CHAPTER

1

Number System, Inequalities and Theory of Equations

- Constant and Variables
- What is Function
- Intervals
- Inequalities
- Generalized Method of Intervals for Solving Inequalities
- Absolute Value of x
- Some Definitions
- Geometrical Meaning of Roots (Zeros) of an Equation
- Key Points in Solving an Equation
- Graphs of Polynomial Functions
- Equations Reducible to Quadratic
- Remainder and Factors Theorem
- Quadratic Equation
- Common Root(S)
- Relation between Coefficient and Roots of n-Degree Equations
- Solving Cubic Equation
- Repeated Roots
- Quadratic Expression in Two Variables
- Finding the Range of a Function Involving Quadratic Expression
- Quadratic Function
- Location of Roots
- Solving Inequalities Using Location of Roots

CONSTANT AND VARIABLES

In *mathematics*, a **variable** is a *value* that may change within the scope of a given problem or set of operations.

In contrast, a **constant** is a value that remains unchanged, though often unknown or undetermined.

Dependent and Independent Variables

Variables are further distinguished as being either a **dependent variable** or an **independent variable**. Independent variables are regarded as inputs to a system and may take on different values freely.

Dependent variables are those values that change as a consequence to changes in other values in the system.

When one value is completely determined by another, or of several others, then it is called a function of the other value or values. In this case the value of the function is a dependent variable and the other values are independent variables. The notation $f(x)$ is used for the value of the function f with x representing the independent variable.

For example, $y = f(x) = 3x^2$, here we can take x as any real value, hence x is independent variable. But value of y depends on value of x, hence y is dependent variable.

WHAT IS FUNCTION

To provide the classical understanding of functions, think of a *function* as a kind of machine. You feed the machine raw materials, and the machine changes the raw materials into a finished product based on a specific set of instructions. The kinds of functions we consider here, for the most part, take in a real number, change it in a formulaic way, and give out a real number (possibly the same as the one it took in). Think of this as an *input–output machine*; you give the function an input, and it gives you an output.

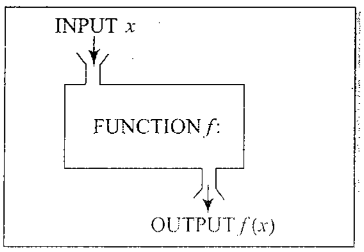


Fig. 1.1

For example, the squaring function takes the input 4 and gives the output value 16. The same squaring function takes the input 1 and gives the output value 1.

A function is always defined as "of a variable" which tells us what to replace in the formula for the function.

For example, $f(x) = 3x + 2$ tells us:

- The function f is a function of x.
- To evaluate the function at a certain number, replace the x with that number.
- Replacing x with that number in the right side of the function will produce the function's output for that certain input.
- In English, the above definition of f is interpreted, "Given *a* number, f will return *two more than the triple of that number*."

Thus, if we want to know the value (or output) of the function at 3:

$$f(x) = 3x + 2$$

$$f(3) = 3(3) + 2 = 11$$

Thus, the value of f at 3 is 11.

Note that $f(3)$ means the value of the dependent variable when "x" takes on the value of 3. So we see that the number 11 is the output of the function when we give the number 3 as the input. We refer to the input as the **argument** of the function (or the **independent variable**), and to the output as the **value** of the function at the given argument (or the **dependent variable**). A good way to think of it is the dependent variable $f(x)$ depends on the value of the independent variable x.

The formal definition of a function states that a function is actually a *rule* that associates elements of one set called the *domain* of the function with the elements of another set called the *range* of the function. For each value, we select from the domain of the function, there exists exactly one corresponding element in the range of the function. The definition of the function tells us which element in the range corresponds to the element we picked from the domain. Classically, the element picked from the domain is pictured as something that is fed into the function and the corresponding element in the range is pictured as the output. Since we "pick" the element in the domain whose corresponding element in the range we want to find, we have control over what element we pick and hence this element is also known as the "independent variable". The element mapped in the range is beyond our control and is "mapped to" by the function. This element is hence also known as the "dependent variable", for it depends on which independent variable we pick. Since the elementary idea of functions is better understood from the classical viewpoint, we shall use it hereafter. However, it is still important to remember the correct definition of functions at all times.

To make it simple, for the function $f(x)$, all of the possible x values constitute the domain, and all of the values $f(x)$ (y on the x-y plane) constitute the range.

Example 1.1 **A function is defined as $f(x) = x^2 - 3x$.**

(i) Find the value of $f(2)$.

(ii) Find the value of x for which $f(x) = 4$.

Sol.

(i) $f(2) = (2)^2 - 3(2) = -2$

(ii) $f(x) = 4$

$\Rightarrow x^2 - 3x = 4 \quad \Rightarrow x^2 - 3x - 4 = 0$

$\Rightarrow (x - 4)(x + 1) = 0 \quad \Rightarrow x = 4$ or -1

This means $f(4) = 4$ and $f(-1) = 4$.

Example 1.2 **If f is linear function and $f(2) = 4, f(-1) = 3$, then find $f(x)$.**

Sol. Let linear function is $f(x) = ax + b$

Given $f(2) = 4 \Rightarrow 2a + b = 4$ (1)

Also $f(-1) = 3 \Rightarrow -a + b = 3$ (2)

Solving (1) and (2) we get $a = \frac{1}{3}$ and $b = \frac{10}{3}$

Hence, $f(x) = \frac{x+10}{3}$

Example 1.3 **A function is defined as $f(x) = \frac{x^2+1}{3x-2}$. Can $f(x)$ take a value 1 for any real x?**

Also find the value/values of x for which $f(x)$ takes the value 2.

Sol. Here $f(x) = \frac{x^2+1}{3x-2} = 1$

$\Rightarrow x^2 + 1 = 3x - 2$

$\Rightarrow x^2 - 3x + 3 = 0.$

Now this equation has no real roots as $D < 0$.

Hence, value of $f(x)$ cannot be 1 for any real x.

For $f(x) = 2$ we have $\frac{x^2+1}{3x-2} = 2$

or $x^2 + 1 = 6x - 4$ or $x^2 - 6x + 5 = 0$

or $(x - 1)(x - 5) = 0$

or $x = 1, 5$

Example 1.4 **Find the values of x for which the following functions are defined. Also find all possible values which functions take.**

(i) $f(x) = \frac{1}{x+1}$ (ii) $f(x) = \frac{x-2}{x-3}$ (iii) $f(x) = \frac{2x}{x-1}$

Sol.

(i) $f(x) = \frac{1}{x+1}$ is defined for all real values of x except when $x + 1 = 0$

Hence, $f(x)$ is defined for $x \in R - \{-1\}$.

Let $y = \frac{1}{x+1}$

Here we cannot find any real x for which $y = \frac{1}{x+1} = 0$

For y other than '0', there exists a real number x.

Hence, $\frac{1}{x+1} \in R - \{0\}$.

(ii) $f(x) = \frac{x-2}{x-3}$ is defined for all real values of x except when $x - 3 = 0$.

Hence, $f(x)$ is defined for $x \in R - \{3\}$

Let $y = \frac{x-2}{x-3}$

Here we cannot find any real x for which $y = \frac{x-2}{x-3} = 1$

Note: *When $\frac{x-2}{x-3} = 1$, we have $x - 2 = x - 3$ or $-2 = -3$ which is absurd.*

For y other than '1' there exists a real number x.

Hence, $\frac{1}{x+1} \in R - \{1\}$.

(iii) $f(x) = \frac{2x}{x-1}$ is defined for all real values of x except when $x - 1 = 0$

Hence, $f(x)$ is defined for $x \in R - \{1\}$

Let $y = \frac{2x}{x-1}$

Here we cannot find any real x for which $y = \frac{2x}{x-1} = 2$

Note: *When $\frac{2x}{x-1} = 2$, we have $2x = 2x - 2$ or $0 = -2$ which is absurd.*

For y other than '2' there exists a real number x.

Hence, $\frac{2x}{x-1} \in R - \{2\}$.

Example 1.5 **If $f(x) = \begin{cases} x^3, & x < 0 \\ 3x-2, & 0 \le x \le 2 \\ x^2+1, & x > 2 \end{cases}$, then find the value of $f(-1) + f(1) + f(3)$.**

Also find the value/values of x for which $f(x) = 2$.

Sol. Here function is differently defined for three different intervals mentioned.

For $x = -1$, consider $f(x) = x^3$

$\Rightarrow f(-1) = -1$

For $x = 1$, consider $f(x) = 3x - 2$

$\Rightarrow f(1) = 1$

For $x = 3$, consider $f(x) = x^2 + 1$

$\Rightarrow f(3) = 10$

$\Rightarrow f(-1) + f(1) + f(3) = -1 + 1 + 10 = 10$

Also when $f(x) = 2$,

for $x^3 = 2$, $x = 2^{1/3}$, which is not possible as $x < 0$.

for $3x - 2 = 2$, $x = 4/3$, which is possible as $0 \le x \le 2$.

For $x^2 + 1 = 2$, $x = \pm 1$, which is not possible as $x > 2$.

Hence, for $f(x) = 2$, we have $x = 4/3$.

INTERVALS

The set of numbers between any two real numbers is called interval. The following are the types of interval.

Closed Interval

$x \in [a, b] \equiv \{x : a \le x \le b\}$

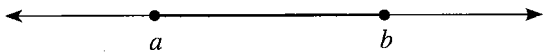


Fig. 1.2

Open Interval

$x \in (a, b)$ or $[a, b] \equiv \{x : a < x < b\}$

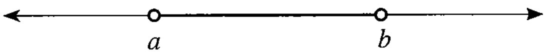


Fig. 1.3

Semi-Open or Semi Closed Interval

$x \in [a, b]$ or $[a, b] = \{x: a \le x < b\}$

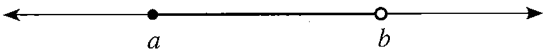


Fig. 1.4

$x \in]a, b]$ or $(a, b] = \{x: a < x \le b\}$

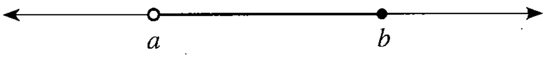


Fig. 1.5

Note:

- *A set of all real numbers can be expressed as* $(-\infty, \infty)$
- $x \in (-\infty, a) \cup (b, \infty) \Rightarrow x \in R - [a, b]$
- $x \in (-\infty, a] \cup [b, \infty) \Rightarrow x \in R - (a, b)$

INEQUALITIES

Some Important Facts about Inequalities

The following are some very useful points to remember:

(i) $a \le b$ either $a < b$ or $a = b$

(ii) $a < b$ and $b < c \Rightarrow a < c$ (transition property)

(iii) $a < b \Rightarrow -a > -b$, i.e., inequality sign reverses if both sides are multiplied by a negative number

(iv) $a < b$ and $c < d \Rightarrow a + c < b + d$ and $a - d < b - c$.

(v) If both sides of inequality are multiplied (or divided) by a positive number, inequality does not change. When both of its sides are multiplied (or divided) by a negative number, inequality gets reversed.

i.e., $a < b \Rightarrow ka < kb$ if $k > 0$ and $ka > kb$ if $k < 0$

(vi) $0 < a < b \Rightarrow a^r < b^r$ if $r > 0$ and $a^r > b^r$ if $r < 0$

(vii) $a + \frac{1}{a} \ge 2$ for $a > 0$ and equality holds for $a = 1$

(viii) $a + \frac{1}{a} \le -2$ for $a < 0$ and equality holds for $a = -1$

(ix) **Squaring an inequality:**

If $a < b$, then $a^2 < b^2$ does not follows always:

Consider the following illustrations:

$2 < 3 \Rightarrow 4 < 9$, but $-4 < 3 \Rightarrow 16 > 9$

Also if $x > 2 \Rightarrow x^2 > 4$, but for $x < 2 \Rightarrow x^2 \ge 0$

If $2 < x < 4 \Rightarrow 4 < x^2 < 16$

If $-2 < x < 4 \Rightarrow 0 \le x^2 < 16$

If $-5 < x < 4 \Rightarrow 0 \le x^2 < 25$

In fact $a < b \Rightarrow a^2 < b^2$ follows only when absolute value of a is less than the absolute value of b or distance of a from zero is less than the distance of b from zero on real number line.

(x) **Law of reciprocal:**

If both sides of inequality have same sign, while taking its reciprocal the sign of inequality gets reversed. i.e., $a > b > 0 \Rightarrow \frac{1}{a} < \frac{1}{b}$ and $a < b < 0 \Rightarrow \frac{1}{a} > \frac{1}{b}$

But if both sides of inequality have opposite sign, then while taking reciprocal sign of inequality does not change, i.e.

$$a < 0 < b \Rightarrow \frac{1}{a} < \frac{1}{b}$$

$$\text{If } x \in [a, b] \Rightarrow \begin{cases} \frac{1}{x} \in \left[\frac{1}{b}, \frac{1}{a}\right], \text{ if } a \text{ and } b \text{ have same sign} \\ \frac{1}{x} \in \left(-\infty, \frac{1}{a}\right] \cup \left[\frac{1}{b}, \infty\right), \text{ if } a \text{ and } b \text{ have opposite signs} \end{cases}$$

Example 1.6 **Find the values of x^2 for the given values of x.**

(i) $x < 2$ **(ii) $x > -1$** **(iii) $x \ge 2$** **(iv) $x < -1$**

Sol.

(i) When $x < 2$ we have $x \in (-\infty, 0) \cup [0, 2)$

for $x \in [0, 2)$, $x^2 \in [0, 4)$

for $x \in (-\infty, 0)$, $x^2 \in (0, \infty)$

$\Rightarrow$ for $x < 2$, $x^2 \in [0, 4) \cup (0, \infty)$

$\Rightarrow$ $x \in [0, \infty)$

(ii) When $x > -1$ we have $x \in (-1, 0) \cup [0, \infty)$

for $x \in (-1, 0)$, $x^2 \in (0, 1)$

for $x \in [0, \infty)$, $x^2 \in [0, \infty)$

$\Rightarrow$ for $x > -1$, $x^2 \in (0, 1) \cup [0, \infty)$

$\Rightarrow$ $x \in [0, \infty)$

(iii) Here $x \in [2, \infty)$

$\Rightarrow$ $x^2 \in [4, \infty)$

(iv) Here $x \in (-\infty, -1)$

$\Rightarrow$ $x^2 \in (1, \infty)$

Example 1.7 **Find the values of $1/x$ for the given values of x.**

(i) $x > 3$ **(ii) $x < -2$** **(iii) $x \in (-1, 3) - \{0\}$**

Sol.

(i) We have $3 < x < \infty$

$\Rightarrow$ $\frac{1}{3} > \frac{1}{x} > \frac{1}{\to \infty}$ ($\to \infty$ means tends to infinity)

$\Rightarrow$ $0 < \frac{1}{x} < \frac{1}{3}$

(ii) We have $-\infty < x < -2$

$\Rightarrow \quad \frac{1}{\to -\infty} > \frac{1}{x} > \frac{1}{-2}$

$\Rightarrow \quad \frac{1}{\to -\infty} > \frac{1}{x} > \frac{1}{-2}$

$\Rightarrow \quad 0 > \frac{1}{x} > -\frac{1}{2}$

(iii) $x \in (-1, 3) - \{0\}$

$\Rightarrow \quad x \in (-1, 0) \cup (0, 3)$

For $x \in (-1, 0)$

$\frac{1}{-1} > \frac{1}{x} > \frac{1}{\to 0^-}$

(here $\to 0^-$ means value of x approaches to 0 from its left hand side or negative side)

$\Rightarrow \quad -1 > \frac{1}{x} > -\infty$

$\Rightarrow \quad -\infty < \frac{1}{x} < -1 \qquad (1)$

For $x \in (0, 3)$

$\frac{1}{\to 0^+} > \frac{1}{x} > \frac{1}{3}$

(here $\to 0^+$ means value of x approaches to 0 from its right hand side or positive side)

$\Rightarrow \quad \infty > \frac{1}{x} > \frac{1}{3}$

$\Rightarrow \quad \frac{1}{3} < \frac{1}{x} < \infty \qquad (2)$

From (1) and (2), $\frac{1}{x} \in (-\infty, -1) \cup \left(\frac{1}{3}, \infty\right)$

Note: *For* $x \in R - \{0\}$, $\frac{1}{x} \in R - \{0\}$

Example 1.8 **Find all possible values of the following expressions:**

(i) $\frac{1}{x^2+2}$ (ii) $\frac{1}{x^2-2x+3}$ (iii) $\frac{1}{x^2-x-1}$

Sol.

(i) We know that $x^2 \ge 0\ x \in R$.

$\Rightarrow \quad x^2 + 2 \ge 2, \forall x \in R$.

or $\quad 2 \le (x^2 + 2) < \infty$

$\Rightarrow \quad \frac{1}{2} \ge \frac{1}{x^2+2} > 0$

$\Rightarrow \quad 0 < \frac{1}{x^2+2} \le \frac{1}{2}$

(ii) $\frac{1}{x^2-2x+3} = \frac{1}{(x-1)^2+2}$

Now we know that $(x-1)^2 \ge 0\ \forall x \in R$.

$\Rightarrow \quad (x-1)^2 + 2 \ge 2\ \forall x \in R$.

or $\quad 2 \le (x-1)^2 + 2 < \infty$

$\Rightarrow \quad \frac{1}{2} \ge \frac{1}{(x-1)^2+2} > 0$

$\Rightarrow \quad \frac{1}{x^2-2x+3} \in \left(0, \frac{1}{2}\right]$

(iii) $\frac{1}{x^2-x-1} = \frac{1}{\left(x-\frac{1}{2}\right)^2-\frac{5}{4}}$

$\left(x-\frac{1}{2}\right)^2 \ge 0, \forall x \in R$

$\Rightarrow \quad \left(x-\frac{1}{2}\right)^2 - \frac{5}{4} \ge -\frac{5}{4}, \forall x \in R$

For $\frac{1}{\left(x-\frac{1}{2}\right)^2-\frac{5}{4}}$, we must have

$\left(x-\frac{1}{2}\right)^2 - \frac{5}{4} \in \left[-\frac{5}{4}, 0\right) \cup (0, \infty)$

$\Rightarrow \quad \frac{1}{\left(x-\frac{1}{2}\right)^2-\frac{5}{4}} \in \left(-\infty, -\frac{4}{5}\right] \cup (0, \infty)$

Example 1.9 **Find all possible values of the following expressions:**

(i) $\sqrt{x^2-4}$ (ii) $\sqrt{9-x^2}$ (iii) $\sqrt{x^2-2x+10}$

Sol.

(i) $\sqrt{x^2-4}$

Least value of square root is 0 when $x^2 = 4$ or $x = \pm 2$. Also $x^2 - 4 \ge 0$

Hence, $\sqrt{x^2-4} \in [0, \infty)$.

(ii) $\sqrt{9-x^2}$

Least value of square root is 0 when $9 - x^2 = 0$ or $x = \pm 3$. Also, the greatest value of $9 - x^2$ is 9 when $x = 0$.

Hence, we have $0 \le 9 - x^2 \le 9 \Rightarrow \sqrt{9-x^2} \in [0, 3]$.

(iii) $\sqrt{x^2-2x+10} = \sqrt{(x-1)^2+9}$

Here, the least value of $\sqrt{(x-1)^2+9}$ is 3 when $x - 1 = 0$.

Also $(x-1)^2 + 9 \ge 9 \Rightarrow \sqrt{(x-1)^2+9} \ge 3$

Hence, $\sqrt{x^2-2x+10} \in [3, \infty)$.

GENERALIZED METHOD OF INTERVALS FOR SOLVING INEQUALITIES

Let $F(x) = (x-a_1)^{k_1}(x-a_2)^{k_2}...(x-a_{n-1})^{k_{n-1}}(x-a_n)^{k_n}$

where $k_1, k_2, \ldots, k_n \in Z$ and $a_1, a_2, \ldots, a_n$ are fixed real numbers satisfying the condition

$$a_1 < a_2 < a_3 < \ldots < a_{n-1} < a_n$$

For solving $F(x) > 0$ or $F(x) < 0$, consider the following algorithm:

- We mark the numbers $a_1, a_2, \ldots, a_n$ on the number axis and put the plus sign in the interval on the right of the largest of these numbers, i.e., on the right of a_n.
- Then we put the plus sign in the interval on the left of a_n if k_n is an even number and the minus sign if k_n is an odd number. In the next interval, we put a sign according to the following rule:
  - When passing through the point a_{n-1} the polynomial $F(x)$ changes sign if k_{n-1} is an odd number. Then we consider the next interval and put a sign in it using the same rule.
- Thus we consider all the intervals. The solution of the inequality $F(x) > 0$ is the union of all intervals in which we have put the plus sign and the solution of the inequality $F(x) < 0$ is the union of all intervals in which we have put the minus sign.

Frequently used Inequalities

(i) $(x-a)(x-b) < 0 \Rightarrow x \in (a, b)$, where $a < b$

(ii) $(x-a)(x-b) > 0 \Rightarrow x \in (-\infty, a) \cup (b, \infty)$, where $a < b$

(iii) $x^2 \le a^2 \Rightarrow x \in [-a, a]$

(iv) $x^2 \ge a^2 \Rightarrow x \in (-\infty, -a] \cup [a, \infty)$

(v) If $ax^2 + bx + c < 0$, $(a > 0) \Rightarrow x \in (\alpha, \beta)$, where α, β $(\alpha < \beta)$ are roots of the equation $ax^2 + bx + c = 0$

(vi) If $ax^2 + bx + c > 0$, $(a > 0) \Rightarrow x \in (-\infty, \alpha) \cup (\beta, \infty)$, where α, β $(\alpha < \beta)$ are roots of the equation $ax^2 + bx + c = 0$

Example 1.10 **Solve $x^2 - x - 2 > 0$.**

Sol. $x^2 - x - 2 > 0$

$\Rightarrow$ $(x-2)(x+1) > 0$

Now $x^2 - x - 2 = 0 \Rightarrow x = -1, 2$.

Now on number line (x-axis) mark $x = -1$ and $x = 2$.

Now when $x > 2$, $x + 1 > 0$ and $x - 2 > 0$

$\Rightarrow$ $(x+1)(x-2) > 0$

when $-1 < x < 2$, $x + 1 > 0$ but $x - 2 < 0$

$\Rightarrow$ $(x+1)(x-2) < 0$

when $x < -1$, $x + 1 < 0$ and $x - 2 < 0$

$\Rightarrow$ $(x+1)(x-2) > 0$

Hence, sign scheme of $x^2 - x - 2$ is

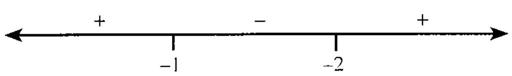


Fig. 1.6

From the figure, $x^2 - x - 2 > 0$, $x \in (-\infty, -1) \cup (2, \infty)$.

Example 1.11 **Solve $x^2 - x - 1 < 0$.**

Sol. Let's first factorize $x^2 - x - 1$.

For that let $x^2 - x - 1 = 0$

$$\Rightarrow \quad x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Now on number line (x-axis) mark $x = \frac{1 \pm \sqrt{5}}{2}$

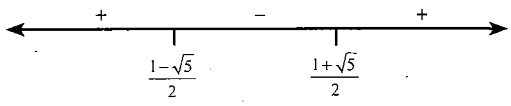


Fig. 1.7

From the sign scheme of $x^2 - x - 1$ which shown in the given figure,

$$x^2 - x - 1 < 0 \Rightarrow x \in \left(\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right)$$

Example 1.12 **Solve $(x-1)(x-2)(1-2x) > 0$.**

Sol. We have $(x-1)(x-2)(1-2x) > 0$

or $\quad -(x-1)(x-2)(2x-1) > 0$

or $\quad (x-1)(x-2)(2x-1) < 0$

On number line mark $x = 1/2, 1, 2$

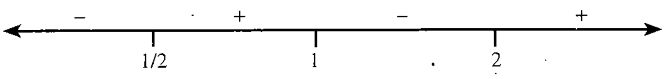


Fig. 1.8

When $x > 2$, all factors $(x-1)$, $(2x-1)$ and $(x-2)$ are positive.

Hence, $(x-1)(x-2)(2x-1) > 0$ for $x > 2$.

Now put positive and negative sign alternatively as shown in figure.

Hence, solution set of $(x-1)(x-2)(1-2x) > 0$ or $(x-1)(x-2)(2x-1) < 0$ is $(-\infty, 1/2) \cup (1, 2)$.

Example 1.13 **Solve $(2x+1)(x-3)(x+7) < 0$.**

Sol. $(2x+1)(x-3)(x+7) < 0$

Sign scheme of $(2x+1)(x-3)(x+7)$ is as follows:

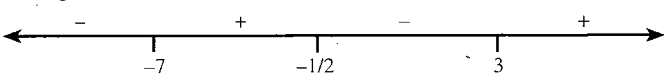


Fig. 1.9

Hence, solution is $(-\infty, -7) \cup (-1/2, 3)$.

Example 1.14 **Solve $\frac{2}{x} < 3$.**

Sol. $\frac{2}{x} < 3$

$\Rightarrow \quad \frac{2}{x} - 3 < 0$ (We cannot crossmultiply with x as x can be negative or positive)

$$\Rightarrow \quad \frac{2-3x}{x} < 0$$

$$\Rightarrow \quad \frac{3x-2}{x} > 0$$

$$\Rightarrow \quad \frac{(x-2/3)}{x} > 0$$

Sign scheme of $\frac{(x-2/3)}{x}$ is as follows:

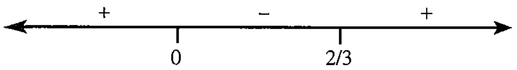


Fig. 1.10

$\Rightarrow \quad x \in (-\infty, 0) \cup (2/3, \infty)$

Example 1.15 **Solve $\frac{2x-3}{3x-5} \geq 3$.**

Sol. $\frac{2x-3}{3x-5} \geq 3$

$$\Rightarrow \quad \frac{2x-3}{3x-5} - 3 \geq 0$$

$$\Rightarrow \quad \frac{2x-3-9x+15}{3x-5} \geq 0$$

$$\Rightarrow \quad \frac{-7x+12}{3x-5} \geq 0$$

$$\Rightarrow \quad \frac{7x-12}{3x-5} \leq 0$$

Sign scheme of $\frac{7x-12}{3x-5}$ is as follows:

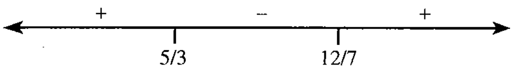


Fig. 1.11

$\Rightarrow \quad x \in (5/3, 12/7]$

$x = 5/3$ is not included in the solutions as at $x = 5/3$ denominator becomes zero.

Example 1.16 **Solve $x > \sqrt{(1-x)}$.**

Sol. Given inequality can be solved by squaring both sides.

But sometimes squaring gives extraneous solutions which do not satisfy the original inequality. Before squaring we must restrict x for which terms in the given inequality are well defined.

$x > \sqrt{(1-x)}$. Here x must be positive.

Here $\sqrt{1-x}$ is defined only when $1 - x \geq 0$ or $x \leq 1$ (1)

Squaring given inequality but sides $x^2 > 1 - x$

$$\Rightarrow x^2 + x - 1 > 0 \Rightarrow \left(x - \frac{-1-\sqrt{5}}{2}\right)\left(x - \frac{-1+\sqrt{5}}{2}\right) > 0$$

$$\Rightarrow x < \frac{-1-\sqrt{5}}{2} \text{ or } x > \frac{-1+\sqrt{5}}{2} \qquad (2)$$

From (1) and (2) $x \in \left(\frac{\sqrt{5}-1}{2}, 1\right]$ (as x is +ve)

Example 1.17 **Solve $\frac{2}{x^2-x+1} - \frac{1}{x+1} - \frac{2x-1}{x^3+1} \leq 0$.**

Sol. $\frac{2}{x^2-x+1} - \frac{1}{x+1} - \frac{2x-1}{x^3+1} \geq 0$

$$\Rightarrow \quad \frac{2(x+1)-(x^2-x+1)-(2x-1)}{(x+1)(x^2-x+1)} \geq 0$$

$$\Rightarrow \quad \frac{-(x^2-x-2)}{(x+1)(x^2-x+1)} \geq 0$$

$$\Rightarrow \quad \frac{-(x-2)(x+1)}{(x+1)(x^2-x+1)} \geq 0$$

$$\Rightarrow \quad \frac{2-x}{x^2-x+1} \geq 0, \text{ where } x \neq 1$$

$\Rightarrow \quad 2 - x \geq 0, x \neq -1$, (as $x^2 - x + 1 > 0$ for $\forall x \in R$)

$\Rightarrow \quad x \leq 2, x \neq -1$

Example 1.18 **Solve $x(x+2)^2(x-1)^5(2x-3)(x-3)^4 \geq 0$.**

Sol. Let $E = x(x+2)^2(x-1)^5(2x-3)(x-3)^4$.

Here for x, $(x-1)$, $(2x-3)$ exponents are odd, hence sign of E changes while crossing $x = 0, 1, 3/2$. Also for $(x+2)$, $(x-3)$ exponents are even, hence sign of E does not change while crossing $x = -2$ and $x = 3$.

Further for $x > 3$, all factors are positive, hence sign of the expression starts with positive sign from the right hand side.

The sign scheme of the expression is as shown in the following figure.

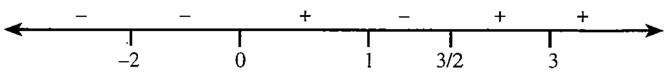


Fig. 1.12

Hence, for $E \geq 0$, we have $x \in [0, 1] \cup [3/2, \infty)$

Example 1.19 **Solve $x(2^x - 1)(3^x - 9)(x - 3) < 0$.**

Sol. Let $E = x(2^x - 1)(3^x - 9)(x - 3)$

Here $2^x - 1 = 0 \Rightarrow x = 0$ and when $3^x - 9 = 0 \Rightarrow x = 2$

Now mark $x = 0, 2$ and 3 on real number line.

Sign of E starts with positive sign from right hand side.

Also at $x = 0$, two factors are 0, x and $2^x - 1$, hence sign of E does not change while crossing $x = 0$.

Sign scheme of E is as shown in the following figure.

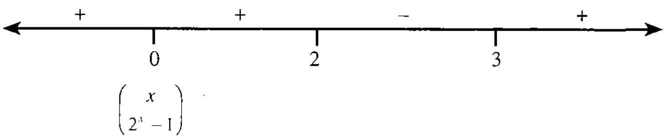


Fig. 1.13

From the figure, we have $E < 0$ for $x \in (2, 3)$.

Example 1.20 **Find all possible values of $\frac{x^2+1}{x^2-2}$.**

Sol. Let $y = \frac{x^2+1}{x^2-2}$

$\Rightarrow \quad yx^2 - 2y = x^2 + 1$

$\Rightarrow \quad x^2 = \frac{2y+1}{y-1}$

Now $x^2 \geq 0 \Rightarrow \frac{2y+1}{y-1} \geq 0$

Now $x^2 \geq 0 \Rightarrow \frac{2y+1}{y-1} \geq 0$

$\Rightarrow \quad y \leq -1/2$ or $y > 1$

Solving Irrational Inequalities

Example 1.21 **Solve $\sqrt{x-2} \geq -1$.**

Sol. We must have $x - 2 \geq 0$ for $\sqrt{x-2}$ to get defined, thus $x \geq 2$.

Now $\sqrt{x-2} \geq -1$, as square roots are always non-negative.

Hence, $x \geq 2$.

Note: *Some students solve it by squaring it both sides for which $x - 2 \geq 1$ or $x \geq 3$ which cause loss of interval [2, 3).*

Example 1.22 **Solve $\sqrt{x-1} > \sqrt{3-x}$.**

Sol. $\sqrt{x-1} > \sqrt{3-x}$ is meaningful if $x - 1 \geq 0$ and $3 - x \geq 0$

or $\quad 1 \leq x \leq 3$ (1)

Also $\quad \sqrt{x-1} > \sqrt{3-x}$

Squaring, we have $x - 1 > 3 - x$

$\Rightarrow \quad x > 2$ (2)

From (1) and (2), we have $2 < x \leq 3$.

Example 1.23 **Solve $x + \sqrt{x} \geq \sqrt{x} - 3$.**

Sol. $x + \sqrt{x} \geq \sqrt{x} - 3$ is meaningful only when $x \geq 0$ (1)

Now $x + \sqrt{x} \geq \sqrt{x} - 3$

$\Rightarrow \quad x \geq -3$ (2)

From (1) and (2), we have $x \geq 0$.

Example 1.24 **Solve $(x^2 - 4)\sqrt{x^2-1} < 0$.**

Sol. $(x^2 - 4)\sqrt{x^2-1} < 0$

We must have $x^2 - 1 \geq 0$

or $\quad (x - 1)(x + 1) \geq 0$

or $\quad x \leq -1$ or $x \geq 1$ (1)

Also $(x^2 - 4)\sqrt{x^2-1} < 0$

$\Rightarrow \quad x^2 - 4 < 0$

$\Rightarrow \quad -2 < x < 2$ (2)

From (1) and (2), we have $x \in (-2, -1] \cup [1, 2)$

ABSOLUTE VALUE OF *x*

Absolute value of any real number x is denoted by $|x|$ (read as modulus of x).

The absolute value is closely related to the notions of *magnitude*, *distance*, and *norm* in various mathematical and physical contexts.

From an *analytic geometry* point of view, the absolute value of a real number is that number's *distance* from zero along the *real number line*, and more generally the absolute value of the difference of two real numbers is the distance between them.

Let's look at the number line:

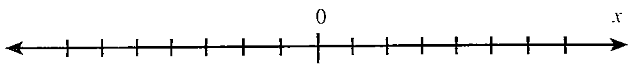


Fig. 1.14

The absolute value of x, denoted "$|x|$" (and which is read as "the absolute value of x"), is the distance of x from zero. This is why absolute value is never negative; absolute value only asks "how far?", not "in which direction?". This means not only that $|3| = 3$, because 3 is three units to the right of zero, but also that $|-3| = 3$, because -3 is three units to the left of zero.

When the number inside the absolute value (the "argument" of the absolute value) was positive anyway, we did not change the sign when we took the absolute value. But when the argument was negative, we did change the sign.

If $x > 0$ (that is, if x is positive), then the value would not change when you take the absolute value. For instance, if $x = 2$, then you have $|x| = |2| = 2 = x$. In fact, for *any* positive value of x (or if x equals zero), the sign would be unchanged, so:

For $x \geq 0$, $|x| = x$

On the other hand, if $x < 0$ (that is, if x is negative), then it will change its sign when you take the absolute value. For instance, if $x = -4$, then $|x| = |-4| = +4 = -(-4) = -x$. In fact, for *any* negative value of x, the sign would have to be changed, so:

For $x < 0$, $|x| = -x$

Thus $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

Also $\sqrt{x^2} = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

i.e., $2 = \sqrt{2^2} = \sqrt{(-2)^2} = \left[(-2)^2\right]^{1/2} = -2$ is absurd as $\sqrt{x^2} = |x|$

$\Rightarrow \sqrt{(-2)^2} = |-2| = 2$

Thus square root exists only for non-negative numbers and its value is also non-negative.

Some students consider $\sqrt{4} = \pm 2$, which is wrong.

In fact $\sqrt{(-4)^2} = |-4| = 4$

$\sqrt{\left(1-\sqrt{2}\right)^2} = |1-\sqrt{2}| = \sqrt{2}-1$ etc.

Also some students write $\sqrt{x^2} = \pm x$ which is wrong, infact,

$$\sqrt{x^2} = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Also $a^2 < b^2 \Rightarrow \sqrt{a^2} < \sqrt{b^2} \Rightarrow |a| < |b|$

Graph of function $f(x) = y = |x|$

x	0	±1	±2	±3
y	0	1	4	9

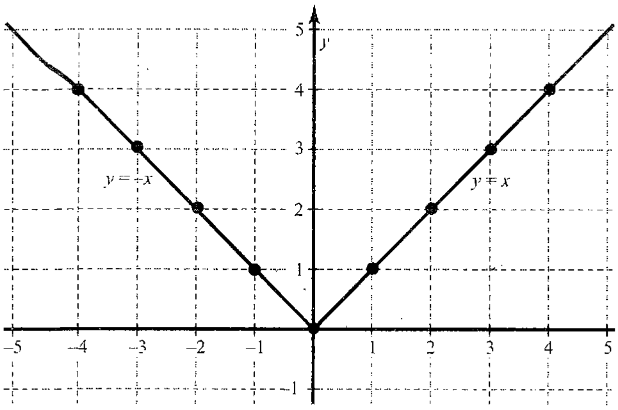


Fig. 1.15

We can see that graph of $y = |x|$ is in 1st and 2nd quadrant only where $y \geq 0$, hence $|x| \geq 0$.

Example 1.25 **Solve the following:**

(i) $|x| = 5$ **(ii) $x^2 - |x| - 2 = 0$**

Sol.

(i) $|x| = 5$, i.e., those points on real number line which are at distance 5 units from "0", which are –5 and 5.

Thus, $|x| = 5 \Rightarrow x = \pm 5$

(ii)

$$x^2 - |x| - 2 = 0$$
$$\Rightarrow |x|^2 - |x| - 2 = 0$$
$$\Rightarrow (|x| - 2)(|x| + 1) = 0$$
$$\Rightarrow |x| = 2\ (\because |x| + 1 \neq 0)$$
$$\Rightarrow x = \pm 2$$

Example 1.26 **Find the value of x for which following expressions are defined:**

(i) $\dfrac{1}{\sqrt{x-|x|}}$ **(ii)** $\dfrac{1}{\sqrt{x+|x|}}$

Sol.

(i) $x - |x| = \begin{cases} x - x = 0, \text{ if } x \geq 0 \\ x + x = 2x, \text{ if } x < 0 \end{cases}$

$\Rightarrow x - |x| \leq 0$ for all x

$\Rightarrow \dfrac{1}{\sqrt{x-|x|}}$ does not take real values for any $x \in R$

$\Rightarrow \dfrac{1}{\sqrt{x-|x|}}$ is not defined for any $x \in R$.

(ii)

$x + |x| = \begin{cases} x + x = 2x, \text{ if } x \geq 0 \\ x - x = 0, \text{ if } x < 0 \end{cases}$

$\Rightarrow \dfrac{1}{\sqrt{x+|x|}}$ is defined only when $x > 0$

What is the geometric meaning of $|x - y|$?

$|x - y|$ is the distance between x and y on the real number line.

Example 1.27 **Solve the following:**

(i) $|x - 2| = 1$ **(ii)** $2|x+1|^2 - |x+1| = 3$

Sol.

(i) $|x - 2| = 1$, i.e., those points on real number line which are distance 1 units from 2.

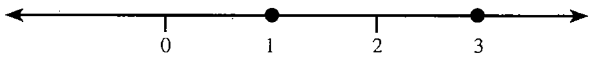


Fig. 1.16

As shown in the figure $x = 1$ and $x = 3$ are at distance 1 units from 2,

Hence, $x = 1$ or $x = 3$.

Thus $|x - 2| = 1$

$\Rightarrow x - 2 = \pm 1$

$\Rightarrow x = 1$ or $x = 3$

(ii)

$$2|x+1|^2 - |x+1| = 3$$
$$\Rightarrow 2|x+1|^2 - |x+1| - 3 = 0$$
$$\Rightarrow 2|x+1|^2 - 3|x+1| + 2|x+1| - 3 = 0$$
$$\Rightarrow (2|x+1| - 3)(|x+1| + 1) = 0$$
$$\Rightarrow 2|x+1| - 3 = 0$$
$$\Rightarrow |x+1| = 3/2$$
$$\Rightarrow x + 1 = \pm 3/2$$
$$\Rightarrow x = 1/2 \text{ or } x = -5/2$$

$|x - a| = \begin{cases} x - a, & x \geq a \\ a - x, & x < a \end{cases}$, where $a > 0$

In general, $|f(x)| = \begin{cases} f(x), & f(x) \geq 0 \\ -f(x), & f(x) < 0 \end{cases}$, where $y = f(x)$ is any real-valued function.

Example 1.28 **Solve the following:**

(i) $|x - 2| = (x - 2)$

(ii) $|x + 3| = -x - 3$

(iii) $|x^2 - x| = x^2 - x$

(iv) $|x^2 - x - 2| = 2 + x - x^2$

Sol.

(i) $|x - 2| = (x - 2)$, if $x - 2 \geq 0$ or $x \geq 2$

(ii) $|x + 3| = -x - 3$, if $x + 3 \leq 0$ or $x \leq -3$

(iii) $|x^2 - x| = x^2 - x$, if $x^2 - x \geq 0$

$\Rightarrow x(x - 1) \geq 0$

$\Rightarrow x \in (-\infty, 0] \cup [1, \infty)$

(iv) $|x^2 - x - 2| = 2 + x - x^2$

$\Rightarrow x^2 - x - 2 \leq 0$

$\Rightarrow (x - 2)(x + 1) \leq 0$

$\Rightarrow -1 \leq x \leq 2$

Example 1.29 **Solve $1 - x = \sqrt{x^2 - 2x + 1}$.**

Sol. $1 - x = \sqrt{x^2 - 2x + 1}$

$\Rightarrow 1 - x = \sqrt{(x-1)^2}$

$\Rightarrow 1 - x = |x - 1|$

$\Rightarrow 1 - x \geq 0$

$\Rightarrow x \leq 1$

Example 1.30 **Solve $|3x - 2| = x$.**

Sol. $|3x - 2| = x$

Case (i)

When $3x - 2 \geq 0$ or $x \geq 2/3$

For which we have $3x - 2 = x$ or $x = 1$.

Case (ii)

When $3x - 2 < 0$ or $x < 2/3$

For which we have $2 - 3x = x$ or $x = 1/2$.

Hence, solution set is $\{1/2, 1\}$.

Example 1.31 **Solve $|x| = x^2 - 1$.**

Sol. $x^2 - 1 = |x|$

$\Rightarrow x^2 - 1 = x$ when $x \geq 0$

or $x^2 - 1 = -x$ when $x < 0$

$$x^2 - x - 1 = 0 \Rightarrow x = \frac{1+\sqrt{5}}{2} \text{ (as } x \geq 0)$$

$$x^2 + x - 1 = 0 \Rightarrow x = \frac{-1-\sqrt{5}}{2} \text{ (as } x < 0)$$

Example 1.32 **Solve**

$$\sqrt{x+3-4\sqrt{x-1}} + \sqrt{x+8-6\sqrt{x-1}} = 1$$

Sol. $\sqrt{x+3-4\sqrt{x-1}} + \sqrt{x+8-6\sqrt{x-1}} = 1$

$\Rightarrow \sqrt{x-1-4\sqrt{x-1}+4} + \sqrt{x-1-6\sqrt{x-1}+9} = 1$

$\Rightarrow \sqrt{|\sqrt{x-1}-2|^2} + \sqrt{|\sqrt{x-1}-3|^2} = 1$

$\Rightarrow |\sqrt{x-1}-2| + |\sqrt{x-1}-3| = 1$

$\Rightarrow |\sqrt{x-1}-2| + |\sqrt{x-1}-3| = (\sqrt{x-1}-2) - (\sqrt{x-1}-3)$

$\Rightarrow \sqrt{x-1}-2 \geq 0$ and $\sqrt{x-1}-3 \leq 0$

$\Rightarrow 2 \leq \sqrt{x-1} \leq 3$

$\Rightarrow 4 \leq x-1 \leq 9$

$\Rightarrow 5 \leq x \leq 10$

Example 1.33 **Prove that**

$$\sqrt{x^2+2x+1} - \sqrt{x^2-2x+1} = \begin{cases} -2, & x < -1 \\ 2x, & -1 \leq x \leq 1 \\ 2, & x > 1 \end{cases}$$

Sol. $\sqrt{x^2+2x+1} - \sqrt{x^2-2x+1}$

$$= \sqrt{(x+1)^2} - \sqrt{(x-1)^2}$$

$$= |x+1| - |x-1|$$

$$= \begin{cases} -x-1-(1-x), & x < -1 \\ x+1-(1-x), & -1 \leq x \leq 1 \\ x+1-(x-1), & x > 1 \end{cases}$$

$$= \begin{cases} -2, & x < -1 \\ 2x, & -1 \leq x \leq 1 \\ 2, & x > 1 \end{cases}$$

Example 1.34

(i) For $2 < x < 4$, find the values of $|x|$.

(ii) For $-3 \leq x \leq -1$, find the values of $|x|$.

(iii) For $-3 \leq x < 1$, find the values of $|x|$.

(iv) For $-5 < x < 7$, find the values of $|x - 2|$.

(v) For $1 \leq x \leq 5$, find the values of $|2x - 7|$.

Sol.

(i) $2 < x < 4$

Here values on real number line whose distance lies between 2 and 4.

Here values of x are positive $\Rightarrow |x| \in (2, 4)$

(ii) $-3 \leq x \leq -1$

Here values on real number line whose distance lies between 1 and 3 or at distance 1 or 3.

$\Rightarrow 1 \leq |x| \leq 3$

(iii) $-3 \leq x < 1$

For $-3 \leq x < 0$, $|x| \in (0, 3]$

For $0 \leq x < 1$, $|x| \in [0, 1)$

So for $-3 \leq x < 1$, $|x| \in [0, 1) \cup (0, 3]$ or $x \in [0, 3]$

(iv) $-5 < x < 7$

$\Rightarrow -7 < x - 2 < 5$

$\Rightarrow 0 \leq |x - 2| < 7$

(v) $1 \le x \le 5$

$\Rightarrow 2 \le 2x \le 10$

$\Rightarrow -5 \le 2x - 7 \le 3$

$\Rightarrow |2x - 7| \in [0, 5]$

Example 1.35 **For $x \in R$, find all possible values of**

(i) $|x - 3| - 2$ **(ii) $4 - |2x + 3|$**

Sol.

(i) We know that $|x - 3| \ge 0\ \forall\ x \in R$

$\Rightarrow\ |x - 3| - 2 \ge -2$

$\Rightarrow\ |x - 3| - 2 \in [-2, \infty)$

(ii) We know that $|2x + 3| \ge 0\ \forall\ x \in R$

$\Rightarrow\ -|2x + 3| \le 0$

$\Rightarrow\ 4 - |2x + 3| \le 4$

or $4 - |2x + 3| \in (-\infty, 4]$

Example 1.36 **Find all possible values of**

(i) $\sqrt{|x|-2}$ **(ii) $\sqrt{3-|x-1|}$** **(iii) $\sqrt{4-\sqrt{x^2}}$**

Sol.

(i) $\sqrt{|x|-2}$

We know that square roots are defined for non-negative values only.

It implies that we must have $|x| - 2 \ge 0$.

$\Rightarrow \sqrt{|x|-2} \ge 0$

(ii) $\sqrt{3-|x-1|}$ is defined when $3 - |x - 1| \ge 0$

But the maximum value of $3 - |x - 1|$ is 3 when $|x - 1|$ is 0.

Hence, for $\sqrt{3-|x-1|}$ to get defined, $0 \le 3 - |x - 1| \le 3$.

$\Rightarrow\ \sqrt{3-|x-1|} \in \left[0, \sqrt{3}\right]$

Alternatively, $|x - 1| \ge 0$

$\Rightarrow\ -|x - 1| \le 0$

$\Rightarrow\ 3 - |x - 1| \le 3$

But for $\sqrt{3-|x-1|}$ to get defined, we must have

$0 \le 3 - |x - 1| \le 3 \Rightarrow 0 \le \sqrt{3-|x-1|} \le \sqrt{3}$

(iii) $\sqrt{4-\sqrt{x^2}} = \sqrt{4-|x|}$

$|x| \ge 0$

$\Rightarrow\ -|x| \le 0$

$\Rightarrow\ 4 - |x| \le 4$

But for $\sqrt{4-|x|}$ to get defined $0 \le 4 - |x| \le 4$

$\Rightarrow\ 0 \le \sqrt{4-|x|} \le 2$

Example 1.37 **Solve $|x - 3| + |x - 2| = 1$.**

Sol. $|x - 3| + |x - 2| = 1$

$\Rightarrow\ |x - 3| + |x - 2| = (3 - x) + (x - 2)$

$\Rightarrow\ x - 3 \le 0$ and $x - 2 \ge 0$

$\Rightarrow\ x \le 3$ and $x \ge 2$

$\Rightarrow\ 2 \le x \le 3$

Inequalities Involving Absolute Value

(i) $|x| \le a$ (where $a > 0$)

It implies those values of x on real number line which are at distance a or less than a.

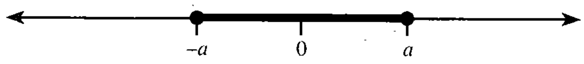


Fig. 1.17

$\Rightarrow -a \le x \le a$

e.g. $|x| \le 2 \Rightarrow -2 \le x \le 2$

$|x| < 3 \Rightarrow -3 < x < 3$

In general, $|f(x)| \le a$ (where $a > 0$) $\Rightarrow -a \le f(x) \le a$.

(ii) $|x| \ge a$ (where $a > 0$)

It implies those values of x on real number line which are at distance a or more than a

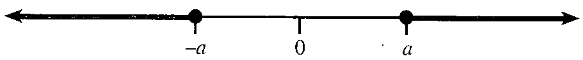


Fig. 1.18

$\Rightarrow x \le -a$ or $x \ge a$

e.g. $|x| \ge 3 \Rightarrow x \le -3$ or $x \ge 3$.

$|x| > 2 \Rightarrow x < -2$ or $x > 2$

In general, $|f(x)| \ge a \Rightarrow f(x) \le -a$ or $f(x) \ge a$.

(iii) $a \le |x| \le b$ (where $a, b > 0$)

It implies those value of x on real number line which are at distance equal a or b or between a and b.

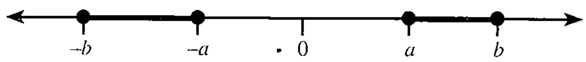


Fig. 1.19

$\Rightarrow [-b, -a] \cup [a, b]$

e.g. $2 \le |x| \le 4 \Rightarrow x \in [-4, -2]\ [2, 4]$

(iv) $|x + y| < |x| + |y|$ if x and y have opposite signs.

$|x - y| < |x| + |y|$ if x and y have same sign.

$|x + y| = |x| + |y|$ if x and y have same sign or at least one of x and y is zero.

$|x - y| = |x| + |y|$ if x and y have opposite signs or at least one of x and y is zero.

Example 1.38 **Solve $x^2 - 4|x| + 3 < 0$.**

Sol. $x^2 - 4|x| + 3 < 0$

$\Rightarrow\ (|x| - 1)(|x| - 3) < 0$

$\Rightarrow\ 1 < |x| < 3$

$\Rightarrow\ -3 < x < -1$ or $1 < x < 3$

$\Rightarrow\ x \in (-3, -1) \cup (1, 3)$

Example 1.39 **Solve $0 < |x| < 2$.**

Sol. We know that $|x| \ge 0,\ \forall\ x \in R$

But given $|x| > 0 \Rightarrow x \ne 0$

Now $0 < |x| < 2$

$\Rightarrow\ x \in (-2, 2), x \ne 0$

$\Rightarrow \quad x \in (-2, 2) - \{0\}$

Example 1.40 **Solve $|3x - 2| < 4$.**

Sol. $|3x-2| < 4$

$\Rightarrow \quad -4 < 3x - 2 < 4$

$\Rightarrow \quad -2 < 3x < 6$

$\Rightarrow \quad -2/3 < x < 2$

Example 1.41 **Solve $1 \le |x-2| \le 3$.**

Sol. $1 \le |x-2| \le 3$

$\Rightarrow \quad -3 \le x - 2 \le -1$ or $1 \le x - 2 \le 3$

$\Rightarrow \quad -1 \le x \le 1$ or $3 \le x \le 5$

$\Rightarrow \quad x \in [-1, 1] \cup [3, 5]$

Example 1.42 **Solve $0 < |x-3| \le 5$.**

Sol. $0 < |x-3| \le 5$

$\Rightarrow \quad -5 \le x - 3 < 0$ or $0 < x - 3 \le 5$

$\Rightarrow \quad -2 \le x < 3$ or $3 < x \le 8$

$\Rightarrow \quad x \in [-2, 3) \cup (3, 8]$

Example 1.43 **Solve $||x-1| - 2| < 5$.**

Sol. $||x-1| - 2| < 5$

$\Rightarrow \quad -5 < |x-1| - 2 < 5$

$\Rightarrow \quad -3 < |x-1| < 7$

$\Rightarrow \quad |x-1| < 7$

$\Rightarrow \quad -7 < x - 1 < 7$

$\Rightarrow \quad -6 < x < 8$

Example 1.44 **Solve $|x-3| \ge 2$.**

Sol. $|x-3| \ge 2$

$\Rightarrow \quad x - 3 \le -2$ or $x - 3 \ge 2$

$\Rightarrow \quad x \le 1$ or $x \ge 5$

Example 1.45 **Solve $||x| - 3| > 1$.**

Sol. $||x| - 3| > 1$

$\Rightarrow \quad |x| - 3 < -1$ or $|x| - 3 > 1$

$\Rightarrow \quad |x| < 2$ or $|x| > 4$

$\Rightarrow \quad -2 < x < 2$ or $x < -4$ or $x > 4$

Example 1.46 **Solve $|x-1| + |x-2| \ge 4$.**

Sol. Let $f(x) = |x-1| + |x-2|$

A.	**B.** $f(x)$	**C.** $f(x) \ge 4$	**D.** $A \cap C$
$x < 1$	$1 - x + 2 - x = 3 - 2x$	$3 - 2x \ge 4 \Rightarrow x \le -2/3$	$x \le -1/2$
$1 \le x \le 2$	$x - 1 + 2 - x = 1$	$1 \ge 4$, not possible	
$x > 2$	$x - 1 + x - 2 = 2x - 3$	$2x - 3 \ge 4 \Rightarrow x \ge 7/2$	$x \ge 7/2$

Hence, solutions is $x \in (-\infty, -1/2] \cup [7/2, \infty)$.

Example 1.47 **Solve $|x+1| + |2x-3| = 4$.**

Sol. Let $f(x) = |x+1| + |2x-3|$

A.	**B.** $f(x)$	**C.** $f(x) \ge 4$	**D.** $A \cap C$
$x < -1$	$-1 - x + 3 - 2x$	$2 - 3x = 4 \Rightarrow x = -2/3$	No such x exists
$-1 \le x \le 3/2$	$x + 1 + 3 - 2x$	$4 - x = 4 \Rightarrow x = 0$	$x = 0$
$x > 3/2$	$x + 1 + 2x - 3$	$3x - 2 = 4 \Rightarrow x = 2$	$x = 2$

Hence, solutions set is $\{0, 2\}$

Example 1.48 **Solve $|x| + |x-2| = 2$.**

Sol. We have $|x| + |x-2| = 2$

$\Rightarrow \quad |x| + |x-2| = x - (x-2)$

$\Rightarrow \quad x(x-2) \le 0$

$\Rightarrow \quad 0 \le x \le 2$

Example 1.49 **Solve $|2x-3| + |x-1| = |x-2|$.**

Sol. $|2x-3| + |x-1| = |(2x-3) - (x-1)|$

$\Rightarrow \quad (2x-3)(x-1) \le 0$

$\Rightarrow \quad 1 \le x \le 3/2$

Example 1.50 **Solve $|x^2 + x - 4| = |x^2 - 4| + |x|$.**

Sol. $|x^2 + x - 4| = |x^2 - 4| + |x|$

$\Rightarrow \quad x(x^2 - 4) \ge 0$

$\Rightarrow \quad x(x-2)(x+2) \ge 0$

$\Rightarrow \quad x \in [-2, 0] \cup [2, \infty)$

Example 1.51 **If $|\sin x + \cos x| = |\sin x| + |\cos x|$ ($\sin x, \cos x \ne 0$), then in which quadrant does x lie?**

Sol. Here we have $|\sin x + \cos x| = |\sin x| + |\cos x|$.
It implies that $\sin x$ and $\cos x$ must have the same sign.
Therefore, x lies in the first or third quadrant.

Example 1.52 **Is $|\tan x + \cot x| < |\tan x| + |\cot x|$ true for any x ? If it is true, then find the values of x.**

Sol. Since $\tan x$ and $\cot x$ have always the same sign, $|\tan x + \cot x| < |\tan x| + |\cot x|$ does not hold true for any value of x.

Example 1.53 **Solve $\left|\dfrac{x-3}{x+1}\right| \le 1$.**

Sol. $\left|\dfrac{x-3}{x+1}\right| \le 1$

$\Rightarrow \quad -1 \le \dfrac{x-3}{x+1} \le 1$

$\Rightarrow \quad \dfrac{x-3}{x+1} - 1 \le 0$ and $0 \le \dfrac{x-3}{x+1} + 1$

$\Rightarrow \quad \dfrac{-4}{x+1} \le 0$ and $0 \le \dfrac{2x-2}{x+1}$

$\Rightarrow \quad x > -1$ and $\{x < -1$ or $x \geq 1\}$
$\Rightarrow \quad x \geq 1$

Example 1.54 **Solve $|x^2 - 2x| + |x - 4| > |x^2 - 3x + 4|$.**

Sol. We have $|x^2 - 2x| + |4 - x| > |x^2 - 2x + 4 - x|$
$\Rightarrow \quad (x^2 - 2x)(4 - x) < 0$
$\Rightarrow \quad x(x - 2)(x - 4) > 0$
$\Rightarrow \quad x \in (0, 2) \cup (4, \infty)$

Concept Application Exercise 1.1

1. If $f(x) = \begin{cases} x+3, & x<1 \\ x^2, & 1 \leq x \leq 3 \\ 2-3x, & x>3 \end{cases}$, then which of the following is greatest ?
$f(0), f(3), f(4), f(2)$
2. If $f(x)$ is quadratic function such that $f(0) = -4$, $f(1) = -5$ and $f(-1) = -1$, then find the value of $f(3)$.
3. Find the value of x^2 for the following values of x:
(i) $[-5, -1]$ **(ii)** $(3, 6)$
(iii) $(-2, 3]$ **(iv)** $(-3, \infty)$ **(v)** $(-\infty, 4)$
4. Find the values of $1/x$ for the following values of x:
(i) $(2, 5)$ **(ii)** $[-5, -1)$
(iii) $(3, \infty)$ **(iv)** $(-\infty, -2]$
(v) $[-3, 4]$
5. Which of the following is always true?
(a) If $a < b$, then $a^2 < b^2$
(b) If $a < b$, then $\frac{1}{a} > \frac{1}{b}$
(c) If $a < b$, then $|a| < |b|$
6. Find the values of x which satisfy the inequalities simultaneously:
(i) $-3 < 2x - 1 < 19$ **(ii)** $-1 \leq \frac{2x+3}{5} \leq 3$
7. Find all the possible values which the following expressions take.
(i) $\frac{2-5x}{3x-4}$
(ii) $\sqrt{x^2 - 7x + 6}$
(iii) $\frac{x^2 - x - 6}{x - 3}$
8. Solve $\frac{x(3-4x)(x+1)}{(2x-5)} < 0$.
9. Solve $\frac{(2x+3)(4-3x)^3(x-4)}{(x-2)^2 x^5} \leq 0$.
10. Solve $\frac{(x-3)(x+5)(x-7)}{|x-4|(x+6)} \leq 0$.
11. Find all possible values of $f(x) = \frac{1-x^2}{x^2+3}$.
12. Solve $\frac{2x}{2x^2 + 5x + 2} > \frac{1}{x+1}$.
13. Solve (i) $\frac{\sqrt{x-1}}{x-2} < 0$ (ii) $\sqrt{x-2} \leq 3$
14. Which of the following equations has maximum number of real roots?
(i) $x^2 - |x| - 2 = 0$
(ii) $x^2 - 2|x| + 3 = 0$
(iii) $x^2 - 3|x| + 2 = 0$
(iv) $x^2 + 3|x| + 2 = 0$
15. Find the number of solutions of the system of equation $x + 2y = 6$ and $|x - 3| = y$.
16. Find the values of x for which $f(x) = \sqrt{\frac{1}{|x-2| - (x-2)}}$ is defined.
17. Find all values of x for which $f(x) = x + \sqrt{x^2}$.
18. Solve $\left|\frac{x+2}{x-1}\right| = 2$.
19. If $|x^2 - 7| \leq 9$, then find the values of x.
20. Find the values of x for which $\sqrt{5 - |2x - 3|}$ is defined.
21. Solve $||x - 2| - 3| < 5$.
22. Which of the following is/are true?
(a) If $|x + y| = |x| + |y|$, then points (x, y) lie in 1st or 3rd quadrant or any of the x-axis or y-axis.
(b) If $|x + y| < |x| + |y|$, then points (x, y) lie in 2nd or 4th quadrant.
(c) If $|x - y| = |x| + |y|$, then points (x, y) lie in 2nd or 4th quadrant.
23. Solve $|x^2 - x - 2| + |x + 6| = |x^2 - 2x - 8|$.
24. Solve $|x| = 2x - 1$.
25. Solve $|2^x - 1| + |2^x + 1| = 2$.
26. Solve $|x^2 - 4x + 3| = x + 1$.
27. Solve $|x^2 - 1| + |x^2 - 4| > 3$.
28. Solve $|x - 1| - |2x - 5| = 2x$.

SOME DEFINITIONS

Real Polynomial

Let $a_0, a_1, a_2, \ldots, a_n$ be real numbers and x is a real variable. Then, $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ is called a real polynomial of real variable x with real coefficients.

Complex Polynomial

If $a_0, a_1, a_2, \ldots, a_n$ are complex numbers and x is a varying complex number, then $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ is called a complex polynomial or a polynomial of complex coefficients.

Rational Expression or Rational Function

An expression of the form

$$\frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials in x is called a rational expression.

In the particular case when $Q(x)$ is a non-zero constant,

$$\frac{P(x)}{Q(x)}$$

reduces to a polynomial. Thus every polynomial is a rational expression but the converse is not true. Some of the examples are as follows:

(1) $\frac{x^2-5x+4}{x-2}$ (2) x^2-5x+4

(3) $\frac{1}{x-2}$ (4) $x+\frac{1}{x}$, i.e., $\frac{x^2+1}{x}$.

Degree of a Polynomial

A polynomial $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, real or complex, is a polynomial of degree n, if $a_n \neq 0$.

The polynomials $2x^3 - 7x^2 + x + 5$ and $(3-2i)x^2 - ix + 5$ are polynomials of degree 3 and 2, respectively

A polynomial of second degree is generally called a quadratic polynomial, and polynomials of degree 3 and 4 are known as cubic and bi-quadratic polynomials, respectively.

Polynomial Equation

If $f(x)$ is a polynomial, then $f(x) = 0$ is called a polynomial equation.

If $f(x)$ is a quadratic polynomial, then $f(x) = 0$ is called a quadratic equation. The general form of a quadratic equation is $ax^2 + bx + c = 0$, $a \neq 0$. Here, x is the variable and a, b and c are called coefficients, real or imaginary.

Roots of an Equation

The values of the variable satisfying a given equation are called its roots.

Thus, $x = \alpha$ is a root of the equation $f(x) = 0$, if $f(\alpha) = 0$. For example, $x = 1$ is a root of the equation $x^3 - 6x^2 + 11x - 6 = 0$, because $1^3 - 6 \times 1^2 + 11 \times 1 - 6 = 0$.

Similarly, $x = \omega$ and $x = \omega^2$ are roots of the equation $x^2 + x + 1 = 0$ as they satisfy it (where ω is the complex cube root of unity).

Solution Set

The set of all roots of an equation, in a given domain, is called the solution set of the equation.

For example, the set {1, 2, 3} is the solution set of the equation $x^3 - 6x^2 + 11x - 6 = 0$.

Solving an equation means finding its solution set. In other words, solving an equation is the process of obtaining all its roots.

Example 1.55 **If $x = 1$ and $x = 2$ are solutions of the equation $x^3 + ax^2 + bx + c = 0$ and $a + b = 1$, then find the value of b.**

Sol. Since $x = 1$ is a root of the given equation it satisfies the equation.

Hence, putting $x = 1$ in the given equation, we get

$$a + b + c = -1 \quad (1)$$

but given that

$$a + b = 1 \quad (2)$$

$\Rightarrow$ $c = -2$

Now put $x = 2$ in the given equation, we have

$8 + 4a + 2b - 2 = 0$

$\Rightarrow$ $6 + 2a + 2(a + b) = 0$

$\Rightarrow$ $6 + 2a + 2 = 0$

$\Rightarrow$ $a = -4$

$\Rightarrow$ $b = 5$

Example 1.56 **Let $f(x) = ax^2 + bx + c$, where $a, b, c \in R$ and $a \neq 0$. It is known that $f(5) = -3f(2)$ and that 3 is a root of $f(x) = 0$, then find the other root of $f(x) = 0$.**

Sol. $f(x) = ax^2 + bx + c$

Given that $f(5) = -3f(2)$

$25a + 5b + c = -3(4a + 2b + c)$

or $37a + 11b + 4c = 0$ (1)

Also $x = 3$ satisfies $f(x) = 0$

$\therefore$ $9a + 3b + c = 0$ (2)

or $36a + 12b + 4c = 0$ (3)

[Multiplying Eq. (2) by 4]

Subtracting (3) from (1), we have

$a - b = 0$

$\Rightarrow$ $a = b$ $\Rightarrow$ In (2) put $b = a$,

$\Rightarrow$ $12a + c = 0$ or $c = -12a$

Hence, equation $f(x) = 0$ becomes

$ax^2 + ax - 12a = 0$

or $x^2 + x - 12 = 0$

or $(x-3)(x+4) = 0$ or $x = -4, 3$

Example 1.57 **A polynomial in x of degree three vanishes when $x = 1$ and $x = -2$, and has the values 4 and 28 when $x = -1$ and $x = 2$, respectively. Then find the value of polynomial when $x = 0$.**

Sol. From the given data $f(x) = (x-1)(x+2)(ax+b)$

Now $f(-1) = 4$ and $f(2) = 28$

$\Rightarrow$ $(-1-1)(-1+2)(-a+b) = 4$

and $(2-1)(2+2)(2a+b) = 28$

$\Rightarrow$ $a - b = 2$ and $2a + b = 7$

Solving, $a = 3$ and $b = 1$

$\Rightarrow$ $f(x) = (x-1)(x+2)(3x+1)$

$\Rightarrow$ $f(0) = -2$

Example 1.58 **If $(1-p)$ is a root of quadratic equation $x^2 + px + (1-p) = 0$, then find its roots.**

Sol. Since $(1-p)$ is the root of quadratic equation

$$x^2 + px + (1-p) = 0 \quad (1)$$

So $(1-p)$ satisfies the above equation

$\therefore$ $(1-p)^2 + p(1-p) + (1-p) = 0$

$\therefore$ $(1-p)[1 - p + p + 1] = 0$

$\therefore$ $(1-p)(2) = 0$

$\Rightarrow$ $p = 1$

On putting this value of p in Eq. (1), we get

$x^2 + x = 0$

$\Rightarrow$ $x(x+1) = 0$

$\Rightarrow$ $x = 0, -1$

Example 1.59 **The quadratic polynomial $p(x)$ has the following properties:**

$p(x)$ can be positive or zero for all real numbers

$p(1) = 0$ and $p(2) = 2$.

Then find the quadratic polynomial.

Sol. $p(x)$ is positive or zero for all real numbers

also $p(1) = 0$

then we have $p(x) = k(x - 1)^2$, where $k > 0$

Now $p(2) = 2$

$\Rightarrow$ $k = 2$

$\therefore$ $p(x) = 2(x - 1)^2$

GEOMETRICAL MEANING OF ROOTS (ZEROS) OF AN EQUATION

We know that that a real number k is a zero of the polynomial $f(x)$ if $f(k) = 0$. But why are the zeroes of a polynomial so important? To answer this, first we will see the *geometrical* representations of polynomials and the geometrical meaning of their zeroes.

We know that graph of the linear function $y = f(x) = ax + b$ is a straight line.

Consider the function $f(x) = x + 3$.

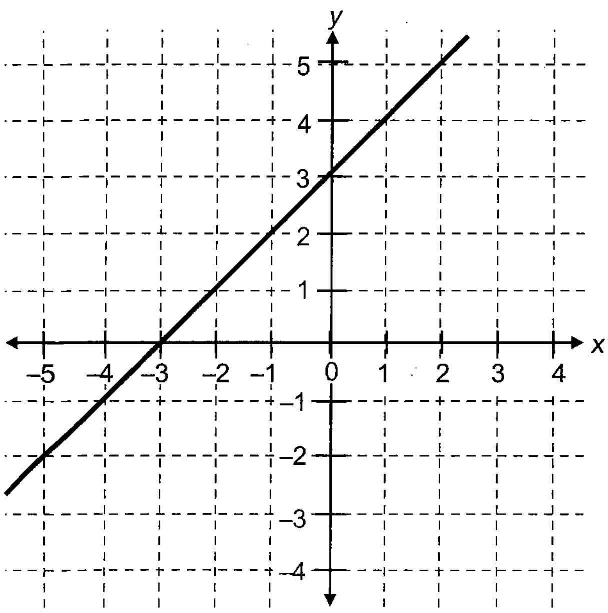


Fig. 1.20

Now we can see that this graph cuts the x-axis at $x = -3$, where value of $y = 0$ or we can say $x + 3 = 0$ (or $y = 0$) when value of $x = -3$. Thus, $x = -3$ which is a root (zero) of equation $x + 3 = 0$ is actually the value of x where graph of $y = f(x) = x + 3$ intersects the x-axis.

Consider the function $f(x) = x^2 - x - 2$, now for $f(x) = 0$ or $x^2 - x - 2 = 0$, we have $(x - 2)(x + 1) = 0$ or $x = -1$ or $x = 2$. Then graph of $f(x) = x^2 - x - 2$ cuts the x-axis at two values of x, $x = -1$ and $x = 2$.

Following is the graph of $y = f(x)$.

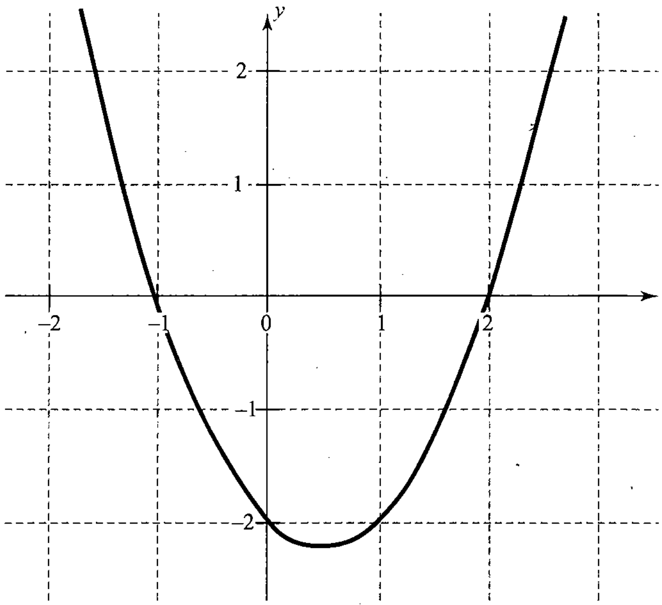


Fig. 1.21

Consider the function $f(x) = x^3 - 6x^2 + 11x - 6$, now for $f(x) = 0$ we have $(x - 1)(x - 2)(x - 3) = 0$ or $x = 1, 2, 3$. Then graph of $y = f(x)$ cuts x-axis at three values of x, $x = 1, 2, 3$.

Following is the graph of $y = f(x)$.

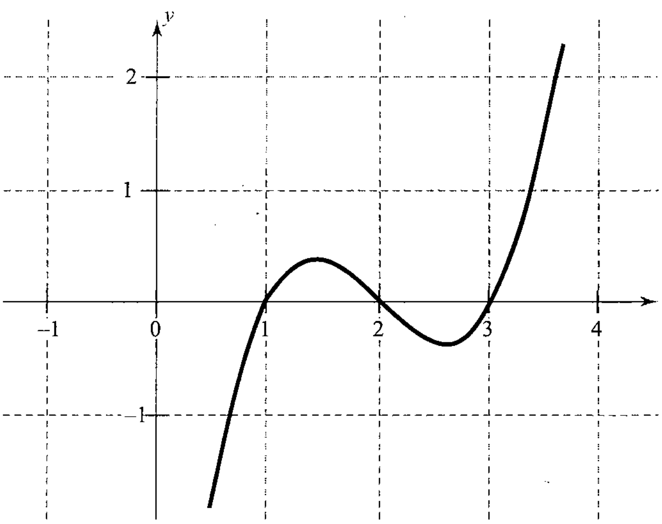


Fig. 1.22

Consider the function $f(x) = (x^2 - 3x + 2)(x^2 - x + 1)$, now for $f(x) = 0$ we have $x = 1$ or $x = 2$, as $x^2 - x + 1 = 0$ is not possible for any real value of x. Hence, $f(x) = 0$ has only two real roots and cuts x-axis for only two values of x, $x = 1$ and $x = 2$.

Following is the graph of $y = f(x)$.

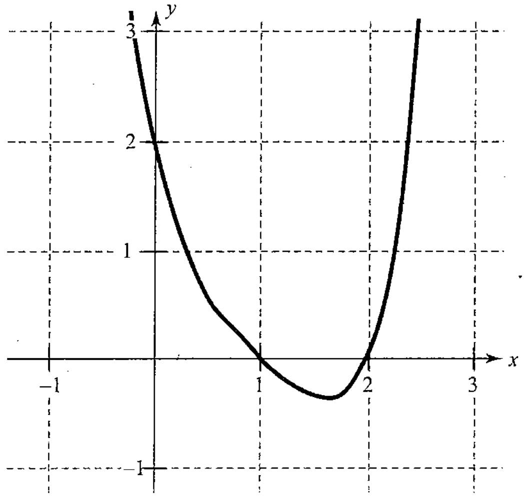


Fig. 1.23

Thus, roots of equation $f(x) = 0$ are actually those values of x where graph $y = f(x)$ meets x-axis.

Roots (Zeros) of the Equation $f(x) = g(x)$

Now we know that zeros of the equation $f(x) = 0$ are the x-coordinates of the points where graph of $y = f(x)$ intersect the x-axis, where $y = 0$ or zeros are x-coordinate of the point of intersection of $y = f(x)$ and $y = 0$ (x-axis)

Consider the equation $x + 5 = 2$.

Let's draw the graph of $y = x + 5$ and $y = 2$, which are as shown in the following figure.

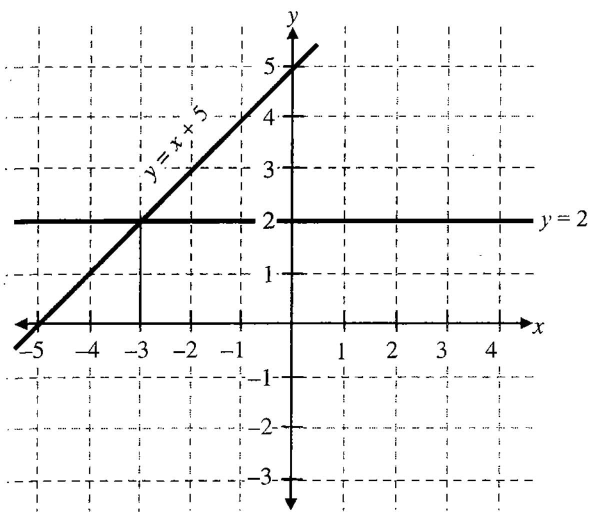


Fig. 1.24

Graph of $y = 2$ is a line parallel to x-axis at height 2 unit above x-axis. Now in the figure, we can see that graphs of $y = x + 5$ and $y = 2$ intersect at point $(-3, 2)$ where value of $x = -3$.

Also from $x + 5 = 2$, we have $x = 2 - 5$ or $x = -3$, which is a root of the equation $x + 2 = 5$. Thus root of the equation $x + 5 = 2$ occurs at point of intersection of graphs $y = x + 5$ and $y = 2$.

Consider the another example $x^2 - 2x = 2 - x$. Let's draw the graph of $y = x^2 - 2x$ and $y = 2 - x$ as shown in the following figure.

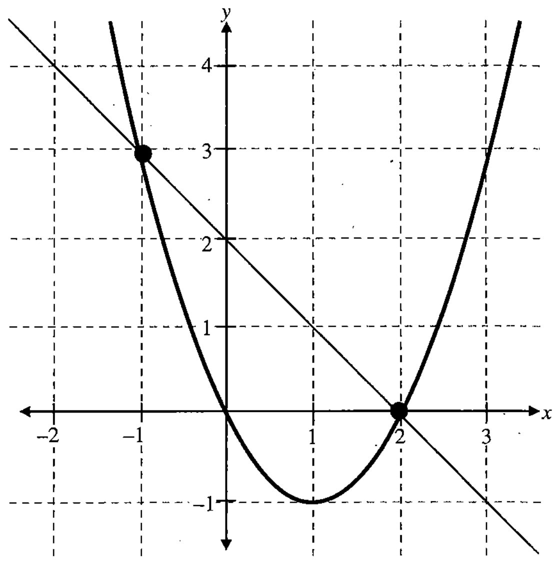


Fig. 1.25

Now in the figure, we can see that graphs of $y = x^2 - 2x$ and $y = 2 - x$ intersect at points $(-1, 3)$ and $(2, 0)$ or where values of x are $x = -1$ and $x = 2$, which are in fact zeros or roots of the equation $x^2 - 2x = 2 - x$ or $x^2 - x - 2 = 0$.

The given equation simplifies to $x^2 - x - 2 = 0$. So one can also locate the roots of the same equation by plotting the graph of $y = x^2 - x - 2$, then the roots of equation are x-coordinates of points where graph of $y = x^2 - x - 2$ intersects with the x-axis (where $y = 0$), as shown in the following figure.

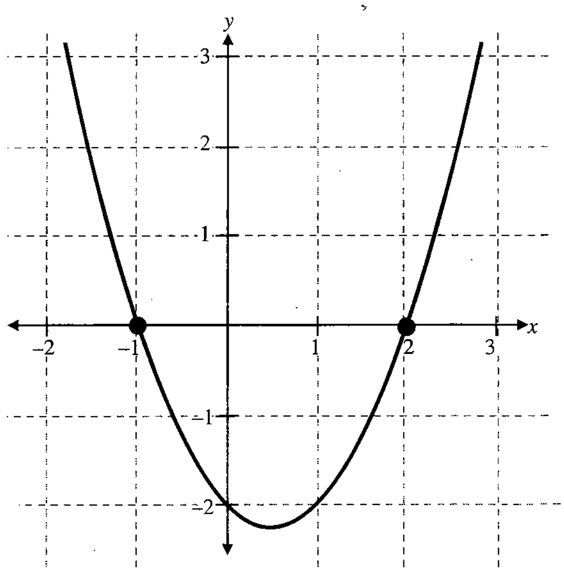


Fig. 1.26

From the above discussion we understand that roots of the equation $f(x) = g(x)$ are the x-coordinate of the points of intersection of graphs $y = f(x)$ and $y = g(x)$.

Example 1.60 **In how many points graph of $y = x^3 - 3x^2 + 5x - 3$ intersect x-axis?**

Sol. Number of point in which $y = x^3 - 3x^2 + 5x - 3$ intersect the x-axis is same as number of real roots of the equation $x^3 - 3x^2 + 5x - 3 = 0$.

Now we can see that $x = 1$ satisfies the equation, hence one root of the equation is $x = 1$.

Now dividing $x^3 - 3x^2 + 5x - 3$ by $x - 1$, we have quotient $x^2 - 2x + 3$.

Hence equation reduces to $(x - 1)(x^2 - 2x + 3) = 0$.

Now $x^2 - 2x + 3 = 0$ or $(x - 1)^2 + 2 = 0$ is not true for any real value of x.

Hence, the only root of the equation is $x = 1$.

Therefore, the graph of $y = x^3 - 3x^2 + 5x - 3$ cuts the x-axis in one point only.

Example 1.61 **In the following diagram, the graph of $y = f(x)$ is given.**

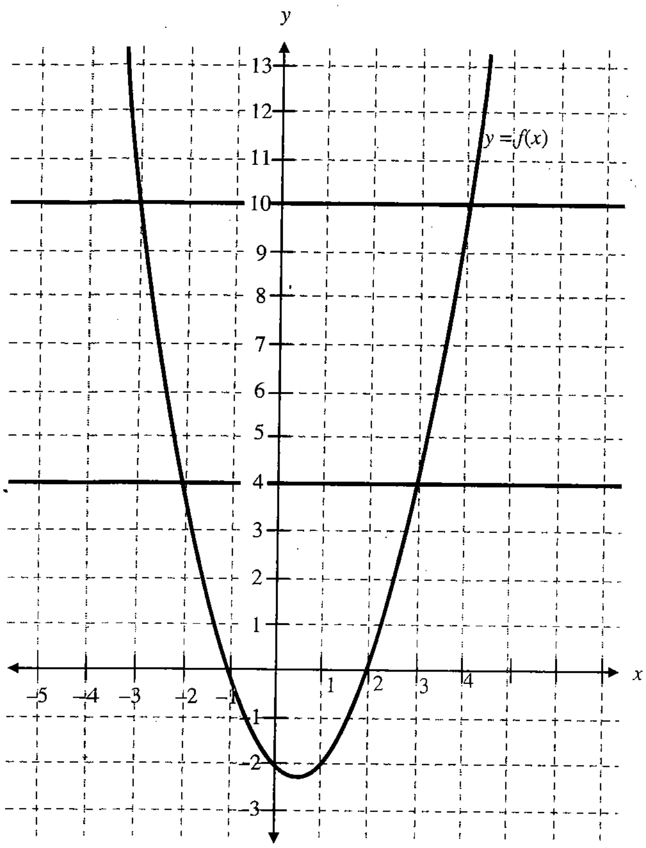


Fig. 1.27

Answer the following questions:

(a) what are the roots of the $f(x) = 0$?

(b) what are the roots of the $f(x) = 4$?

(c) what are the roots of the $f(x) = 10$?

Sol.

(a) The root of the equation $f(x) = 0$ occurs for the values of x where the graphs of $y = f(x)$ and $y = 0$ intersect.

From the diagram, for these point of intersection $x = -1$ and $x = 2$. Hence, roots of the equation $f(x) = 0$ are $x = -1$ and $x = 2$.

(b) The root of the equation $f(x) = 4$ occurs for the values of x where the graphs of $y = f(x)$ and $y = 4$ intersect.

From the diagram, for these point of intersection $x = -2$ and $x = 3$. Hence, roots of the equation $f(x) = 0$ are $x = -2$ and $x = 3$.

(c) Also roots of the equation $f(x) = 10$ are -3 and 4.

Example 1.62 **Which of the following pair of graphs intersect?**

(i) $y = x^2 - x$ and $y = 1$

(ii) $y = x^2 - 2x + 3$ and $y = \sin x$

(iii) $y = x^2 - x + 1$ and $y = x - 4$

Sol. $y = x^2 - x$ and $y = 1$ intersect if $x^2 - x = 1 \Rightarrow x^2 - x - 1 = 0$, which has real roots.

$y = x^2 - 2x + 3$ and $y = \sin x$ intersect if $x^2 - 2x + 3 = \sin x$ or $(x - 1)^2 + 2 = \sin x$, which is not possible as L.H.S. has the least value 2, while R.H.S. has the maximum value 1.

$y = x^2 - x + 1$ and $y = x - 4$ intersect if $x^2 - x + 1 = x - 4$ or $x^2 - 2x + 5 = 0$, which has non-real roots. Hence, graphs do not intersect.

Example 1.63 **Prove that graphs $y = 2x - 3$ and $y = x^2 - x$ never intersect.**

Sol. $y = 2x - 3$ and $y = x^2 - x$ intersect only when $x^2 - x = 2x - 3$ or $x^2 - 3x + 3 = 0$

Now discriminant $D = (-3)^2 - 4(3) = -3 < 0$

Hence, roots of the equation are not real, or we can say that there is no real number for which $2x - 3$ and $x^2 - x$ are equal (or $y = 2x - 3$ and $y = x^2 - x$ intersect).

Hence, proved.

KEY POINTS IN SOLVING AN EQUATION

Domain of Equation

It is a set of the values of independent variables x for which each function used in the equation is defined, i.e., it takes up finite real values. In other words, the final solution obtained while solving any equation must satisfy the domain of the expression of the parent equation.

Example 1.64 **Solve $\dfrac{x^2 - 2x - 3}{x + 1} = 0$.**

Sol. Equation $\dfrac{x^2 - 2x - 3}{x + 1} = 0$ is solvable over $R - \{-1\}$

Now $\dfrac{x^2 - 2x - 3}{x + 1} = 0$

$\Rightarrow$ $x^2 - 2x - 3 = 0$ or $(x - 3)(x + 1) = 0$

$\Rightarrow$ $x = 3$ (as $x \in R - \{-1\}$)

Example 1.65 **Solve $(x^3 - 4x)\sqrt{x^2 - 1} = 0$.**

Sol. Given equation is solvable for $x^2 - 1 \geq 0$

or $x \in (-\infty, -1]\ [1, \infty)$

$(x^3 - 4x)\sqrt{x^2 - 1} = 0$

$\Rightarrow$ $x(x - 2)(x + 2)\sqrt{x^2 - 1} = 0$

$\Rightarrow$ $x = 0, -2, 2, -1, 1$

But $x \in (-\infty, -1] \cup [1, \infty)$

$\Rightarrow$ $x = \pm 1, \pm 2$

Example 1.66 **Solve** $\dfrac{2x-3}{x-1}+1=\dfrac{6x-x^2-6}{x-1}$.

Sol. $\dfrac{2x-3}{x-1}+1=\dfrac{6x-x^2-6}{x-1}, x \neq 1$

$\Rightarrow$ $\dfrac{3x-4}{x-1}=\dfrac{6x-x^2-6}{x-1}, x \neq 1$

$\Rightarrow$ $3x - 4 = 6x - x^2 - 6, x \neq 1$

$\Rightarrow$ $x^2 - 3x + 2 = 0, x \neq 1$

$\Rightarrow$ $x = 2$

Extraneous Roots

While simplifying the equation, the domain of the equation may expand and give the extraneous roots.
For example, consider the equation $\sqrt{x} = x - 2$.
For solving, we first square it
so $\sqrt{x} = x - 2$

$\Rightarrow$ $x = (x-2)^2$ [on squaring both sides]

$\Rightarrow$ $x^2 - 5x + 4 = 0$

$\Rightarrow$ $(x-1)(x-4) = 0$

$\Rightarrow$ $x = 1, 4$

We observe that $x = 4$ satisfies the given equation but $x = 1$ does not satisfy it.
Hence, $x = 4$ is the only solution of the given equation.
The domain of actual equation is $[2, \infty)$.
While squaring the equation, domain expands to R, which gives extra root $x = 1$.

Loss of Root

Cancellation of common factors from both sides of equation leads to loss of root.
For example, consider an equation $x^2 - 2x = x - 2$

$\Rightarrow$ $x(x-2) = x - 2$

$\Rightarrow$ $x = 1$

Here we have cancelled factor $x - 2$ which causes the loss of root, $x = 2$
The correct way of solving is

$x^2 - 2x = x - 2$

$\Rightarrow$ $x^2 - 3x + 2 = 0$

$\Rightarrow$ $(x-1)(x-2) = 0$

$\Rightarrow$ $x = 1$ and $x = 2$.

GRAPHS OF POLYNOMIAL FUNCTIONS

When the polynomial function is written in standard form, $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, $(a_n \neq 0)$, the leading term is $a_n x^n$. In other words, the leading term is the term that the variable has its highest exponent. The degree of a term of a polynomial function is the exponent on the variable. The degree of the polynomial is the largest degree of all of its terms.

For drawing the graph of the polynomial function, we consider the following tests.

Test 1: Leading Co-efficient

If n is odd and the leading coefficient a_n is positive, then the graph falls to the left and rises to the right:

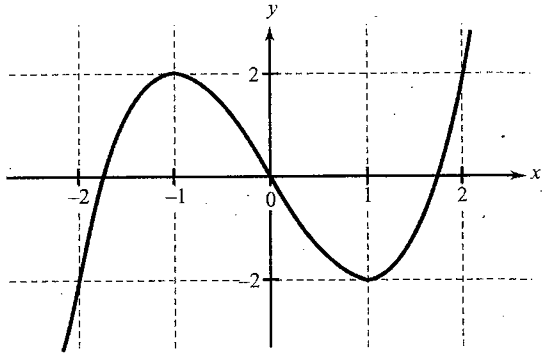


Fig. 1.28

If n is odd and the leading coefficient a_n is negative, the graph rises to the left and falls to the right.

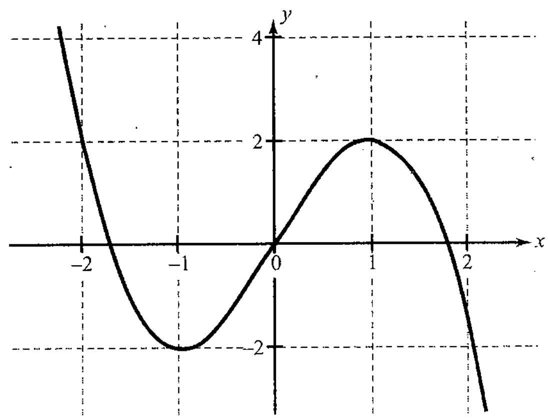


Fig. 1.29

If n is even and the leading coefficient a_n is positive, the graph rises to the left and to the right.

Fig. 1.30

If n is even and the leading coefficient a_n is negative, the graph falls to the left and to the right

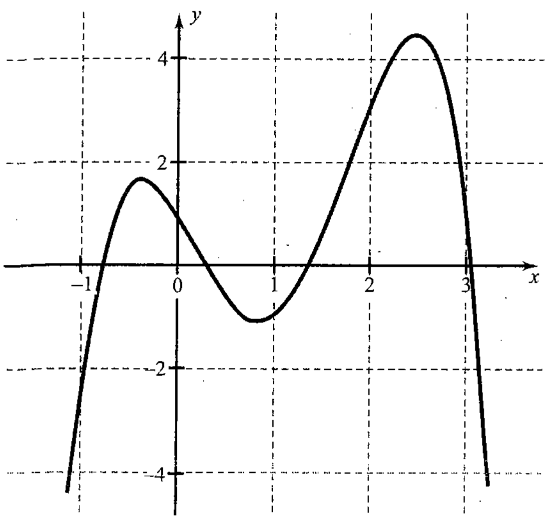


Fig. 1.31

Test 2: Roots (Zeros) of Polynomial

In other words, when a polynomial function is set equal to zero and has been completely factored and each different factor is written with the highest appropriate exponent, depending on the number of times that factor occurs in the product, the exponent on the factor that the zero is a solution for it gives the multiplicity of that zero.

The exponent indicates how many times that factor would be written out in the product, this gives us a multiplicity.

Multiplicity of Zeros and the x-Intercept

If r is a zero of even multiplicity:

This means the graph touches the x-axis at r and turns around.

This happens because the sign of $f(x)$ does not change from one side to the other side of r.

See the graph of $f(x) = (x - 2)^2 (x - 1)(x + 1)$.

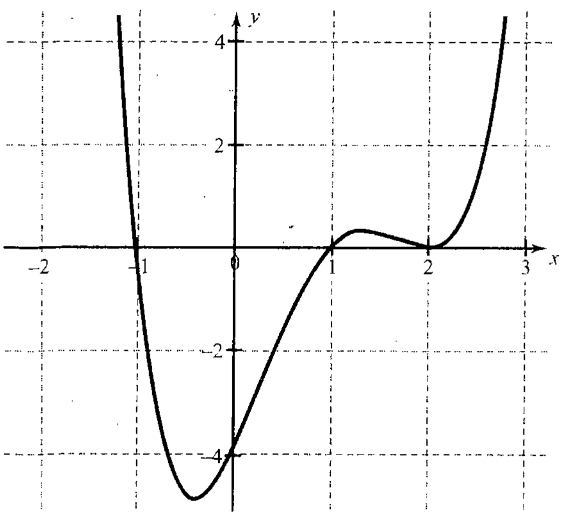


Fig. 1.32

If r is a zero of odd multiplicity:

This means the graph crosses (also touches if exponent is more than 1) the x-axis at r. This happens because the sign of $f(x)$ changes from one side to the other side or r.

See the graph of $f(x) = (x - 1)(3x - 2)(x - 3)^3$

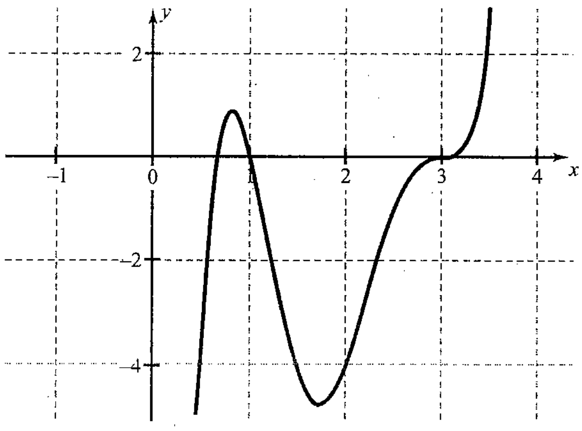


Fig. 1.33

Thus, in general, polynomial function graphs consist of a smooth line with a series of hills and valleys. The hills and valleys are called **turning points**. The maximum possible number of turning points is one less than the degree of the polynomial. The point where graph has turning point, derivative of function $f(x)$ becomes zero, which provides point of local minima or local maxima. Knowledge of derivative provides great help in drawing the graph of the function, hence finding its point of intersection with x-axis or roots of the equation $f(x) = 0$. Also we know that geometrically the derivative of function at any point of the graph of the function is equal to the slope of tangent at that point to the curve.

Consider the following graph of the function $y = f(x)$ as shown in the following figure.

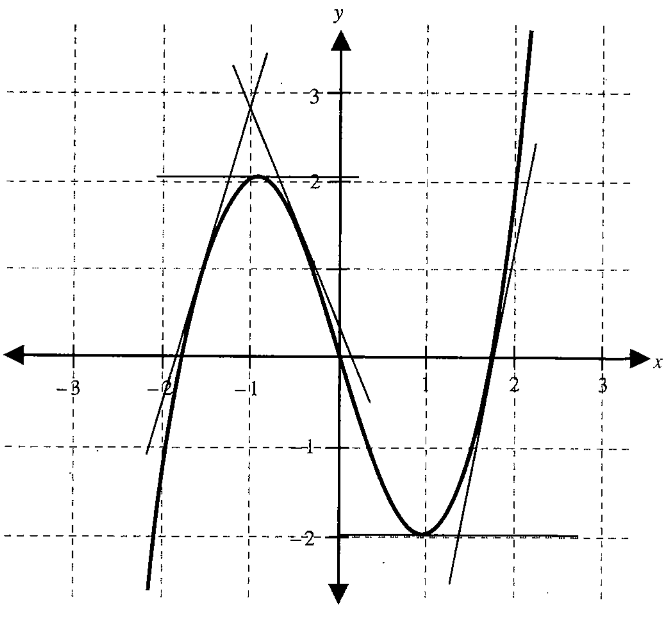


Fig. 1.34

In the figure, we can see that tangent to the curve at point for which $x < -1$ and $x > 1$ makes acute angle with the positive direction of x-axis, hence derivative is positive for these points. For $-1 < x < 1$, tangent to the curve makes obtuse angle with the positive direction of x-axis, hence derivative is negative at these points. At $x = -1$ and $x = 1$, tangent is parallel to x-axis, where derivative is zero.

Here $x = -1$ is called point of maxima, where derivative changes sign from positive to negative (from left to right), and $x = 1$ is called point of minima, where derivative changes sign from negative to positive (from left to right).

At point of maxima and minima, derivative of the function is zero.

Example 1.67 **Using differentiation method check how many roots of the equation $x^3 - x^2 + x - 2 = 0$ are real?**

Sol. Let $y = f(x) = x^3 - x^2 + x - 2$

$$\Rightarrow \quad \frac{dy}{dx} = 3x^2 - 2x + 1$$

Let $3x^2 - 2x + 1 = 0$, now this equation has non-real roots, i.e., derivative never becomes zero or graph of $y = f(x)$ has no turning point.

Also when $x \to \infty, f(x) \to \infty$ and when $x \to -\infty, f(x) \to -\infty$

Further $3x^2 - 2x + 1 > 0 \ \forall \ x \in R$

Thus graph of the function is as shown in the following figure.

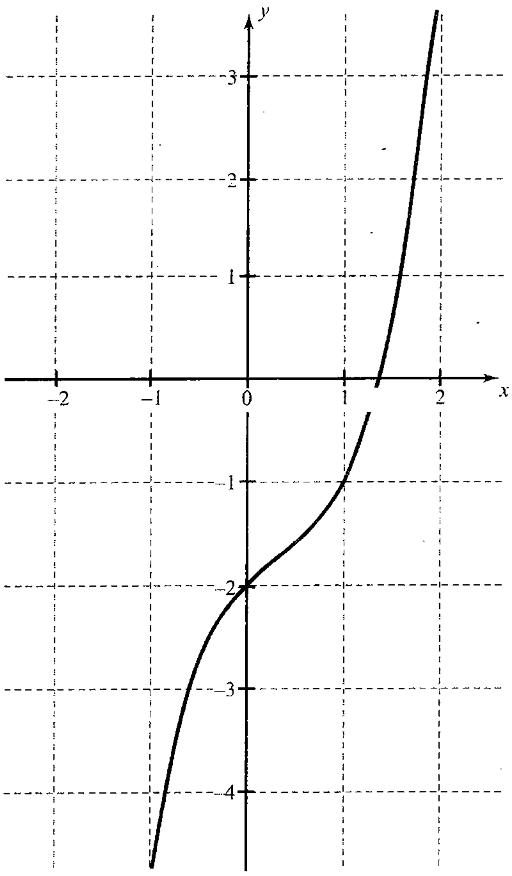


Fig. 1.35

Also $f(0) = -2$, hence graph cuts the x-axis for some positive value of x.

Hence, the only root of the equation is positive.

Thus we can see that differentiation and then graph of the function is much important in analyzing the equation.

Example 1.68 **Analyze the roots of the following equations:**

(i) $2x^3 - 9x^2 + 12x - (9/2) = 0$

(ii) $2x^3 - 9x^2 + 12x - 3 = 0$

Sol.

(i) Let $f(x) = 2x^3 - 9x^2 + 12x - (9/2)$

Then $f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x - 1)(x - 2)$

Now $f'(x) = 0 \Rightarrow x = 1$ and $x = 2$.

Hence, graph has turn at $x = 1$ and at $x = 2$.

Also $f(1) = 2 - 9 + 12 - (9/2) > 0$

and $f(2) = 16 - 36 + 24 - (9/2) < 0$

Hence, graph of the function $y = f(x)$ is as shown in the following figure.

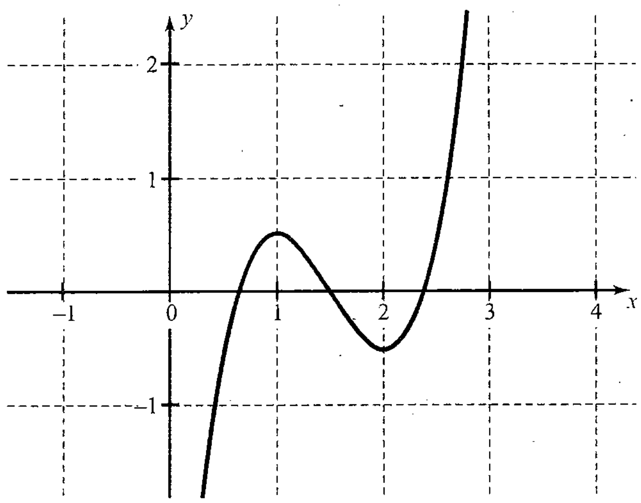


Fig. 1.36(a)

As shown in the figure, graph cuts x-axis at three distinct point.

Hence, equation $f(x) = 0$ has three distinct roots.

(ii) For $2x^3 - 9x^2 + 12x - 3 = 0$, $f(x) = 2x^3 - 9x^2 + 12x - 3$

$f'(x) = 0 \Rightarrow x = 1$ and $x = 2$

Also $f(1) = 2 - 9 + 12 - 3 = 2$ and $f(2) = 16 - 36 + 24 - 3 = 1$

Hence, graph of $y = f(x)$ is as shown in the following figure.

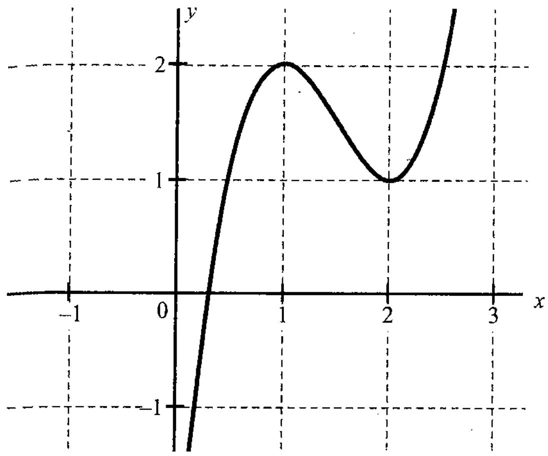


Fig. 1.36(b)

Thus from the graph, we can see that $f(x) = 0$ has only one real root, though $y = f(x)$ has two turning points.

Example 1.69 **Find how many roots of the equation $x^4 + 2x^2 - 8x + 3 = 0$ are real.**

Sol. Let $f(x) = x^4 + 2x^2 - 8x + 3$

$\Rightarrow f'(x) = 4x^3 + 4x - 8 = 4(x-1)(x^2+x+2)$

Now $f'(x) = 0 \Rightarrow x = 1$

Hence graph of $y = f(x)$ has only one turn (maxima/minima).

Now $f(1) = 1 + 2 - 8 + 3 < 0$

Also when $x \to \pm\infty, f(x) \to \infty$

Then graph of the function is as shown in the following figure.

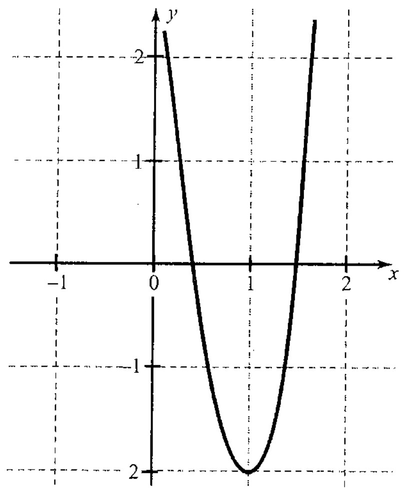


Fig. 1.37

Hence, equation $f(x) = 0$ has only two real roots.

EQUATIONS REDUCIBLE TO QUADRATIC

Example 1.70 **Solve $\sqrt{5x^2-6x+8} - \sqrt{5x^2-6x-7} = 1$.**

Sol. Let $5x^2 - 6x = y$. Then,

$$\sqrt{5x^2-6x+8} - \sqrt{5x^2-6x-7} = 1$$

$$\Rightarrow \sqrt{y+8} - \sqrt{y-7} = 1$$

$$\Rightarrow (\sqrt{y+8} - \sqrt{y-7})^2 = 1$$

$$\Rightarrow y = \sqrt{y^2+y-56}$$

$$\Rightarrow y^2 = y^2 + y - 56$$

$$\Rightarrow y = 56$$

$$\Rightarrow 5x^2 - 6x = 56 \quad [\because y = 5x^2 - 6x]$$

$$\Rightarrow 5x^2 - 6x - 56 = 0$$

$$\Rightarrow (5x+14)(x-4) = 0$$

$$\Rightarrow x = 4, \frac{-14}{5}$$

Clearly, both the values satisfy the given equation. Hence, the roots of the given equation are 4 and –14/5.

Example 1.71 **Solve $(x^2 - 5x + 7)^2 - (x-2)(x-3) = 1$.**

Sol. We have,

$$(x^2 - 5x + 7)^2 - (x-2)(x-3) = 1$$

$$\Rightarrow (x^2 - 5x + 7)^2 - (x^2 - 5x + 7) = 0$$

$$\Rightarrow y^2 - y = 0, \text{ where } -y = x^2 - 5x + 7$$

$$\Rightarrow y(y-1) = 0$$

$$\Rightarrow y = 0, 1$$

Now,

$$y = 0$$

$$\Rightarrow x^2 - 5x + 7 = 0$$

$$\Rightarrow x = \frac{5 \pm \sqrt{25-28}}{2} = \frac{5 \pm \sqrt{-3}}{2} = \frac{5 \pm i\sqrt{3}}{2}$$

where $i = \sqrt{-1}$

and

$$y = 1$$

$$\Rightarrow x^2 - 5x + 6 = 0$$

$$\Rightarrow (x-3)(x-2) = 0$$

$$\Rightarrow 3, 2$$

Hence, the roots of the equation are 2, 3, $(5+i\sqrt{3})/2$ and $(5-i\sqrt{3})/2$.

Example 1.72 **Solve the equation $4^x - 5 \times 2^x + 4 = 0$.**

Sol. We have,

$$4^x - 5 \times 2^x + 4 = 0$$

$$\Rightarrow (2^x)^2 - 5(2^x) + 4 = 0$$

$$\Rightarrow y^2 - 5y + 4 = 0, \text{ where } y = 2^x$$

$$\Rightarrow (y-4)(y-1) = 0$$

$$\Rightarrow y = 1, 4$$

$$\Rightarrow 2^x = 1, 2^x = 4$$

$$\Rightarrow 2^x = 2^0, 2^x = 2^2$$

$$\Rightarrow x = 0, 2$$

Hence, the roots of the given equation are 0 and 2.

Example 1.73 **Solve the equation $12x^4 - 56x^3 + 89x^2 - 56x + 12 = 0$.**

Sol. The given equation is

$$12x^4 - 56x^3 + 89x^2 - 56x + 12 = 0$$

Dividing by x^2, we get

$$12x^2 - 56x + 89 - \frac{56}{x} + \frac{12}{x^2} = 0$$

$$\Rightarrow \quad 12\left(x^2 + \frac{1}{x^2}\right) - 56\left(x + \frac{1}{x}\right) + 89 = 0$$

$$\Rightarrow \quad 12\left[\left(x + \frac{1}{x}\right)^2 - 2\right] - 56\left(x + \frac{1}{x}\right) + 89 = 0$$

$$\Rightarrow \quad 12\left(x + \frac{1}{x}\right)^2 - 56\left(x + \frac{1}{x}\right) + 65 = 0$$

$$\Rightarrow \quad 12y^2 - 56y + 65 = 0, \text{ where } y = x + \frac{1}{x}$$

$$\Rightarrow \quad 12y^2 - 26y - 30y + 65 = 0$$

$$\Rightarrow \quad (6y - 13)(2y - 5) = 0$$

$$\Rightarrow \quad y = \frac{13}{6} \text{ or } y = \frac{5}{2}$$

If $y = 13/6$, then

$$x + \frac{1}{x} = \frac{13}{6}$$

$$\Rightarrow \quad 6x^2 - 13x + 6 = 0$$

$$\Rightarrow \quad (3x - 2)(2x - 3) = 0$$

$$\Rightarrow \quad x = \frac{2}{3}, \frac{3}{2}$$

If $y = 5/2$, then

$$x + \frac{1}{x} = \frac{5}{2}$$

$$\Rightarrow \quad 2x^2 - 5x + 2 = 0$$

$$\Rightarrow \quad (x - 2)(2x - 1) = 0$$

$$\Rightarrow \quad x = 2, \frac{1}{2}$$

Hence, the roots of the given equation are 2, 1/2, 2/3, 3/2.

Example 1.74 **Solve the equation $3^{x^2-x} + 4^{x^2-x} = 25$.**

Sol. We have,

$$3^{x^2-x} + 4^{x^2-x} = 25$$

$$\Rightarrow \quad 3^{x^2-x} + 4^{x^2-x} = 3^2 + 4^2$$

$$\Rightarrow \quad x^2 - x = 2$$

$$\Rightarrow \quad x^2 - x - 2 = 0$$

$$\Rightarrow \quad (x - 2)(x + 1) = 0$$

$$\Rightarrow \quad x = -1, 2$$

Hence, the roots of the given equation are −1 and 2.

Example 1.75 **Solve the equation $(x - 1)^4 + (x - 5)^4 = 82$.**

Sol. Let

$$y = \frac{(x-1) + (x-5)}{2} = x - 3$$

$$\Rightarrow \quad x = y + 3$$

Putting $x = y + 3$ in the given equation, we obtain

$$(y + 2)^4 + (y - 2)^4 = 82$$

$$\Rightarrow \quad (y^2 + 4y + 4)^2 + (y^2 - 4y + 4)^2 = 82$$

$$\Rightarrow \quad \{(y^2 + 4)^2 + 4y\}^2 + \{(y^2 + 4) - 4y\}^2 = 82$$

$$\Rightarrow \quad 2\{(y^2 + 4)^2 + 16y^2\} = 82$$

$$[\because (a + b)^2 + (a - b)^2 = 2(a^2 + b^2)]$$

$$\Rightarrow \quad y^4 + 8y^2 + 16 + 16y^2 = 41$$

$$\Rightarrow \quad y^4 + 24y^2 - 25 = 0$$

$$\Rightarrow \quad (y^2 + 25)(y^2 - 1) = 0$$

$$\Rightarrow \quad y^2 + 25 = 0, y^2 - 1 = 0$$

$$\Rightarrow \quad y = \pm 5i, y = \pm 1 \qquad (\text{where } i = \sqrt{-1})$$

$$\Rightarrow \quad x - 3 = \pm 5i, x - 3 = \pm 1$$

$$\Rightarrow \quad x = 3 \pm 5i, x = 4, 2 \qquad [\because y = x - 3]$$

Hence, the roots of the given equation are $3 \pm 5i$, 2 and 4.

Example 1.76 **Solve the equation $(x + 2)(x + 3)(x + 8) \times (x + 12) = 4x^2$.**

Sol. $(x + 2)(x + 3)(x + 8)(x + 12) = 4x^2$

$$\Rightarrow \quad \{(x + 2)(x + 12)\}\{(x + 3)(x + 8)\} = 4x^2$$

$$\Rightarrow \quad (x^2 + 14x + 24)(x^2 + 11x + 24) = 4x^2$$

Dividing throughout by x^2, we get

$$\left(x + 14 + \frac{24}{x}\right)\left(x + 11 + \frac{24}{x}\right) = 4$$

$$\Rightarrow \quad (y + 14)(y + 11) = 4, \text{ where } x + \frac{24}{x} = y$$

$$\Rightarrow \quad y^2 + 25y + 154 = 4$$

$$\Rightarrow \quad y^2 + 25y + 150 = 0$$

$$\Rightarrow \quad (y + 15)(y + 10) = 0$$

$$\Rightarrow \quad y = -15, -10$$

If $y = -15$, then

$$x + \frac{24}{x} = -15$$

$$\Rightarrow \quad x^2 + 15x + 24 = 0$$

$$\Rightarrow \quad x = \frac{-15 - \sqrt{129}}{2}$$

If $y = -10$, then

$$x + \frac{24}{x} = -10$$

$$\Rightarrow \quad x^2 + 10x + 24 = 0$$

$$\Rightarrow \quad (x + 4)(x + 6) = 0$$

$$\Rightarrow \quad x = -4, -6$$

Hence, the roots of the given equation are −4, −6, $(-15 - \sqrt{129})/2$.

Example 1.77 **Evaluate $\sqrt{6 + \sqrt{6 + \sqrt{6 + \cdots \infty}}}$.**

Sol. Let $x = \sqrt{6 + \sqrt{6 + \sqrt{6 + \cdots \infty}}}$. Then,

$$x = \sqrt{6 + x}$$

$$\Rightarrow \quad x^2 = 6 + x$$

$$\Rightarrow \quad x^2 - x - 6 = 0$$

$$\Rightarrow \quad (x - 3)(x + 2) = 0$$

$$\Rightarrow \quad x = 3 \text{ or } x = -2$$

But, the given expression is positive. So, $x = 3$. Hence, the value of the given expression is 3.

Example 1.78 **Solve $\sqrt{x+5}+\sqrt{x+21}=\sqrt{6x+40}$.**

Sol. $\sqrt{x+5}+\sqrt{x+21}=\sqrt{6x+40}$

$\Rightarrow (\sqrt{x+5}+\sqrt{x+21})^2=6x+40$

$\Rightarrow (x+5)+(x+21)+2\sqrt{(x+5)(x+21)}=6x+40$

$\Rightarrow \sqrt{(x+5)(x+21)}=2x+7$

$\Rightarrow (x+5)(x+21)=(2x+7)^2$

$\Rightarrow 3x^2+2x-56=0$

$\Rightarrow (3x+14)(x-4)=0$

$\Rightarrow x=4$ or $x=-14/3$

Clearly, $x=-14/3$ does not satisfy the given equation. Hence, $x=4$ is the only root of the given equation.

Concept Application Exercise 1.2

1. Prove that graph of $y=x^2+2$ and $y=3x-4$ never intersect.
2. In how many points the line $y+14=0$ cuts the curve whose equation is $x(x^2+x+1)+y=0$?
3. Consider the following graphs:

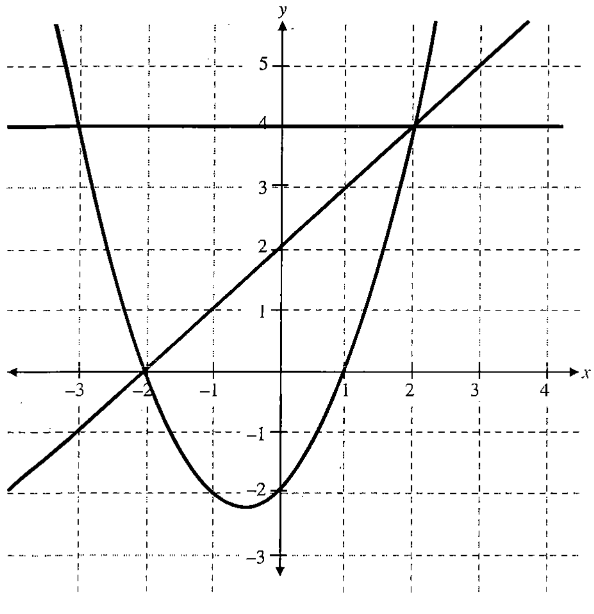


Fig. 1.38

Answer the following questions:
 (i) sum of roots of the equation $f(x)=0$
 (ii) product of roots of the equation $f(x)=4$
 (iii) the absolute value of the difference of the roots of equation $f(x)=x+2$

4. Solve $\dfrac{x^2+3x+2}{x^2-6x-7}=0$.
5. Solve $\sqrt{x-2}+\sqrt{4-x}=2$.
6. Solve $\sqrt{x-2}(x^2-4x-5)=0$.
7. Solve the equation $x(x+2)(x^2-1)=-1$.
8. Find the value of $2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cdots\infty}}}$.
9. Solve $4^x+6^x=9^x$.
10. Solve $3^{2x^2-7x+7}=9$.
11. Find the number of real roots of the equation $(x-1)^2+(x-2)^2+(x-3)^2=0$.
12. Solve $\sqrt{3x^2-7x-30}+\sqrt{2x^2-7x-5}=x+5$.
13. If $x=\sqrt{7+4\sqrt{3}}$, prove that $x+1/x=4$
14. Solve $\sqrt{5x^2-6x+8}-\sqrt{5x^2-6x-7}=1$.
15. Solve $\sqrt{x^2+4x-21}+\sqrt{x^2-x-6}=\sqrt{6x^2-5x-39}$.
16. How many roots of the equation $3x^4+6x^3+x^2+6x+3=0$ are real ?
17. Find the value of k if $x^3-3x+a=0$ has three real distinct roots.
18. Analyze the roots of the equation $(x-1)^3+(x-2)^3+(x-3)^3+(x-4)^3+(x-5)^3=0$ by differentiation method.
19. In how many points the graph of $f(x)=x^3+2x^2+3x+4$ meets x-axis.

REMAINDER AND FACTOR THEOREMS

Remainder Theorem

The remainder theorem states that if a polynomial $f(x)$ is divided by a linear function $x-k$, then the remainder is $f(k)$.

Proof:

In any division,

Dividend = Divisor × Quotient + Remainder

Let $Q(x)$ be the quotient and R be the remainder. Then,

$$f(x)=(x-k)\,Q(x)+R$$

$$\Rightarrow f(k)=(k-k)\,Q(x)+R=0+R=R$$

Note: *If a n-degree polynomial is divided by a m-degree polynomial, then the maximum degree of the remainder polynomial is m – 1.*

Example 1.79 **Find the remainder when x^3+4x^2-7x+6 is divided by $x-1$.**

Sol. Let $f(x)=x^3+4x^2-7x+6$. The remainder when $f(x)$ is divided by $x-1$ is

$f(1)=1^3+4\times(1)^2-7+6=4$

Example 1.80 **If the expression $ax^4+bx^3-x^2+2x+3$ has remainder $4x+3$ when divided by x^2+x-2, find the value of a and b.**

Sol. Let $f(x)=ax^4+bx^3-x^2+2x+3$.

Now, $x^2+x-2=(x+2)\times(x-1)$.

Given, $f(-2)=a(-2)^4+b(-2)^3-(-2)^2+2(-2)+3$

$= 4(-2) + 3$

$\Rightarrow \quad 16a - 8b - 4 - 4 + 3 = -5$

$\Rightarrow \quad 2a - b = 0 \qquad (1)$

Also,

$f(1) = a + b - 1 + 2 + 3 = 4(1) + 3$

$\Rightarrow \quad a + b = 3 \qquad (2)$

From (1) and (2), $a = 1$, $b = 2$.

Factor Theorem

Factor Theorem Is a Special Case of Remainder Theorem

Let,

$f(x) = (x - k)\, Q(x) + R$

$\Rightarrow \quad f(x) = (x - k)\, Q(x) + f(k)$

When $f(k) = 0$, $f(x) = (x - k)\, Q(x)$. Therefore, $f(x)$ is exactly divisible by $x - k$.

Example 1.81 **Given that $x^2 + x - 6$ is a factor of $2x^4 + x^3 - ax^2 + bx + a + b - 1$, find the values of a and b.**

Sol. We have,

$x^2 + x - 6 = (x + 3)(x - 2)$

Let,

$f(x) = 2x^4 + x^3 - ax^2 + bx + a + b - 1$

Now,

$f(-3) = 2(-3)^4 + (-3)^3 - a(-3)^2 - 3b + a + b - 1 = 0$

$\Rightarrow \quad 134 - 8a - 2b = 0$

$\Rightarrow \quad 4a + b = 67 \qquad (1)$

$\Rightarrow \quad f(2) = 2(2)^4 + 2^3 - a(2)^2 + 2b + a + b - 1 = 0$

$\Rightarrow \quad 39 - 3a + 3b = 0$

$\Rightarrow \quad a - b = 13 \qquad (2)$

From (1) and (2), $a = 16$, $b = 3$.

Example 1.82 **Use the factor theorem to find the value of k for which $(a + 2b)$, where $a, b \neq 0$ is a factor of $a^4 + 32b^4 + a^3b(k + 3)$.**

Sol. Let $f(a) = a^4 + 32b^4 + a^3b(k + 3)$. Now,

$f(-2b) = (-2b)^4 + 32b^4 + (-2b)^3b(k + 3) = 0$

$\Rightarrow \quad 48b^4 - 8b^4(k + 3) = 0$

$\Rightarrow \quad 8b^4[6 - (k + 3)] = 0$

$\Rightarrow \quad 8b^4(3 - k) = 0$

Since $b \neq 0$, so, $3 - k = 0$ or $k = 3$.

Example 1.83 **If c, d are the roots of the equation $(x - a)(x - b) - k = 0$, prove that a, b are the roots of the equation $(x - c)(x - d) + k = 0$.**

Sol. Since c and d are the roots of the equation $(x - a)(x - b) - k = 0$, therefore,

$(x - a)(x - b) - k = (x - c)(x - d)$

$\Rightarrow \quad (x - a)(x - b) = (x - c)(x - d) + k$

$\Rightarrow \quad (x - c)(x - d) + k = (x - a)(x - b)$

Clearly, a and b are roots of the equation $(x - a)(x - b) = 0$. Hence, a, b are roots of $(x - c)(x - d) + k = 0$.

Concept Application Exercise 1.3

1. Given that the expression $2x^3 + 3px^2 - 4x + p$ has a remainder of 5 when divided by $x + 2$, find the value of p.
2. Determine the value of k for which $x + 2$ is a factor of $(x + 1)^7 + (2x + k)^3$.
3. Find the value of p for which $x + 1$ is a factor of $x^4 + (p - 3)x^3 - (3p - 5)x^2 + (2p - 9)x + 6$.
   Find the remaining factors for this value of p.
4. If $x^2 + ax + 1$ is a factor of $ax^3 + bx + c$, then find the conditions.
5. If $f(x) = x^3 - 3x^2 + 2x + a$ is divisible by $x - 1$, then find the remainder when $f(x)$ is divided by $x - 2$.
6. If $f(x) = x^3 - x^2 + ax + b$ is divisible by $x^2 - x$, then find the value of $f(2)$.

Identity

A relation which is true for every value of the variable is called an identity.

Example 1.84 **If $(a^2 - 1)x^2 + (a - 1)x + a^2 - 4a + 3 = 0$ be an identity in x, then find the value of a.**

Sol. The given relation is satisfied for all real values of x, so all the coefficients must be zero. Then,

$$\left.\begin{aligned} a^2 - 1 = 0 &\Rightarrow a = \pm 1 \\ a - 1 = 0 &\Rightarrow a = 1 \\ a^2 - 4a + 3 = 0 &\Rightarrow a = 1, 3 \end{aligned}\right\} \text{ common value of } a \text{ is } 1$$

Example 1.85 **Show that $\dfrac{(x+b)(x+c)}{(b-a)(c-a)} + \dfrac{(x+c)(x+a)}{(c-b)(a-b)} + \dfrac{(x+a)(x+b)}{(a-c)(b-c)} = 1$ is an identity.**

Sol. Given relation is

$$\frac{(x+b)(x+c)}{(b-a)(c-a)} + \frac{(x+c)(x+a)}{(c-b)(a-b)} + \frac{(x+a)(x+b)}{(a-c)(b-c)} = 1 \qquad (1)$$

When $x = -a$,

$$\text{L.H.S.} = \frac{(b-a)(c-a)}{(b-a)(c-a)} = 1 = \text{R.H.S.}$$

Similarly, when $x = -b$,

$$\text{L.H.S.} = \frac{(c-b)(a-b)}{(c-b)(a-b)} = 1 = \text{R.H.S.}$$

When $x = -c$,

$$\text{L.H.S.} = \frac{(a-c)(b-c)}{(a-c)(b-c)} = 1 = \text{R.H.S.}$$

Thus, the highest power of x occurring in relation (1) is 2 and this relation is satisfied by three distinct values a, b and c of x; therefore, it is an equation but an identity.

Example 1.86 **A certain polynomial $P(x)$, $x \in R$ when divided by $x - a$, $x - b$ and $x - c$ leaves remainders a, b and c, respectively. Then find the remainder when $P(x)$ is divided by $(x - a)(x - b)(x - c)$ where a, b, c are distinct.**

Sol. By remainder theorem, $P(a) = a$, $P(b) = b$ and $P(c) = c$.

Let the required remainder be $R(x)$. Then,

$$P(x) = (x - a)(x - b)(x - c)\,Q(x) + R(x)$$

where $R(x)$ is a polynomial of degree at most 2. We get $R(a) = a$, $R(b) = b$ and $R(c) = c$. So, the equation $R(x) - x = 0$ has three roots a, b and c. But its degree is at most 2. So, $R(x) - x$ must be zero polynomial (or identity). Hence $R(x) = x$.

QUADRATIC EQUATION

Quadratic Equation with Real Coefficients

Consider the quadratic equation

$$ax^2 + bx + c = 0 \quad (1)$$

where $a, b, c \in R$ and $a \neq 0$.

Roots of the equation are given by

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Now, we observe that the nature of the roots depend upon the value of the quantity $b^2 - 4ac$. This quantity is generally denoted by D and is known as the discriminant of the quadratic equation [Eq.(1)].

We also observe the following results:

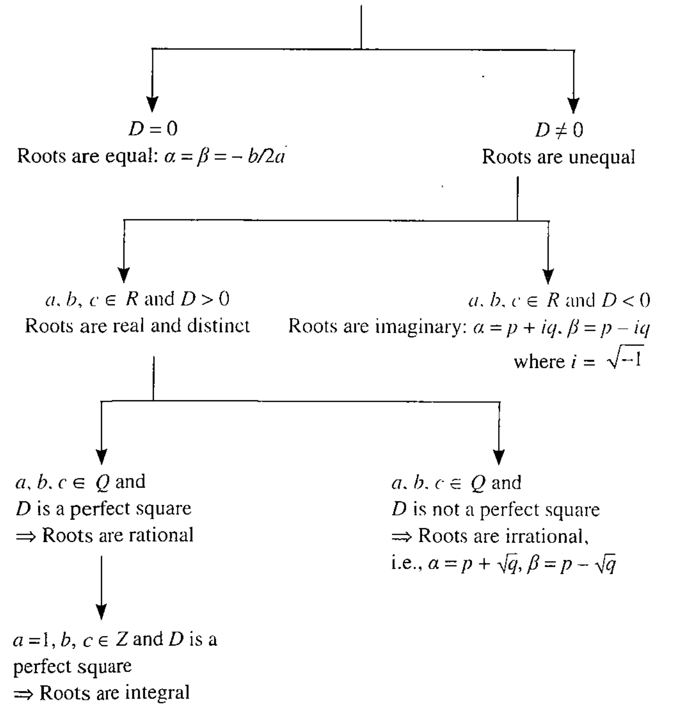


Fig. 1.39

Note:

- *If $a, b, c \in Q$ and $b^2 - 4ac$ is positive but not a perfect square, then roots are irrational and they always occur in conjugate pair like $2 + \sqrt{3}$ and $2 - \sqrt{3}$. However, if a, b, c are irrational numbers and $b^2 - 4ac$ is positive but not a perfect square, then the roots may not occur in conjugate pairs. For example, the roots of the equation $x^2 - (5 + \sqrt{2})x + 5\sqrt{2} = 0$ are 5 and $\sqrt{2}$, which do not form a conjugate pair.*
- *If $b^2 - 4ac < 0$, then roots of equations are complex. If a, b and c are real then complex roots occur in conjugate pair such as of the form $p + iq$ and $p - iq$. If all the coefficients are not real then complex roots may not conjugate.*

Example 1.87 **If $a, b, c \in R^+$ and $2b = a + c$, then check the nature of roots of equation $ax^2 + 2bx + c = 0$.**

Sol. Given equation is $ax^2 + 2bx + c = 0$. Hence,

$$D = 4b^2 - 4ac$$
$$= (a + c)^2 - 4ac$$
$$= (a - c)^2 > 0$$

Thus, the roots are real and distinct.

Example 1.88 **If the roots of the equation $a(b - c)x^2 + b(c - a)x + c(a - b) = 0$ are equal, show that $2/b = 1/a + 1/c$.**

Sol. Since the roots of the given equations are equal, therefore its discriminant is zero, i.e.,

$$b^2(c - a)^2 - 4a(b - c)c(a - b) = 0$$
$$\Rightarrow b^2(c^2 + a^2 - 2ac) - 4ac(ba - ca - b^2 + bc) = 0$$
$$\Rightarrow a^2b^2 + b^2c^2 + 4a^2c^2 + 2b^2ac - 4a^2bc - 4abc^2 = 0$$
$$\Rightarrow (ab + bc - 2ac)^2 = 0$$
$$\Rightarrow ab + bc - 2ac = 0$$
$$\Rightarrow ab + bc = 2ac$$
$$\Rightarrow \frac{1}{c} + \frac{1}{a} = \frac{2}{b} \quad \text{[Dividing both sides by } abc]$$
$$\Rightarrow \frac{2}{b} = \frac{1}{a} + \frac{1}{c}$$

Example 1.89 **Prove that the roots of the equation $(a^4 + b^4)x^2 + 4abcdx + (c^4 + d^4) = 0$ cannot be different, if real.**

Sol. The discriminant of the given equation is

$$D = 16a^2b^2c^2d^2 - 4(a^4 + b^4)(c^4 + d^4)$$
$$= -4[(a^4 + b^4)(c^4 + d^4) - 4a^2b^2c^2d^2]$$
$$= -4[a^4c^4 + a^4d^4 + b^4c^4 + b^4d^4 - 4a^2b^2c^2d^2]$$
$$= -4[(a^4c^4 + b^4d^4 - 2a^2b^2c^2d^2) + (a^4d^4 + b^4c^4 - 2a^2b^2c^2d^2)]$$
$$= -4[(a^2c^2 - b^2d^2)^2 + (a^2d^2 - b^2c^2)^2] \quad (1)$$

Since roots of the given equation are real, therefore

$$D \geq 0$$
$$\Rightarrow -4[(a^2c^2 - b^2d^2)^2 + (a^2d^2 - b^2c^2)^2] \geq 0$$

$\Rightarrow \quad (a^2c^2 - b^2d^2)^2 + (a^2d^2 - b^2c^2)^2 \le 0$

$\Rightarrow \quad (a^2c^2 - b^2d^2)^2 + (a^2d^2 - b^2c^2)^2 = 0 \quad (2)$

(since sum of two positive quantities cannot be negative)

From (1) and (2), we get $D = 0$. Hence, the roots of the given quadratic equation are not different, if real.

Example 1.90 **If the roots of the equation $x^2 - 8x + a^2 - 6a = 0$ are real distinct, then find all possible values of a.**

Sol. Since the roots of the given equation are real and distinct, we must have

$D > 0$

$\Rightarrow \quad 64 - 4(a^2 - 6a) > 0$

$\Rightarrow \quad 4[16 - a^2 + 6a] > 0$

$\Rightarrow \quad -4(a^2 - 6a - 16) > 0$

$\Rightarrow \quad a^2 - 6a - 16 < 0$

$\Rightarrow \quad (a - 8)(a + 2) < 0$

$\Rightarrow \quad -2 < a < 8$

Hence, the roots of the given equation are real if a lies between -2 and 8.

Example 1.91 **Find the quadratic equation with rational coefficients whose one root is $1/(2 + \sqrt{5})$.**

Sol. If the coefficients are rational, then irrational roots occur in conjugate pair. Given that if one root is $\alpha = 1/(2 + \sqrt{5}) = \sqrt{5} - 2$, then the other root is $\beta = 1/(2 - \sqrt{5}) = -(2 + \sqrt{5})$.

Sum of roots $\alpha + \beta = -4$ and product of roots $\alpha\beta = -1$. Thus, required equation is $x^2 + 4x - 1 = 0$.

Example 1.92 **If $f(x) = ax^2 + bx + c$, $g(x) = -ax^2 + bx + c$, where $ac \ne 0$, then prove that $f(x)\, g(x) = 0$ has at least two real roots.**

Sol. Let D_1 and D_2 be discriminants of $ax^2 + bx + c = 0$ and $-ax^2 + bx + c = 0$, respectively. Then,

$D_1 = b^2 - 4ac,\ D_2 = b^2 + 4ac$

Now,

$ac \ne 0 \Rightarrow$ either $ac > 0$ or $ac < 0$

If $ac > 0$, then $D_2 > 0$. Therefore, roots of $-ax^2 + bx + c = 0$ are real.

If $ac < 0$, then $D_1 > 0$. Therefore, roots of $ax^2 + bx + c = 0$ are real.

Thus, $f(x)\, g(x)$ has at least two real roots.

Example 1.93 **If $a, b, c \in R$ such that $a + b + c = 0$ and $a \ne c$, then prove that the roots of $(b + c - a)x^2 + (c + a - b)x + (a + b - c) = 0$ are real and distinct.**

Sol. Given equation is

$(b + c - a)x^2 + (c + a - b)x + (a + b - c) = 0$

or

$(-2a)x^2 + (-2b)x + (-2c) = 0$

or

$ax^2 + bx + c = 0$

$\Rightarrow \quad D = b^2 - 4ac$

$= (-c - a)^2 - 4ac$

$= (c - a)^2$

> 0

Hence, roots are real and distinct.

Example 1.94 **If $\cos\theta$, $\sin\phi$, $\sin\theta$ are in G.P., then check the nature of roots of $x^2 + 2\cot\phi \cdot x + 1 = 0$.**

Sol. We have,

$\sin^2\phi = \cos\theta \sin\theta$

The discriminant of the given equation is

$$D = 4\cot^2\phi - 4$$

$$= 4\left[\frac{\cos^2\varphi - \sin^2\varphi}{\sin^2\varphi}\right]$$

$$= \frac{4(1 - 2\sin^2\varphi)}{\sin^2\varphi}$$

$$= \frac{4(1 - 2\sin\theta\cos\theta)}{\sin^2\varphi}$$

$$= \left[\frac{2(\sin\theta - \cos\theta)}{\sin\varphi}\right]^2 \ge 0$$

Example 1.95 **If a, b and c are odd integers, then prove that roots of $ax^2 + bx + c = 0$ cannot be rational.**

Sol. Discriminant $D = b^2 - 4ac$. Suppose the roots are rational. Then, D will be a perfect square.

Let $b^2 - 4ac = d^2$. Since a, b and c are odd integers, d will be odd. Now,

$b^2 - d^2 = 4ac$

Let $b = 2k + 1$ and $d = 2m + 1$. Then

$b^2 - d^2 = (b - d)(b + d)$

$= 2(k - m)\, 2(k + m + 1)$

Now, either $(k - m)$ or $(k + m + 1)$ is always even. Hence $b^2 - d^2$ is always a multiple of 8. But, $4ac$ is only a multiple of 4 (not of 8), which is a contradiction. Hence, the roots of $ax^2 + bx + c = 0$ cannot be rational.

Quadratic Equations with Complex Coefficients

Consider the quadratic equation $ax^2 + bx + c = 0$, where a, b, c are complex numbers and $a \ne 0$. Roots of equation are given by

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Here nature of roots should not be analyzed by sign of $b^2 - 4ac$.

Note: *In case of quadratic equations with real coefficients, imaginary (complex) roots always occur in conjugate pairs. However, it is not true for quadratic equations with complex coefficients. For example, the equation $4x^2 - 4ix - 1 = 0$ has both roots equal to $1/(2i)$.*

Concept Application Exercise 1.4

1. Find the values of a for which the roots of the equation $x^2 + a^2 = 8x + 6a$ are real.
2. Find the condition if the roots of $ax^2 + 2bx + c = 0$ and $bx^2 - 2\sqrt{acx} + b = 0$ are simultaneously real.
3. If $a < c < b$, then check the nature of roots of the equation $(a-b)^2 x^2 + 2(a+b-2c)x + 1 = 0$.
4. If $a + b + c = 0$ then check the nature of roots of the equation $4ax^2 + 3bx + 2c = 0$ where $a, b, c \in R$.
5. Find the greatest value of a non-negative real number λ for which both the equations $2x^2 + (\lambda - 1)x + 8 = 0$ and $x^2 - 8x + \lambda + 4 = 0$ have real roots.

Relations Between Roots and Coefficients

Let α and β be the roots of quadratic equation $ax^2 + bx + c = 0$. Then by factor theorem,

$$ax^2 + bx + c = a(x-\alpha)(x-\beta) = a(x^2 - (\alpha+\beta)x + \alpha\beta)$$

Comparing coefficients, we have $\alpha + \beta = -b/a$ and $\alpha\beta = c/a$. Thus, we find that

$$\alpha + \beta = \frac{-b}{a} = -\frac{\text{coeff of } x}{\text{coeff of } x^2} \text{ and } \alpha\beta = \frac{c}{a} = \frac{\text{constant term}}{\text{coeff of } x^2}$$

Also, if sum of roots is S and product is P, then quadratic equation is given by $x^2 - Sx + P = 0$.

Example 1.96 **Form a quadratic equation whose roots are –4 and 6.**

Sol. We have sum of the roots, $S = -4 + 6 = 2$ and, product of the roots, $P = -4 \times 6 = -24$. Hence, the required equation is

$$x^2 - Sx + P = 0$$

$$\Rightarrow \quad x^2 - 2x - 24 = 0$$

Example 1.97 **Form a quadratic equation with real coefficients whose one root is 3 – 2*i*.**

Sol. Since the complex roots always occur in pairs, so the other root is $3 + 2i$. The sum of the roots is $(3 + 2i) + (3 - 2i) = 6$. The product of the roots is $(3 + 2i)(3 - 2i) = 9 - 4i^2 = 9 + 4 = 13$. Hence, the equation is

$$x^2 - Sx + P = 0$$

$$\Rightarrow \quad x^2 - 6x + 13 = 0$$

Example 1.98 **If roots of the equation $ax^2 + bx + c = 0$ are α and β, find the equation whose roots are**

(i) $\frac{1}{\alpha}, \frac{1}{\beta}$

(ii) $-\alpha, -\beta$

(iii) $\frac{1-\alpha}{1+\alpha}, \frac{1-\beta}{1+\beta}$

Sol. Here in all cases functions of α and β are symmetric.

(i) Let $\frac{1}{\alpha} = y \Rightarrow \alpha = \frac{1}{y}$

Now α is a root of the equation $ax^2 + bx + c = 0$

$$\Rightarrow \quad a\alpha^2 + b\alpha + c = 0$$

$$\frac{a}{y^2} + \frac{b}{y} + c = 0$$

$$\Rightarrow \quad cy^2 + by + a = 0$$

Hence, the required equation is $cx^2 + bx + a = 0$.

We get same equation if we start with $1/\beta$.

(ii) Let $-\alpha = y \Rightarrow \alpha = -y$

Now α is root of the equation $ax^2 + bx + c = 0$

$$\Rightarrow \quad a\alpha^2 + b\alpha + c = 0$$

$$\Rightarrow \quad a(-y)^2 + b(-y) + c = 0$$

Hence, the required equation is $ax^2 - bx + c = 0$.

(iii) Let $\frac{1-\alpha}{1+\alpha} = y \Rightarrow \alpha = \frac{1-y}{1+y}$

Now α is root of the equation $ax^2 + bx + c = 0$

$$\Rightarrow \quad a\alpha^2 + b\alpha + c = 0$$

$$\Rightarrow \quad a\left(\frac{1-y}{1+y}\right)^2 + b\left(\frac{1-y}{1+y}\right) + c = 0$$

Hence required equation is $a(1-x)^2 + b(1-x^2) + c(1+x)^2 = 0$.

Example 1.99 **If *a*, *b* and *c* are in A.P. and one root of the equation $ax^2 + bx + c = 0$ is 2, then find the other root.**

Sol. Let α be the other root. Then,

$$4a + 2b + c = 0 \text{ and } 2b = a + c$$

$$\Rightarrow \quad 5a + 2c = 0$$

$$\Rightarrow \quad \frac{c}{a} = \frac{-5}{2}$$

Now,

$$2 \times \alpha = \frac{c}{a} = -\frac{5}{2}$$

$$\therefore \quad \alpha = -\frac{5}{4}$$

Example 1.100 **If the roots of the quadratic equation $x^2 + px + q = 0$ are tan 30° and tan 15°, respectively, then find the value of $2 + q - p$.**

Sol. The equation $x^2 + px + q = 0$ has roots tan 30° and tan 15°. Therefore,

$$\tan 30° + \tan 15° = -p \qquad (1)$$

$$\tan 30° \tan 15° = q \qquad (2)$$

Now,

$$\tan 45° = \tan(30° + 15°)$$

$$\Rightarrow \quad 1 = \frac{\tan 30 + \tan 15}{1 - \tan 30 \tan 15}$$

$$\Rightarrow \quad 1 = \frac{-p}{1-q} \quad \text{[Using (1) and (2)]}$$

$$\Rightarrow \quad 1 - q = -p \Rightarrow q - p = 1$$

$\Rightarrow \quad 2 + q - p = 3$

Example 1.101 **If the sum of the roots of the equation $1/(x + a) + 1/(x + b) = 1/c$ is zero, then prove that the product of the roots is $(-1/2)(a^2 + b^2)$.**

Sol. We have,

$$\frac{1}{x+a} + \frac{1}{x+b} = \frac{1}{c}$$

$\Rightarrow \quad x^2 + (a + b - 2c)x + (ab - bc - ca) = 0$

Let α, β be the roots of this equation. Then,

$\alpha + \beta = -(a + b - 2c)$ and $\alpha\beta = ab - bc - ca$

It is given that

$\alpha + \beta = 0$

$\Rightarrow \quad -(a + b - 2c) = 0$

$\Rightarrow \quad c = \dfrac{a+b}{2} \qquad (1)$

$\therefore \quad \alpha\beta = ab - bc - ca$

$= ab - c(a + b)$

$= ab - \left(\dfrac{a+b}{2}\right)(a + b) \quad$ [Using (1)]

$= \dfrac{2ab - (a+b)^2}{2} = -\dfrac{1}{2}(a^2 + b^2)$

Example 1.102 **Solve the equation $x^2 + px + 45 = 0$. It is given that the squared difference of its roots is equal to 144.**

Sol. Let α, β be the roots of the equation $x^2 + px + 45 = 0$. Then,

$\alpha + \beta = -p \qquad (1)$

$\alpha\beta = 45 \qquad (2)$

It is given that

$(\alpha - \beta)^2 = 144$

$\Rightarrow \quad (\alpha + \beta)^2 - 4\alpha\beta = 144$

$\Rightarrow \quad p^2 - 4 \times 45 = 144$

$\Rightarrow \quad p^2 = 324$

$\Rightarrow \quad p = \pm 18$

Substituting $p = 18$ in the given equation, we obtain

$x^2 + 18x + 45 = 0$

$\Rightarrow \quad (x + 3)(x + 15) = 0$

$\Rightarrow \quad x = -3, -15$

Substituting $p = -18$ in the given equation, we obtain

$x^2 + 18x + 45 = 0$

$\Rightarrow \quad (x - 3)(x - 15) = 0$

$\Rightarrow \quad x = 3, 15$

Hence, the roots of the given equation are $-3, -15$ or $3, 15$.

Example 1.103 **If the ratio of the roots of the equation $x^2 + px + q = 0$ are equal to the ratio of the roots of the equation $x^2 + bx + c = 0$, then prove that $p^2c = b^2q$.**

Sol. Let α, β be the roots of $x^2 + px + q = 0$ and γ, δ be the roots of the equation $x^2 + bx + c = 0$. Then,

$\alpha + \beta = -p, \alpha\beta = q \qquad (1)$

$\gamma + \delta = -b, \gamma\delta = c \qquad (2)$

We have,

$$\frac{\alpha}{\beta} = \frac{\gamma}{\delta} \Rightarrow \frac{\alpha+\beta}{\alpha-\beta} = \frac{\gamma+\delta}{\gamma-\delta}$$

[Using componendo and dividendo]

$$\Rightarrow \quad \frac{(\alpha-\beta)^2}{(\alpha+\beta)^2} = \frac{(\gamma-\delta)^2}{(\gamma+\delta)^2}$$

$$\Rightarrow \quad \frac{(\alpha+\beta)^2 - 4\alpha\beta}{(\alpha+\beta)^2} = \frac{(\gamma+\delta)^2 - 4\gamma\delta}{(\gamma+\delta)^2}$$

$$\Rightarrow \quad 1 - \frac{4\alpha\beta}{(\alpha+\beta)^2} = 1 - \frac{4\gamma\delta}{(\gamma+\delta)^2}$$

$$\Rightarrow \quad \frac{\alpha\beta}{(\alpha+\beta)^2} = \frac{\gamma\delta}{(\gamma+\delta)^2}$$

$$\Rightarrow \quad \frac{q}{p^2} = \frac{c}{b^2}$$

$\Rightarrow \quad p^2c = b^2q$

Example 1.104 **If $\sin\theta$, $\cos\theta$ be the roots of $ax^2 + bx + c = 0$, then prove that $b^2 = a^2 + 2ac$.**

Sol. We have,

$$\sin\theta + \cos\theta = -\frac{b}{a}, \sin\theta\cos\theta = \frac{c}{a}$$

Now, we know that

$\sin^2\theta + \cos^2\theta = 1$

$\Rightarrow \quad (\sin\theta + \cos\theta)^2 - 2\sin\theta\cos\theta = 1$

$$\Rightarrow \quad \frac{b^2}{a^2} = 1 + 2\frac{c}{a} \Rightarrow b^2 = a^2 + 2ac$$

Example 1.105 **If a and b ($\neq 0$) are the roots of the equation $x^2 + ax + b = 0$, then find the least value of $x^2 + ax + b$ ($x \in R$).**

Sol. Since a and b are the roots of the equation $x^2 + ax + b = 0$, so

$a + b = -a, ab = b$

Now,

$ab = b \Rightarrow (a - 1)b = 0 \Rightarrow a = 1 \quad (\because b \neq 0)$

Putting $a = 1$ in $a + b = -a$, we get $b = -2$. Hence,

$x^2 + ax + b = x^2 + x - 2 = (x + 1/2)^2 - 1/4 - 2$

$= (x + 1/2)^2 - 9/4$

which has a minimum value $-9/4$.

Example 1.106 **If the sum of the roots of the equation $(a + 1)x^2 + (2a + 3)x + (3a + 4) = 0$ is -1, then find the product of the roots.**

Sol. Let α, β be roots of the equation $(a + 1)x^2 + (2a + 3)x + (3a + 4) = 0$. Then,

$$\alpha + \beta = -1 \Rightarrow -\left(\frac{2a+3}{a+1}\right) = -1 \Rightarrow a = -2$$

Now, product of the roots is $(3a + 4)/(a + 1) = (-6 + 4)/(-2 + 1) = 2$.

Example 1.107 **Find the value of 'a' for which one root of the quadratic equation $(a^2 - 5a + 3)x^2 + (3a - 1)x + 2 = 0$ is twice as large as the other.**

Sol. Let the roots be α and 2α. Then,

$$\alpha + 2\alpha = \frac{1-3a}{a^2-5a+3}, \alpha \times 2\alpha = \frac{2}{a^2-5a+3}$$

$$\Rightarrow 2\left[\frac{1}{9}\frac{(1-3a)^2}{(a^2-5a+3)^2}\right] = \frac{2}{a^2-5a+3}$$

$$\Rightarrow \frac{(1-3a)^2}{a^2-5a+3} = 9 \Rightarrow 9a^2 - 6a + 1 = 9a^2 - 45a + 27$$

$$\Rightarrow 39a = 26 \Rightarrow a = \frac{2}{3}$$

Example 1.108 **If the difference between the roots of the equation $x^2 + ax + 1 = 0$ is less than $\sqrt{5}$, then find the set of possible values of a.**

Sol. If α, β are roots of $x^2 + ax + 1 = 0$, then

$$|\alpha - \beta| < \sqrt{5}$$

$$\Rightarrow \sqrt{(\alpha+\beta)^2 - 4\alpha\beta} < \sqrt{5}$$

$$\Rightarrow \sqrt{a^2-4} < \sqrt{5}$$

$$\Rightarrow \sqrt{a^2-4} < \sqrt{5}$$

$$\Rightarrow a^2 - 4 < 5$$

$$\Rightarrow a^2 < 9$$

$$\Rightarrow -3 < a < 3$$

$$\therefore a \in (-3, 3)$$

Example 1.109 **Find the values of the parameter a such that the roots α, β of the equation $2x^2 + 6x + a = 0$ satisfy the inequality $\alpha/\beta + \beta/\alpha < 2$.**

Sol. We have $\alpha + \beta = -3$ and $\alpha\beta = a/2$. Now,

$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} < 2$$

$$\Rightarrow \frac{\alpha^2+\beta^2}{\alpha\beta} < 2$$

$$\Rightarrow \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta} < 2$$

$$\Rightarrow \frac{9-a}{a/2} < 2$$

$$\Rightarrow \frac{9-a}{a} < 1$$

$$\Rightarrow \frac{9-a}{a} - 1 < 0$$

$$\Rightarrow \frac{9-2a}{a} < 0$$

$$\Rightarrow \frac{2a-9}{a} > 0$$

$$\Rightarrow a < 0 \text{ or } a > 9/2$$

Example 1.110 **If the harmonic mean between roots of $(5 + \sqrt{2})x^2 - bx + 8 + 2\sqrt{5} = 0$ is 4, then find the value of b.**

Sol. Let α, β be the roots of the given equation whose H.M. is 4. Then,

$$4 = \frac{2\alpha\beta}{\alpha+\beta}$$

$$\Rightarrow 4 = 2 \times \frac{\frac{8+2\sqrt{5}}{5+\sqrt{2}}}{\frac{b}{5+\sqrt{2}}}$$

$$\Rightarrow 2 = \frac{8+2\sqrt{5}}{b} \Rightarrow b = 4 + \sqrt{5}$$

Example 1.111 **If α, β are the roots of the equation $2x^2 - 3x - 6 = 0$, find the equation whose roots are $\alpha^2 + 2$ and $\beta^2 + 2$.**

Sol. Since α, β are roots of the equation $2x^2 - 3x - 6 = 0$, so

$$\alpha + \beta = 3/2 \text{ and } \alpha\beta = -3$$

$$\Rightarrow \alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = \frac{9}{4} + 6 = \frac{33}{4}$$

Now,

$$(\alpha^2 + 2) + (\beta^2 + 2) = (\alpha^2 + \beta^2) + 4 = \frac{33}{4} + 4 = \frac{49}{4}$$

and

$$(\alpha^2 + 2)(\beta^2 + 2) = \alpha^2\beta^2 + 2(\alpha^2 + \beta^2) + 4$$

$$= (3)^2 + 2\left(\frac{33}{4}\right) + 4$$

$$= \frac{59}{2}$$

So, the equation whose roots are $\alpha^2 + 2$ and $\beta^2 + 2$ is

$$x^2 - x[(\alpha^2 + 2) + (\beta^2 + 2)] + (\alpha^2 + 2)(\beta^2 + 2) = 0$$

$$\Rightarrow x^2 - \frac{49}{4}x + \frac{59}{2} = 0$$

$$\Rightarrow 4x^2 - 49x + 118 = 0$$

Example 1.112 **If $\alpha \neq \beta$ and $\alpha^2 = 5\alpha - 3$ and $\beta^2 = 5\beta - 3$, find the equation whose roots are α/β and β/α.**

Sol. We have $\alpha^2 = 5\alpha - 3$ and $\beta^2 = 5\beta - 3$. Hence, α, β are roots of $x^2 = 5x - 3$, i.e., $x^2 - 5x + 3 = 0$. Therefore,

$$\alpha + \beta = 5 \text{ and } \alpha\beta = 3$$

Now,

$$S = \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2+\beta^2}{\alpha\beta}$$

$$= \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta}$$

$$= \frac{25-6}{3} = \frac{19}{3}$$

and

$$P = \frac{\alpha}{\beta}\frac{\beta}{\alpha} = 1$$

So, the required equation is

$$x^2 - Sx + P = 0$$

$$\Rightarrow \quad x^2 - \frac{19}{3}x + 1 = 0$$

$$\Rightarrow \quad 3x^2 - 19x + 3 = 0$$

Example 1.113 **If α, β are the roots of the equation $ax^2 + bx + c = 0$, then find the roots of the equation $ax^2 - bx(x - 1) + c(x - 1)^2 = 0$ in terms of α and β.**

Sol. $ax^2 - bx(x - 1) + c(x - 1)^2 = 0$

$$\Rightarrow \quad \frac{ax^2}{(1-x)^2} + \frac{bx}{1-x} + c = 0 \qquad (1)$$

Now, α is a root of $ax^2 + bx + c = 0$. Then let

$$\alpha = \frac{x}{1-x}$$

$$\Rightarrow \quad x = \frac{\alpha}{\alpha+1}$$

Hence, the roots of (1) are $\alpha/(1 + \alpha)$, $\beta/(1 + \beta)$.

Concept Application Exercise 1.5

1. If the product of the roots of the equation $(a + 1)x^2 + (2a + 3)x + (3a + 4) = 0$ is 2, then find the sum of roots.
2. Find the value of a for which the sum of the squares of the roots of the equation $x^2 - (a - 2)x - a - 1 = 0$ assumes the least value.
3. If x_1 and x_2 are the roots of $x^2 + (\sin\theta - 1)x - 1/2\cos^2\theta = 0$, then find the maximum value of $x_1^2 + x_2^2$.
4. If $\tan\theta$ and $\sec\theta$ are the roots of $ax^2 + bx + c = 0$, then prove that $a^4 = b^2(4ac - b^2)$.
5. If the roots of the equation $x^2 - bx + c = 0$ be two consecutive integers, then find the value of $b^2 - 4c$.
6. If the roots of the equation $12x^2 - mx + 5 = 0$ are in the ratio 2:3, then find the value of m.
7. If α, β are the roots of $x^2 + px + 1 = 0$ and γ, δ are the roots of $x^2 + qx + 1 = 0$, then prove that $q^2 - p^2 = (\alpha - \gamma)(\beta - \gamma)(\alpha + \delta) \times (\beta + \delta)$.
8. If the equation formed by decreasing each root of $ax^2 + bx + c = 0$ by 1 is $2x^2 + 8x + 2 = 0$, find the condition.
9. If α, β be the roots of $x^2 - a(x - 1) + b = 0$, then find the value of $1/(\alpha^2 - a\alpha) + 1/(\beta^2 - b\beta) + 2/a + b$.
10. If α, β are roots of $375x^2 - 25x - 2 = 0$ and $s_n = \alpha^n + \beta^n$, then find the value of $\lim_{n\to\infty} \sum_{r=1}^{n} s_r$.
11. If α and β are the roots of the equation $2x^2 + 2(a + b)x + a^2 + b^2 = 0$, then find the equation whose roots are $(\alpha + \beta)^2$ and $(\alpha - \beta)^2$.
12. If the sum of the roots of an equation is 2 and sum of their cubes is 98, then find the equation.
13. Let α, β be the roots of $x^2 + bx + 1 = 0$. Then find the equation whose roots are $-(\alpha + 1/\beta)$ and $-(\beta + 1/\alpha)$.

COMMON ROOT(S)

Condition for One Common Root

Let us find the condition that the quadratic equations $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ may have a common root. Let α be the common root of the given equations. Then,

$$a_1\alpha^2 + b_1\alpha + c_1 = 0$$

and

$$a_2\alpha^2 + b_2\alpha + c_2 = 0$$

Solving these two equations by cross-multiplication, we have

$$\frac{\alpha^2}{b_1c_2 - b_2c_1} = \frac{\alpha}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

$$\Rightarrow \quad \alpha^2 = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} \quad \text{(from first and third)}$$

and

$$\alpha = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1} \quad \text{(from second and third)}$$

$$\Rightarrow \quad \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} = \left(\frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)^2$$

$$\Rightarrow \quad (c_1a_2 - c_2a_1)^2 = (b_1c_2 - b_2c_1)(a_1b_2 - a_2b_1)$$

This condition can easily be remembered by cross-multiplication method as shown in the following figure.

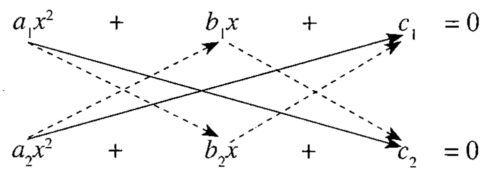


Fig. 1.40

(Bigger cross product)^2
= Product of the two smaller crosses

This is the condition required for a root to be common to two quadratic equations. The common root is given by

$$\alpha = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$$

or

$$\alpha = \frac{b_1c_2 - b_2c_1}{c_1a_2 - c_2a_1}$$

Note: *The common root can also be obtained by making the coefficient of x^2 common to the two given equations and then subtracting the two equations. The other roots of the given equations can be determined by using the relations between their roots and coefficients.*

Condition for Both the Common Roots

Let α, β be the common roots of the quadratic equations $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$. Then, both the equations are identical, hence,

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Note:

- *If two quadratic equations with real coefficients have a non-real complex common root then both the roots will be common, i.e. both the equations will be the same. So the coefficients of the corresponding powers of x will have proportional values.*
- *If two quadratic equations with rational coefficients have a common irrational root $p + \sqrt{q}$ then both the roots will be common, i.e. no two different quadratic equations with rational coefficients can have a common irrational root $p + \sqrt{q}$.*

Example 1.114 **Determine the values of m for which the equations $3x^2 + 4mx + 2 = 0$ and $2x^2 + 3x - 2 = 0$ may have a common root.**

Sol. Let α be the common root of the equations $3x^2 + 4mx + 2 = 0$ and $2x^2 + 3x - 2 = 0$. Then, α must satisfy both the equations. Therefore,

$$3\alpha^2 + 4m\alpha + 2 = 0$$
$$2\alpha^2 + 3\alpha - 2 = 0$$

Using cross-multiplication method, we have

$$(-6 - 4)^2 = (9 - 8m)(-8m - 6)$$
$$\Rightarrow \quad 50 = (8m - 9)(4m + 3)$$
$$\Rightarrow \quad 32m^2 - 12m - 77 = 0$$
$$\Rightarrow \quad 32m^2 - 56m + 44m - 77 = 0$$
$$\Rightarrow \quad 8m(4m - 7) + 11(4m - 7) = 0$$
$$\Rightarrow \quad (8m + 11)(4m - 7) = 0$$
$$\Rightarrow \quad m = -\frac{11}{8}, \frac{7}{4}$$

Example 1.115 **If $x^2 + 3x + 5 = 0$ and $ax^2 + bx + c = 0$ have common root/roots and $a, b, c \in N$, then find the minimum value of $a + b + c$.**

Sol. The roots of $x^2 + 3x + 5 = 0$ are non-real. Thus given equations will have two common roots. We have,

$$\frac{a}{1} = \frac{b}{3} = \frac{c}{5} = \lambda$$
$$\Rightarrow \quad a + b + c = 9\lambda$$

Thus minimum value of $a + b + c$ is 9.

Example 1.116 **If $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$ have a common root and a, b and c are non-zero real numbers then find the value of $(a^3 + b^3 + c^3)/abc$.**

Sol. Given that $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$ have a common root. Hence,

$$(bc - a^2)^2 = (ab - c^2)(ac - b^2)$$
$$\Rightarrow \quad b^2c^2 + a^4 - 2a^2bc = a^2bc - ab^3 - ac^3 + b^2c^2$$
$$\Rightarrow \quad a^4 + ab^3 + ac^3 = 3a^2bc$$
$$\Rightarrow \quad \frac{a^3 + b^3 + c^3}{abc} = 3$$

Example 1.117 ***a, b, c* are positive real numbers forming a G.P. If $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root, then prove that d/a, e/b, f/c are in A.P.**

Sol. For first equation $D = 4b^2 - 4ac = 0$ (as given a, b, c are in G.P.). The equation has equal roots which are equal to $-b/a$ each. Thus, it should also be the root of the second equation. Hence,

$$d\left(\frac{-b}{a}\right)^2 + 2e\left(\frac{-b}{a}\right) + f = 0$$
$$\Rightarrow \quad d\frac{b^2}{a^2} - 2\frac{be}{a} + f = 0$$
$$\Rightarrow \quad d\frac{ac}{a^2} - 2\frac{be}{a} + f = 0 \qquad (\because b^2 = ac)$$
$$\Rightarrow \quad \frac{d}{a} + \frac{f}{c} = 2\frac{eb}{ac} = 2\frac{e}{b}$$

Example 1.118 **If the equations $x^2 + ax + 12 = 0$, $x^2 + bx + 15 = 0$ and $x^2 + (a + b)x + 36 = 0$ have a common positive root, then find the values of a and b.**

Sol. We have,

$$x^2 + ax + 12 = 0 \qquad (1)$$
$$x^2 + bx + 15 = 0 \qquad (2)$$

Adding (1) and (2), we get

$$2x^2 + (a + b)x + 27 = 0$$

Now subtracting it from the third given equation, we get

$$x^2 - 9 = 0 \Rightarrow x = 3, -3$$

Thus, common positive root is 3. Hence,

$$9 + 12 + 3a = 0$$
$$\Rightarrow \quad a = -7 \text{ and } 9 + 3b + 15 = 0$$
$$\Rightarrow \quad b = -8$$

Example 1.119 **The equations $ax^2 + bx + a = 0$ and $x^3 - 2x^2 + 2x - 1 = 0$ have two roots common. Then find the value of $a + b$.**

Sol. By observation, $x = 1$ is a root of equation $x^3 - 2x^2 + 2x - 1 = 0$. Thus we have

$$(x - 1)(x^2 - x + 1) = 0$$

Now roots of $x^2 - x + 1 = 0$ are non-real.

Then equation $ax^2 + bx + a = 0$ has both roots common with $x^2 - x + 1 = 0$. Hence, we have

$$\frac{a}{1} = \frac{b}{-1} = \frac{a}{1}$$

or $\quad a + b = 0$

Concept Application Exercise 1.6

1. If $x^2 + ax + bc = 0$ and $x^2 + bx + ca = 0$ $(a \neq b)$ have a common root, then prove that their other roots satisfy the equation $x^2 + cx + ab = 0$.

2. Find the condition that the expressions $ax^2 + bxy + cy^2$ and $a_1x^2 + b_1xy + c_1y^2$ may have factors $y - mx$ and $my - x$, respectively.
3. If $a, b, c \in R$ and equations $ax^2 + bx + c = 0$ and $x^2 + 2x + 9 = 0$ have a common root, then find $a:b:c$.
4. Find the condition on a, b, c, d such that equations $2ax^3 + bx^2 + cx + d = 0$ and $2ax^2 + 3bx + 4c = 0$ have a common root.
5. Let $f(x)$, $g(x)$ and $h(x)$ be the quadratic polynomials having positive leading coefficients and real and distinct roots. If each pair of them has a common root, then find the roots of $f(x) + g(x) + h(x) = 0$.

RELATION BETWEEN COEFFICIENT AND ROOTS OF *n*-DEGREE EQUATIONS

- Let α and β be roots of quadratic equation $ax^2 + bx + c = 0$. Then by factor theorem

$$ax^2 + bx + c = a(x - \alpha)(x - \beta)$$
$$= a(x^2 - (\alpha + \beta)x + \alpha\beta)$$

Comparing coefficients, we have

$$\alpha + \beta = -b/a \text{ and } \alpha\beta = c/a$$

- Let α, β, γ are roots of cubic equation $ax^3 + bx^2 + cx + d = 0$. Then,

$$ax^3 + bx^2 + cx + d = a(x - \alpha)(x - \beta)(x - \gamma)$$
$$= a(x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \alpha\gamma)x - \alpha\beta\gamma)$$

Comparing coefficients, we have

$$\alpha + \beta + \gamma = -b/a$$
$$\alpha\beta + \beta\gamma + \alpha\gamma = c/a$$
$$\alpha\beta\gamma = -d/a$$

- If $\alpha, \beta, \gamma, \delta$ are roots of $ax^4 + bx^3 + cx^2 + dx + e = 0$, then

$$\alpha + \beta + \gamma + \delta = -b/a$$
$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = c/a$$
(sum of product taking two at a time)
$$\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta = -d/a$$
(sum of product taking three at a time)
$$\alpha\beta\gamma\delta = e/a$$

In general, if $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are the roots of equation $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$, then sum of the roots is

$$\alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_n = -\frac{a_1}{a_0}$$

Sum of the product taken two at a time is

$$\left.\begin{array}{r}\alpha_1\alpha_2 + \alpha_1\alpha_3 + \cdots + \alpha_1\alpha_n \\ \cdots + \alpha_2\alpha_3 + \cdots + \alpha_2\alpha_n \\ \cdots + \alpha_{n-1}\alpha_n\end{array}\right\} = \frac{a_2}{a_0}$$

Sum of the product taken three at a time is $-a_3/a_0$ and so on.
Product of all the root is

$$\alpha_1\alpha_2\alpha_3 \ldots \alpha_n = (-1)^n \frac{a_n}{a_0}$$

Note:
- *A polynomial equation of degree n has n roots (real or imaginary).*
- *If all the coefficients are real then the imaginary roots occur in conjugate pairs, i.e., number of imaginary roots is always even.*
- *If the degree of a polynomial equation is odd, then the number of real roots will also be odd. It follows that at least one of the roots will be real.*

SOLVING CUBIC EQUATION

By using factor theorem together with some intelligent guessing, we can factorise polynomials of higher degree.

In summary, to solve a cubic equation of the form $ax^3 + bx^2 + cx + d = 0$,

1. obtain one factor $(x - \alpha)$ by trial and error
2. factorize $ax^3 + bx^2 + cx + d = 0$ as $(x - \alpha)(hx^2 + kx + s) = 0$
3. solve the quadratic expression for other roots

Example 1.120 **If α, β, γ are the roots of the equation $x^3 + 4x + 1 = 0$, then find the value of $(\alpha + \beta)^{-1} + (\beta + \gamma)^{-1} + (\gamma + \alpha)^{-1}$.**

Sol. For the given equation $\alpha + \beta + \gamma = 0$,
$$\alpha\beta + \beta\gamma + \alpha\gamma = 4, \quad \alpha\beta\gamma = -1$$
Now,
$$(\alpha + \beta)^{-1} + (\beta + \gamma)^{-1} + (\gamma + \alpha)^{-1} = (-\gamma)^{-1} + (-\alpha)^{-1} + (-\beta)^{-1}$$
$$= -\frac{\alpha\beta + \beta\gamma + \alpha\gamma}{\alpha\beta\gamma}$$
$$= -\frac{4}{(-1)}$$
$$= 4$$

Example 1.121 **Let $\alpha + i\beta$ ($\alpha, \beta \in R$) be a root of the equation $x^3 + qx + r = 0$, $q, r \in R$. Find a real cubic equation, independent of α and β, whose one root is 2α.**

Sol. If $\alpha + i\beta$ is a root then, $\alpha - i\beta$ will also be a root. If the third root is γ, then
$$(\alpha + i\beta) + (\alpha - i\beta) + \gamma = 0$$
$$\Rightarrow \quad \gamma = -2\alpha$$
But γ is a root of the given equation $x^3 + qx + r = 0$. Hence,
$$(-2\alpha)^3 + q(-2\alpha) + r = 0$$
$$\Rightarrow \quad (2\alpha)^3 + q(2\alpha) - r = 0$$
Therefore, 2α is a root of $t^3 + qt - r = 0$, which is independent of α and β.

Example 1.122 **In equation $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$ if two of its roots are equal in magnitude but opposite in sign, find the roots.**

Sol. Given that $\alpha + \beta = 0$ but $\alpha + \beta + \gamma + \delta = 2$. Hence,
$$\gamma + \delta = 2$$
Let $\alpha\beta = p$ and $\gamma\delta = q$. Therefore, given equation is equivalent to $(x^2 + p)(x^2 - 2x + q) = 0$. Comparing the coefficients, we get

$p + q = 4$, $-2p = 6$, $pq = -21$. Therefore, $p = -3$, $q = 7$ and they satisfy $pq = -21$. Hence,

$$(x^2 - 3)(x^2 - 2x + 7) = 0$$

Therefore, the roots are $\pm\sqrt{3}$ and $1 \pm i\sqrt{6}$. (where $i = \sqrt{-1}$)

Example 1.123 **Solve the equation $x^3 - 13x^2 + 15x + 189 = 0$ if one root exceeds the other by 2.**

Sol. Let the roots be α, $\alpha + 2$, β. Sum of roots is $2\alpha + \beta + 2 = 13$.

$$\therefore \quad \beta = 11 - 2\alpha \quad (1)$$

Sum of the product of roots taken two at a time is

$$\alpha(\alpha + 2) + (\alpha + 2)\beta + \beta\alpha = 15$$

or

$$\alpha^2 + 2\alpha + 2(\alpha + 1)\beta = 15 \quad (2)$$

Product of the roots is

$$\alpha\beta(\alpha + 2) = -189 \quad (3)$$

Eliminating β from (1) and (2), we get

$$\alpha^2 + 2\alpha + 2(\alpha + 1)(11 - 2\alpha) = 15$$

or

$$3\alpha^2 - 20\alpha - 7 = 0$$

$$\therefore \quad (\alpha - 7)(3\alpha + 1) = 0$$

$$\therefore \quad \alpha = 7 \text{ or } -\frac{1}{3}$$

$$\therefore \quad \beta = -3, \frac{35}{3}$$

Out of these values, $\alpha = 7$, $\beta = -3$ satisfy the third relation $\alpha\beta(\alpha + 2) = -189$, i.e., $(-21)(9) = -189$. Hence, the roots are 7, 7 + 2, –3 or 7, 9, –3.

REPEATED ROOTS

In equation $f(x) = 0$, where $f(x)$ is a polynomial function, and if it has roots $\alpha, \alpha, \beta, \ldots$ or α is a repeated root, then $f(x) = 0$ is equivalent to $(x - \alpha)^2(x - \beta)\cdots = 0$, from which we can conclude that $f'(x) = 0$ or $2(x - \alpha)[(x - \beta)\cdots] + (x - \alpha)^2[(x - \beta)\cdots]' = 0$ or $(x - \alpha)[2\{(x - \beta)\cdots\} + (x - \alpha)\{(x - \beta)\cdots\}'] = 0$ has root α.

Thus if α root occurs twice in equation then it is common in equations $f(x) = 0$ and $f'(x) = 0$.

Similarly, if root α occurs thrice in equation, then it is common in the equations $f(x) = 0$, $f'(x) = 0$ and $f''(x) = 0$.

Example 1.124 **If $x - c$ is a factor of order m of the polynomial $f(x)$ of degree n ($1 < m < n$), then find the polynomials for which $x = c$ is a root.**

Sol. From the given information we have $f(x) = (x - c)^m g(x)$, where $g(x)$ is polynomial of degree $n - m$. Then $x = c$ is common root for the equations $f(x) = 0$, $f^1(x) = 0$, $f^2(x) = 0$, …, $f^{m-1}(x) = 0$, where $f^r(x)$ represents r^{th} derivative of $f(x)$ w.r.t. x.

Example 1.125 **If $a_1x^3 + b_1x^2 + c_1x + d_1 = 0$ and $a_2x^3 + b_2x^2 + c_2x + d_2 = 0$ have a pair of repeated roots common, then prove that**

$$\begin{vmatrix} 3a_1 & 2b_1 & c_1 \\ 3a_2 & 2b_2 & c_2 \\ a_2b_1 - a_1b_2 & c_1a_2 - c_2a_1 & d_1a_2 - d_2a_1 \end{vmatrix} = 0$$

Sol. If $f(x) = a_1x^3 + b_1x^2 + c_1x + d_1 = 0$ has roots α, α, β, then $g(x) = a_2x^3 + b_2x^2 + c_2x + d_2 = 0$ must have roots α, α, γ. Hence,

$$a_1\alpha^3 + b_1\alpha^2 + c_1\alpha + d_1 = 0 \quad (1)$$

$$a_2\alpha^3 + b_2\alpha^2 + c_2\alpha + d_2 = 0 \quad (2)$$

Now, α is also a root of equations $f'(x) = 3a_1x^2 + 2b_1x + c_1 = 0$ and $g'(x) = 3a_2x^2 + 2b_2x + c_2 = 0$. Therefore,

$$3a_1\alpha^2 + 2b_1\alpha + c_1 = 0 \quad (3)$$

$$3a_2\alpha^2 + 2b_2\alpha + c_2 = 0 \quad (4)$$

Also, from $a_2 \times (1) - a_1 \times (2)$, we have

$$(a_2b_1 - a_1b_2)\alpha^2 + (c_1a_2 - c_2a_1)\alpha + d_1a_2 - d_2a_1 = 0 \quad (5)$$

Eliminating α from (3), (4) and (5), we have

$$\begin{vmatrix} 3a_1 & 2b_1 & c_1 \\ 3a_2 & 2b_2 & c_2 \\ a_2b_1 - a_1b_2 & c_1a_2 - c_2a_1 & d_1a_2 - d_2a_1 \end{vmatrix} = 0$$

Concept Application Exercise 1.7

1. If $b^2 < 2ac$, then prove that $ax^3 + bx^2 + cx + d = 0$ has exactly one real root.
2. If two roots of $x^3 - ax^2 + bx - c = 0$ are equal in magnitude but opposite in signs, then prove that $ab = c$.
3. If α, β and γ are the roots of $x^3 + 8 = 0$, then find the equation whose roots are α^2, β^2 and γ^2.
4. If α, β, γ are the roots of the equation $x^3 - px + q = 0$, then find the cubic equation whose roots are $\alpha/(1 + \alpha)$, $\beta/(1 + \beta)$, $\gamma/(1 + \gamma)$.
5. If the roots of equation $x^3 + ax^2 + b = 0$ are α_1, α_2 and α_3 ($a, b \neq 0$), then find the equation whose roots are

$$\frac{\alpha_1\alpha_2 + \alpha_2\alpha_3}{\alpha_1\alpha_2\alpha_3}, \frac{\alpha_2\alpha_3 + \alpha_3\alpha_1}{\alpha_1\alpha_2\alpha_3}, \frac{\alpha_1\alpha_3 + \alpha_1\alpha_2}{\alpha_1\alpha_2\alpha_3}$$

QUADRATIC EXPRESSION IN TWO VARIABLES

The general quadratic expression $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ can be factorized into two linear factors. Given quadratic expression is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c \quad (1)$$

Corresponding equation is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

or

$$ax^2 + 2(hy + g)x + by^2 + 2fy + c = 0 \quad (2)$$

$$\therefore \quad x = \frac{-2(hy + g) \pm \sqrt{4(hy + g)^2 - 4a(by^2 + 2fy + c)}}{2a}$$

$$\Rightarrow \quad x = \frac{-(hy + g) \pm \sqrt{h^2y^2 + g^2 + 2ghy - aby^2 - 2afy - ac}}{a}$$

$$\Rightarrow ax + hy + g = \pm\sqrt{h^2y^2 + g^2 + 2ghy - aby^2 - 2afy - ac} \quad (3)$$

Now, expression (1) can be resolved into two linear factors if $(h^2 - ab)y^2 + 2(gh - af)y + g^2 - ac$ is a perfect square and $h^2 - ab > 0$. But $(h^2 - ab)y^2 + 2(gh - af)y + g^2 - ac$ will be a perfect square if

$\Rightarrow$ $4(gh-af)^2-4(h^2-ab)(g^2-ac)=0$ and $h^2-ab>0$

$\Rightarrow$ $g^2h^2+a^2f^2-2afgh-h^2g^2+abg^2+ach^2-a^2bc=0$

and

$h^2-ab>0$

$\Rightarrow$ $abc+2fgh-af^2-bg^2-ch^2=0$

and

$h^2-ab>0$

This is the required condition.

Example 1.126 **Find the values of m for which the expression $2x^2+mxy+3y^2-5y-2$ can be resolved into two rational linear factors.**

Sol. We know that $ax^2+2hxy+by^2+2gx+2fy+c$ can be resolved into two linear factors if and only if

$$abc+2fgh-af^2-bg^2-ch^2=0$$

Given expression is

$$2x^2+mxy+3y^2-5y-2 \quad (1)$$

Here, $a=2$, $h=m/2$, $b=3$, $g=0, f=-5/2$, $c=-2$. Therefore, expression $2x^2+mxy+3y^2-5y-2$ will have two linear factors if and only if

$$abc+2fgh-af^2-bg^2-ch^2=0$$

$$\Rightarrow \quad 2\times 3(-2)+2\left(\frac{-5}{2}\right)(0)\left(\frac{m}{2}\right) -2\left(\frac{-5}{2}\right)^2-3\times 0^2-(-2)\left(\frac{m}{2}\right)^2=0$$

$$\Rightarrow \quad -12-\frac{25}{2}+\frac{m^2}{2}=0$$

$$\Rightarrow \quad m^2=49 \Rightarrow m=\pm 7$$

Example 1.127 **Find the linear factors of $2x^2-y^2-x+xy+2y-1$.**

Sol. Given expression is

$$2x^2-y^2-x+xy+2y-1 \quad (1)$$

Its corresponding equation is

$$2x^2-y^2-x+xy+2y-1=0$$

or

$$2x^2-(1-y)x-(y^2-2y+1)=0$$

$$\therefore \quad x=\frac{1-y\pm\sqrt{(1-y)^2+4.2(y^2-2y+1)}}{4}$$

$$=\frac{1-y\pm\sqrt{(1-y)^2+8(y-1)^2}}{4}$$

$$=\frac{1-y\pm\sqrt{9(1-y)^2}}{4}$$

$$=\frac{1-y\pm 3(1-y)}{4}$$

$$=1-y,-\frac{1-y}{2}$$

Hence, the required linear factors are $(x+y-1)$ and $(2x-y+1)$.

FINDING THE RANGE OF A FUNCTION INVOLVING QUADRATIC EXPRESSION

In this section, some examples are given to illustrate the range of a function involving quadratic expression.

Example 1.128 **Find the range of the function $f(x)=x^2-2x-4$.**

Sol. Let

$$x^2-2x-4=y$$

$$\Rightarrow \quad x^2-2x-4-y=0$$

Now if x is real, then

$$D\ge 0$$

$$\Rightarrow \quad (-2)^2-4(1)(-4-y)\ge 0$$

$$\Rightarrow \quad 4+16+4y\ge 0$$

$$\Rightarrow \quad y\ge -5$$

Hence range of $f(x)$ is $[-5,\infty)$.

Alternative method:

$$f(x)=x^2-2x-4$$

$$=(x-1)^2-5$$

$$\ge -5$$

Hence, range is

$$[-5,\infty)$$

Example 1.129 **Find the least value of $\dfrac{(6x^2-22x+21)}{(5x^2-18x+17)}$ for real x.**

Sol. Let,

$$\frac{6x^2-22x+21}{5x^2-18x+17}=y$$

$$\Rightarrow \quad (6-5y)x^2-2x(11-9y)+21-17y=0$$

Since x is real

$$4(11-9y)^2-4(6-5y)(21-17y)\ge 0$$

$$\Rightarrow \quad -4y^2+9y-5\ge 0$$

$$\Rightarrow \quad 4y^2-9y+5\le 0$$

$$\Rightarrow \quad 4(y-1)(y-5/4)\le 0$$

$$\Rightarrow \quad 1\le y\le 5/4$$

Hence, the least value of the given expression is 1.

Example 1.130 **Prove that if the equation $x^2+9y^2-4x+3=0$ is satisfied for real value of x and y, then x must lie between 1 and 3 and y must lie between $-1/3$ and $1/3$.**

Sol. Given equation is

$$x^2+9y^2-4x+3=0 \quad (1)$$

$$\Rightarrow \quad x^2-4x+9y^2+3=0$$

Since x is real,

$$(-4)^2-4(9y^2+3)\ge 0$$

$$\Rightarrow \quad 16-4(9y^2+3)\ge 0$$

$$\Rightarrow \quad 4-9y^2-3\ge 0$$

$$\Rightarrow \quad 9y^2-1\le 0$$

$$\Rightarrow \quad 9y^2\le 1$$

$$\Rightarrow \quad y^2\le\frac{1}{9}$$

$$\Rightarrow \quad -\frac{1}{3}\le y\le\frac{1}{3} \quad (2)$$

Equation (1) can also be written as

$$9y^2+0y+x^2-4x+3=0 \quad (3)$$

Since y is real, so

$$0^2 - 4.9(x^2 - 4x + 3) \geq 0$$

or

$$x^2 - 4x + 3 \leq 0 \qquad (4)$$

or

$$(x - 3)(x - 1) \leq 0$$

or

$$1 \leq x \leq 3$$

Example 1.131 **Find the domain and the range of**

$$f(x) = \sqrt{3 - 2x - x^2}.$$

Sol. $f(x) = \sqrt{3 - 2x - x^2}$ is defined if

$$3 - 2x - x^2 \geq 0$$

$$\Rightarrow x^2 + 2x - 3 \leq 0$$

$$\Rightarrow (x - 1)(x + 3) \leq 0$$

$$\Rightarrow x \in [-3, 1]$$

Also, $f(x) = \sqrt{4 - (x+1)^2}$ has maximum value when $x + 1 = 0$. Hence range is $[0, 2]$.

Example 1.132 **Find the domain and range of**

$$f(x) = \sqrt{x^2 - 3x + 2}.$$

Sol. $x^2 - 3x + 2 \geq 0$

$$\Rightarrow (x - 1)(x - 2) \geq 0$$

$$\Rightarrow x \in (-\infty, 1] \cup [2, \infty)$$

Now,

$$f(x) = \sqrt{x^2 - 3x + 2}$$

$$= \sqrt{\left(x - \frac{3}{2}\right)^2 + 2 - \frac{9}{4}}$$

$$= \sqrt{\left(x - \frac{3}{2}\right)^2 - \frac{1}{4}}$$

Now, the least permissible value of $(x - 3/2)^2 - 1/4$ is 0 when $(x - 3/2) = \pm 1/2$. Hence, the range is $[0, \infty)$.

Concept Application Exercise 1.8

1. Find the range of $f(x) = x^2 - x - 3$.
2. Find the range of

   (i) $f(x) = \dfrac{x^2 + 34x - 71}{x^2 + 2x - 7}$

   (ii) $f(x) = \dfrac{x^2 - x + 1}{x^2 + x + 1}$
3. Find the range of $f(x) = \sqrt{x - 1} + \sqrt{5 - x}$.
4. Find the range of the function $f(x) = 6^x + 3^x + 6^{-x} + 3^{-x} + 2$.
5. Find the domain and range of $f(x) = \sqrt{x^2 - 4x + 6}$.

QUADRATIC FUNCTION

Let $f(x) = ax^2 + bx + c$, where $a, b, c, \in R$ and $a \neq 0$. We have,

$$f(x) = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right]$$

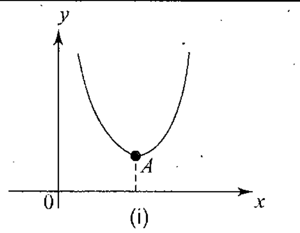
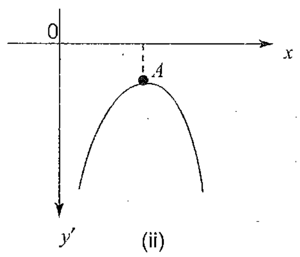
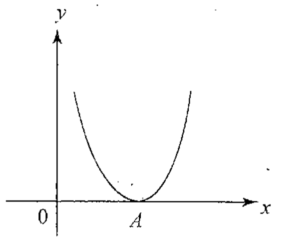
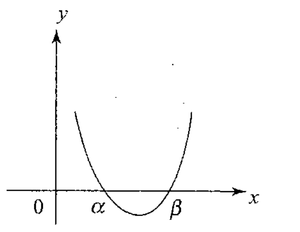
$$= a\left[x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + \frac{c}{a} - \frac{b^2}{4a^2}\right]$$

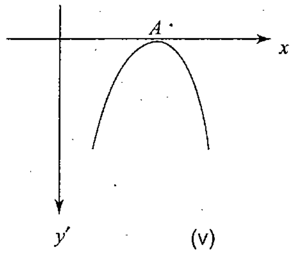
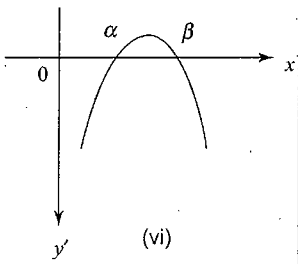
$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a^2}\right]$$

$$\Rightarrow \left(y + \frac{D}{4a}\right) = a\left(x + \frac{b}{2a}\right)^2$$

Thus $y = f(x)$ represents a parabola whose axis is parallel to y-axis and vertex is $A(-b/2a, -D/4a)$. For some values of x, $f(x)$ may be positive, negative or zero and for $a > 0$, the parabola opens upwards and for $a < 0$, the parabola opens downwards. This gives the following cases:

1. $a > 0$ and $D < 0$, so $f(x) > 0$, $\forall x \in R$, i.e., $f(x)$ is positive for all values of x. Range of function is $[-D/(4a), \infty)$. $x = -b/(2a)$ is a point of minima.	 (i)
2. $a < 0$ and $D < 0$ so $f(x) < 0$, $\forall x \in R$, i.e., $f(x)$ is negative for all values of x. Range of function is $(-\infty, -D/(4a)]$. $x = -b/(2a)$ is a point of maxima.	 (ii)
3. $a > 0$ and $D = 0$, so $f(x) \geq 0$, $\forall x \in R$, i.e., $f(x)$ is positive for all values of x except at vertex where $f(x) = 0$.	 (iii)
4. $a > 0$ and $D > 0$. Let $f(x) = 0$ have two real roots α and β. If $\alpha < \beta$, then $f(x) > 0$, $\forall x \in (-\infty, \alpha) \cup (\beta, \infty)$ and $f(x) < 0$, $\forall x \in (\alpha, \beta)$.	 (iv)

5. $a < 0$ and $D = 0$, so $f(x) \le 0$, $\forall x \in R$, i.e., $f(x)$ is negative for all values of x except at vertex where $f(x) = 0$.	
6. $a < 0$ and $D > 0$. Let $f(x) = 0$ have two roots α and β $(\alpha < \beta)$. Then $f(x) < 0$, $\forall x \in (-\infty, \alpha) \cup (\beta, \infty)$ and $f(x) > 0$, $\forall x \in (\alpha, \beta)$.	

Note: *If $f(x) \ge 0$, $\forall x \in R$, then $a > 0$ and $D \le 0$ and if $f(x) \le 0$, $\forall x \in R$, then $a < 0$ and $D \le 0$.*

Example 1.133 **What is the minimum height of any point on the curve $y = x^2 - 4x + 6$ above the x-axis?**

Sol. $y = x^2 - 4x + 6$

$= (x - 2)^2 + 2$

Now $(x - 2)^2 \ge 0$ for all real x.

Then $(x - 2)^2 + 2 \ge 2$ for all real x.

Hence, the minimum value of y (or $x^2 - 4x + 6$) is 2, which is the height of the graph above the x-axis.

Example 1.134 **What is the maximum height of any point on the curve $y = -x^2 + 6x - 5$ above the x-axis?**

Sol. $y = -x^2 + 6x - 5$

$= 4 - (x - 3)^2$

Now $(x - 3)^2 \ge 0$ for all real x,

Hence, the maximum value of y (or $-x^2 + 6x - 5$) is 4, which is the height of the graph above x-axis.

Example 1.135 **Find the largest natural number 'a' for which the maximum value of $f(x) = a - 1 + 2x - x^2$ is smaller than the minimum value of $g(x) = x^2 - 2ax + 10 - 2a$.**

Sol. $f(x) = a - 1 + 2x - x^2$

$= a - (x^2 - 2x + 1)$

$= a - (x - 1)^2$

Hence, the maximum value of $f(x)$ is "a" when $(x - 1)^2 = 0$ or $x = 1$

$g(x) = x^2 - 2ax + 10 - 2a$

$= (x - a)^2 + 10 - 2a - a^2$

Hence, the minimum value of $g(x)$ is $10 - 2a - a^2$ when $(x - a)^2 = 0$ or $x = a$.

Now given that maximum of $f(x)$ is smaller than the minimum of $g(x)$

$a < -a^2 + 10 - 2a$

$\Rightarrow a^2 + 3a - 10 < 0$

$\therefore (a + 5)(a - 2) < 0$

The largest natural number $a = 1$.

Example 1.136 **Find the least value of n such that $(n - 2)x^2 + 8x + n + 4 > 0$, $\forall x \in R$, where $n \in N$.**

Sol. $(n - 2)x^2 + 8x + n + 4 > 0$, $\forall x \in R$

$\Rightarrow 64 - 4(n - 2)(n + 4) < 0$ and $n - 2 > 0$

$\Rightarrow 16 - (n^2 + 2n - 8) < 0$ and $n > 2$

$\Rightarrow n^2 + 2n - 24 > 0$ and $n > 2$

$\Rightarrow (n + 6)(n - 4) > 0$ and $n > 2$

$\Rightarrow n > 4$ as $n \in N$ and $n > 2$

$\Rightarrow n \ge 5$

Hence, the least value of n is 5.

Example 1.137 **If the inequality $(mx^2 + 3x + 4)/(x^2 + 2x + 2) < 5$ is satisfied for all $x \in R$, then find the values of m.**

Sol. We have,

$x^2 + 2x + 2 = (x + 1)^2 + 1 > 0$, $\forall x \in R$

Therefore,

$$\frac{mx^2 + 3x + 4}{x^2 + 2x + 2} < 5$$

$\Rightarrow (m - 5)x^2 - 7x - 6 < 0$, $\forall x \in R$

$\Rightarrow m - 5 < 0$ and $D < 0$

$\Rightarrow m < 5$ and $49 + 24(m - 5) < 0$

$\Rightarrow m < \frac{71}{24}$

Example 1.138 **If $f(x) = (a_1x + b_1)^2 + (a_2x + b_2)^2 + \cdots + (a_nx + b_n)^2$, then prove that $(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \le (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2)$.**

Sol. Given,

$$f(x) = (a_1x + b_1)^2 + (a_2x + b_2)^2 + \cdots + (a_nx + b_n)^2 \quad (1)$$

or

$$f(x) = (a_1^2 + a_2^2 + \cdots + a_n^2)x^2 + 2(a_1b_1 + a_2b_2 + \cdots + a_nb_n)x + (b_1^2 + b_2^2 + \cdots + b_n^2) \quad (2)$$

From (1), $f(x) \ge 0$, $\forall x \in R$. Hence, from (2), we have

$$(a_1^2 + a_2^2 + \cdots + a_n^2)x^2 + 2(a_1b_1 + a_2b_2 + \cdots + a_nb_n)x + (b_1^2 + b_2^2 + \cdots + b_n^2) \ge 0 \ \forall x \in R$$

Discriminant of its corresponding equation is

$D \le 0$ ($\because$ coefficient of x^2 is positive)

$$\Rightarrow 4(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \le 4(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2)$$

$$\Rightarrow (a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \le (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2)$$

Example 1.139 **If c is positive and $2ax^2 + 3bx + 5c = 0$ does not have any real roots, then prove that $2a - 3b + 5c > 0$.**

Sol. Given $c > 0$ and $2ax^2 + 3bx + 5c = 0$ does not have real roots. Let

$$f(x) = 2ax^2 + 3bx + 5c$$

$\Rightarrow$ $f(x) > 0, \forall x \in R$, if $a > 0$ or $f(x) < 0\ \forall x \in R$, if $a < 0$

But

$5c = f(0) > 0$

$\Rightarrow$ $f(x) > 0, \forall x \in R$

$\Rightarrow$ $2ax^2 + 3bx + 5c > 0, \forall x \in R$

$\Rightarrow$ $2a - 3b + 5c > 0$ (for $x = -1$)

Example 1.140 **If $ax^2 + bx + 6 = 0$ does not have distinct real roots, then find the least value of $3a + b$.**

Sol. Given equation $ax^2 + bx + 6 = 0$ does not have distinct real roots. Hence,

$\Rightarrow$ $f(x) = ax^2 + bx + 6 \le 0, \forall x \in R$, if $a < 0$

or

$f(x) = ax^2 + bx + 6 \ge 0, \forall x \in R$, if $a > 0$

But

$f(0) = 6 > 0$

$\Rightarrow$ $f(x) = ax^2 + bx + 6 \ge 0, \forall x \in R$

$\Rightarrow$ $f(3) = 9a + 3b + 6 \ge 0$

$\Rightarrow$ $3a + b \ge -2$

Therefore, the least value of $3a + b$ is -2.

Example 1.141 **A quadratic trinomial $P(x) = ax^2 + bx + c$ is such that the equation $P(x) = x$ has no real roots. Prove that in this case the equation $P(P(x)) = x$ has no real roots either.**

Sol. Since the equation $ax^2 + bx + c = x$ has no real roots, the expression $P(x) - x = ax^2 + (b - 1)x + c$ assumes values of one sign $\forall x \in R$, say $P(x) - x > 0$. Then

$$P(P(x_0)) - P(x_0) > 0$$

for any $x = x_0$, i.e., $P(x_0) > x_0$ and hence $P(P(x_0)) > x_0$. Therefore, x_0 cannot be a root of the 4^{th} degree equation $P(P(x)) = x$.

Example 1.142 **Prove that for real values of x the expression $(ax^2 + 3x - 4)/(3x - 4x^2 + a)$ may have any value provided a lies between 1 and 7.**

Sol. Let,

$$y = \frac{ax^2 + 3x - 4}{3x - 4x^2 + a}$$

$\Rightarrow$ $(a + 4y)x^2 + (3 - 3y)x - 4 - ay = 0$

Now, x is real. So,

$D \ge 0$

$\Rightarrow$ $9(1 - y)^2 + 4(a + 4y)(4 + ay) \ge 0$

$\Rightarrow$ $(9 + 16a)y^2 + (-18 + 4a^2 + 64)y + (9 + 16a) \ge 0$, $\forall y \in R$ ($\because$ y takes any real value)

$\Rightarrow$ $9 + 16a > 0$ and $(4a^2 + 46)^2 - 4(9 + 16a)^2 \le 0$

$\Rightarrow$ $a > -\frac{9}{16}$ and $(4a^2 + 46 - 18 - 32a)(4a^2 + 46 + 18 + 32a) \le 0$

$\Rightarrow$ $a > -\frac{9}{16}$ and $(a^2 - 8a + 7)(a^2 + 8a + 16) \le 0$

$\Rightarrow$ $a > -\frac{9}{16}$ and $1 \le a \le 7$ or $a = -4$

$\Rightarrow$ $1 \le a \le 7$

Example 1.143 **Let a, b and c be real numbers such that $a + 2b + c = 4$. Find the maximum value of $(ab + bc + ca)$.**

Sol. Given,

$$a + 2b + c = 4 \Rightarrow a = 4 - 2b - c$$

Let,

$ab + bc + ca = x \Rightarrow a(b + c) + bc = x$

$\Rightarrow$ $(4 - 2b - c)(b + c) + bc = x$

$\Rightarrow$ $4b + 4c - 2b^2 - 2bc - bc - c^2 + bc = x$

$\Rightarrow$ $2b^2 - 4b + 2bc - 4c + c^2 + x = 0$

$\Rightarrow$ $2b^2 + 2(c - 2)b - 4c + c^2 + x = 0$

Since $b \in R$, so

$4(c - 2)^2 - 4 \times 2(-4c + c^2 + x) \ge 0$

$\Rightarrow$ $c^2 - 4c + 4 + 8c - 2c^2 - 2x \ge 0$

$\Rightarrow$ $c^2 - 4c + 2x - 4 \le 0$

Since $c \in R$, so

$16 - 4(2x - 4) \ge 0 \Rightarrow x \le 4$

$\therefore$ $\max(ab + bc + ac) = 4$

Example 1.144 **Prove that for all real values of x and y, $x^2 + 2xy + 3y^2 - 6x - 2y \ge -11$.**

Sol. Let,

$x^2 + 2xy + 3y^2 - 6x - 2y + 11 \ge 0, \forall x, y \in R$

$\Rightarrow$ $x^2 + (2y - 6)x + 3y^2 - 2y + 11 \ge 0, \forall x \in R$

$\Rightarrow$ $(2y - 6)^2 - 4(3y^2 - 2y + 11) \le 0, \forall y \in R$

$\Rightarrow$ $(y - 3)^2 - (3y^2 - 2y + 11) \le 0, \forall y \in R$

$\Rightarrow$ $2y^2 + 4y + 2 \ge 0, \forall y \in R$

$\Rightarrow$ $(y + 1)^2 \ge 0, \forall y \in R$, which is always true

Concept Application Exercise 1.9

1. If $f(x) = \sqrt{x^2 + ax + 4}$ is defined for all x, then find the values of a.
2. If $ax^2 + bx + c = 0$, $a, b, c \in R$ has no real zeros, and if $c < 0$, then which of the following is true?
   (i) $a < 0$
   (ii) $a + b + c > 0$
   (iii) $a > 0$
3. If $ax^2 + bx + c = 0$ has imaginary roots and $a + c < b$, then prove that $4a + c < 2b$.

4. Let $x, y, z \in R$ such that $x + y + z = 6$ and $xy + yz + zx = 7$. Then find the range of values of x, y and z.
5. If x is real and $(x^2 + 2x + c)/(x^2 + 4x + 3c)$ can take all real values, then show that $0 \le c \le 1$.
6. If $x \in R$, and a, b, c are in ascending or descending order of magnitude, show that $(x - a)(x - c)/(x - b)$ (where $x \neq b$) can assume any real value.
7. Find the complete set of values of a such that $(x^2 - x)/(1 - ax)$ attains all real values.
8. If the quadratic equation $ax^2 + bx + 6 = 0$ does not have real roots and $b \in R^+$, then prove that
$$a > \max\left\{\frac{b^2}{24}, b - 6\right\}$$
9. If x be real and the roots of the equation $ax^2 + bx + c = 0$ are imaginary, then prove that $a^2x^2 + abx + ac$ is always positive.

LOCATION OF ROOTS

In some problems, we want the roots of the equation $ax^2 + bx + c = 0$ to lie in a given interval. For this we impose conditions on a, b and c.

1. $\alpha, \beta > 0$

Conditions:

(a) sum of roots, $\alpha + \beta > 0$

(b) product of roots, $\alpha\beta > 0$

(c) $D \ge 0$

Graphically:

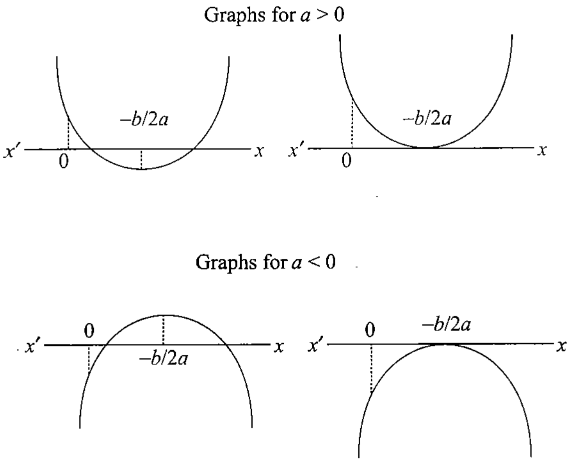


Fig. 1.41

Conditions:

(a) $af(0) > 0$ ($\because$ when $a > 0, f(0) > 0$ and when $a < 0, f(0) < 0$)

(b) $-b/2a > 0$

(c) $D \ge 0$

2. $\alpha, \beta < 0$

Conditions:

(a) sum of roots, $\alpha + \beta < 0$

(b) product of roots, $\alpha\beta > 0$

(c) $D \ge 0$

Graphically:

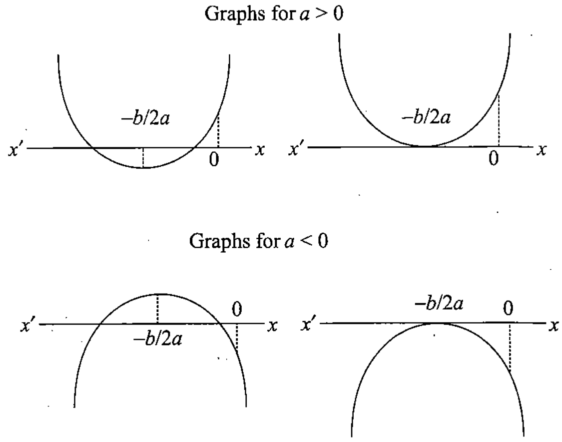


Fig. 1.42

Conditions:

(a) $af(0) > 0$

(b) $-b/2a < 0$

(c) $D \ge 0$

3. $\alpha < 0 < \beta$ (roots of opposite sign)

Product of roots, $\alpha\beta < 0$

Note That when $\alpha\beta = \frac{C}{a} 0,\ ac < 0$

$\Rightarrow\ D = b^2 - 4ac > 0.$

Graphically:

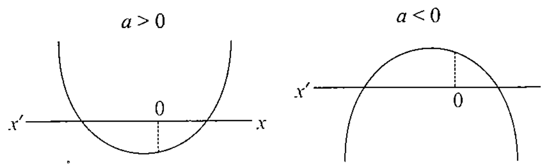


Fig. 1.43

When $a > 0, f(0) < 0$ and when $a < 0$, then $f(0) > 0$

$\Rightarrow\ af(0) < 0.$

4. $\alpha, \beta > k$

Graphically:

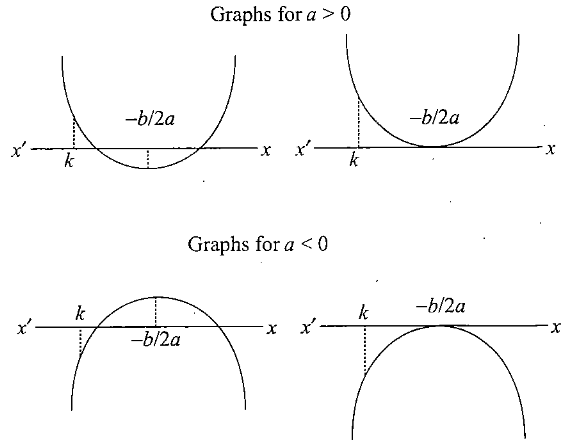


Fig. 1.44

Conditions:
(a) $af(k) > 0$ ($\because$ when $a > 0, f(k) > 0$ and when $a < 0, f(k) < 0$)
(b) $-b/2a > k$
(c) $D \geq 0$

5. $\alpha, \beta < k$

Graphically:

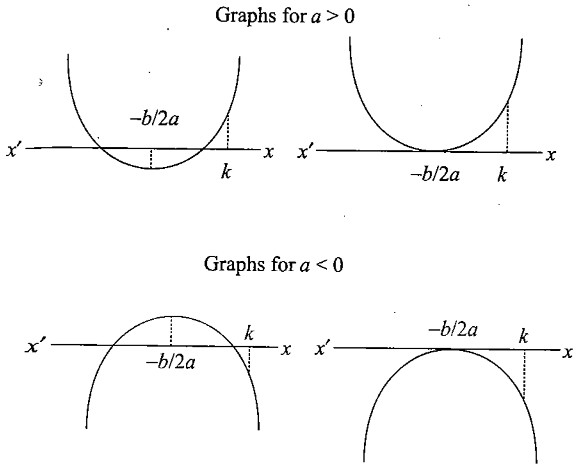


Fig. 1.45

Conditions:
(a) $af(k) > 0$
($\because$ when $a > 0, f(k) > 0$ and when $a < 0, f(k) < 0$)
(b) $-b/(2a) < k$
(c) $D \geq 0$

6. $\alpha < k < \beta$ (one root is smaller than k and other root is greater than k)

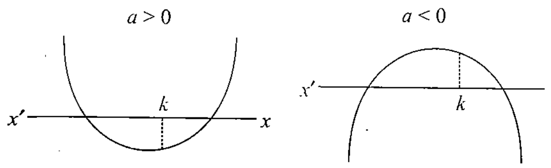


Fig. 1.46

When $a > 0, f(k) < 0$ and when $a < 0$, then $f(k) > 0$
$\Rightarrow \quad af(k) < 0.$

7. Exactly one root lying in (k_1, k_2)

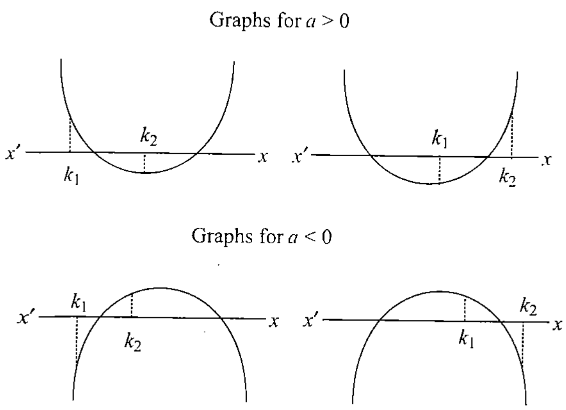


Fig. 1.47

From the graphs, we can see that $f(k_1)$ and $f(k_2)$ have opposite sign. Hence, $f(k_1)f(k_2) < 0$.

8. Both the roots lying in the interval (k_1, k_2).

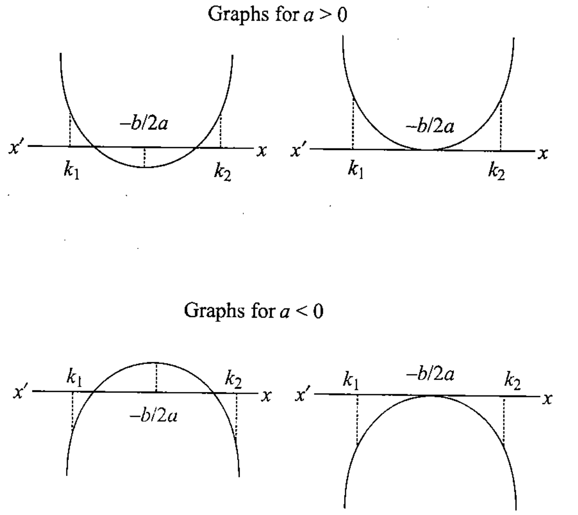


Fig. 1.48

From the graphs,
(a) $af(k_1) > 0$ and $af(k_2) > 0$
(b) $k_1 < -b/(2a) < k_2$
(c) $D \geq 0$

9. One root is smaller than k_1 and other root is greater than k_2. In this case k_1 and k_2 lie between the roots.

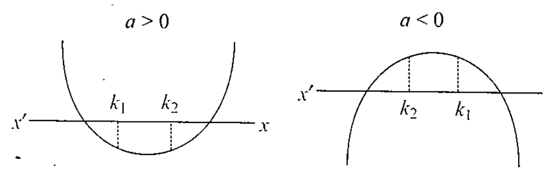


Fig. 1.49

From the graphs, $af(k_1) < 0$ and $af(k_2) < 0$.

Example 1.145 **Let $x^2 - (m - 3)x + m = 0$ ($m \in R$) be a quadratic equation. Find the values of m for which the roots are**

(i) **real and distinct**
(ii) **equal**
(iii) **not real**
(iv) **opposite in sign**
(v) **equal in magnitude but opposite in sign**
(vi) **positive**
(vii) **negative**
(viii) **such that at least one is positive**
(ix) **one root is smaller than 2 and the other root is greater than 2**
(x) **both the roots are greater than 2**

(xi) **both the roots are smaller than 2**

(xii) **exactly one root lies in the interval (1, 2)**

(xiii) **both the roots lie in the interval (1, 2)**

(xiv) **at least one root lies in the interval (1, 2)**

(xv) **one root is greater than 2 and the other root is smaller than 1**

Sol. Let $f(x) = x^2 - (m - 3)x + m = 0$

(i)

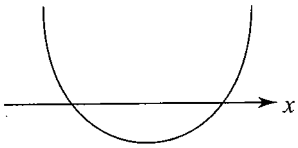


Fig. 1.50

Both the roots are real and distinct. So,

$D > 0$

$\Rightarrow \quad (m - 3)^2 - 4m > 0$

$\Rightarrow \quad m^2 - 10m + 9 > 0$

$\Rightarrow \quad (m - 1)(m - 9) > 0$

$\Rightarrow \quad m \in (-\infty, 1) \cup (9, \infty)$

(ii)

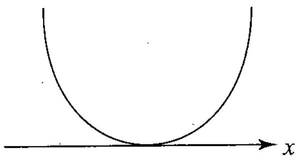


Fig. 1.51

Both the roots are equal. So,

$D = 0 \Rightarrow m = 9$ or $m = 1$

(iii)

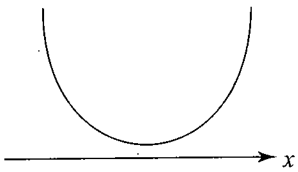


Fig. 1.52

Both the roots are imaginary. So,

$D < 0$

$\Rightarrow \quad (m - 1)(m - 9) < 0$

$\Rightarrow \quad m \in (1, 9)$

(iv)

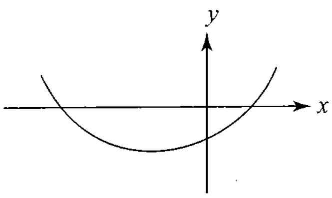


Fig. 1.53

The roots are opposite in sign. Hence, the product of roots is negative. So,

$m < 0 \Rightarrow m \in (-\infty, 0)$

(v)

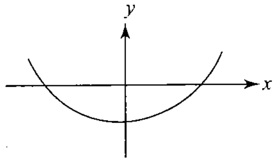


Fig. 1.54

Roots are equal in magnitude but opposite in sign. Hence, sum of roots is zero as well as $D \geq 0$. So,

$m \in (-\infty, 1) \cup (9, \infty)$ and $m - 3 = 0$, i.e., $m = 3$

$\Rightarrow$ no such m exists, so $m \in \phi$.

(vi)

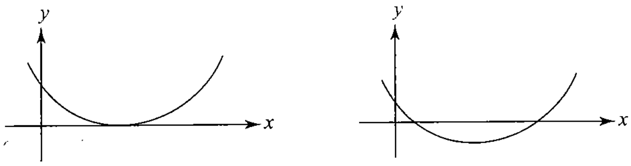


Fig. 1.55

Both the roots are positive. Hence, $D \geq 0$ and both the sum and the product of roots are positive. So,

$m - 3 > 0$, $m > 0$ and $m\ (-\infty, 1] \cup [9, \infty)$

$m \in [9, \infty)$

(vii)

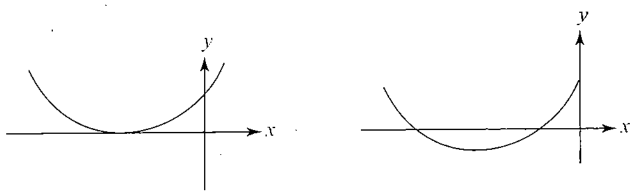


Fig. 1.56

Both the roots are negative. Hence, $D \geq 0$, and sum is negative but product is positive. So,

$m - 3 < 0$, $m > 0$, $m \in (-\infty, 1] \cup [9, \infty)$

$\Rightarrow \quad m \in (0, 1]$

(viii) At least one root is positive. Hence, either one root is positive or both roots are positive. So,

$m \in (-\infty, 0) \cup [9, \infty)$

(ix)

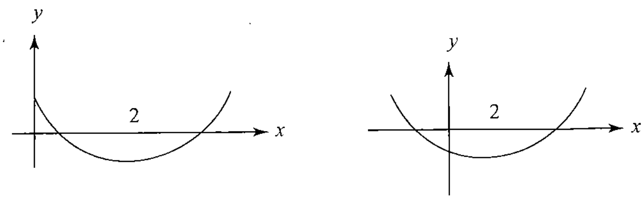


Fig. 1.57

One root is smaller than 2 and the other root is greater than 2, i.e., 2 lies between the roots. So,

$f(2) < 0$

$\Rightarrow \quad 4 - 2(m - 3) + m < 0$

$\Rightarrow \quad m > 10$

(x)

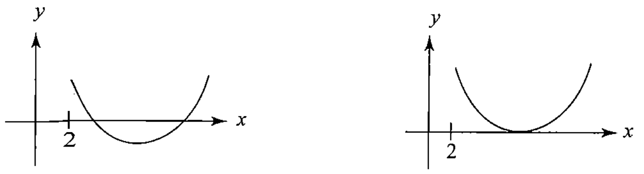


Fig. 1.58

Both the roots are greater than 2. So,

$$f(2) > 0,\ D \geq 0,\ -\frac{b}{2a} > 2$$

$\Rightarrow \quad m < 10$ and $m \in (-\infty, 1] \cup [9, \infty)$ and $m - 3 > 4$

$\Rightarrow \quad m \in [9, 10)$

(xi)

Fig. 1.59

Both the roots are smaller than 2. So,

$$f(2) > 0,\ D \geq 0,\ -\frac{b}{2a} < 2$$

$\Rightarrow \quad m \in (-\infty, 1]$

(xii)

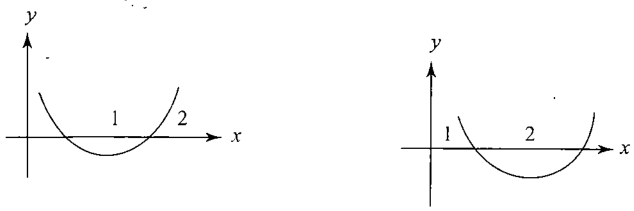


Fig. 1.60

Exactly one root lies in (1, 2). So,

$f(1)f(2) < 0$

$\Rightarrow \quad 4(10 - m) < 0$

$\Rightarrow \quad m \in (10, \infty)$

(xiii) Both the roots lie in the interval (1, 2). Then,

$D \geq 0 \Rightarrow (m - 1)(m - 9) \geq 0 \Rightarrow m \leq 1$ or $m \geq 9$ (1)

Also

$f(1) > 0$ and $f(2) > 0 \Rightarrow 10 > m$ (2)

and

$1 < -\dfrac{b}{2a} < 2 \Rightarrow 5 < m < 7$ (3)

Thus, no such m exists.

(xiv) **Case I:** Exactly one root lies in (1, 2). So,

$f(1)f(2) < 0 \Rightarrow m > 10$

Case II: Both the roots lie in (1, 2). So, from (xiii), $m \in \phi$. Hence, $m \in (10, \infty)$.

(xv) For one root greater than 2 and the other root smaller than 1,

$f(1) < 0$ (1)

$f(2) < 0$ (2)

From (1), $f(1) < 0$, but $f(1) = 4$, which is not possible. Thus, no such m exists.

Example 1.146 **Find the values of a for which the equation $\sin^4 x + a \sin^2 x + 1 = 0$ will have a solution.**

Sol. Let

$t = \sin^2 x \Rightarrow t \in [0, 1]$

Hence, $t^2 + at + 1 = 0$ should have at least one solution in [0, 1]. Since product of roots is positive and equal to one, $t^2 + at + 1 = 0$ must have exactly one root in [0, 1]. Hence,

$f(1) < 0$

$\Rightarrow \quad 2 + a < 0$

$\Rightarrow \quad a \in (-\infty, -2)$

Example 1.147 **If $(x^2 + x + 2)^2 - (a - 3)(x^2 + x + 1)(x^2 + x + 2) + (a - 4)(x^2 + x + 1)^2 = 0$ has at least one root, then find the complete set of values of a.**

Sol. Let,

$$t = x^2 + x + 1 \Rightarrow t \in \left[\frac{3}{4}, \infty\right)$$

Hence,

$(t + 1)^2 - (a - 3)\, t(t + 1) + (a - 4)\, t^2 = 0$

$\Rightarrow \quad t^2 + 2t + 1 - (a - 3)(t^2 + t) + (a - 4)t^2 = 0$

$\Rightarrow \quad t(2 - a + 3) + 1 = 0$

$\Rightarrow \quad t = \dfrac{1}{(a - 5)}$

$\Rightarrow \quad \dfrac{1}{a - 5} \geq \dfrac{3}{4}$

$\Rightarrow \quad \dfrac{19 - 3a}{(a - 5)} \geq 0$

$\Rightarrow \quad a \in \left(5, \dfrac{19}{3}\right]$

Example 1.148 **If α is a real root of the quadratic equation $ax^2 + bx + c = 0$ and β is a real root of $-ax^2 + bx + c = 0$,**

then show that there is a root γ of the equation $(a/2)x^2 + bx + c = 0$ which lies between α and β.

Sol. Let,

$$f(x) = \frac{a}{2}x^2 + bx + c$$

$$\Rightarrow \quad f(\alpha) = \frac{a}{2}\alpha^2 + b\alpha + c$$

$$= a\alpha^2 + b\alpha + c - \frac{a}{2}\alpha^2$$

$$= -\frac{a}{2}\alpha^2 \qquad (\because \alpha \text{ is a root of } ax^2 + bx + c = 0)$$

$$f(\beta) = \frac{a}{2}\beta^2 + b\beta + c$$

$$= -a\beta^2 + b\beta + c + \frac{3}{2}a\beta^2$$

$$= \frac{3}{2}a\beta^2 \qquad (\because \beta \text{ is a root of } -ax^2 + bx + c = 0)$$

Now,

$$f(\alpha)f(\beta) = \frac{-3}{4}a^2\alpha^2\beta^2 < 0$$

Hence, $f(x) = 0$ has one real root between α and β.

Example 1.149 **For what real values of a do the roots of the equation $x^2 - 2x - (a^2 - 1) = 0$ lie between the roots of the equation $x^2 - 2(a + 1)x + a(a - 1) = 0$.**

Sol. $x^2 - 2x - (a^2 - 1) = 0$ (1)

$x^2 - 2(a + 1)x + a(a - 1) = 0$ (2)

From Eq. (1),

$$x = \frac{2 \pm \sqrt{4 + 4(a^2 - 1)}}{2} = 1 \pm a$$

Now, roots of Eq. (1) lie between roots of Eq. (2). Hence, graphs of expressions for Eqs. (1) and (2) are as follows:

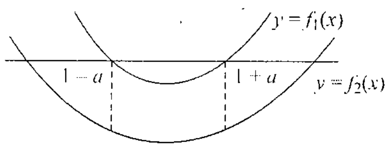


Fig. 1.61

$$f_1(x) = x^2 - 2x - (a^2 - 1)$$

$$f_2(x) = x^2 - 2(a + 1)x + a(a - 1)$$

From the graph, we have

$f_2(1 - a) < 0$ and $f_2(1 + a) < 0$

$\Rightarrow \quad (1 - a)^2 - 2(a + 1)(1 - a) + a(a - 1) < 0$

$\Rightarrow \quad (1 - a)[(1 - a) - 2a - 2 - a] < 0$

$\Rightarrow \quad (1 - a)(-4a - 1) < 0$

$\Rightarrow \quad (a - 1)(4a + 1) < 0$

$\Rightarrow \quad -\frac{1}{4} < a < 1$ (3)

and

$\Rightarrow \quad (1 + a)^2 - 2(a + 1)(a + 1) + a(a - 1) < 0$

$\Rightarrow -(a + 1)^2 + a(a - 1) < 0$

$\Rightarrow \quad -a^2 - 2a - 1 + a^2 - a < 0$

$\Rightarrow \quad 3a + 1 > 0$

$\Rightarrow \quad a > -\frac{1}{3}$ (4)

From (3) and (4), the required values of a lies in the range $-1/4 < a < 1$.

SOLVING INEQUALITIES USING LOCATION OF ROOTS

Example 1.150 **Find the value of a for which $ax^2 + (a - 3)x + 1 < 0$ for at least one positive real x.**

Sol. Let $f(x) = ax^2 + (a - 3)x + 1$

Case I:

If $a > 0$, then $f(x)$ will be negative only for those values of x which lie between the roots. From the graphs, we can see that $f(x)$ will be less than zero for at least one positive real x, when $f(x) = 0$ has distinct roots and at least one of these roots is a positive real root.

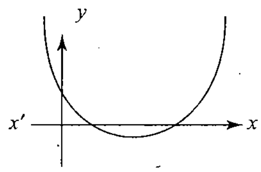


Fig. 1.62

Since $f(0) = 1 > 0$, the favourable graph according to the question is shown in the figure given above. From the graph, we can see that both the roots are non-negative. For this,

(i) $D > 0 \Rightarrow (a - 3)^2 - 4a > 0$

$\Rightarrow \quad a < 1$ or $a > 9$ (1)

(ii) sum > 0 and product ≥ 0

$\Rightarrow \quad -(a - 3) > 0$ and $1/a > 0$

$\Rightarrow \quad 0 < a < 3$ (2)

From (1) and (2), we have

$a \in (0, 1)$

Case II: $a < 0$

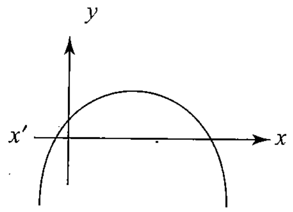


Fig. 1.63

Since $f(0) = 1 > 0$, then graph is as shown in the figure, which shows that $ax^2 + (a - 3)x + 1 < 0$, for at least one positive x.

Case III: $a = 0$

If $a = 0$,

$$f(x) = -3x + 1$$

$$\Rightarrow f(x) < 0, \forall x > 1/3$$

Thus, from all the cases, the required set of values of a is $(-\infty, 1)$.

Example 1.151 **If $x^2 + 2ax + a < 0 \ \forall x \in [1, 2]$, then find the values of a.**

Sol. Given,

$$x^2 + 2ax + a < 0, \forall x \in [1, 2]$$

Hence, 1 and 2 lie between the roots of the equation $x^2 + 2ax + a = 0$,

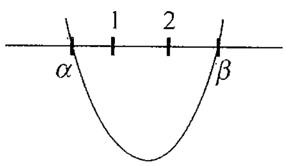


Fig. 1.64

$$\Rightarrow f(1) < 0 \text{ and } f(2) < 0$$

$$\Rightarrow 1 + 2a + a < 0, 4 + 4a + a < 0$$

$$\Rightarrow a < -\frac{1}{3}, a < -\frac{4}{5}$$

$$\Rightarrow a \in \left(-\infty, -\frac{4}{5}\right)$$

Example 1.152 **If $(y^2 - 5y + 3)(x^2 + x + 1) < 2x$ for all $x \in R$, then find the interval in which y lies.**

Sol. $(y^2 - 5y + 3)(x^2 + x + 1) < 2x, \forall x \in R$

$$\Rightarrow y^2 - 5y + 3 < \frac{2x}{x^2 + x + 1} \quad (\because \ x^2 + x + 1 > 0 \ \forall x \in R)$$

L.H.S. must be less than the least value of R.H.S. Now lets find the range of R.H.S.

Let

$$\frac{2x}{x^2 + x + 1} = p$$

$$\Rightarrow px^2 + (p - 2)x + p = 0$$

Since x is real,

$$(p - 2)^2 - 4p^2 \geq 0$$

$$\Rightarrow -2 \leq p \leq \frac{2}{3}$$

The minimum value of $2x/(x^2 + x + 1)$ is -2. So,

$$y^2 - 5y + 3 < -2$$

$$\Rightarrow y^2 - 5y + 5 < 0$$

$$\Rightarrow y \in \left(\frac{5 - \sqrt{5}}{2}, \frac{5 + \sqrt{5}}{2}\right)$$

Example 1.153 **Find the values of a for which $4^t - (a - 4)2^t + (9/4)a < 0, \forall t \in (1, 2)$.**

Sol. Let $2^t = x$ and $f(x) = x^2 - (a - 4)x + (9/4)a$. We want $f(x) < 0, \forall x \in (2^1, 2^2)$, i.e., $\forall x \in (2, 4)$.

(i) Since coefficient of x^2 in $f(x)$ is positive, $f(x) < 0$ for some x only when roots of $f(x) = 0$ are real and distinct. So,

$$D > 0$$

$$\Rightarrow a^2 - 17a + 16 > 0 \quad (1)$$

(ii) Since we want $f(x) < 0 \ \forall x \in (2, 4)$, one of the roots of $f(x) = 0$ should be smaller than 2 and the other must be greater than 4, i.e.,

$$f(2) < 0 \text{ and } f(4) < 0$$

$$\Rightarrow a < -48 \text{ and } a > 128/7$$

which is not possible. Hence, no such a exists.

Concept Application Exercise 1.10

1. Find the values of a if $x^2 - 2(a - 1)x + (2a + 1) = 0$ has positive roots.
2. If the equation $(a - 5)x^2 + 2(a - 10)x + a + 10 = 0$ has roots of opposite sign, then find the values of a.
3. If both the roots of $x^2 - ax + a = 0$ are greater than 2, then find the values of a.
4. If both the roots of $ax^2 + ax + 1 = 0$ are less than 1, then find exhaustive range of values of a.
5. If both the roots of $x^2 + ax + 2 = 0$ lies in the interval (0, 3), then find exhaustive range of values of a.
6. If α, β are the roots of $x^2 - 3x + a = 0$, $a \in R$ and $\alpha < 1 < \beta$, then find the values of a.
7. If α is the root (having the least absolute value) of the equation $x^2 - bx - 1 = 0$ $(b \in R^+)$, then prove that $-1 < \alpha < 0$.
8. If $a < b < c < d$, then show that the quadratic equation $\mu(x - a)(x - c) + \lambda(x - b)(x - d) = 0$ has real roots for all real μ and λ.

EXERCISES

Subjective Type

Solutions on page 1.60

1. Solve the following:

$$\left(\sqrt{x^2-5x+6}+\sqrt{x^2-5x+4}\right)^{x/2}+\left(\sqrt{x^2-5x+6}-\sqrt{x^2-5x+4}\right)^{x/2}=2^{\frac{x+4}{4}}$$

2. Show that the equation $A^2/(x-a)+B^2/(x-b)+C^2/(x-c)+\cdots+H^2/(x-h)=k$ has no imaginary root, where $A, B, C, \ldots, H$ and $a, b, c, \ldots, h$ and $k \in R$.

3. Given that a, b, c are distinct real numbers such that the expression ax^2+bx+c, bx^2+cx+a and cx^2+ax+b are always non-negative. Prove that the quantity $(a^2+b^2+c^2)/(ab+bc+ca)$ can never lie in $(-\infty, 1] \cup [4, \infty)$.

4. Find the number of quadratic equations, which are unchanged by squaring their roots.

5. If α and β are the roots of $x^2-p(x+1)-c=0$, then show that $(\alpha+1)(\beta+1)=1-c$. Hence, prove that

$$\frac{\alpha^2+2\alpha+1}{\alpha^2+2\alpha+c}+\frac{\beta^2+2\beta+1}{\beta^2+2\beta+c}=1$$

6. If α, β are the roots of the equation $ax^2+bx+c=0$ and $S_n=\alpha^n+\beta^n$, then show that $aS_{n+1}+bS_n+cS_{n-1}=0$ and hence find S_5.

7. If α be a root of the equation $4x^2+2x-1=0$, then prove that $4\alpha^3-3\alpha$ is the other root.

8. If $(ax^2+bx+c)y+(a'x^2+b'x+c')=0$ and x is a rational function of y, then prove that $(ac'-a'c)^2=(ab'-a'b)\times(bc'-b'c)$.

9. If the roots of the equation $x^2-ax+b=0$ are real and differ by a quantity which is less than c $(c>0)$, then show that b lies between $(a^2-c^2)4$ and $a^2/4$.

10. The equation $ax^2+bx+c=0$ has real and positive roots. Prove that the roots of the equation $a^2x^2+a(3b-2c)x+(2b-c)(b-c)+ac=0$ are real and positive.

11. If $x^2+px-444p=0$ has integral roots where p is a prime number, then find the value(s) of p.

12. If a and c are odd prime numbers and $ax^2+bx+c=0$ has rational roots where $b \in I$. Prove that one root of the equation will be independent of a, b, c.

13. If $2x^2-3xy-2y^2=7$, then prove that there will be only two integral pairs (x, y) satisfying the above relation.

14. Let $a, b \in N$ and $a>1$. Also p is a prime number. If $ax^2+bx+c=p$ for two distinct integral values of x, then prove that $ax^2+bx+c \neq 2p$ for any integral value of x.

15. Show that minimum value of $(x+a)(x+b)/(x+c)$, where $a>c$, $b>c$ is $(\sqrt{a-c}+\sqrt{b-c})^2$ for real values of $x>-c$.

16. If $x \in R$, then prove that maximum value of $2(a-x)(x+\sqrt{x^2+b^2})$ is a^2+b^2.

17. If $f(x)=x^3+bx^2+cx+d$ and $f(0), f(-1)$ are odd integers, prove that $f(x)=0$ cannot have all integral roots.

18. Find the values of k for which

$$\left|\frac{x^2+kx+1}{x^2+x+1}\right|<2, \forall x \in R$$

19. Solve the equation $\sqrt{a(2^x-2)+1}=1-2^x, x \in R$.

20. For $a<0$, determine all real roots of the equation $x^2-2a|x-a|-3a^2=0$.

21. Find the integral part of the greatest root of equation $x^3-10x^2-11x-100=0$.

22. Find the values of a for which all the roots of the equation $x^4-4x^3-8x^2+a=0$ are real.

Objective Type

Solutions on page 1.64

Each question has four choices a, b, c and d, out of which *only one* is correct. Find the correct answer.

1. If $x=2+2^{2/3}+2^{1/3}$, then the value of x^3-6x^2+6x is
   a. 3 **b.** 2 **c.** 1 **d.** –2

2. The least value of the expression $x^2+4y^2+3z^2-2x-12y-6z+14$ is
   a. 1 **b.** no least value **c.** 0 **d.** none of these

3. Number of positive integers n for which n^2+96 is a perfect square is
   a. 8 **b.** 12 **c.** 4 **d.** infinite

4. If $x, y \in R$ satisfy the equation $x^2+y^2-4x-2y+5=0$, then the value of the expression $[(\sqrt{x}-\sqrt{y})^2+4\sqrt{xy}]/(x+\sqrt{xy})$ is
   a. $\sqrt{2}+1$ **b.** $\frac{\sqrt{2}+1}{2}$
   c. $\frac{\sqrt{2}-1}{2}$ **d.** $\frac{\sqrt{2}+1}{\sqrt{2}}$

5. The number of real roots of the equation $x^2-3|x|+2=0$ is
   a. 2 **b.** 1 **c.** 4 **d.** 3

6. If $x=1+i$ is a root of the equation $x^3-ix+1-i=0$, then the other real root is
   a. 0 **b.** 1 **c.** –1 **d.** none of these

7. The number of roots of the equation $\sqrt{x-2}(x^2-4x+3)=0$ is
   a. three **b.** four
   c. one **d.** two

8. The curve $y=(\lambda+1)x^2+2$ intersects the curve $y=\lambda x+3$ in exactly one point, if λ equals
   a. $\{-2, 2\}$ **b.** $\{1\}$

c. {−2}   d. {2}

9. If the expression $x^2 + 2(a + b + c)x + 3(bc + ca + ab)$ is a perfect square, then
a. $a = b = c$ b. $a = \pm b = \pm c$
c. $a = b \neq c$ d. none of these

10. If one root of the equation $ax^2 + bx + c = 0$ is square of the other, then $a(c - b)^3 = cX$, where X is
a. $a^3 - b^3$ b. $a^3 + b^3$
c. $(a - b)^3$ d. none of these

11. If $x^2 + px + 1$ is a factor of the expression $ax^3 + bx + c$, then
a. $a^2 - c^2 = ab$ b. $a^2 + c^2 = -ab$
c. $a^2 - c^2 = -ab$ d. none of these

12. Sum of the non-real roots of $(x^2 + x - 2)(x^2 + x - 3) = 12$ is
a. −1   b. 1
c. −6   d. 6

13. If $(ax^2 + c)y + (a'x^2 + c') = 0$ and x is a rational function of y and ac is negative, then
a. $ac' + a'c = 0$ b. $a/a' = c/c'$
c. $a^2 + c^2 = a'^2 + c'^2$ d. $aa' + cc' = 1$

14. Let p and q be roots of the equation $x^2 - 2x + A = 0$ and let r and s be the roots of the equation $x^2 - 18x + B = 0$. If $p < q < r < s$ are in arithmetic progression, then the values of A and B are
a. 3, −77   b. 3, 77
c. −3, −77   d. −3, 77

15. The number of irrational roots of the equation $4x/(x^2 + x + 3) + 5x/(x^2 - 5x + 3) = -3/2$ is
a. 4   b. 0
c. 1   d. 2

16. Let a, b and c be real numbers such that $4a + 2b + c = 0$ and $ab > 0$. Then the equation $ax^2 + bx + c = 0$ has
a. complex roots   b. exactly one root
c. real roots   d. none of these

17. If α, β are the roots of the equation $x^2 - 2x + 3 = 0$. Then the equation whose roots are $P = \alpha^3 - 3\alpha^2 + 5\alpha - 2$ and $Q = \beta^3 - \beta^2 + \beta + 5$ is
a. $x^2 + 3x + 2 = 0$ b. $x^2 - 3x - 2 = 0$
c. $x^2 - 3x + 2 = 0$ d. none of these

18. If α, β be the roots of the equation $2x^2 - 35x + 2 = 0$, then the value of $(2\alpha - 35)^3 (2\beta - 35)^3$ is equal to
a. 8   b. 1
c. 64   d. none of these

19. If a, b, c are three distinct positive real numbers, then the number of real roots of $ax^2 + 2b|x| - c = 0$ is
a. 0   b. 4
c. 2   d. none of these

20. If $p(q - r)x^2 + q(r - p)x + r(p - q) = 0$ has equal roots, then $2/q =$
a. $\frac{1}{p} + \frac{1}{r}$ b. $p + r$
c. $p^2 + r^2$ d. $\frac{1}{p^2} + \frac{1}{r^2}$

21. Let $a \neq 0$ and $p(x)$ be a polynomial of degree greater than 2. If $p(x)$ leaves remainders a and $-a$ when divided respectively by $x + a$ and $x - a$, the remainder when $p(x)$ is divided by $x^2 - a^2$ is
a. $2x$ b. $-2x$
c. x d. $-x$

22. The quadratic $x^2 + ax + b + 1 = 0$ has roots which are positive integers, then $(a^2 + b^2)$ can be equal to
a. 50   b. 37
c. 61   d. 19

23. The sum of values of x satisfying the equation $(31 + 8\sqrt{15})^{x^2-3} + 1 = (32 + 8\sqrt{15})^{x^2-3}$ is
a. 3   b. 0
c. 2   d. none of these

24. If α, β be the roots of the equation $ax^2 + bx + c = 0$, then value of $(a\alpha^2 + c)/(a\alpha + b) + (a\beta^2 + c)/(a\beta + b)$ is
a. $\frac{b(b^2 - 2ac)}{4a}$ b. $\frac{b^2 - 4ac}{2a}$
c. $\frac{b(b^2 - 2ac)}{a^2 c}$ d. none of these

25. A quadratic equation whose product of roots x_1 and x_2 is equal to 4 and satisfying the relation $x_1/(x_1 - 1) + x_2/(x_2 - 1) = 2$ is
a. $x^2 - 2x + 4 = 0$ b. $x^2 + 2x + 4 = 0$
c. $x^2 + 4x + 4 = 0$ d. $x^2 - 4x + 4 = 0$

26. If $a, b, c, d \in R$, then the equation $(x^2 + ax - 3b)(x^2 - cx + b)(x^2 - dx + 2b) = 0$ has
a. 6 real roots   b. at least 2 real roots
c. 4 real roots   d. 3 real roots

27. If α and β are the roots of the equation $x^2 + px + q = 0$, and α^4 and β^4 are the roots of $x^2 - rx + q = 0$, then the roots of $x^2 - 4qx + 2q^2 - r = 0$ are always
a. both non-real   b. both positive
c. both negative   d. opposite in sign

28. If the roots of the equation $(a - 1)(x^2 + x + 1)^2 = (a + 1)(x^4 + x^2 + 1)$ are real and distinct then the value of $a \in$
a. $(-\infty, 3]$ b. $(-\infty, -2) \cup (2, \infty)$
c. $[-2, 2]$ d. $[-3, \infty)$

29. If $b_1 b_2 = 2(c_1 + c_2)$, then at least one of the equations $x^2 + b_1 x + c_1 = 0$ and $x^2 + b_2 x + c_2 = 0$ has
a. imaginary roots   b. real roots
c. purely imaginary roots   d. none of these

30. The integral values of m for which the roots of the equation $mx^2 + (2m - 1)x + (m - 2) = 0$ are rational are given by the expression [where n is integer]

a. n^2 b. $n(n + 2)$
c. $n(n + 1)$ d. none of these

31. Suppose A, B, C are defined as $A = a^2b + ab^2 - a^2c - ac^2$, $B = b^2c + bc^2 - a^2b - ab^2$ and $C = a^2c + ac^2 - b^2c - bc^2$, where $a > b > c > 0$ and the equation $Ax^2 + Bx + C = 0$ has equal roots, then a, b, c are in

a. A.P. b. G.P.
c. H.P. d. A.G.P.

32. The coefficient of x in the equation $x^2 + px + q = 0$ was wrongly written as 17 in place of 13 and the roots thus found was −2 and −15. Then the roots of the correct equation are

a. −3, 10 b. −3, −10
c. 3, −10 d. none of these .

33. If $a(p + q)^2 + 2bpq + c = 0$ and $a(p + r)^2 + 2bpr + c = 0$ $(a \neq 0)$, then

a. $qr = p^2$ b. $qr = p^2 + \frac{c}{a}$
c. $qr = -p^2$ d. none of these

34. If the roots of the equation $ax^2 - bx + c = 0$ are α, β then the roots of the equation $b^2cx^2 - ab^2x + a^3 = 0$ are

a. $\frac{1}{\alpha^3+\alpha\beta}, \frac{1}{\beta^3+\alpha\beta}$ b. $\frac{1}{\alpha^2+\alpha\beta}, \frac{1}{\beta^2+\alpha\beta}$
c. $\frac{1}{\alpha^4+\alpha\beta}, \frac{1}{\beta^4+\alpha\beta}$ d. none of these

35. If α and β, α and γ, α and δ are the roots of the equations $ax^2 + 2bx + c = 0$, $2bx^2 + cx + a = 0$ and $cx^2 + ax + 2b = 0$, respectively, where a, b and c are positive real numbers, then $\alpha + \alpha^2 =$

a. abc b. $a + 2b + c$
c. −1 d. 0

36. $x^2 - xy + y^2 - 4x - 4y + 16 = 0$ represents

a. a point b. a circle
c. a pair of straight lines d. none of these

37. If α, β be the non-zero roots of $ax^2 + bx + c = 0$ and α^2, β^2 be the roots of $a^2x^2 + b^2x + c^2 = 0$, then a, b, c are in

a. G.P. b. H.P.
c. A.P. d. none of these

38. If the roots of the equation $ax^2 + bx + c = 0$ are of the form $(k+1)/k$ and $(k+2)/(k+1)$, then $(a + b + c)^2$ is equal to

a. $2b^2 - ac$ b. Σa^2
c. $b^2 - 4ac$ d. $b^2 - 2ac$

39. If α, β are the roots of $ax^2 + bx + c = 0$ and $\alpha + h, \beta + h$ are the roots of $px^2 + qx + r = 0$, then $h =$

a. $-\frac{1}{2}\left(\frac{a}{b}-\frac{p}{q}\right)$ b. $\left(\frac{b}{a}-\frac{q}{p}\right)$
c. $\frac{1}{2}\left(\frac{b}{a}-\frac{q}{p}\right)$ d. none of these

40. If α, β be the roots of the equation $(x - a)(x - b) + c = 0$ $(c \neq 0)$, then the roots of the equation $(x - c - \alpha)(x - c - \beta) = c$ are

a. $a + c$ and $b + c$ b. $a - c$ and $b - c$
c. a and $b + c$ d. $a + c$ and b

41. If α, β are the roots of $ax^2 + c = bx$, then the equation $(a + cy)^2 = b^2y$ in y has the roots

a. $\alpha\beta^{-1}, \alpha^{-1}\beta$ b. α^{-2}, β^{-2}
c. α^{-1}, β^{-1} d. α^2, β^2

42. If the roots of the equation, $x^2 + 2ax + b = 0$, are real and distinct and they differ by at most $2m$, then b lies in the interval

a. $(a^2, a^2 + m^2)$ b. $(a^2 - m^2, a^2)$
c. $[a^2 - m^2, a^2)$ d. none of these

43. If the ratio of the roots of $ax^2 + 2bx + c = 0$ is same as the ratio of the $px^2 + 2qx + r = 0$, then

a. $\frac{2b}{ac}=\frac{q^2}{pr}$ b. $\frac{b}{ac}=\frac{q}{pr}$
c. $\frac{b^2}{ac}=\frac{q^2}{pr}$ d. none of these

44. If one root of $x^2 - x - k = 0$ is square of the other, then $k =$

a. $2\pm\sqrt{5}$ b. $2\pm\sqrt{3}$
c. $3\pm\sqrt{2}$ d. $5\pm\sqrt{2}$

45. If α and β be the roots of the equation $x^2 + px - 1/(2p^2) = 0$ where $p \in R$. Then the minimum value of $\alpha^4 + \beta^4$ is

a. $2\sqrt{2}$ b. $2-\sqrt{2}$
c. 2 d. $2+\sqrt{2}$

46. If α, β are real and α^2, β^2 are the roots of the equation $a^2x^2 + x + 1 - a^2 = 0$ $(a > 1)$, then $\beta^2 =$

a. a^2 b. $1-\frac{1}{a^2}$
c. $1 - a^2$ d. $1 + a^2$

47. If α and β are the roots of the equation $x^2 - ax + b = 0$ and $A_n = \alpha^n + \beta^n$, then which of the following is true?

a. $A_{n+1} = aA_n + bA_{n-1}$ b. $A_{n+1} = bA_n + aA_{n-1}$
c. $A_{n+1} = aA_n - bA_{n-1}$ d. $A_{n+1} = bA_n - aA_{n-1}$

48. The value of m for which one of the roots of $x^2 - 3x + 2m = 0$ is double of one of the roots of $x^2 - x + m = 0$ is

a. −2 b. 1
c. 2 d. none of these

49. If the equations $ax^2 + bx + c = 0$ and $x^3 + 3x^2 + 3x + 2 = 0$ have two common roots, then

a. $a = b = c$ b. $a = b \neq c$
c. $a = -b = c$ d. none of these

50. Number of values of a for which equations $x^3 + ax + 1 = 0$ and $x^4 + ax^2 + 1 = 0$ have a common root

a. 0 b. 1
c. 2 d. infinite

51. Let $p(x) = 0$ be a polynomial equation of the least possible degree, with rational coefficients, having $\sqrt[3]{7}+\sqrt[3]{49}$ as one of its roots. Then the product of all the roots of $p(x) = 0$ is

a. 56 **b.** 63
c. 7 **d.** 49

52. If $\alpha, \beta, \gamma, \sigma$ are the roots of the equation $x^4 + 4x^3 - 6x^2 + 7x - 9 = 0$, then the value of $(1 + \alpha^2)(1 + \beta^2)(1 + \gamma^2)(1 + \sigma^2)$ is

a. 9 **b.** 11
c. 13 **d.** 5

53. If $(m_r, 1/m_r)$, $r = 1, 2, 3, 4$ be four pairs of values of x and y that satisfy the equation $x^2 + y^2 + 2gx + 2fy + c = 0$, then value of $m_1 m_2 m_3 m_4$ is

a. 0 **b.** 1
c. -1 **d.** none of these

54. If roots of an equation $x^n - 1 = 0$ are $1, a_1, a_2, \ldots, a_{n-1}$, then the value of $(1 - a_1)(1 - a_2)(1 - a_3)\cdots(1 - a_{n-1})$ will be

a. n **b.** n^2
c. n^n **d.** 0

55. If $\tan\theta_1, \tan\theta_2, \tan\theta_3$ are the real roots of the $x^3 - (a + 1)x^2 + (b - a)x - b = 0$, where $\theta_1 + \theta_2 + \theta_3 \in (0, \pi)$, then $\theta_1 + \theta_2 + \theta_3$ is equal to

a. $\pi/2$ **b.** $\pi/4$
c. $3\pi/4$ **d.** π

56. If α, β, γ are the roots of $x^3 - x^2 - 1 = 0$ then the value of $(1+\alpha)/(1-\alpha)+(1+\beta)/(1-\beta)+(1+\gamma)/(1-\gamma)$ is equal to

a. -5 **b.** -6
c. -7 **d.** -2

57. Let r, s and t be the roots of the equation, $8x^3 + 1001x + 2008 = 0$. The value of $(r + s)^3 + (s + t)^3 + (t + r)^3$ is

a. 251 **b.** 751
c. 735 **d.** 753

58. If x be real, then $x/(x^2-5x+9)$ lies between

a. -1 and $-1/11$ **b.** 1 and $-1/11$
c. 1 and 1/11 **d.** none of these

59. If x is real, then the maximum value of $(3x^2+9x+17)/(3x^2+9x+7)$ is

a. 1/4 **b.** 41
c. 1 **d.** 17/7

60. If $a, b \in R$, $a \neq 0$ and the quadratic equation $ax^2 - bx + 1 = 0$ has imaginary roots then $(a + b + 1)$ is

a. positive **b.** negative
c. zero **d.** dependent on the sign of b

61. If the expression $[mx-1+(1/x)]$ is non-negative for all positive real x, then the minimum value of m must be

a. $-1/2$ **b.** 0
c. 1/4 **d.** 1/2

62. Suppose that $f(x)$ is a quadratic expression positive for all real x. If $g(x) = f(x) + f'(x) + f''(x)$, then for any real x (where $f'(x)$ and $f''(x)$ represent 1st and 2nd derivative respectively)

a. $g(x) < 0$ **b.** $g(x) > 0$
c. $g(x) = 0$ **d.** $g(x) \geq 0$

63. Let $f(x) = ax^2 - bx + c^2$, $b \neq 0$ and $f(x) \neq 0$ for all $x \in R$. Then

a. $a + c^2 < b$ **b.** $4a + c^2 > 2b$
c. $9a - 3b + c^2 < 0$ **d.** none of these

64. x_1 and x_2 are the roots of $ax^2 + bx + c = 0$ and $x_1 x_2 < 0$. Roots of $x_1(x - x_2)^2 + x_2(x - x_1)^2 = 0$ are

a. real and of opposite sign **b.** negative
c. positive **d.** non-real

65. If a, b, c, d are four consecutive terms of an increasing A.P. then the roots of the equation $(x - a)(x - c) + 2(x - b)(x - d) = 0$ are

a. non-real complex **b.** real and equal
c. integers **d.** real and distinct

66. If roots of $x^2 - (a - 3)x + a = 0$ are such that at least one of them is greater than 2, then

a. $a \in [7, 9]$ **b.** $a \in [7, \infty)$
c. $a \in [9, \infty)$ **d.** $a \in [7, 9)$

67. Let $f(x) = ax^2 + bx + c$, $a, b, c \in R$. If $f(x)$ takes real values for real values of x and non-real values for non-real values of x, then

a. $a = 0$
b. $b = 0$
c. $c = 0$
d. nothing can be said about a, b, c.

68. All the values of m for which both the roots of the equation $x^2 - 2mx + m^2 - 1 = 0$ are greater than -2 but less than 4, lie in the interval

a. $-2 < m < 0$ **b.** $m > 3$
c. $-1 < m < 3$ **d.** $1 < m < 4$

69. If the roots of the quadratic equation $(4p - p^2 - 5)x^2 - (2p - 1)x + 3p = 0$ lie on either side of unity, then the number of integral values of p is

a. 1 **b.** 2
c. 3 **d.** 4

70. The interval of a for which the equation $\tan^2 x - (a - 4)\tan x + 4 - 2a = 0$ has at least one solution $\forall\, x \in [0, \pi/4]$

a. $a \in (2, 3)$ **b.** $a \in [2, 3]$
c. $a \in (1, 4)$ **d.** $a \in [1, 4]$

71. The range of a for which the equation $x^2 + ax - 4 = 0$ has its smaller root in the interval $(-1, 2)$ is

a. $(-\infty, -3)$ **b.** $(0, 3)$

c. $(0, \infty)$ d. $(-\infty, -3) \cup (0, \infty)$

72. If both roots of the equation $ax^2 + x + c - a = 0$ are imaginary and $c > -1$, then

a. $3a > 2 + 4c$ b. $3a < 2 + 4c$

c. $c < a$ d. none of these

73. The set of all possible real values of a such that the inequality $(x - (a - 1))(x - (a^2 + 2)) < 0$ holds for all $x \in (-1, 3)$ is

a. $(0, 1)$ b. $(\infty, -1]$

c. $(-\infty, -1)$ d. $(1, \infty)$

74. Consider the equation $x^2 + 2x - n = 0$, where $n \in N$ and $n \in [5, 100]$. Total number of different values of 'n' so that the given equation has integral roots is

a. 8 b. 3

c. 6 d. 4

75. Total number of values of a so that $x^2 - x - a = 0$ has integral roots, where $a \in N$ and $6 \le a \le 100$, is equal to

a. 2 b. 4

c. 6 d. 8

76. Total number of integral values of 'a' so that $x^2 - (a + 1)x + a - 1 = 0$ has integral roots is equal to

a. 1 b. 2

c. 4 d. none of these

77. The number of values of k for which $[x^2 - (k - 2)x + k^2] \times [x^2 + kx + (2k - 1)]$ is a perfect square is

a. 2 b. 1

c. 0 d. none of these

78. If α, β are the roots of $x^2 + px + q = 0$ and $x^{2n} + p^n x^n + q^n = 0$ and if $(\alpha/\beta), (\beta/\alpha)$ are the roots of $x^n + 1 + (x + 1)^n = 0$, then $n (\in N)$

a. must be an odd integer b. may be any integer

c. must be an even integer d. cannot say anything

79. The number of positive integral solutions of $x^4 - y^4 = 3789108$ is

a. 0 b. 1

c. 2 d. 4

80. If $xy = 2(x + y)$, $x \le y$ and $x, y \in N$, then the number of solutions of the equation are

a. two b. three

c. no solution d. infinitely many solutions

81. If α, β, γ are such that $\alpha + \beta + \gamma = 2$, $\alpha^2 + \beta^2 + \gamma^2 = 6$, $\alpha^3 + \beta^3 + \gamma^3 = 8$, then $\alpha^4 + \beta^4 + \gamma^4$ is

a. 18 b. 10

c. 15 d. 36

82. The number of integral values of a for which the quadratic equation $(x + a)(x + 1991) + 1 = 0$ has integral roots are

a. 3 b. 0

c. 1 d. 2

83. The number of real solutions of the equation $(9/10)^x = -3 + x - x^2$ is

a. 2 b. 0

c. 1 d. none of these

84. If a, b and c are real numbers such that $a^2 + b^2 + c^2 = 1$, then $ab + bc + ca$ lies in the interval

a. $[1/2, 2]$ b. $[-1, 2]$

c. $[-1/2, 1]$ d. $[-1, 1/2]$

85. If the equation $\cot^4 x - 2 \operatorname{cosec}^2 x + a^2 = 0$ has at least one solution then, sum of all possible integral values of a is equal to

a. 4 b. 3

c. 2 d. 0

86. Let x, y, z, t be real numbers $x^2 + y^2 = 9$, $z^2 + t^2 = 4$ and $xt - yz = 6$. Then the greatest value of $P = xz$ is

a. 2 b. 3

c. 4 d. 6

87. If a, b, c be distinct positive numbers, then the nature of roots of the equation $1/(x-a) + 1/(x-b) + 1/(x-c) = 1/x$ is

a. all real and distinct

b. all real and at least two are distinct

c. at least two real

d. all non-real

88. If $(b^2 - 4ac)^2 (1 + 4a^2) < 64a^2$, $a < 0$, then maximum value of quadratic expression $ax^2 + bx + c$ is always less than

a. 0 b. 2

c. -1 d. -2

89. For $x^2 - (a + 3)|x| + 4 = 0$ to have real solutions, the range of a is

a. $(-\infty, -7] \cup [1, \infty)$ b. $(-3, \infty)$

c. $(-\infty, -7]$ d. $[1, \infty)$

90. If the quadratic equation $4x^2 - 2(a + c - 1)x + ac - b = 0$ $(a > b > c)$

a. both roots are greater than a

b. both roots are less than c

c. both roots lie between $c/2$ and $a/2$

d. exactly one of the roots lies between $c/2$ and $a/2$

91. If the equation $x^2 + ax + b = 0$ has distinct real roots and $x^2 + a|x| + b = 0$ has only one real root, then which of the following is true

a. $b = 0, a > 0$ b. $b = 0, a < 0$

c. $b > 0, a < 0$ d. $b < 0, a > 0$

92. The equation $2^{2x} + (a - 1)2^{x+1} + a = 0$ has roots of opposite signs then exhaustive set of values of a is

a. $a \in (-1, 0)$ **b.** $a < 0$
c. $a \in (-\infty, 1/3)$ **d.** $a \in (0, 1/3)$

93. If the equation $|x^2 + bx + c| = k$ has four real roots, then

a. $b^2 - 4c > 0$ and $0 < k < \frac{4c - b^2}{4}$

b. $b^2 - 4c < 0$ and $0 < k < \frac{4c - b^2}{4}$

c. $b^2 - 4c > 0$ and $k > \frac{4c - b^2}{4}$

d. none of these

94. $P(x)$ is a polynomial with integral coefficients such that for four distinct integers a, b, c, d; $P(a) = P(b) = P(c) = P(d) = 3$. If $P(e) = 5$ (e is an integer), then

a. $e = 1$ **b.** $e = 3$
c. $e = 4$ **d.** no real value of e

95. The number of integral values of x satisfying $\sqrt{-x^2 + 10x - 16} < x - 2$ is

a. 0 **b.** 1
c. 2 **d.** 3

96. If $x^2 + ax - 3x - (a + 2) = 0$ has real and distinct roots, then minimum value of $(a^2 + 1)/(a^2 + 2)$ is

a. 1 **b.** 0
c. $\frac{1}{2}$ **d.** $\frac{1}{4}$

97. The set of values of a for which $(a - 1)x^2 - (a + 1)x + a - 1 \geq 0$ is true for all $x \geq 2$

a. $(-\infty, 1)$ **b.** $\left(1, \frac{7}{3}\right)$
c. $\left(\frac{7}{3}, \infty\right)$ **d.** none of these

98. The value of the expression $x^4 - 8x^3 + 18x^2 - 8x + 2$ when $x = 2 + \sqrt{3}$

a. 2 **b.** 1
c. 0 **d.** 3

Multiple Correct Answers Type

Solutions on page 1.72

Each question has four choices a, b, c and d, out of which *one or more* answers are correct.

1. If $x, y \in R$ and $2x^2 + 6xy + 5y^2 = 1$, then

a. $|x| \leq \sqrt{5}$ **b.** $|x| \geq \sqrt{5}$
c. $y^2 \leq 2$ **d.** $y^2 \leq 4$

2. If the equation whose roots are the squares of the roots of the cubic $x^3 - ax^2 + bx - 1 = 0$ is identical with the given cubic equation, then

a. $a = 0, b = 3$
b. $a = b = 0$
c. $a = b = 3$
d. a, b are roots of $x^2 + x + 2 = 0$

3. If the equation $ax^2 + bx + c = 0$, $a, b, c \in R$ have non-real roots, then

a. $c(a - b + c) > 0$ **b.** $c(a + b + c) > 0$
c. $c(4a - 2b + c) > 0$ **d.** none of these

4. If $c \neq 0$ and the equation $p/(2x) = a/(x + c) + b/(x - c)$ has two equal roots, then p can be

a. $(\sqrt{a} - \sqrt{b})^2$ **b.** $(\sqrt{a} + \sqrt{b})^2$
c. $a + b$ **d.** $a - b$

5. If the equations $4x^2 - x - 1 = 0$ and $3x^2 + (\lambda + \mu)x + \lambda - \mu = 0$ have a root common then the rational values of λ and μ are

a. $\lambda = \frac{-3}{4}$ **b.** $\lambda = 0$
c. $\mu = \frac{3}{4}$ **d.** $\mu = 0$

6. If the equation $ax^2 + bx + c = 0$ ($a > 0$) has two real roots α and β such that $\alpha < -2$ and $\beta > 2$, then which of the following statements is/are true?

a. $a - |b| + c < 0$ **b.** $c < 0, b^2 - 4ac > 0$
c. $4a - 2|b| + c < 0$ **d.** $9a - 3|b| + c < 0$

7. If the following figure shows the graph of $f(x) = ax^2 + bx + c$, then

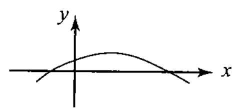


Fig. 1.65

a. $ac < 0$ **b.** $bc > 0$
c. $ab > 0$ **d.** $abc < 0$

8. If $\cos x - y^2 - \sqrt{y - x^2 - 1} \geq 0$, then

a. $y \geq 1$ **b.** $x \in R$
c. $y = 1$ **d.** $x = 0$

9. The value of x satisfying the equation $2^{2x} - 8 \times 2^x = -12$ is

a. $1 + \frac{\log 3}{\log 2}$ **b.** $\frac{1}{2}\log 6$
c. $1 + \log\frac{3}{2}$ **d.** 1

10. If α, β are the roots of the quadratic equation $ax^2 + bx + c = 0$ then which of the following expression will be the symmetric function of roots

a. $\left|\log\frac{\alpha}{\beta}\right|$ **b.** $\alpha^2\beta^5 + \beta^2\alpha^5$
c. $\tan(\alpha - \beta)$ **d.** $\left(\log\frac{1}{\alpha}\right)^2 + (\log\beta)^2$

11. If the quadratic equation $ax^2 + bx + c = 0$ ($a > 0$) has $\sec^2\theta$ and $\text{cosec}^2\theta$ as its roots, then which of the following must hold good?

a. $b + c = 0$ **b.** $b^2 - 4ac \geq 0$
c. $c \geq 4a$ **d.** $4a + b \geq 0$

12. Let $a, b, c \in Q^+$ satisfying $a > b > c$. Which of the following statement(s) hold true for the quadratic polynomial $f(x) = (a + b - 2c)x^2 + (b + c - 2a)x + (c + a - 2b)$?

a. The mouth of the parabola $y = f(x)$ opens upwards
b. Both roots of the equation $f(x) = 0$ are rational
c. x-coordinate of vertex of the graph is positive
d. Product of the roots is always negative

13. The graph of the quadratic trinomial $y = ax^2 + bx + c$ has its vertex at $(4, -5)$ and two x-intercepts one positive and one negative. Which of the following holds good?

a. $a > 0$ **b.** $b < 0$
c. $c < 0$ **d.** $8a = b$

14. If the roots of the equation, $x^3 + px^2 + qx - 1 = 0$ form an increasing G.P., where p and q are real, then

a. $p + q = 0$
b. $p \in (-3, \infty)$
c. one of the root is unity
d. one root is smaller than 1 and one root is greater than 1

15. If $(\sin \alpha)\, x^2 - 2x + b \geq 2$, for all real values of $x \leq 1$ and $\alpha \in (0, \pi/2) \cup (\pi/2, \pi)$, then possible real values of b is/are

a. 2 **b.** 3 **c.** 4 **d.** 5

16. If every pair from among the equations $x^2 + ax + bc = 0$, $x^2 + bx + ca = 0$ and $x^2 + cx + ab = 0$ has a common root, then

a. the sum of the three common roots is $-1/2\,(a+b+c)$
b. the sum of the three common roots is $2(a + b + c)$
c. the product of the three common roots is abc
d. the product of the three common roots is $a^2b^2c^2$

17. If a, b, c are in G.P. then the roots of the equation $ax^2 + bx + c = 0$ are in the ratio

a. $\frac{1}{2}(-1+i\sqrt{3})$ **b.** $\frac{1}{2}(1-i\sqrt{3})$
c. $\frac{1}{2}(-1-i\sqrt{3})$ **d.** $\frac{1}{2}(1+i\sqrt{3})$

18. If $ax^2 + (b - c)x + a - b - c = 0$ has unequal real roots for all $c \in R$, then

a. $b < 0 < a$ **b.** $a < 0 < b$
c. $b < a < 0$ **d.** $b > a > 0$

19. Given that α, γ are roots of the equation $Ax^2 - 4x + 1 = 0$, and β, δ the roots of the equation of $Bx^2 - 6x + 1 = 0$, such that α, β, γ and δ are in H.P., then

a. $A = 3$ **b.** $A = 4$
c. $B = 2$ **d.** $B = 8$

20. If the equations $x^2 + px + q = 0$ and $x^2 + p'x + q' = 0$ have a common root, then it must be equal to

a. $\frac{pq' - p'q}{q - q'}$ **b.** $\frac{q - q'}{p' - p}$
c. $\frac{p' - p}{q - q'}$ **d.** $\frac{pq' - p'q}{p - p'}$

21. If $x^3 + 3x^2 - 9x + c$ is of the form $(x - \alpha)^2 (x - \beta)$, then c is equal to

a. 27 **b.** –27 **c.** 5 **d.** –5

22. If the equations $x^2 + bx - a = 0$ and $x^2 - ax + b = 0$ have a common root, then

a. $a + b = 0$ **b.** $a = b$
c. $a - b = 1$ **d.** $a + b = 1$

23. If $(x^2+ax+3)/(x^2+x+a)$ takes all real values for possible real values of x, then

a. $4a^3 + 39 < 0$ **b.** $4a^3 + 39 \geq 0$
c. $a \geq \frac{1}{4}$ **d.** $a < \frac{1}{4}$

24. If $\cos^4 \theta + \alpha$, $\sin^4 \theta + \alpha$ are the roots of the equation $x^2 + 2bx + b = 0$ and $\cos^2 \theta + \beta$, $\sin^2 \theta + \beta$ are the roots of the equation $x^2 + 4x + 2 = 0$, then values of b are

a. 2 **b.** –1 **c.** –2 **d.** 1

25. If the roots of the equation $x^2 + ax + b = 0$ are c and d, then roots of the equation $x^2 + (2c + a)x + c^2 + ac + b = 0$ are

a. c **b.** $d - c$ **c.** $2c$ **d.** 0

26. If $a, b, c \in R$ and $abc < 0$, then the equation $bcx^2 + 2(b + c - a)x + a = 0$, has

a. both positive roots
b. both negative roots
c. real roots
d. one positive and one negative root

27. For the quadratic equation $x^2 + 2(a + 1)x + 9a - 5 = 0$, which of the following is/are true?

a. If $2 < a < 5$, then roots are of opposite sign.
b. If $a < 0$, then roots are of opposite sign.
c. If $a > 7$, then both roots are negative.
d. If $2 \leq a \leq 5$, then roots are unreal.

28. Let $P(x) = x^2 + bx + c$, where b and c are integer. If $P(x)$ is a factor of both $x^4 + 6x^2 + 25$ and $3x^4 + 4x^2 + 28x + 5$, then

a. $P(x) = 0$ has imaginary roots
b. $P(x) = 0$ has roots of opposite sign
c. $P(1) = 4$
d. $P(1) = 6$

30. If $|ax^2 + bx + c| \leq 1$ for all x in $[0, 1]$, then

a. $|a| \leq 8$ **b.** $|b| > 8$
c. $|c| \leq 1$ **d.** $|a| + |b| + |c| \leq 17$

31. Let $f(x) = ax^2 + bx + c$. Consider the following diagram. Then

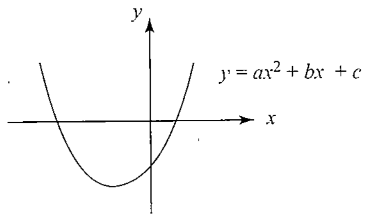


Fig. 1.66

a. $c < 0$ b. $b > 0$
c. $a + b - c > 0$ d. $abc < 0$

Reasoning Type

Solutions on page 1.76

Each question has four choices a, b, c and d, out of which *only one* is correct. Each question contains STATEMENT 1 and STATEMENT 2.

a. Both the statements are TRUE and STATEMENT 2 is the correct explanation of Statement 1.
b. Both the statements are TRUE but STATEMENT 2 is NOT the correct explanation of STATEMENT 1.
c. STATEMENT 1 is TRUE and STATEMENT 2 is FALSE.
d. STATEMENT 1 is FALSE and STATEMENT 2 is TRUE.

1. Consider the function $f(x) = \log_e (ax^3 + (a + b) x^2 + (b + c) x + c)$.

   Statement 1: Domain of the functions is $(-1, \infty) \sim \{-(b/2a)\}$, where $a > 0$, $b^2 - 4ac = 0$.

   Statement 2: $ax^2 + bx + c = 0$ has equal roots when $b^2 - 4ac = 0$.

2. **Statement 1:** If $a > 0$ and $b^2 - ac < 0$, then domain of the function $f(x) = \sqrt{ax^2+2bx+c}$ is R.

   Statement 2: If $b^2 - ac < 0$, then $ax^2 + 2bx + c = 0$ has imaginary roots.

3. **Statement 1:** If equations $ax^2 + bx + c = 0$ and $x^2 - 3x + 4 = 0$ have exactly one root common, then at least one of a, b, c is imaginary.

   Statement 2: If a, b, c are not all real, then equation $ax^2 + bx + c = 0$ can have one root real and one root imaginary.

4. **Statement 1:** If $\cos^2 \pi/8$ is a root of the equation $x^2 + ax + b = 0$ where $a, b \in Q$, then ordered pair (a, b) is $[-1, (1/8)]$.

   Statement 2: If $a + mb = 0$ and m is irrational, then $a, b = 0$.

5. **Statement 1:** If $a^2 + b^2 + c^2 < 0$, then if roots of the equation $ax^2 + bx + c = 0$ are imaginary, then they are not complex conjugates.

   Statement 2: Equation $ax^2 + bx + c = 0$ has complex conjugate roots when a, b, c are real.

6. **Statement 1:** Equation $ix^2+(i-1)x-(1/2)-i=0$ has imaginary roots.

   Statement 2: If $a = i$, $b = i - 1$ and $c=-(1/2)-i$, then $b^2 - 4ac < 0$.

7. **Statement 1:** If $f(x)$ is a quadratic polynomial satisfying $f(2) + f(4) = 0$. If unity is a root of $f(x) = 0$, then the other root is 3.5.

   Statement 2: If $g(x) = px^2 + qx + r = 0$ has roots α, β, then $\alpha + \beta = - q/p$ and $\alpha\beta=(r/p)$.

8. Let $f(x) = -x^2 + (a + 1) x + 5$.

   Statement 1: $f(x)$ is positive for some $\alpha < x < \beta$ and for all $a \in R$.

   Statement 2: $f(x)$ is positive for all $x \in R$ and for some real a.

9. Let a, b, c be real such that $ax^2 + bx + c = 0$ and $x^2 + x + 1 = 0$ have a common root.

   Statement 1: $a = b = c$

   Statement 2: Two quadratic equations with real coefficients cannot have only one imaginary root common.

10. **Statement 1:** The equation $(x - p) (x - r) + \lambda (x - q) (x - s) = 0$, where $p < q < r < s$, has non-real roots.

    Statement 2: The equation $px^2 + qx + r = 0$ $(p, q, r \in R)$ has non-real roots if $q^2 - 4pr < 0$.

11. **Statement 1:** If $px^2 + qx + r = 0$ is a quadratic equation $(p, q, r \in R)$ such that its roots are α, β and $p + q + r < 0$, $p - q + r < 0$ and $r > 0$, then $[\alpha] + [\beta] = -1$, where $[\cdot]$ denotes greatest integer function.

    Statement 2: If for any two real numbers a and b, function $f(x)$ is such that $f(a) f(b) < 0 \Rightarrow f(x)$ has at least one real root lying in (a, b).

12. **Statement 1:** If $0<\alpha<(\pi/4)$, then the equation $(x - \sin \alpha) \times (x - \cos \alpha) - 2 = 0$ has both roots in $(\sin \alpha, \cos \alpha)$.

    Statement 2: If $f(a)$ and $f(b)$ possess opposite signs, then there exist at least one solution of the equation $f(x) = 0$ in open interval (a, b).

13. **Statement 1:** If all real values of x obtained from the equation $4^x - (a - 3) 2^x + (a - 4) = 0$ are non-positive, then $a \in (4, 5]$.

    Statement 2: If $ax^2 + bx + c$ is non-positive for all real values of x, then $b^2 - 4ac$ must be negative or zero and 'a' must be negative.

14. **Statement 1:** If $(a^2 - 4) x^2 + (a^2 - 3a + 2) x + (a^2 - 7a + 10) = 0$ is an identity, then the value of a is 2.

    Statement 2: If $a - b = 0$, then $ax^2 + bx + c = 0$ is an identity.

15. **Statement 1:** If the roots of $x^5 - 40x^4 + Px^3 + Qx^2 + Rx + S = 0$ are in G.P. and sum of their reciprocal is 10, then $|S| = 64$.

    Statement 2: $x_1 x_2 x_3 x_4 x_5 = -S$, where x_1, x_2, x_3, x_4, x_5 are the roots of given equation.

16. **Statement 1:** If a, b, c, a_1, b_1, c_1 are rational and equations $ax^2 + 2bx + c = 0$ and $a_1x^2 + 2b_1x + c_1 = 0$ have one and only one root in common, then both $b^2 - ac$ and $b_1^2 - a_1c_1$ must be perfect squares.

Statement 2: If two quadratic equations with rational coefficients have a common irrational root $p+\sqrt{q}$, then both roots will be common.

17. **Statement 1:** If $a, b, c \in Z$ and $ax^2 + bx + c = 0$ has an irrational root, then $|f(\lambda)| \geq 1/q^2$, where $\lambda \in (\lambda = \frac{p}{q}; p, q \in Z)$ and $f(x) = ax^2 + bx + c$.

Statement 2: If $a, b, c \in Q$ and $b^2 - 4ac$ is positive but not a perfect square, then roots of equation $ax^2 + bx + c = 0$ are irrational and always occur in conjugate pair like $2 + \sqrt{3}$ and $2 - \sqrt{3}$.

18. **Statement 1:** The number of values of a for which $(a^2 - 3a + 2)x^2 + (a^2 - 5a + 6)x + a^2 - 4 = 0$ is an identity in x is 2.

Statement 2: If $a = b = c = 0$, then equation $ax^2 + bx + c = 0$ is an identity in x.

19. **Statement 1:** If roots of the equation $x^2 - bx + c = 0$ are two consecutive integers, then $b^2 - 4c = 1$.

Statement 2: If a, b, c are odd integer, then the roots of the equation $4abc\, x^2 + (b^2 - 4ac)x - b = 0$ are real and distinct.

20. Let $ax^2 + bx + c = 0, a \neq 0$ $(a, b, c \in R)$ has no real roots and $a + b + 2c = 2$.

Statement 1: $ax^2 + bx + c > 0, \forall x \in R$.

Statement 2: $a + b$ is positive.

21. Consider a general expression of degree 2 in two variables as $f(x, y) = 5x^2 + 2y^2 - 2xy - 6x - 6y + 9$.

Statement 1: $f(x, y)$ can be resolved into two linear factors over real coefficients.

Statement 2: If we compare $f(x, y)$ with $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$, we have $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$.

22. **Statement 1:** The equation $x^2 + (2m + 1)x + (2n + 1) = 0$, where m and n are integers cannot have any rational roots.

Statement 2: The quantity $(2m + 1)^2 - 4(2n + 1)$, where $m, n \in I$ can never be a perfect square.

Linked Comprehension Type

Solutions on page 1.78

Based upon each paragraph, some multiple choice questions have to be answered. Each question has four choices a, b, c and d, out of which *only one* is correct.

For Problems 1–3

Consider an unknown polynomial which when divided by $(x - 3)$ and by $(x - 4)$ leaves remainders as 2 and 1, respectively. Let $R(x)$ be the remainder when this polynomial is divided by $(x - 3)(x - 4)$.

1. If equation $R(x) = x^2 + ax + 1$ has two distinct real roots, then exhaustive values of a are

a. $(-2, 2)$ **b.** $(-\infty, -2) \cup (2, \infty)$
c. $(-2, \infty)$ **d.** all real numbers

2. If $R(x) = px^2 + (q - 1)x + 6$ has no distinct real roots and $p > 0$, then least value of $3p + q$ is

a. -2 **b.** $2/3$
c. $-1/3$ **d.** none of these

3. Range of $f(x) = [R(x)]/(x^2 - 3x + 2)$ is

a. $[-2, 2]$
b. $(-\infty, -2-\sqrt{3}] \cup [-2+\sqrt{3}, \infty)$
c. $(-\infty, -7-4\sqrt{3}] \cup [-7+4\sqrt{3}, \infty)$
d. none of these

For Problems 4–6

Consider the quadratic equation $ax^2 - bx + c = 0, a, b, c \in N$, which has two distinct real roots belonging to the interval $(1, 2)$.

4. The least value of a is

a. 4 **b.** 6 **c.** 7 **d.** 5

5. The least value of b is

a. 10 **b.** 11 **c.** 13 **d.** 15

6. The least value of c is

a. 4 **b.** 6 **c.** 7 **d.** 5

For Problems 7–9

Consider the equation $x^4 + 2ax^3 + x^2 + 2ax + 1 = 0$, where $a \in R$. Also range of function $f(x) = x + 1/x$ is $(-\infty, -2] \cup [2, \infty)$.

7. If equation has at least two distinct positive real roots then all possible values of a are

a. $(-\infty, -1/4)$ **b.** $(5/4, \infty)$
c. $(-\infty, -3/4)$ **d.** none of these

8. If equation has at least two distinct negative real roots, then all possible values of a are

a. $(3/4, \infty)$ **b.** $(-5/4, \infty)$
c. $(-\infty, 1/4)$ **d.** none of these

9. If exactly two roots are positive and two roots are negative, then number of integral values of a is

a. 2 **b.** 1 **c.** 0 **d.** 3

For Problems 10–12

Let $f(x) = x^2 + b_1x + c_1$, $g(x) = x^2 + b_2x + c_2$. Let the real roots of $f(x) = 0$ be α, β and real roots of $g(x) = 0$ be $\alpha + h, \beta + h$. The least value of $f(x)$ is $-1/4$. The least value of $g(x)$ occurs at $x = 7/2$

10. The least value of $g(x)$ is

a. $-\frac{1}{4}$ **b.** -1 **c.** $-\frac{1}{3}$ **d.** $-\frac{1}{2}$

11. The value of b_2 is

a. –5 **b.** 9 **c.** –8 **d.** –7

12. The roots of $f(x) = 0$ are

a. 3, –4 **b.** –3, 4 **c.** 3, 4 **d.** –3, –4

For Problems 13–15

In the given figure, vertices of ΔABC lie on $y = f(x) = ax^2 + bx + c$. The ΔABC is right angled isosceles triangle whose hypotenuse $AC = 4\sqrt{2}$ units.

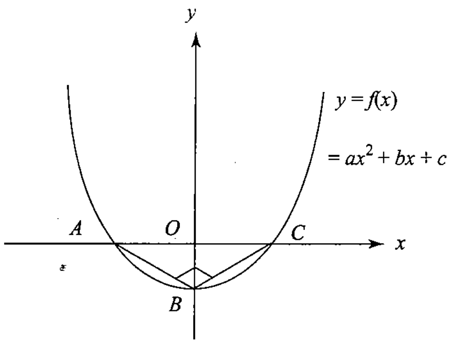


Fig. 1.67

13. $y = f(x)$ is given by

a. $y = x^2 - 2\sqrt{2}$ **b.** $y = x^2 - 12$

c. $y = \frac{x^2}{2} - 2$ **d.** $y = \frac{x^2}{2\sqrt{2}} - 2\sqrt{2}$

14. Minimum value of $y = f(x)$ is

a. –4 **b.** –2

c. $-2\sqrt{2}$ **d.** none of these

15. Number of integral values of k for which one root of $f(x) = 0$ is more than k and other less than k

a. 6 **b.** 4 **c.** 5 **d.** 7

For Problems 16–18

Consider the inequality $9^x - a3^x - a + 3 \le 0$, where 'a' is a real parameter.

16. The given inequality has at least one negative solution for $a \in$

a. $(-\infty, 2)$ **b.** $(3, \infty)$

c. $(-2, \infty)$ **d.** $(2, 3)$

17. The given inequality has at least one positive solution for $a \in$

a. $(-\infty, -2)$ **b.** $[3, \infty)$

c. $(2, \infty)$ **d.** $[-2, \infty)$

18. The given inequality has at least one real solution for $a \in$

a. $(-\infty, 3)$ **b.** $[2, \infty)$

c. $(3, \infty)$ **d.** $[-2, \infty)$

For Problems 19–21

Consider the inequation $x^2 + x + a - 9 < 0$.

19. The values of the real parameter 'a' so that the given inequation has at least one positive solution:

a. $(-\infty, 37/4)$ **b.** $(-\infty, \infty)$

c. $(3, \infty)$ **d.** $(-\infty, 9)$

20. The values of the real parameter 'a' so that the given inequation has at least one negative solution:

a. $(-\infty, 9)$ **b.** $(37/4, \infty)$

c. $\left(-\infty, \frac{37}{4}\right)$ **d.** none of these

21. The values of the real parameter 'a' so that the given inequation is true $\forall\, x \in (-1, 3)$:

a. $(-\infty, -3)$ **b.** $(-3, \infty)$

c. $[9, \infty)$ **d.** $(-\infty, 37/4)$

For Problems 22–24

'$af(\mu) < 0$' is the necessary and sufficient condition for a particular real number μ to lie between the roots of a quadratic equation $f(x) = 0$, where $f(x) = ax^2 + bx + c$. Again if $f(\mu_1)f(\mu_2) < 0$, then exactly one of the roots will lie between μ_1 and μ_2.

22. If $|b| > |a + c|$, then

a. one root of $f(x) = 0$ is positive, the other is negative

b. exactly one of the roots of $f(x) = 0$ lies in $(-1, 1)$

c. 1 lies between the roots of $f(x) = 0$

d. both the roots of $f(x) = 0$ are less than 1

23. If $a(a + b + c) < 0 < (a + b + c)c$, then

a. one root is less than 0, the other is greater than 1

b. exactly one of the roots lies in $(0, 1)$

c. both the roots lie in $(0, 1)$

d. at least one of the roots lies in $(0, 1)$

24. If $(a + b + c)c < 0 < a(a + b + c)$, then

a. one root is less than 0, the other is greater than 1

b. one root lies in $(-\infty, 0)$ and other in $(0, 1)$

c. both the roots lie in $(0, 1)$

d. one root lies in $(0, 1)$ and other in $(1, \infty)$

For Problems 25–26

The real numbers x_1, x_2, x_3 satisfying the equation $x^3 - x^2 + \beta x + \gamma = 0$ are in A.P.

25. All possible values of β are

a. $\left(-\infty, \frac{1}{3}\right)$ **b.** $\left(-\infty, -\frac{1}{3}\right)$

c. $\left(\frac{1}{3}, \infty\right)$ **d.** $\left(-\frac{1}{3}, \infty\right)$

26. All possible values of γ are

a. $\left(-\frac{1}{9}, \infty\right)$ **b.** $\left(-\frac{1}{27}, +\infty\right)$

c. $\left(\frac{2}{9}, +\infty\right)$ **d.** none of these

Matrix-Match Type

Solutions on page 1.81

Each question contains statements given in two columns, which have to be matched. Statements a, b, c, d in column I have to be matched with statements p, q, r, s in column II. If the correct matches are a → p, a → s, b → q, b → r, c → p, c → q and d → s, then the correctly bubbled 4 × 4 matrix should be as follows:

	p	q	r	s
a	(p)	(q)	(r)	(s)
b	(p)	(q)	(r)	(s)
c	(p)	(q)	(r)	(s)
d	(p)	(q)	(r)	(s)

1. Match the following for the equation $x^2 + a\,|x| + 1 = 0$, where a is a parameter.

Column I	Column II
a. No real roots	**p.** $a < -2$
b. Two real roots	**q.** ϕ
c. Three real roots	**r.** $a = -2$
d. Four distinct real roots	**s.** $a \geq 0$

2.

Column I	Column II
a. $y = \frac{x^2 - 2x + 4}{x^2 + 2x + 4}, x \in R$, then y can be	**p.** 1
b. $y = \frac{x^2 - 3x - 2}{2x - 3}, x \in R$, then y can be	**q.** 4
c. $y = \frac{2x^2 - 2x + 4}{x^2 - 4x + 3}, x \in R$, then y can be	**r.** -3
d. $x^2 - (a - 3)x + 2 < 0, \forall x \in (-2, 3)$, then a can be	**s.** -10

3.

Column I	Column II
a. If a, b, c and d are four zero real number such that $(d + a - b)^2 + (d + b - c)^2 = 0$ and the roots of the equation $a(b - c)x^2 + b(c - a)x + c(a - b) = 0$ are real and equal then	**p.** $a + b + c = 0$
b. If the roots of the equation $(a^2 + b^2)x^2 - 2b(a + c)x + (b^2 + c^2) = 0$ are real and equal, then	**q.** a, b, c are in A.P.
c. If the equation $ax^2 + bx + c = 0$ and $x^3 - 3x^2 + 3x - 1 = 0$ have a common real root, then	**r.** a, b, c are in G.P.
d. Let a, b, c be positive real numbers such that the expression $bx^2 + \left(\sqrt{(a+c)^2 + 4b^2}\right)x + (a + c)$ is non-negative $\forall x \in R$, then	**s.** a, b, c are in H.P.

4.

Column I (Number of positive integers for which)	Column II
a. one root is positive and the other is negative for the equation $(m - 2)x^2 - (8 - 2m)x - (8 - 3m) = 0$	**p.** 0
b. exactly one root of equation $x^2 - m(2x - 8) - 15 = 0$ lies in interval (0, 1)	**q.** infinite
c. the equation $x^2 + 2(m + 1)x + 9m - 5 = 0$ has both roots negative	**r.** 1
d. the equation $x^2 + 2(m - 1)x + m + 5 = 0$ has both roots lying on either sides of 1	**s.** 2

5.

Column I	Column II
a. If $x^2 + ax + b = 0$ has roots α, β and $x^2 + px + q = 0$ has roots $-\alpha, \gamma$, then	**p.** $(1 - bq)^2 = (a - pb)(p - aq)$
b. If $x^2 + ax + b = 0$ has roots α, β and $x^2 + px + q = 0$ has roots $1/\alpha$, γ, then	**q.** $(4 - bq)^2 = (4a + 2pb)(-2p - aq)$
c. If $x^2 + ax + b = 0$ has roots α, β and $x^2 + px + q = 0$ has roots $-2/\alpha, \gamma$, then	**r.** $(1 - 4bq)^2 = (a + 2bp)(-2p - 4aq)$
d. If $x^2 + ax + b = 0$ has roots α, β and $x^2 + px + q = 0$ has roots $-1/(2\alpha), \gamma$, then	**s.** $(q - b)^2 = (aq + bp)(p - a)$

Integer Type

Solutions on page 1.83

1. Let 'a' is a real number satisfying $a^3 + \frac{1}{a^3} = 18$. Then the value of $a^4 + \frac{1}{a^4} - 39$ is.

2. Let $P(x) = \frac{5}{3} - 6x - 9x^2$ and $Q(y) = -4y^2 + 4y + \frac{13}{2}$. If there exist unique pair of real numbers (x, y) such that $P(x)\,Q(y) = 20$, then the value of $(6x + 10y)$ is.

3. Let $P(x) = x^3 - 8x^2 + cx - d$ be a polynomial with real coefficients and with all its roots being distinct positive integers. Then number of possible value of 'c' is.

4. Let α_1, β_1 are the roots of $x^2 - 6x + p = 0$ and α_2, β_2 are the roots of $x^2 - 54x + q = 0$. If $\alpha_1, \beta_1, \alpha_2, \beta_2$ form an increasing G.P., then sum of the digits of the value of $(q - p)$ is.

5. If the equation $2x^2 + 4xy + 7y^2 - 12x - 2y + t = 0$ where 't' is a parameter has exactly one real solution of the form (x, y). Then the sum of $(x + y)$ is equal to.

6. Polynomial $P(x)$ contains only terms of odd degree. When $P(x)$ is divided by $(x - 3)$, the remainder is 6. If $P(x)$ is divided by $(x^2 - 9)$, then the remainder is $g(x)$. Then the value of $g(2)$ is.

7. If set of values of 'a' for which $f(x) = ax^2 - (3 + 2a)x + 6$, $a \neq 0$ is positive for exactly three distinct negative integral values of x is $(c, d]$, then the value of $(c^2 + 4|d|)$ is equal to.

8. Given α and β are the roots of the quadratic equation $x^2 - 4x + k = 0$ $(k \neq 0)$. If $\alpha\beta$, $\alpha\beta^2 + \alpha^2\beta$, $\alpha^3 + \beta^3$ are in geometric progression, then the value of $7k/2$ equals.

9. If the equation $x^2 + 2(\lambda + 1)x + \lambda^2 + \lambda + 7 = 0$ has only negative roots, then the least value of λ equals.

10. Let $P(x) = x^4 + ax^3 + bx^2 + cx + d$ be a polynomial such that $P(1) = 1$, $P(2) = 8$, $P(3) = 27$, $P(4) = 64$, then the value of $P(5)$ is divisible by prime number.

11. If $\sqrt{\sqrt{\sqrt{x}}} = \sqrt[4]{\sqrt[4]{\sqrt[4]{3x^4 + 4}}}$, then the value of x^4 is.

12. Number of positive integers x for which $f(x) = x^3 - 8x^2 + 20x - 13$ is a prime number is.

13. If equation $x^4 - (3m + 2)x^2 + m^2 = 0$ $(m > 0)$ has four real solutions which are in A.P., then the value of 'm' is.

14. The quadratic polynomial $p(x)$ has the following properties: $p(x) \geq 0$ for all real numbers, $p(1) = 0$ and $p(2) = 2$. Find the value of $p(3)$ is.

15. $f: R \to R, f(x) = \frac{3x^2 + mx + n}{x^2 + 1}$. If the range of this function is $[-4, 3)$, then find the value of $|m + n|$ is.

16. If a and b are positive numbers and each of the equations $x^2 + ax + 2b = 0$ and $x^2 + 2bx + a = 0$ has real roots, then the smallest possible value of $(a + b)$ is.

17. Suppose a, b, c are the roots of the cubic $x^3 - x^2 - 2 = 0$. Then the value of $a^3 + b^3 + c^3$ is.

18. Given that $x^2 - 3x + 1 = 0$, then the value of the expression $y = x^9 + x^7 + x^{-9} + x^{-7}$ is divisible by prime number.

19. Suppose $a, b, c \in I$ such that greatest common divisor of $x^2 + ax + b$ and $x^2 + bx + c$ is $(x + 1)$ and the least common multiple of $x^2 + ax + b$ and $x^2 + bx + c$ is $(x^3 - 4x^2 + x + 6)$. Then the value of $|a + b + c|$ is equal to.

20. If the roots of the cubic, $x^3 + ax^2 + bx + c = 0$ are three consecutive positive integers. Then the value of $\frac{a^2}{b+1}$ is equal to.

21. If $x + y + z = 12$ and $x^2 + y^2 + z^2 = 96$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 36$. Then the value $x^3 + y^3 + z^3$ is divisible by prime number.

22. Let α and β be the solutions of the quadratic equation $x^2 - 1154x + 1 = 0$, then the value of $\sqrt[4]{\alpha} + \sqrt[4]{\beta}$ is equal to.

23. If $a^2 - 4a + 1 = 4$, then the value of $\frac{a^3 - a^2 + a - 1}{a^2 - 1}$ $(a^2 \neq 1)$ is equal to.

24. The function $f(x) = ax^3 + bx^2 + cx + d$ has three positive roots. If the sum of the roots of $f(x)$ is 4, the largest possible integral values of c/a is.

25. Let $x^2 + y^2 + xy + 1 \geq a(x + y)$ $\forall\, x, y \in R$, then the number of possible integer(s) in the range of a is.

26. a, b and c are all different and non-zero real numbers in arithmetic progression. If the roots of quadratic equation $ax^2 + bx + c = 0$ are α and β such that $\frac{1}{\alpha} + \frac{1}{\beta}$, $\alpha + \beta$ and $\alpha^2 + \beta^2$ are in geometric progression, then the value of a/c will be.

27. All the values of k for which the quadratic polynomial $f(x) = -2x^2 + kx + k^2 + 5$ has two distinct zeroes and only one of them satisfying $0 < x < 2$, lie in the interval (a, b). The value of $(a + 10b)$ is.

28. The quadratic equation $x^2 + mx + n = 0$ has roots which are twice those of $x^2 + px + m = 0$ and m, n and $p \neq 0$. Then the value of n/p is.

29. a, b, c are reals such that $a + b + c = 3$ and $\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} = \frac{10}{3}$. The value of $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}$ is.

30. Let a, b and c be real numbers which satisfy the equations $a + \frac{1}{bc} = \frac{1}{5}$, $b + \frac{1}{ac} = \frac{-1}{15}$ and $c + \frac{1}{ab} = \frac{1}{3}$. The value of $\frac{c-b}{c-a}$ is equal to.

31. If a, b, c are non-zero real numbers, then the minimum value of the expression $\left(\frac{(a^4 + 3a^2 + 1)(b^4 + 5b^2 + 1)(c^4 + 7c^2 + 1)}{a^2 b^2 c^2}\right)$ is not divisible by prime number.

32. If $a, b \in R$ such that $a + b = 1$ and $(1 - 2ab)(a^3 + b^3) = 12$. The value of $(a^2 + b^2)$ is equal to.

33. If the cubic $2x^3 - 9x^2 + 12x + k = 0$ has two equal roots then maximum value of $|k|$ is

34. Let a, b and c be distinct non zero real numbers such that $\frac{1-a^3}{a} = \frac{1-b^3}{b} = \frac{1-c^3}{c}$. The value of $(a^3 + b^3 + c^3)$, is

Archives

Solutions on page 1.86

Subjective Type

1. Solve for x: $4^x - 3^{x-1/2} = 3^{x+1/2} - 2^{2x-1}$. **(IIT-JEE, 1978)**

2. Solve for x: $\sqrt{x+1} - \sqrt{x-1} = 1$. **(IIT-JEE, 1978)**

3. Solve the following equation for x: $2\log_x a + \log_{ax} a + 3\log_{a^2x} a = 0, a > 0$. **(IIT-JEE, 1978)**

4. Show that the square of $(\sqrt{26-15\sqrt{3}})/(5\sqrt{2}-\sqrt{38+5\sqrt{3}})$ is a rational number. **(IIT-JEE, 1978)**

5. Find all integers x for which $(5x - 1) < (x + 1)^2 < (7x - 3)$. **(IIT-JEE, 1978)**

6. If α, β are the roots of $x^2 + px + q = 0$ and γ, δ are the roots of $x^2 + rx + s = 0$, evaluate $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)$ in terms of p, q, r and s. Deduce the condition that the equation has a common root. **(IIT-JEE, 1979)**

7. Show that for any triangle with sides a, b and c, $3(ab + bc + ca) < (a + b + c)^2 < 4(bc + ca + ab)$. When are the first two expressions equal? **(IIT-JEE, 1979)**

8. Let $y = \sqrt{((x+1)(x-3))/(x-2)}$. Find all the real values of x for which y takes real values. **(IIT-JEE, 1980)**

9. For what values of m, does the system of equations $3x + my = m$, $2x - 5y = 20$ has solution satisfying the conditions $x > 0, y > 0$. **(IIT-JEE, 1980)**

10. Find the solution set of the system

$$x + 2y + z = 1$$
$$2x - 3y - w = 2$$
$$x \geq 0, y \geq 0, z \geq 0, w \geq 0$$ **(IIT-JEE, 1980)**

11. Show that the equation $e^{\sin x} - e^{-\sin x} - 4 = 0$ has no real solution. **(IIT-JEE, 1982)**

12. mn squares of equal size are arranged to form a rectangle of dimension m by n, where m and n are natural numbers. Two square will be called 'neighbours' if they have exactly one common side. A natural is written in each square such that the number written in any square is the arithmetic mean of the numbers written in its neighouring squares. Show that this is possible only if all the numbers used are equal. **(IIT-JEE, 1982)**

13. If one root of the quadratic equation $ax^2 + bx + c = 0$ is equal to the n^{th} power of the other, then show that

$$(ac^n)^{\frac{1}{n+1}} + (a^n c)^{\frac{1}{n+1}} + b = 0$$ **(IIT-JEE, 1983)**

14. Find all real values of x which satisfy $x^2 - 3x + 2 > 0$ and $x^2 - 3x - 4 \leq 0$. **(IIT-JEE, 1983)**

15. Solve for $x: (5+2\sqrt{6})^{x^2-3} + (5-2\sqrt{6})^{x^2-3} = 10$. **(IIT-JEE, 1985)**

16. For $a \leq 0$, determine all real roots of the equation $x^2 - 2a|x - a| - 3a^2 = 0$ **(IIT-JEE, 1985)**

17. Find the set of all x for which $2x/(2x^2+5x+2) > 1/(x+1)$. **(IIT-JEE, 1987)**

18. Solve $|x^2 + 4x + 3| + 2x + 5 = 0$. **(IIT-JEE, 1988)**

19. Let a, b, c be real. If $ax^2 + bx + c = 0$ has two real roots α and β, where $\alpha < -1$ and $\beta > 1$, then show that

$$1 + \frac{c}{a} + \left|\frac{b}{a}\right| < 0$$ **(IIT-JEE, 1995)**

20. The real numbers x_1, x_2, x_3 satisfying the equation $x^3 - x^2 + bx + \gamma = 0$ are in A.P. Find the intervals in which β and γ lie. **(IIT-JEE, 1996)**

21. Let S be a square of unit area. Consider any quadrilateral, which has one vertex on each side of S. If a, b, c and d denote the lengths of the sides of the quadrilateral, prove that $2 \leq a^2 + b^2 + c^2 + d^2 \leq 4$. **(IIT-JEE, 1997)**

22. Let $f(x) = Ax^2 + Bx + C$, where A, B, C are real numbers. Prove that if $f(x)$ is an integer whenever x is an integer, then the numbers $2A$, $A + B$ and C are all integers. Conversely, prove that if the number $2A$, $A + B$ and C are all integers, then $f(x)$ is an integer whenever x is an integer. **(IIT-JEE, 1998)**

23. If α, β are the roots of $ax^2 + bx + c = 0$ $(a \neq 0)$ and $\alpha + \delta, \beta + \delta$ are the roots of $Ax^2 + Bx + C = 0$ $(A \neq 0)$ for some constant δ, then prove that $(b^2 - 4ac)/a^2 = (B^2 - 4AC)/A^2$. **(IIT-JEE, 2000)**

24. Let a, b, c be real numbers with $a \neq 0$ and let α, β be the roots of the equations $ax^2 + bx + c = 0$. Express the roots of $a^3x^2 + abcx + c^3 = 0$ in terms of α, β. **(IIT-JEE, 2001)**

25. If $x^2 + (a-b)x + (1-a-b) = 0$ where $a, b \in R$, then find the values of a for which equation has unequal real roots for all values of b. **(IIT-JEE, 2003)**

26. Let a and b be the roots of the equation $x^2 - 10cx - 11d = 0$ and those of $x^2 - 10ax - 11b = 0$ are c, d. Then find the value of $a+b+c+d$, when $a \neq b \neq c \neq d$. **(IIT-JEE, 2006)**

Objective Type

Fill in the blanks

1. The coefficient of x^{99} in the polynomial $(x-1)(x-2)\cdots(x-100)$ is ________. **(IIT-JEE, 1982)**

2. If $2 + i\sqrt{3}$ is a root of the equation $x^2 + px + q = 0$, where p and q are real, then $(p, q) =$ (________, ________).

3. If the product of the roots of the equation $x^2 - 3kx + 2e^{2\ln k} - 1 = 0$ is 7, then the roots are real for = ________.

4. If the quadratic equations $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ $(a \neq b)$ have a common root, then the numerical value of $a + b$ is ________. **(IIT-JEE, 1986)**

5. If $x<0, y<0, x+y+(x/y)=(1/2)$ and $(x+y)(x/y)=-(1/2)$, then $x =$ ________ and $y =$ ________. **(IIT-JEE, 1982)**

6. The sum of all real roots of the equation $|x-2|^2 + |x-2| - 2 = 0$ is ________. **(IIT-JEE, 1997)**

7. The solution of the equation $\log_7 \log_5(\sqrt{x+5}+\sqrt{x}) = 0$ is ________. **(IIT-JEE, 1986)**

True or false

1. The equation $2x^2 + 3x + 1 = 0$ has an irrational root. **(IIT-JEE, 1983)**

2. If $a < b < c < d$, then the roots of the equation $(x-a)(x-c) + 2(x-b)(x-d) = 0$ are real and distinct. **(IIT-JEE, 1984)**

3. If $n_1, n_2, \ldots, n_p$ are p positive integers, whose sum is an even number, then the number of odd integers among them is odd. **(IIT-JEE, 1985)**

4. If $P(x) = ax^2 + bx + c$ and $Q(x) = -ax^2 + dx + c$, where $ac \neq 0$, then $P(x)\,Q(x) = 0$ has at least two real roots. **(IIT-JEE, 1985)**

Multiple choice questions with one correct answer

1. If l, m, n are real $l \neq m$, then the roots of the equation $(l-m)x^2 - 5(l+m)x - 2(l-m) = 0$ are

a. real and equal **b.** complex
c. real and unequal **d.** none of these

(IIT-JEE, 1979)

2. If x, y and z are real and different and $u = x^2 + 4y^2 + 9z^2 - 6yz - 3zx - 2xy$, then u is always

a. non-negative **b.** zero
c. non-positive **d.** none of these

(IIT-JEE, 1979)

3. If $a > 0$, $b > 0$ and $c > 0$ then the roots of the equation $ax^2 + bx + c = 0$

a. are real and negative **b.** have positive real parts
c. have negative real parts **d.** none of these

(IIT-JEE, 1979)

4. Both the roots of the equation $(x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b) = 0$ are always

a. positive **b.** real
c. negative **d.** none of these

(IIT-JEE, 1980)

5. If $(x^2 + px + 1)$ is a factor of $(ax^3 + bx + c)$, then

a. $a^2 + c^2 = -ab$ **b.** $a^2 - c^2 = -ab$
c. $a^2 - c^2 = ab$ **d.** none of these

(IIT-JEE, 1980)

6. The number of real solutions of the equation $|x|^2 - 3|x| + 2 = 0$ is

a. 4 **b.** 1 **c.** 2 **d.** 0 **(IIT-JEE, 1982)**

7. Two towns A and B are 60 km apart. A school is to be built to serve 150 students in town A and 50 students in town B. If the total distance to be travelled by all 200 students is to be as small as possible, then the school be built at

a. town B **b.** 45 km from town A
c. town A **d.** 45 km from town B

(IIT-JEE, 1982)

8. The largest interval for which $x^{12} - x^9 + x^4 - x + 1 > 0$ is

a. $-4 < x \leq 0$ **b.** $0 < x < 1$
c. $-100 < x < 100$ **d.** $-\infty < x < \infty$

(IIT-JEE, 1982)

9. The equation $x - 2/(x-1) = 1 - 2/(x-1)$ has

a. no root **b.** one root
c. two equals roots **d.** infinitely many roots

(IIT-JEE, 1984)

10. If $a^2 + b^2 + c^2 = 1$, then $ab + bc + ca$ lies in the interval

a. $\left[\frac{1}{2}, 2\right]$ **b.** $[-1, 2]$

c. $\left[-\frac{1}{2}, 1\right]$

d. $\left[-1, \frac{1}{2}\right]$

(IIT-JEE, 1984)

11. If α and β are the roots of $x^2 + px + q = 0$ and α^4, β^4 are the roots of $x^2 - rx + s = 0$, then the equation $x^2 - 4qx + 2q^2 - r = 0$ has always

a. one positive and one negative root

b. two positive roots

c. two negative roots

d. cannot say anything **(IIT-JEE, 1989)**

12. Let a, b, c be real numbers, $a \neq 0$. If α is a root of $a^2x^2 + bx + c = 0$. β is the root of $a^2x^2 - bx - c = 0$ and $0 < \alpha < \beta$, then the equation $a^2x^2 + 2bx + 2c = 0$ has a root γ that always satisfies

a. $\gamma = \frac{\alpha+\beta}{2}$

b. $\gamma = \alpha + \frac{\beta}{2}$

c. $\gamma = \alpha$

d. $\alpha < \gamma < \beta$

(IIT-JEE, 1989)

13. The number of solutions of the equation $\sin(e^x) = 5^x + 5^{-x}$ is

a. 0

b. 1

c. 2

d. infinitely many

14. Let α, β be the roots of the equation $(x - a)(x - b) = c, c \neq 0$. Then the roots of the equation $(x - \alpha)(x - \beta) + c = 0$ are

a. a, c

b. b, c

c. a, b

d. $a + c, b + c$

(IIT-JEE, 1992)

15. The number of points of intersection of two curves $y = 2 \sin x$ and $y = 5x^2 + 2x + 3$ is

a. 0

b. 1

c. 2

d. ∞ **(IIT-JEE, 1994)**

16. If p, q, r are +ve and are in A.P., in the roots of quadratic equation $px^2 + qx + r = 0$ are all real for

a. $\left|\frac{r}{p} - 7\right| \geq 4\sqrt{3}$

b. $\left|\frac{p}{r} - 7\right| \geq 4\sqrt{3}$

c. all p and r

d. no p and r

(IIT-JEE, 1994)

17. The equation $\sqrt{x+1} - \sqrt{x-1} = \sqrt{4x-1}$ has

a. no solution

b. one solution

c. two solutions

d. more than two solutions

(IIT-JEE, 1997)

18. If the roots of the equation $x^2 - 2ax + a^2 + a - 3 = 0$ are real and less than 3, then

a. $a < 2$

b. $2 \leq a \leq 3$

c. $3 < a \leq 4$

d. $a > 4$ **(IIT-JEE, 1999)**

19. If α and β $(\alpha < \beta)$ are the roots of the equation $x^2 + bx + c = 0$, where $c < 0 < b$, then

a. $0 < \alpha < \beta$

b. $\alpha < 0 < \beta < |\alpha|$

c. $\alpha < \beta < 0$

d. $\alpha < 0 < |\alpha| < \beta$

(IIT-JEE, 2000)

20. If $b > a$, then the equation $(x - a)(x - b) - 1 = 0$ has

a. both roots in (a, b)

b. both roots in $(-\infty, a)$

c. both roots in $(b, +\infty)$

d. one root in $(-\infty, a)$ and the other in $(b, +\infty)$

(IIT-JEE, 2000)

21 For the equation $3x^2 + px + 3 = 0, p > 0$, if one of the root is square of the other, then p is equal to

a. 1/3 **b.** 1 **c.** 3 **d.** 2/3 **(IIT-JEE, 2000)**

22. Let $f(x) = (1 + b^2)x^2 + 2bx + 1$ and let $m(b)$ be the minimum value of $f(x)$. As b varies, the range of $m(b)$ is

a. $[0, 1]$

b. $\left(0, \frac{1}{2}\right]$

c. $\left[\frac{1}{2}, 1\right]$

d. $(0, 1]$

(IIT-JEE, 2001)

23. Let α, β be the roots of $x^2 - x + p = 0$ and γ, δ be roots of $x^2 - 4x + q = 0$. If $\alpha, \beta, \gamma, \delta$ are in G.P., then the integral values of p and q, respectively are

a. $-2, -32$ **b.** $-2, 3$ **c.** $-6, 3$ **d.** $-6, -32$

(IIT-JEE, 2001)

24. The set of all real numbers x for which $x^2 - |x + 2| + x > 0$ is

a. $(-\infty, -2)$

b. $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

c. $(-\infty, -1) \cup (1, \infty)$

d. $(\sqrt{2}, \infty)$

(IIT-JEE, 2002)

25. If $f(x) = x^2 + 2bx + 2c^2$ and $g(x) = -x^2 - 2cx + b^2$ are such that $\min f(x) > \max g(x)$, then the relation between b and c is

a. no relation

b. $0 < c < b/2$

c. $|c| < |b|\sqrt{2}$

d. $|c| > |b|\sqrt{2}$ **(IIT-JEE, 2003)**

26. For all x, $x^2 + 2ax + 10 - 3a > 0$, then the interval in which a lies is

a. $a < -5$

b. $-5 < a < 2$

c. $a > 5$

d. $2 < a < 5$

(IIT-JEE, 2004)

27. If one root is square of the other root of the equation $x^2 + px + q = 0$, then the relation between p and q is

a. $p^3 - q(3p - 1) + q^2 = 0$

b. $p^3 - q(3p + 1) + q^2 = 0$

c. $p^3 + q(3p - 1) + q^2 = 0$

d. $p^3 + q(3p + 1) + q^2 = 0$ **(IIT-JEE, 2004)**

28. Let α, β be the roots of the quadratic equation $ax^2 + bx + c = 0$ and $\Delta = b^2 - 4ac$. If $\alpha + \beta, \alpha^2 + \beta^2, \alpha^3 + \beta^3$ are in G.P. Then

a. $\Delta = 0$ **b.** $\Delta \neq 0$ **c.** $b\Delta = 0$ **d.** $c\Delta = 0$

(IIT-JEE, 2005)

29. Let a, b, c be the sides of a triangle, where $a \neq b \neq c$ and $\lambda \in R$. If the roots of the equation $x^2 + 2(a + b + c)x + 3\lambda(ab + bc + ca) = 0$ are real. Then

a. $\lambda < \frac{4}{3}$ **b.** $\lambda > \frac{5}{3}$

c. $\lambda \in \left(\frac{1}{3}, \frac{5}{3}\right)$ **d.** $\lambda \in \left(\frac{4}{3}, \frac{5}{3}\right)$ **(IIT-JEE, 2006)**

30. Let α, β be the roots of the equation $x^2 - px + r = 0$ and $\alpha/2, 2\beta$ be the roots of the equation $x^2 - qx + r = 0$. Then the value of r is

a. $\frac{2}{9}(p-q)(2q-p)$ **b.** $\frac{2}{9}(q-p)(2p-q)$

c. $\frac{2}{9}(q-2p)(2q-p)$ **d.** $\frac{2}{9}(2p-q)(2q-p)$

(IIT-JEE, 2007)

31. Let p and q be real numbers such that $p \neq 0$, $p^3 \neq q$ and $p^3 \neq -q$. If α and β are non-zero complex numbers satisfying $\alpha + \beta = -p$ and $\alpha^3 + \beta^3 = q$, then a quadratic equation having α/β and β/α as its roots is

a. $(p^3 + q)x^2 - (p^3 + 2q)x + (p^3 + q) = 0$

b. $(p^3 + q)x^2 - (p^3 - 2q)x + (p^3 + q) = 0$

c. $(p^3 - q)x^2 - (5p^3 - 2q)x + (p^3 - q) = 0$

d. $(p^3 - q)x^2 - (5p^3 + 2q)x + (p^3 - q) = 0$

(IIT-JEE, 2010)

32. A value of b for which the equations $x^2 + bx - 1 = 0$, $x^2 + x + b = 0$ have one root in common is

a. $-\sqrt{2}$ **b.** $-i\sqrt{3}$ **c.** $\sqrt{2}$ **d.** $\sqrt{3}$

(IIT-JEE, 2011)

33. Let α and β be the roots of $x^2 - 6x - 2 = 0$, with $\alpha > \beta$. If $a_n = \alpha^n - \beta^n$ for $n \geq 1$, then the value of $\frac{a_{10} - 2a_8}{2a_9}$ is

a. 1 **b.** 2 **c.** 3 **d.** 4

(IIT-JEE, 2011)

Multiple choice questions with one or more than one correct answer

1. For real x, then function $(x-a)(x-b)/(x-c)$ will assume all real values provided

a. $a > b > c$ **b.** $a < b < c$

c. $a > c > b$ **d.** $a < c < b$ **(IIT-JEE, 1984)**

2. If S is the set of all real x such that $(2x-1)/(2x^3+3x^2+x)$ is positive, then S contains

a. $\left(-\infty, -\frac{3}{2}\right)$ **b.** $\left(-\frac{3}{2}, -\frac{1}{4}\right)$

c. $\left(-\frac{1}{4}, \frac{1}{2}\right)$ **d.** $\left(\frac{1}{2}, 3\right)$

e. none of these **(IIT-JEE, 1986)**

3. The equation $x^{\frac{3}{4}(\log_2 x)^2 + \log_2 x - \frac{5}{4}} = \sqrt{2}$ has

a. at least one real solution

b. exactly three solutions

c. exactly one irrational solution

d. complex roots **(IIT-JEE, 1989)**

Assertion and reasoning

1. Let a, b, c, p, q be real numbers. Suppose α, β are the roots of the equation $x^2 + 2px + q = 0$ and $1/\beta$, are the roots of the equation $ax^2 + 2bx + c = 0$, where $\beta^2 \notin \{-1, 0, 1\}$.

Statement 1: $(p^2 - q)(b^2 - ac) \geq 0$

Statement 2: $b \neq pa$ or $c \neq qa$

a. Statement 1 is true, statement 2 is true; statement 2 is a correct explanation for statement 1.

b. Statement 1 is true, statement 2 is true; statement 2 is not a correct explanation for statement 1.

c. Statement 1 is true, statement 2 is false.

d. Statement 1 is false, statement 2 is true.

(IIT-JEE, 2008)

Integer type

1. The number of distinct real roots of $x^4 - 4x^3 + 12x^2 + x - 1 = 0$ is. **(IIT-JEE, 2011)**

ANSWERS AND SOLUTIONS

Subjective Type

1. Let,

$$(\sqrt{x^2-5x+6}+\sqrt{x^2-5x+4})^{x/2}=A$$

and

$$(\sqrt{x^2-5x+6}-\sqrt{x^2-5x+4})^{x/2}=B$$

We have,

$$A+B=2^{\frac{x+4}{4}}=2\times 2^{\frac{x}{4}} \text{ and } AB=2^{\frac{x}{2}}$$

Therefore,

$$(A-B)^2=(A+B)^2-4AB$$
$$=4\times 2^{\frac{x}{2}}-4\times 2^{\frac{x}{2}}=0$$

$\Rightarrow \quad A=B$

Since powers in A and B are same, either the power is equal to zero or the bases are the same. Hence,

$$x=0 \text{ or } \sqrt{x^2-5x+6}+\sqrt{x^2-5x+4}=\sqrt{x^2-5x+6}-\sqrt{x^2-5x+4}$$

$\Rightarrow \quad x=0$ or $\sqrt{x^2-5x+4}=0$

$\Rightarrow \quad x=0$ or $x=4$ or 1

2. Suppose one root of the equation is $u+iv$, then the other root would be $u-iv$. Hence,

$$\Rightarrow \frac{A^2}{(u-a)+iv}+\frac{B^2}{(u-b)+iv}+\cdots+\frac{H^2}{(u-h)+iv}=k \quad (1)$$

and

$$\frac{A^2}{(u-a)-iv}+\frac{B^2}{(u-b)-iv}+\cdots+\frac{H^2}{(u-h)-iv}=k \quad (2)$$

From (1) − (2), we get

$$iv\left[\frac{A^2}{(u-a)^2+v^2}+\frac{B^2}{(u-b)^2+v^2}+\cdots+\frac{H^2}{(u-h)^2+v^2}\right]=0$$

This is possible only when $v=0$, and for this case there is no imaginary root.

3. Given, $a\neq b\neq c$, $a, b, c \in R$. Now,

$ax^2+bx+c\geq 0$

$\Rightarrow \quad b^2-4ac\leq 0$ and $a>0$ (1)

$bx^2+cx+a\geq 0$

$\Rightarrow \quad c^2-4ab\leq 0$ and $b>0$ (2)

$cx^2+ax+b\geq 0$

$\Rightarrow \quad a^2-4bc\leq 0$ and $c>0$ (3)

Equality cannot hold simultaneously in (1), (2) and (3).

$$\therefore \quad \frac{a^2+b^2+c^2}{ab+bc+ca}<4 \quad (4)$$

Now,

$(a-b)^2+(b-c)^2+(c-a)^2>0$ ($\because$ a, b, c are distinct)

$$\Rightarrow \quad \frac{a^2+b^2+c^2}{ab+bc+ca}>1 \quad (5)$$

From (4) and (5), $(a^2+b^2+c^2)/(ab+bc+ca)$ can never lie in $(-\infty, 1]\cup[4,\infty)$.

4. Let α, β be the root of a quadratic and α^2, β^2 be the roots of another quadratic. Since the quadratics remain same, we have

$\alpha+\beta=\alpha^2+\beta^2$ (1)

$\alpha\beta=\alpha^2\beta^2$ (2)

Now,

$\alpha\beta=\alpha^2\beta^2$

$\Rightarrow \quad \alpha\beta(1-\alpha\beta)=0$

$\Rightarrow \quad \alpha=0$ or $\beta=0$ or $\alpha\beta=1$

Case I:

When $\alpha=0$, from (1),

$\alpha+\beta=\alpha^2+\beta^2$

$\Rightarrow \quad \beta=\beta^2$

$\Rightarrow \quad \beta(1-\beta)=0$

$\Rightarrow \quad \beta=0$ or $\beta=1$

Thus, we get two sets of values of α and β, viz., $\alpha=0, \beta=0$ and $\alpha=0, \beta=1$

Case II:

When $\beta=0$, from (1),

$\alpha+\beta=\alpha^2+\beta^2$

$\Rightarrow \quad \alpha=\alpha^2$

$\Rightarrow \quad \alpha(1-\alpha)=0$

$\Rightarrow \quad \alpha=0$ or $\alpha=1$

Thus, we get two sets of values of α and β, viz., $\alpha=0, \beta=0$ and $\alpha=1, \beta=0$.

Case III:

When $\alpha\beta=1$, from (1),

$\alpha+\beta=\alpha^2+\beta^2$

$$\Rightarrow \quad \alpha+\frac{1}{\alpha}=\alpha^2+\frac{1}{\alpha^2} \qquad \left[\because \alpha\beta=1\Rightarrow\beta=\frac{1}{\alpha}\right]$$

$$\Rightarrow \quad \alpha+\frac{1}{\alpha}=\left(\alpha+\frac{1}{\alpha}\right)^2-2$$

$$\Rightarrow \quad \left(\alpha+\frac{1}{\alpha}\right)^2-\left(\alpha+\frac{1}{\alpha}\right)-2=0$$

$$\Rightarrow \quad y^2-y-2=0, \text{ where } y=\alpha+\frac{1}{\alpha}$$

$\Rightarrow \quad (y-2)(y+1)=0$

$\Rightarrow \quad y=2$ or $y=-1$

Now,

$y=2$

$$\Rightarrow \quad \alpha+\frac{1}{\alpha}=2$$

$\Rightarrow \quad \alpha=1$

Similarly,

$y=-1$

$$\Rightarrow \quad \alpha+\frac{1}{\alpha}=-1$$

$\Rightarrow \quad \alpha^2+\alpha+1=0$

$\Rightarrow \quad \alpha = \dfrac{-1 \pm i\sqrt{3}}{2} = \omega, \omega^2$ (where ω, ω^2 are cube roots of unity. See in Chapter 3 for more details)

When

$$\alpha = 1, \alpha\beta = 1 \Rightarrow \beta = 1$$

When

$$\alpha = \omega, \alpha\beta = 1 \Rightarrow \beta = \frac{1}{\omega} = \omega^2$$

When

$$\alpha = \omega^2, \alpha\beta = 1 \Rightarrow \beta = \frac{1}{\omega^2} = \omega$$

Thus, when $\alpha\beta = 1$, we get two sets of values of α and β, viz.,

$$\alpha = 1, \beta = 1; \alpha = \omega, \beta = \omega^2.$$

Hence, there are four sets of values of α and β, viz. $\alpha = 0$, $\beta = 0$; $\alpha = 1, \beta = 0$; $\alpha = 1, \beta = 1$ and $\alpha = \omega, \beta = \omega^2$. Consequently, there are four quadratic equations, which do not change by squaring their roots.

5. The given equation is $x^2 - px - (p + c) = 0$.

$\therefore \quad \alpha + \beta = p, \alpha\beta = -(p + c)$

So,

$$(\alpha + 1)(\beta + 1)$$
$$= \alpha\beta + (\alpha + \beta) + 1$$
$$= -(p + c) + p + 1$$
$$= 1 - c \qquad (1)$$

Now,

$$\frac{\alpha^2+2\alpha+1}{\alpha^2+2\alpha+c} + \frac{\beta^2+2\beta+1}{\beta^2+2\beta+c}$$

$$= \frac{(\alpha+1)^2}{(\alpha+1)^2-(1-c)} + \frac{(\beta+1)^2}{(\beta+1)^2-(1-c)}$$

$$= \frac{(\alpha+1)^2}{(\alpha+1)^2-(\alpha+1)(\beta+1)} + \frac{(\beta+1)^2}{(\beta+1)^2-(\alpha+1)(\beta+1)} \quad \text{[Using (1)]}$$

$$= \frac{\alpha+1}{\alpha-\beta} + \frac{\beta+1}{\beta-\alpha} = \frac{(\alpha+1)-(\beta+1)}{\alpha-\beta} = 1$$

6. Since α, β are the roots of equation $ax^2 + bx + c = 0$, therefore,

$$a\alpha^2 + b\alpha + c = 0$$
$$a\beta^2 + b\beta + c = 0$$
$$\alpha + \beta = -\frac{b}{a}, \alpha\beta = \frac{c}{a}$$

Given, $S_n = \alpha^n + \beta^n$. Now,

$$aS_{n+1} + bS_n + cS_{n-1}$$
$$= a(\alpha^{n+1} + \beta^{n+1}) + b(\alpha^n + \beta^n) + c(\alpha^{n-1} + \beta^{n-1})$$
$$= \alpha^{n-1}(a\alpha^2 + b\alpha + c) + \beta^{n-1}(a\beta^2 + b\beta + c)$$
$$= \alpha^{n-1} \times 0 + \beta^{n-1} \times 0$$
$$= 0$$

$$\therefore \quad S_{n+1} = -\frac{b}{a}S_n - \frac{c}{a}S_{n-1} \qquad (1)$$

Putting $n = 4$ in (1), we get

$$S_5 = -\frac{b}{a}S_4 - \frac{c}{a}S_3$$

$$= -\frac{b}{a}\left(-\frac{b}{a}S_3 - \frac{c}{a}S_2\right) - \frac{c}{a}S_3 \quad \text{[From (1), when } n = 3]$$

$$= \left(\frac{b^2}{a^2} - \frac{c}{a}\right)S_3 + \frac{bc}{a^2}S_2$$

$$= \left(\frac{b^2}{a^2} - \frac{c}{a}\right)\left(-\frac{b}{a}S_2 - \frac{c}{a}S_1\right) + \frac{bc}{a^2}S_2$$

$$= \left(-\frac{b^3}{a^3} + \frac{2abc}{a^3}\right)S_2 - \left(\frac{b^2}{a^2} - \frac{c}{a}\right)\frac{c}{a}S_1$$

$$= -\frac{b(b^2-2ac)}{a^3}\left[\left(-\frac{b}{a}\right)^2 - \frac{2c}{a}\right] - \left(\frac{b^2-ac}{a^2}\right)\frac{c}{a}\left(-\frac{b}{a}\right)$$

$$[\because S_2 = \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta \text{ and } S_1 = \alpha + \beta = -\frac{b}{a}]$$

$$= -\frac{b}{a^5}(b^2 - 2ac)^2 + \frac{(b^2-ac)bc}{a^4}$$

7. $4x^2 + 2x - 1 = 0$

$$\therefore \quad \alpha + \beta = -\frac{1}{2}, \alpha\beta = -\frac{1}{4} \qquad (1)$$

Also, $4\alpha^2 + 2\alpha - 1 = 0$ as α is a root, and we have to prove that $\beta = 4\alpha^3 - 3\alpha$. Now,

$$4\alpha^3 - 3\alpha = 4\alpha^2\alpha - 3\alpha$$
$$= \alpha(1 - 2\alpha) - 3\alpha$$
$$= -2\alpha^2 - 2\alpha$$
$$= -\frac{1}{2}[4\alpha^2 + 4\alpha]$$
$$= -\frac{1}{2}[1 - 2\alpha + 4\alpha]$$
$$= -\frac{1}{2}(1 + 2\alpha) = -\frac{1}{2} - \alpha = \beta$$

Now,

$$a + \beta = -\frac{1}{2} \quad \text{[From (1)]}$$

Hence, the other root β is $4\alpha^3 - 3\alpha$.

8. The given equation can be written as $(ay + a')x^2 + (by + b')x + (cy + c') = 0$. Since roots are rational, therefore, D is a perfect square. Hence, $(by + b')^2 - 4(ay + a')(cy + c')$ is a perfect square. That is, $y^2(b^2 - 4ac) + [2bb' - 4(ac' + a'c)]y + (b'^2 - 4a'c')$ is a perfect square. In other words, the roots of the above equation are equal so that $D = 0$.

$\therefore \quad [2bb' - 4(ac' + a'c)]^2 - 4(b^2 - 4ac)(b'^2 - 4a'c') = 0$

On simplifying, we get the required result.

9. Given roots are real and distinct. So,

$$a^2 - 4b > 0 \Rightarrow b < a^2/4 \qquad (1)$$

Again α and β differ by a quantity less than c ($c > 0$). Hence,

$$|\alpha - \beta| < c \text{ or } (\alpha - \beta)^2 < c^2$$
$$\Rightarrow \quad (\alpha + \beta)^2 - 4\alpha\beta < c^2$$
$$\Rightarrow \quad a^2 - 4b < c^2$$
$$\Rightarrow \quad \frac{a^2 - c^2}{4} < b \qquad (2)$$
$$\Rightarrow \quad \frac{a^2 - c^2}{4} < b < \frac{a^2}{4} \quad \text{[From (1) and (2)]}$$

10. Given $ax^2 + bx + c = 0$ has real and positive roots. Then,

$$b^2 - 4ac \geq 0 \quad (1)$$

Sum of roots is

$$-b/a > 0 \text{ or } b/a < 0 \quad (2)$$

i.e., a and b have opposite signs.

Product of root is

$$c/a > 0 \quad (3)$$

i.e., a and c have same sign.

Now, for equation $a^2x^2 + a(3b - 2c)x + (2b - c)(b - c) + ac = 0$, we have

$$D = a^2(3b - 2c)^2 - 4a^2[(2b - c)(b - c) + ac]$$
$$= a^2[9b^2 - 12bc + 4c^2 - 4(2b^2 - 3bc + c^2 + ac)]$$
$$= a^2[9b^2 - 12bc + 4c^2 - 8b^2 + 12bc - 4c^2 - 4ac]$$
$$= a^2(b^2 - 4ac) \geq 0 \quad \text{[Using (1)]}$$

Hence, the roots are real. Also, sum of roots,

$$\frac{-a(3b-2c)}{a^2} = -\left(\frac{3b}{a} - \frac{2c}{a}\right) > 0 \quad \text{[Using (2) and (3)]}$$

Product of roots,

$$\frac{(2b-c)(b-c)+ac}{a^2} = \left(2\frac{b}{a} - \frac{c}{a}\right)\left(\frac{b}{a} - \frac{c}{a}\right) + \frac{c}{a} > 0 \quad \text{[Using (2) and (3)]}$$

Hence, the roots are positive.

11. $$x = \frac{-p - \sqrt{p^2 + 4 \times 444p}}{2}$$

Since $p = 2$ does not give the integral roots, so D must be a perfect square of an odd integer, i.e.,

$$D^2 = p^2 + 1776p = p(p + 1776)$$

Since D is perfect square, hence $p + 1776$ must be a multiple of p, i.e., 1776 must be a multiple of p. Now, $1776 = 2^4 \times 3 \times 37$ hence $p = 2$ or 3 or 37.

(i) If $p = 2$, then $p(p + 1776) = 2(3 + 1776) = 3556 = 4 \times 7 \times 127$, which is not a perfect square.

(ii) If $p = 3$, then $p(p + 1776) = 3(3 + 1776) = 5337$, which is not a perfect square as its last digit is 7.

(iii) If $p = 37$, then $p(p + 1776) = 37(37 + 1776) = 37^2 \times 7^2$, which is odd. Hence, $p = 37$.

12. Consider the equation

$$ax^2 + bx + c = 0$$

Since the roots are rational, discriminant of the given quadratic equation will be a perfect square. Hence,

$$b^2 - 4ac = \lambda^2, \lambda \in I$$
$$\Rightarrow b^2 - \lambda^2 = 4ac \Rightarrow (b + \lambda)(b - \lambda) = 4ac$$

Then, we have the following possibilities:

$$b + \lambda = 2a, b - \lambda = 2c$$
$$\Rightarrow b + \lambda = 2c, b - \lambda = 2a$$
$$\Rightarrow b + \lambda = -2a, b - \lambda = -2c$$
$$\Rightarrow b + \lambda = -2c, b - \lambda = -2a$$

(as $b + \lambda$ and $b - \lambda$ both should be even)

Solving the above cases, we get

$$b = \pm(a + c), \lambda = \pm(a - c)$$

Hence, the roots are $-(b-\lambda)/2a$. Clearly, one of the roots is $-1/2$ (put the value of λ and simplify to get $-1/2$).

13. We are given that $(2x + y)(x - 2y) = 7$. Since x and y are to be integers, hence, L.H.S. is the product of two integers and R.H.S. is also the product of two integers, viz, 7 and 1, or 1 and 7, or -7 and -1, or -1 and -7. Hence, we can choose

$$2x + y = 7 \text{ and } x - 2y = 1 \quad (1)$$
$$2x + y = 1 \text{ and } x - 2y = 7 \quad (2)$$
$$2x + y = -7 \text{ and } x - 2y = -1 \quad (3)$$
$$2x + y = -1 \text{ and } x - 2y = -7 \quad (4)$$

Solving them as usual we find only (1) and (3) give integral solutions as 3, 1 for (1) and $-3, -1$ for (3). Both (2) and (4) when solved do not give integral values of x and y.

14. Given $ax^2 + bx + c - p = 0$ has integral roots. Let α, β be the roots. Then,

$$ax^2 + bx + c - p = a(x - \alpha)(x - \beta) \quad (1)$$

Now from $ax^2 + bx + c = 2p$, we have

$$ax^2 + bx + c - p = p$$
$$\Rightarrow a(x - \alpha)(x - \beta) = p \quad \text{[Using (1)]}$$

In above equation, L.H.S. has three factors but R.H.S. is prime number, which is contradiction. Hence, $ax^2 + bx^2 + c = 2p$ cannot have integral roots.

15. Given expression is $(x+a)(x+b)/(x+c)$. Let $x + c = y$. Then,

$$\frac{(x+a)(x+b)}{(x+c)} = \frac{(y+(a-c))(y+(b-c))}{y}$$
$$= \frac{y^2 + [(a-c)+(b-c)]y + (a-c)(b-c)}{y}$$
$$= y + \frac{(a-c)(b-c)}{y} + (a-c) + (b-c)$$
$$= \left[\sqrt{y} - \sqrt{\frac{(a-c)(b-c)}{y}}\right]^2 + \left[\sqrt{a-c} + \sqrt{b-c}\right]^2$$
$$\geq \left[\sqrt{a-c} + \sqrt{b-c}\right]^2$$

Hence, the least value is $\left[\sqrt{a-c} + \sqrt{b-c}\right]^2$.

16. Let

$$t = x + \sqrt{x^2 + b^2} \quad (1)$$
$$\Rightarrow \sqrt{x^2 + b^2} - x = \frac{b^2}{t} \quad (2)$$
$$\therefore 2x = t - \frac{b^2}{t} \quad \text{[Subtracting (2) from (1)]}$$
$$\Rightarrow x = \frac{1}{2}\left(\frac{t^2 - b^2}{t}\right)$$

Now,

$$y = 2(a - x)t$$
$$= 2\left(a - \left(\frac{t^2 - b^2}{2t}\right)\right)t$$
$$= (2at - t^2 + b^2)$$
$$= b^2 - (t^2 - 2at + a^2 - a^2)$$
$$= a^2 + b^2 - (t - a)^2$$
$$\Rightarrow y \leq a^2 + b^2$$

17. $f(0) = d, f(-1) = -1 + b - c + d$

$\Rightarrow$ $d =$ odd and $-1 + b - c + d =$ odd

$\Rightarrow$ $b - c = 1 +$ odd $- d$

$= (1 +$ odd$) - ($odd$) =$ even $-$ odd $=$ odd (1)

Thus, both d and $b - c$ are odd.

If possible let the three roots α, β, γ be all integers. Now,

$$\alpha\beta\gamma = -\frac{d}{1} = -d = \text{negative odd integer} \quad (2)$$

$\Rightarrow$ α, β, γ are three integers whose product is odd

$\Rightarrow$ α, β, γ all are odd

Again

$\alpha + \beta + \gamma = -b$ and $\alpha\beta + \beta\gamma + \alpha\gamma = c$ (3)

$\Rightarrow$ b and c both will be odd

$\Rightarrow$ $(b - c)$ will be even which contradicts with (1)

Hence, the three roots cannot be all integers.

18. We have,

$|x| < a \Rightarrow -a < x < +a$

Therefore, the given inequality implies

$$-2 < \frac{x^2+kx+1}{x^2+x+1} < 2 \quad (1)$$

Now, $x^2 + x + 1 = (x+1/x)^2 + (3/4)$ is positive for all values of x.

Multiplying (1) by $x^2 + x + 1$, we get

$-2(x^2 + x + 1) < x^2 + kx + 1 < 2(x^2 + x + 1)$

This yields two inequalities, viz.,

$3x^2 + (2 + k)x + 3 > 0$

and

$x^2 + (2 - k)x + 1 > 0$

For these quadratic expressions to be positive for all values of x, their discriminants must be negative. Hence,

$(2 + k)^2 - 36 < 0$ and $(2 - k)^2 - 4 < 0$ (2)

$\Rightarrow$ $(k + 8)(k - 4) < 0$ and $k(k - 4) < 0$ (3)

$\Rightarrow$ $-8 < k < 4$ and $0 < k < 4$

For both these conditions to be satisfied, $0 < k < 4$.

19. Given equation is

$$\sqrt{a(2^x - 2)+1} = 1 - 2^x \quad (1)$$

$\Rightarrow$ $\sqrt{a(y-2)+1} = 1 - y$, where $y = 2^x > 0$ (2)

$\Rightarrow$ $a(y - 2) + 1 = (1 - y)^2 = 1 - 2y + y^2$

$\Rightarrow$ $y^2 - (2 + a)y + 2a = 0$

$$\Rightarrow \quad y = \frac{(2+a) - \sqrt{(2+a)^2 - 8a}}{2}$$

$$= \frac{2+a-(2-a)}{2} = 2, a$$

$y = 2$ does not satisfy Eq. (2) because in that case R.H.S. of Eq. (2) is negative and L.H.S. is positive. When $y = a$, from (2),

$\sqrt{a(a-2)+1} = 1 - a$

$\Rightarrow$ $\sqrt{(a-1)^2} = 1 - a$ or $|a - 1| = 1 - a$

$\Rightarrow$ $a - 1 \le 0 \Rightarrow a \le 1$

$\Rightarrow$ $0 < a \le 1$ $[\because y > 0]$

Now,

$y = a \Rightarrow 2^x = a \Rightarrow x = \log_2 a$,

where $0 < a \le 1$. When $a > 1$, given equation has no solution.

20. Case I:

Suppose $x \ge a$. Then the given equation becomes

$x^2 - 2a(x - a) - 3a^2 = 0$

$\Rightarrow$ $x^2 - 2ax - a^2 = 0$

$$\Rightarrow x = \frac{2a \pm 2\sqrt{2}a}{2} = (1 \pm \sqrt{2})a$$

As $a < 0$ and $1 + \sqrt{2} > 1$, so $(1 + \sqrt{2})a < a$, therefore $x \ne (1 + \sqrt{2})a$.

Next, as $1 - \sqrt{2} < 1$, so $(1 - \sqrt{2})a > a$, therefore $x = (1 - \sqrt{2})a$.

Case II:

Suppose $x < a$. Then the given equation becomes

$x^2 - 2a(a - x) - 3a^2 = 0$

$\Rightarrow$ $x^2 + 2ax - 5a^2 = 0$

$$\Rightarrow \quad x = \frac{-2a - \sqrt{24a}}{2} = (-1 \pm \sqrt{6})a$$

As $a < 0$, $-1 - \sqrt{6} < 0 < 1$, so $(-1 - \sqrt{6})a > a$, therefore $x \ne (-1 - \sqrt{6})a$ ($\because x < a$). Next, as $a < 0$, $-1 + \sqrt{6} > 1$ and $(-1 + \sqrt{6})a < a$, therefore, $x = (-1 + \sqrt{6})a = (\sqrt{6} - 1)a$.

21. Given equation is

$x^3 - 10x^2 - 11x - 100 = 0$

Let

$f(x) = x^3 - 10x^2 - 11x - 100$

$\Rightarrow f'(x) = 3x^2 - 20x - 11$

For $3x^2 - 20x - 11 = 0$, we have

$$x = \frac{20 - \sqrt{400+132}}{6} = \frac{10 - \sqrt{133}}{3}$$

Hence, graph of $y = f(x)$ is

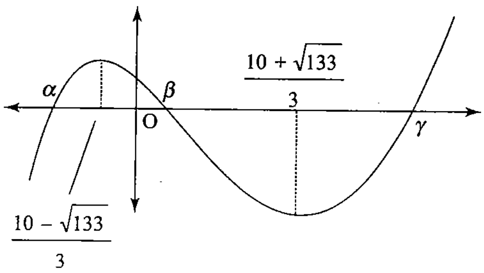


Fig. 1.68

Now, $(10 + \sqrt{133})/3 \cong 7.16$.

$f(8) = 8^3 - 10(8)^2 - 11(8) - 100 < 0$

$f(9) = 9^3 - 10(9)^2 - 11(9) - 100 < 0$

$f(10) = 10^3 - 10(10)^2 - 11(10) - 100 < 0$

$f(11) = 11^3 - 10(11)^2 - 11(11) - 100 < 0$

$f(12) = 12^3 - 10(12)^2 - 11(12) - 100 > 0$

$\Rightarrow$ $\gamma \in (11, 12)$

$\Rightarrow$ $[\gamma] = 11$

22. Given,

$x^4 - 4x^3 - 8x^2 + a = 0$

Let,

$f(x) = x^4 - 4x^3 - 8x^2$

$\Rightarrow \quad f'(x) = 4x^3 - 12x^2 - 16x = 4x(x^2 - 3x - 4) = 4x(x-4)(x+1)$

Hence, graph of $y = f(x)$ is

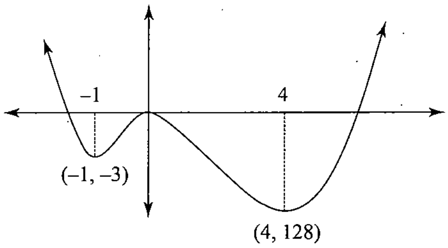


Fig. 1.69

Now, graph of $y = f(x) + a$ moves up or down depending on values of a. It is clear that if equation $f(x) + a = 0$ has four real roots, then

$0 \le a < 3.$

Objective Type

1. b. Given,

$x - 2 = 2^{2/3} + 2^{1/3}$

Cubing both sides, we get

$(x-2)^3 = 2^2 + 2 + 3 \times 2^{2/3} \times 2^{1/3}(x-2) = 6 + 6(x-2)$

or

$x^3 - 6x^2 + 12x - 8 = -6 + 6x$

$\therefore \quad x^3 - 6x^2 + 6x = 2$

2. a. Let,

$f(x, y, z) = x^2 + 4y^2 + 3z^2 - 2x - 12y - 6z + 14$

$= (x-1)^2 + (2y-3)^2 + 3(z-1)^2 + 1$

For the least value of $f(x, y, z)$,

$x - 1 = 0.\ 2y - 3 = 0$ and $z - 1 = 0$

$\therefore \quad x = 1, y = 3/2, z = 1$

Hence the least value of $f(x, y, z)$ is $f(1, 3/2, 1) = 1$.

3. c. Let m be a positive integer for which

$n^2 + 96 = m^2$

$\Rightarrow \quad m^2 - n^2 = 96 \Rightarrow (m+n)(m-n) = 96$

$\Rightarrow \quad (m+n)\{(m+n) - 2n\} = 96$

$\Rightarrow \quad m + n$ and $m - n$ must be both even

As $96 = 2 \times 48$ or 4×24 or 6×16 or 8×12, hence, number of solutions is 4.

4. d. $f(x, y) = (x-2)^2 + (y-1)^2 = 0$

$\Rightarrow \quad x = 2$ and $y = 1$

$\therefore \quad E = \dfrac{(\sqrt{2}-1)^2 + 4\sqrt{2}}{2+\sqrt{2}} = \dfrac{(\sqrt{2}+1)^2}{\sqrt{2}(\sqrt{2}+1)} = \dfrac{\sqrt{2}+1}{\sqrt{2}}$

5. c. We have,

$|x|^2 - 3|x| + 2 = 0$

$\Rightarrow \quad (|x| - 1)(|x| - 2) = 0$

$\Rightarrow \quad |x| = 1, 2 \Rightarrow x = \pm 1, \pm 2$

6. c. Clearly, $x = -1$ satisfies the equation.

7. d. Given equation is satisfied by $x = 1, 2, 3$. But for $x = 1$, $\sqrt{x-2}$ is not defined. Hence, number of roots is 2 and the roots are $x = 2$ and 3.

8. c. As $(\lambda + 1)x^2 + 2 = \lambda x + 3$ has only one solution, so

$D = 0$

$\Rightarrow \quad \lambda^2 - 4(\lambda + 1)(-1) = 0$

$\Rightarrow \quad \lambda^2 + 4\lambda + 4 = 0$

$\Rightarrow \quad (\lambda + 2)^2 = 0$

$\therefore \quad \lambda = -2$

9. a. Given quadratic expression is $x^2 + 2(a + b + c)x + 3(bc + ca + ab)$. this quadratic expression will be a perfect square if the discriminant of its corresponding equation is zero. Hence,

$4(a+b+c)^2 - 4 \times 3(bc + ca + ab) = 0$

$\Rightarrow \quad (a+b+c)^2 - 3(bc + ca + ab) = 0$

$\Rightarrow \quad a^2 + b + c^2 + 2ab + 2bc + 2ca - 3(bc + ca + ab) = 0$

$\Rightarrow \quad a^2 + b^2 + c^2 - ab\ \ bc - ca = 0$

$\Rightarrow \quad \frac{1}{2}[2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca] = 0$

$\Rightarrow \quad \frac{1}{2}[(a^2+b^2-2ab)+(b^2+c^2-2bc)+(c^2+a^2-2ca)]=0$

$\Rightarrow \quad \frac{1}{2}[(a-b)^2 + (b-c)^2 + (c-a)^2] = 0$

which is possible only when $(a-b)^2 = 0$, $(b-c)^2 = 0$ and $(c-a)^2 = 0$, i.e., $a = b = c$.

10. c. If one root is square of the other root of the equation $ax^2 + bx + c = 0$, then

$\beta = \alpha^2 \Rightarrow \alpha^2 + \alpha = -b/a$ and $\alpha^2 a = c/a$

By eliminating α, we get

$b^3 + ac^2 + a^2c = 3abc$

which can be written in the form $a(c-b)^3 = c(a-b)^3$.

Alternative solution:

Let the roots be 2 and 4. Then the equation is $x^2 - 6x + 8 = 0$.

Here obviously,

$$X = \frac{a(c-b)^3}{c} = \frac{1(14)^3}{8} = \frac{14}{2} \times \frac{14}{2} \times \frac{14}{2} = 7^3$$

which is given by $(a-b)^3 = 7^3$.

11. a. Given that $x^2 + px + 1$ is a factor of $ax^3 + bx + c$. Then let $ax^3 + bx + c = (x^2 + px + 1)(ax + \lambda)$, where λ is a constant. Then equating the coefficients of like powers of x on both sides, we get

$0 = ap + \lambda,\ b = p\lambda + a,\ c = \lambda$

$\Rightarrow \quad p = -\dfrac{\lambda}{a} = -\dfrac{c}{a}$

Hence,

$b = \left(-\dfrac{c}{a}\right)c + a$

or

$ab = a^2 - c^2$

12. a. Put $x^2 + x = y$, so that Eq. (1) becomes

$(y-2)(y-3) = 12$

$\Rightarrow \quad y^2 - 5y - 6 = 0$

$\Rightarrow \quad (y-6)(y+1) = 0 \Rightarrow y = 6, -1$

When $y = 6$, we get

$x^2 + x - 6 = 0$

$\Rightarrow \quad (x+3)(x-2) = 0$ or $x = -3, 2$

When $y = -1$, we get

$x^2 + x + 1 = 0$

which has non-real roots and sum of roots is -1.

13. b. Given,

$(ax^2 + c)y + (a'x^2 + c') = 0$

or

$x^2(ay + a') + (cy + c') = 0.$

If x is rational, then the discriminant of the above equation must be a perfect square. Hence,

$0 - 4(ay + a')(cy + c')$ must be a perfect square

$\Rightarrow -acy^2 - (ac' + a'c)y - a'c'$ must be a perfect square

$\Rightarrow (ac' + a'c)^2 - 4ac\,a'c' = 0 \quad [\because D = 0]$

$\Rightarrow (ac' - a'c)^2 = 0$

$\Rightarrow ac' = a'c$

$\Rightarrow \dfrac{a}{a'} = \dfrac{c}{c'}$

14. d. Let the four numbers in A.P. be $p = a - 3d$, $q = a - d$, $r = a + d$, $s = a + 3d$. Therefore,

$p + q = 2, r + s = 18$

Given that $pq = A$, $rs = B$.

$\therefore \; p + q + r + s = 4a = 20$

$\Rightarrow a = 5$

Now,

$p + q = 2 \Rightarrow 10 - 4d = 2$

$r + s = 18 \Rightarrow 10 + 4d = 18$

$\therefore \; d = 2$

Hence, the numbers are $-1, 3, 7, 11$.

$pq = A = -3, rs = B = 77$

15. d. Here, $x = 0$ is not a root. Divide both the numerator and denominator by x and put $x + 3/x = y$ to obtain

$$\frac{4}{y+1} + \frac{5}{y-5} = -\frac{3}{2} \Rightarrow y = -5, 3$$

$x + 3/x = -5$ has two irrational roots and $x + 3/x = 3$ has imaginary roots.

16. c. Clearly, $x = 2$ is a root of the equation and imaginary roots always occur in pairs. Therefore, the other root is also real.

17. c. Given, α, β are roots of equation

$x^2 - 2x + 3 = 0$

$\Rightarrow \alpha^2 - 2\alpha + 3 = 0 \quad (1)$

and

$\beta^2 - 2\beta + 3 = 0 \quad (2)$

$\Rightarrow \alpha^2 = 2\alpha - 3 \Rightarrow \alpha^3 = 2\alpha^2 - 3\alpha$

$\Rightarrow P = (2\alpha^2 - 3\alpha) - 3\alpha^2 + 5\alpha - 2$

$= -\alpha^2 + 2\alpha - 2 = 3 - 2 = 1, \quad$ [Using (1)]

Similarly, we have $Q = 2$.

Now, sum of roots is 3 and product of roots is 2. Hence, the required equation is $x^2 - 3x + 2 = 0$.

18. c. Since α, β are the roots of the equation $2x^2 - 35x + 2 = 0$, therefore,

$$2\alpha^2 - 35\alpha = -2 \text{ or } 2\alpha - 35 = \frac{-2}{\alpha}$$

and

$$2\beta^2 - 35\beta = -2 \text{ or } 2\beta - 35 = \frac{-2}{\beta}$$

Now,

$$(2\alpha - 35)^3(2\beta - 35)^3 = \left(\frac{-2}{\alpha}\right)^3\left(\frac{-2}{\beta}\right)^3$$

$$= \frac{8 \times 8}{\alpha^3\beta^3} = \frac{64}{1} = 64 \quad (\because \alpha\beta = 1)$$

19. c. Here a, b, c are positive. So,

$|x| = -b + \sqrt{b^2 + ac}$

Hence, x has two real values, neglecting $|x| = -b - \sqrt{b^2 + ac}$, as $|x| \geq 0$.

20. a. Since $p(q - r) + q(r - p) + r(p - q) = 0$, so one root is 1 and the other root is $r(p - q)/[p(q - r)]$. Since both the roots are equal, we have

$$\frac{rp - rq}{pq - pr} = 1$$

$\Rightarrow rp - rq = pq - pr$

$\Rightarrow 2rp = q(p + r)$

$\Rightarrow \dfrac{2}{q} = \dfrac{p + r}{pr} = \dfrac{1}{p} + \dfrac{1}{r}$

21. d. We are given that $p(-a) = a$ and $p(a) = -a$

[since when a polynomial $f(x)$ is divided by $x - a$, remainder is $f(a)$]. Let the remainder, when $p(x)$ is divided by $x^2 - a^2$, be $Ax + B$. Then,

$p(x) = Q(x)(x^2 - a^2) + Ax + B \quad (1)$

where $Q(x)$ is the quotient. Putting $x = a$ and $-a$ in (1), we get

$p(a) = 0 + Aa + B \Rightarrow -a = Aa + B \quad (2)$

and

$p(-a) = 0 - aA + B \Rightarrow a = -aA + B \quad (3)$

Solving (2) and (3), we get

$B = 0$ and $A = -1$

Hence, the required remainder is $-x$.

22. a. $x^2 + ax + b + 1 = 0$ has positive integral roots α and β. Hence,

$(\alpha + \beta) = -a$ and $\alpha\beta = b + 1$

$\Rightarrow (\alpha + \beta)^2 + (\alpha\beta - 1)^2 = a^2 + b^2$

$\Rightarrow a^2 + b^2 = (\alpha^2 + 1)(\beta^2 + 1).$

$\Rightarrow a^2 + b^2$ can be equal to 50 (since other options have prime numbers)

23. b. $(31 + 8\sqrt{15})^{x^2-3} + 1 = (32 + 8\sqrt{15})^{x^2-3}$

$\Rightarrow (31 + 8\sqrt{15})^{x^2-3} + 1^{x^2-3} = (32 + 8\sqrt{15})^{x^2-3}$

$\Rightarrow x^2 - 3 = 1$ or $x = \pm 2$ $[\because a^n + b^n = (a + b)^n]$

24. c. $a\alpha^2 + c = -b\alpha$, $a\alpha + b = -\dfrac{c}{\alpha}$

Hence, the given expression is

$$\frac{b}{c}(\alpha^2 + \beta^2) = \frac{b(b^2 - 2ac)}{a^2c}$$

25. a. We have,

$x_1 x_2 = 4$

$\Rightarrow x_2 = \dfrac{4}{x_1}$

$\therefore \; \dfrac{x_1}{x_1 - 1} + \dfrac{x_2}{x_2 - 1} = 2$

$$\Rightarrow \frac{x_1}{x_1 - 1} + \frac{\frac{4}{x_1}}{\frac{4}{x_1} - 1} = 2$$

$$\Rightarrow \frac{x_1}{x_1 - 1} + \frac{4}{4 - x_1} = 2$$

$\Rightarrow$ $4x_1 - x_1^2 + 4x_1 - 4 = 2(x_1 - 1)(4 - x_1)$

$\Rightarrow$ $x_1^2 - 2x_1 + 4 = 0$

$\Rightarrow$ $x^2 - 2x + 4 = 0$

26. b. The discriminants of the given equations are $D_1 = a^2 + 12b$, $D_2 = c^2 - 4b$ and $D_3 = d^2 - 8b$.

$\therefore$ $D_1 + D_2 + D_3 = a^2 + c^2 + d^2 \geq 0$

Hence, at least one of D_1, D_2, D_3 is non-negative. Therefore, the equation has at least two real roots.

27. d. α, β are roots of $x^2 + px + q = 0$. Hence,

$\alpha + \beta = -p$

$\alpha\beta = q$

Now,

α^4, β^4 are roots $x^2 - px + q = 0$. Hence,

$\alpha^4 + \beta^4 = r, \alpha^4\beta^4 = q$

Now, for equation $x^2 - 4qx + 2q^2 - r = 0$, product of roots is

$2q^2 - r = 2(\alpha\beta)^2 - (\alpha^4 + \beta^4)$

$= -(\alpha^2 - \beta^2)^2$

< 0

As product of roots is negative, so the roots must be real.

28. b. $x^4 + x^2 + 1 = (x^2 + 1)^2 - x^2$

$= (x^2 + x + 1)(x^2 - x + 1)$

$$x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} \neq 0 \ \forall x$$

Therefore, we can cancel this factor and we get

$(a - 1)(x^2 - x + 1) = (a + 1)(x^2 - x + 1)$

or

$x^2 - ax + 1 = 0$

It has real and distinct roots if $D = a^2 - 4 > 0$.

29. b. Let D_1 and D_2 be discriminants of $x^2 + b_1x + c_1 = 0$ and $x^2 + b_2x + c_2 = 0$, respectively. Then,

$D_1 + D_2 = b_1^2 - 4c_1 + b_2^2 - 4c_2$

$= (b_1^2 + b_2^2) - 4(c_1 + c_2)$

$= b_1^2 + b_2^2 - 2b_1b_2$ $\quad [\because b_1 b_2 = 2(c_1 + c_2)]$

$= (b_1 - b_2)^2 \geq 0$

$\Rightarrow$ $D_1 \geq 0$ or $D_2 \geq 0$ or D_1 and D_2 both are positive

Hence, at least one of the equations has real roots.

30.c. Discriminant $D = (2m - 1)^2 - 4(m - 2)m = 4m + 1$ must be perfect square. Hence,

$4m + 1 = k^2$, say for some $k \in I$

$$\Rightarrow \quad m = \frac{(k-1)(k+1)}{4}$$

Clearly, k must be odd. Let $k = 2n + 1$.

$$\therefore \quad m = \frac{2n(2n+2)}{4} = = n(n+1), n \in I$$

31.c. $A = a(b - c)(a + b + c)$

$B = b(c - a)(a + b + c)$

$C = c(a - b)(a + b + c)$

Now,

$Ax^2 + Bx + C = 0$

$\Rightarrow$ $(a + b + c)\{a(b - c)x^2 + b(c - a)x + c(a - b)\} = 0$

Given that roots are equal. Hence,

$D = 0$

$\Rightarrow$ $b^2(c - a)^2 - 4ac(b - c)(a - b) = 0$

$\Rightarrow$ $b^2c^2 - 2ab^2c + b^2a^2 - 4a^2bc + 4acb^2 + 4a^2c^2 - 4abc^2 = 0$

$\Rightarrow$ $b^2c^2 + b^2a^2 + 4a^2c^2 + 2ab^2c - 4a^2bc - 4abc^2 = 0$

$\Rightarrow$ $(bc + ab - 2ac)^2 = 0$

$\Rightarrow$ $bc + ab = 2ac$

$$\Rightarrow \quad \frac{1}{a} + \frac{1}{c} = \frac{2}{b}$$

$\Rightarrow$ a, b, c are in H.P.

32. b. Correct equation is

$x^2 + 13x + q = 0$ (1)

Incorrect equation is

$x^2 + 17x + q = 0$ (2)

Given that roots of Eq. (1) are -2 and -15. Therefore, product of the roots of incorrect equation is $q = (-2)(-15) = 30$. From (1), the correct equation is

$x^2 + 13x + 30 = 0$

$\therefore$ $x = -3, -10$

33. b. Given,

$a(p + q)^2 + 2bpq + c = 0$ and $a(p + r)^2 + 2bpr + c = 0$

$\Rightarrow q$ and r satisfy the equation $a(p + x)^2 + 2bpx + c = 0$

$\Rightarrow q$ and r are the roots of

$ax^2 + 2(ap + bp)x + c + ap^2 = 0$

$$\Rightarrow \quad qr = \text{product of roots } = \frac{c + ap^2}{a} = p^2 + \frac{c}{a}$$

34. b. Multiplying the given equation by c/a^3, we get

$$\frac{b^2c^2}{a^3}x^2 - \frac{b^2c}{a^2}x + c = 0$$

$$\Rightarrow \quad a\left(\frac{bc}{a^2}x\right)^2 - b\left(\frac{bc}{a^2}\right)x + c = 0$$

$$\Rightarrow \quad \frac{bc}{a^2}x = \alpha, \beta$$

$\Rightarrow$ $(\alpha + \beta)\alpha\beta\, x = \alpha, \beta$

$$\Rightarrow \quad x = \frac{1}{(\alpha+\beta)\alpha}, \frac{1}{(\alpha+\beta)\beta}$$

35. c. Since α is root of all equations

$a\alpha^2 + 2b\alpha + c = 0$

$2b\alpha^2 + c\alpha + a = 0$

$c\alpha^2 + a\alpha + 2b = 0$

Adding we get $(a + 2b + c)(\alpha^2 + \alpha + 1) = 0$

$a + 2b + c \neq 0$ as $a, b, c > 0$

$\Rightarrow$ $\alpha^2 + \alpha + 1 = 0$ or $\alpha^2 + \alpha = -1$

36. a. Given equation is

$x^2 - (y + 4)x + y^2 - 4y + 16 = 0$

Since x is real, so,

$D \geq 0$

$\Rightarrow$ $(y + 4)^2 - 4(y^2 - 4y + 16) \geq 0$

$\Rightarrow$ $-3y^2 + 24y - 48 \geq 0$

$\Rightarrow$ $y^2 - 8y + 16 \leq 0$

$\Rightarrow$ $(y - 4)^2 \leq 0$

$\Rightarrow$ $y - 4 = 0$

$\Rightarrow$ $y = 4$

Since the equation is symmetric in x and y, therefore $x = 4$ only.

37. a. α, β are roots of $ax^2 + bx + c = 0$. Hence,

$$\alpha + \beta = -\frac{b}{a} \quad (1)$$

$$\alpha\beta = \frac{c}{a} \quad (2)$$

α^2, β^2 are roots of $a^2x^2 + b^2x + c^2 = 0$. Hence,

$$\alpha^2 + \beta^2 = -\frac{b^2}{a^2} \quad (3)$$

$$\alpha^2\beta^2 = \frac{c^2}{a^2} \quad (4)$$

Now, from (3),

$$(\alpha + \beta)^2 - 2\alpha\beta = -\frac{b^2}{a^2}$$

$$\Rightarrow \left(\frac{-b}{a}\right)^2 - 2\frac{c}{a} = \frac{-b^2}{a^2}$$

$$\Rightarrow 2\frac{b^2}{a^2} = \frac{2c}{a}$$

$\Rightarrow$ $b^2 = ac \Rightarrow a, b, c$ are in G.P.

38. c. We have,

$$\frac{k+1}{k} + \frac{k+2}{k+1} = -\frac{b}{a} \quad (1)$$

and

$$\frac{k+1}{k}\,\frac{k+2}{k+1} = \frac{c}{a}$$

$$\Rightarrow \frac{k+2}{k} = \frac{c}{a} \text{ or } \frac{2}{k} = \frac{c}{a} - 1 = \frac{c-a}{a} \text{ or } k = \frac{2a}{c-a} \quad (2)$$

Now, eliminate k. Putting the value of k in Eq. (1), we get

$$\frac{c+a}{2a} + \frac{2c}{c+a} = -\frac{b}{a}$$

$\Rightarrow$ $(c + a)^2 + 4ac = -2b(a + c)$

$\Rightarrow$ $(a + c)^2 + 2b(a + c) = -4ac$

Adding b^2 to both sides, we have

$(a + b + c)^2 = b^2 - 4ac$

39. c. We have,

$$\alpha + \beta = -\frac{b}{a}, \alpha\beta = \frac{c}{a}$$

$$\alpha + h + \beta + h = -\frac{q}{p}, (\alpha + h)(\beta + h) = \frac{r}{p}$$

$$\Rightarrow \alpha + \beta + 2h = -\frac{q}{p}$$

$$\Rightarrow -\frac{b}{a} + 2h = \frac{-q}{p} \quad \left[\because a + b = -\frac{b}{a}\right]$$

$$\Rightarrow h = \frac{1}{2}\left(\frac{b}{a} - \frac{q}{p}\right)$$

40. a. Given, α, β are roots of the equation $(x - a)(x - b) + c = 0$.

Then, by factor theorem,

$(x - a)(x - b) + c = (x - \alpha)(x - \beta)$

Replacing x by $x - c$,

$(x - c - a)(x - c - b) + c = (x - c - \alpha)(x - c - \beta)$

$\Rightarrow (x - c - \alpha)(x - c - \beta) - c = [x - (c + a)][x - (c + b)]$

Then, again by factor theorem roots of the equation $(x - c - \alpha)(x - c - \beta) - c = 0$ are $a + c$ and $b + c$.

41. b. $ax^2 - bx + c = 0$

$$\alpha + \beta = \frac{b}{a}, \alpha\beta = \frac{c}{a}$$

Also,

$(a + cy)^2 = b^2y$

$\Rightarrow$ $c^2y^2 - (b^2 - 2ac)y + a^2 = 0$

$$\Rightarrow \left(\frac{c}{a}\right)^2 y^2 - \left(\left(\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right)\right)y + 1 = 0$$

$\Rightarrow$ $(\alpha\beta)^2 y^2 - (\alpha^2 + \beta^2)y + 1 = 0$

$\Rightarrow$ $y^2 - (\alpha^{-2} + \beta^{-2})y + \alpha^{-2}\beta^{-2} = 0$

$\Rightarrow$ $(y - \alpha^{-2})(y - \beta^{-2}) = 0$

Hence the roots are α^{-2}, β^{-2}.

42. c. Let the roots be α, β.

$\therefore$ $\alpha + \beta = -2a$ and $\alpha\beta = b$

Given,

$|\alpha - \beta| \le 2m$

$\Rightarrow$ $|\alpha - \beta|^2 \le (2m)^2$

$\Rightarrow$ $(\alpha + \beta)^2 - 4ab \le 4m^2$

$\Rightarrow$ $4a^2 - 4b \le 4m^2$

$\Rightarrow$ $a^2 - m^2 \le b$ and discriminant $D > 0$ or $4a^2 - 4b > 0$

$\Rightarrow$ $a^2 - m^2 \le b$ and $b < a^2$

Hence, $b \in [a^2 - m^2, a^2)$.

43. c. Let roots of the equation $ax^2 + 2bx + c = 0$ be α and β and roots of the equation $px^2 + 2qx + r = 0$ be γ and δ. Given,

$$\frac{\alpha}{\beta} = \frac{\gamma}{\delta} \Rightarrow \frac{\alpha}{\gamma} = \frac{\beta}{\delta}$$

$$\Rightarrow \frac{\alpha + \beta}{\gamma + \delta} = \sqrt{\frac{\alpha\beta}{\gamma\delta}}$$

$$\Rightarrow \frac{-\frac{2b}{a}}{-\frac{2q}{p}} = \sqrt{\frac{\frac{c}{a}}{\frac{r}{p}}}$$

$$\Rightarrow \frac{b^2}{ac} = \frac{q^2}{pr}$$

44. a. Let α and α^2 be the roots of $x^2 - x - k = 0$. Then,

$\alpha + \alpha^2 = 1$ and $\alpha^3 = -k$

$\Rightarrow$ $(-k)^{1/3} + (-k)^{2/3} = 1$

$\Rightarrow$ $-k^{1/3} + k^{2/3} = 1$

$\Rightarrow$ $(k^{2/3} - k^{1/3})^3 = 1$

$\Rightarrow$ $k^2 - k - 3k(k^{2/3} - k^{1/3}) = 1$

$\Rightarrow$ $k^2 - k - 3k(1) = 1$

$\Rightarrow$ $k^2 - 4k - 1 = 0$

$\Rightarrow$ $k = 2 \pm \sqrt{5}$

45. d. Here,

$$\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2$$

$$= \{(\alpha + \beta)^2 - 2\alpha\beta\}^2 - 2(\alpha\beta)^2$$

$$= \left(p^2 + \frac{1}{p^2}\right)^2 - \frac{1}{2p^4}$$

$$= p^4 + \frac{1}{2p^4} + 2$$

$$= \left(p^2 - \frac{1}{\sqrt{2}p^2}\right)^2 + 2 + \sqrt{2} \ge 2 + \sqrt{2}$$

Thus, the minimum value of $\alpha^4 + \beta^4$ is $2 + \sqrt{2}$.

46. b. We have,

$$x = \frac{-1 \pm \sqrt{1 - 4a^2(1 - a^2)}}{2a^2}$$

$$= \frac{-1 \pm (2a^2 - 1)}{2a^2}$$

$$= 1 - \frac{1}{a^2} \text{ or } - a^2$$

$$\Rightarrow \quad \beta^2 = 1 - \frac{1}{a^2}$$

47. c. $A_{n+1} = \alpha^{n+1} + \beta^{n+1}$

$= \alpha^{n+1} + \alpha^n\beta + \beta^{n+1} + \alpha\beta^n - \alpha^n\beta - \alpha\beta^n$

$= \alpha^n(\alpha + \beta) + \beta^n(\beta + \alpha) - \alpha\beta(\alpha^{n-1} + \beta^{n-1})$

$= \alpha^n(\alpha + \beta) + \beta^n(\beta + \alpha) - \alpha\beta(\alpha^{n-1} + \beta^{n-1})$

$= (\alpha + \beta)(\alpha^n + \beta^n) - \alpha\beta(\alpha^{n-1} + \beta^{n-1})$

$= aA_n - bA_{n-1}$

48. a. Let α be the root of $x^2 - x + m = 0$ and 2α be the root of $x^2 - 3x + 2m = 0$. Then,

$\alpha^2 - \alpha + m = 0$ and $4\alpha^2 - 6\alpha + 2m = 0$

Eliminating α, $m^2 = -2m \Rightarrow m = 0, m = -2$

49. a. By observation $x = -2$ satisfies equation $x^3 + 3x^2 + 3x + 2 = 0$ then we have $(x + 2)(x^2 + x + 1) = 0$

$x^2 + x + 1 = 0$ has non-real roots.

Since non-real roots occur in conjugate pair, $x^2 + x + 1 = 0$ and $ax^2 + bx + c = 0$ are identical

$\Rightarrow a = b = c$

50. b. Given equations are

$x^3 + ax + 1 = 0$

or

$x^4 + ax^2 + x = 0$ (1)

and

$x^4 + ax^2 + 1 = 0$ (2)

From (1) – (2), we get $x = 1$. Thus, $x = 1$ is the common roots. Hence,

$1 + a + 1 = 0 \Rightarrow a = -2$

51. a. $x = \sqrt[3]{7} + \sqrt[3]{49}$

$\Rightarrow x^3 = 7 + 49 + 3\sqrt[3]{7}\ \sqrt[3]{49}(\sqrt[3]{7} + \sqrt[3]{49}) = 56 + 21x$

$\Rightarrow x^3 - 21x - 56 = 0$

Therefore, the product of roots is 56.

52. c. Since $\alpha, \beta, \gamma, \sigma$ are the roots of the given equation, therefore

$x^4 + 4x^3 - 6x^2 + 7x - 9 = (x - \alpha)(x - \beta)(x - \gamma)(x - \sigma)$

Putting $x = i$ and then $x = -i$, we get

$1 - 4i + 6 + 7i - 9 = (i - \alpha)(i - \beta)(i - \gamma)(i - \sigma)$

and

$1 + 4i + 6 - 7i - 9 = (-i - \alpha)(-i - \beta)(-i - \gamma)(-i - \sigma)$

Multiplying these two equations, we get

$(-2 + 3i)(-2 - 3i) = (1 + \alpha^2)(1 + \beta^2)(1 + \gamma^2)(1 + \sigma^2)$

$\Rightarrow 13 = (1 + \alpha^2)(1 + \beta^2)(1 + \gamma^2)(1 + \sigma^2)$

53. b. If $[m_r, (1/m_r)]$ satisfy the given equation $x^2 + y^2 + 2gx + 2fy + c = 0$, then

$$m_r^2 + \frac{1}{m_r^2} + 2gm_r + \frac{2f}{m_r} + c = 0$$

$\Rightarrow m_r^4 + 2gm_r^3 + cm_r^2 + 2fm_r + 1 = 0$

Now, roots of given equation are m_1, m_2, m_3, m_4. The product of roots

$$m_1 m_2 m_3 m_4 = \frac{\text{constant term}}{\text{coefficient of } m_r^4} = \frac{1}{1} = 1$$

54. a. Clearly,

$x^n - 1 = (x - 1)(x - a_1)(x - a_2) \cdots (x - a_{n-1})$

$$\Rightarrow \frac{x^n - 1}{x - 1} = (x - a_1)(x - a_2) \cdots (x - a_{n-1})$$

$\Rightarrow 1 + x + x^2 + \cdots + x^{n-1} = (x - a_1)(x - a_2) \cdots (x - a_{n-1})$

$\Rightarrow n = (1 - a_1)(1 - a_2) \cdots (1 - a_{n-1})$ [putting $x = 1$]

55. b. $\tan\theta_1 + \tan\theta_2 + \tan\theta_3 = (a + 1)$

$\Sigma \tan\theta_1 \tan\theta_2 = (b - a)$

$\tan\theta_1 \tan\theta_2 \tan\theta_3 = b$

$$\therefore \tan(\theta_1 + \theta_2 + \theta_3) = \frac{\Sigma\tan\theta_1 - \Pi\tan\theta_1}{1 - \Sigma\tan\theta_1\tan\theta_2}$$

$$= \frac{a + 1 - b}{1 - (b - a)} = 1$$

$$\Rightarrow \theta_1 + \theta_2 + \theta_3 = \frac{\pi}{4}$$

56. a. $\Sigma\alpha = 1, \Sigma\alpha\beta = 0, \alpha\beta\gamma = 1$

$$\Sigma\frac{1 + \alpha}{1 - \alpha} = -\Sigma\frac{-\alpha + 1 - 2}{1 - \alpha} = \Sigma\left(\frac{2}{1 - \alpha} - 1\right)$$

$$= 2\Sigma\frac{1}{1 - \alpha} - 3$$

Now,

$$\frac{1}{(x - \alpha)} + \frac{1}{(x - \beta)} + \frac{1}{(x - \gamma)} = \frac{3x^2 - 2x}{x^3 - x^2 - 1}$$

$$\Rightarrow \frac{1}{1 - \alpha} + \frac{1}{1 - \beta} + \frac{1}{1 - \gamma} = \frac{3 - 2}{1 - 1 - 1} = -1$$

$$\Rightarrow \frac{1 + \alpha}{1 - \alpha} = -5$$

57. d. Equation $8x^3 + 1001x + 2008 = 0$ has roots r, s and t.

$$r + s + r = 0, rst = -\frac{2008}{8} = -251$$

Now, let $r + s = A, s + t = B, t + r = C$.

$\therefore A + B + C = 2(r + s + t) = 0$

Hence,

$A^3 + B^3 + C^3 = 3ABC$

$\therefore (r + s)^3 + (s + t)^3 + (t + r)^3$

$= 3(r + s)(s + t)(t + r)$

$= 3(r + s + t - t)(s + t + r - r)(t + r + s - s)$

$= -3rst$ (as $r + s + t = 0$)

$= 3(251) = 753$

58. b. Let,

$$\frac{x}{x^2 - 5x + 9} = y$$

$\Rightarrow yx^2 - 5yx + 9y = x$

$\Rightarrow\ yx^2 - (5y+1)x + 9y = 0$

Now, x is real, so

$D \geq 0$

$\Rightarrow\ (-(5y+1))^2 - 4\cdot y\cdot(9y) \geq 0$

$\Rightarrow\ -11y^2 + 10y + 1 \geq 0$

$\Rightarrow\ 11y^2 - 10y - 1 \leq 0$

$\Rightarrow\ (11y+1)(y-1) \leq 0$

$\Rightarrow\ -\frac{1}{11} \leq y \leq 1$

59. b. Let,

$$y = \frac{3x^2 + 9x + 17}{3x^2 + 9x + 7}$$

$\Rightarrow\ 3x^2(y-1) + 9x(y-1) + 7y - 17 = 0$

Since x is real, so,

$D \geq 0$

$\Rightarrow\ 81(y-1)^2 - 4 \times 3(y-1)(7y-17) \geq 0$

$\Rightarrow\ (y-1)(y-41) \leq 0 \Rightarrow 1 \leq y \leq 41$

Therefore, the maximum value of y is 41.

60. a. $D = b^2 - 4a < 0 \Rightarrow a > 0$

Therefore the graph is concave upwards.

$f(x) > 0,\ \forall\ x \in R$

$\Rightarrow\ f(-1) > 0$

$\Rightarrow\ a + b + 1 > 0$

61. c. We know that $ax^2 + bx + c \geq 0,\ \forall\ x \in R$,

if $a > 0$ and $b^2 - 4ac \leq 0$. So,

$$mx - 1 + \frac{1}{x} \geq 0 \Rightarrow \frac{mx^2 - x + 1}{x} \geq 0$$

$\Rightarrow\ mx^2 - x + 1 \geq 0$ as $x > 0$.

Now,

$mx^2 - x + 1 \geq 0$ if $m > 0$ and $1 - 4m \leq 0$

$\Rightarrow\ m > 0$ and $m \geq 1/4$

Thus, the minimum value of m is 1/4.

62. b. Let $f(x) = ax^2 + bx + c$ be a quadratic expression such that $f(x) > 0$ for all $x \in R$. Then, $a > 0$ and $b^2 - 4ac < 0$. Now,

$g(x) = f(x) + f'(x) + f''(x)$

$\Rightarrow\ g(x) = ax^2 + x(b + 2a) + (b + 2a + c)$

Discriminant of $g(x)$ is

$D = (b + 2a)^2 - 4a(b + 2a + c)$

$= b^2 - 4a^2 - 4ac$

$= (b^2 - 4ac) - 4a^2$

$< 0\quad (\because b^2 - 4ac < 0)$

Therefore, $g(x) > 0$ for all $x \in R$.

63. b. Here, $ax^2 - bx + c^2 = 0$ does not have real roots. So,

$D < 0 \Rightarrow b^2 - 4ac^2 < 0 \Rightarrow a > 0$

Therefore, $f(x)$ is always positive. So,

$f(2) > 0 \Rightarrow 4a - 2b + c^2 > 0$

64. a. $x_1(x - x_2)^2 + x_2(x - x_1)^2 = 0$

$\Rightarrow\ x^2(x_1 + x_2) - 4x\,x_1x_2 + x_1x_2(x_1 + x_2) = 0$

$D = 16(x_1x_2)^2 - 4x_1x_2(x_1 + x_2)^2 > 0\ (\because x_1x_2 < 0)$

The product of roots is $x_1x_2 < 0$. Thus, the roots are real and of opposite signs.

65. d. Given that $a < b < c < d$. Let

$f(x) = (x-a)(x-c) + 2(x-b)(x-d)$

$\Rightarrow\ f(b) = (b-a)(b-c) < 0$

and

$f(d) = (d-a)(d-c) > 0$

Hence, $f(x) = 0$ has one root in (b, d). Also, $f(a)f(c) < 0$. So the other root lies in (a, c). Hence, roots of the equation are real and distinct.

66. c. $x^2 - (a-3)x + a = 0$

$\Rightarrow\ D = (a-3)^2 - 4a$

$= a^2 - 10a + 9$

$= (a-1)(a-9)$

Case I:

Both the roots are greater than 2.

$$D \geq 0, f(2) > 0, -\frac{B}{2A} > 2$$

$$\Rightarrow\ (a-1)(a-9) \geq 0;\ 4 - (a-3)2 + a > 0;\ \frac{a-3}{2} > 2$$

$\Rightarrow\ a \in (-\infty, 1] \cup [9, \infty);\ a < 10;\ a > 7$

$\Rightarrow\ a \in [9, 10)$ (1)

Case II:

One root is greater than 2 and the other root is less than or. equal to 2. Hence,

$f(2) \leq 0$

$\Rightarrow\ 4 - (a-3)2 + a \leq 0$

$\Rightarrow\ a \geq 10$ (2)

From (1) and (2),

$a \in [9, 10) \cup [10, \infty) \Rightarrow a \in [9, \infty)$

67. a. Suppose $a \neq 0$. We rewrite $f(x)$ as follows:

$$f(x) = a\left\{x^2 + \frac{b}{a}x + \frac{c}{a}\right\}$$

$$= a\left\{\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right\}$$

$$f\left(-\frac{b}{2a} + i\right) = a\left\{\left(-\frac{b}{2a} + i + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right\}$$

$$= a\left\{-1 + \frac{4ac - b^2}{4a^2}\right\},$$ which is a real number

This is against the hypothesis. Therefore, $a = 0$.

68. c. The given equation is

$x^2 - 2mx + m^2 - 1 = 0$

$\Rightarrow\ (x - m)^2 - 1 = 0$

$\Rightarrow\ (x - m + 1)(x - m - 1) = 0$

$\Rightarrow\ x = m - 1, m + 1$

From given condition,

$m - 1 > -2$ and $m + 1 < 4$

$\Rightarrow\ m > -1$ and $m < 3$

Hence, $-1 < m < 3$.

69. b. Note that coefficient of x^2 is $(4p - p^2 - 5) < 0$. Therefore the graph is concave downward. According to the question, 1 must lie between the roots. Hence,

$f(1) > 0$

$\Rightarrow \quad 4p - p^2 - 5 - 2p + 1 + 3p > 0$

$\Rightarrow \quad -p^2 + 5p - 4 > 0$

$\Rightarrow \quad p^2 - 5p + 4 < 0$

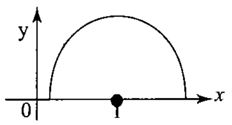


Fig. 1.70

$\Rightarrow \quad (p - 4)(p - 1) < 0$

$\Rightarrow \quad 1 < p < 4$

$\Rightarrow \quad p \in \{2, 3\}$

70. b. $\tan x = \dfrac{a - 4 - \sqrt{(a-4)^2 - 4(4-2a)}}{2}$

$= \dfrac{a - 4 - a}{2} = a - 2, -2$

$\therefore \quad \tan x = a - 2 \qquad (\because \tan x \neq -2)$

$\because \quad x \in \left[0, \dfrac{\pi}{4}\right]$

$\therefore \quad 0 \leq a - 2 \leq 1$

$\Rightarrow \quad 2 \leq a \leq 3$

71. a. Clearly, $f(-1) > 0, f(2) < 0$. Now,

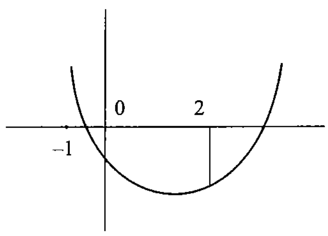


Fig. 1.71

$f(0) = -4 < 0$

$\Rightarrow \quad f(-1) = 1 - a - 4 > 0$ and $f(2) = 4 + 2a - 4 < 0$

$\Rightarrow \quad a < -3$ and $a < 0$

$\Rightarrow \quad a \in (-\infty, -3)$

72. b. Let,

$f(x) = ax^2 + x + c - a$

$f(1) = c + 1 > 0 \qquad (\because c > -1)$

Therefore, given expression is positive $\forall\, x \in R$. So,

$f\left(\dfrac{1}{2}\right) > 0$

$\Rightarrow \quad \dfrac{a}{4} + \dfrac{1}{2} + c - a > 0$

$\Rightarrow \quad 4c - 3a + 2 > 0$

$\Rightarrow \quad 4c + 2 > 3a$

73. b.

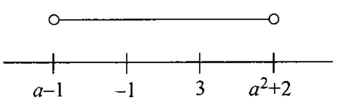


Fig. 1.72

We have,

$a - 1 \leq -1$ and $a^2 + 2 \geq 3$

$a \leq 0$ and $a^2 \geq 1$

Hence, $a \leq -1$.

74. a. $x^2 + 2x - n = 0 \Rightarrow (x + 1)^2 = n + 1$

$\Rightarrow \quad x = -1 \pm \sqrt{n+1}$

Thus, $n + 1$ should be a perfect square. Now,

$n \in [5, 100] \Rightarrow n + 1 \in [6, 101]$

Perfect square values of $n + 1$ are 9, 16, 25, 36, 49, 64, 81, 100. Hence, number of values is 8.

75. d. $x^2 - x - a = 0$, $D = 1 + 4a =$ odd

D must be perfect square of some odd integer. Let

$D = (2\lambda + 1)^2$

$\Rightarrow \quad 1 + 4a = 1 + 4\lambda^2 + 4\lambda$

$\Rightarrow \quad a = \lambda(\lambda + 1)$.

Now,

$a \in [6, 100]$

$\Rightarrow \quad a = 6, 12, 20, 30, 42, 56, 72, 90$

Thus a can attain eight different values.

76. a. $x^2 - (a + 1)x + a - 1 = 0$

$\Rightarrow \quad (x - a)(x - 1) = 1$

Now, $a \in I$ and we want x to be an integer. Hence,

$x - a = 1, x - 1 = 1$ or $x - a = -1, x - 1 = -1$

$\Rightarrow \quad a = 1$ in both cases

77. b. For given situation, $x^2 - (k - 2)x + k^2 = 0$ and $x^2 + kx + 2k - 1 = 0$ should have both roots common or each should have equal roots. If both roots are common, then

$\dfrac{1}{1} = \dfrac{-(k-2)}{k} = \dfrac{k^2}{2k-1}$

$\Rightarrow \quad k = -k + 2$ and $2k - 1 = k^2 \Rightarrow k = 1$

If both the equations have equal roots, then

$(k - 2)^2 - 4k^2 = 0$ and $k^2 - 4(2k - 1) = 0$

$\Rightarrow \quad (3k - 2)(-k - 2) = 0$ and $k^2 - 8k + 4 = 0$ (no common value)

Therefore, $k = 1$ is the only possible value.

78. c. We have,

$\alpha + \beta = -p$ and $\alpha\beta = q \qquad (1)$

Also, since α, β are the roots of $x^{2n} + p^n x^n + q^n = 0$, we have

$\alpha^{2n} + p^n\alpha^n + q^n = 0$ and $\beta^{2n} + p^n\beta^n + q^n = 0$

Subtracting the above relations, we get

$(\alpha^{2n} - \beta^{2n}) + p^n(\alpha^n - \beta^n) = 0$

$\therefore \quad \alpha^n + \beta^n = -p^n \qquad (2)$

Given, α/β or β/α is a root of $x^n + 1 + (x + 1)^n = 0$. So,

$(\alpha/\beta)^n + 1 + [(\alpha/\beta) + 1]^n = 0$

$\Rightarrow \quad (\alpha^n + \beta^n) + (\alpha + \beta)^n = 0$

$\Rightarrow \quad -p^n + (-p)^n = 0$ [Using (1) and (2)]

It is possible only when n is even.

79. a. Since R.H.S. is an even integer. So L.H.S. is also an even integer. So, either both x and y are even integers, or both of them are odd integers. Now,

$x^4 - y^4 = (x - y)(x + y)(x^2 + y^2)$

$\Rightarrow$ $x - y, x + y, x^2 + y^2$ must be an even integer

Therefore, $(x - y)(x + y)(x^2 + y^2)$ must be divisible by 8. But R.H.S. is not divisible by 8. Hence, the given equation has no solution.

80. a. $xy = 2(x + y) \Rightarrow y(x - 2) = 2x$

$\therefore \quad y = \frac{2x}{x-2} = 2 + \frac{4}{x-2} \Rightarrow x = 3, 4 \; (x \neq 6 \text{ as } x < y)$

By trial, $x = 3, 4, 6$. Then $y = 6, 4, 3$. But $x \le y$. Therefore, $x = 3, 4$ and $y = 6, 4$ are two solutions.

81. a. We have,

$(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\beta\gamma + \gamma\alpha + \alpha\beta)$

$\Rightarrow \; 4 = 6 + 2(\beta\gamma + \gamma\alpha + \alpha\beta)$

$\Rightarrow \; \beta\gamma + \gamma\alpha + \alpha\beta = -1$

Also,

$\alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma = (\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2 - \beta\gamma - \gamma\alpha - \alpha\beta)$

$\Rightarrow \; 8 - 3\alpha\beta\gamma = 2(6 + 1)$

$\Rightarrow \; 3\alpha\beta\gamma = 8 - 14 = -6$ or $\alpha\beta\gamma = -2$

Now,

$(\alpha^2 + \beta^2 + \gamma^2)^2 = \Sigma\alpha^4 + 2\Sigma\alpha^2\beta^2$

$= \Sigma\alpha^4 + 2[(\Sigma\alpha\beta)^2 - 2\alpha\beta\gamma(\Sigma\alpha)]$

$\Rightarrow \; \Sigma\alpha^4 = 36 - 2[(-1)^2 - 2(-2)(2)] = 18$

82. d. $(x + a)(x + 1991) + 1 = 0$

$\Rightarrow \; (x + a)(x + 1991) = -1$

$\Rightarrow \; (x + a) = 1$ and $x + 1991 = -1$

$\Rightarrow \; a = 1993$

or $\; x + a = -1$ and $x + 1991 = 1 \; \Rightarrow \; a = 1989$

83. b. Let $f(x) = -3 + x - x^2$. Then $f(x) < 0$ for all x, because coefficient of x^2 is less than 0 and $D < 0$. Thus, L.H.S. of the given equation is always positive whereas the R.H.S. is always less than zero. Hence, there is no solution.

84. c. Put $ab + bc + ca = t$. Now,

$(a + b + c)^2 = a^2 + b^2 + c^2 + 2t$

$\Rightarrow \; (a + b + c)^2 = 1 + 2t$

$\Rightarrow \; 1 + 2t \ge 0$

$\Rightarrow \; -\frac{1}{2} \le t$

Again,

$(a - b)^2 + (b - c)^2 + (c - a)^2 = 2 - 2t$

$\Rightarrow \; 2 - 2t \ge 0$

$\Rightarrow \; t \le 1$

$\Rightarrow \; -\frac{1}{2} \le t \le t \le 1$

85. d. $\cot^4 x - 2(1 + \cot^2 x) + a^2 = 0$

$\Rightarrow \; \cot^4 x - 2\cot^2 x + a^2 - 2 = 0$

$\Rightarrow \; (\cot^2 x - 1)^2 = 3 - a^2$

Now, for at least one solution

$3 - a^2 \ge 0$

$\Rightarrow \; a^2 - 3 \le 0$

$\therefore \; a \in [-\sqrt{3}, \sqrt{3}]$

Integral values are $-1, 0, 1$.

$\therefore$ sum = 0

86. b. $x = 3\cos\theta$; $y = 3\sin\theta$

$z = 2\cos\phi$; $t = 2\sin\phi$

$\therefore \; 6\cos\theta\sin\phi - 6\sin\theta\cos\phi = 6$

$\Rightarrow \; \sin(\phi - \theta) = 1$

$\Rightarrow \; \phi = 90° + \theta$

$\Rightarrow \; P = xz = -6\sin\theta\cos\theta = -3\sin 2\theta$

$\Rightarrow \; P_{max} = 3$

87. a. The equation on simplifying gives

$$x(x - b)(x - c) + x(x - c)(x - a) + x(x - a)(x - b) - (x - a)(x - b)(x - c) = 0 \quad (1)$$

Let,

$f(x) = x(x - b)(x - c) + x(x - c)(x - a) + x(x - a)(x - b) - (x - a)(x - b)(x - c)$

We can assume without loss of generality that $a < b < c$. Now,

$f(a) = a(a - b)(a - c) > 0$

$f(b) = b(b - c)(b - a) < 0$

$f(c) = c(c - a)(c - b) > 0$

So, one root of (1) lies in (a, b) and one root in (b, c). Obviously the third root must also be real.

88. b. $$\frac{(b^2 - 4ac)^2}{16a^2} < \frac{4}{1 + 4a^2} \quad (1)$$

Now,

$$\max(ax^2 + bx + c) = -\frac{b^2 - 4ac}{4a}$$

Also,

$$\frac{-2}{\sqrt{1 + 4a^2}} < -\frac{b^2 - 4ac}{4a} < \frac{2}{\sqrt{1 + 4a^2}} \quad \text{[From (1)]}$$

So, maximum value is always less than 2 (when $a \to 0$).

89. d.

$$a = \frac{x^2 + 4}{|x|} - 3$$

$$= |x| + \frac{4}{|x|} - 3 = \left(\sqrt{|x|} - \frac{2}{\sqrt{|x|}}\right) + 1$$

$\Rightarrow \; a \ge 1$

90. d. Here, $f(x) = (2x - a)(2x - c) + (2x - b)$. So,

$$f\left(\frac{a}{2}\right) = a - b, \; f\left(\frac{c}{2}\right) = c - b$$

Now,

$$f\left(\frac{a}{2}\right) f\left(\frac{c}{2}\right) = (a - b)(c - b) < 0 \; (a > b > c)$$

Hence, exactly one of the roots lies between $c/2$ and $a/2$.

91. a. Since the equation $x^2 + ax + b = 0$ has distinct real roots and $x^2 + a|x| + b = 0$ has only one real root, so one root of the equation $x^2 + ax + b = 0$ will be zero and other root will be negative. Hence, $b = 0$ and $a > 0$.

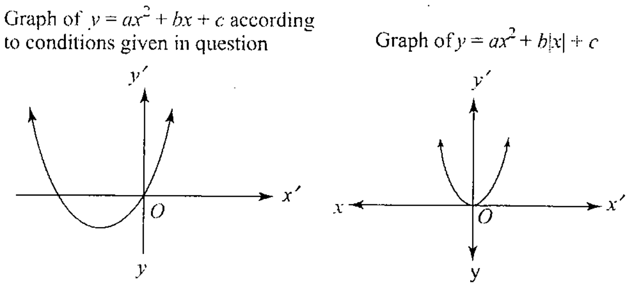


Fig. 1.73

92. c. The given equation is

$$2^{2x} + (a-1)2^{x+1} + a = 0$$

or

$$t^2 + 2(a-1)t + a = 0, \text{ where } 2^x = t$$

Now, $t = 1$ should lie between the roots of this equation.

$$\therefore \quad 1 + 2(a-1) + a < 0 \Rightarrow a < \frac{1}{3}$$

93. a.

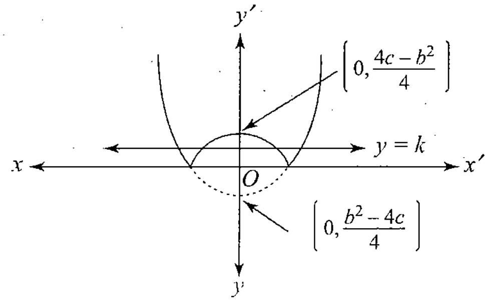


Fig. 1.74

For the equation to have four real roots, the line $y = k$ must intersect $y = |x^2 + bx + c|$ at four points.

$$\therefore \quad D > 0 \text{ and } k \in \left(0, \frac{-D}{4}\right)$$

94. d. $P(a) = P(b) = P(c) = P(d) = 3$

$\Rightarrow$ $P(x) = 3$ has a, b, c, d as its roots

$\Rightarrow$ $P(x) - 3 = (x-a)(x-b)(x-c)(x-d)\,Q(x)$

[$\because$ $Q(x)$ has integral coefficient]

Given $P(e) = 5$, then

$$(e-a)(e-b)(e-c)(e-d)\,Q(e) = 5$$

This is possible only when at least three of the five integers $(e-a)$, $(e-b)$, $(e-c)$, $(e-d)$, $Q(e)$ are equal to 1 or −1. Hence, two of them will be equal, which is not possible. Since a, b, c, d are distinct integers, therefore $P(e) = 5$ is not possible.

95. d. $\sqrt{-x^2 + 10x - 16} < x - 2$

We must have

$-x^2 + 10x - 16 \geq 0$

$\Rightarrow$ $x^2 - 10x + 16 \leq 0$

$\Rightarrow$ $2 \leq x \leq 8$ (1)

Also,

$-x^2 + 10x - 16 < x^2 - 4x + 4$

$\Rightarrow$ $2x^2 - 14x + 20 > 0$

$\Rightarrow$ $x^2 - 7x + 10 > 0$

$\Rightarrow$ $x > 5$ or $x < 2$ (2)

From (1) and (2),

$5 < x \leq 8 \Rightarrow x = 6, 7, 8$

96. c. $D > 0 \Rightarrow (a-3)^2 + 4(a+2) > 0$

$\Rightarrow$ $a^2 - 6a + 9 + 4a + 8 > 0$

$\Rightarrow$ $a^2 - 2a + 17 > 0$

$\Rightarrow$ $a \in R$

$$\therefore \quad \frac{a^2+1}{a^2+2} = 1 - \frac{1}{a^2+2} \geq \frac{1}{2}$$

97. c. Given,

$(a-1)x^2 - (a+1)x + a - 1 \geq 0$

$\Rightarrow$ $a(x^2 - x + 1) - (x^2 + x + 1) \geq 0$

$$\Rightarrow \quad a \geq \frac{x^2 + x + 1}{x^2 - x + 1}$$

$$= 1 + \frac{2x}{x^2 - x + 1}$$

$$= 1 + \frac{2}{x + \frac{1}{x} - 1} \quad (1)$$

Let $y = x + 1/x$. Now, y is increasing in $[2, \infty)$. Hence,

$$1 + \frac{2}{x + \frac{1}{x} - 1} \in \left(1, \frac{7}{3}\right]$$

For all $x \geq 2$, Eq. (1) should be true. Hence, $a > 7/3$.

98. b. $x = 2 + \sqrt{3}$

$\Rightarrow$ $(x-2)^2 = 3$

$\Rightarrow$ $x^2 - 4x + 1 = 0$ (1)

$\Rightarrow$ $(x-2)^4 = 9$

$\Rightarrow$ $x^4 - 8x^3 + 24x^2 - 32x + 16 = 9$

$\Rightarrow$ $x^4 - 8x^3 + 18x^2 - 8x + 2 + 6(x^2 - 4x + 1) - 1 = 0$

Using (1), we get

$$x^4 - 8x^3 + 18x^2 - 8x + 2 = 1$$

Multiple Correct Answers Type

1. a, c. $2x^2 + 6xy + 5y^2 = 1$ (1)

Equation (1) can be rewritten as

$2x^2 + (6y)x + 5y^2 - 1 = 0$

Since x is real,

$36y^2 - 8(5y^2 - 1) \geq 0$

$\Rightarrow$ $y^2 \leq 2$

$\Rightarrow$ $-\sqrt{2} \leq y \leq \sqrt{2}$

Equation (1) can also be rewritten as

$5y^2 + (6x)y + 2x^2 - 1 = 0$

Since y is real,

$36x^2 - 20(2x^2 - 1) \geq 0$

$\Rightarrow$ $36x^2 - 40x^2 + 20 \geq 0$

$\Rightarrow$ $-4x^2 \geq -20$

$\Rightarrow$ $x^2 \leq 5$

$\Rightarrow$ $-\sqrt{5} \leq x \leq \sqrt{5}$

2. b, c, d.

Given equation is $x^3 - ax^2 + bx - 1 = 0$. If roots of the equation be α, β, γ, then

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$

$$= a^2 - 2b$$

$$\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma)$$

$= b^2 - 2a$

$\alpha^2\beta^2\gamma^2 = 1$

So, the equation whose roots are $\alpha^2, \beta^2, \gamma^2$ is given by

$$x^3 - (a^2 - 2b)x^2 + (b^2 - 2a)x - 1 = 0$$

It is identical to

$$x^3 - ax^2 + bx - 1 = 0$$

$\Rightarrow \quad a^2 - 2b = a$ and $b^2 - 2a = b$

Eliminating b, we get

$$\frac{(a^2-a)^2}{4} - 2a = \frac{a^2-a}{2}$$

$\Rightarrow \quad a\{a(a-1)^2 - 8 - 2(a-1)\} = 0$

$\Rightarrow \quad a(a^3 - 2a^2 - a - 6) = 0$

$\Rightarrow \quad a(a-3)(a^2 + a + 2) = 0$

$\Rightarrow \quad a = 0$ or $a = 3$ or $a^2 + a + 2 = 0$

which gives $b = 0$ or $b = 3$ or $b^2 + b + 2 = 0$. So, $a = b = 0$ or $a = b = 3$ or a, b are roots of $x^2 + x + 2 = 0$.

3. a, b, c.

Since the roots of $ax^2 + bx + c = 0$ are non-real, so, $f(x) = ax^2 + bx + c$ will have same sign for every value of x. Hence,

$f(0) = c, f(1) = a + b + c, f(-1) = a - b + c$

$f(-2) = 4a - 2b + c$

$\Rightarrow \quad c(a + b + c) > 0, c(a - b + c) > 0, c(4a - 2b + c) > 0$

4. a, b.

We can write the given equation as

$$\frac{p}{2x} = \frac{(a+b)x + c(b-a)}{x^2 - c^2}$$

$\Rightarrow \quad p(x^2 - c^2) = 2(a + b)x^2 - 2c(a - b)x$

$\Rightarrow \quad (2a + 2b - p)x^2 - 2c(a - b)x + pc^2 = 0$

For this equation to have equal roots,

$c^2(a - b)^2 - pc^2(2a + 2b - p) = 0$

$\Rightarrow \quad (a - b)^2 - 2p(a + b) + p^2 = 0 \qquad [\because c^2 \neq 0]$

$\Rightarrow \quad [p - (a + b)]^2 = (a + b)^2 - (a - b)^2 = 4ab$

$\Rightarrow \quad p - (a + b) = \pm 2\sqrt{ab}$

$\Rightarrow \quad p = a + b \pm 2\sqrt{ab} = (\sqrt{a} \pm \sqrt{b})^2$

5. a, d.

Roots of $4x^2 - x - 1 = 0$ are irrational. So, one root common implies both roots are common. Therefore,

$$\frac{4}{3} = \frac{-1}{\lambda + \mu} = \frac{-1}{\lambda - \mu}$$

$$\Rightarrow \lambda = \frac{-3}{4}, \ \mu = 0$$

6. a, b, c.

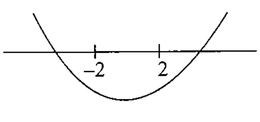


Fig. 1.75

$f(x) = ax^2 + bx + c$

$f(0) = c < 0, D > 0 \Rightarrow b^2 - 4ac > 0$

$f(1) < 0$ and $f(-1) < 0$

$\Rightarrow \quad a - |b| + c < 0$

$f(2) < 0$ and $f(-2) < 0$

$\Rightarrow \quad 4a - 2|b| + c < 0$

Nothing can be said about $f(3)$ or $f(-3)$, whether it is positive or negative.

7. a, b, d.

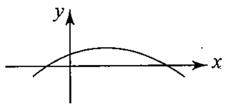


Fig. 1.76

From the graph,

$f(0) = c > 0 \qquad (1)$

Also, the graph is concave downward. Hence,

$a < 0 \qquad (2)$

Further, abscissa of the vertex,

$\dfrac{b}{2a} \qquad (3)$

From (1), (2), (3),

$ac < 0, ab < 0$ and $bc > 0$

8. c, d.

$\cos x - y^2 - \sqrt{y - x^2 - 1} \geq 0 \qquad (1)$

Now, $\sqrt{y - x^2 - 1}$ is defined when $y - x^2 - 1 \geq 0$ or $y \geq x^2 + 1$. So minimum value of y is 1. From (1),

$\cos x - y^2 \geq \sqrt{y - x^2 - 1}$

where $\cos x - y^2 \leq 0$ [as when $\cos x$ is maximum (=1) and y^2 is minimum (=1), so $\cos x - y^2$ is maximum]. Also,

$\sqrt{y - x^2 - 1} \geq 0$

Hence,

$\cos x - y^2 = \sqrt{y - x^2 - 1} = 0$

$\Rightarrow \quad y = 1$ and $\cos x = 1, y = x^2 + 1$

$\Rightarrow \quad x = 0, y = 1$

9. a, d.

$2^x = t$

$t^2 - 8t + 12 = 0$

$(t - 6)(t - 2) = 0$

$2^x = 6 \Rightarrow x = \log_2 6 = 1 + \dfrac{\log 3}{\log 2}$

$2^x = 2 \Rightarrow x = 1$

10. a, b, d.

Symmetric functions are those which do not change by interchanging α and β.

11. a, b, c.

$\sec^2\theta + \text{cosec}^2\theta = \sec^2\theta.\text{cosec}^2\theta$

Sum of the roots is equal to their product and the roots are real. Hence,

$$-\frac{b}{a} = \frac{c}{a}$$

$$\Rightarrow \quad b + c = 0$$

Also $\quad b^2 - 4ac \geq 0$

$$\Rightarrow \quad c^2 - 4ac \geq 0$$

$$\Rightarrow \quad c(c - 4a) \geq 0$$

$$\Rightarrow \quad c - 4a \geq 0 \quad (\because \quad c > 0)$$

Further

$$b^2 + 4ab \geq 0$$

$$\Rightarrow \quad b + 4a \leq 0 \quad (\because \quad b < 0)$$

12. a, b, c.

$$f(x) = Ax^2 + Bx + C$$

$$A = a + b - 2c = (a - c) + (b - c) > 0$$

$$\Rightarrow \quad A > 0$$

Hence, the graph is concave upwards. Also, $x = 1$ is obvious solution; therefore, both roots are rational.

$$b + c - 2a = \underbrace{(b - a)}_{-\text{ve}} + \underbrace{(c - a)}_{-\text{ve}} < 0$$

$$\Rightarrow \quad B < 0$$

$$\therefore \quad \text{vertex} = -\frac{B}{2A} > 0$$

Hence, abscissa of the vertex is positive. Option (d) need not be correct as with $a = 5, b = 4, c = 2, P < 0$ and with $a = 6, b = 3, c = 2, P > 0$.

13. a, b, c.

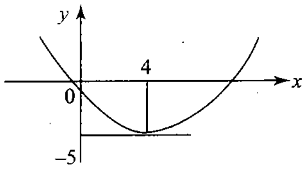


Fig. 1.77

From figure,

$$a > 0$$

$$-\frac{b}{2a} = 4 \Rightarrow -\frac{b}{2a} > 0$$

$$\therefore \quad b < 0$$

$$f(0) = c < 0$$

Also,

$$-\frac{b}{2a} = 4 \Rightarrow 8a + b = 0$$

14. a, c, d.

Let the roots be $a/r, a, ar$, where $a > 0, r > 1$. Now,

$$a/r + a + ar = -p \quad (1)$$

$$a(a/r) + a(ar) + (ar)(a/r) = q \quad (2)$$

$$(a/r)(a)(ar) = 1 \quad (3)$$

$$\Rightarrow \quad a^3 = 1$$

$$\Rightarrow \quad a = 1$$

Hence, (c) is correct. From (1), putting $a = 1$, we get

$$-p - 3 > 0 \quad \left(\because r + \frac{1}{r} > 2\right)$$

$$\Rightarrow \quad p < -3$$

Hence, (b) is not correct. Also,

$$1/r + 1 + r = -p \quad (4)$$

From (2), putting $a = 1$, we get

$$1/r + r + 1 = q \quad (5)$$

From (4) and (5), we have

$$-p = q \Rightarrow p + q = 0$$

Hence, (a) is correct. Now, as $r > 1$

$$a/r = 1/r < 1$$

and

$$ar = r > 1$$

Hence, (d) is correct.

15. a, b.

Given, $(\sin \alpha)\, x^2 - 2x + b \geq 2$. Let $f(x) = (\sin \alpha)x^2 - 2x + b - 2$. Abscissa of the vertex is given by

$$x = \frac{1}{\sin \alpha} > 1$$

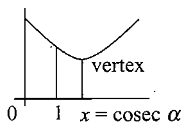


Fig. 1.78

The graph of $f(x) = (\sin \alpha)x^2 - 2x + b - 2, \forall\, x \leq 1$, is shown in the figure. Therefore, minimum of $f(x) = (\sin \alpha)x^2 - 2x + b - 2$ must be greater than zero but minimum is at $x = 1$. That is,

$$\sin \alpha - 2 + b - 2 \geq 0, b \geq 4 - \sin \alpha, \alpha \in (0, \pi)$$

16. a, c.

Since each pair has common root, let the roots be α, β for Eq. (1); β, γ for Eq. (2) and γ, α for Eq. (3). Therefore,

$$\alpha + \beta = -a, \alpha\beta = bc$$

$$\beta + \gamma = -b, \beta\gamma = ca$$

$$\gamma + \alpha = -c, \gamma\alpha = ab$$

Adding, we get

$$2(\alpha + \beta + \gamma) = -(a + b + c)$$

$$\Rightarrow \quad \alpha + \beta + \gamma = -\frac{1}{2}(a + b + c)$$

Also by multiplying product of roots, we have

$$\alpha^2\beta^2\gamma^2 = a^2b^2c^2 \Rightarrow \alpha\beta\gamma = abc$$

17. a, c.

Given,

$$b^2 = ac$$

$$\Rightarrow \left(\frac{b}{a}\right)^2 = \frac{c}{a}$$

$$\Rightarrow \left(-\frac{b}{a}\right)^2 = \frac{c}{a}$$

$$\Rightarrow (\alpha + \beta)^2 = \alpha\beta$$

$$\Rightarrow \alpha^2 + \beta^2 + \alpha\beta = 0$$

$$\Rightarrow \left(\frac{\alpha}{\beta}\right)^2 + \left(\frac{\alpha}{\beta}\right) + 1 = 0$$

$$\Rightarrow \frac{\alpha}{\beta} = \frac{-1 \pm \sqrt{3}i}{2} \text{ (where } i = \sqrt{-1})$$

18. c, d.

We have,

$$D = (b-c)^2 - 4a(a-b-c) > 0$$

$$\Rightarrow b^2 + c^2 - 2bc - 4a^2 + 4ab + 4ac > 0$$

$$\Rightarrow c^2 + (4a - 2b)c - 4a^2 + 4ab + b^2 > 0 \text{ for all } c \in R$$

Discriminant of the above expression in c must be negative. Hence,

$$(4a - 2b)^2 - 4(-4a^2 + 4ab + b^2) < 0$$

$$\Rightarrow 4a^2 - 4ab + b^2 + 4a^2 - 4ab - b^2 < 0$$

$$\Rightarrow a(a-b) < 0$$

$$\Rightarrow a < 0 \text{ and } a - b > 0 \text{ or } a > 0 \text{ and } a - b < 0$$

$$\Rightarrow b < a < 0 \text{ or } b > a > 0$$

19. a, d.

Since $\alpha, \beta, \gamma, \delta$ are in H.P., hence $1/\alpha, 1/\beta, 1/\gamma, 1/\delta$ are in A.P. and they may be taken as $a - 3d, a - d, a + d, a + 3d$. Replacing x by $1/x$, we get the equation whose roots are $1/\alpha, 1/\beta, 1/\gamma, 1/\delta$. Therefore, equation $x^2 - 4x + A = 0$ has roots $a - 3d, a + d$ and equation $x^2 - 6x + B = 0$ has roots $a - d, a + 3d$. Sum of the roots is

$$2(a - d) = 4, 2(a + d) = 6$$

$$\therefore a = 5/2, d = 1/2$$

Product of the roots is

$$(a - 3d)(a + d) = A = 3$$

$$(a - d)(a + 3d) = B = 8$$

20. a, b.

Equations $x^2 + px + q = 0$ and $x^2 + p'x + q' = 0$ have a common root. Therefore,

$$(q - q')^2 = (pq' - p'q)(p' - p) \quad (1)$$

Subtracting two equations, we have

$$x = \frac{q - q'}{p' - p}$$

Also using (1),

$$x = \frac{q - q'}{p' - p} = \frac{pq' - p'q}{q - q'}$$

21. b, c.

$f(x) = x^3 + 3x^2 - 9x + c$ is of the form $(x - \alpha)^2 (x - \beta)$, showing that α is a double root so that $f'(x) = 0$ has also one root α, i.e., $3x^2 + 6x - 9 = 0$ has one root α. Hence, $x^2 + 2x - 3 = 0$ or $(x + 3)(x - 1) = 0$ has the root α which can be either -3 or 1. If $\alpha = 1$, then $f(x) = 0$ gives $c - 5 = 0$ or $c = 5$. If $\alpha = -3$, then $f(x) = 0$ gives

$$-27 + 27 + 27 + c = 0$$

$$\therefore c = -27$$

22. a, c.

If α be the common root, then

$$\alpha^2 + b\alpha - a = 0 \text{ and } \alpha^2 - a\alpha + b = 0$$

Subtracting,

$$\alpha(b + a) - (a + b) = 0$$

$$\Rightarrow (a + b)(\alpha - 1) = 0$$

$$\Rightarrow a + b = 0 \text{ or } \alpha = 1$$

When $\alpha = 1$, then from any equation we have $a - b = 1$.

23. a, d.

Let,

$$\frac{x^2 + ax + 3}{x^2 + x + a} = y$$

$$\Rightarrow x^2(1 - y) - x(y - a) + 3 - ay = 0$$

$$\because x \in R$$

$$(y - a)^2 - 4(1 - y)(3 - ay) \geq 0$$

$$\Rightarrow (1 - 4a)y^2 + (2a + 12)y + a^2 - 12 \geq 0 \quad (1)$$

Now, (1) is true for all $y \in R$, if $1 - 4a > 0$ and $D \leq 0$. Hence,

$$a < \frac{1}{4} \text{ and } 4(a + 6)^2 - 4(a^2 - 12)(1 - 4a) \leq 0$$

$$\Rightarrow a < \frac{1}{4} \text{ and } 4a^3 - 36a + 48 \leq 0$$

$$\Rightarrow a < \frac{1}{4} \text{ and } 4a^3 \leq 36a - 48$$

$$\Rightarrow 4a^3 < 36\left(\frac{1}{4}\right) - 48$$

$$\Rightarrow 4a^3 + 39 < 0 \quad \left[\because a < \frac{1}{4}\right]$$

24. a, b.

Here,

$$\cos^2\theta - \sin^2\theta = \cos 2\theta$$

$$\Rightarrow \cos^4\theta - \sin^4\theta = \cos 2\theta$$

$$\Rightarrow (-2b)^2 - 4b = (-4)^2 - 4 \times 2$$

(since L.H.S. is difference of roots of first equation and R.H.S. is difference of roots of second equation)

$$\Rightarrow 4b^2 - 4b = 16 - 8 = 8$$

$$\Rightarrow 4b^2 - 4b - 8 = 0$$

$$\Rightarrow b^2 - b - 2 = 0$$

$$\Rightarrow (b + 1)(b - 2) = 0$$

$$\Rightarrow b = 2, -1$$

25. b, d.

Let $f(x) = x^2 + ax + b$. Then,

$$x^2 + (2c + a)x + c^2 + ac + b = f(x + c)$$

Thus, the roots of $f(x + c) = 0$ will be $0, d - c$.

26. c, d.

Product of roots is

$$\frac{a}{bc} < 0 \quad [\because abc < 0]$$

Hence, roots are real and of opposite sign.

27. b, c, d.

Given equation is

$$x^2 + 2(a + 1)x + 9a - 5 = 0$$

$$D = 4(a + 1)^2 - 4(9a - 5) = 4(a - 1)(a - 6)$$

$$\therefore \quad D \ge 0 \Rightarrow a \le 1 \text{ or } a \ge 6 \Rightarrow \text{roots are real}$$

If $a < 0$, then $9a - 5 < 0$. Hence, the products of roots is less than 0. So, the roots are of opposite sign. If $a > 7$, then sum of roots is $-2(a + 1) < 0$. Product of roots is greater than 0.

28. a, c.

Since $P(x)$ divides both of them, hence $P(x)$ also divides

$$(3x^4 + 4x^2 + 28x + 5) - 3(x^4 + 6x^2 + 25)$$

$$= -14x^2 + 28x - 70$$

$$= -14(x^2 - 2x + 5)$$

which is a quadratic. Hence,

$$P(x) = x^2 - 2x + 5 \Rightarrow P(1) = 4$$

30. a., d.

On putting $x = 0$, 1 and 1/2, we get

$$-1 \le c \le 1 \quad (1)$$

$$-1 \le a + b + c \le 1 \quad (2)$$

$$-4 \le a + 2b + 4c \le 4 \quad (3)$$

From (1), (2), (3), we get

$$|b| \le 8 \text{ and } |a| \le 8$$

$$\Rightarrow \quad |a| + |b| + |c| \le 17$$

31. a, b, c, d.

Let $f(x) = ax^2 + bx + c$.

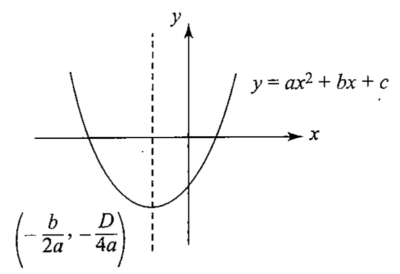


Fig. 1.79

From the diagram, we can see that $a > 0$, $c < 0$ and $-[b(2a)] < 0$. Hence, $b > 0$.

$$\therefore \quad a + b - c > 0$$

Reasoning Type

1. b. We must have

$$ax^3 + (a + b)x^2 + (b + c)x + c > 0$$

$$\Rightarrow \quad ax^2(x + 1) + bx(x + 1) + c(x + 1) > 0$$

$$\Rightarrow \quad (x + 1)(ax^2 + bx + c) > 0$$

$$\Rightarrow \quad a(x + 1)\left(x + \frac{b}{2a}\right)^2 > 0 \text{ as } b^2 = 4ac$$

$$\Rightarrow \quad x > -1 \text{ and } x \ne -\frac{b}{2a}$$

2. b. If $a > 0$, then graph of $y = ax^2 + 2bx + c$ is concave upward. Also if $b^2 - ac < 0$, then the graph always lies above x-axis; hence, $ax^2 + 2bx + c > 0$ for all real values of x. Thus, domain of function $f(x) = \sqrt{ax^2 + 2bx + c}$ is R.

If $b^2 - ac < 0$, then $ax^2 + 2bx + c = 0$ has imaginary roots. Then the graph of $y = ax^2 + 2bx + c$ never cuts x-axis, or y is either always positive or always negative. Hence, both the statements are correct but statement 2 is not correct explanation of statement 1.

3. a. $ax^2 + bx + c = 0$ has two complex conjugate roots only if all the coefficients are real. If all the coefficients are not real then it is not necessary that both the roots are imaginary. Hence, statement 2 is true.

Now, equation $x^2 - 3x + 4 = 0$ has two complex conjugate roots. If $ax^2 + bx + c = 0$ has all coefficients real, then there will be two common roots. But if there is only one root common, then at least one of a, b, c must be non-real.

Thus, both the statements are true and statement 2 is correct explanation of statement 1.

4. a. $$\cos\frac{\pi}{4} = 2\cos^2\frac{\pi}{8} - 1$$

$$\Rightarrow \quad \cos^2\frac{\pi}{8} = \left(\frac{1}{\sqrt{2}} + 1\right)\frac{1}{2}$$

$$\Rightarrow \quad \cos^4\frac{\pi}{8} = \frac{1}{4}\left(\frac{1}{2} + 1 + \frac{2}{\sqrt{2}}\right) = \left(\frac{3}{2} + \sqrt{2}\right)\frac{1}{4}$$

$$\Rightarrow \quad \frac{1}{4}\left(\frac{3}{2} + \sqrt{2}\right) + \frac{a}{2}\left(\frac{1}{\sqrt{2}} + 1\right) + b = 0$$

($\because \cos^2 \pi/8$ a root of equation)

$$\Rightarrow \quad \Rightarrow\left(\frac{3}{8} + \frac{a}{2} + b\right) + \sqrt{2}\left(\frac{1}{4} + \frac{a}{4}\right) = 0$$

Since a and b are rational, so

$$\frac{1}{4} + \frac{a}{4} = 0, \frac{3}{8} + \frac{a}{2} + b = 0$$

$$\Rightarrow \quad a = -1, b = \frac{1}{8}$$

Thus, both the statements are correct and statement 2 is correct explanation of statement 1.

5. a. If $a^2 + b^2 + c^2 < 0$, then all a, b, c are not real or at least one of a, b, c is imaginary number. Hence roots of equation $ax^2 + bx + c = 0$ has no complex conjugate roots, even though the roots are complex. Hence statement 1 is true. Statement 2 is obviously true (see the theory). Also, statement 2 is correct explanation of statement 1.

6. b. $ix^2 + (i-1)x - \frac{1}{2} - i = 0$

$$\Rightarrow \quad x = \frac{-(i-1) \pm \sqrt{(i-1)^2 - 4(i)\left(-\frac{i}{2} - i\right)}}{2i} = \frac{-(i-1) \pm \sqrt{-4}}{2i}$$

Thus, roots are imaginary. Also, we have $b^2 - 4ac = -4 < 0$, but this is not the correct reason for which roots are imaginary as coefficients of the equation are imaginary.

Hence, both the statements are correct but statement 2 is not correct explanation of statement 1.

7. a. $f(x) = (x-1)(ax+b)$

$f(2) = 2a + b$

$f(4) = 3(4a + b) = 12a + 3b$

$f(2) + f(4) = 14a + 4b = 0$

$\Rightarrow \frac{-b}{a} = 3.5$

Now, sum of roots is $(a-b)/a = 1 - (b/a) = 1 + 3.5 = 4.5$. Hence, the other root is 3.5.

8. c. Here, $f(x)$ is a downward parabola.

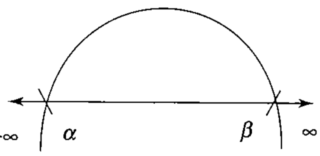


Fig. 1.80

$D = (a+1)^2 + 20 > 0$

From the graph, clearly, statement 1 is true but statement 2 is false.

9. a. $x^2 + x + 1 = 0$

$D = -3 < 0$

Therefore, $x^2 + x + 1 = 0$ and $ax^2 + bx + c = 0$ have both the roots common. Hence,

$a = b = c$

10. d. Statement 2 is obviously true. Let,

$f(x) = (x-p)(x-r) + \lambda(x-q)(x-s) = 0$

Then,

$f(p) = \lambda(p-q)(p-s)$

$f(r) = \lambda(r-q)(r-s)$

$\Rightarrow f(p)f(r) < 0$

Hence, there is a root between p and r. Thus, statement 1 is false.

11. a. Given equation is

$px^2 + qx + r = 0$

Let,

$f(x) = px^2 + qx + r$

$f(0) = r > 0$

$f(1) = p + q + r < 0$

$f(-1) = p - q + r < 0$

Hence, one root lies in $(-1, 0)$ and the other in $(0, 1)$.

$\therefore \quad [\alpha] = -1$ and $[\beta] = 0$

$\Rightarrow \quad [\alpha] + [\beta] = -1$

Therefore, statement 2 is true and is correct explanation of statement 1.

12. d. Let $f(x) = (x - \sin\alpha)(x - \cos\alpha) - 2$. Then,

$f(\sin\alpha) = -2 < 0, f(\cos\alpha) = -2 < 0$

Also, as $0 < a < \pi/4$, hence, $\sin\alpha < \cos\alpha$. Therefore, equation $f(x) = 0$ has one root in $(-\infty, \sin\alpha)$ and other in $(\cos\alpha, \infty)$.

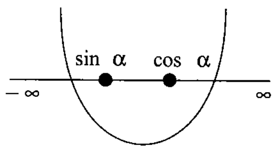


Fig. 1.81

13. b. The equation can be written as

$(2^x)^2 - (a-3)\,2^x + (a-4) = 0$

$\Rightarrow \quad 2^x = 1$ and $2^x = a - 4$

We have,

$x \le 0$ and $2^x = a - 4 \quad$ [$\because$ x is non-positive]

$\therefore \quad 0 < a - 4 \le 1 \Rightarrow 4 < a \le 5$

$\therefore \quad a \in (4, 5]$

14. c. Clearly, Statement 1 is true but Statement 2 is false, since, $ax^2 + bx + c = 0$ is an identity when $a = b = c = 0$.

15. d. Roots of the equation $x^5 - 40x^4 + Px^3 + Qx^2 + Rx + S = 0$ are in G.P. Let the roots be a, ar, ar^2, ar^3, ar^4. Therefore,

$$a + ar + ar^2 + ar^3 + ar^4 = 40 \quad (1)$$

and

$$\frac{1}{a} + \frac{1}{ar} + \frac{1}{ar^2} + \frac{1}{ar^3} + \frac{1}{ar^4} = 10 \quad (2)$$

From (1) and (2),

$$ar^2 = \pm 2 \quad (3)$$

Now, the product of roots is $a^5r^{10} = (ar^2)^5 = \pm 32$.

$\therefore \quad |S| = 32$

16. a. Given equations are

$$ax^2 + 2bx + c = 0 \quad (1)$$

$$a_1x^2 + 2b_1x + c_1 = 0 \quad (2)$$

Since Eqs. (1) and (2) have only one common root and a, b, c, a_1, b_1, c_1 are rational, therefore, common root cannot be imaginary or irrational (as irrational roots occur in conjugate pair when coefficients are rational, and complex roots always occur in conjugate pair).

Hence, the common root must be rational. Therefore, both the roots of Eqs. (1) and (2) will be rational. Therefore, $4(b^2 - ac)$ and $4(b_1^2 - a_1c_1)$ must be perfect squares (squares of rational numbers). Hence, $b^2 - ac$ and $b_1^2 - a_1c_1$ must be perfect squares.

17. a. Let $f(x) = ax^2 + bx + c$. Since coefficient are integers and one root is irrational, so both the roots are irrational. Hence, for any $\lambda \in Q$,

$f(\lambda) \neq 0 \Rightarrow |f(\lambda)| > 0$

$\Rightarrow \left|\frac{ap^2}{q^2} + \frac{bp}{q} + c\right| > 0,$ where $\lambda = \frac{p}{q}$, $p, q \in Z$

$\Rightarrow \frac{1}{q^2}\left|ap^2 + bpq + cq^2\right| > 0$

Now, $a, b, c, p, q \in I$. Hence,

$|ap^2 + bpq + cq^2| \geq 1$

$\Rightarrow |f(\lambda)| \geq \frac{1}{q^2}$

18. d. $a^2 - 3a + 2 = 0 \Rightarrow a = 1, 2$

$a^2 - 5a + 6 = 0 \Rightarrow a = 2, 3,$

$a^2 - 4 = 0 \Rightarrow a = \pm 2$

Therefore, $a = 2$ is the only solution.

Hence, statement 1 is false. Statement 2 is true by definition.

19. b. According to statement 1, given equation is

$x^2 - bx + c = 0$

Let α, β be two roots such that

$|\alpha - \beta| = 1$

$\Rightarrow (\alpha + \beta)^2 - 4\alpha\beta = 1$

$\Rightarrow b^2 - 4c = 1$

According to statement 2, given equation is $4abc\, x^2 + (b^2 - 4ac)x - b = 0$. Hence,

$D = (b^2 - 4ac)^2 + 16ab^2c$

$= (b^2 + 4ac)^2 > 0$

Hence, roots are real and unequal.

20. c. $f(x) = ax^2 + bx + c$

Given,

$f(0) + f(1) = 2$

$\Rightarrow f(x) > 0\ \forall x \in R$

Hence, statement 1 is true. Let,

$f(x) = x^2 - x + 1$

$a + b = 0$

Hence, statement 2 is false.

21. d. $f(x, y) = (2x - y)^2 + (x + y - 3)^2$ (1)

Therefore, statement 1 is false as it represents a point (1, 2).

22. a. $D = \underbrace{\underbrace{(2m+1)^2}_{\text{odd}} - \underbrace{4(2n+1)}_{\text{even}}}_{\text{odd}}$

For rational root, D must be a perfect square. As D is odd, let D be perfect square of $2l + 1$, where $l \in Z$.

$(2m + 1)^2 - 4(2n + 1) = (2l + 1)^2$

$\Rightarrow (2m + 1)^2 - (2l + 1)^2 = 4(2n + 1)$

$\Rightarrow [(2m + 1) + (2l + 1)]\,[2(m - l)] = 4(2n + 1)$

$\Rightarrow (m + l + 1)(m - l) = (2n + 1)$ (1)

R.H.S. of (1) is always odd but L.H.S. is always even. Hence, D cannot be a perfect square. So, the roots cannot be rational.

Hence, statement 1 is true, statement 2 is true and statement 2 is correct explanation for statement 1.

Linked Comprehension Type

For Problems 1–3

1. d, 2. c, 3. c.

Sol. Let unknown polynomial be $P(x)$. Let $Q(x)$ and $R(x)$ be the quotient and remainder, respectively, when it is divided by $(x - 3)(x - 4)$. Then,

$P(x) = (x - 3)(x - 4)Q(x) + R(x)$

Then, we have

$R(x) = ax + b$

$\Rightarrow P(x) = (x - 3)(x - 4)Q(x) + ax + b$

Given that $P(3) = 2$ and $P(4) = 1$. Hence,

$3a + b = 2$ and $4a + b = 1$

$\Rightarrow a = -1$ and $b = 5$

$\Rightarrow R(x) = 5 - x$

1. $5 - x = x^2 + ax + 1 \Rightarrow x^2 + (a + 1)x - 4 = 0$

Given that roots are real and distinct.

$\therefore D > 0 \Rightarrow (a + 1)^2 + 16 > 0$

which is true for all real x.

2. $-x + 5 = px^2 + (q - 1)x + 6 \Rightarrow px^2 + qx + 1 = 0$

Now, $p > 0$ and equation has no distinct real roots or equation has real and equal or imaginary roots. Then,

$px^2 + qx + 1 \geq 0,\ \forall x \in R$

$\Rightarrow f(3) \geq 0 \Rightarrow 9p + 3q + 1 \geq 0 \Rightarrow 3p + q \geq -1/3$

Hence, the least value of $3p + q$ is $-1/3$.

3. $f(x) = y = \frac{-x + 5}{x^2 - 3x + 2}$

$\Rightarrow yx^2 + (1 - 3y)x + 2y - 5 = 0$

Now, x is real, then

$D \geq 0$

$\Rightarrow (1 - 3y)^2 - 4y(2y - 5) \geq 0$

$\Rightarrow \quad y^2 + 14y + 1 \geq 0$

$\Rightarrow \quad y \in \left(-\infty, \frac{-14-\sqrt{192}}{2}\right] \cup \left[\frac{-14+\sqrt{192}}{2}, \infty\right)$

$(-\infty, -7-4\sqrt{3}] \cup [-7+4\sqrt{3}, \infty)$

For Problems 4–6

4. d, 5. b, 6. b.

Sol. $ax^2 - bx + c = 0$

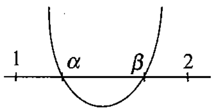


Fig. 1.82

Let $f(x) = ax^2 - bx + c$ be the corresponding quadratic expression and α, β be the roots of $f(x) = 0$. Then,

$f(x) = a(x-\alpha)(x-\beta)$

Now,

$af(1) > 0, af(2) > 0, 1 < \frac{b}{2a} < 2, b^2 - 4ac > 0$

$\Rightarrow \quad a(1-\alpha)(1-\beta) > 0, a(2-\alpha)(2-\beta) > 0, 2a < b < 4a,$

$b^2 - 4ac > 0$

$\Rightarrow \quad a^2(1-\alpha)(1-\beta)(2-\alpha)(2-\beta) > 0$

$\Rightarrow \quad a^2(\alpha-1)(2-\alpha)(\beta-1)(2-\beta) > 0$

As $f(1)$ and $f(2)$ both are integers and $f(1) > 0$, and $f(2) > 0$, so

$f(1)\ f(2) > 0$

$\Rightarrow \quad f(1)\ f(2) \geq 1$

$\Rightarrow \quad 1 \leq a^2(\alpha-1)(2-\alpha)(\beta-1)(2-\beta)$

Now,

$$\frac{(\alpha-1)+(2-\alpha)}{2} \geq ((\alpha-1)(2-\alpha))^{1/2}$$

$\Rightarrow \quad (\alpha-1)(2-\alpha) \leq \frac{1}{4}$

Similarly,

$(\beta-1)(2-\beta) \leq \frac{1}{4}$

$\Rightarrow \quad (\alpha-1)(2-\alpha)(\beta-1)(2-\beta) < \frac{1}{16}$

As $\alpha \neq \beta$, so

$a^2 > 16 \Rightarrow a \geq 5$

$\Rightarrow \quad b^2 > 20c$ and $b > 10 \Rightarrow b \geq 11$

Also,

$b^2 > 100 \Rightarrow c > 5 \Rightarrow c \geq 6$

For Problems 7–9

7. c, 8. a, 9. c.

Sol. Given equation is

$$x^4 + 2ax^3 + x^2 + 2ax + 1 = 0 \quad (1)$$

or

$$\left(x^2 + \frac{1}{x^2}\right) + 2a\left(x + \frac{1}{x}\right) + 1 = 0$$

$\Rightarrow \quad \left(x + \frac{1}{x}\right)^2 + 2a\left(x + \frac{1}{x}\right) - 1 = 0$

$\Rightarrow \quad t^2 + 2at - 1 = 0 \quad (2)$

where $t = x + (1/x)$. Now,

$\left(x + \frac{1}{x}\right) \geq 2$

or

$\left(x + \frac{1}{x}\right) \leq -2$

$\therefore \quad t \geq 2$ or $t \leq -2$

7. Now, Eq. (1) will have at least two positive roots, when at least one root of Eq. (2) will be greater than 2. From Eq. (2),

$D = 4a^2 - 4(-1) = 4(1 + a^2) > 0, \forall a \in R \quad (3)$

Let the roots of Eq. (2) be α, β. If $\alpha, \beta \leq 2$, then

$\Rightarrow \quad f(2) \geq 0$ and $\frac{-B}{2A} < 2$

$\Rightarrow \quad 4 + 4a - 1 \geq 0$ and $-\frac{2a}{2} < 2$

$\Rightarrow \quad a \geq -\frac{3}{4}$ and $a > -2$

$\Rightarrow \quad a \geq -\frac{3}{4}$

Therefore, at least one root will be greater than 2. Then,

$a < -\frac{3}{4} \quad (4)$

Combining (3) and (4), we get

$a < -\frac{3}{4}$

Hence, at least one root will be positive if $a \in [-\infty, -(3/4)]$.

8. Now, Eq. (1) will have at least two roots negative, when at least one root of Eq. (2) will be less than -2. If $\alpha, \beta \geq -2$, then

$f(-2) \geq 0$ and $-\frac{B}{2A} > -2$

$\therefore \quad 4 - 4a - 1 \geq 0$ and $-\frac{2a}{2} > -2$

$\therefore \quad a \leq \frac{3}{4}$ and $a < 2$

$\therefore \quad a \leq \frac{3}{4} \quad (5)$

Combining (3) and (5), at least one root will be less than -2 for Eq. (2) if

$a > \frac{3}{4}$

$\therefore \quad a \in \left(\frac{3}{4}, \infty\right)$

9. If exactly two roots are positive, then other two roots are negative. Then – 2 and 2 must lie between the roots. So,

$f(-2) < 0$ and $f(2) < 0$

$\Rightarrow$ $a > 3/4$ and $a < -3/4$

Hence, no such values of a exist.

For Problems 10–12

10. a, 11. d, 12. c.

Sol. $(\beta - \alpha) = ((\beta + h) - (\alpha + h))$

$(\beta + \alpha)^2 - 4\alpha\beta = [(\beta + h) + (\alpha + h)]^2 - 4(\beta + h)(\alpha + h)$

$(-b_1)^2 - 4c_1 = (-b_2)^2 - 4c_2$

$D_1 = D_2$ (i)

The least value of $f(x)$ is

$$-\frac{D_1}{4} = -\frac{1}{4} \Rightarrow D_1 = 1 \text{ and } D_2 = 1$$

Therefore, the least value of

$$g(x) \text{ is } -\frac{D_2}{4} = -\frac{1}{4}$$

The least value of $g(x)$ occurs at

$$-\frac{b_2}{2} = \frac{7}{2} \Rightarrow b_2 = -7$$

$$\Rightarrow b_2^2 - 4c_2 = D_2$$

$$\Rightarrow 49 - 4c_2 = 1 \Rightarrow \frac{48}{4} = c_2 \Rightarrow c_2 = 12$$

$$\Rightarrow x^2 - 7x + 12 = 0 \Rightarrow x = 3, 4$$

For Problems 13–15

13. d, 14. c, 15. c.

Sol.

13. $\because$ $AC = 4\sqrt{2}$

$$\therefore AB = BC = \frac{4\sqrt{2}}{\sqrt{2}} = 4 \text{ units}$$

$$OB = \sqrt{4^2 - (2\sqrt{2})^2} = 2\sqrt{2}$$

$$\therefore A(-2\sqrt{2}, 0), B(2\sqrt{2}, 0), C(0, -2\sqrt{2})$$

Since $y = ax^2 + bx + c$ passes through A, B and C, we get

$$y = \frac{x^2}{2\sqrt{2}} - 2\sqrt{2}$$

14. Minimum value of $y = x^2/(2\sqrt{2}) - 2\sqrt{2}$ is $-2\sqrt{2}$ at $x = 0$.

15. $f(x) = 0$

$$\Rightarrow x^2/(2\sqrt{2}) - 2\sqrt{2} = 0 \text{ or } x = \pm 2\sqrt{2}$$

Therefore, number of integral values of k for which k lies in

$(-2\sqrt{2}, 2\sqrt{2})$ is 5.

For Problems 16–18

16. d, 17. c, 18. b.

Sol.

Given that $9^x - a3^x - a + 3 \leq 0$

Let $t = 3^x$. Then,

$$t^2 - at - a + 3 \leq 0$$

or

$$t^2 + 3 \leq a(t + 1) \quad (1)$$

where $t \in R^+$, $\forall x \in R$

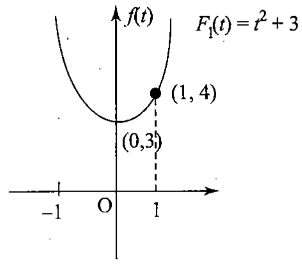


Fig. 1.83

Let $f_1(t) = t^2 + 3$ and $f_2(t) = a(t + 1)$.

16. For $x < 0$, $t \in (0, 1)$. That means (1) should have at least one solution in $t \in (0, 1)$. From (1), it is obvious that $a \in R^+$. Now $f_2(t) = a(t + 1)$ represents a straight line. It should meet the curve $f_1(t) = t^2 + 3$, at least once in $t \in (0, 1)$.

$$f_1(0) = 3, f_1(1) = 4, f_2(0) = a, f_2(1) = 2a$$

If $f_1(0) = f_2(0)$, Then $a = 3$; if $f_1(1) = f_2(1)$, then $a = 2$. Hence, the required range is $a \in (2, 3)$.

17. For at least one positive solution, $t \in (1, \infty)$. That means graphs of $f_1(t) = t^2 + 3$ and $f_2(t) = a(t + 1)$ should meet at least once in $t \in (1, \infty)$. If $a = 2$, both the curves touch each other at (1, 4). Hence, the required range is $a \in (2, \infty)$.

18. In this case both graphs should meet at least once in $t \in (0, \infty)$. For $a = 2$ both the curves touch, hence, the required range is $a \in [2, \infty)$.

For Problems 19–21

19. d, 20. c, 21. a.

Sol.

Let $f(x) = x^2 + x + a - 9$.

$x^2 + x + a - 9 < 0$ has at least one positive solution, then either both the roots of equation $x^2 + x + a - 9 = 0$ are non-negative or 0 lies between the roots.

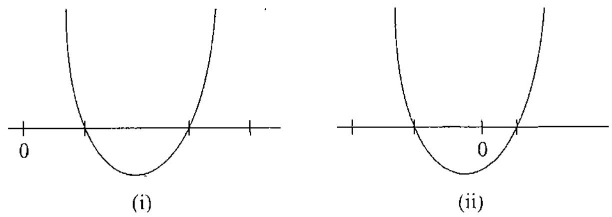


Fig. 1.84

Now sum of roots $= -\frac{1}{2}$, hence case I is not possible. For case II,

$f(O) < 0 \Rightarrow a - 9 < 0 \Rightarrow a < 9$

20.

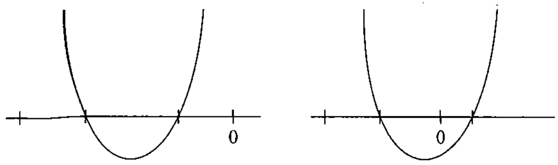


Fig. 1.85

If $x^2 + x + a - 9 < 0$ has at least one negative solution, then either both the roots of equation $x^2 + x + a - 9 = 0$ are non-positive or 0 lies between the roots.

For case **I**, sum of roots is $-1/2 < 0$. Product of roots is $a - 9 > 0 \Rightarrow a \geq 9$ and

$$D > 0 \Rightarrow 1 - 4(a - 9) > 0 \Rightarrow a < \frac{37}{4}$$

Hence, $9 \leq a < 37/4$.

For case **II** $f(0) < 0 \Rightarrow a < 9 \Rightarrow a \in \left(-\infty, \frac{37}{4}\right)$

21. If $x^2 + x + a - 9 < 0$ is true $\forall x \in (-1, 3)$, then $f(-1) < 0$ and $f(3) < 0$.

$\therefore \quad 1 - 1 + a - 9 < 0$ and $9 + 3 + a - 9 < 0$

$\Rightarrow \quad a < 9$ and $a < -3$

$\Rightarrow \quad a < -3$

For Problems 22–24

22. b, 23. a, 24. b.

Sol.

22. $b^2 > (a + c)^2$

$\Rightarrow \quad (a + c - b)(a + c + b) < 0$

$\Rightarrow \quad f(-1) f(1) < 0$

So, there is exactly one root in $(-1, 1)$.

23. $af(1) < 0$ and $f(0) f(1) > 0$

$\Rightarrow \quad af(1) < 0$ and $af(0) < 0$

Hence, both the numbers 0 and 1 lie between the roots.

24. $f(0) f(1) < 0$ and $af(1) > 0$

$\Rightarrow \quad f(0) f(1) < 0$ and $af(0) < 0$

Hence, exactly one root lies in (0, 1) and 0 lies between the roots.

For Problems 25–26

25. a, 26. b.

Sol. From the question, the real roots of $x^3 - x^2 + \beta x + \gamma = 0$ are x_1, x_2, x_3 and they are in A.P. As x_1, x_2, x_3 are in A.P., let $x_1 = a - d$, $x_2 = a$, $x_3 = a + d$. Now,

$$x_1 + x_2 + x_3 = -\frac{-1}{1} = 1$$

$\Rightarrow \quad a - d + a + a + d = 1$

$\Rightarrow \quad a = \frac{1}{3}$ (1)

$$x_1x_2 + x_2x_3 + x_3x_1 = \frac{\beta}{1} = \beta$$

$\Rightarrow \quad (a - d)\, a + a\,(a + d) + (a + d)(a - d) = \beta$ (2)

$$x_1x_2x_3 = -\frac{\gamma}{1} = -\gamma$$

$\Rightarrow \quad (a - d)\, a\,(a + d) = -\gamma$ (3)

From (1) and (2), we get

$3a^2 - d^2 = \beta$

$\Rightarrow \quad \Rightarrow 3\frac{1}{9} - d^2 = \beta$, so $\beta = \frac{1}{3} - d^2 < \frac{1}{3}$

From (1) and (3), we get

$$\frac{1}{3}\left(\frac{1}{9} - d^2\right) = -\gamma$$

$\Rightarrow \quad \gamma = \frac{1}{3}\left(d^2 - \frac{1}{9}\right) > \frac{1}{3}\left(-\frac{1}{9}\right) = -\frac{1}{27}$

$$\gamma \in \left(-\frac{1}{27}, +\infty\right)$$

Matrix-Match Type

1. a → s; b → r; c → q; d → p.

Obviously when $a \geq 0$, we have no roots as all the terms are followed by +ve sign. Also for $a = -2$, we have

$x^2 - 2|x| + 1 = 0$

or

$|x| - 1 = 0 \Rightarrow x = \pm 1$

Hence, the equation has two roots.

Also when $a < -2$, for given equation

$$|x| = \frac{-a \pm \sqrt{a^2 - 4}}{2} > 0$$

Hence, the equation has four roots as $|-a| > \sqrt{a^2 - 4}$. Obviously, the equation has no three real roots for any value of a.

2. a → p; b → p, q, r, s; c → p, q, s; d → r, s.

a. $y = \dfrac{x^2 - 2x + 4}{x^2 + 2x + 4}$

$\Rightarrow \quad x^2y + 2xy + 4y = x^2 - 2x + 4$

$\Rightarrow \quad (y - 1)x^2 + 2(y + 1)x + 4(y - 1) = 0$

$D \geq 0$

$\Rightarrow \quad 4(y + 1)^2 - 16(y - 1)^2 \geq 0$

$\Rightarrow \quad (y + 1)^2 - (2y - 2)^2 \geq 0$

$\Rightarrow \quad (3y - 1)(3 - y) \geq 0$

$\Rightarrow \quad (3y - 1)(y - 3) \leq 0 \Rightarrow y \in \left[\frac{1}{3}, 3\right]$

$\Rightarrow \quad \{1\} \Rightarrow P$

b. $y = \dfrac{x^2 - 3x - 2}{2x - 3}$

$\Rightarrow \quad x^2 - 3x - 2 = 2xy - 3y$

$\Rightarrow \quad x^2 - (3 + 2y)x + (3y - 2) = 0$

$D \geq 0$

$\Rightarrow$ $(3 + 2y)^2 - 4(3y - 2) \geq 0$

$\Rightarrow$ $9 + 4y^2 + 12y - 12y + 8 \geq 0$

$\Rightarrow$ $4y^2 + 17 \geq 0$

which is always true. Hence,

$y \in R \Rightarrow \{1, 4, -3, -10\} \Rightarrow p, q, r, s$

c. $y = \dfrac{2x^2 - 2x + 4}{x^2 - 4x + 3}$

$\Rightarrow$ $x^2 y - 4xy + 3y = 2x^2 - 2x + 4$

$(y - 2) x^2 + 2(1 - 2y)x + 3y - 4 = 0$

$D \geq 0$

$4(1 - 2y)^2 - 4(y - 2)(3y - 4) \geq 0$

$\Rightarrow$ $1 + 4y^2 - 4y - (3y^2 - 10y + 8) \geq 0$

$\Rightarrow$ $y^2 + 6y - 7 \geq 0$

$\Rightarrow$ $(y + 7)(y - 1) \geq 0$

$\Rightarrow$ $y \geq 1$ or $y \leq -7$

$\Rightarrow$ $\{1, 4, -10\} \Rightarrow p, q, s$

d. $f(x) = x^2 - (a - 3)x + 2 < 0, \forall x \in [-2, -1]$

$\Rightarrow$ $f(-2) < 0$ and $f(-1) < 0$

$\Rightarrow$ $4 + 2(a - 3) + 2 < 0$ and $1 + (a - 2) + 2 < 0$

$\Rightarrow$ $a < 0$ and $a < -1$

$\Rightarrow$ $a < -1$

$\Rightarrow$ $a \in \{-10, -3\}$

3. a → q, r, s; b → r; c → p; d → q.

a. $d + a - b = 0$ and $d + b - c = 0$

$d = b - a$ and $d = c - b$

$\therefore b - a = c - b \Rightarrow 2b = a + b \Rightarrow a, b, c$ are in A.P.

Also $x = 1$ satisfies the second equation. Therefore, the other root is also 1. Product of roots is 1.

$\therefore c(a - b) = a(b - c) \Rightarrow b = \dfrac{2ac}{a + c} \Rightarrow a, b, c$ are in H.P.

Therefore, a, b, c are in A.P. and a, b, c in H.P. Hence, a, b, c are in G.P.

b. $(a^2 + b^2) x^2 - 2b(a + c)x + (b^2 + c^2) = 0$

The roots are real and equal. Hence,

$4b^2 (a + c)^2 - 4(a^2 + b^2)(b^2 + c^2) = 0$

$\Rightarrow$ $b^2 (a^2 + c^2 + 2ac) - (a^2b^2 + a^2c^2 + b^4 + b^2c^2) = 0$

$\Rightarrow$ $b^2a^2 + b^2c^2 + 2ab^2c - a^2b^2 - a^2c^2 - b^4 - b^2c^2 = 0$

$\Rightarrow$ $2ab^2c - a^2c^2 - b^4 = 0 \Rightarrow (b^2 - ac)^2 = 0$

Hence, $b^2 = ac$. Thus a, b, c are in G.P.

c. $(x - 1)^3 = 0 \Rightarrow x = 1$ is the common root. Hence, $a + b + c = 0$.

d. $(a + c)^2 + 4b^2 - 4b(a + c) \leq 0, \forall x \in R$

$\Rightarrow$ $((a + c) - 2b)^2 \leq 0$

$\Rightarrow$ $a + c = 2b$

$\Rightarrow$ a, b, c in A.P.

4. a → r; b → r; c → q; d → p.

a. $(m - 2)x^2 - (8 - 2m)x - (8 - 3m) = 0$ has roots of opposite signs. The product of roots is

$-\dfrac{8 - 3m}{m - 2} < 0$

$\Rightarrow$ $\dfrac{3m - 8}{m - 2} < 0$

$\Rightarrow$ $2 < m < 8/3$

b. Exactly one root of equation $x^2 - m(2x - 8) - 15 = 0$ lies in interval (0, 1).

$f(0)f(1) < 0$

$\Rightarrow$ $(0 - m(-8) - 15)(1 - m(-6) - 15) < 0$

$\Rightarrow$ $(8m - 15)(6m - 15) < 0$

$\Rightarrow$ $15/8 < m < 15/6$

c. $x^2 + 2(m + 1)x + 9m - 5 = 0$ has both roots negative. Hence, sum of roots is

$-2(m + 1) < 0$ or $m > -1$ (1)

Product of roots is

$9m - 5 > 0 \Rightarrow m > 5/9$ (2)

Discriminant,

$D \geq 0 \Rightarrow 4(m + 1)^2 - 4(9m - 5) \geq 0$

$\Rightarrow$ $m^2 - 7m + 6 \geq 0$

$\Rightarrow$ $m \leq 1$ or $m \geq 6$ (3)

Hence, for (1), (2) and (3), we get

$m \in \left(\dfrac{5}{9}, 1\right] \cup [6, \infty)$

d. $f(x) = x^2 + 2(m - 1)x + m + 5 = 0$ has one root less than 1 and the other root greater than 1. Hence,

$f(1) < 0$

$\Rightarrow$ $1 + 2(m - 1) + m + 5 < 0$

$\Rightarrow$ $m < -4/3$

5. a → s; b → p; c → q; d → r.

a. $x^2 + ax + b = 0$ has root α. Hence,

$\alpha^2 + a\alpha + b = 0$ (1)

$x^2 + px + q = 0$ has roots $-\alpha, \gamma$. Hence,

$\alpha^2 - p\alpha + q = 0$ (2)

Eliminating α from (1) and (2), we get

$(q - b)^2 = (aq + bp)(-p - a)$

$\Rightarrow$ $(q - b)^2 = -(aq + bp)(p + a)$

b. $x^2 + ax + b = 0$ has root is α, β. Hence,

$\alpha^2 + a\alpha + b = 0$ (1)

$x^2 + px + q = 0$ has root $1/\alpha$. Hence,

$q\alpha^2 + p\alpha + 1 = 0$ (2)

Eliminating α from (1) and (2), we get

$(1 - bq)^2 = (a - pb)(p - aq)$

c. $x^2 + ax + b = 0$ has roots α, β. Hence,

$\alpha^2 + a\alpha + b = 0$ (1)

$x^2 + px + q = 0$ has roots $-2/\alpha, \gamma$. Hence,

$q\alpha^2 - 2p\alpha + 4 = 0$ (2)

Eliminating α from (1) and (2), we get

$(4 - bq)^2 = (4a + 2pb)(-2p - aq)$

d. $x^2 + ax + b = 0$ has roots α, β. Hence,

$\alpha^2 + a\alpha + b = 0$ (1)

$x^2 + px + q = 0$ has roots $-1/2\alpha, \gamma$. Hence,

$4q\alpha^2 - 2p\alpha + 1 = 0$ (2)

Eliminating α from (1) and (2), we get

$(1 - 4bq)^2 = (a + 2bp)(-2p - 4aq)$

Integer Type

1.(8) Let $\left(a + \frac{1}{a}\right) = t$

$\Rightarrow a^3 + \frac{1}{a^3} = 18$

$t^3 - 3t - 18 = 0$ (1)

$t = 3$ satisfies (1)

hence factorizing (1)

$(t-3)(t^2 + 3t + 6) = 0$

$t = 3$ only is the solution

$\therefore \quad a + \frac{1}{a} = 3 \Rightarrow a^2 + \frac{1}{a^2} = 7 \Rightarrow a^4 + \frac{1}{a^4} = 47$

2. (3) We have $P(x) = \frac{5}{3} - 6x - 9x^2 = -(3x+1)^2 + \frac{8}{3}$

$\Rightarrow P_{max} = \frac{8}{3}$

Similarly, $Q(y) = -4y^2 + 4y + \frac{13}{2} = -(2y-1)^2 + \frac{15}{2}$

$\Rightarrow Q_{max} = \frac{15}{2}$

Now, $P_{max} \times Q_{max} = \frac{8}{3} \times \frac{15}{2} = 20$

So $(x, y) \equiv \left(-\frac{1}{3}, \frac{1}{2}\right)$

Hence, $6x + 10y = 6\left(\frac{-1}{3}\right) + 10\left(\frac{1}{2}\right) = -2 + 5 = 3$

3. (2) We have $x_1 + x_2 + x_3 = 8$

$x_1 x_2 x_3 = d$

$x_1x_2 + x_2x_3 + x_3x_1 = c$

Possible roots 1, 2, 5 or 1, 3, 4

$\therefore \quad d = 10$ or $d = 12$

$\Rightarrow \quad c = 2 + 10 + 5 = 17$ or $3 + 12 + 4 = 19$

Hence, $d = 10$ and $c = 17$ or $d = 12$ and $c = 19$

4.(9) Let $\alpha_1 = A, \beta_1 = AR, \alpha_2 = AR^2, \beta_2 = AR^3$

we have $\alpha_1 + \beta_1 = 6 \Rightarrow A(1 + R) = 6$ (1)

$\alpha_1\beta_1 = p \Rightarrow A^2 R = p$ (2)

Also $\alpha_2 + \beta_2 = 54 \Rightarrow AR^2(1 + R) = 54$ (3)

$\alpha_2\beta_2 = q \Rightarrow A^2R^5 = q$ (4)

Now, on dividing Eq. (3) by Eq. (1), we get

$\frac{AR^2(1+R)}{A(1+R)} = \frac{54}{6} = 9 \Rightarrow R^2 = 9$

$\therefore \quad R = 3$ (As it is an increasing G.P.)

$\therefore$ On putting $R = 3$ in Eq. (1), we get

$A = \frac{6}{4} = \frac{3}{2}$

$\therefore \quad p = A^2 R = \frac{9}{4} \times 3 = \frac{27}{4}$ and $q = A^2R^5 = \frac{9}{4} \times 243 = \frac{2187}{4}$

Hence, $q - p = \frac{2187 - 27}{4} = \frac{2160}{4} = 540$

5.(3) $2x^2 + 4x(y-3) + 7y^2 - 2y + t = 0$

$D = 0$ (for one solution)

$\Rightarrow 16(y-3)^2 - 8(7y^2 - 2y + t) = 0$

$\Rightarrow 2(y-3)^2 - (7y^2 - 2y + t) = 0$

$\Rightarrow 2(y^2 - 6y + 9) - (7y^2 - 2y + t) = 0$

$\Rightarrow -5y^2 - 10y + 18 - t = 0$

$\Rightarrow 5y^2 + 10y + t - 18 = 0$

Again $D = 0$ (for one solution)

$\Rightarrow 100 - 20(t - 18) = 0$

$\Rightarrow 5 - t + 18 = 0$

$\Rightarrow t = 23$

for $t = 23$; $5y^2 + 10y + 5 = 0$

$(y+1)^2 = 0 \Rightarrow y = -1$

for $y = -1$; $2x^2 - 16x + 32 = 0$

$x^2 - 8x + 16 = 0$

$x = 4 \Rightarrow x + y = 3$

6. (4) As $P(x)$ is an odd function

Hence, $P(-x) = -P(x) \Rightarrow P(-3) = -P(3) = -6$

Let $P(x) = Q(x^2 - 9) + ax + b$

(where Q is quotient and $(ax + b) = g(x)$ = remainder)

Now $P(3) = 3a + b = 6$ (1)

$P(-3) = -3a + b = -6$ (2)

Hence, $b = 0$ and $a = 2$

Hence, $g(x) = 2x \Rightarrow g(2) = 4$

7. (4) $f(x) = ax^2 - (3 + 2a)x + 6$

$= (ax - 3)(x - 2)$

Here, roots of the equation $f(x) = 0$ are 2 and $3/a$, and $f(0) = 6$.

$f(x)$ should be positive for exactly three negative integral values of x.

Therefore, graph of $f(x)$ must be a downward parabola passing through $x = 2$ and $x = 3/a$ and $-4 \le \frac{3}{a} < -3$

$\therefore \quad a \in \left(-1, -\frac{3}{4}\right]$

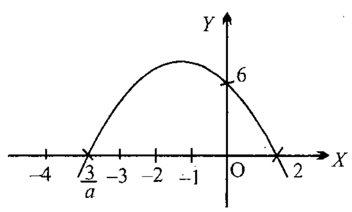


Fig. 1.86

$\therefore \quad c = -1, d = -\frac{3}{4}$

$\Rightarrow \quad c^2 + 4|d| = 1 + 3 = 4$

8. (8) Given $\alpha\beta$; $\alpha\beta(\alpha+\beta)$; $\alpha^3+\beta^3$ are in G.P.

$\alpha+\beta = 4$; $\alpha\beta = k; \alpha\beta^2 + \alpha^2\beta = \alpha\beta(\alpha+\beta) = 4k$

$\alpha^3+\beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$

$= 64 - 3k(4) = 4(16-3k)$

$\therefore$ k ; $4k$; $4(16-3k)$ are in G.P.

$16k^2 = 4k(16-3k)$

$4k(4k - 16 + 3k) = 0$

$k = 0;\ k = \frac{16}{7}$

9. (6)

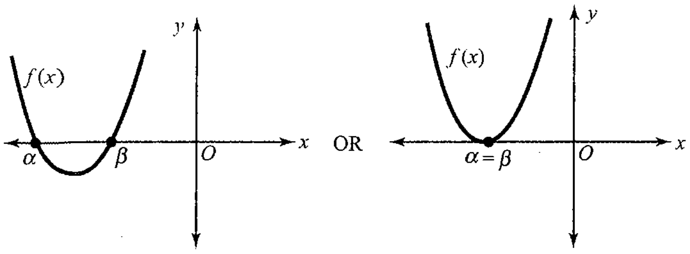


Fig. 1.87

Let $f(x) = x^2 + 2(\lambda+1)x + \lambda^2 + \lambda + 7$

If both roots of $f(x) = 0$ are negative, then

$D = b^2 - 4ac = 4(\lambda+1)^2 - 4(\lambda^2+\lambda+7) \geq 0 \Rightarrow \lambda - 6 \geq 0$

$\Rightarrow \lambda \in [6, \infty)$ (1)

Sum of roots $= -2(\lambda+1) < 0$

$\Rightarrow \lambda \in (-1, \infty)$ (2)

and product of roots $= \lambda^2 + \lambda + 7 > 0\ \forall\ \lambda \in R$ (3)

$\therefore$ From (1), (2), (3), we get $\lambda \in [6, \infty)$

(As (1), (2), (3) must be satisfied simultaneously.)

Hence, the least value of $\lambda = 6$.

10. (3) Clearly, $P(x) - x^3 = 0$ has roots 1, 2, 3, 4.

$\therefore\ P(x) - x^3 = (x-1)(x-2)(x-3)(x-4)$

$\Rightarrow P(x) = (x-1)(x-2)(x-3)(x-4) + x^3$

Hence, $P(5) = 1 \times 2 \times 3 \times 4 + 125 = 129$

11. (4) $x^{1/8} = (3x^4+4)^{1/64} \Rightarrow x^8 = 3x^4 + 4 \Rightarrow x^4 = 4$

12. (3) $f(x) = (x-1)(x^2 - 7x + 13)$

for $f(x)$ to be prime at least one of the factors must be prime.

Hence, $x - 1 = 1 \Rightarrow x = 2$ or

$x^2 - 7x + 13 = 1 \Rightarrow x^2 - 7x + 12 = 0 \Rightarrow x = 3$ or 4

$\Rightarrow x = 2, 3, 4$

13. (6) Let the roots be $a-3d, a-d, a+d, a+3d$

Sum of roots $= 4a = 0 \Rightarrow a = 0$

Hence, roots are $-3d, -d, d, 3d$.

Product of roots $= 9d^4 = m^2 \Rightarrow d^2 = \frac{m}{3}$ (1)

Again $\sum x_1x_2 = 3d^2 - 3d^2 - 9d^2 - d^2 - 3d^2 + 3d^2 = -10d^2$

$= -(3m+2)$

$\Rightarrow \frac{10m}{3} = 3m + 2 \Rightarrow 10m = 9m + 6$

$\Rightarrow \quad m = 6$

14. (8) As $P(1) = 0$

and $p(x) \geq 0$ hence let $p(x) = k(x-1)^2, k > 0$

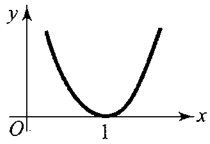


Fig. 1.88

$p(2) = k = 2 \Rightarrow k = 2$

$\therefore\ p(x) = 2(x-1)^2 \Rightarrow p(3) = 8$

15. (4) $y = \frac{3x^2 + mx + n}{x^2 + 1}$

$x^2(y-3) - mx + y - n = 0$

$x \in R$

$D \geq 0$

$\Rightarrow m^2 - 4(y-3)(y-n) \geq 0$

$\Rightarrow m^2 - 4(y^2 - ny - 3y + 3n) \geq 0$

$4y^2 - 4y(n+3) + 12n - m^2 \leq 0$ (1)

Also given $(y+4)(y-3) \leq 0$

$y^2 + y - 12 \leq 0$ (2)

Compare (1) and (2), we get $\frac{4}{1} = -\frac{4(n+3)}{1} = \frac{12n - m^2}{-12}$

$\Rightarrow m = 0$ and $n = -4$

16. (6) $a^2 \geq 8b$ and $4b^2 \geq 4a$

Now $b^2 \geq a \Rightarrow b^4 \geq a^2 \geq 8b \quad (a > 0, b > 0)$

$\therefore \Rightarrow \quad b^3 \geq 8 \Rightarrow b \geq 2$ (1)

Again $a^2 \geq 8b$ and $b \geq 2$

$\Rightarrow \quad a^2 \geq 16$

$\Rightarrow \quad a \geq 4$ (2)

From (1) and (2), $(a+b)_{\text{least}} = 6$.

17. (7) Given $a + b + c = 1$ (1)

$ab + bc + ca = 0$ (2)

$abc = 2$ (3)

Now $(a+b+c)^2 = 1$

$a^2 + b^2 + c^2 + 2\sum ab = 1$

$\therefore \quad a^2 + b^2 + c^2 = 1$

Now, $a^3 + b^3 + c^3 - 3abc = (a+b+c)[\sum a^2 - \sum ab]$

$= 1(1-0) = 1$

$a^3 + b^3 + c^3 = 1 + 3abc = 1 + 3 \times 2 = 7$

18.(3) The given equation $x + \frac{1}{x} = 3$

$\therefore\ x^2 + \frac{1}{x^2} = 7 \Rightarrow x^4 + \frac{1}{x^4} = 47 \Rightarrow x^8 + \frac{1}{x^8} = (47)^2 - 2$

$\therefore\ x^8 + x^{-8} = 2207$ (1)

Now $E = x^9 + x^7 + x^{-9} + x^{-7}$

$= x^8\left(x + \frac{1}{x}\right) + x^{-9} + x^{-7} = x^8\left(x + \frac{1}{x}\right) + x^{-8}\left(x + \frac{1}{x}\right)$

$E = \left(x + \frac{1}{x}\right)(x^8 + x^{-8})$ (2)

Substitute the value of $x^8 + x^{-8} = 2207$ from (1) and $x + \frac{1}{x} = 3$

$E = (3)(2207) = 6621$

19.(6) $x^2 + ax + b \equiv (x+1)(x+b) \Rightarrow b + 1 = a$ (1)

also $x^2 + bx + c \equiv (x+1)(x+c) \Rightarrow c + 1 = b$

or $b + 1 = c + 2$ (2)

hence $b + 1 = a = c + 2$

also $(x+1)(x+b)(x+c) \equiv x^3 - 4x^2 + x + 6$

$\Rightarrow x^3 + (1+b+c)x^2 + (b+bc+c)x + bc \equiv x^3 - 4x^2 + x + 6$

$\Rightarrow 1 + b + c = -4$

$\Rightarrow 2c + 2 = -4 \Rightarrow c = -3; b = -2$ and $a = -1$

$\Rightarrow a + b + c = -6$

20.(3) $n, n+1, n+2$

sum $= 3(n+1) = -a$

$\therefore a^2 = 9(n+1)^2$

sum of the roots taken 2 at a time $= +b$

$\therefore n(n+1) + (n+1)(n+2) + (n+2)n + 1 = b + 1$

$n^2 + n + n^2 + 3n + 2 + n^2 + 2n + 1 = b + 1$

$\therefore b + 1 = 3n^2 + 6n + 3$

$b + 1 = 3(n+1)^2 = \dfrac{a^2}{3}; \quad \therefore \dfrac{a^2}{b+1} = 3$

21.(2) $(x+y+z)^2 = 144$ (given)

$\Rightarrow \sum x^2 + 2\sum xy = 144$

$\Rightarrow 96 + 2\sum xy = 144 \quad \Rightarrow \sum xy = 24$

again $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 36 \quad \Rightarrow xyz = \dfrac{24}{36} = \dfrac{2}{3}$

now $x^3 + y^3 + z^3 - 3xyz = (x+y+z)\left(\sum x^2 - \sum xy\right)$

$\Rightarrow \sum x^3 - 2 = (12)(96 - 24) = (12)(72) = 864$

$\Rightarrow \sum x^3 = 866$

22.(6) $\alpha + \beta = 1154$ and $\alpha\beta = 1$

$(\sqrt{\alpha} + \sqrt{\beta})^2 = \alpha + \beta + 2\sqrt{\alpha\beta} = 1154 + 2 = 1156 = (34)^2$

$\Rightarrow \sqrt{\alpha} + \sqrt{\beta} = 34$

Again $(\alpha^{1/4} + \beta^{1/4})^2 = \sqrt{\alpha} + \sqrt{\beta} + 2(\alpha\beta)^{1/4} = 34 + 2 = 36$

$\alpha^{1/4} + \beta^{1/4} = 6$

23.(4) Given $a^2 - 4a + 1 = 4 \Rightarrow a^2 + 1 = 4(1+a)$

$y = \dfrac{(a-1)(1+a^2)}{a^2-1} = \dfrac{a^2+1}{a+1} = \dfrac{4(a+1)}{a+1} = 4$

24.(5) Let $ax^3 + bx^2 + cx + d = 0$ has roots p, q, r

$pq + qr + rp = \dfrac{c}{a}$ (1)

but $pq + qr + rp \le p^2 + q^2 + r^2$

$= (p+q+r)^2 - 2\sum pq$

$\therefore 3(pq + qr + rp) \le (p+q+r)^2 = 16$

$\therefore 3\dfrac{c}{a} \le 16 \Rightarrow \dfrac{c}{a} \le \dfrac{16}{3} \Rightarrow$ largest possible integral value of $\dfrac{c}{a}$ is 5

25.(3) $x^2 + x(y-a) + y^2 - ay + 1 \ge 0 \; x \in R$

$\Rightarrow (y-a)^2 - 4(y^2 - ay + 1) \le 0$

$\Rightarrow -3y^2 + 2ay + a^2 - 4 \le 0$

$\therefore 3y^2 - 2ay + 4 - a^2 \ge 0 \quad y \in R$

$D \le 0$

$\Rightarrow 4a^2 - 4 . 3(4 - a^2) \le 0 \Rightarrow a^2 - 3(4 - a^2) \le 0 \Rightarrow 4a^2 - 12 \le 0$

$\therefore$ range of $a \in (-\sqrt{3}, \sqrt{3}) \Rightarrow$ number of integer $\{-1, 0, 1\}$

26.(3) We have $(\alpha+\beta)^2 = \left(\dfrac{1}{\alpha} + \dfrac{1}{\beta}\right)(\alpha^2 + \beta^2)$

$\Rightarrow (\alpha+\beta)^2 = \left(\dfrac{1}{\alpha} + \dfrac{1}{\beta}\right)[(\alpha+\beta)^2 - 2\alpha\beta]$

Substituting $\alpha + \beta = -\dfrac{b}{a}$ and $\alpha\beta = \dfrac{c}{a}$ we have

$\dfrac{b^2}{a^2} = \dfrac{-b}{c}\left(\dfrac{b^2}{a^2} - \dfrac{2c}{a}\right)$

$\Rightarrow cb^2 + b(b^2 - 2ac) = 0$

$b \ne 0, \quad \therefore bc + b^2 - 2ac = 0$

a, b, c are in AP, $\quad \therefore b = \dfrac{a+c}{2}$

$\therefore$ we have $\dfrac{(a+c)c}{2} + \left(\dfrac{a+c}{2}\right)^2 - 2ac = 0$

$\Rightarrow a^2 - 4ac + 3c^2 = 0 \quad \Rightarrow (a-c)(a-3c) = 0$

$a \ne c \quad \therefore a = 3c \quad \therefore \dfrac{a}{c} = 3$

27.(7) For two distinct roots, $D > 0$ i.e., $k^2 + 8(k^2 + 5) > 0$ which is always true

Also let $f(x) = -2x^2 + kx + k^2 + 5 = 0$

But $f(0) > 0$ and $f(2) < 0$

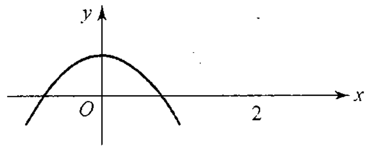


Fig. 1.89

$-8 + 2k + k^2 + 5 < 0 \quad \Rightarrow k^2 + 2k - 3 < 0$

$\Rightarrow (k+3)(k-1) < 0$

$k \in (-3, 1) \quad \Rightarrow a = -3; b = 1 \quad \Rightarrow a + 10b = -3 + 10 = 7$

28.(8) $x^2 + mx + n = 0$ with roots 2α, 2β and $x^2 + px + m = 0$ with roots α, β

$2(\alpha + \beta) = -m$ (1)

$4\alpha\beta = n$ (2)

and $\alpha + \beta = -p$ (3)

$\alpha\beta = m$ (4)

$\therefore$ (1) and (3) $\Rightarrow 2p = m$

and (2) and (4) $\Rightarrow 4m = n$

$\Rightarrow \dfrac{n}{p} = \dfrac{4m}{m/2} = 8$

29.(7) Let $E = \dfrac{a}{b+c} + \dfrac{b}{c+a} + \dfrac{c}{a+b}$

now $3 + E = \dfrac{a}{b+c} + 1 + \dfrac{b}{c+a} + 1 + \dfrac{c}{a+b} + 1$

$\Rightarrow 3 + E = (a+b+c)\left[\dfrac{1}{b+c} + \dfrac{1}{c+a} + \dfrac{1}{a+b}\right]$

$\Rightarrow 3 + E = 3 \times \frac{10}{3} = 10 \quad \Rightarrow E = 7$

30.(3) We have $abc + 1 = \frac{bc}{5} = \frac{-ac}{15} = \frac{ab}{3}$

$\Rightarrow \frac{a}{b} = -3$ and $\frac{c}{b} = -5$

Now $\frac{c-b}{c-a} = \frac{\frac{c}{b}-1}{\frac{c}{b}-\frac{a}{b}} = \frac{-5-1}{-5-(-3)} = 3$

31.(2) We have $\left(\frac{a^4+3a^2+1}{a^2}\right)\left(\frac{b^4+5b^2+1}{b^2}\right)\left(\frac{c^4+7c^2+1}{c^2}\right)$

$= \left(a^2+\frac{1}{a^2}+3\right)\left(b^2+\frac{1}{b^2}+5\right)\left(c^2+\frac{1}{c^2}+7\right)$

$= \left(\left(a-\frac{1}{a}\right)^2+5\right)\left(\left(b-\frac{1}{b}\right)^2+7\right)\left(\left(c-\frac{1}{c}\right)^2+9\right)$

32.(3) Let $a^2 + b^2 = x$

$1 - 2ab = (a + b)^2 - 2ab = a^2 + b^2 = x(a + b = 1)$;

also $ab = \frac{1-x}{2}$

and $a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$

$a^3 + b^3 = x - ab$ (1)

but $ab = \frac{1-x}{2}$

Hence $a^3 + b^3 = x - \frac{1-x}{2} = \frac{3x-1}{2}$

Hence the equation

$(1 - 2ab)(a^3 + b^3) = 12$, becomes

$x\left(\frac{3x-1}{2}\right) = 12$

$\Rightarrow 3x^2 - x - 24 = 0$

$\Rightarrow (x - 3)(3x + 8) = 0$

$\Rightarrow x = 3$ or $x = -\frac{8}{3}$ (not possible) as $x = a^2 + b^2 \not< 0$

$\therefore x = 3 \quad \Rightarrow a^2 + b^2 = 3$

33.(5) We have $2x^3 - 9x^2 + 12x + k = 0$

Let the roots are α, α, β

$2\alpha + \beta = \frac{9}{2}$ (1)

$\alpha^2 + 2\alpha\beta = \frac{12}{2} = 6$ (2)

are $\alpha^2\beta = -\frac{k}{2}$ (3)

putting $\beta = \left(\frac{9}{2} - 2\alpha\right)$ from (1) in (2)

$\alpha^2 + 2\alpha\left(\frac{9}{2} - 2\alpha\right) = 6 \quad \Rightarrow \alpha^2 + 9\alpha - 4\alpha^2 = 6$

$\Rightarrow 3\alpha^2 - 9\alpha + 6 = 0$

$\Rightarrow \alpha^2 - 3\alpha + 2 = 0$

$\Rightarrow (\alpha - 2)(\alpha - 1) = 0 \quad \Rightarrow \alpha = 2$ or 1

if $\alpha = 2$ then $\beta = \frac{1}{2}$; if $\alpha = 1$ then $\beta = \frac{5}{2}$

$\therefore k = -2(4)\frac{1}{2} = -4$ or $k = -2(1^2)\left(\frac{5}{2}\right) = -5$

34.(3) Let $\frac{1-a^3}{a} = \frac{1-b^3}{b} = \frac{1-c^3}{c} = k \Rightarrow \frac{1-x^3}{x} = k,$

where x takes 3 values a, b and c.

$\Rightarrow x^3 + kx - 1 = 0$ has roots a, b, c

Now $a + b + c = 0$ (1)

$abc = 1$ (2)

Hence $a^3 + b^3 + c^3 = 3abc = 3$

Archives

Subjective Type

1. $4^x - 3^{x-\frac{1}{2}} = 3^{x+\frac{1}{2}} - 2^{2x-1}$

$\Rightarrow 4^x - \frac{3^x}{\sqrt{3}} = 3^x\sqrt{3} - \frac{4^x}{2}$

$\Rightarrow \frac{3}{2}4^x = 3^x\left(\sqrt{3} + \frac{1}{\sqrt{3}}\right)$

$\Rightarrow \frac{3}{2}4^x = 3^x\frac{4}{\sqrt{3}}$

$\Rightarrow \frac{4^{x-1}}{4^{1/2}} = \frac{3^{x-1}}{\sqrt{3}}$

$\Rightarrow 4^{x-3/2} = 3^{x-3/2}$

$\Rightarrow \left(\frac{4}{3}\right)^{x-3/2} = 1$

$\Rightarrow x - \frac{3}{2} = 0$

$\Rightarrow x = 3/2$

2. We have,

$\sqrt{x+1} = 1 + \sqrt{x-1}$

Squaring both sides, we get

$x + 1 = 1 + x - 1 + 2\sqrt{x-1}$

$\Rightarrow 1 = 2\sqrt{x-1}$

$\Rightarrow 1 = 4(x - 1)$

$\Rightarrow x = 5/4$

3. Given $a > 0$. So we have two cases: $a \neq 1$ and $a = 1$. Also, it is clear that $x > 0$ and $x \neq 1$, $ax \neq 1$, $a^2x \neq 1$.

Case I: If $a > 0, \neq 1$, then given equation can be simplified as

$$\frac{2}{\log_a x} + \frac{1}{1+\log_a x} + \frac{3}{2+\log_a x} = 0$$

Putting $\log_a x = y$, we get

$$2(1+y)(2+y) + y(2+y) + 3y(1+y) = 0$$

$\Rightarrow$ $6y^2 + 11y + 4 = 0$

$\Rightarrow$ $y = -4/3$ and $-1/2$

$\Rightarrow$ $\log_a x = -4/3$ and $\log_a x = -1/2$

$\Rightarrow$ $x = a^{-4/3}$ and $x = a^{-1/2}$

Case II: If $a = 1$, The then equation becomes

$$2\log_x 1 + \log_x 1 + 3\log_x 1 = 5\log_x 1 = 0$$

which is true $\forall\, x > 0, \neq 1$. Hence, solution is

$$\begin{cases} x > 0, \neq 1, \text{ if } a = 1 \\ x = a^{-1/2}, a^{-4/3}, \text{ if } a > 0, \neq 1 \end{cases}$$

4. Let,

$$x = \frac{\sqrt{26-15\sqrt{3}}}{5\sqrt{2} - \sqrt{38+5\sqrt{3}}}$$

$$\Rightarrow x^2 = \frac{26-15\sqrt{3}}{50+38+5\sqrt{3}-10\sqrt{76+10\sqrt{3}}}$$

$$\Rightarrow x^2 = \frac{26-15\sqrt{3}}{88+5\sqrt{3}-10\sqrt{75+1+10\sqrt{3}}}$$

$$\Rightarrow x^2 = \frac{26-15\sqrt{3}}{88+5\sqrt{3}-10\sqrt{(5\sqrt{3})^2+(1)^2+2\times 5\sqrt{3}\times 1}}$$

$$\Rightarrow x^2 = \frac{26-15\sqrt{3}}{88+5\sqrt{3}-10\sqrt{(5\sqrt{3}+1)^2}}$$

$$\Rightarrow x^2 = \frac{26-15\sqrt{3}}{88+5\sqrt{3}-10(5\sqrt{3}+1)}$$

$$\Rightarrow x^2 = \frac{26-15\sqrt{3}}{78-45\sqrt{3}}$$

$$= \frac{26-15\sqrt{3}}{3(26-15\sqrt{3})} = \frac{1}{3},$$

which is a rational number

5. There are two parts: $(5x-1) < (x+1)^2$ and $(x+1)^2 < (7x-3)$.
Taking first part:

$(5x-1) < (x+1)^2$

$\Rightarrow$ $5x - 1 < x^2 + 2x + 1$

$\Rightarrow$ $x^2 - 3x + 2 > 0$

$\Rightarrow$ $(x-1)(x-2) > 0$

$\Rightarrow$ $x < 1$ or $x > 2$ (1)

Taking second part:

$(x+1)^2 < (7x-3)$

$\Rightarrow$ $x^2 - 5x + 4 < 0$

$\Rightarrow$ $(x-1)(x-4) < 0$

$\Rightarrow$ $1 < x < 4$ (2)

From (1) and (2), taking common values of x, we get $2 < x < 4$.

Then, integral value of x is 3 only.

6. α, β are the roots of $x^2 + px + q = 0$.

$\therefore$ $\alpha + \beta = -p$, $\alpha\beta = q$

γ, δ are the roots of $x^2 + rx + s = 0$.

$\therefore$ $\gamma + \delta = -r$, $\gamma\delta = s$

Now,

$$E = (\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)$$
$$= [\alpha^2 - (\gamma+\delta)\alpha + \gamma\delta][\beta^2 - (\gamma+\delta)\beta + \gamma\delta]$$
$$= [\alpha^2 + r\alpha + s][\beta^2 + r\beta + s]$$

Also $\alpha^2 + p\alpha + q = 0$ and $\beta^2 + p\beta + q = 0$

$$\Rightarrow E = [(r-p)\alpha + (s-q)][(r-p)\beta + (s-q)]$$
$$= (r-p)^2\alpha\beta + (r-p)(s-q)(\alpha+\beta) + (s-q)^2$$
$$= q(r-p)^2 - p(r-p)(s-q) + (s-q)^2$$

Now if the equations $x^2 + px + q = 0$ and $x^2 + rx + s = 0$ have a common root say α, then

$\alpha^2 + p\alpha + q = 0$

and

$\alpha^2 + r\alpha + s = 0$

$\Rightarrow$ $(q-s)^2 = (r-p)(ps-qr)$,

which is the required condition.

7. We know that for sides a, b, c of a triangle,

$(a-b)^2 \geq 0$

$\Rightarrow$ $a^2 + b^2 \geq 2ab$ (1)

Similarly,

$b^2 + c^2 \geq 2bc$ (2)

$c^2 + a^2 \geq 2ca$ (3)

Adding the three inequalities, we get

$2(a^2 + b^2 + c^2) \geq 2(ab + bc + ca)$

$\Rightarrow$ $a^2 + b^2 + c^2 \geq ab + bc + ca$

Adding $2(ab + bc + ca)$ to both sides, we get

$(a+b+c)^2 \geq 3(ab + bc + ca)$

or

$3(ab + bc + ca) \leq (a+b+c)^2$ (4)

Also,

$c < a + b$ (triangle inequality)

$\Rightarrow$ $c^2 < ac + bc$ (5)

Similarly,

$b^2 < ab + bc$ (6)

$a^2 < ab + ca$ (7)

Adding (4), (5) and (6), we get

$a^2 + b^2 + c^2 < 2(ab + bc + ca)$

Adding $2(ab + bc + ca)$ to both sides, we get

$$(a + b + c)^2 < 4(ab + bc + ca) \quad (8)$$

Combining (4) and (8), we get

$$3(ab + bc + ca) \le (a + b + c)^2 < 4(ab + bc + ca)$$

First two expressions will be equal for $a = b = c$.

8. $y = \sqrt{((x+1)(x-3))/(x-2)}$ will be real if

$$((x+1)(x-3))/(x-2) \ge 0$$

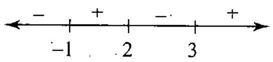


Fig. 1.90

From the sign scheme of $((x+1)(x-3))/(x-2)$, we have

$$x \in [-1, 2) \cup [3, \infty)$$

9. The given equations are

$$3x + my - m = 0$$

and

$$2x - 5y - 20 = 0$$

Solving these equations, we get

$$x = \frac{25m}{2m+15},\ y = \frac{2m-60}{2m+15}$$

For $x > 0$,

$$\frac{25m}{2m+15} > 0$$

$$\Rightarrow \quad m < -\frac{15}{2} \text{ or } m > 0 \quad (1)$$

For $y > 0$,

$$\frac{2(m-30)}{2m+15} > 0$$

$$\Rightarrow \quad m < -\frac{15}{2} \text{ or } m > 30 \quad (2)$$

Combining (1) and (2), we get the common values of m,

$$m < -\frac{15}{2} \text{ or } m > 30$$

$$\therefore \quad m \in \left(-\infty, \frac{-15}{2}\right) \cup (30, \infty)$$

10. The given system is

$$x + 2y + z = 1 \quad (1)$$

$$2x - 3y - \omega = 2 \quad (2)$$

where $x, y, z, \omega \ge 0$.

Multiplying Eq. (1) by 2 and subtracting from Eq. (2), we get

$$7y + 2z + \omega = 0$$

$$\Rightarrow \quad \omega = -(7y + 2z)$$

Now, $x, y, z, \omega \ge 0$

$$\Rightarrow \quad y = z = \omega = 0$$

$$\Rightarrow \quad x = 1$$

11. $e^{\sin x} - e^{-\sin x} - 4 = 0$

Let $e^{\sin x} = y$. Then the equation becomes

$$y - \frac{1}{y} - 4 = 0$$

$$\Rightarrow \quad y^2 - 4y - 1 = 0$$

$$\Rightarrow \quad y = 2 + \sqrt{5}, 2 - \sqrt{5}$$

But y is real +ve number. Hence,

$$\therefore \quad y \ne 2 - \sqrt{5}$$

$$\Rightarrow \quad y = 2 + \sqrt{5}$$

$$\Rightarrow \quad e^{\sin x} = 2 + \sqrt{5}$$

$$\Rightarrow \sin x = \log_e (2 + \sqrt{5})$$

But

$$2 + \sqrt{5} > e$$

$$\Rightarrow \quad \log_e (2 + \sqrt{5}) > \log_e e$$

$$\Rightarrow \quad \log_e (2 + \sqrt{5}) > 1$$

$$\Rightarrow \quad \sin x > 1,$$

which is not possible. Therefore the given equation has no real solution.

12. For any square there can be at most four neighbouring squares.

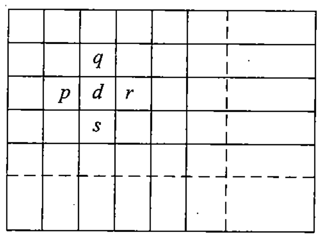


Fig. 1.91

Let for a square having largest number d, p, q, r, s be written. Then according to the question,

$$p + q + r + s = 4d$$

$$\Rightarrow \quad (d - p) + (d - q) + (d - r) + (d - s) = 0$$

Sum of four positive numbers can be zero only if these are zeros individually. Therefore,

$$d - p = d - q = d - r = d - s = 0$$

$$\Rightarrow \quad p = q = r = s = d$$

Hence, all the numbers written are same.

13. Let α, β be the roots of equation $ax^2 + bx + c = 0$. Given that $\beta = \alpha^n$. Also, $\alpha + \beta = -b/a$, $\alpha\beta = c/a$. Now,

$$\alpha\beta = \frac{c}{a} \Rightarrow \alpha\,\alpha^n = \frac{c}{a} \Rightarrow \alpha = \left(\frac{c}{a}\right)^{\frac{1}{n+1}}$$

$$\alpha + \beta = -b/a \Rightarrow \alpha + \alpha^n = \frac{-b}{a}$$

or

$$\left(\frac{c}{a}\right)^{\frac{1}{n+1}} + \left(\frac{c}{a}\right)^{\frac{n}{n+1}} = \frac{-b}{a}$$

$$\Rightarrow \quad a\left(\frac{c}{a}\right)^{\frac{1}{n+1}} + a\left(\frac{c}{a}\right)^{\frac{n}{n+1}} + b = 0$$

$$\Rightarrow \quad a^{\frac{n}{n+1}} c^{\frac{1}{n+1}} + a^{\frac{1}{n+1}} c^{\frac{n}{n+1}} + b = 0$$

$$\Rightarrow (a^n c)^{\frac{1}{n+1}} + (a c^n)^{\frac{1}{n+1}} + b = 0$$

14. $x^2 - 3x + 2 > 0$, $x^2 - 3x - 4 \le 0$

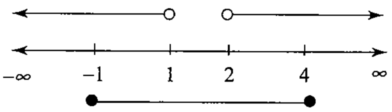


Fig. 1.92

$\Rightarrow$ $(x-1)(x-2) > 0$ and $(x-4)(x+1) \le 0$

$\Rightarrow$ $x \in (-\infty, 1) \cup (2, \infty)$ and $x \in [-1, 4]$.

Therefore, the common solution is $[-1, 1) \cup (2, 4]$

15. $(5+2\sqrt{6})(5-2\sqrt{6}) = 25 - 24 = 1$

$$\Rightarrow 5 - 2\sqrt{6} = \frac{1}{5+2\sqrt{6}}$$

Hence, the given equation is

$$(5+2\sqrt{6})^{x^2-3} + \frac{1}{(5+2\sqrt{6})^{x^2-3}} = 10$$

$\Rightarrow$ $y + \dfrac{1}{y} = 10$, where $y = (5+2\sqrt{6})^{x^2-3}$

$\Rightarrow$ $y^2 - 10y + 1 = 0$

$$\Rightarrow y = \frac{10 \pm \sqrt{100-4}}{2}$$

$\Rightarrow$ $y = 5 \pm 2\sqrt{6}$

$\Rightarrow$ $(5+2\sqrt{6})^{x^2-3} = 5 \pm 2\sqrt{6}$

$\Rightarrow$ $(5+2\sqrt{6})^{x^2-3} = (5+2\sqrt{6})$

or

$$(5+2\sqrt{6})^{x^2-3} = \frac{1}{5+2\sqrt{6}}$$

$\Rightarrow$ $(5+2\sqrt{6})^{x^2-3} = (5+2\sqrt{6})^1$ or $(5+2\sqrt{6})^{x^2-3} = (5+2\sqrt{6})^{-1}$

$\Rightarrow$ $x^2 - 3 = 1$ or $x^2 - 3 = -1$

$\Rightarrow$ $x^2 = 4$ or $x^2 = 2$

$\Rightarrow$ $x = \pm 2$ or $\pm\sqrt{2}$

16. The given equation is

$x^2 - 2a|x - a| - 3a^2 = 0$

Case I: If $x - a \ge 0$, then $|x - a| = x - a$. Hence, the equation becomes

$x^2 - 2a(x - a) - 3a^2 = 0$

$\Rightarrow$ $x^2 - 2ax - a^2 = 0$

$$\Rightarrow x = \frac{2a \pm \sqrt{4a^2 + 4a^2}}{2} \Rightarrow a \pm a\sqrt{2}$$

Case II: If $x - a < 0$, then $|x - a| = -(x - a)$. Hence, the equation becomes

$x^2 + 2a(x - a) - 3a^2 = 0$

$\Rightarrow$ $x^2 + 2ax - 5a^2 = 0$

$$\Rightarrow x = \frac{-2a \pm \sqrt{4a^2 + 20a^2}}{2}$$

$$\Rightarrow x = \frac{-2a \pm 2a\sqrt{6}}{2}$$

$x = -a \pm a\sqrt{6}$

Thus, the solution set is $\{a \pm a\sqrt{2}, -a \pm a\sqrt{6}\}$.

17. We are given

$$\frac{2x}{2x^2+5x+2} > \frac{1}{x+1}$$

$$\Rightarrow \frac{2x}{2x^2+5x+2} - \frac{1}{x+1} > 0$$

$$\Rightarrow \frac{2x^2 + 2x - 2x^2 - 5x - 2}{(2x^2+5x+2)(x+1)} > 0$$

$$\Rightarrow \frac{-3x-2}{(2x+1)(x+1)(x+2)} > 0$$

$$\Rightarrow \frac{(3x+2)}{(x+1)(x+2)(2x+1)} < 0$$

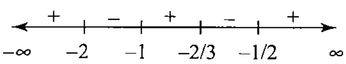


Fig. 1.93

From the sign scheme, solution is $x \in (-2, -1) \cup (-2/3, -1/2)$.

18. The given equation is

$|x^2 + 4x + 3| + 2x + 5 = 0$

Case I:

$x^2 + 4x + 3 \ge 0$

$\Rightarrow$ $(x+1)(x+3) \ge 0$

$\Rightarrow$ $x \in (-\infty, -3] \cup [-1, \infty)$ (1)

Then, given equation becomes

$x^2 + 6x + 8 = 0$

$\Rightarrow$ $(x+4)(x+2) = 0$

$\Rightarrow$ $x = -4, -2$

But $x = -2$ does not satisfy (1); hence rejected. Therefore, $x = -4$ is the only solution.

Case II:

$x^2 + 4x + 3 < 0$

$\Rightarrow$ $(x+1)(x+3) < 0$

$\Rightarrow$ $x \in (-3, -1)$ (2)

Then, given equation becomes

$-(x^2 + 4x + 3) + 2x + 5 = 0$

$\Rightarrow$ $-x^2 - 2x + 2 = 0$

$\Rightarrow$ $x^2 + 2x - 2 = 0$

$$\Rightarrow x = \frac{-2 \pm \sqrt{4+8}}{2}$$

$\Rightarrow \quad x = -1 + \sqrt{3}, -1 - \sqrt{3}$

out of which $x = -1-\sqrt{3}$ satisfies (2). Thus, $x = -4, -1-\sqrt{3}$.

19. Let $f(x) = x^2 + (b/a)x + (c/a)$. According to the question, we have the following graph.

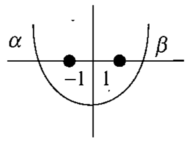


Fig. 1.94

From graph, $f(-1) < 0$ and $f(1) < 0$. So,

$$1 + \frac{c}{a} - \frac{b}{a} < 0 \text{ and } 1 + \frac{c}{a} + \frac{b}{a} < 0 \Rightarrow 1 + \frac{c}{a} + \left|\frac{b}{a}\right| < 0$$

20. Refer to problems 25–26 of linked comprehension type.

21.

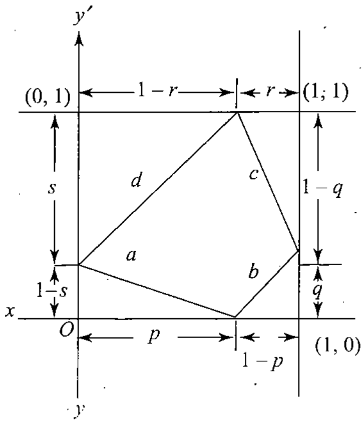


Fig. 1.95

$$a^2 = p^2 + (1-s)^2$$
$$b^2 = (1-p)^2 + q^2$$
$$c^2 = (1-q)^2 + r^2$$
$$d^2 = (1-r)^2 + s^2$$

$\therefore \quad a^2 + b^2 + c^2 + d^2 = [p^2 + (1-p)^2] + [q^2 + (1-q)^2] + [r^2 + (1-r)^2] + [s^2 + (1-s)^2]$, where $p, q, r, s \in [0, 1]$

Now consider the function

$y = x^2 + (1-x)^2, 0 \le x \le 1$

$\Rightarrow \quad y = 2x^2 - 2x + 1$

which has vertex $(1/2, (1/2))$.

Hence, minimum value is $1/2$ when $x = 1/2$ and maximum value is at $x = 1$, which is 1. Therefore, minimum value of $a^2 + b^2 + c^2 + d^2$ is $1/2 + 1/2 + 1/2 + 1/2 = 2$ and maximum value is $1 + 1 + 1 + 1 = 4$.

22. Let us consider the integral values of x as 0, 1, −1. Then $f(0)$, $f(1)$ and $f(-1)$ are all integers. Therefore, C, $A + B + C$ and $A - B + C$ are all integers.

Therefore, C is integer and hence, $A + B$ is an integer and also $A - B$ is an integer.

$$2A = (A + B) + (A - B)$$

Therefore, $2A$, $A + B$ and C are all integers. Conversely, let $n \in I$. Then,

$$f(n) = An^2 + Bn + C = 2A\left[\frac{n(n-1)}{2}\right] + (A+B)n + C$$

Now, A, $A + B$ and C are all integers and

$$\frac{n(n-1)}{2} = \frac{\text{Even number}}{2} = \text{integer}$$

Therefore, $f(n)$ is also an integer.

23. We know that

$$(\alpha - \beta)^2 = [(\alpha + \delta) - (\beta + \delta)]^2$$

$\Rightarrow \quad (\alpha + \beta)^2 - 4\alpha\beta = (\alpha + \delta + \beta + \delta)^2 - 4(\alpha + \delta)(\beta + \delta) \quad (1)$

Now, here

$$\alpha + \beta = -\frac{b}{a}, \alpha\beta = \frac{c}{a}$$

and

$$(\alpha + \delta) + (\beta + \delta) = -\frac{B}{A}, (\alpha + \delta)(\beta + \delta) = \frac{C}{A}$$

$\Rightarrow \quad \dfrac{b^2 - 4ac}{a^2} = \dfrac{B^2 - 4AC}{A^2} \quad$ [From (1)]

24. $\alpha + \beta = \dfrac{-b}{a}, \alpha\beta = \dfrac{c}{a}$

Roots of the equation $a^3x^2 + abcx + c^3 = 0$ are

$$x = \frac{-abc \pm \sqrt{(abc)^2 - 4a^3c^2}}{2a^3}$$

$$= \left(-\frac{b}{a}\right)\left(\frac{c}{a}\right) \pm \frac{\sqrt{\left(\frac{b}{a}\right)^2\left(\frac{c}{a}\right)^2 - 4\left(\frac{c}{a}\right)^3}}{2}$$

$$= \frac{(\alpha + \beta)(\alpha\beta) \pm \sqrt{(\alpha + \beta)^2(\alpha\beta)^2 - 4(\alpha\beta)^3}}{2}$$

$$= (\alpha\beta)\frac{[(\alpha + \beta) \pm \sqrt{(\alpha - \beta)^2}]}{2}$$

$$= \alpha\beta\frac{[(\alpha + \beta) \pm (\alpha - \beta)]}{2}$$

$$= \alpha^2\beta, \alpha\beta^2$$

25. The given equation is

$x^2 + (a - b)x + (1 - a - b) = 0, a, b \in R$

For this equation to have unequal real roots $\forall\, b$,

$D > 0$

$\Rightarrow \quad (a - b)^2 - 4(1 - a - b) > 0$

$\Rightarrow \quad a^2 + b^2 - 2ab - 4 + 4a + 4b > 0$

$\Rightarrow \quad b^2 + b(4 - 2a) + a^2 + 4a - 4 > 0 \quad (1)$

which is a quadratic expression in b, and it will be true $\forall\, b \in R$. Then its discriminant will be less than 0. Hence,

$(4 - 2a)^2 - 4(a^2 + 4a - 4) < 0$

$\Rightarrow \quad (2 - a)^2 - (a^2 + 4a - 4) < 0$

$\Rightarrow \quad 4 - 4a + a^2 - a^2 - 4a + a < 0$

$\Rightarrow \quad -8a + 8 < 0$

$\Rightarrow \quad a > 1$

26. Roots of $x^2 - 10cx - 11d = 0$ are a and b. Hence,

$a + b = 10c$ and $ab = -11d$

c and d are the roots of $x^2 - 10ax - 11b = 0$. Hence,

$$c + d = 10a \text{ and } cd = -11b$$

$\Rightarrow$ $a + b + c + d = 10(a + c)$ and $abcd = 121\, bd$

$\Rightarrow$ $b + d = 9(a + c)$ and $ac = 121$

Also, we have

$$a^2 - 10ac - 11d = 0 \text{ and } c^2 - 10ac - 11b = 0$$

$\Rightarrow$ $a^2 + c^2 - 20ac - 11(b + d) = 0$

$\Rightarrow$ $(a + c)^2 - 22 \times 121 - 99(a + c) = 0$

$\Rightarrow$ $a + c = 121$ or -22

For $a + c = -22$, we get $a = c$. Rejecting these values, we have $a + c = 121$. Therefore,

$$a + b + c + d = 10(a + c) = 1210$$

Objective Type

Fill in the blanks

1. Given polynomial is

$$(x - 1)(x - 2)(x - 3) \cdots (x - 100)$$

$$= x^{100} - (1 + 2 + 3 + \cdots + 100)\, x^{99} + (\cdots)x^{98} \cdots$$

Hence, coefficient of x^{99} is

$$-(1 + 2 + 3 + \cdots + 100) = \frac{-100 \times 101}{2}$$

$$= -5050$$

2. As p and q are real and one root is $2 + i\sqrt{3}$, so the other root must be $2 - i\sqrt{3}$. Then,

$$p = -(\text{sum of roots}) = -4$$

$$q = \text{product of roots} = (2 + i\sqrt{3})(2 - i\sqrt{3}) = 4 + 3 = 7$$

3. Given equation is

$$x^2 - 3kx + 2e^{2\ln k} - 1 = 0$$

$\Rightarrow$ $x^2 - 3kx + (2k^2 - 1) = 0$

Here, product of roots is $2k^2 - 1$.

$\therefore$ $2k^2 - 1 = 7 \Rightarrow k^2 = 4 \Rightarrow k = 2, -2$

Now for real roots, we must have

$$D \ge 0$$

$\Rightarrow$ $9k^2 - 4(2k^2 - 1) \ge 0$

$\Rightarrow$ $k^2 + 4 \ge 0$

which is true for all k. Thus, $k = 2, -2$. But for $k = -2$, $\ln k$ is not defined. Therefore, rejecting $k = -2$, we get $k = 2$.

4. By observation, one root is $x = 1$,

$\Rightarrow$ $a + b = -1$

5. Given $x < 0$, $y < 0$.

$$x + y + \frac{x}{y} = \frac{1}{2} \text{ and } (x + y)\frac{x}{y} = -\frac{1}{2}$$

Let,

$$x + y = a \text{ and } \frac{x}{y} = b \qquad (1)$$

Therefore, we get

$$a + b = \frac{1}{2},\ ab = -\frac{1}{2}$$

Solving these two, we get

$$a + \left(-\frac{1}{2a}\right) = \frac{1}{2}$$

$\Rightarrow$ $2a^2 - a - 1 = 0$

$\Rightarrow$ $a = 1, -1/2$

$\Rightarrow$ $b = -1/2, 1$

$\therefore$ (1) $\Rightarrow x + y = 1$ and $\frac{x}{y} = -\frac{1}{2}$

or

$$x + y = -\frac{1}{2} \text{ and } \frac{x}{y} = 1$$

But $x, y < 0$

$\therefore$ $x + y < 0 \Rightarrow x + y = \frac{-1}{2}$ and $\frac{x}{y} = 1$

On solving, we get $x = -1/4$ and $y = -1/4$.

6. $|x - 2|^2 + |x - 2| - 2 = 0$

$\Rightarrow$ $(|x - 2| + 2)(|x - 2| - 1) = 0$

$\Rightarrow$ $|x - 2| - 1 = 0$

$\Rightarrow$ $x - 2 = \pm 1$

$\Rightarrow$ $x = 1, 3$

Therefore, the sum of the roots is $3 + 1 = 4$.

7. $\log_7 \log_5 (\sqrt{x+5} + \sqrt{x}) = 0$

$\Rightarrow$ $\log_5 (\sqrt{x+5} + \sqrt{x}) = 1$

$\Rightarrow$ $(\sqrt{x+5} + \sqrt{x}) = 5$

$\Rightarrow$ $x + 5 = 25 + x - 10\sqrt{x}$

$\Rightarrow$ $2 = \sqrt{x}$

$\Rightarrow$ $x = 4$

which satisfies the given equation

True or false

1. False.

$$2x^2 + 3x + 1 = 0$$

$\Rightarrow$ $(2x + 1)(x + 1) = 0$

$\Rightarrow$ $x = -1, -1/2$, both are rational

2. True. Given equation is

$$(x - a)(x - c) + 2(x - b)(x - d) = 0$$

Let,

$$f(x) = (x - a)(x - c) + 2(x - b)(x - d)$$

$$f(b) = (b - a)(b - c) < 0$$

$$f(d) = (d - a)(d - c) > 0$$

Thus,

$$f(b) f(d) < 0$$

Therefore, one root lies between b and d; hence the roots are real.

3. False. Consider $N = n_1 + n_2 + n_3 + \cdots + n_p$, where N is an even number. Let k numbers among these p numbers be odd, then $p - k$ are even numbers.

Now, sum of $p - k$ even numbers is even and for N to be an even number, sum of k odd numbers must be even, which is possible only when k is even.

4. True. We have $P(x) = ax^2 + bx + c$, for which

$$D_1 = b^2 - 4ac \qquad (1)$$

and $Q(x) = -ax^2 + dx + c$, for which

$$D_2 = d^2 + 4ac \qquad (2)$$

Given that $ac \neq 0$. Following two cases are possible.

If $ac > 0$, then from Eq. (2), D_2 is +ve $\Rightarrow Q(x)$ has real roots.

If $ac < 0$, then from Eq. (1), D_1 is +ve $\Rightarrow P(x)$ has real roots.
Thus, $P(x)\,Q(x) = 0$ has at least two real roots.

Multiple choice questions with one correct answer

1. c. l, m, n are real and $l \neq m$. Given equation is

$$(l - m)x^2 - 5(l + m)x - 2(l - m) = 0$$
$$D = 25(l + m)^2 + 8(l - m)^2 > 0,\ l, m \in R$$

Therefore, the roots are real and unequal.

2. a. $u = x^2 + 4y^2 + 9z^2 - 6yz - 3zx - 2xy$

$$= \frac{1}{2}\left[2x^2 + 8y^2 + 18z^2 - 12yz - 6zx - 4xy\right]$$
$$= \frac{1}{2}\left[(x^2 - 4xy + 4y^2) + (4y^2 + 9z^2 - 12yz) + (x^2 + 9z^2 - 6zx)\right]$$
$$= \frac{1}{2}\left[(x - 2y)^2 + (2y - 3z)^2 + (3z - x)^2\right] \geq 0$$

Hence, u is always non-negative.

3. c. As $a, b, c > 0$, so a, b, c should be real (note that other relation is not defined in the set of complex numbers). Therefore, the roots of equation are either real or complex conjugate.

Let α, β be the roots of $ax^2 + bx + c = 0$. Then,

$$\alpha + \beta = -\frac{b}{a} = -\text{ve and } \alpha\beta = \frac{c}{a} = +\text{ve}$$

Hence, either both α, β are –ve (if roots are real) or both α, β have –ve real part (if roots are complex conjugate).

4. b. The given equation is

$$(x - b)(x - c) + (x - a)(x - c) + (x - a)(x - b) = 0$$
$$\Rightarrow 3x^2 - 2(a + b + c)x + (ab + bc + ca) = 0$$
$$D = 4(a + b + c)^2 - 12(ab + bc + ca)$$
$$= 4[a^2 + b^2 + c^2 - ab - bc - ca]$$
$$= 2[(a - b)^2 + (b - c)^2 + (c - a)^2] \geq 0,\ \forall\ a, b, c$$

Therefore, the roots of the given equation are always real.

5. c.

$$\begin{array}{r|l} & ax - ap \\ x^2 + px + 1 & ax^3 + bx + c \\ & ax^3 + apx^2 + ax \\ \hline & -apx^2 + (b - a)x + c \\ & -apx^2 - ap^2x - ap \\ \hline & (b - a + ap^2)x + c + ap \end{array}$$

Now, remainder must be zero. Hence,

$$b - a + ap^2 = 0 \text{ and } c + ap = 0$$
$$\Rightarrow p = -\frac{c}{a} \text{ and } p^2 = \frac{a - b}{a}$$
$$\Rightarrow \left(\frac{-c}{a}\right)^2 = \frac{a - b}{a}$$
$$\Rightarrow c^2 = a^2 - ab$$
$$\Rightarrow a^2 - c^2 = ab$$

6. a. $|x|^2 - 3|x| + 2 = 0$

$$\Rightarrow (|x| - 2)(|x| - 1) = 0$$
$$\Rightarrow |x| = 1 \text{ or } 2$$
$$\Rightarrow x = \pm 1, \pm 2$$

Hence, there are four real solutions.

7. c. Let the distance of the school from A be x. Therefore, the distance of the school from B is $60 - x$. The total distance covered by 200 students is

$$[150x + 50(60 - x)] = [100x + 3000]$$

This is minimum when $x = 0$. Hence, the school should be at town A.

8. d. Given expression is

$$x^{12} - x^9 + x^4 - x + 1 = f(x)$$

For $x < 0$, put $x = -y$, where $y > 0$. Thus, we get

$$f(x) = y^{12} + y^9 + y^4 + y + 1 > 0 \text{ for } y > 0$$

For $0 < x < 1$,

$$x^9 < x^4 \Rightarrow -x^9 + x^4 > 0$$

Also,

$$1 - x > 0 \text{ and } x^{12} > 0$$
$$\Rightarrow x^{12} - x^9 + x^4 + 1 - x > 0 \Rightarrow f(x) > 0$$

For $x > 1$,

$$f(x) = x(x^3 - 1)(x^8 + 1) + 1 > 0$$

So $f(x) > 0$ for $-\infty < x < \infty$.

9. a. Given equation is

$$x - \frac{2}{x - 1} = 1 - \frac{2}{x - 1}$$

Clearly, $x \neq 1$ for the given equation to be defined if $x - 1 \neq 0$. We can cancel the common term $-2/(x - 1)$ on both sides to get $x = 1$, but it is not possible. So, given equation has no roots.

10. c. Given that

$$a^2 + b^2 + c^2 = 1 \quad (1)$$

We know that

$$(a + b + c)^2 \geq 0$$
$$\Rightarrow a^2 + b^2 + c^2 + 2ab + 2bc + 2ca \geq 0$$
$$\Rightarrow 2(ab + bc + ca) \geq -1 \quad \text{[Using (1)]}$$
$$\Rightarrow ab + bc + ca \geq -1/2 \quad (2)$$

Also, we know that

$$\frac{1}{2}\left[(a - b)^2 + (b - c)^2 + (c - a)^2\right] \geq 0$$
$$\Rightarrow a^2 + b^2 + c^2 - ab\ \ bc - ca \geq 0$$
$$\Rightarrow ab + bc + ca \leq 1 \quad \text{[Using (1)]} \quad (3)$$

Combining (2) and (3), we get

$$-1/2 \leq ab + bc + ca \leq 1$$
$$\Rightarrow ab + bc + ca \in [-1/2, 1]$$

11. a. α, β are roots of $x^2 + px + q = 0$. Hence,

$$\alpha + \beta = -p \text{ and } \alpha\beta = q$$

α^4, β^4 are roots of $x^2 - rx + s = 0$. Hence,

$$\alpha^4 + \beta^4 = r,\ \alpha^4\beta^4 = q$$

Now for equation $x^2 - 4qx + 2q^2 - r = 0$, the product of roots is

$$2q^2 - r = 2(\alpha\beta)^2 - (\alpha^4 + \beta^4)$$
$$= -(\alpha^2 - \beta^2)^2$$
$$< 0$$

Therefore, the product of roots is negative. So, the roots must be real and of opposite signs.

12. d. We know that if $f(\alpha)$ and $f(\beta)$ are of opposite signs then there must be a value γ between α and β such that $f(\gamma) = 0$. Hence, a, b, c are real numbers and $a \neq 0$. As α is a root of $a^2x^2 + bx + c = 0$, so

$a^2\alpha^2 + b\alpha + c = 0$ (1)

Also, β is a root of $a^2x^2 - bx - c = 0$, so

$a^2\beta^2 - b\beta - c = 0$ (2)

Now, let $f(x) = a^2x^2 + 2bx + 2c$. Then,

$$\begin{aligned} f(\alpha) &= a^2\alpha^2 + 2b\alpha + 2c \\ &= a^2\alpha^2 + 2(b\alpha + c) \\ &= a^2\alpha^2 + 2(-a^2\alpha^2) \quad \text{[Using (1)]} \\ &= -a^2\alpha^2 < 0 \end{aligned}$$

and

$$\begin{aligned} f(\beta) &= a^2\beta^2 + 2b\beta + 2c \\ &= a^2\beta^2 + 2(b\beta + c) \\ &= a^2\beta^2 + 2(a^2\beta^2) \quad \text{[Using (2)]} \\ &= 3a^2\beta^2 > 0 \end{aligned}$$

Since $f(\alpha)$ and $f(\beta)$ are of opposite signs and γ is a root of equation $f(x) = 0$, therefore, γ must lie between α and β. Thus, $\alpha < \gamma < \beta$.

13. a. The given equation is $\sin(e^x) = 5^x + 5^{-x}$. We know that 5^x and 5^{-x} both are +ve real numbers.

Now, $5^x + 5^{-x} = \left(\sqrt{5^x} - \sqrt{5^{-x}}\right)^2 + 2 \ge 2$

But L.H.S. $= \sin(e^x) \le 1$

Hence, no solution.

14. c. α, β are roots of the equation $(x - a)(x - b) = c, c \ne 0$.

$\therefore \quad (x - a)(x - b) - c = (x - \alpha)(x - \beta)$

$\Rightarrow \quad (x - \alpha)(x - \beta) + c = (x - a)(x - b)$

Hence, the roots of $(x - \alpha)(x - \beta) + c = 0$ are a and b.

15. a. Minimum value of $5x^2 + 2x + 3$ is

$$-\frac{D}{4a} = -\frac{(2)^2 - 4(5)(3)}{4(5)} > 2$$

where maximum value of $2\sin x$ is 2. Therefore, the two curves do not meet at all.

16. b. For real roots,

$q^2 - 4pr \ge 0$

$\Rightarrow \left(\dfrac{p+r}{2}\right)^2 - 4pr \ge 0$ $\quad (\because p, q, r$ are in A.P.)

$\Rightarrow p^2 + r^2 - 14pr \ge 0$

$\Rightarrow \dfrac{p^2}{r^2} - 14\dfrac{p}{r} + 1 \ge 0$

$\Rightarrow \left(\dfrac{p}{r} - 7\right)^2 - 48 \ge 0$

$\Rightarrow \left|\dfrac{p}{r} - 7\right| \ge 4\sqrt{3}$

17. a. The given equation is

$$\sqrt{x+1} - \sqrt{x-1} = \sqrt{4x-1}$$

Squaring both sides, we get

$$x + 1 + x - 1 - 2\sqrt{x^2 - 1} = 4x - 1$$

$\Rightarrow -2\sqrt{x^2 - 1} = 2x - 1$

Again squaring both sides, we get

$\Rightarrow 4(x^2 - 1) = 4x^2 - 4x + 1$

$\Rightarrow -4x = -5$

$\Rightarrow x = 5/4$

Substituting this value of x in given equation, we get

$$\sqrt{\frac{5}{4} + 1} - \sqrt{\frac{5}{4} - 1} = \sqrt{4 \times \frac{5}{4} - 1}$$

$\Rightarrow \dfrac{3}{2} - \dfrac{1}{2} = 2$ (not satisfied)

Therefore, 5/4 is not a solution of given equation. Hence, the given equation has no solution.

18. a. If both the roots of a quadratic equation $ax^2 + bx + c = 0$ are less than k, then $af(k) > 0$, $-b/2a < k$ and $D \ge 0$. Now,

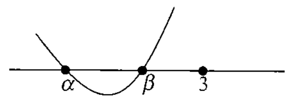


Fig. 1.96

$f(x) = x^2 - 2ax + a^2 + a - 3$

$\Rightarrow f(3) > 0, a < 3, -4a + 12 \ge 0$

$\Rightarrow a^2 - 5a + 6 > 0, a < 3, -4a + 12 \ge 0$

$\Rightarrow a < 2$ or $a > 3, a < 3, a \le 3$

$\Rightarrow a < 2$

19. b. Here $D = b^2 - 4c > 0$ because $c < 0 < b$. So, roots are real and unequal. Now,

$\alpha + \beta = -b < 0$ and $\alpha\beta = c < 0$

Therefore, one root is positive and the other root is negative, the negative root being numerically bigger. As $\alpha < \beta$, so α is the negative root while β is the positive root. So, $|\alpha| > \beta$ and $\alpha < 0 < \beta < |\alpha|$.

20. d. Given equation is

$(x - a)(x - b) - 1 = 0$

Let $f(x) = (x - a)(x - b) - 1$. Then,

$f(a) = -1$ and $f(b) = -1$

Also, graph of $f(x)$ is concave upward; hence, a and b lie between the roots. Also, if $b > a$, then one root lies in $(-\infty, a)$ and the other root lies in $(b, +\infty)$.

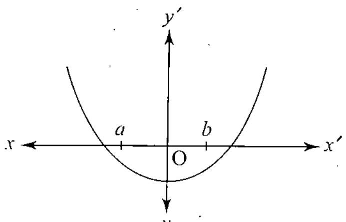


Fig. 1.97

21. c. Let α, α^2 be the roots of $3x^2 + px + 3 = 0$. Now,

$S = \alpha + \alpha^2 = -p/3, p = \alpha^3 = 1$

$\Rightarrow \alpha = 1, \omega, \omega^2 \quad \left(\text{where } \omega = \dfrac{-1 + \sqrt{3}i}{2}\right)$

$\alpha + \alpha^2 = -p/3 \Rightarrow \omega + \omega^2 = -p/3$

$\Rightarrow -1 = -p/3 \Rightarrow p = 3$

22. d. Minimum value of $f(x) = (1 + b^2)x^2 + 2bx + 1$ is

$$m(b) = -\frac{(2b)^2 - 4(1 + b^2)}{4(1 + b^2)} = \frac{1}{1 + b^2}$$

Clearly, $m(b)$ has range (0, 1].

23. a. Clearly, $\alpha + \beta = 1, \alpha\beta = p, \gamma + \delta = 4, \gamma\delta = q$ $(p, q \in I)$.

Since $\alpha, \beta, \gamma, \delta$ are in G.P. (with common ratio r), so

$$\alpha + \alpha r = 1, \alpha(r^2 + r^3) = 4$$

$\Rightarrow \quad \alpha(1 + r) = 1, \alpha r^2 (1 + r) = 4$

$\Rightarrow \quad r^2 \times 1 = 4 \Rightarrow r^2 = 4 \Rightarrow r = 2, -2$

If $\quad r = 2,$

$$\alpha + 2\alpha = 1 \Rightarrow \alpha = \frac{1}{3}$$

If $\quad r = -2,$

$$\alpha - 2\alpha = 1 \Rightarrow \alpha = -1$$

But $p = \alpha\beta \in I$

$\therefore \quad r = -2$ and $\alpha = -1$

$\Rightarrow \quad p = -2,$

$$q = \alpha^2 r^5 = 1\,(-2)^5 = -32$$

24. b. $x^2 - |x + 2| + x > 0$

$\Rightarrow \quad x^2 + 2 > |x + 2|$

Let us draw the graphs of $y = x^2 + 2$ and $y = |x + 2|$.

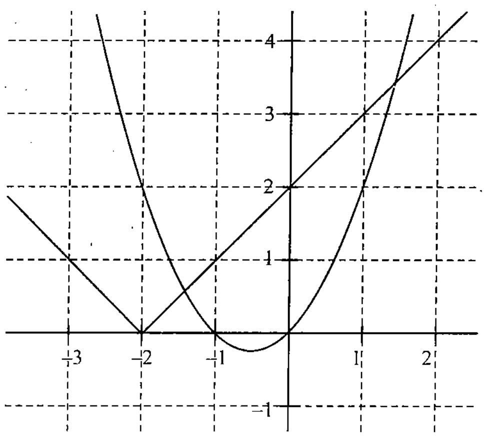


Fig. 1.98

Solving $y = |x + 2|$ and $y = x^2 + x$ for their points of intersection, we have

$$x + 2 = x^2 + x \Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$$

Hence, solution of $x^2 + 2 > |x + 2|$ is $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$.

25. d. $f(x) = x^2 + 2bx + 2c^2$

$\quad = (x + b)^2 + 2c^2 - b^2$

$g(x) = -x^2 - 2cx + b^2$

$\quad = -(x + c)^2 + b^2 + c^2$

Given that

$\min f(x) > \max g(x)$

$\Rightarrow \quad 2c^2 - b^2 > b^2 + c^2$

$\Rightarrow \quad c^2 > 2b^2$

$\Rightarrow \quad |c| > |b|\sqrt{2}$

26. b. $\quad x^2 + 2ax + 10 - 3a > 0, \forall x \in R$

$\Rightarrow \quad D < 0$

$\Rightarrow \quad 4a^2 - 4(10 - 3a) < 0$

$\Rightarrow \quad a^2 + 3a - 10 < 0$

$\Rightarrow \quad (a + 5)(a - 2) < 0$

$\Rightarrow \quad a \in (-5, 2)$

27. a. α and α^2 are the roots of the equation $x^2 + px + q = 0$. Hence,

$$\alpha + \alpha^2 = -p \qquad (1)$$

and

$$\alpha\alpha^2 = q \Rightarrow \alpha^3 = q \qquad (2)$$

Cubing (1),

$$\alpha^3 + \alpha^6 + 3\alpha\,\alpha^2(\alpha + \alpha^2) = -p^3$$

$\Rightarrow \quad q + q^2 + 3q(-p) = -p^3$

$\Rightarrow \quad p^3 + q^2 - q\,(3p - 1) = 0$

28. d. $\alpha + \beta, \alpha^2 + \beta^2, \alpha^3 + \beta^3$ are in G.P. Hence,

$$(\alpha^2 + \beta^2)^2 = (\alpha + \beta)(\alpha^3 + \beta^3)$$

$\Rightarrow \quad \alpha\beta(\alpha - \beta)^2 = 0$

$\Rightarrow \quad c\Delta = 0$

29. a. a, b, c are sides of a triangle and $a \neq b \neq c$.

$\therefore \quad |a - b| < |c| \Rightarrow c^2 + b^2 - 2ab < c^2$

Similarly, we have

$$b^2 + c^2 - 2bc < a^2$$

and

$$c^2 + a^2 - 2ca < b^2$$

On adding, we get

$$a^2 + b^2 + c^2 < 2(ab + bc + ca)$$

$\Rightarrow \quad \dfrac{a^2 + b^2 + c^2}{ab + bc + ca} < 2 \qquad (1)$

Since the roots of the given equation are real, therefore

$$(a + b + c)^2 - 3\lambda(ab + bc + ca) \geq 0$$

$\Rightarrow \quad \dfrac{a^2 + b^2 + c^2}{ab + bc + ca} > 3\lambda - 2 \qquad (2)$

From (1) and (2), we get

$$3\lambda - 2 < 2 \Rightarrow \lambda < \frac{4}{3}$$

30. d. α, β are the roots of $x^2 - px + r = 0$. Hence,

$$\alpha + \beta = p \qquad (1)$$

and

$$\alpha\beta = r \qquad (2)$$

Also, $\alpha/2, 2\beta$ are the roots of $x^2 - qx + r = 0$. Hence,

$$\frac{\alpha}{2} + 2\beta = q \qquad (3)$$

or

$$\alpha + 4\beta = 2q$$

Solving (1) and (3) for α and β, we get

$$\beta = \frac{1}{3}(2q - p) \text{ and } \alpha = \frac{2}{3}(2p - q)$$

Substituting values of α and β, in Eq. (2), we get

$$\frac{2}{9}(2p - q)(2q - p) = r$$

31. b. $\alpha^3 + \beta^3 = q$

$\Rightarrow \quad (\alpha + \beta) - 3\alpha\beta(\alpha + \beta) = q$

$\Rightarrow \quad -p^3 + 3p\alpha\beta = q \Rightarrow \alpha\beta = \dfrac{q + p^3}{3p}$

Required equation is

$$x^2-\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)x+\frac{\alpha}{\beta}\cdot\frac{\beta}{\alpha}=0$$

$$x^2-\frac{(\alpha^2+\beta^2)}{\alpha\beta}x+1=0$$

$$\Rightarrow\quad x^2-\left(\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}\right)x+1=0$$

$$\Rightarrow\quad x^2-\frac{p^2-2\left(\dfrac{p^3+q}{3p}\right)}{\dfrac{p^3+q}{3p}}x+1=0$$

$$\Rightarrow\quad (p^3+q)x^2-(3p^3-2p^3-2q)x+(p^3+q)=0$$

$$\Rightarrow\quad (p^3+q)x^2-(p^3-2q)x+(p^3+q)=0.$$

32. b. $x^2+bx-1=0$

$x^2+x+b=0$ (1)

Common root is $(b-1)x-1-b=0$

$$\Rightarrow\quad x=\frac{b+1}{b-1}$$

This value of x satisfies equation (1)

$$\Rightarrow\quad \frac{(b+1)^2}{(b-1)^2}+\frac{b+1}{b-1}+b=0$$

$$\Rightarrow\quad b=\sqrt{3}\,i,-\sqrt{3}\,i,0$$

33. c. $a_n=\alpha^n-\beta^n$

Also $\alpha^2-6\alpha-2=0$

Multiply with α^8 on both sides

$\Rightarrow\ \alpha^{10}-6\alpha^9-2\alpha^8=0$ (1)

similarly $\beta^{10}-6\beta^9-2\beta^8=0$ (2)

Subtracting (2) from (1) we have

$\alpha^{10}-\beta^{10}-6(\alpha^9-\beta^9)=2(\alpha^8-\beta^8)$

$$\Rightarrow a_{10}-6a_9=2a_8\Rightarrow\ \frac{a_{10}-2a_8}{2a_9}=3.$$

Multiple choice questions with one or more than one correct answer

1. c, d. Let,

$$y=\frac{(x-a)(x-b)}{(x-c)}$$

$\Rightarrow\ (x-c)y=x^2-(a+b)x+ab$

$\Rightarrow\ x^2-(a+b+y)x+ab+cy=0$

Since x is real, so

$D\geq 0$

$\Rightarrow\ (a+b+y)^2-4(ab+cy)\geq 0,\ \forall\, x\in R$

$\Rightarrow\ y^2+2y(a+b-2c)+(a-b)^2\geq 0,\ \forall\, x\in R$

$\Rightarrow\ 4(a+b-2c)^2-4(a-b)^2<0$

$\Rightarrow\ (a+b-2c+a-b)(a+b-2c-a+b)<0$

$\Rightarrow\ 4(a-c)(b-c)<0$

$\Rightarrow\ a-c<0$ and $b-c>0$ or $a-c>0$ and $b-c<0$

$\Rightarrow\ a<c<b$ or $a>c>b$

2. a, d. We have,

$$f(x)=\frac{2x-1}{2x^3+3x^2+x}=\frac{2x-1}{x(2x+1)(x+1)}$$

Critical points are $x=1/2, 0, -1/2, -1$.

On number line by sign scheme method, we have

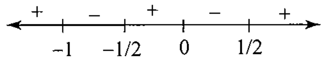


Fig. 1.99

For $f(x)>0$, $x\in(-\infty,-1)\cup(-1/2,0)\cup(1/2,\infty)$. Clearly, S contains $(-\infty,-3/2)$ and $(1/2,3)$.

3. a, b, c.

$$x^{\frac{3}{4}(\log_2 x)^2+\log_2 x-\frac{5}{4}}=\sqrt{2}$$

$$\Rightarrow\quad \left(\frac{3}{4}(\log_2 x)^2+\log_2 x-\frac{5}{4}\right)\log_2 x=\log_2\sqrt{2}$$

(taking logarithm both sides on base 2)

$$\Rightarrow\quad \left(\frac{3}{4}t^2+t-\frac{5}{4}\right)t=\frac{1}{2}\ \text{(putting }\log_2 x=t)$$

$\Rightarrow\ 3t^3+4t^2-5t-2=0$

$\Rightarrow\ 3t^3-3t^2+7t^2-7t+2t-2=0$

$\Rightarrow\ (3t^2+7t+2)(t-1)=0$

$\Rightarrow\ (3t+1)(t+2)(t-1)=0$

$$\Rightarrow\quad t=\log_2 x=1,-2,-\frac{1}{3}$$

$$\Rightarrow\quad x=2,2^{-2},2^{-\frac{1}{3}}$$

Assertion and Reasoning

1. b. Suppose the roots are imaginary. Then

$$\beta=\bar{\alpha}\ \text{ and }\ \frac{1}{\beta}=\bar{\alpha}\Rightarrow\beta=\frac{1}{\beta}$$

which is not possible. The roots are real, so

$(p^2-q)(b^2-ac)\geq 0$

Hence, statement 1 is correct.

Also, $-2b/a=\alpha+\beta$ and $\alpha/\beta=c/a$, $\alpha+\beta=-2p$, $\alpha\beta=q$. If $\beta=1$, then

$\alpha=q\Rightarrow c=qa$ (which is not possible)

Also,

$$s+1=\frac{-2b}{a}\Rightarrow -2p=\frac{-2b}{a}\Rightarrow b=ap\quad\text{(which is not possible)}$$

Hence, statement 2 is correct, but it is not correct explanation of statement 1.

Integer type

1. b. Let $f(x)=x^4-4x^3+12x^2+x-1=0$

$f'(x)=4x^3-12x^2+24x+1$

$f''(x)=12x^2-24x+24=12(x^2-2x+2)>0$

$\Rightarrow\ f''(x)=0$ has imaginary roots

$\Rightarrow\ f(x)=0$ has maximum 2 distinct real roots.