6

Permutations and Combinations

6.01 Fundamental principle of multiplication

"If an event A can occur in m different ways and f, following it another event B can ocur in n different ways but both events A and B are not related, i.e. event B in n different ways does not depend on result of event A occurs. Then the total number of occurrence of the events A and B given order is $m \times n$."

Illustrative Examples

Example 1 : A classroom contains 10 boys and 8 girls. For any event, a teacher wants to select one boy and one girl. The number of ways in which one boy and one girl can be chosen will be $10 \times 8 = 80$.

Example 2 : Let the two questions A and B can be solved in 2 and 3 ways, thus the total number of ways of solving them will be $2 \times 3 = 6$.

Example 3 : Three persons stay in Ajmer and there are 6 hotels for their stay. If each one of them stays in different hotel, then in how many ways they can stay in Ajmer?

Solution : First person stay in any one of 6 hotels so this work can be done in 6 ways. If that person stays in a special hotel then second person can choose any hotel from 5 hence, there are 5 ways for second person. Now, when second person have selected his hotel so for third person there are 4 hotels, so he can do this work in four ways. Hence third person has 4 method to select hotel. Hence, for stay for three persons the total methods to select the hotels will be

$6 \times 5 \times 4 = 120$.

Example 4 : Let 5 trains run between Jodhpur and Jaipur. A person travels from Jodhpur to Jaipur by one train and goes back from Jaipur to Jodhpur by another train. In how many ways he can travel?

Solution : There are two parts of travelling-

(i) Jodhpur to Jaipur

(ii) Jaipur to Jodhpur

A passenger can travel in 5 different ways in the first part and 4 different ways in the second part, therefore the required ways will $5 \times 4 = 20$.

6.02 Fundamental principle of addition :

Let an event A, occurs in *m different ways and an event B*, occurs in n different ways and both the events can not occur together, then event A or B (atleast one of the two) will occur in $(m+n)$ different ways.

Illustrative Examples

Example 5 : There are 10 boys and 8 girls in a classroom. The teacher selects one boy or one girl for the event. The number of ways in which the selection can be done in $10 + 8 = 18$ ways.

Example 6 : Let the two questions A and B be solved in 2 and 3 different ways respectively, then A or B (atleast one of the two) can be solved in $2 + 3 = 5$ ways.

6.03 Factorial :

The product of first n natural numbers is termed as factorial n and denoted as $n!$ or $\lfloor n$. Hence

$n! = 1.2.3.......(n-1).n$

e.g. $3! = 1.2.3 = 6, \; 4! = 1.2.3.4 = 24, \; 5! = 1.2.3.4.5 = 120$

$$n! = 1.2.3.....(n-1)n$$
$$= \{1.2.3.....(n-1)\}n = \{(n-1)!\}n$$

Similarly $n! = n \times (n-1)!$

Example : $8! = 8 \times (7)!, \; 5! = 5 \times (4)!$

Note : 1. $0! = 1.$

2. Factorial is not defined for negative integers or fraction. Hence it is defined for whole numbers only.

6.04 Permutation :

Taking some or all things from the given number of things, the number in which they can be arranged in different ways of each order from them is called as permutation.

Illustrative Examples

Example 7 : If A, B, C are three objects then there will be 6 ways of chosing two objects out of three i.e.

AB, BC, CA, BA, CB, AC

here $AB \neq BA, \; AC \neq CA$ and $BC \neq CB$ i.e. the order is important. Changing the order a new permutation is formed.

Example 8 : If we choose 3 letters from the four given letters a, b, c, d. There are 24 permutations given below.

abc	*abd*	*bcd*	*acd*
acb	*adb*	*bdc*	*adc*
bca	*bda*	*cbd*	*cad*
bac	*bad*	*cdb*	*cda*
cab	*dab*	*dcb*	*dac*
cba	*dba*	*dbc*	*dca*

We get the above permutations after selecting 3 out of four letters and arranging them in all possible order.

6.05 Number of Permutations :

Theorem 1 : The numbers of permutations of n different objects taken r objects at a time is

$$n(n-1)(n-2).....\{n-(r-1)\}$$

i.e. : $${}^nP_r = n(n-1)(n-2).....(n-r+1).$$

Proof : We have n different objects and we have to fill r places. The first place can be filled in n ways then the number of ways for first place is n, following which the second place can be filled by remaining $(n-1)$ ways. Then the number of ways for second place is $(n-1)$. Hence by fundamental principle of multiplication the first two places have filled by $n\,(n-1)$ ways. Then the third place can be filled by remaining $(n-2)$ ways. The number of ways for third place is $(n-2)$ objects. Hence the first three places have filled by $n\,(n-1)\,(n-2)$ ways. Similarly this process is going on r places will filled by $n(n-1)(n-2)(n-3).....[n-(r-1)]$ ways. This number is denoted by ${}^n\mathrm{P}_\mathrm{r}$. Hence the number of permutations of n different objects selecting r objects at a time is

$${}^n\mathrm{P}_\mathrm{r} = n(n-1)(n-2).....(n-r+1)$$

Remark : There are some restrictions for the use of objects. Which is illursrated by the following examples.

Corollory 1 : $\quad {}^nP_r = \dfrac{n!}{(n-r)!}$

Proof : We know from theorem 1 that

$${}^nP_r = n(n-1)(n-2).....(n-r+1)$$

$$= \frac{n(n-1)(n-2).....\{n-(r-1)\}(n-r)\{n-(r+1)\}.....3.2.1}{(n-r)\{n-(r+1)\}.....3.2.1} = \frac{n!}{(n-r)!}.$$

Corollory 2 : The number of permutations of n objects taken all at a time is $n!$

Proof : From theorem 1, $\quad {}^nP_n = n(n-1)(n-2).....(n-n+1) = n(n-1)(n-2).....1 = n!$

Corollory 3 : $0! = 1$

From corollory 1.

Proof : We know that, $\quad {}^nP_r = \dfrac{n!}{(n-r)!}$

Putting $r = n$, $\quad {}^nP_n = \dfrac{n!}{(n-n)!}$

or $\quad n! = \dfrac{n!}{0!}$ $\quad$ or $\quad 0! = 1.$ $\quad$ [From corollory 2, $\because {}^nP_n = n!$]

Illustrative Examples

Example 9 : Prove the following

(i) ${}^nP_r = n.{}^{n-1}P_{r-1} = n(n-1).{}^{n-2}P_{r-2}$ $\quad$ **(ii)** ${}^nP_0 = 1$

(iii) ${}^nP_n = {}^nP_{n-1}$ $\quad$ **(iv)** ${}^nP_r = (n-r+1).{}^nP_{r-1}$

Solution : (i) ${}^nP_r = \dfrac{n!}{(n-r)!}$

$$= \frac{n(n-1)!}{\{(n-1)-(r-1)\}!} = n.{}^{n-1}P_{r-1}$$ (using corollary 1) ..... (1)

by (1) ${}^nP_r = \dfrac{n(n-1)(n-2)!}{\{(n-2)-(r-2)\}!} = n(n-1).{}^{n-2}P_{r-2}$

(ii) ${}^nP_0 = \dfrac{n!}{(n-0)!} = \dfrac{n!}{n!} = 1$ $\quad$ **(iii)** ${}^nP_n = n!$ $\quad$ [From corollory 2]

${}^nP_{n-1} = \dfrac{n!}{\{n-(n-1)\}!} = \dfrac{n!}{1!} = n! \quad \therefore \quad {}^nP_n = {}^nP_{n-1}$

(iv) R.H.S. $= (n-r+1).{}^nP_{r-1}$

$$= \frac{(n-r+1)\,n!}{\{n-(r-1)\}!} = \frac{(n-r+1)\,n!}{(n-r+1)!} = \frac{(n-r+1)\,n!}{(n-r+1)(n-r)!} = \frac{n!}{(n-r)!} = {}^nP_r =$$ R.H.S.

Example 10 : If ${}^{10}P_r = 5040$ then find the value of $\boldsymbol{r}$

Solution : ${}^{10}P_r = 5040$ (Given)

$$\Rightarrow \frac{10!}{(10-r)!} = 10 \times 504$$

$$\Rightarrow \frac{10!}{(10-r)!} = 10 \times 9 \times 8 \times 7$$

$$\Rightarrow \frac{10!}{(10-r)!} = \frac{10 \times 9 \times 8 \times 7 \times 6!}{6!}$$

$$\Rightarrow \frac{10!}{(10-r)!} = \frac{10!}{6!}$$

$$\Rightarrow (10-r)! = 6! \Rightarrow 10 - r = 6 \Rightarrow r = 4$$

Alternative method : ${}^{10}P_r = 5040$ (Given)

$\Rightarrow 10.{}^{9}P_{r-1} = 5040 \quad \Rightarrow {}^{9}P_{r-1} = 504 \qquad \left[\because {}^{n}P_r = n.{}^{n-1}P_{r-1}\right]$

$\Rightarrow 9.{}^{8}P_{r-2} = 504 \quad \Rightarrow {}^{8}P_{r-2} = 56$

$\Rightarrow 8.{}^{7}P_{r-3} = 56 \quad \Rightarrow {}^{7}P_{r-3} = 7$

$\Rightarrow 7.{}^{6}P_{r-4} = 7 \quad \Rightarrow {}^{6}P_{r-4} = 1 \Rightarrow r - 4 = 0 \Rightarrow r = 4 \qquad \left[\because {}^{n}P_0 = 1\right]$

Example 11 : Find the value of n if :

(i) ${}^{n}P_5 : {}^{n}P_3 = 2:1$ **(ii)** $2.{}^{5}P_3 = {}^{n}P_4$

Proof : **(i)** $\frac{{}^{n}P_5}{{}^{n}P_3} = \frac{2}{1}$ (Given)

$$\Rightarrow \frac{n!}{(n-5)!} \times \frac{(n-3)!}{n!} = \frac{2}{1} \quad \Rightarrow \frac{(n-3)(n-4)(n-5)!}{(n-5)!} = 2$$

$$\Rightarrow (n-3)(n-4) = 2 \quad \Rightarrow (n-3)(n-4) = (5-3)(5-4)$$

(Comparing both sides)

$$\Rightarrow n = 5$$

(ii) $2.{}^{5}P_3 = {}^{n}P_4$ (Given)

$$\Rightarrow {}^{n}P_4 = 2.{}^{5}P_3$$

$$\Rightarrow \frac{n!}{(n-4)!} = 2.\frac{5!}{2!}$$

$$\Rightarrow \frac{n(n-1)(n-2)(n-3)(n-4)!}{(n-4)!} = 5!$$

$$\Rightarrow n(n-1)(n-2)(n-3) = 5.4.3.2.1$$

$$\Rightarrow n(n-1)(n-2)(n-3) = 5(5-1)(5-2)(5-3)$$

$\Rightarrow$ On comparing both the sides we have $n = 5$

Example 12: How many words can be formed using the word **DELHI** if

(i) All the letters are taken ....

(ii) Any three letters are taken .

(iii) The word starts with D.

(iv) The word starts with D and ends with I .

(v) Both the vowels occur together (vi) the word starts with a vowel .

Proof : Since all the letters are distinct in the word DELHI

(i) If all s letters taken, then total number of permutations (words) will be

$$= {}^5P_5 = 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120.$$

(ii) If 3 letters taken out of 5 letters, then numbers of words.

$$= {}^5P_3 = \frac{5!}{2!} = 5 \times 4 \times 3 = 60$$

(iii) The word starts with D i.e. place of D is fixed so we need to arrange remaining 4 letters only, the number of words will be

$${}^4P_4 = 4! = 4 \times 3 \times 2 \times 1 = 24$$

This can be understand by making box :

1	4	3	2	1	$= 1 \times 4 \times 3 \times 2 \times 1 = 24$

↑D
(Fix)

(iv) The word starts with D and ends with I Then D & I are have fixed place so we need to arrange only 3 letters, so the number of words are,

$${}^3P_3 = 3! = 3 \times 2 \times 1 = 6$$

This can be understand by making box :

1	3	2	1	1	$= 1 \times 3 \times 2 \times 1 \times 1 = 6$

↑D ↑I
(Fix) (Fix)

(v) Keeping the vowels (EI) together and assuming as one letter we have now four letters which can be arranged in ${}^4P_4 = 4! = 24$ ways but (EI) can be arranged in $2! = 2$ ways thus the required number of ways will be $= 2 \times 24 = 48$

(vi) As the word starts with vowel we can fix either E or I and the remaining 3 letters can be arranged in ${}^3P_3 = 3! = 6$ ways but (EI) can be arranged in $2! = 2$ ways thus the required number of ways will be $= 2 \times 6 = 12$

It can be easily understood by following table :

E	I	3	2	1	$= 3 \times 2 \times 1 = 6$
I	E	3	2	1	$= 3 \times 2 \times 1 = 6$

Example 13 : Find the number of 4-digit numbers that can be formed using the digits 1, 2, 3, 4, 5, 6

(i) If no digit is repeated.

(ii) If digit can be repeated.

Proof : (i) Numbers by taking 4 digits from 6 will be

$${}^6P_4 = \frac{6!}{2!} = 6 \times 5 \times 4 \times 3 = 360$$

This can be understood by making the following blocks :

3	4	5	6	$= 3 \times 4 \times 5 \times 6 = 360$

(i) Since the repetition of digit is not allowed the one's place can be filled with any 6 of digits in 6 ways. Now tens place can be fixed by 5 digits in 5 ways. So hundred's and thousand's place can be filled by 4 and 3 ways respectively.

(ii) Since the repetition of digit is allowed, so each of the one's, ten's, hundred's place can be filled with any 6 of digits in 6 ways. Now tens place can be fixed by 5 digits in 5 ways. So hundred's and thousand's can be filled in 6 ways.

$$6 \times 6 \times 6 \times 6 = 1296.$$

This can be understood by making block :

6	6	6	6

$= 6 \times 6 \times 6 \times 6 = 1296.$

Example 14 : How many numbers can be formedbetween 6000 and 7000 using digits 1, 2, 3, 4, 5, 6, 7, 8 if no digit is repeated and in which how many numbers will be divisible by 5.

Proof : Every number is from 6000 and 7000 are of 4 digits and this should be start from 6. Hence we need to arrange 3 digits.

1, 2, 3, 4, 5, 7, 8 from remaining seven because digits can not be repeated.

Hence, counting of required number =

$$= {}^7P_3 = \frac{7!}{4!} = 7 \times 6 \times 5 = 210$$

This can be understand by following ways :

1	7	6	5

$= 1 \times 7 \times 6 \times 5 = 210$

↑6

(fixed)

How, we see at another part only those will be divisible by 5, which ends with 5. So, for 4 digit number the starting digit is 6 and at last 5 will be placed. So, we need to select 2 digits from remaining 6 digits.

Hence the counting of numbers $= {}^6P_2 = \frac{6!}{4!} = 6 \times 5 = 30$

This can be understood by making block :

1	6	5	1

$= 1 \times 6 \times 5 \times 1 = 30$

↑6 ↑5

(fixed) (fixed)

6.06 Permutations of those when all the objects are not all distinct :

Theorem 2 : Let total n things, where p objects are alike of one kind q are alike of second kind, r are alike the third and remaining are different objects, than the number of permutations of all objects are taken altogether is given by $\frac{n!}{p!q!r!}$

Proof : Let total n things. In which p are alike of one type, q are alike second and r is of third type remaining are of various types. Let the required number of permutations are x. If we take one from these permutations are interchange p by q so new permutations are $p!$

Similarly, if we change q and r equal things by q and r unequal things, so permutations are $q!$ and $r!$. If All changes are done altogether so we get $p! \times q! \times r!$ permutations from given permutations.

Now, because all things are different and all are taken together number of permutation will be $n!$. Then

$$x \times p! \times q! \times r! = n! \qquad \therefore \quad x = \frac{n!}{p!q!r!}$$

Note : Let there be n number of objects of which p objects are of one kind, q are of second kind and r is of third kind also if $p + q + r = n$ then the number of permutations taking all the objects can be found using the formula given in theorem 2.

6.07 Circular permutations :

We have studied in row permutation of the objects which is known as linear permutations. In this type of permutations n objects are arranged in a circular way keeping one object fix the arrangements of remaining objects are done as per linear permutation.

Theorem 3 : n different objects are arranged in a circular permutation in $(n-1)!$ ways.

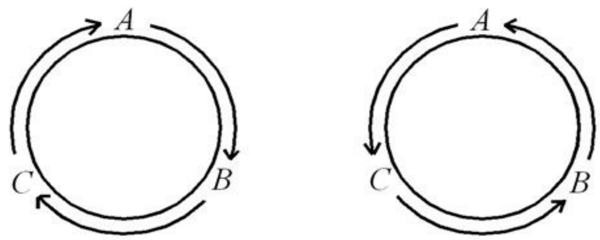
Proof : There is no end in circular permutation. So we keep in mind of the position of objects. If one object is fixed on a circular table (say) then remaining objects will be arranged in $(n-1)$ ways hence the total number of arrangements will be $(n-1)!$ using fundamental principle of counting.

Remark : In the above theorem, the clockwise order and anticlockwise order have considered as different permutations.

6.08 Difference between clockwise and anticlockwise permutations:

There are six different permutations of the letters A, B, C

ABC, ACB, BAC, BCA, CAB, CBA



Clockwise Anticlockwise

Figure 6.1

(i) Since the order of ABC, BCA, CAB Clockwise is same hence linear permutations are considered as circular permutation in a single permutation.

(ii) Since the order of ACB, CBA, BAC Anticlockwise is same hence there linear permutations are considered as circular permutation in a single permutation. Hence the total circular permutations of letters A, B, C is 2. If we consider no difference between clockwise and anticlockwise circular permutation. So the total permutations of letters A, B, C is 1.

If the clockwise and anticlockwise arrangements are considered as one arrangement then the total number of permutations of n different objects will be given by $\frac{(n-1)!}{2}$

Note : Garlands made of flowers or beads are some of the examples where the arrangements are considered as one single arrangement. Here is no difference in both order. Hence circular permutation is divided by 2.

Illustrative Examples

Example 15 : Find the number of arrangements taking the letters of the word **MATHEMATICS.**

Proof : Here the number of letters are 11, in this M occurs twice, A twice and T also occurs twice so the possible permutations are

$$\frac{11!}{2!\,2!\,2!}$$

Example 16 : In how many ways can a group of 7 people sit around a circular table?

Proof : let us first fix the position of one person then remaining 6 persons can be made to sit in 6! ways therefore the arrangements are

$$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720\,.$$

Example 17 : In how many ways can 7 different beads can be arranged to make a necklace?

Proof : The possible arragements are

$$\frac{(7-1)!}{2}=\frac{6!}{2}=\frac{720}{2}=360 \; .$$

Example 18 : In how many ways can 10 people be made to sit around a circular table, so that no two people are together?

Proof : 10 people can be made to sit around a circular table in 9! ways but here the clockwise and anticlockwise arrangements are considered as one arrangement thus the required ways are $9!/2$

Example 19 : In how many ways can 5 girls and 5 boys be seated around the circular table so that no two girls are together??

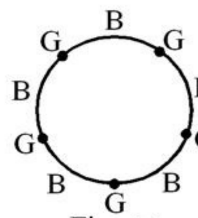


Fig. 6.2

Proof : Let us first seat the 5 girls. This can be done in 5! ways.

Hence, the total way to sit 5 boys are 4!. There are 5! ways to sit girls between 5 places between boys. Hence, total numbers $4!\times 5!$

Example 20 : How many words can be formed from BHARAT , in which B and H will not come together?

Proof : Total letters in BHARAT are 6. Letter A comes two times. Hence total numbers of permutations are,

$\frac{6!}{2!}=360$. Placing B and H together, we can consider as (BH) a letter. Hence, remaining letters are 5 in which 2 letters are of A. Letter (BH) occurs two times, we can take it as (BH) or (HB), so these can be arranged as 2!. So totals numbers of words for B and H comes together.

$$=2!\times\frac{5!}{2!}=120$$

So, number of words in which B and H not come together,

$$=360-120=240$$

Exercise 6.1

1. Find the value of n when :

   (i) ${}^{n-1}P_3 : {}^{n+1}P_3 = 5 : 12$ (ii) ${}^{n}P_6 = 10.{}^{n}P_5$

   (iii) ${}^{56}P_{n+6} : {}^{54}P_{n+3} = 30800$ (iv) ${}^{6+n}P_2 : {}^{6-n}P_2 = 56 : 12$

2. Find the numbers of words formed from the letter of the word ALLAHABAD ?
3. How many words can be formed from the letters of the word TRIANGLE ? How many of these will begin the with T and end with E ?
4. How many numbers lying between 3000 and 4000 and divisible by 5 is formed by using the digits 1, 2, 3, 4, 5, 6 when no digit is repeated ?
5. How many six digit numbers can be formed by using the digits 0, 1, 2, 3, 4, 5 ?
6. How many three digit numbers less then 1000 can be formed using the digit 1, 2, 3, 4, 5, 6 when the digits are not repeated ?
7. In how many ways can 15 members of a committee sit on a round table so that Secretary and Deputy Secretary sit on the either side of the President ?
8. There are 15 station on a Railway line. How many tickets of different class should be printed so that a passenger can buy a ticket of any station from another ?

9. In how many ways can 10 different beads be arranged to form a necklace so that 4 beads are always together ?

10. How many numbers greater than or equal to 6000 and less than 7000 can be formed using the digits 0, 1, 2, ... 9 which are divisible by 5 and the digits can be repeated?

11. How many different words can be formed from the letter of the word SCHOOL so that no two O's are together?

6.09 Combinations :

The study of placing the given things in various orders or arrangements is done in section 6.04. Sometimes we are intrested to select a few things from given things, we are not intresting in the order of selection. For example a student wants to select 3 books at a time from a library, a company wants to select 3 persons from 10.

Definition : Combination is a selection of objects or people taken some or all at a time. (order is not important). n objects taken r at a time is given by ${}^nC_r$ or $C(n,r)$ Clearly ${}^nC_r$ is defined if $0 \le r \le n$.

Example : Let two letters are to be selected out of the three given letters A, B and C

A, B; B, C; C, A.

Here they should not be written as

A,B; B,A; B,C,; C,B; C,A; A,C.

As the selection is important and not the arrangement of letters.

6.10 Difference between Permutation and Combination :

In permutation order is important whereas in combination order is not important. For example AB and BA are considered as different arrangements in permutation whereas in combination it is considered as one selection. Thus the number of permutation is greater than combination.

Generally combination involves selection of terms, committees, making words from the letters where permutation involves arrangement of letters and numbers.

Theorem 4 : Combination of n different objects taken r objects at a time is given by

$$\frac{n!}{(n-r)!\ r!} \text{ i.e. } {}^nC_r = \frac{n!}{(n-r)!\ r!}$$

Proof : Corresponding to each combination of ${}^nC_r$ we have $r!$ permutations because r objects in every combination can be rearranged in $r!$ ways. Hence the total number of permutations of n different things taken r at a time are $r! \times {}^nC_r$. Which is same as ${}^nP_r$. Thus

$$r! \times {}^nC_r = {}^nP_r$$

or $${}^nC_r = \frac{{}^nP_r}{r!} = \frac{n!}{(n-r)!\ r!} \qquad \left[\because\ {}^nP_r = \frac{n!}{(n-r)!}\right]$$

Note : 1. $${}^nC_r = \frac{n!}{(n-r)!\ r!}$$

or $${}^nC_r = \frac{n(n-1)(n-2).....(n-r+1)(n-r)(n-r-1).....3.2.1}{\{(n-r)(n-r-1).....3.2.1\}\{1.2.3......r\}}$$

or $${}^nC_r = \frac{n(n-1)......(n-r+1)}{1.2......r}$$

2. $${}^nC_n = \frac{n!}{(n-n)!\ n!} = \frac{n!}{0!\ n!} = 1 \qquad [\because 0! = 1]$$

and $\quad {}^nC_0 = \dfrac{n!}{(n-0)!\ 0!} = \dfrac{n!}{n!\ 0!} = 1 \qquad \therefore \qquad {}^nC_n = {}^nC_0 = 1$

6.10 Properties of ${}^nC_r$:

Property I : ${}^nC_r = {}^nC_{n-r} \qquad (0 \le r \le n)$

Proof: R.H.S. $= {}^nC_{n-r} = \dfrac{n!}{\{n-(n-r)\}!\,(n-r)!} = \dfrac{n!}{r!\,(n-r)!} = {}^nC_r =$ L.H.S.

Property II : ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r \qquad (0 \le r \le n)$

Proof: L.H.S. $= {}^nC_r + {}^nC_{r-1}$

$$= \frac{n!}{r!\ (n-r)!} + \frac{n!}{(r-1)!\,\{n-(r-1)\}!}$$

$$= \frac{n!}{r!\ (n-r)!} + \frac{n!}{(r-1)!\ (n-r+1)!}$$

$$= \frac{n!}{(r-1)!\ (n-r)!}\left(\frac{1}{r} + \frac{1}{n-r+1}\right)$$

$$= \frac{n!}{(r-1)!\ (n-r)!}\left\{\frac{n+1}{r(n-r+1)}\right\}$$

$$= \frac{(n+1)!}{r!\ (n-r+1)!} = {}^{n+1}C_r = \text{R.H.S.}$$

Property III : ${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$

Proof: ${}^nC_x = {}^nC_y$

$\Rightarrow \quad {}^nC_x = {}^nC_y = {}^nC_{n-y} \qquad [\because\ {}^nC_y = {}^nC_{n-y}$ by **I**]

$\Rightarrow \quad x = y$

or $\quad x = n - y$

$\Rightarrow \quad x = y$

or $\quad x + y = n$

Illustrative Examples

Example 21 : Find the value of expression ${}^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3$

Solution : ${}^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3 = {}^{47}C_4 + {}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3 + {}^{48}C_3 + {}^{47}C_3$

$= \left({}^{47}C_4 + {}^{47}C_3\right) + {}^{48}C_3 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3$

$= \left({}^{48}C_4 + {}^{48}C_3\right) + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3 \qquad \left[\because\ {}^{47}C_4 + {}^{47}C_3 = {}^{48}C_4\right]$

$= \left({}^{49}C_4 + {}^{49}C_3\right) + {}^{50}C_3 + {}^{51}C_3 \qquad \left[\because\ {}^{48}C_4 + {}^{48}C_3 = {}^{49}C_4\right]$

$$= \left({}^{50}C_4 + {}^{50}C_3\right) + {}^{51}C_3 \qquad \left[\because\ {}^{49}C_4 + {}^{49}C_3 = {}^{50}C_4\right]$$

$$= {}^{51}C_4 + {}^{51}C_3 \qquad \left[\because\ {}^{50}C_4 + {}^{50}C_3 = {}^{51}C_4\right]$$

$$= {}^{52}C_4\ . \qquad \left[\because\ {}^{51}C_4 + {}^{51}C_3 = {}^{52}C_4\right]$$

Example 22 : If ${}^{n}C_{15} = {}^{n}C_{8}$ then find the value ${}^{n}C_{21}$.

Solution : ${}^{n}C_{15} = {}^{n}C_{8} \Rightarrow n = 15 + 8 = 23$ $\quad \left[\because\ {}^{n}C_{x} = {}^{n}C_{y} \Rightarrow x + y = n\right]$

$\therefore \quad {}^{n}C_{21} = {}^{23}C_{21}$

$$= \frac{23!}{21!\ (23-21)!} = \frac{23!}{21!\ 2!}$$

$$= \frac{23 \times 22}{2} = 23 \times 11 = 253.$$

Example 23 : If ${}^{10}C_{x} = {}^{10}C_{x+4}$ then find the value of x

Solution : ${}^{10}C_{x} = {}^{10}C_{x+4} \Rightarrow x + x + 4 = 10 \Rightarrow 2x = 6 \Rightarrow x = 3.$

Note : Combinations under some restrictions are given below :

(i) Out of n different objects r where p objects are always taken. In this to form a group of r objects out of (n-p) objects only (r-p) objects are selected. Therefore required selection will be ${}^{n-p}C_{r-p}$

(ii) If objects p are never taken then out of remaining (n-p) objects are selected from the r objects, thus the selection is given by ${}^{n-p}C_{r}$

(iii) Sometimes the use of permutation and Combinations both are required. In such cases firstly we find combination and then we find permutation.

Example 24 : Find the number of diagonals in an n sided polygon.

Solution : There are n verties in a n sided polygon, thus the number of lines joining two vertices will be $= {}^{n}C_{2} = \frac{n(n-1)}{2}$, but this involves the n sides of the polygon also, thus the number of diagonals are

$$= \frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}$$

Example 25 : How many words with or without meaning taking 3 consonants and 2 vowels can be formed using 5 consonants and 4 vowels?

Solution : 3 Consonants and 2 vowels can be selected in ${}^{5}C_{3} \times {}^{4}C_{2}$ ways. Now a five letter word can be arranged in 5! ways, hence the total number of words are ${}^{5}C_{3} \times {}^{4}C_{2} \times 5!$.

Example 26 : A group consists of 5 women and 8 men. In how ways can a team of 6 members be selected if the team has

(i) Only 2 men **(ii)** Only 2 women

(iii) Atleast 2 women **(iv)** Atleast 2 men

Solution : (i) 2 men can be selected in ${}^{8}C_{2}$ ways remaining 4 women can be selected in ${}^{5}C_{4}$ ways.

$\therefore$ required number of ways $= {}^{8}C_{2} \times {}^{5}C_{4} = 28 \times 5 = 140$

(ii) 2 women can be selected in ${}^5C_2$ ways and remaining 4 men can be selected in ${}^8C_4$ ways

$\therefore$ required number of ways $= {}^5C_2 \times {}^8C_4 = 10 \times 70 = 700$

(iii) Atleast 2 women can be selected as-

2 women and 4 men : 3 women and 3 men, 4 women and 2 men, 5 women and 1 men hence their selection will be done as,

$${}^5C_2 \times {}^8C_4;\ {}^5C_3 \times {}^8C_3;\ {}^5C_4 \times {}^8C_2;\ or\ {}^5C_5 \times {}^8C_1 \text{ ways}$$

$\therefore$ required number of ways

$$= {}^5C_2 \times {}^8C_4 + {}^5C_3 \times {}^8C_3 + {}^5C_4 \times {}^8C_2 + {}^5C_5 \times {}^8C_1$$
$$= 700 + 560 + 140 + 8 = 1408.$$

(iv) Atleast 2 men can be selected as-

$${}^8C_2 \times {}^5C_4 + {}^8C_3 \times {}^5C_3 + {}^8C_4 \times {}^5C_2 + {}^8C_5 \times {}^5C_1 + {}^8C_6 = 1436 \text{ ways}$$

Exercise 6.2

1. Find the value of n if :

   (i) ${}^{2n}C_3 : {}^nC_3 = 11 : 1$ (ii) ${}^{20}C_{n-2} = {}^{20}C_{n+2}$ (iii) ${}^nC_{10} = {}^nC_{15}$
2. Find the value of ${}^{50}C_{11} + {}^{50}C_{12} + {}^{51}C_{13} - {}^{52}C_{13}$
3. There are 3, 4 and 5 points respectively on the sides AB, BC, CA, of triangle ABC. Find the number of triangles formed using these points.
4. There are two white, three black and four red balls in a box. Find the number of ways in which three balls can be drawn with atleast one black ball in it.
5. Find the number of signels generated from 6 different coloured flags.

   [Hint : ${}^6C_1 \times 1! + {}^6C_2 \times 2! + {}^6C_3 \times 3! + {}^6C_4 \times 4! + {}^6C_5 \times 5! + {}^6C_6 \times 6!$]
6. If the number of diagonals in a polygon is 44, then find its numbers of sides.
7. How many numbers can be formed using 4 digits at a time with the digits 1, 2, 3, 4, 5, 6. The numbers must contain the digits 4 and 5.
8. In how many ways can six '+' and four '-' sysbols can be placed in a row if no two '-' signs are together ?
9. A college committee is to be formed with 5 students and 2 professors when 8 students and 5 proffessors are avaible. How many such committees can be formed ?
10. In how many ways can one select a cricket team of eleven players from 14 players in which only 4 players can bowl, if each cricket team of 11 must include atleast 2 bowlers ?

Miscellaneous Exercise 6

1. If ${}^nP_{n-2} = 60$ then the value of n:

   (A) 2 (B) 4 (C) 5 (D) 3
2. ${}^nP_r \div {}^nC_r$ is equal to:

   (A) $n!$ (B) $(n-r)!$ (C) $\frac{1}{r!}$ (D) $r!$
3. In how many ways 5 people can sit on a round table -

   (A) 120 (B) 24 (C) 60 (D) 12
4. How many words with or without meaning can be formed using the word BHILWARA ?

(A) $\frac{8!}{2!}$ (B) $8!$ (C) $7!$ (D) $\frac{6!}{2!}$

5. ${}^{47}C_4 + \sum_{r=1}^{5} {}^{52-r}C_3$ is equal to:

(A) ${}^{51}C_4$ (B) ${}^{52}C_4$ (C) ${}^{53}C_4$ (D) None of these

6. The value of ${}^{61}C_{57} - {}^{60}C_{56}$

(A) ${}^{61}C_{58}$ (B) ${}^{60}C_{57}$ (C) ${}^{60}C_{58}$ (D) ${}^{60}C_{56}$

7. If ${}^{15}C_{3r} = {}^{15}C_{r+3}$ then the value of r is :

(A) 5 (B) 4 (C) 3 (D) 2

8. There are 6 points on the circumference of a circle. The number of lines joining these points will be -

(A) 30 (B) 15 (C) 12 (D) 20

9. How many words can be formed using the letters of word BHOPAL ?

(A) 124 (B) 240 (C) 360 (D) 720

10. There are 4 points on the circumference of a circle. Find the number of trangles that can be formed with these points ?

(A) 4 (B) 6 (C) 8 (D) 12

11. If ${}^{n}C_9 = {}^{n}C_7$, Then find ${}^{n}C_{16}$

12. Find the value of n

(i) ${}^{2n}C_2 : {}^{n}C_2 = 12:1$ (ii) ${}^{2n}C_3 : {}^{n}C_3 = 11:1$

13. Find the number of chords joining 11 points on the circle.

14. Determine the number of 5 card combinations out of a deck of 52 cards if there is exactly one ace in each combination.

15. A plane has n points in which m points are collinear. Find the number of triangles formed by these points.

16. Find the number of diagonals in a dicagon.

17. A train has five vacant seats. In how many ways can 3 passengers be made to sit on these seats.

18. A group of 7 is to be formed from 6 boys and 4 girls. Find the number of ways in which the group is formed if the group contains maximum number of boys.

19. In a seminar of 8 peoples if every person shakes hand with one another only once then find the total number of hand shakes ?

20. In how many ways 6 men and 6 women can sit on a round table if no two women sit together ?

21. In how many ways can the letters of the word ASSASSINATION be arranged so that all the S's are together ?

Important Points

1. **Principle of multiplication :** If an event can occur in m different ways, following which another event can occur in n different ways, then the total number of occurrence of the events $m \times n$
2. **Principle of Addition :** If a work is done in m and n different ways then total work can be done in $m + n$ ways.
3. **Permutation :** A permutation is an arrangement in a definite order of a number of objects taken some or

all at a time.

4. $n! = n(n-1)(n-2).....3.2.1$; where $n \in N$ $\quad 0! = 1$; $\quad (-n)!$ is not defined

5. The number of permutation of n objects taken r at a time, where repeation is not allowed is $= {}^nP_r = n(n-1)(n-2)......(n-r+1)$

6. ${}^nP_r = \frac{n!}{(n-r)!}$; ${}^nP_n = n!$

7. The number of permutation of n objects taken all a time, where p objects are alike of first kind, q objects are alike of the second kind, r objects are alike of third kind. Remaining (or $p+q+r=n$) objects are of the different kind is $\frac{n!}{p!\ q!\ r!}$

8. The number of circular permutation of n objects will be $(n-1)!$ permutation. If the clockwise and anticlockwise arrangements are considered as one arrangement, then the total number of permutation will be given by $\frac{(n-1)!}{2}$.

9. Combination : Taking all or some things together and not considering the order, the various group obtain which is called combination.

10. The number of combinations of n different things taken r at a time, denoted by ${}^nC_r$ is given by ${}^nC_r = \frac{n!}{r!\ (n-r)!}$

11. ${}^nC_n = {}^nC_0 = 1$; ${}^nC_r = {}^nC_{n-r}$; ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$;
${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$

12. The number of diagonals in a n sided polygon is $\frac{n(n-3)}{2}$

Answers

Exercise 6.1

1.	(i) 8	(ii) 15	(iii) 41	(iv) 2
2.	$\frac{9!}{2!4!}$	3. 40320, 720		4. 12
5.	600	6. ${}^6P_3 = \frac{\lfloor 6}{\lfloor 3} = 120$		7. $12! \times 2!$
8.	210	10. 200		11. $\frac{6!}{2!} - 5! = 240$

Exercise 6.2

1.	(i) 6	(ii) 10		(iii) 25		
2.	0		3. 205	4. 64	5. 1956	
6.	11	7.	144	8. 35	9. 560	10. 360

Miscellaneous Exercise 6

(1) C	(2) D	(3) B	(4) A	(5) B
(6) B	(7) C	(8) B	(9) D	(10) A
11. 1	12. (i) 5 ; (ii) 6	13. 55	14. 778320	
15. $^{n}C_3 - ^{m}C_3$	16. 35	17. 60	18. 100	
19. $^{8}C_2 = 28$	20. $5! \times 6! = 86400$	21. 151200		