y_1$$
$$u_{22}x_2 + u_{23}x_3 = y_2$$
$$u_{33}x_3 = y_3$$

which can be solved by backward substitution.

We shall now describe a scheme for computing the matrices L and U, and illustrate the procedure with a matrix of order 3. From the relation (ii), we obtain

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Multiplying the vertices on the left and equating the corresponding elements of both sides we get

$$u_{11} = a_{11},\ u_{12} = a_{12},\ u_{13} = a_{13}$$
$$\ell_{21}u_{11} = a_{21}$$

or $$\ell_{21} = \frac{a_{21}}{u_{11}},$$

$$\ell_{21}u_{12} + u_{22} = a_{22}$$

$\Rightarrow$ $$u_{22} = a_{22} - \ell_{21}u_{12}$$

$$\ell_{31}u_{13} + u_{23} = a_{23}$$

$\Rightarrow$ $$u_{23} = a_{23} - \ell_{21}u_{13}$$

$$\ell_{31}u_{11} = a_{31}$$

$\Rightarrow$ $$\ell_{31} = \frac{a_{31}}{u_{11}}$$

$$\ell_{31}u_{12} = \ell_{32}u_{22} = a_{32}$$

$$\Rightarrow \quad \ell_{32} = \frac{a_{32} - l_{31}u_{12}}{u_{22}}$$

Lastly, $\ell_{31}\, u_{13} + \ell_{32}\, u_{23} + u_{33} = a_{33}$

$$\Rightarrow \quad u_{33} = a_{33} - \ell_{31}u_{13} - \ell_{32}u_{23}$$

$\therefore$ the variables are solved in the following

order u_{11}, u_{12}, u_{13}

then $\ell_{21}, u_{22}, u_{23}$

lastly, $\ell_{31}, \ell_{32}, u_{33}$

Example:

Solve the equations

$$2x + 3y + z = 9,$$
$$x + 2y + 3z = 6$$
$$3x + y + 2z = 8$$

by the factorisation method.

Solution:

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

clearly $u_{11} = 2,\ u_{12} = 3,\ u_{13} = 1$

also $\ell_{21}u_{11} = 1$, so that $\ell_{21} = 1/2$

$$\ell_{21}u_{12} + u_{22} = 2$$

$$\Rightarrow \quad u_{22} = 2 - \ell_{21}u_{12} = 1/2$$

$$\ell_{21}u_{13} + u_{23} = 3$$

from which we obtain $u_{23} = 5/2$

$$\ell_{31}\, u_{11} = 3$$

$$\Rightarrow \quad \ell_{31} = 3/2$$

$$\ell_{31}\, u_{12} + \ell_{32}\, u_{22} = 1$$

$$\Rightarrow \quad \ell_{32} = -7$$

$$\ell_{31}\, u_{13} + \ell_{32}\, u_{23} + u_{23} = 2$$

$$\Rightarrow \quad u_{33} = 18$$

It follows that

$$A = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix}$$

and hence the given system of equations can be written as

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

or as

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

solving this system by forward substitution, we get

$$y_i = 9, \frac{y_1}{2} + y_2 = 6$$

$$\Rightarrow \qquad y_2 = \frac{3}{2}$$

$$\frac{3}{2}y_1 - 7y_2 + y_3 = 8 \text{ or } y_3 = 5$$

Hence the solution of the original system is given by

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ \frac{3}{2} \\ 5 \end{bmatrix}$$

which when solved by back substitution process.

$$x = \frac{35}{18};\ y = \frac{29}{18};\ z = \frac{5}{18}$$

Note: The Crout's triangularisation method is very similar to Dolittle's method except that in Crout's method

$$L = \begin{bmatrix} \ell_{11} & 0 & 0 \\ \ell_{21} & \ell_{22} & 0 \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}.$$

Also the order of solving the unknowns in Crout's method is column wise instead of row wise i.e., we solve : $\ell_{11}, \ell_{21}, \ell_{31}$ then $u_{12}, \ell_{22}, \ell_{32}$ then u_{13}, u_{23} and ℓ_{33}. There is no particular advantage of Crout's method over ittle's method and hence either method can be used for triangularisation.

.2 Gauss Seidel Method

In the first equation of (ii), we substitute the first approximation

$(x_1^{(1)}, x_2^{(1)}, \ldots\ldots\ldots\ldots x_n^{(1)})$ into right hand side and denote the result as $x_1^{(2)}$.

In the second equation we substitute $(x_1^{(2)}, x_2^{(1)}, \ldots\ldots\ldots\ldots x_n^{(1)})$ and denote the result as $x_2^{(2)}$.

In the third approximation we substitute $(x_1^{(2)}, x_2^{(2)}, x_3^{(1)} \ldots\ldots\ldots x_n^{(1)})$ and call the result as $x_3^{(2)}$. In this manner, we complete the first stage of iteration and the entire process is repeated till the values of $x_1, x_2 \ldots\ldots x_n$ are obtained to the accuracy required. It is clear therefore that this method uses an improved component as soon as it is available and it is called the method of **"Successive displacements"** or **"Gauss-Seidel method"**.

Note: It can be shown that the Gauss-Seidel method converges twice as fast as the **"Jacobi method"**.

6.3 Numerical Solutions of Nonlinear Algebraic and Transcendental Equations by Bisection, Regula-Falsi, Secant and Newton-Raphson Methods

In scientific and engineering work, a frequently occurring problem is to find the roots of equations of the form

$$f(x) = 0 \qquad \ldots \text{(i)}$$

If $f(x)$ is a quadratic, cubic or biquadratic expression then algebraic formula are available for expressing the roots in terms of the coefficients. On the other hand when $f(x)$ is a polynomial of higher degree or on expression involving transcendental functions e.g., $1 + \cos x - 5x$, $x \tan x - \cosh x$, $e^x - \sin x$ etc. Algebraic methods are not available and recourse must be taken to find the roots by approximate methods.

There are some numerical methods for the solutions of equations of the form (1), where $f(x)$ is algebraic or transcendental or a combinations of both.

6.3.1 Roots of Algebraic Equations

Let $p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n$ be a rational integral function of x of n dimensions, and let us denote it by $f(x)$; then $f(x) = 0$ is the general type of a rational integral equation of the n^{th} degree.

Dividing throughout by p_0, we see that without any loss of generality we may take

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n = 0$$

as the type of a rational integral equation of n^{th} degree.

1. Unless otherwise stated the coefficients $p_1, p_2, \ldots p_n$ will always be supposed rational.
2. Any value of x which makes $f(x)$ vanish is called a root of the equation $f(x) = 0$.
3. When $f(x)$ is divided by $x - a$ without remainder, a is a root of the equation $f(x) = 0$.
4. We shall assume that every equation of the form $f(x) = 0$ has a root, real or imaginary.
5. Every equation of the n^{th} degree has n roots, and no more.

Proof: Denote the given equation by $f(x) = 0$, where

$$f(x) = p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots\ldots + p_n$$

The equation $f(x) = 0$ has a root, real or imaginary; let this be denoted by a_1; then $f(x)$ is divisible by $x - a_1$, so that

$$f(x) = (x - a_1)\,\phi_1(x)$$

where $\phi_1(x)$ is a rational integral function of $n - 1$ dimensions. Again, the equation $\phi_1(x) = 0$ has a root, real or imaginary; let this be denoted by a_2; then $\phi_1(x)$ is divisible by $x - a_2$, so that

$$\phi_1(x) = (x - a_2)\,\phi_2(x)$$

where $\phi_2(x)$ is a rational integral function of $n - 2$ dimensions,

Thus $$f(x) = p_0(x - a_1)(x - a_2)\,\phi_2(x)$$

Proceeding in this way, we obtain,

$$f(x) = p_0(x - a_1)(x - a_2)\ldots\ldots(x - a_n).$$

Hence the equation $f(x) = 0$ has n roots, since $f(x)$ vanishes when x has any of the values $a_1, a_2, \dots a_n$.

6. Also the equation cannot have more than n roots; for if x has any value different from any of the quantities $a_1, a_2, a_3, \dots a_n$, all the factors on the right are different from zero, and therefore $f(x)$ cannot vanish for that value of x.

7. In the above investigation some of the quantities $a_1, a_2, a_3, \dots a_n$ may be equal; in this case, however, we shall suppose that the equation has still n roots, although these are not all different.

8. In an equation with real coefficients imaginary roots occur in pairs.

   Suppose that $f(x) = 0$ is an equation with real coefficients, and suppose that it has an imaginary root $a + ib$; we shall show that $a - ib$ is also a roots. The factor $f(x)$ corresponding to these two roots is

$$(x - a - ib)\,(x - a + ib), \text{ or } (x - a)^2 + b^2.$$

   Suppose that $a = ib$, $c = id$, $e = ig$, ... are the imaginary roots of the equation $f(x) = 0$, and that $f(x)$ is the product of the quadratic factors corresponding to these imaginary roots; then

$$f(x) = \{(x - a)^2 + b^2\}\,\{(x - e)^2 + d^2\}\{(x - e)^2 + g^2\} \dots$$

   Now each of these factors is positive for every real value of x; hence $\phi(x)$ is always positive for real values of x.

9. We may show that in an equation with rational coefficients, surd roots enter in pairs; that is, if a $+\sqrt{b}$ is a root then a $-\sqrt{b}$ is also a root.

Example:

Solve the equation $6x^4 - 13x^3 - 35x^2 - x + 3 = 0$, having given that one root is $2 - \sqrt{3}$.

Solution:

Since $2 - \sqrt{3}$ is a root, we know that $2 + \sqrt{3}$ is also a root, and corresponding to this pair of roots we have the quadratic factor $x^2 - 4x + 1$.

Also $6x^2 + 13x^3 - 35x^2 - x + 3 = (x^2 - 4x + 1)(6x^2 + 11x + 3)$;

hence the other roots are obtained from

$$6x^2 + 11x + 3 = 0,$$

or

$$(3x + 1)(2x + 3) = 0$$

thus the roots are $-\frac{1}{3}, -\frac{3}{2}, 2 + \sqrt{3}, 2 - \sqrt{3} = 0$

To determine the nature of some of the roots of an equation it is not always necessary to solve it; for ınce, the truth of the following statements will be readily admitted.

1. If the coefficients are all positive, the equation has no positive root; thus the equation $x^5 + x^3 + 2x + 1 = 0$ cannot have a positive root.
2. If the coefficients of the even powers of x are all of one sign, and the coefficients of the odd powers are all of the contrary sign, the equation has no negative roots; thus the equation $x^7 + x^5 - 2x^4 + x^3 - 3x^2 + 7x - 5 = 0$ cannot have a negative root
3. If the equation contains only even powers of x and the coefficients are all of the same sign, the equation has no real root; thus the equation $2x^8 + 3x^4 + x^2 + 7 = 0$ cannot have a real root.
4. If the equation contains only odd powers of x, and the coefficients are all of the same sign, the equation has no real root except $x = 0$; thus the equation $x^9 + 2x^5 + 3x^3 + x = 0$ has no real root except $x = 0$.

All the foregoing results are included in the theorem of the next article, which is known as Descartes' Rule of Signs.

6.3.2 Descarte's Rule of Signs

An equation $f(x) = 0$ cannot have more positive roots than there are changes of sign in $f(x)$, and cannot have more negative roots than there are changes of sign in $f(-x)$.

i.e. number of real positive roots $\leq$ number of sign changes in $f(x)$

and number of real negative roots $\leq$ number of sign changes in $f(-x)$.

Example:

Consider the equation $x^9 + 5x^3 - x^3 + 7x + 2 = 0$.

Solution:

Here there are two changes of sign, therefore there are at most two positive roots.

Again $f(-x) = -x^9 + 5x^3 + x^3 - 7x + 2$, and here there are three changes of sign, therefore the given equation has at most three negative roots, and therefore it must have at least four imaginary roots, since total number of roots is nine, it being a ninth degree polynomial.

6.3.3 Numerical Methods for Root Finding

We shall study four numerical methods, all of which are iterative (trial and error methods) for root finding i.e. solving $f(x) = 0$.

1. Bisection Method
2. Regula-Falsi Method
3. Secant Method
4. Newton-Raphson Method

6.3.3.1 Bisection Method

This method is based on the intermediate value theorem which states that if a function $f(x)$ is continuous between a and b, and $f(a)$ and $f(b)$ are of opposite signs then there exists at least one root between a and b for definiteness.

Let $f(a)$ be negative, and $f(b)$ be positive (see figure below). Then the root lies between a and b and let its approximate value be given by $x_0 = (a + b)/2$.

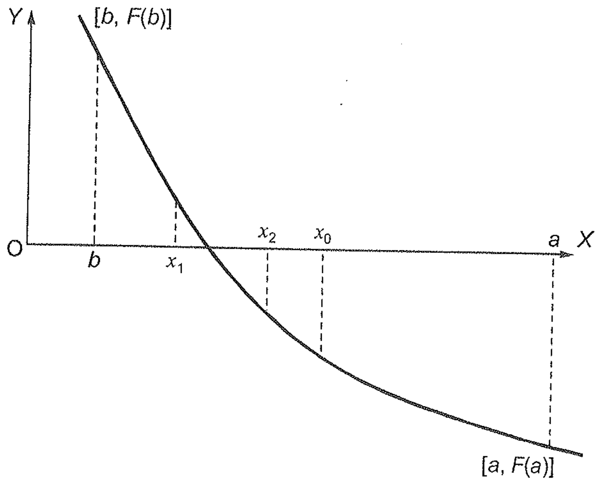
If $f(x_0) = 0$, we conclude that x_0 is a root of the equation $f(x_0) = 0$, otherwise the root lies either x_0 and b or between x_0 and a depending on whether $f(x_0)$ is negative or positive. We designate this new interval as $[a_1, b_1]$ whose length is $|b - a|/2$.

As before this is bisected at x_1 and the new interval will be exactly half the length of the previous one. We repeat this process until the latest interval is as small as desired say ϵ. It is clear that the interval width is reduced by a factor of one-half at each step and at the end of the n^{th} step, the new interval will be $[a_n, b_n]$ of length $|b - a|/2^n$.

We then have $\dfrac{|b-a|}{2^n} \leq \epsilon$ which gives on simplification $n \geq \dfrac{\log_e\left(\dfrac{|b-a|}{\epsilon}\right)}{\log_e 2}$. ... (i)

Inequality (i) gives the number of iterations required to achieve an accuracy ϵ.

This method can be shown graphically as follows:



The iteration equation for bisection method is $x_2 = \dfrac{x_0 + x_1}{2}$ or more generally, $x_{n+1} = \dfrac{x_{n-1} + x_n}{2}$.

xample:

Find a real root of the equation $f(x) = x^3 - x - 1 = 0$.

olution:

Since $f(1)$ is negative and $f(2)$ is positive, a root lies between 1 and 2 and therefore we take $x_0 = 3/2$.

Then $f(x_0) = \dfrac{27}{8} - \dfrac{3}{2} = \dfrac{15}{8}$ which is positive. Hence the root lies between 1 and 1.5 and we obtain

$x_1 = (1 + 1.5)/2 = 1.25$ we find $f(x_1) = -19/64$, which is negative. We therefore, conclude that the root lies between 1.25 and 1.5. It follows that $x_2 = (1.25+1.5)/2 = 1.375$.

The procedure is repeated and the successive approximations are $x_3 = 1.3125$, $x_4 = 1.34375$, $x_5 = 1.328125$; etc.

2 Regula-Falsi Method

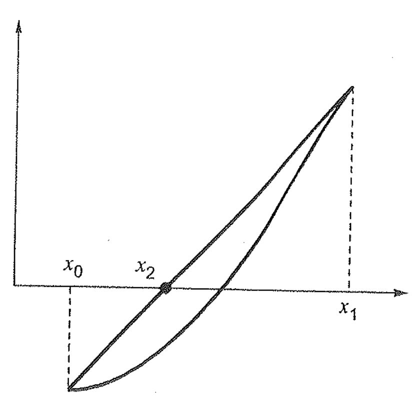
The method starts by taking two guess values x_0 and x_1 for the root, just like the bisection method, such $(x_0)\ f(x_1) < 0$. The iteration formula for Regula-Falsi method is different from bisection method and it is

$$x_2 = \frac{f_1 x_0 - f_0 x_1}{f_1 - f_0}$$

or more generally

$$x_{n+1} = \frac{f_n x_{n-1} - f_{n-1} x_n}{f_n - f_{n-1}}$$

Graphically this can be shown as drawing a chord between and (x_1, f_1) and seeing that the point of interaction of this with x axis is x_2, as shown in Figure.



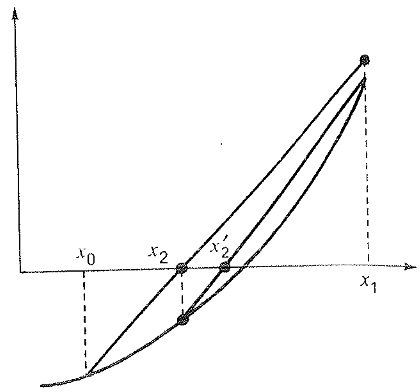
In the next iteration, the root is either between x_0 and x_2 or en x_1 and x_2.

So x_2 replaces either x_0 or x_1 depending on whether $f(x_0)$ 0 or $f(x_1)\ f(x_2 < 0)$.

If $f(x_0)\, f(x_2) < 0$ then x_1 replaced by x_2, else x_0 replaced by x_2. And the iteration is again continued and the new value of x'_2 in indicated by x_2 is figure below.

This is illustrated graphically as follows:

The process is continued until we get as close to the root as desired. Like bisection method, Regula-Falsi method is 100% reliable and the root will always be found, since always x_0 and x_1 are taken on either side of the root i.e. root is kept trapped between x_0 and x_1 in both bisection as well as Regula-Falsi methods.



Both Bisection and Regula-Falsi methods are (first order convergence or linear convergent), as compared with secant and Newton-Raphson methods which have convergence rates of 1.62 and 2 respectively i.e. Newton-Raphson method is quadratic convergent.

6.3.3.3 Secant Method

The Secant method proceeds similarly to Regula-Falsi method in the sense that it also requires two starting guess values, but the difference is that $f(x_0)\, f(x_1)$ need not be negative i.e. at any stage of iteration we do not ensure that the root is between x_0 and x_1. However, Secant method uses the same iteration equation as Regula-Falsi method.

$$x_2 = \frac{f_1 x_0 - f_0 x_1}{f_1 - f_0}$$

or more generally

$$x_{n+1} = \frac{f_n x_{n-1} - f_{n-1} x_n}{f_n - f_{n-1}}$$

In Secant method, once the value of x_2 is obtained, to proceed to the next iteration, x_0 is always replaced by x_1 and x_1 is always replaced by x_2. This is the only and primary difference between Regula-Falsi and Secant method. Geometrically, both Regula-Falsi and Secant methods find x_2 by same way, that is by drawing the chord from (x_0, f_0) to (x_1, f_1) and intersection of this chord with x axis is x_2. The advantage of the Secant method is that it is faster than both the Bisection and Regula-Falsi method as it has a convergence order of 1.62. However, the disadvantage is that, Secant method is not 100% reliable, since the equation

$$x_2 = \frac{f_1 x_0 - f_0 x_1}{f_1 - f_0}$$

will fail if $f_1 = f_0$, which may happen since no effort is made to keep f_1 and f_0 to be of opposite signs as it is done in case of Regula-Falsi method, which uses the same iteration equation.

6.3.3.4 Newton-Raphson Method

This method is generally used to improve the result obtained by one of the previous method. Let x_0 be an approximate root of $f(x) = 0$ and let $x_1 = x_0 + h$ be the correct root so that $f(x_1) = 0$. Expanding $f(x_0 + h)$ by Taylor's series we obtain

$$f(x) = f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + \ldots = 0$$

Neglecting the second and higher order derivatives we have $f(x_0) + h\, f'(x_0) = 0$

which gives

$$h = -\frac{f(x_0)}{f'(x_0)}$$

A better approximation than x_0 is therefore given by x_1, where

$$x_1 = x_0 + h = x_0 - \frac{f(x_0)}{f'(x_0)}$$

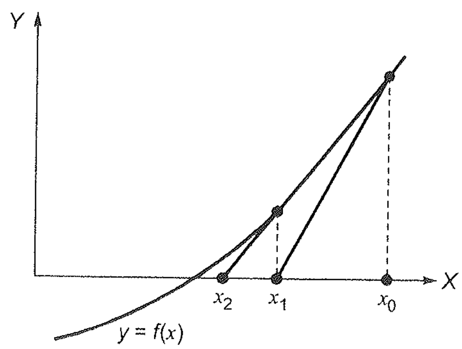
Successive approximations are given by $x_2, x_3, \ldots x_{n+1}$,

where $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{... (i)}$$

which is **Newton Raphson** formula.

$$\epsilon_{n+1} = \approx \frac{1}{2} \epsilon_n^2 \frac{f''(\xi)}{f'(\xi)} \quad \text{... (ii)}$$

So that the **Newton Raphson** process has a second order or dratic convergence.



Geometrically, in Newton-Raphson method a tangent to curve is vn at point $[x_0, f(x_0)]$ and the point of intersection of this tangent and is is taken as x_1 which is the next value of the iterate ofcourse x_1 is er to root than x_0. It can be used for solving both algebraic and scendental equations and it can also be used when the roots are iplex.

The method converges rapidly to the root with a second order vergence. The number of significant digits in root which are correct, bles, after each iteration of N-R method.

Following is a list of Common Newton Raphson iterative problems alongwith the Newton-Raphson iteration ation, for solving that problem.

1. The inverse of b, is the root of the equation $f(x) = \frac{1}{x} - b = 0$

   Iteration Equation: $$x_{n+} = x_n(2 - bx_n)$$

2. The inverse square root b, is the root of equation $f(x) = \frac{1}{x^2} - b = 0$

   Iteration Equation: $$x_{n+1} = \frac{1}{2} x_n \left(3 - bx_n^2\right)$$

3. The p^{th} root of a given number N, is root of equation $f(x) = x^p - N = 0$

   Iteration Equation: $$x_{n+r} = \frac{(p-1)x_n^p + N}{px_n^{p-1}}$$

Note: The order of Bisection, Regular Falsi and Secant Method and Newton Raphson Method are given w:

Sl. No.	Method	Order
1.	Bisection	1
2.	Regula Falsi	1
3.	Secant Method	1.62
4.	Newton Raphson	2

6.4 Numerical Integration (Quadrature) by Trapezoidal and Simpson's Rules

The general problem of numerical integration may be stated as follows. Given a set of data points (x_0, y_0), (x_1, y_1), ...(x_n, y_n). of a function $y = f(x)$, where $f(x)$ is not known explicitly it is required to compute the value of the definite integral,

$$I = \int_a^b y dx \qquad \text{... (i)}$$

As in the case of numerical differentiation, we replace $f(x)$ by an interpolating polynomial $\phi(x)$ and obtain on integration an approximate value of the definite integral. Thus, different integration formulas can be obtained depending upon the type of interpolation formula used.

Let the interval $[a, b]$ be divided into n equal subintervals such that

$$a = x_0 < x_1 < x_2 < \ldots\ldots\ldots < x_n = b.$$

Clearly,

$$x_n = x_0 + nh$$

Hence, the integral becomes, $I = \int_{x_0}^{x_n} y dx$

Approximating y by **Newton's Forward Difference** formula, we obtain,

$$I = \int_{x_0}^{x_n} [y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{6}\Delta^3 y_0 + \ldots\ldots]dx$$

Since $x = x_0 + ph$, $dx = hdp$ and hence the above integral becomes

$$h\int_0^n \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{6}\Delta^3 y_0 + \ldots\right] dp$$

which gives on simplification

$$\int_{x_0}^{x_n} y dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \ldots\right]$$

This is known as General formula, we can obtain different integration formulas by putting $n = 1, 2, 3, \ldots$ etc. We derive here a few of these formulae but it should be remarked that the **Trapezoidal and Simpson's 1/3 rules** are found to give sufficient accuracy for use in practical problems.

The following table shows how Δy_0, Δy_1, $\Delta^2 y_0$ are derived from (x_0, y_0), (x_1, y_1), (x_2, y_2) etc.

x_0	y_0		
		Δy_0	
x_1	y_1		$\Delta^2 y_0$
		Δy_1	
x_2	y_2		

$$\Delta y_0 = y_1 - y_0$$

$$\Delta y_1 = y_2 - y_1$$

and

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = y_2 - 2y_1 + y_0$$

6.4.1 Trapezoidal Rule

Setting $n = 1$ in the general formula, all differences higher than the first will become zero and we obtain;

$$\int_{x_0}^{x_1} y dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right] = h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right] = \frac{h}{2}[y_0 + y_1] \qquad \text{... (i)}$$

For the next interval $[x_1, x_2]$, we deduce similarly

$$\int_{x_1}^{x_2} y dx = \frac{h}{2}[y_1 + y_2] \quad \text{... (ii)}$$

and so on. For the last interval $[x_{n-1}, x_n]$, we have

$$\int_{x_{n-1}}^{x_n} y dx = \frac{h}{2}[y_{n-1} + y_n] \quad \text{... (iii)}$$

combining all these expressions, we obtain the rule

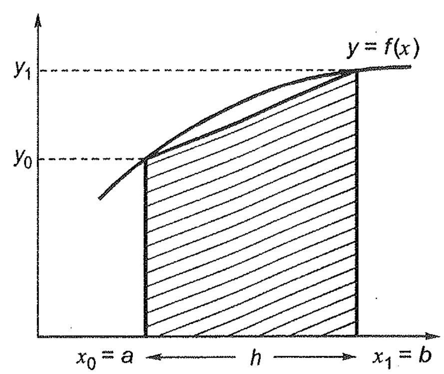
$$\int_{x_0}^{x_n} y dx = \frac{h}{2}[y_0 + 2(y_1 + y_2 + ... + y_{n-1}) + y_n]$$

which is known as trapezoidal rule.

The geometrical significance of this rule is that the curve $y = f(x)$ is replaced by n straight lines joining the ·oints (x_0, y_0) and (x_1, y_1); (x_1, y_1) and (x_2, y_2)....; (x_{n-1}, y_{n-1}) and (x_n, y_n).

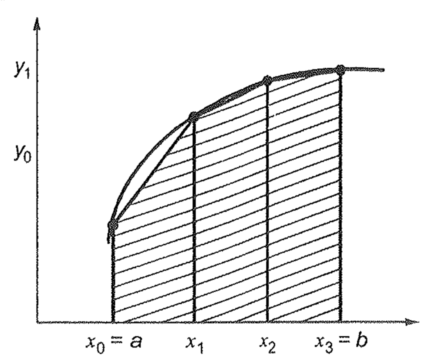
The area bounded by the curve $y = f(x)$, the ordinates $x = x_0$ and $x = x_n$ and the x-axis is then approximately quivalent to the sum of the areas of n **Trapeziums** obtained.

Simple Trapezoidal Rule:



$$\text{Shaded Area} = \text{Area of Trapezium} \cong \int_a^b f(x)dx$$

Compound Trapezoidal Rule (with 4 pts and 3 intervals):



$$\text{Shaded Area} = \text{Sum of Area of 3 trapezium} \cong \int_a^b f(x)dx$$

·.4.2 Simpson's Rules

·.4.2.1 Simpson's 1/3 Rule

This rule is obtained by putting $n = 2$ in general formula i.e., by replacing the curve by $n/2$ arcs of second legree polynomials or parabolas. We have been,

$$\int_{x_0}^{x_2} ydx = 2h\left[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right]$$

$$= \frac{h}{3}\left[y_0 + (y_1 - y_0) + \frac{1}{6}(y_2 - 2y_1 + y_0)\right]$$

$$= \frac{h}{3}[y_0 + 4y_1 + y_2]$$

Similarly, $$\int_{x_2}^{x_4} ydx = \frac{h}{3}[y_2 + 4y_3 + y_4]$$

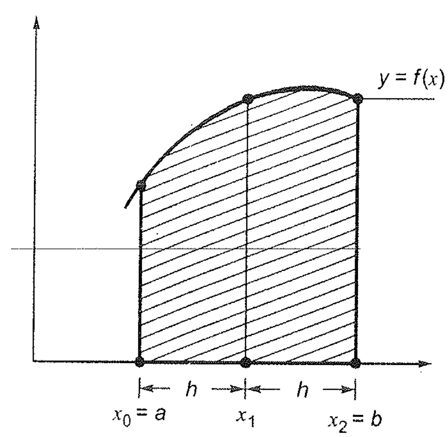
and finally $$\int_{x_{n-2}}^{x_n} ydx = \frac{h}{3}[y_{n-2} + 4y_{n-1} + y_n]$$

Summing up we obtain,

$$\int_{x_2}^{x_n} ydx = \frac{h}{3}[y_0 + 4(y_1 + y_3 + y_5 + \ldots + y_n) + 2(y_2 + y_4 + \ldots + y_{n-2}) + y_n]$$

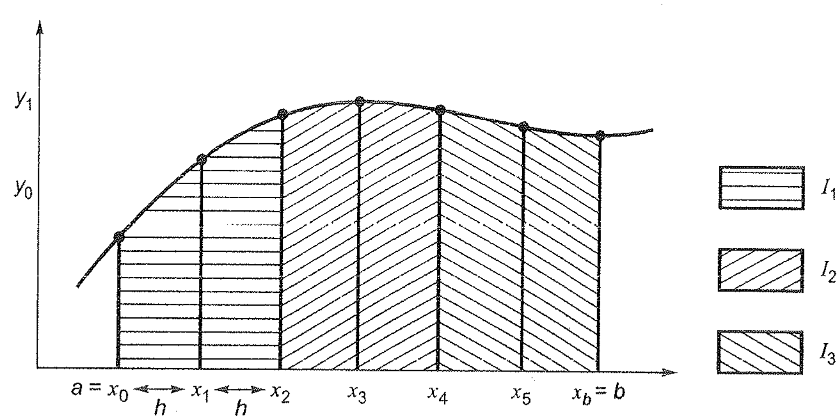
which is known as **"Simpson's 1/3 rule"** or simply **"Simpson's rule"**. It should be noted that this rule requires the divisions of the whole range into an even number of subintervals of width h.

Simple Simpson's Rule:



Shaded Area $= \int_a^b f(x)dx$

Compound Simpson's Rule: (7 points or 6 intervals)



$$I = \int f(x)dx = I_1 + I_2 + I_3$$

6.4.2.2 Simpson's 3/8 Rule

Setting $n = 3$ in general formula we observe that all differences higher than the third will become zero and we obtain,

$$\int_{x_0}^{x_3} y dx = 3h\left[y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right]$$

$$= 3h\left[y_0 + \frac{3}{2}(y_1 - y_0) + \frac{3}{4}(y_2 - 2y_1 + y_0) + \frac{1}{8}(y_3 - 3y_2 + 3y_1 - y_0)\right]$$

$$= \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + y_3\right]$$

Similarly,

$$\int_{x_3}^{x_6} y dx = \frac{3h}{8}\left[y_3 + 3y_4 + 3y_5 + y_6\right]$$

and so on. Summing up all these, we obtain,

$$\int_{x_0}^{x_n} y dx = \frac{3h}{8}[(y_0 + 3y_1 + 3y_2 + y_3) + (y_3 + 3y_4 + 3y_5 + y_6) + ... + y_{n-3} + y_{n-2} + y_{n-1} + y_n]$$

$$\int_{x_0}^{x_n} y dx = \frac{3h}{8}[y_0 + 3y_1 + 3y_2 + 2y_3 + 3y_4 + 3y_5 + 2y_6 + 2y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n]$$

This rule called **"Simpson's 3/8 rule"**, is not so accurate as Simpson's rule.

Example:

Evaluate, $I = \int_0^1 \frac{1}{1+x} dx$ correct to three decimal places using (i) Trapezoidal- rule and (ii) Simpson's rules (take $h = 0.5$) and check which rule is more accurate.

Solution:

We solve this question by both the **Trapezoidal** and **Simpson's** rules with $h = 0.5$.
The value of x and y are tabulated below.

X	0	0.5	1.0
$y = \frac{1}{1+x}$	1.0000	0.6667	0.5

(a) Trapezoidal rule gives:

$$I = \frac{1}{4}[1.0000 + 2(0.6667) + 0.5] = 0.7084$$

(b) Simpson's rule gives:

$$\frac{1}{6}[1.0000 + 4(0.6667) + 0.5] = 0.6945.$$

Note that the exact answer for this problem by analytical integration method

$$I = \int_0^1 \frac{1}{1+x} dx = \left[\log_e(1+x)\right]_0^1 = \log_e 2 = 0.6931$$

Clearly, Simpson's rule is closer to the answer and has less error compared to trapezoidal rule.

6.4.3 Truncation Error Formulae for Trapezoidal and Simpson's Rule

Let h be the step size used in integration.

The truncation error formula for simple trapezoidal rule with 2 pts is given by

$$T_E = -\frac{h^3}{12} f''(\xi)$$

For composite trapezoidal rule with N_i intervals.

$$T_{E(\max)} = -\frac{h^3}{12} N_i f''(\xi)$$

The absolute T_E bound for simple trapezoidal rule is given by

$$|T_E|_{\text{bound}} = \max\left|-\frac{h^3}{12} f''(\xi)\right|$$

$$= \frac{h^3}{12}\max|f''(\xi)| \qquad \text{where, } x_0 \le \xi \le x_n$$

For Composite rule also similarly,

$$|T_E|_{\text{bound}} = \max\left|-\frac{h^3}{12} N_i\, f''(\xi)\right|$$

$$= \frac{h^3}{12} N_i \max|f''(\xi)| \qquad \text{where, } x_0 \le \xi \le x_n$$

The truncation error for simple Simpson's rule with 3 pts is given by

$$T_E = -\frac{h^5}{90} f^{iv}(\xi)$$

For composite Simpson's rule with N_i intervals, the truncation error bound is given by

$$T_{E(\max)} = -\frac{h^5}{90} f^{iv}(\xi) N_{si}$$

where, N_{si} is number of Simpson's intervals.

Since, $$N_{si} = \frac{N_i}{2}$$

So, $$T_{E(\max)} = -\frac{h^5}{90}\left(\frac{N}{2}\right) f^{iv}(\xi)$$

The absolute truncation error bound for simple Simpson's rule is given by,

$$|T_E|_{\text{bound}} = \max\left|-\frac{h^5}{90} f^{iv}(\xi)\right|$$

$$= \frac{h^5}{90}\max\left|f^{iv}(\xi)\right| \qquad \text{where, } x_0 \le \xi \le x_n$$

The absolute truncation error bound for composite Simpson's rule with N intervals is given by,

$$|T_E|_{\text{bound}} = \max\left|-\frac{h^5}{90}\left(\frac{N}{2}\right) f^{iv}(\xi)\right| = \frac{h^5}{90}\left(\frac{N_i}{2}\right)\max\left|f^{iv}(\xi)\right|$$

$$= \frac{h^5}{180} N_i \max\left|f^{iv}(\xi)\right| \qquad \text{where, } x_0 \le \xi \le x_n$$

In all these formulae, $N_i = (b-a)/h$ (where a and b are the limits of integration) and $N_i = N_{pt} - 1$ (where N_{pt} is the number of pts used in the integration). Since T_E for simple trapezoidal rule is proportional to h^2, it is a third order method. i.e. $TE = O(h^2)$. Since T_E for simple Simpson's rule is proportional to h^4, it is a fifth order method. i.e. $TE = O(h^4)$.

Important Note:

1. Trapezoidal rule gives exact results while integrating polynomials upto degree = 1.
2. Simpson's rule gives exact results while integrating polynomials upto degree = 3.

6.5 Numerical Solution of Ordinary Differential Equations

6.5.1 Introduction

Analytical methods of solution are applicable only to a limited class of differential equations. Frequently differential equations appearing in physical problems do not belong to any of these familiar types and one is obliged to resort to numerical methods. These methods are of even greater importance when we realise that computing machines are now available which reduce the time taken to do numerical computation considerably.

A number of numerical methods are available for the solution of first order differential equations of the form:

$$\frac{dy}{dx} = f(x, y), \text{ given } y(x_0) = y_0. \qquad \text{... (i)}$$

These methods yield solutions either as a power series in x from which the values of y can be found by direct substitution, or as a set of values of x and y. The method of Picard and Taylor series belong to the former class of solutions whereas those of Euler, Runge-Kutta, Milne, Adams-Bashforth etc. belong to the latter class. In these later methods, the values of y are calculated in short steps for equal intervals of x and are therefore, termed as step-by-step methods.

Euler and Runge-Kutta methods are used for computing y over a limited range of x-values whereas Milne and Adams-Bashforth method may be-applied for finding y over a wider range of x-values. These later methods require starting values which are found by Picard's or Taylor series or Runge-Kutta methods.

The initial condition in (i) is specified at the point x_0. Such problems in which all the initial conditions are given at the initial point only are called initial value problems. But there are problems where conditions are given at two or more points. These are known as boundary value problems. In this chapter, we shall study three methods common used for solution of first order differential equations, namely.

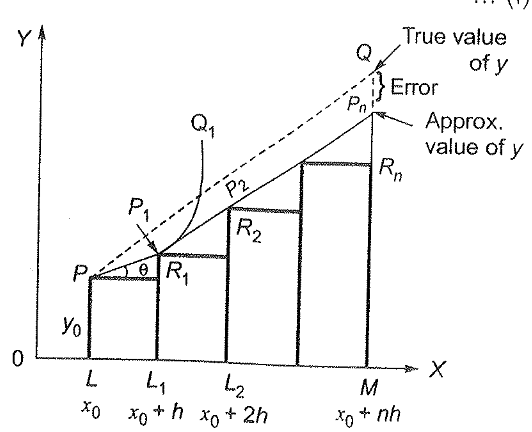
1. Euler's Method
2. Modified Euler's Method
3. Runge-Kutta Method of Fourth Order (Classical Runge-Kutta Method)

6.5.2 Euler's Method

Consider the equation,

$$\frac{dy}{dx} = f(x, y) \qquad \text{... (i)}$$

given that $y(x_0) = y_0$. Its curve of solution through $P(x_0, y_0)$ is shown in Fig. Now we have to find the ordinate of any other point Q on this curve.



Let us divide LM into n sub-intervals each of width h at L_1, L_2, ... so that h is quiet small. In the interval LL_1, we approximate the curve by the tangent at P. If the ordinate through L_1 meets this tangent in $P_1(x_0 + h, y_1)$, then

$$y = L_1P_1 = LP + R_1P_1$$
$$= y_0 + PR1 \tan\theta$$

$$= y_0 + h\left(\frac{dy}{dx}\right)_P$$

$$= y_0 + h\, f(x_0, y_0)$$

Let P_1Q_1 be the curve of solution of (i) through P_1 and let its tangent at P_1 meet the ordinate through $L2$ in $P_2(x_0 + 2h, y_2)$. Then repeating this process n times, we finally reach an approximation MP_n of MQ given by

$$y_{n+1} + f(x_0 + (n-1)h, y_{n-1})$$

In general we may write

$$y_{i+1} = y_i + h\, f(x_i, y_i)$$

This is Euler's method of finding an approximate solution of (i).

Obs. In Euler's method, we approximate the curve of solution by the tangent in each interval, i.e. by a sequence of short lines. Unless h is small, the error is bound to be quiet significant. This sequence of lines may also deviate considerably from the curve of solution. Hence there is a modification of this method which is given in the next section, called modified Euler's method, which is more accurate.

Example:

Using Euler's method, find an approximate value of y corresponding to $x = 1$, given that $dy/dx = x + y$ and $y = 1$ when $x = 0$.

Solution:

We take $n = 10$ and $h = 0.1$ which is sufficiently small. The various calculations are arranged as follows:

x	y	$x + y = dy/dx$	old $y + 0.1(dy/dx) =$ new y
0.0	1.00	1.00	$1.00 + 0.1(1.00) = 1.10$
0.1	1.10	1.20	$1.10 + 0.1(1.20) = 1.22$
0.2	1.22	1.42	$1.22 + 0.1(1.42) = 1.36$
0.3	1.36	1.66	$1.36 + 0.1(1.66) = 1.53$
0.4	1.53	1.93	$1.53 + 0.1(1.93) = 1.72$
0.5	1.72	2.22	$1.72 + 0.1(2.22) = 1.94$
0.6	1.94	2.54	$1.94 + 0.1(2.54) = 2.19$
0.7	2.19	2.89	$2.19 + 0.1(2.89) - 2.48$
0.8	2.48	3.89	$2.48 + 0.1(3.89) = 2.81$
0.9	2.81	3.71	$2.81 + 0.1(3.71) = 3.1$
1.0	3.18		

Thus the required approximate value of y is 3.18 at $x = 1.0$.

Obs. In this example, the true value of y from its exact solution at $x = 1$ is

$$y = 2e^x - x - 1$$

$$2e^1 - 1 - 1 = 3.44$$

whereas by Euler's method $y = 3.18$. In the above solution, had we chosen $n = 20$, the accuracy would have been considerably increased but at the expense of double the labour of computation. Euler's method is no doubt very simple, but cannot be considered as one of the best.

Example:

Given $\frac{dy}{dx} = \frac{y-x}{y+x}$ with initial condition $y = 1$ at $x = 0$; find y for $x = 0.1$ by Euler's method.

Solution:

We divide the interval (0, 0.1) into five steps i.e. we take $n = 5$, $h = \frac{b-a}{n} = \frac{0.1-0}{5} = 0.02$. The various calculations are arranged as follows:

x	y	$x + y = dy/dx$	old $y + h(dy/dx) =$ new y
0.00	1.0000	1.0000	$1.0000 + 0.02(1.0000) = 1.0200$
0.02	1.0200	0.9615	$1.0200 + 0.200 + 0.02(9615) = 1.0392$
0.04	1.0392	0.926	$1.0392 + 0.02(926) = 1.0577$
0.06	1.0577	0.893	$1.0577 + 0.02(893) = 1.0756$
0.08	1.0756	0.862	$1.0756 + 0.02(802) = 1.0928$
0.10	1.0928		

Hence the required approximate value of $y = 1.0928$.

6.5.3 Modified Euler's Method

In Euler's method $\quad y_{i+1} = y_i + h\, f(x_i, y_i)$

In Backward Euler's method $\quad y_{i+1} = y_i + h\, f(x_{i+1}, y_{i+1}) \quad$ ... (i)

A numerical method where y_{i+1} appears on LHS and RHS of the iterative equation is called an implicit method. So Backward Euler's method is an Implicit method, while Euler's method is explicit since y_{i+1} appears only on left side of iterative equation.

In Backward Euler's method, we need to rearrange and solve (i) for y_{i+1} before proceeding further.

Example:

Using Backward Euler's Method find an approximate value of y corresponding to $x = 0.2$, given that $dy/dx = x + y$ and $y = 1$ when $x = 0$, use step size $h = 0.1$.

$$y_{i+1} = y_i + h\, f(x_{i+1}, y_{i+1})$$
$$y_{i+1} = y_i + h(x_{i+1} + y_{i+1})$$

Solution:

Solving for y_{i+1} we get, $\quad y_{i+1} = \frac{y_i + h\, x_{i+1}}{1-h}$

Now the calculations are shown below:

i	x_i	y_i	Comments
0	0.0	1.00	Initial condition given
1	0.1	1.122	$y_1 = \frac{y_0 + h\, x_1}{1-h} = \frac{1 + 0.1 \times 0.1}{1 - 0.1} = 1.122$
2	0.2	1.2689	$y_2 = \frac{y_1 + h\, x_2}{1-h} = \frac{1.122 + 0.1 \times 0.2}{1 - 0.1} = 1.2689$

So, the approximate value of y at $x = 0.2$ is 1.2689.

Notice that this same problem when solved by forward Euler's method, gave a slightly different answer for y which was $y = 1.22$ at $x = 0.2$.

The advantage of Backward Euler's method is its stability. Backward Euler's method is more stable compared to forward Euler's method.

A method is stable if the effect of any single fixed round off error is bounded, independent of the number of mesh points.

6.5.4 Runge-Kutta Method

The Taylor's series method of solving differential equations numerically is restricted by the labour involved in finding the higher order derivatives. However there is a class of methods known as Runge-Kutta methods which do not require the calculations of higher order derivatives. These methods agree with Taylor's series solution upto the terms in h^r, where r differs from method to method and is called the order of that method. Euler's method Modified Euler's method and Runge's method are the Runge-Kutta methods of the first, second and third order respectively.

The fourth-order Runge-Kutta method is most commonly used and is often referred to as 'Runge-Kutta' method' or classical Runge-Kutta method.

Working rule for finding the increment k of y corresponding to an increment h of x by Runge-Kutta method from

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0 \text{ is as follows:}$$

Calculate successively

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right)$$

and

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

Finally compute

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

which gives the required approximate value $y_1 = y_0 + k$.

(Note that k is the weighted mean of k_1, k_2, k_3 and k_4).

Obs. One of the advantages of these methods is that the operation is identical whether the differential equation is linear or non-linear.

Example:

Apply Runge-Kutta fourth order method to find an approximate value of y when $x = 0.2$ given that $dy/dx = x + y$ and $y = 1$ when $x = 0$.

Solution:

Here, $x_0 = 0$, $y_0 = 1$, $h = 0.2$, $f(x_0, y_0) = 1$

$\therefore$

$$k_1 = hf(x_0, y_0) = 0.2 \times 1 = 0.2$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = 0.2 \times f(0.1, 1.1) = 0.2400$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right)$$

$$= 0.2 \times f(0.1, 1.12) = 0.2440$$

and

$$k_4 = hf(x_0+h, y_0+ k_3)$$

$$= 0.2 \times f(0.2, 1.244) = 0.2888$$

$$\therefore \quad k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.2000 + 0.4800 + 0.4880 + 0.2888)$$

$$= \frac{1}{6} \times (1.4568) = 0.2428$$

Now, $y_1 = y_0 + k$

$$= 1 + 0.2428 = 1.2428$$

Hence the required approximate value of y is 1.2428.

6.5.5 Stability Analysis

If the effect of round off error remains bounded as $j \to \infty$, with a fixed step size, then the method is said to be stable; otherwise unstable. Unstable methods will diverge away from solution and cause overflow error.

Using a general single step method equation

$$y_{j+1} = E \, . \, y \qquad \dots \text{(i)}$$

Condition for absolute stability is

$$|E| \le 1$$

Using a test equation $y' = \lambda y$,

let us find the condition for stability for Euler's method.

Euler's method equation is $y_{j+1} = y_j + hf(x_j, y_j)$

$$= y_j + h\lambda y_j$$

$$= (1 + h\lambda) y_j$$

Now, comparing with (i) we get

$$E = 1 + h\lambda$$

Condition for stability if $|E| < 1$

$$|1 + h\lambda| < 1$$

$$-1 < 1 + h\lambda < 1$$

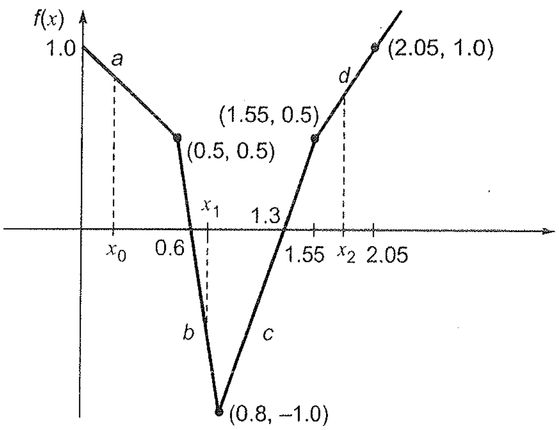
So, condition for stability is

$$-2 < \lambda h < 0$$

■■■■

Previous GATE and ESE Questions

Q.1 A piecewise linear function $f(x)$ is plotted using thick solid lines in the figure below (the plot is drawn to scale).



If we use the Newton-Raphson method to find the roots of $f(x) = 0$ using $x0$, $x1$ and $x2$ respectively as initial guesses, the roots obtained would be

(a)1.3, 0.6 and 0.6 respectively
(b)0.6, 0.6 and 1.3 respectively
(c)1.3, 1.3 and 0.6 respectively
(d)1.3, 0.6 and 1.3 respectively

[CS, GATE-2003, 2 marks]

Q.2 The accuracy of Simpson's rule quadrature for a step size h is

(a) $O(h^2)$ (b) $O(h^3)$
(c) $O(h^4)$ (d) $O(h^5)$

[ME, GATE-2003, 1 mark]

Statement for Linked Answer Questions 3 and 4.
Given $a > 0$, we wish to calculate the reciprocal value $\frac{1}{a}$ by Newton-Raphson method for $f(x) = 0$.

Q.3 The Newton Raphson algorithm for the function will be

(a) $x_{k+1} = \frac{1}{2}\left(x_k + \frac{a}{x_k}\right)$

(b) $x_{k+1} = \left(x_k + \frac{a}{2}x_k^2\right)$

(c) $x_{k+1} = 2x_k - ax_k^2$

(d) $x_{k+1} = x_k - \frac{a}{2}x_k^2$

[CE, GATE-2005, 2 marks]

Q.4 For $a = 7$ and starting with $x_0 = 0.2$, the first two iterations will be

(a) 0.11, 0.1299 (b) 0.12, 0.1392
(c) 0.12, 0.1416 (d) 0.13, 0.1428

[CE, GATE-2005, 2 marks]

Q.5 Starting from $x_0 = 1$, one step of Newton-Raphson method in solving the equation $x^3 + 3x - 7 = 0$ gives the next value (x_1) as

(a) $x_1 = 0.5$ (b) $x_1 = 1.406$
(c) $x_1 = 1.5$ (d) $x_1 = 2$

[ME, GATE-2005, 2 marks]

Q.6 Match **List-I** with **List-II** and select the correct answer using the codes given below the lists:

List-I
A. Newton-Raphson method
B. Rung-kutta method equations
C. Simpson's Rule equations
D. Gauss elimination

List-II
1. Solving nonlinear equations
2. Solving simultaneous linear equations
3. Solving ordinary differential
4. Numerical integration
5. Interpolation
6. Calculation of Eigenvalues

Codes:

	A	B	C	D
(a)	6	1	5	3
(b)	1	6	4	3
(c)	1	3	4	2
(d)	5	3	4	1

[EC, GATE-2005, 2 marks]

Q.7 A 2nd degree polynomial, $f(x)$ has values of 1, 4 and 15 at $x = 0$, 1 and 2, respectively. The integral $\int_0^2 f(x)dx$ is to be estimated by applying the trapezoidal rule to this data. What is the error (defined as "true value - approximate value") in the estimate?

(a) $-\frac{4}{3}$ (b) $-\frac{2}{3}$

(c) 0 (d) $\frac{2}{3}$

[CE, GATE-2006, 2 marks]

Q.8 The differential equation $(dy/dx) = 0.25\, y^2$ is to be solved using the backward (implicit) Euler's method with the boundary condition $y = 1$ at $x = 0$ and with a step size of 1. What would be the value of y at $x = 1$?

(a) 1.33 (b) 1.67

(c) 2.00 (d) 2.33

[CE, GATE-2006, 1 mark]

Q.9 Given that one root of the equation

$$x^3 - 10x^2 + 31x - 30 = 0 \text{ is } 5,$$

the other two roots are

(a) 2 and 3 (b) 2 and 4

(c) 3 and 4 (d) –2 and –3

[CE, GATE-2007, 2 marks]

Q.10 The following equation needs to be numerically solved using the Newton-Raphson method.

$$x^3 + 4x - 9 = 0$$

The iterative equation for this purpose is (k indicates the iteration level)

(a) $x_{k+1} = \dfrac{2x_k^3 + 9}{3x_k^2 + 4}$

(b) $x_{k+1} = \dfrac{3x_k^2 + 4}{2x_k^2 + 9}$

(c) $x_{k+1} = x_k - 3x_k^2 + 4$

(d) $x_{k+1} = \dfrac{4x_k^2 + 3}{9x_k^2 + 2}$

[CE, GATE-2007, 2 marks]

Q.11 The equation $x^3 - x^2 + 4x - 4 = 0$ is to be solved using the Newton-Raphson method. If $x = 2$ is taken as the initial approximation of the solution, then the next approximation using this method will be

(a) $\frac{2}{3}$ (b) $\frac{4}{3}$

(c) 1 (d) $\frac{3}{2}$

[EC, GATE-2007, 2 marks]

Q.12 Consider the series $x_{n+1} = \dfrac{x_n}{2} + \dfrac{9}{8x_n}$, $x_0 = 0.5$ obtained from the Newton-Raphson method. The series converges to

(a) 1.5 (b) $\sqrt{2}$

(c) 1.6 (d) 1.4

[CS, GATE-2007, 2 marks]

Q.13 A calculator has accuracy up to 8 digits after decimal place. The value of $\int_0^{2\pi} \sin x\, dx$ when evaluated using this calculator by trapezoidal method with 8 equal intervals, to 5 significant digits is

(a) 0.00000 (b) 1.0000

(b) 0.00500 (d) 0.00025

[ME, GATE-2007, 2 marks]

Q.14 The differential equation $(dx/dt) = [(1 - x)/\tau]$ is discretised using Euler's numerical integration method with a time step $\Delta T > 0$. What is the maximum permissible value of ΔT to ensure stability of the solution of the corresponding discrete time equation?

(a) 1 (b) $\tau/2$

(c) τ (d) 2τ

[EE, GATE-2007, 2 marks]

Q.15 Equation $e^x - 1 = 0$ is required to be solved using Newton's method with an initial guess $x_0 = -1$. Then, after one step of Newton's method, estimate x_1 of the solution will be given by

(a) 0.71828 (b) 0.36784

(c) 0.20587 (d) 0.00000

[EE, GATE-2008, 2 marks]

Q.16 The recursion relation to solve $x = e^{-x}$ using Newton-Raphson method is

(a) $x_{n+1} = e^{-x_n}$

(b) $x_{n+1} = x_n - e^{-x_n}$

(c) $x_{n+1} = (1 + x_n)\dfrac{e^{-x_n}}{1 + e^{-x_n}}$

(d) $x_{n+1} = \dfrac{x_n^2 - e^{-x_n}(1 + x_n) - 1}{x_n - e^{-x_n}}$

[EC, GATE-2008, 2 marks]

Q.17 The Newton-Raphson iteration $x_{n+1} = \frac{1}{2}\left(x_n + \frac{R}{x_n}\right)$ can be used to compute the

(a) square of R (b) reciprocal of R

(c) square root of R (d) logarithm of R

[CS, GATE-2008, 2 marks]

Q.18 The minimum number of equal length subintervals needed to approximate $\int_1^2 xe^x\,dx$ to an accuracy of at least $1/3 \times 10^{-6}$ using the trapezoidal rule is

(a) $1000e$ (b) 1000

(c) $100e$ (d) 100

[CS, GATE-2008, 2 marks]

Q.19 Let $x^2 - 117 = 0$. The iterative steps for the solution using Newton-Raphson's method is given by

(a) $x_{k+1} = \frac{1}{2}\left(x_k + \frac{117}{x_k}\right)$

(b) $x_{k+1} = x_k - \frac{117}{x_k}$

(c) $x_{k+1} = x_k - \frac{x_k}{117}$

(d) $x_{k+1} = x_k - \frac{1}{2}\left(x_k + \frac{117}{x_k}\right)$

[EE, GATE-2009, 2 marks]

Q.20 Newton-Raphson method is used to compute a root of the equation $x^2 - 13 = 0$ with 3.5 as the initial value. The approximation after one iteration is

(a) 3.575 (b) 3.677

(c) 3.667 (d) 3.607

[CS, GATE-2010, 1 mark]

Q.21 The table below gives values of a function $F(x)$ obtained for values of x at intervals of 0.25.

x	0	0.25	0.5	0.75	1.0
$F(x)$	1	0.9412	0.8	0.64	0.50

The value of the integral of the function between the limits 0 to 1 using Simpson's rule is

(a) 0.7854 (b) 2.3562

(c) 3.1416 (d) 7.5000

[CE, GATE-2010, 2 marks]

Q.22 Torque exerted on a flywheel over a cycle is listed in the table. Flywheel energy (in J per unit cycle) using Simpson's rule is

Angle (degree)	0	60	120	180	240	300	360
Torque (N m)	0	1066	−323	0	323	−355	0

(a) 542 (b) 993

(c) 1444 (d) 1986

[ME, GATE-2010, 2 marks]

Q.23 Consider a differential equation $\frac{dy(x)}{dx} - y(x) = x$ with the initial condition y(0) = 0. Using Euler's first order method with a step size of 0.1, the value of y(0.3) is

(a) 0.01 (b) 0.031

(c) 0.0631 (d) 0.1

[EC, GATE-2010, 2 marks]

Q.24 The matrix $[A] = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$ is decomposed into a product of a lower triangular matrix $x[L]$ and an upper triangular matrix $[U]$. The properly decomposed $[L]$ and $[U]$ matrices respectively are

(a) $\begin{bmatrix} 1 & 0 \\ 4 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 0 \\ 4 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 0 \\ 4 & -3 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}$

[EE, GATE-2011, 2 marks]

Q.25 The square root of a number N is to be obtained by applying the Newton Raphson iterations to the equation $x^2 - N = 0$. If i denotes the iteration index, the correct iterative scheme will be

(a) $x_{i+1} = \frac{1}{2}\left(x_i + \frac{N}{x_i}\right)$

(b) $x_{i+1} = \frac{1}{2}\left(x_i^2 + \frac{N}{x_i^2}\right)$

(c) $x_{i+1} = \frac{1}{2}\left(x_i + \frac{N^2}{x_i}\right)$

(d) $x_{i+1} = \frac{1}{2}\left(x_i - \frac{N}{x_i}\right)$

[CE, GATE-2011, 2 marks]

26 Roots of the algebraic equation $x^3 + x^2 + x + 1 = 0$ are

(a) $(+1, +j, -j)$ (b) $(+1, -1, +1)$
(c) $(0, 0, 0)$ (d) $(-1, +j, -j)$

[EE, GATE-2011, 1 marks]

27 Solution of the variables x_1 and x_2 for the following equations is to be obtained by employing the Newton-Raphson iterative method

equation (i) $10x_2 \sin x_1 - 0.8 = 0$

equation (ii) $10x_2^2 - 10x_2 \cos x_1 - 0.6 = 0$

Assuming the initial values $x_1 = 0.0$ and $x_2 = 1.0$, the Jacobian matrix is

(a) $\begin{bmatrix} 10 & -0.8 \\ 0 & -0.6 \end{bmatrix}$ (b) $\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & -0.8 \\ 10 & -0.6 \end{bmatrix}$ (d) $\begin{bmatrix} 10 & 0 \\ 10 & -10 \end{bmatrix}$

[EE, GATE-2011, 2 marks]

28 A numerical solution of the equation $f(x) = x + \sqrt{x} - 3 = 0$ can be obtained using Newton-Raphson method. If the starting value is $x = 2$ for the iteration, the value of x that is to be used in the next step is

(a) 0.306 (b) 0.739
(c) 1.694 (d) 2.306

[EC, GATE-2011, 2 marks]

9 The integral $\int_1^3 \frac{1}{x} dx$, when evaluated by using Simpson's 1/3 rule on two equal subintervals each of length 1, equals

(a) 1.000 (b) 1.098
(c) 1.111 (d) 1.120

[ME, GATE-2011, 2 marks]

0 The bisection method is applied to compute a zero of the function $f(x) = x^4 - x^3 - x^2 - 4$ in the interval [1, 9]. The method converges to a solution after ____ iterations.

(a) 1 (b) 3
(c) 5 (d) 7

[CS, GATE-2012, 2 marks]

Q.31 The estimate of $\int_{0.5}^{1.5} \frac{dx}{x}$ Obtained using Simpson's rule with three-point function evaluation exceeds the exact value by

(a) 0.235 (b) 0.068
(c) 0.024 (d) 0.012

[CE, GATE-2012, 1 mark]

Q.32 The error in $\left.\frac{d}{dx} f(x)\right|_{x=x_0}$ for a continuous function estimated with $h = 0.03$ using the central difference formula $\left.\frac{d}{dx} f(x)\right|_{x=x_0} = \frac{f(x_0 + h) - f(x_0 - h)}{2h}$, is 2×10^{-3}. The values of x_0 and $f(x_0)$ are 19.78 and 500.01, respectively. The corresponding error in the central difference estimate for $h = 0.02$ is approximately

(a) 1.3×10^{-4} (b) 3.0×10^{-4}
(c) 4.5×10^{-4} (d) 9.0×10^{-4}

[CE, GATE-2012, 2 marks]

Q.33 When the Newton-Raphson method is applied to solve the equation $f(x) = x^3 + 2x - 1 = 0$, the solution at the end of the first iteration with the initial guess value as $x_0 = 1.2$ is

(a) –0.82 (b) 0.49
(c) 0.705 (d) 1.69

[EE, GATE-2013, 2 Marks]

Q.34 The magnitude of the error (correct to two decimal places) in the estimation of following integral using simpson 1/3 rule. Take the step length as 1

$$\int_0^4 (x^4 + 10) dx$$

[CE, GATE-2013, 2 Mark]

Q.35 Match the correct pairs

Numerical Integration Scheme	Order of Fitting Polynomial
P. Simpson's 3/8 Rule	1. First
Q. Trapezoidal Rule	2. Second
R. Simpson's 1/3 Rule	3. Third

(a) P-2, Q-1, R-3 (b) P-3, Q-2, R-1
(c) P-1, Q-2, R-3 (d) P-3, Q-1, R-2

[ME, GATE-2013, 1 Mark]

Q.36 While numerically solving the differential equation $\frac{dy}{dx} + 2xy^2 = 0$, $y(0) = 1$ using Euler's predictor-corrector (improved Euler-Cauchy) with a step size of 0.2, the value of y after the first step is

(a) 1.00 (b) 1.03
(c) 0.97 (d) 0.96

[IN, GATE-2013 : 2 marks]

Q.37 Match the application to appropriate numerical method.

Application

P1: Numerical integration
P2: Solution to a transcendental equation
P3: Solution to a system of linear equations
P4: Solution to a differential equation
M1: Newton-Raphson Method
M2: Runge-Kutta Method
M3: Simpson's 1/3-rule
M4: Gauss Elimination Method

(a) P1—M3, P2—M2, P3—M4, P4—M1
(b) P1—M3, P2—M1, P3—M4, P4—M2
(c) P1—M4, P2—M1, P3—M3, P4—M2
(d) P1—M2, P2—M1, P3—M3, P4—M4

[EC, GATE-2014 : 1 Mark]

Q.38 The real root of the equation $5x - 2\cos x - 1 = 0$ (up to two decimal accuracy) is ______.

[ME, GATE-2014 : 2 Marks]

Q.39 The function $f(x) = e^x - 1$ is to be solved using Newton-Raphson method. If the initial value of x_0 is taken as 1.0, then the absolute error observed at 2nd iteration is ______ .

[EE, GATE-2014 : 2 Marks]

Q.40 In the Newton-Raphson method, an initial guess of $x_0 = 2$ is made and the sequence $x_0, x_1, x_2 \ldots$ is obtained for the function

$$0.75x^3 - 2x^2 - 2x + 4 = 0$$

Consider the statements

(I) $x_3 = 0$.
(II) The method converges to a solution in a finite number of iterations.

Which of the following is TRUE?

(a) Only I (b) Only II
(c) Both I and II (d) Neither I nor II

[CS, GATE-2014 (Set-2) : 2 Marks]

Q.41 The value of $\int_{2.5}^{4} ln(x)dx$ calculated using the Trapezoidal rule with five subintervals is ____.

[ME, GATE-2014 : 2 Marks]

Q.42 The definite integral $\int_{1}^{3} \frac{1}{x}dx$ is evaluated using trapezoidal rule with a step size of 1. The correct answer is ________.

[ME, GATE-2014 : 1 Mark]

Q.43 Using the trapezoidal rule, and dividing the interval of integration into three equal subintervals, the definite integral $\int_{-1}^{+1}|x|dx$ is _______.

[ME, GATE-2014 : 2 Marks]

Q.44 With respect to the numerical evaluation of the definite integral $K = \int_a^b x^2 dx$, where a and b are given, which of the following statements is/are TRUE?

(I) The value of K obtained using the trapezoidal rule is always greater than or equal to the exact value of the definite integral.
(II) The value of K obtained using the Simpson's rule is always equal to the exact value of the definite integral

(a) I only (b) II only
(b) Both I and II (d) Neither I nor II

[CS, GATE-2014 : 2 Marks]

Q.45 Consider an ordinary differential equation $\frac{dx}{dt} = 4t + 4$. If $x = x_0$ at $t = 0$, the increment in x calculated using Runge-Kutta fourth order multi-step method with a step size of $\Delta t = 0.2$ is

(a) 0.22 (b) 0.44
(c) 0.66 (d) 0.88

[ME, GATE-2014 : 2 Marks]

Q.46 In the LU decomposition of the matrix $\begin{bmatrix} 2 & 2 \\ 4 & 9 \end{bmatrix}$, if the diagonal elements of U are both 1, then the lower diagonal entry l_{22} of L is ___________.

[CS, GATE-2015 : 1 Mark]

.47 If a continuous function $f(x)$ does not have a root in the interval $[a, b]$, then which one of the following statements is TRUE?

(a) $f(a) \cdot f(b) = 0$ (b) $f(a) \cdot f(b) < 0$
(c) $f(a) \cdot f(b) > 0$ (d) $f(a)/f(b) \leq 0$

[EE, GATE-2015 : 1 Mark]

48 The quadratic equation $x^2 - 4x + 4 = 0$ is to be solved numerically, starting with the initial guess $x_0 = 3$. The Newton-Raphson method is applied once to get a new estimate and then the Secant method is applied once using the initial guess and this new estimate. The estimated value of the root after the application of the Secant method is ________.

[CE, GATE-2015 : 2 Marks]

49 In Newton-Raphson iterative method, the initial guess value (x_{ini}) is considered as zero while finding the roots of the equation: $f(x) = -2 + 6x - 4x^2 + 0.5x^3$. The correction, Δx, to be added to x_{ini} in the first iteration is ______.

[CE, GATE-2015 : 1 Mark]

50 Newton-Raphson method is used to find the roots of the equation, $x^3 + 2x^2 + 3x - 1 = 0$. If the initial guess is $x_0 = 1$, then the value of x after 2nd iteration is ________.

[ME, GATE-2015 : 2 Marks]

51 The Newton-Raphson method is used to solve the equation $f(x) = x^3 - 5x^2 + 6x - 8 = 0$. Taking the initial guess as $x = 5$, the solution obtained at the end of the first iteration is_____.

[EC, GATE-2015 : 2 Marks]

52 The secant method is used to find the root of an equation $f(x) = 0$. It is started from two distinct estimates x_a and x_b for the root. It is an iterative procedure involving linear interpolation to a root. The iteration stops if $f(x_b)$ is very small and then x_b is the solution. The procedure is given below. Observe that there is an expression which is missing and is marked by ? Which is the suitable expression that is to be put in place of ? So that it follows all steps of the secant method?

Secant

Initialize: x_a, x_b, ε, N
// ε = convergence indicator
$f_b = f(x_b)$
// N = maximum number of iterations
$i = 0$
while ($i < N$ and $|f_b| > \varepsilon$) do
  $i = i + 1$// update counter
   x_t = ?// missing expression for
      // intermediate value
   $x_a = x_b$// reset x_a
   $x_b = x_t$// reset x_b
  $f_b = f(x_b)$// function value at new x_b
end while
if $|f_b| > \varepsilon$ then // loop is terminated with $i = N$
write "Non-convergence"
else
write "return x_b"
end if

(a) $x_b - (f_b - f(x_a))\, f_b / (x_b - x_a)$
(b) $x_a - (f_a - f(x_a))\, f_a / (x_b - x_a)$
(c) $x_b - (x_b - x_a)\, f_b / (f_b - f(x_a))$
(d) $x_a - (x_b - x_a)\, f_a / (f_b - f(x_a))$

[CS, GATE-2015 : 2 Marks]

Q.53 The integral $\int_{x_1}^{x_2} x^2 dx$ with $x_2 > x_1 > 0$ is evaluated analytically as well as numerically using a single application of the trapezoidal rule. If I is the exact value of the integral obtained analytically and J is the approximate value obtained using the trapezoidal rule, which of the following statements is correct about their relationship?

(a) $J > I$
(b) $J < I$
(c) $J = I$
(d) Insufficient data to determine the relationship

[CE, GATE-2015 : 1 Mark]

Q.54 For step-size, $\Delta x = 0.4$, the value of following integral using Simpson's 1/3 rule is ________.

$$\int_0^{0.8} \left(0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5\right) dx$$

[CE, GATE-2015 : 2 Marks]

Q.55 Using a unit step size, the volume of integral $\int_1^2 x \ln x \, dx$ by trapezoidal rule is ________.

[ME, GATE-2015 : 1 Mark]

Q.56 Simpson's $\frac{1}{3}$ rule is used to integrate the function $f(x) = \frac{3}{5}x^2 + \frac{9}{5}$ between $x = 0$ and $x = 1$ using the least number of equal sub-intervals. The value of the integral is ________.

[ME, GATE-2015 : 1 Mark]

Q.57 The values of function $f(x)$ at 5 discrete points are given below:

x	0	0.1	0.2	0.3	0.4
$f(x)$	0	10	40	90	160

Using Trapezoidal rule step size of 0.1, the value of $\int_0^{0.4} f(x)dx$ is ______.

[ME, GATE-2015 : 2 Marks]

Q.58 The velocity v (in kilometer/minute) of a motorbike which starts from rest, is given at fixed intervals of time t (in minutes) as follows:

t	2	4	6	8	10	12	14	16	18	20
v	10	18	25	29	32	20	11	5	2	0

The approximate distance (in kilometers) rounded to two places of decimals covered in 20 minutes using Simpson's 1/3rd rule is ______.

[CS, GATE-2015 : 2 Marks]

Q.59 Gauss Seidel method is used to solve the following equations (as per the given order):

$x_1 + 2x_2 + 3x_3 = 1;$

$2x_1 + 3x_2 + x_3 = 1;$

$3x_1 + 2x_2 + x_3 = 1$

Assuming initial guess as $x_1 = x_2 = x_3 = 0$, the value of x_3 after the first iteration is _____.

[ME, GATE-2016 : 2 Marks]

Q.60 Solve the equation $x = 10\cos(x)$ using the Newton-Raphson method. The initial guess is $x = \frac{\pi}{4}$. The value of the predicted root after the first iteration, up to second decimal, is_______.

[ME, GATE-2016 : 1 Mark]

Q.61 The root of the function $f(x) = x^3 + x - 1$ obtained after first iteration on application of Newton Raphson scheme using an initial guess of $x_0 = 1$ is

(a) 0.682 (b) 0.686

(c) 0.750 (d) 1.000

[ME, GATE-2016 : 1 Mark]

Q.62 Newton-Raphson method is to be used to find foot of equation $3x - e^x + \sin x = 0$. If the initial trail value of the roots is taken as 0.333, the next approximation for the root would be __________.

[CE, GATE-2016 : 1 Mark]

Q.63 Numerical integration using trapezoidal rule gives the best result for a single variable function, which is

(a) linear (b) parabolic

(c) logarithmic (d) hyperbolic

[ME, GATE-2016 : 1 Mark]

Q.64 The error in numerically computing the integral $\int_0^{\pi}(\sin x + \cos x)dx$ using the trapezoidal rule with three intevals of equal length between 0 and π is ________.

[ME, GATE-2016 : 2 Marks]

Q.65 The ordinary differential equation

$$\frac{dx}{dt} = -3x + 2, \text{ with } x(0) = 1$$

is to be solved using the forward Euler method. The largest time step that can be used to solve the equation without making the numerical solution unstable is ________.

[EC, GATE-2016 : 2 Marks]

Q.66 Consider the first order initial value problem $y' = y + 2x - x^2$, $y(0) = 1$, $(0 \le x < \infty$ with exact solution $y(x) = x^2 + e^x$. For $x = 0.1$, the percentage difference between the exact solution and the solution obtained using a single iteration of the second-order Runge-Kutta method with step-size $h = 0.1$ is ______.

[EC, GATE-2016 : 1 Mark]

Q.67 P(0, 3), Q(0.5, 4) and R(1, 5) are three points on the curve defined by $f(x)$. Numerical integration is carried out using both Trapezoidal rule and Simpson's rule within limits $x = 0$ and $x = 1$ for the curve. The difference between the two results will be

(a) 0 (b) 0.25

(c) 0.5 (d) 1

[ME, GATE-2017 : 2 Marks]

Q.68 The following table lists an n^{th} order polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots\ldots a_1 x + a_0$ and the forward difference evaluated at equally spaced values of x. The order of the polynomial is

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
–0.4	1.7648	–0.2965	0.089	–0.03
–0.3	1.4683	–0.2075	0.059	–0.0228
–0.2	1.2608	–0.1485	0.0362	–0.0156
–0.1	1.1123	–0.1123	0.0206	–0.0084
0	1	–0.0917	0.0122	–0.0012
0.1	0.9083	–0.0795	0.011	0.006
0.2	0.8288	–0.0685	0.017	0.0132

(a) 1 (b) 2
(c) 3 (d) 4

[IN, GATE-2017 : 2 Marks]

Q.69 Only one of the real roots of $f(x) = x^6 - x - 1$ lies in the interval $1 \le x \le 2$ and bisection method is used to find its value. For achieving an accuracy of 0.001, the required minimum number of iterations is ________ . (Give the answer up to two decimal places.)

[EE, GATE-2017 : 2 Marks]

Q.70 Starting with $x = 1$, the solution of the equation $x^3 + x = 1$, after two iterations of Newton-Raphson's method (up to two decimal places) is ______.

[EC, GATE-2017 : 2 Marks]

Q.71 Consider the equation $\frac{du}{dt} = 3t^2 + 1$ with $u = 0$ at $t = 0$. This is numerically solved by using the forward Euler method with a step size. $\Delta t = 2$. The absolute error in the solution in the end of the first time step is __________.

[CE, GATE-2017 : 2 Marks]

Q.72 The quadratic equation $2x^2 - 3x + 3 = 0$ is to be solved numerically starting with an initial guess as $x_0 = 2$. The new estimate of x after the first iteration using Newton-Raphson method is __________.

[CE, GATE-2018 : 1 Mark]

Q.73 An explicit forward Euler method is used to numerically integrate the differential equation

$$\frac{dy}{dt} = y$$

using a time step of 0.1. With the initial condition $y(0) = 1$,the value of y(1) computed by this method is ______(correct to two decimal places).

[ME, GATE-2018 : 2 Marks]

Q.74 Consider $p(s) = s^3 + a_2 s^2 + a_1 s + a_0$ with all real coefficients. It is known that its derivative $p'(s)$ has no real roots. The number of real roots of $p(s)$ is

(a) 0 (b) 1
(c) 2 (d) 3

[EC, GATE-2018 : 1 Mark]

Q.75 What is the cube root of 1468 to 3 decimal places?

(a) 11.340 (b) 11.353
(c) 11.365 (d) 11.382

[ESE Prelims-2018]

Q.76 What is the value of $(1525)^{0.2}$ to 2 decimal places?

(a) 4.33 (b) 4.36
(c) 4.38 (d) 4.30

[ESE Prelims-2018]

Answers Numerical Methods

1. (d) 2. (c) 3. (c) 4. (b) 5. (c) 6. (c) 7. (a) 8. (c) 9. (a)
10. (a) 11. (b) 12. (a) 13. (a) 14. (d) 15. (a) 16. (c) 17. (c) 18. (a)
19. (a) 20. (d) 21. (a) 22. (b) 23. (b) 24. (d) 25. (a) 26. (d) 27. (b)
28. (c) 29. (c) 30. (b) 31. (d) 32. (d) 33. (c) 34. (0.5) 35. (d) 36. (d)
37. (b) 38. (0.54) 39. (0.25) 40. (a) 41. (1.75) 42. (1.16) 43. (1.11) 44. (c) 45. (d)
46. (5) 47. (c) 48. (2.33) 49. (0.33) 50. (0.3) 51. (4.29) 52. (c, d) 53. (a) 54. (1.37)
55. (0.69) 56. (2) 57. (22) 58. (309) 59. (1.55) 60. (1.56) 61. (c) 62. (0.36) 63. (a)
64. (0.19) 65. (0.66) 66. (0.06) 67. (a) 68. (d) 69. (10) 70. (0.69) 71. (8) 72. (1)
73. (2.59) 74. (b) 75. (c) 76. (a)

1. (d)

Starting from x_0, slope of line a

$$= \frac{1-0.5}{0-0.5} = -1$$

y-intercept = 1

Eqn. of a is $y = mx + c = -1x + 1$

This line will cut x axis (i.e., $y = 0$), at $x = 1$

Since $x = 1$ is > than $x = 0.8$, a perpendicular at $x = 1$ will cut the line c and not line b.

∴ root will be 1.3

Starting from x_1,

the perpendicular at x_1 is cutting line b and root will be 0.6.

Starting from x_2,

$$\text{Slope of line } d = \frac{1-0.5}{2.05-1.55} = 1$$

Equation of d is $y - 0.5 = 1(x - 1.55)$

i.e. $y = x - 1.05$

This line will cut x axis at $x = 1.05$

Since, $x = 1.05$ is > than $x = 0.8$, the perpendicular at $x = 1.05$ will cut the line c and not line b. The root will be therefore equal to 1.3.

So starting from x_0, x_1 and x_2 the roots will be respectively 1.3, 0.6 and 1.3.

3. (c)

To calculate $\frac{1}{a}$ using N-R method,

set up the equation as

$$x = \frac{1}{a}$$

i.e. $\frac{1}{x} = a$

$\Rightarrow \frac{1}{x} - a = 0$

i.e. $f(x) = \frac{1}{x} - a = 0$

Now $f'(x) = -\frac{1}{x^2}$

$$f(x_k) = \frac{1}{x_k} - a$$

$$f'(x_k) = -\frac{1}{x_k^2}$$

For N-R method,

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$\Rightarrow \quad x_{k+1} = x_k - \frac{(1/x_k - a)}{-\frac{1}{x_k^2}}$$

Simplifying which we get

$$x_{k+1} = 2x_k - ax_k^2$$

4. (b)

For $a = 7$ the iteration equation, becomes

$$x_{k+1} = 2x_k - 7x_k^2$$

with $x_0 = 0.2$

$$x_1 = 2x_0 - 7x_0^2 = 2 \times 0.2 - 7(0.2)^2 = 0.12$$

and

$$x_2 = 2x_1 - 7x_1^2 = 2 \times 0.12 - 7(0.12)^2 = 0.1392$$

5. (c)

From Newton–Raphson method

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \qquad \text{... (i)}$$

Given function is

$$f(x) = x^3 + 3x - 7$$

and $f'(x) = 3x^2 + 3$

Putting $x_0 = 1$,

$$f(x_0) = f(1) = (1)^3 + 3 \times (1) - 7 = -3$$

$$f'(x_0) = f'(1) = 3 \times (1)^2 + 3 = 6$$

Substituting x_0, $f(x_0)$ and $f'(x_0)$ values into (i) we get,

$$\therefore \quad x_1 = 1 - \left(\frac{-3}{6}\right) \times 1 = 1.5$$

7. (a)

$f(x) = 1, 4, 15$

at $x = 0, 1$ and 2 respectively

$$\int_0^2 f(x)\,dx = \frac{h}{2}(f_1 + 2f_2 + f_3)$$

(3 point Trapezoidal Rule)

here $h = 1$

$$\therefore \int_0^2 f(x)\,dx = \frac{1}{2}(1 + 2 \times 4 + 15) = 12$$

∴ Approximate value by Trapezoidal Rule = 12

Since $f(x)$ is second degree polynomial, let

$$f(x) = a_0 + a_1 x + a_2 x^2$$

$$f(0) = 1$$
$$\Rightarrow a_0 + 0 + 0 = 1$$
$$\Rightarrow a_0 = 1$$
$$f(1) = 4$$
$$\Rightarrow a_0 + a_1 + a_2 = 4$$
$$\Rightarrow 1 + a_1 + a_2 = 4$$
$$\Rightarrow a_1 + a_2 = 3 \quad \text{... (i)}$$
$$f(2) = 15$$
$$\Rightarrow a_0 + 2a_1 + 4a_2 = 15$$
$$\Rightarrow 1 + 2a_1 + 4a_2 = 15$$
$$\Rightarrow 2a_1 + 4a_2 = 14 \quad \text{... (ii)}$$

Solving (i) and (ii)

$$a_1 = -1 \text{ and } a_2 = 4$$
$$\therefore f(x) = 1 - x + 4x^2$$

Now, exact value of

$$\int_0^2 f(x)dx = \int_0^2 (1 - x + 4x^2)\, dx$$

$$= \left[x - \frac{x^2}{2} + \frac{4x^3}{3}\right]_0^2 = \frac{32}{3}$$

Error = Exact – Approximate value

$$= \frac{32}{3} - 12 = -\frac{4}{3}$$

(c)

$$\frac{dy}{dx} = 0.25y^2 \qquad (y = 1 \text{ at } x = 0)$$
$$h = 1$$

Iterative equation for backward (implicit) Euler methods for above equation would be

$$y_{k+1} = y_k + h\, f(x_{k+1}, y_{k+1})$$
$$y_{k+1} = y_k + h \times 0.25\, y_{k+1}^2$$
$$\Rightarrow 0.25h y_{k+1}^2 - y_{k+1} + y_k = 0$$

putting $k = 0$ in above equation

$$0.25h\, y_1^2 - y_1 + y_0 = 0$$

since, $y_0 = 1$ and $h = 1$

$$0.25\, y_1^2 - y_1 + 1 = 0$$

$$\Rightarrow y_1 = \frac{1 \pm \sqrt{1-1}}{2 \times 0.25} = 2$$
$$\Rightarrow y_1 = 2$$

(a)

Since 5 is a root, $f(x)$ is divisible by $x - 5$. Now dividing $f(x)$ by $x - 5$ we get

$$x - 5\overline{)\,x^3 - 10x^2 + 31x - 30\,}(x^2 - 5x + 6$$
$$\underline{x^3 - 5x^2}$$
$$-5x^2 + 31x - 30$$
$$\underline{-5x^2 + 25x}$$
$$6x - 30$$
$$\underline{6x - 30}$$
$$0$$

$$\therefore x^3 - 10x^2 + 31x - 30 = 0$$
$$\Rightarrow (x - 5)(x^2 - 5x + 6) = 0$$

Roots of $x^2 - 5x + 6$ are 2 and 3.

$\therefore$ The other two roots are 2 and 3.

10. (a)

$$f(x) = x^3 + 4x - 9 = 0$$
$$f'(x) = 3x^2 + 4$$

N–R equation for iteration is,

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$
$$f(x_k) = x_k^3 + 4x_k - 9$$
$$f'(x_k) = 3x_k^2 + 4$$
$$x_{k+1} = x_k - \frac{(x_k^3 + 4x_k - 9)}{(3x_k^2 + 4)}$$
$$= \frac{(3x_k^3 + 4x_k) - (x_k^3 + 4x_k - 9)}{3x_k^2 + 4}$$
$$x_{k+1} = \frac{2x_k^3 + 9}{3x_k^2 + 4}$$

11. (b)

Here, $x_0 = 2$

$$f(x) = x^3 - x^2 + 4x - 4$$
$$f'(x) = 3x^2 - 2x + 4$$
$$f(x_0) = f(2) = 8$$
$$f'(x_0) = f'(2) = 12$$
$$\Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{8}{12} = \frac{4}{3}$$

12. (a)

Given, $x_{n+1} = \dfrac{x_n}{2} + \dfrac{9}{8x_n}$, $x_0 = 0.5$

as $n \to \infty$, when the series converges

$$x_{n+1} = x_n = \alpha = \text{root of equation}$$
$$\alpha = \frac{\alpha}{2} + \frac{9}{8\alpha}$$
$$\alpha = \frac{4\alpha^2 + 9}{8\alpha}$$

$$\Rightarrow \quad 8\alpha^2 = 4\alpha^2 + 9$$

$$\Rightarrow \quad \alpha^2 = \frac{9}{4}$$

$$\alpha = \frac{3}{2} = 1.5$$

13. (a)

$$h = \frac{2\pi - 0}{8} = \frac{\pi}{4}$$

$$y_0 = \sin(0) = 0$$

$$y_1 = \sin\left(\frac{\pi}{4}\right) = 0.70710$$

$$y_2 = \sin\left(\frac{\pi}{2}\right) = 1$$

$$y_3 = \sin\left(\frac{3\pi}{4}\right) = 0.7010$$

$$y_4 = \sin(\pi) = 0$$

$$y_5 = \sin\left(\frac{5\pi}{4}\right) = -0.70710$$

$$y_6 = \sin\left(\frac{6\pi}{4}\right) = -1$$

$$y_7 = \sin\left(\frac{7\pi}{4}\right) = -0.70710$$

$$y_8 = \sin\left(\frac{8\pi}{4}\right) = 0$$

i	x	$y = \sin x$
0	0	0
1	$\frac{\pi}{4}$	0.70710
2	$\frac{\pi}{2}$	1
3	$\frac{3\pi}{4}$	0.70710
4	π	0
5	$\frac{5\pi}{4}$	–0.70710
6	$\frac{6\pi}{4}$	–1
7	$\frac{7\pi}{4}$	–0.70710
8	2π	0

Trapezoidal rule

$$\int_{x_0}^{x_0+nh} f(x)\cdot dx = \frac{h}{2}[(y_0 + y_n + 2(y_1 + y_2 + \ldots + y_{n-1})]$$

$$\int_0^{2\pi} \sin x \cdot dx = \frac{\pi}{8} \times [(0+0) + 2(0.70710 + 1 + 0.70710 + 0 - 0.70710 - 0.70710)] = 0.00000$$

14. (d)

Here, $\frac{dx}{dt} = \frac{1-x}{\tau}$

Here, $f(x, y) = \frac{1-x}{\tau}$

Euler's Method Equation is

$$x_{j+1} = x_j + h\,f(x_j, y_j)$$

$$\Rightarrow \quad x_{j+1} = x_j + h\left(\frac{1-x_j}{\tau}\right)$$

$$\Rightarrow \quad x_{j+1} = \left(1 - \frac{h}{\tau}\right)x_j + \frac{h}{\tau}$$

For stability $\left|1 - \frac{h}{\tau}\right| < 1$

$$\Rightarrow -1 \le 1 - \frac{h}{\tau} \le 1$$

Since, $h = \Delta T$ here,

$$-1 \le 1 - \frac{\Delta T}{\tau} < 1$$

$$\Rightarrow \quad \Delta T < 2\tau$$

So, maximum permissible value of ΔT is 2τ.

15. (a)

Here $f(x) = e^x - 1$

$f'(x) = e^x$

The newton Raphson iterative equation is

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$f(x_i) = e^{x_i} - 1$$

$$f'(x_i) = e^{x_i}$$

$$\therefore \quad x_{i+1} = x_i - \frac{e^{x_i} - 1}{e^{x_i}}$$

i.e.

$$x_{i+1} = \frac{x_i\,e^{x_i} - (e^{x_i} - 1)}{e^{x_i}}$$

$$= \frac{e^{x_i}(x_i - 1) + 1}{e^{x_i}}$$

Now put $i = 0$

$$x_1 = \frac{e^{x_0}(x_0 - 1) + 1}{e^{x_0}}$$

Put $x_0 = -1$ as given,

$$x_1 = [e^{-1}(-2) + 1]/e^{-1}$$

$$= 0.71828$$

16. (c)

The given equation to be solved is

$$x = e^{-x}$$

Which can be rewritten as

$$f(x) = x - e^{-x} = 0$$

$$f'(x) = 1 + e^{-x}$$

The Newton-Raphson iterative formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Here $f(x_n) = x_n - e^{-x_n}$

$f'(x_n) = 1 + e^{-x_n}$

$\therefore$ The Newton-Raphson iterative formula is

$$x_{n+1} = x_n - \frac{x_n - e^{-x_n}}{1 + e^{-x_n}}$$

$$= \frac{e^{-x_n}x_n + e^{-x_n}}{1 + e^{-x_n}} = (1 + x_n)\frac{e^{-x_n}}{1 + e^{-x_n}}$$

17. (c)

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{R}{x_n}\right)$$

at convergence

$$x_{n+1} = x_n = \alpha$$

$$\alpha = \frac{1}{2}\left(\alpha + \frac{R}{\alpha}\right)$$

$$2\alpha = \alpha + \frac{R}{\alpha} = \frac{\alpha^2 + R}{\alpha}$$

$$2\alpha^2 = \alpha^2 + R$$

$$\Rightarrow \quad \alpha^2 = R$$

$$\alpha = \sqrt{R}$$

So, this iteration will compute the square roof of R. Correct choice is (c).

18. (a)

Here, the function being integrated is

$$f(x) = xe^x$$

$$f'(x) = xe^x + e^x = e^x(x+1)$$

$$f''(x) = xe^x + e^x + e^x = e^x(x+2)$$

Since, both e^x and x are increasing functions of x, maximum value of $f''(\xi)$ in interval $1 \le \xi \le 2$, occurs at $\xi = 2$.

So,

$$\max|f''(\xi)| = e^2(2+2) = 4e^2$$

Truncation Error for trapezoidal rule = TE (bound)

$$= \frac{h^3}{12}\max|f''(\xi)| * N_i$$

where N_i is number of subintervals

$$N_i = \frac{b-a}{h}$$

$$\therefore \quad T_{\epsilon(\text{bound})} = \frac{h^3}{12}\max|f''(\xi)| * \frac{b-a}{h}$$

$$= \frac{h^2}{12}(b-a)\max|f''(\xi)|\, 1 \le \xi \le 2$$

$$= \frac{h^2}{12}(2-1)(4e^2) = \frac{h^2}{3}e^2$$

Now putting

$$T_{\epsilon(\text{bound})} = \frac{1}{3} \times 10^{-6}$$

We get

$$\frac{h^2}{3}e^2 = \frac{1}{3} \times 10^{-6}$$

$$\Rightarrow \quad h^2 = \frac{10^{-6}}{e^2}$$

$$\Rightarrow \quad h = \frac{10^{-3}}{e}$$

Now, Number of Intervals = N_i

$$= \frac{b-a}{h} = \frac{2-1}{(10^{-3}/e)} = 1000e$$

19. (a)

$$x_{K+1} = x_K - \frac{f(x_K)}{f'(x_K)} = x_K - \frac{x_K^2 - 117}{2x_K}$$

$$= \frac{1}{2}\left[x_K + \frac{117}{x_K}\right]$$

20. (d)

The equation is $f(x) = x^2 - 13 = 0$

Newton-Raphson iteration equation is

$$x_1 = x_0 - \left[\frac{f(x_0)}{f'(x_0)}\right]$$

$$f(x_0) = x_0^2 - 13$$

$$f'(x_0) = 2x_0$$

$$\therefore \quad x_1 = x_0 - \left[\frac{x_0^2 - 13}{2x_0}\right] = \frac{x_0^2 + 13}{2x_0}$$

put $\quad x_0 = 3.5$ (as given)

$$x_1 = \frac{3.5^2 + 13}{2 \times 3.5} = 3.607$$

$\therefore$ The approximation after one interation = 3.607

21. (a)

$$I = \frac{1}{3}h(f_0 + 4f_i + 2f_2 + 4f_3 + f_4)$$

$$= \frac{1}{3} \times 0.25(1 + 4 \times 0.9412 + 2 \times 0.8 + 4 \times 0.64 + 0.5)$$

$$= 0.7854$$

22. (b)

Flywheel energy = $\int_0^{2\pi} T(\theta)d\theta$, where $T(\theta)$ is torque exerted.

The integral by using simpson's rule is

$$I = \frac{h}{3}(f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + 4f_5 + f_6)$$

$$h = 60 \text{ degrees} = \frac{\pi}{3} \text{ radians}$$

$$I = \frac{1}{3} \times \frac{\pi}{3} \times \left[\begin{matrix}0 + 4 \times 1066 + 2(-323) \\ +4(0) + 2(323) + 4(-355) + 0\end{matrix}\right] = 993$$

23. (b)

$$\frac{dy}{dx} - y = x, \quad y(0) = 0$$

step size = $h = 0.1$

Euler's first order formula is

$$y_{i+1} = y_i + hf(x_i, y_i)$$

$$y_1 = y_0 + hf(x_0, y_0)$$

Here, $x_0 = 0,\ y_0 = y(x_0) = y(0) = 0$

$x_1 = x_0 + h = 0 + 0.1 = 0.1$

$$f(x, y) = \frac{dy}{dx} = y + x$$

$\Rightarrow \quad y_1 = y_0 + hf(x_0, y_0)$

$= 0 + 0.1 \times f(0, 0)$

$= 0 + 0.1 \times (0 + 0) = 0$

Now, $x_1 = 0.1,\ y_1 = 0$

$x_2 = x_0 + 2h = 0 + 2 \times 0.1 = 0.2$

$\Rightarrow \quad y_2 = y_1 + hf(x_1, y_1)$

$= 0 + 0.1 \times f(0.1, 0)$

$= 0 + 0.1\,(0.1 + 0) = 0.01$

Now, $x_2 = 0.2,\ y_2 = 0.01$

$x_3 = x_0 + 3h = 0 + 3 \times 0.1 = 0.3$

$\Rightarrow \quad y_3 = y_2 + hf(x_2, y_2)$

$= 0.01 + 0.1 \times f(0.2, 0.01) = 0.01$

$+ 0.1\,(0.2 + 0.01) = 0.031$

$\therefore$ at $x_3 = 0.3,\ y_3 = 0.031.$

24. (d)

Let us try Dolittle's decomposition by putting

$l_{11} = 1$ and $l_{22} = 1$

$$\begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$$

$u_{11} = 2,\ u_{12} = 1$

$l_{21}\, u_{11} = 4$

$$\Rightarrow \quad l_{21} = \frac{4}{2} = 2$$

$l_{21}\, u_{12} + u_{22} = -1$

$\Rightarrow 2 \times 1 + u_{22} = -1$

$\Rightarrow \quad u_{22} = -3$

So one possible breakdown is

$$\begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & -3 \end{bmatrix}$$

But this is not any of the choices given.

So let us do Crout's decomposition, by putting $u_{11} = 1$ and $u_{22} = 1$

$$\begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix}\begin{bmatrix} 1 & u_{12} \\ 0 & 1 \end{bmatrix}$$

$l_{11} = 2,\ l_{11}u_{12} = 1$

$$\Rightarrow \quad u_{12} = \frac{1}{2} = 0.5$$

$l_{21} = 4,\ l_{21}\, u_{12} + l_{22} = -1$

$$\Rightarrow 4 \times \frac{1}{2} + l_{22} = -1$$

$\Rightarrow \quad l_{22} = -3$

$$\text{So,} \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}$$

25. (a)

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \left(\frac{x_i^2 - N}{2x_i}\right)$$

$$= \frac{x_i^2 + N}{2x_i} = \frac{1}{2}\left[x_i + \frac{N}{x_i}\right]$$

26. (d)

-1 is one of the roots since

$(-1)^3 + (-1)^2 + (-1) + 1 = 0$

By polynomial division

$$\frac{x^3 + x^2 + x + 1}{\{x - (-1)\}} = x^2 + 1$$

$\Rightarrow x^3 + x^2 + x + 1 = (x^2 + 1)(x + 1)$

So roots are $(-1, +j, -j)$

27. (b)

$$u(x_1, x_2) = 10x_2 \sin x_1 - 0.8 = 0$$

$$v(x_1, x_2) = 10x_2^2 - 10x_2 \cos x_1 - 0.6 = 0$$

The Jacobian matrix is

$$\begin{bmatrix} \frac{\partial u}{\partial x_1} & \frac{\partial u}{\partial x_2} \\ \frac{\partial v}{\partial x_1} & \frac{\partial v}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 10x_2 \cos x_1 & 10 \sin x_1 \\ 10x_2 \sin x_1 & 20x_2 - 10\cos x_1 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

28. (c)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$x = 2, \quad f(x_0) = 2 + \sqrt{2} - 3 = \sqrt{2} \quad 1$

$$f'(x) = 1 + \frac{1}{2\sqrt{x}}$$

$$f'(x_0) = 1 + \frac{1}{2\sqrt{2}}$$

Then, $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{\sqrt{2} - 1}{1 + \frac{1}{2\sqrt{2}}}$$

$\Rightarrow \quad x_1 = 1.694$

29. (c)

$$I = \int_1^3 \frac{1}{x}\,dx$$

x	$f(x) = \frac{1}{x}$
1	1
2	$\frac{1}{2}$
3	$\frac{1}{3}$

$$I = \frac{h}{3}(f_o + 4f_1 + f_2)$$

$$= \frac{1}{3}\left(1 + 4 \times \frac{1}{2} \times \frac{1}{3}\right) = 1.111$$

30. (b)

If bisection method is applied to given problem with $x_0 = 1$ and $x_1 = 9$

After 1 iteration $x_2 = \frac{1+9}{2} = 5$

Now since $f(x_1)\, f(x_2) > 0$, x_2 replaces x_1

Now, $x_0 = 1$ and $x_1 = 5$

and after 2nd iteration $x_2 = \frac{1+5}{2} = 3$

Now since $f(x_1)\, f(x_2) > 0$, x_2 replaces x_1 and $x_0 = 1$ and $x_1 = 3$ and after 3rd iteration

$$x_2 = \frac{1+3}{2} = 2$$

Now $f(x_2) = f(2)$

$$= 2^4 - 2^3 - 2^2 - 4 = 0$$

So the method converges exactly to the root in 3 iterations.

31. (d)

Exact value of

$$\int_{0.5}^{1.5} \frac{dx}{x} = [\log x]_{0.5}^{1.5}$$

$$= \log(1.5) - \log(0.5) = 1.0986$$

Approximate value by Simpson's rule with 3pts is

$$I = \frac{h}{3}(f(0) + 4f(1) + f(2))$$

$$n_i = n_{pt} - 1 = 3 - 1 = 2$$

(n_{pt} is the number of pts and n_i is the number of intervals)

Here $h = \frac{b-a}{n_i} = \frac{1.5-0.5}{2} = 0.5$

The table is

i	x_i	f_i
0	0.5	$\frac{1}{0.5}$
1	1.0	$\frac{1}{1}$
2	1.5	$\frac{1}{1.5}$

$$I = \frac{0.5}{3}\left(\frac{1}{0.5} + 4 \times 1 + \frac{1}{1.5}\right) = 1.1111$$

So the estimate exceeds the exact value by

Approximate value – Exact value

$$= 1.1111 - 1.0986$$

$$= 0.012499 \approx 0.012$$

32. (d)

Error in central difference formula is $0(h^2)$

This means,

$$\text{error} \propto h^2$$

If error for $h = 0.03$ is 2×10^{-3}

then

Error for $h = 0.02$ is approximately

$$2 \times 10^{-3} \times \frac{(0.02)^2}{(0.03)^2} \simeq 9 \times 10^{-4}$$

33. (c)

$$f'(x) = 3x^2 + 2$$

$$f'(x_0) = 3(1.2)^2 + 2 = 6.32$$

$$f(x_0) = (1.2)^3 + 2 \times 1.2 - 1 = 3.128$$

$$f'(x_1) = x_0 - \frac{f(x_0)}{f'(x_1)} = 1.2 - \frac{3.728}{6.32}$$

$$= 0.705$$

34. Sol.

x	0	1	2	3	4
y	10	11	26	91	266

Using Simpson's Rule, the estimated value of the integral $\int_0^4 (x^4 + 10)\,dx$

$$= \frac{1}{3}\left[(10 + 266) + 2(26) + 4(11 + 91)\right] = 245.33$$

The exact value of integral

$$\int_0^4 (x^4 + 10)\,dx = \left[\frac{x^5}{5} + 10x\right]_0^4$$

$$= \frac{4^5}{5} + 10 \times 4 = 244.8$$

∴ Magnitude of error = |exact value – estimated value|

$$= |244.8 - 245.33| = 0.53$$

36. (d)

$$\frac{dy}{dx} + 2xy^2 = 0$$

$$\therefore \quad \frac{dy}{dx} = -2xy^2$$

after one iteration

$$y_1^* = y_0 + h[-2x_0\, y_0^2]$$

$$= 1 + 0.2\,[-2 \times 0 \times 1^2] = 1 + 0 = 1$$

$$y_1 = y_0 + \frac{1}{2}\times(0.2)[-2x_0 y_0^2 - 2x_1 y_1^*]$$
$$= 1 + 0.1[-(2\times 0\times 1^2) - (2\times 0.2\times 1)]$$
$$= 1 + 0.1\,[-0 - 0.4] = 1 - 0.04 = 0.96$$

38. Sol.

$$f(x) = 5x - 2\cos x - 1$$
$$f'(x) = 5 + 2\sin x$$

By Newton Raphson's equation:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Assuming $x_0 = 1$ (1 rad = 57.32°)

$$\Rightarrow \quad x_1 = 1 - \frac{5\times 1 - 2\cos(57.32°) - 1}{5 + 2\sin(57.32°)} = 0.5632$$

Again,

$$x_2 = 0.5632 - \frac{5\times 0.5632 - 2\cos(32.27°) - 1}{5 + 2\sin(32.27°)}$$
$$= 0.5425$$
$$x_3 = 0.5425 - \frac{5\times 0.5425 - 2\cos(31.09°) - 1}{5 + 2\sin(31.09°)}$$
$$= 0.5424$$

$\therefore$ Real root, $x = 0.54$

39. Sol.

Given, $f(x) = e^x - 1$

or, $f(x_K) = (e^{x_K} - 1)$ and $x_o = 1$

In Newton-Raphson method, we have:

$$x_{K+1} = \left[x_K - \frac{f(x_K)}{f'(x_K)}\right]$$
$$\therefore \quad x_1 = \left[x_o - \frac{f(x_o)}{f'(x_o)}\right] \quad \text{...(i)}$$

Now, $f(x_o) = e^{x_o} - 1 = e^1 - 1 = (e - 1)$

and $f'(x) = e^x$

$\therefore$ $f'(x_0) = e^1 = e$

Putting the values, we get:

$$x_1 = \left[1 - \frac{(e-1)}{e}\right]$$
$$= \left[\frac{e - e + 1}{e}\right] = e^{-1} = 0.367$$

Also, $$x_2 = \left[x_1 - \frac{f(x_1)}{f'(x_1)}\right] \quad \text{...(ii)}$$

Now, $x_1 = \frac{1}{e} = e^{-1}$ and $f(x_1) = (e^{e^{-1}} - 1)$,

$$f'(x_1) = e^{e^{-1}}$$

Putting the values, we get:

$$x_2 = \left[e^{-1} - \frac{(e^{e^{-1}} - 1)}{e^{e^{-1}}}\right]$$
$$= \left[e^{-1} - \frac{(e^{0.37} - 1)}{e^{0.37}}\right] = 0.06$$

Therefore, the absolute error observed at second iteration = 0.06.

Absolute error at any iteration

$$= \left|\frac{\text{Exact value} - \text{Approximate value}}{\text{Exact value}}\right|$$
$$\simeq \left|\frac{\text{New value} - \text{Old value}}{\text{New value}}\right|$$
$$\simeq \left|\frac{0.06 - 0.368}{0.06}\right| = 0.248$$

40. (a)

Compute $x_1, x_2, \ldots$ using the iteration equation

$$x_1 = x_0 - \left[\frac{f(x_0)}{f'(x_0)}\right]$$
$$= x_0 - \left[\frac{0.75x_0^3 - 2x_0^2 - 2x_0 + 4}{2.25x_0^2 - 4x_0 - 2}\right]$$

$$\Rightarrow \quad x_0 = 2,\ x_1 = 0,\ x_2 = 2,\ x_3 = 0, \ldots$$

$x_3 = 0$ is correct but it converges in an infinite steps (i.e. it doesn't converge).

41. Sol.

x	2.5	2.8	3.1	3.4	3.7	4
$y = f(x)$	0.1963	1.0296	1.1314	1.2237	1.3083	1.3863
y_n	y_0	y_1	y_2	y_3	y_4	y_5

$$I = \int_{2.5}^{4} ln(x)dx$$
$$= \frac{h}{2}[(y_0 + y_5) + 2(y_1 + y_2 + y_3 + y_4]$$
$$I = \frac{0.3}{2}[(0.9163 + 1.3863) + 2(1.0296 + 1.1314 + 1.2237 + 1.3083)]$$
$$I = \frac{0.3}{2}\times 11.6886 = 1.7533$$

42. Sol.

x	1	2	3
$y = f(x)$	1	0.5	0.33
y_n	y_0	y_1	y_2

$$I = \int_1^3 \frac{1}{x} dx = \frac{h}{2}[(y_0 + y_2) + 2y_1]$$

$$= \frac{1}{2}[1 + 0.33 + 2 \times 0.5] = \frac{2.33}{2} = 1.165$$

. **Sol.**

$$h = \frac{b-a}{n_i} = \frac{1-(-1)}{3} = \frac{2}{3} = 0.667$$

x	$f(x) = \lvert x \rvert$
−1	1
−0.333	0.333
+0.333	0.333
1	1

$$I = \frac{h}{2}(f_0 + 2f_1 + 2f_2 + f_3)$$

$$= \frac{0.667}{2}(1 + 2 \times 0.333 + 2 \times 0.333 + 1)$$

$$= 1.11$$

. (c)

While computing $K = \int_a^b x^2 dx$

Error = Exact value − Approximate value

For trapezoidal rule

$$\text{Error} = -\frac{h^3}{12} f''(\xi) \times n_i$$

Since h and n_i are always positive, sign of the error is controled only by the sign of $f''(\xi)$.

Here $f(x) = x^2$ so $f''(x) = 2$ which is always positive.

So the sign of the error is always negative. i.e. approximate value always greater than or equal to the exact value of the integral.

So (I) is true.

Similarly for **Simpson's rule**

$$\text{Error} = -\frac{h^5}{90} f''''(\xi) \times n_i$$

Since h and n_i are always positive, sign of the error is controled only by the sign of $f''''(\xi)$.

Here $f(x) = x^2$ so $f''''(x) = 0$

So the error is always 0. i.e. approximate value always equal to the exact value of the integral.

So (II) is true.

Therefore both (I) and (II) are correct.

;. (d)

$$\frac{dx}{dt} = 4t + 4 = f(t_0, x_0)$$

At $t = 0$, $x = x_0$

Irrespective of values of x, $f(t_0, x_0)$ depends on t only.

$$k_1 = h\, f(t_0, x_0) = 0.2x \times 4 = 0.8$$

$$k_2 = hf\left(t_0 + \frac{1}{2}h, x_0 + \frac{k_1}{2}\right)$$

$$= hf(0.1, x_0 + 0.4)$$

$$= 0.2(4 \times 0.1 + 4) = 0.88$$

$$k_3 = hf\left(t_0 + \frac{1}{2}h, x_0 + \frac{k_2}{2}\right)$$

$$= 0.2f(0.1, x_0 + 0.44)$$

$$= 0.2(4 \times 0.1 + 4) = 0.88$$

$$k_4 = hf(t_0 + h, x_0 + k_3)$$

$$= 0.2f(0.2, x_0 + 0.88)$$

$$= 0.2(4 \times 0.2 + 4) = 0.96$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.8 + 2 \times 0.88 + 2 \times 0.88 + 0.96)$$

$$= 0.88$$

46. **Sol.**

$$\begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} 1 & U_{12} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 4 & 9 \end{bmatrix}$$

This is crout's LU decomposition, since diagonal elements of U are 1. So we will setup the equations for the elements of the matrix taken column-wise, as follows

$$L_{11} = 2,\ L_{21} = 4$$

$$L_{11} \times U_{12} = 2$$

$$\Rightarrow U_{12} = 1,\ L_{21} \times U_{12} + L_{22} = 9$$

$$\Rightarrow 4 + L_{22} = 9$$

$$\Rightarrow L_{22} = 5$$

47. (c)

Intermediate value theorem states that if a function is continuous and $f(a) \cdot f(b) < 0$, then surely there is a root in (a, b). The contrapositive of this theorem is that if a function is continuous and has no root in (a, b) then surely $f(a) \cdot f(b) \geq 0$. But since it is given that there is no root in the closed interval $[a, b]$ it means $f(a) \cdot f(b) \neq 0$.

So surely $f(a) \cdot f(b) > 0$ which is choice (c).

48. **Sol.**

$$f(x) = x^2 - 4x + 4$$

$$f'(x) = 2x - 4$$

$$x_0 = 3$$

$$f(3) = 1,\ f'(3) = 2$$

By Newton Rapshon method,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - \frac{1}{2} = \frac{5}{2}$$

$$f\left(\frac{5}{2}\right) = \frac{25}{4} - 10 + 4 = \frac{1}{4}$$

By secant method,

$$x_2 = x_1 - \frac{(x_1 - x_0)f(x_1)}{f(x_1) - f(x_0)}$$

$$= \frac{f_1 x_0 - f_0 x_1}{f_1 - f_0} = \frac{\left(\frac{1}{4} \times 3\right) - \left(1 \times \frac{5}{2}\right)}{\frac{1}{4} - 1}$$

$$= \frac{7}{3}$$

49. Sol.

$$f(x) = -2 + 6x - 4x^2 + 0.5x^3$$

$$f'(x) = 6 - 8x + 1.5x^2$$

$$x_{ini} = 0$$

By Newton Raphson Method,

$$x_1 = x_{ini} - \frac{f(x_{ini})}{f'(x_{ini})} = 0 - \frac{-2}{6}$$

$$\Rightarrow \quad x_1 = \frac{1}{3}$$

$$\therefore \quad \Delta x = x_1 - x_{ini} = \frac{1}{3}$$

50. Sol.

$$f(x) = x^3 + 2x^2 + 3x - 1$$

$$f'(x) = 3x^2 + 4x + 3$$

$$x_0 = 1$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{[5]}{[10]} = 0.5$$

$$x_2 = 0.5 - \frac{[f(0.5)]}{[f'(0.5)]} = 0.5 - \frac{[0.125]}{[5.15]}$$

$$= 0.3043$$

51. Sol.

$$f(x) = x^3 - 5x^2 + 6x - 8$$

$$x_0 = 5$$

$$f'(x) = 3x^2 - 10x + 6$$

$$x_1 = x_0 - \frac{f(x_o)}{f'(x_o)} = 5 - \frac{f(5)}{f'(5)}$$

$$= 5 - \frac{5^3 - 5 \times 5^2 + 6 \times 5 - 8}{3 \times 5^2 - 10 \times 5 + 6}$$

$$= 5 - \frac{22}{31} = 5 - 0.7097 = 4.2903$$

52. (c & d)

Secant method formula is $x_2 = \dfrac{f_1 x_0 - f_0 x_1}{f_1 - f_0}$

i.e. $$x_t = \frac{f_b x_a - f_a x_b}{f_b - f_a}$$

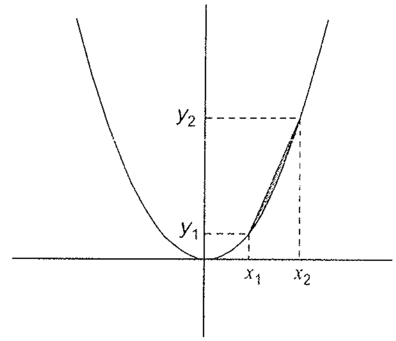
$$x_t = x_b - (x_b - x_a)\, f_b / (f_b - f(x_a))$$

$$= x_a - (x_b - x_a)\, f_a / (f_b - f(x_a))$$

$$= \frac{f_b x_a - f_a x_b}{f_b - f_a}$$

$\therefore$ Both (c) & (d) after simplification reduce to the required formula. So both (c) and (d) are correct.

53. (a)



Exact value is computed by integration which follows thee exact shape of graph while computing the area.

Whereas, in Trapezoidal rule, the lines joining each points are considered straight lines which is not the exact variation of graph all the time like as shown in figure.

$\therefore \quad J > I$

OR

$$\text{Error} = -\frac{h^3}{12} f''(\xi) \times n_i$$

Here, $f(x) = x^2$

$f'(x) = 2x$

$f''(x) = 2 > 0$

Since $f''(x)$ is positive, the error is negative.

Since error = exact – approximate.

$= I - J$

and since error is negative in this case $J > I$ is true.

54. Sol.

x	0	0.4	0.8
$f(x)$	0.2	2.456	0.232

$$a = 0,\ b = 0.8,\ \Delta x = 0.4$$

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

By Simpson's 1/3 Rule

$$y = \int_0^{0.8} f(x)dx$$

$$= \frac{4}{3}[y_0 + 4y_1 + y_2]$$

$$y_0 = y(0) = 0.2$$

$$y_1 = y(0.4) = 2.456$$

$$y_2 = y(0.8) = 0.232$$

$$y(n) = \frac{0.4}{3}(0.2 + 4 \times 2.456 + 0.232)$$

$$= 1.367$$

Sol.

	y_0	y_n
x	1	2
$f(x)$	0	2(ln2)

$$\therefore \quad I = \frac{h}{2}[y_0 + y_n]$$

$$I = \frac{1}{2}[0 + 2\ln 2] = \ln 2 = 0.693$$

Sol.

$$f(x) = \frac{3}{5}x^2 + \frac{9}{5}$$

x	0	0.5	1
$f(x)$	1.8	1.95	2.4

$$\Rightarrow \quad \int_0^1 f(x) = \frac{h}{3}[y_0 + 4y_1 + y_2]$$

$$= \frac{0.5}{3}[1.8 + 4(1.95) + 2.4] = 2$$

Sol.

$$\int_0^{0.4} f(x)dx = \frac{h}{2}[y_0 + 2[y_1 + y_2 + y_3] + y_4]$$

$$= \frac{0.1}{2}[0 + 2[10 + 40 + 90] + 160]$$

$$= 22$$

Sol.

Given that the motorbike starts from rest.

$\therefore$ At $t = 0$, $v = 0$

So the table now becomes

t	2	4	6	8	10	12	14	16	18	20
v	10	18	25	29	32	20	11	5	2	0

h = Table spacing = 2 minutes

So the distance (in kilometers) covered in 20 minutes using Simpson's rule

$$= \int_0^{20} v\,dt$$

$$= \frac{h}{3}(f_0 + 4f_1 + 2f_2 + 4f_3 + \ldots + f_{10})$$

$$= \frac{2}{3}(0 + 4 \times 10 + 2 \times 18 + 4 \times 25 + \ldots + 0)$$

$$= 309.33$$

59. Sol.

The equations are

$x_1 + 2x_2 + 3x_3 = 5$ ...(3)

$2x_1 + 3x_2 + x_3 = 1$ ...(2)

$3x_1 + 2x_2 + x_3 = 3$ ...(1)

By poveting we can write

$3x_1 + 2x_2 + x_3 = 3$

$2x_1 + 3x_2 + x_3 = 1$

$$x_1 = \frac{3 - 2x_2 - x_3}{3} \quad \ldots(1)$$

$$x_2 = \frac{1 - 2x_1 - x_3}{3} \quad \ldots(2)$$

$x_1 + 2x_2 + 3x_3 = 5$

$$x_3 = \frac{5 - x_1 - 2x_2}{3} \quad \ldots(3)$$

Put $x_2 = 0$ $x_3 = 0$ in equation (1) $x_1 = 1$

Put $x_1 = 1$ $x_3 = 0$ in equation (3) $x_2 = -0.333$

Put $x_1 = 1$ $x_2 = -0.333$ in equation (3) $x_3 = 1.555$

$x_3 = 1.555$

60. Sol.

$$f(x) = x - 10\cos x \quad f\left(\frac{\pi}{4}\right)$$

$$= \frac{\pi}{4} - \frac{10}{\sqrt{2}} = -6.286$$

$$f'(x) = 1 + 10\sin x \quad f'\left(\frac{\pi}{4}\right)$$

$$= 1 + \frac{10}{\sqrt{2}} = 8.07$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = \frac{\pi}{4} - \left(\frac{-6.286}{8.07}\right)$$

$$= \frac{\pi}{4} + \frac{6.286}{8.07} = 1.5639$$

61. (c)

$$f(x) = x^3 + x - 1$$

$$f(1) = 1 + 1 - 1 = 1$$

$$f'(x) = 3x^2 + 1$$
$$f'(1) = 3 + 1 = 4$$
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
$$= 1 - \frac{1}{4} = 1 - 0.25 = \mathbf{0.750}$$

62. Sol.

According to Newton-Raphson Method:

$$X_{N+1} = X_N - \frac{f(X_N)}{f'(X_N)}$$
$$f(x) = 3x - e^x + \sin x$$
$$f'(x) = 3 - e^x + \cos x$$
$$\Rightarrow \quad X_1 = X_0 - \frac{f(0.333)}{f'(0.333)}$$
$$= 0.333 - \frac{3 \times 0.333 - e^{0.333} + \sin 0.333}{3 - e^{0.333} + \cos 0.333}$$
$$\therefore \quad X_1 = 0.36$$

63. (a)

Trapezoidal rule gives the best result in single variable function when the function is linear (degree 1).

64. Sol.

x	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π
$f(x)$	1 y_0	1.366 y_1	0.366 y_2	−1 y_3

By Trapezoidal

$$\int_0^{\pi} (\sin x + \cos x)dx$$
$$= \frac{\pi/3}{2}(1 + (-1) + 2(1.366 + 0.366))$$
$$= \frac{\pi}{3}(1.732) = 1.1812$$
$$\int_0^{\pi} (\sin x + \cos x)dx = \int_0^{\pi/2} (\sin x + \cos x)dx + \int_{\pi/2}^{\pi} (\sin x + \cos x)dx$$
$$= (-\cos x + \sin x)\Big|_0^{\pi/2} + (-\cos x + \sin x)\Big|_{\pi/2}^{\pi}$$
$$= [(0 + 1) - (-1 + 0)] + [(1 + 0) - (0 + 1)]$$
$$= 1 + 1 + 1 - 1 = 2$$

Error = Exact value − approx value
= 2 − 1.1812 = 0.187

65. Sol.

$$\frac{dy}{dx} = -3y + 2, \quad y(0) = 1$$

If $|1 - 3h| < 1$, then solution of differential equation is stable.

$$-1 < 1 - 3h < 1$$
$$-2 < -3h < 0$$
$$0 < h < \frac{2}{3}$$
$$h_{max} = \frac{2}{3} = 0.66$$

66. Sol.

$$\frac{dy}{dx} = y + 2x - x^2$$
$$y(0) = 1$$
$$0 \le x \le \infty$$
$$f(x, y) = y + 2x - x^2$$
$$x_0 = 0\ ;\ y_0 = 1\ ;\ h = 0.1$$
$$k_1 = hf(x_0, y_0)$$
$$= 0.1(1 + 2 \times 0 - 0^2) = 0.1$$
$$k_2 = hf(x_0 + h, y_0 + k_1)$$
$$= 0.1((y_0 + k_1) + 2(x_0 + h) - (x_0 + h)^2)$$
$$= 0.1((1 + 0.1) + 2(0.1) - (0.1)^2)$$
$$= 0.129$$
$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$
$$= 1 + \frac{1}{2}(0.1 + 0.129) = 1.1145$$

Exact solution

$$y(x) = x^2 + e^x = (0.1)^2 + e^{0.1} = 1.1152$$

Error = 1.1152 − 1.1145 = 0.00069

% error = 0.06%

67. (a)

x	0	0.5	1
y	3	4	5

By Trapezoidal rule

$$\int_0^1 f(x)dx = \frac{h}{2}[(y_0 + y_2) + 2y_1]$$
$$= \frac{0.5}{2}[(3 + 5) + 2(4)]$$

By Simpson rule

$$\int_0^1 f(x)dx = \frac{h}{3}[(y_0 + y_2) + 4y_1]$$
$$= \frac{0.5}{3}[(3 + 5) + 4(4)]$$

The difference between the two results will be zero.

68. (d)

In the following table by calculating $\Delta^4 f$ we get 7.2×10^{-3} for all the differences. Which is constant for all values. Therefore the order of the polynomial is 4.

69. Sol.

$$f(x) = x^6 - x - 1$$

$$a = 1 \quad b = 2 \quad \epsilon = 0.01 = 10^{-3}$$

The minimum number of iterations by Bisection method is given by

$$\frac{|b-a|}{2^n} < \epsilon$$

$$\frac{2-1}{2^n} < 10^{-3}$$

$$\frac{1}{2^n} < \frac{1}{10^3}$$

$$2^n > 10^3$$

$$2^n > 1000$$

$$\frac{n}{n_2} > ln1000$$

$$n > \frac{ln1000}{ln2}$$

$$n > 9.96$$

$$\therefore \quad n = 10$$

70. Sol.

$$f(x) = x^3 + x - 1$$

$$f(1) = 1$$

$$f'(x) = 3x^2 + 1$$

$$f'(1) = 4$$

By Newton-Raphson method,

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

For $x_0 = 1$,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{1}{4} = 0.75$$

For $x_1 = 0.75$,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.75 - \frac{f(0.75)}{f'(0.75)}$$

$$= 0.75 - \frac{0.171875}{2.6875} = 0.686$$

71. Sol.

$$\frac{du}{dt} = 3t^2 + 1$$

$$f(u, t) = 3t^2 + 1$$

$$u_0 = 0$$

$$t_0 = 0$$

$$\Delta t = 2$$

By Euler's method

$$u_1 = u_0 + hf(u_0 t_0) \qquad t_1 = t_0 + h$$

$$= u_0 + h\left(3t_0^2 + 1\right) \qquad = 0 + 2$$

$$= 0 + 2\left(3(0)^2 + 1\right) \qquad = 2$$

$$= 2$$

After first iteration $u = 2$ when $t = 2$

$$\frac{du}{dt} = 3t^2 + 1$$

$$du = \left(3t^2 + 1\right) dt$$

$$\int du = \int_0^2 \left(3t^2 + 1\right) dt$$

$$u = \left(3\frac{t^3}{3} + t\right)\Bigg|_0^2$$

$$= 8 + 2 = 10$$

Absolute error = Exact value – approx value

$$= 10 - 2$$

$$= 8$$

72. (1)

Given $f(x) = 2x^2 - 3x + 3,\ x_0 = 2$

$$f'(x) = 4x - 3$$

By Newton-Rapshon

$$x_1 =$$

$$x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{2(2)^2 - 3(2) + 3}{4(2) - 3}$$

$$= 2 - \frac{5}{5} = 1$$

73. Sol.

$$y_1 = y_0 + h_f(t_0, y_0)$$

$$= y_0 + hy_0$$

$$= 1 + 0.1\,(1)$$

$$y_1 = 1.1$$

$$y_2 = y_1 + h_f(t_1, y_1)$$

$$= y_1 + h \cdot y_1$$

$$= 1.1 + 0.1(1.1)$$

$$y_2 = 1.21$$

$$y_3 = y_2 + hf(t_2, y_2)$$

$= y_2 + h \cdot y_2$

$= 1.21 + 0.1 \times 1.21$

$y_3 = 1.331$

$y_4 = y_3 + h \cdot f(t_3, y_3)$

$= y_3 + h \cdot y_3$

$= 1.331 + 0.1 \times 1.331$

$y_4 = 1.4641$

$y_5 = y_4 + h \cdot f(t_4, y_4)$

$= y_4 + h \cdot y_4$

$= 1.4641 + 0.1 \times (1.4641)$

$= 1.61051$

$y_6 = y_5 + h \cdot f(t_5, y_5)$

$= y_5 + h \cdot y_5$

$= 1.61051 + 0.1 \times 1.61051$

$y_6 = 1.771561$

$y_7 = y_6 + h \cdot f(t_6, y_6)$

$= y_6 + h \times y_6$

$= 1.771561 + 0.1 \times 1.771561$

$= 1.9487$

$y_8 = y_7 + h.f(t_7, y_7)$

$= y_7 + h \cdot y_7$

$= 1.9487 + 0.1 \times (1.9487)$

$y_8 = 2.14357$

$y_9 = y_8 + h \cdot f(t_8, y_8)$

$= y_8 + h \cdot y_8$

$= 2.14357 + 0.1 \times 2.14357$

$y_9 = 2.3579$

$y_{10} = y_9 + h \cdot f(t_9, y_9) = y_9 + h \cdot y_9$

$= 2.3579 + 0.1 \times (2.3579)$

$y_{10} = 2.5937$

74. (b)

If $p(s)$ has "r" real roots, then $p'(s)$ will have atleast "$r - 1$" real roots.

75. (c)

Let $x^3 = 1468$

$f(x) = x^3 - 1468 = 0$

By Newton Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 1468}{3x_n^2}$$

$$= \frac{2x_n^3 + 1468}{3x_n^2}$$

Start with $x_0 = 11$

$$x_1 = \frac{2(11)^3 + 1468}{3(11)^2} = 11.377$$

$$x_2 = \frac{2(11.377)^3 + 1468}{3(11.377)^2} = 11.365$$

76. (a)

Method-1:

$x = (1525)^{0.2}$

$x = (1525)^{1/5}$

$x^5 = 1525$

By substitution,

$(4.33)^5 = 1522.08$

$(4.36)^5 = 1575.55$

$(4.38)^5 = 1612.02$

$(4.30)^5 = 1470.08$

$\therefore$ $x = 4.33$ is the nearest value.

Method-2: (Newton Raphson method)

$x = (1525)^{0.2}$

$x^5 = 1525$

$f(x) = x^5 - 1525 = 0$

$f'(x) = 5x^4$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow \quad x_{n+1} = \frac{4x_n^5 + 1525}{5x_n^4}$$

Case-1: $x_0 = 4$

$$x_1 = 4 - \frac{f(4)}{f'(4)} = 4.391$$

Note: This cannot be solution as it is not correct upto 2 decimals.

$$x_2 = (4.391) - \frac{f(4.391)}{f'(4.391)} = 4.333$$

$$x_3 = (4.333) - \frac{f(4.333)}{f'(4.333)} = 4.331$$

Note: This is the correct solution upto 2 decimals.