CHAPTER

1

Functions

- Number System and Inequalities
- Function
- Different Types of Functions
- Different Types of Mappings(Functions)
- Even and Odd Functions
- Periodic Functions
- Composite Function
- Inverse Functions
- Indentical Funtion
- Transformation of Graphs

NUMBER SYSTEM AND INEQUALITIES

Number System

Natural Numbers

The set of numbers $\{1, 2, 3, 4, \ldots\}$ is called natural numbers, and is denoted by N, i.e., $N = \{1, 2, 3, \ldots\}$.

Integers

The set of numbers $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ is called integers, and the set is denoted by I or Z.

Here, we represent

a. Positive integers $= \{1, 2, 3, 4, \ldots\}$ = Natural numbers
b. Negative integers $= \{\ldots, -4, -3, -2, -1\}$
c. Non-negative integers (or N_0) $= \{0, 1, 2, 3, 4, \ldots\}$ = Whole numbers
d. Non-positive integers $= \{\ldots, -3, -2, -1, 0\}$

Rational Numbers

A number which can be written as $\frac{a}{b}$, where a and b are integers, $b \neq 0$ and H.C.F. of a and b is 1, is called a rational number, and a set of rational numbers is denoted by Q.

Note:
- *Every integer is a rational number as it could be written as $Q = \frac{a}{b}$ (where $b = 1$).*
- *All recurring decimals are rational numbers; e.g., $n = 0.3333\ldots = 1/3$.*
- *"Two consecutive rational numbers" is meaningless.*
- *The set of rational numbers cannot be expressed in roaster form.*

Irrational Numbers

Those values which could be neither terminated nor expressed as recurring decimals are irrational numbers (i.e., such numbers cannot be expressed in $\frac{a}{b}$ form). Their set is denoted by Q^c (i.e., complement of Q), e.g., $\sqrt{2}, \pi, -\frac{1}{\sqrt{3}}, 2+\sqrt{2}, \ldots$

Note:
- *"Two consecutive irrational numbers" is meaningless.*
- *The set of irrational numbers cannot be expressed in roaster form.*

Real Numbers

The set of numbers that contains both rational and irrational numbers is called real numbers and is denoted by R. As from, the above definitions, it could be shown that real numbers can be expressed on number line with respect to origin as

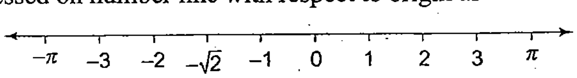


Fig. 1.1

Note:
- *The set R represents the set of continuous values (not discrete values).*
- *Between any two irrational numbers, there exist infinite rational numbers and between two rational numbers there exist infinite irrational numbers.*

Intervals

The set of numbers between any two real numbers is called interval. The following are the types of interval.

Closed Interval

$x \in [a, b] \equiv \{x : a \leq x \leq b\}$

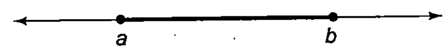


Fig. 1.2

Open Interval

$x \in (a, b)$ or $]a, b[\equiv \{x : a < x < b\}$

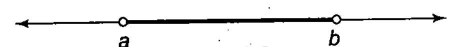
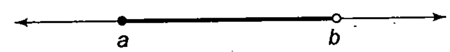


Fig. 1.3

Semi-Open or Semi-Closed Interval

$x \in [a, b[$ or $[a, b) = \{x: a \leq x < b\}$



$x \in]a, b]$ or $(a, b] = \{x: a < x \leq b\}$

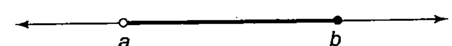


Fig. 1.4

Note:
- *A set of all real numbers can be expressed as $(-\infty, \infty)$*
- *$x \in (-\infty, a) \cup (b, \infty) \Rightarrow x \in R - [a, b]$*
- *$x \in (-\infty, a] \cup [b, \infty) \Rightarrow x \in R - (a, b)$*

Some Facts About Inequalities

The following are some very useful points to remember:

a. $a \leq b \Rightarrow$ either $a < b$ or $a = b$
b. $a < b$ and $b < c \Rightarrow a < c$
c. $a < b \Rightarrow -a > -b$, i.e., the inequality sign reverses if both sides are multiplied by a negative number
d. $a < b$ and $c < d \Rightarrow a + c < b + d$ and $a - d < b - c$
e. $a < b \Rightarrow ka < kb$ if $k > 0$ and $ka > kb$ if $k < 0$
f. $0 \leq a < b \Rightarrow a^r < b^r$ if $r > 0$ and $a^r > b^r$ if $r < 0$
g. $a + \frac{1}{a} \geq 2$ for $a > 0$ and equality holds for $a = 1$
h. $a + \frac{1}{a} \leq -2$ for $a < 0$ and equality holds for $a = -1$
i. If $x > 2 \Rightarrow 0 < \frac{1}{x} < \frac{1}{2}$
j. If $x < -3 \Rightarrow -\frac{1}{3} < \frac{1}{x} < 0$
k. If $x < 2$, then we must consider $-\infty < x < 0$ or $0 < x < 2$ (as for $x = 0$, $\frac{1}{x}$ is not defined), then

$$\lim_{x\to-\infty}\frac{1}{x}>\frac{1}{x}>\lim_{x\to 0-}\frac{1}{x} \text{ or } \lim_{x\to 0^+}\frac{1}{x}>\frac{1}{x}>\frac{1}{2}$$

$$\Rightarrow 0>\frac{1}{x}>-\infty \text{ or } \infty>\frac{1}{x}>\frac{1}{2}$$

$$\Rightarrow \frac{1}{x}\in(-\infty,0)\cup\left(\frac{1}{2},\infty\right)$$

l. Squaring an inequality:
If $a<b$, then $a^2<b^2$ does not follow always:
Consider the following illustrations:
$2<3\Rightarrow 4<9$, but $-4<3\Rightarrow 16>9$
Also if $x>2\Rightarrow x^2>4$, but for $x<2\Rightarrow x^2\ge 0$
If $2<x<4\Rightarrow 4<x^2<16$
If $-2<x<4\Rightarrow 0\le x^2<16$
If $-5<x<4\Rightarrow 0\le x^2<25$

Generalized Method of Intervals

Let $F(x)=(x-a_1)^{k_1}(x-a_2)^{k_2}\ldots(x-a_{n-1})^{k_{n-1}}(x-a_n)^{k_n}$.
Here $k_1, k_2, \ldots k_n \in Z$ and $a_1, a_2, \ldots, a_n$ are fixed real numbers satisfying the condition

$a_1<a_2<a_3<\ldots<a_{n-1}<a_n$

For solving $F(x)>0$ or $F(x)<0$, consider the following algorithm:

- We mark the numbers $a_1, a_2, \ldots, a_n$ on the number axis and put plus sign in the interval on the right of the largest of these numbers, i.e., on the right of a_n.
- Then we put plus sign in the interval on the left of a_n if k_n is an even number and minus sign if k_n is an odd number. In the next interval, we put a sign according to the following rule:
  - When passing through the point a_{n-1} the polynomial $F(x)$ changes sign if k_{n-1} is an odd number. Then we consider the next interval and put a sign in it using the same rule.
- Thus we consider all the intervals. The solution of the inequality $F(x)>0$ is the union of all intervals in which we put plus sign and the solution of the inequality $F(x)<0$ is the union of all intervals in which we put minus sign.

Frequently Used Inequalities

a. $(x-a)(x-b)<0\Rightarrow x\in(a,b)$, where $a<b$
b. $(x-a)(x-b)>0\Rightarrow x\in(-\infty,a)\cup(b,\infty)$, where $a<b$
c. $x^2\le a^2\Rightarrow x\in[-a,a]$
d. $x^2\ge a^2\Rightarrow x\in(-\infty,-a]\cup[a,\infty)$
e. If $ax^2+bx+c<0$, $(a>0)\Rightarrow x\in(\alpha,\beta)$, where α, β $(\alpha<\beta)$ are the roots of the equation $ax^2+bx+c=0$
f. If $ax^2+bx+c>0$, $(a>0)\Rightarrow x\in(-\infty,\alpha)\cup(\beta,\infty)$, where α, β $(\alpha<\beta)$ are the roots of the equation $ax^2+bx+c=0$

Example 1.1 Solve $(2x+1)(x-3)(x+7)<0$.

Sol. $(2x+1)(x-3)(x+7)<0$

Sign scheme of $(2x+1)(x-3)(x+7)$ is as follows:

$-$ | -7 | $+$ | $-1/2$ | $-$ | 3 | $+$

Fig. 1.5

Hence, solution is $(-\infty,-7)\cup(-1/2,3)$.

Example 1.2 Solve $\frac{2}{x}<3$.

Sol. $\frac{2}{x}<3$

$\Rightarrow \frac{2}{x}-3<0$ (we cannot cross multiply with x, as x can be negative or positive)

$\Rightarrow \frac{2-3x}{x}<0$

$\Rightarrow \frac{3x-2}{x}>0$

$\Rightarrow \frac{(x-2/3)}{x}>0$

Sign scheme of $\frac{(x-2/3)}{x}$ is as follows:

$+$ | 0 | $-$ | $2/3$ | $+$

Fig. 1.6

$\Rightarrow x\in(-\infty,0)\cup(2/3,\infty)$.

Example 1.3 Solve $\frac{2x-3}{3x-5}\ge 3$.

Sol. $\frac{2x-3}{3x-5}\ge 3$

$\Rightarrow \frac{2x-3}{3x-5}-3\ge 0$

$\Rightarrow \frac{2x-3-9x+15}{3x-5}\ge 0$

$\Rightarrow \frac{-7x+12}{3x-5}\ge 0$

$\Rightarrow \frac{7x-12}{3x-5}\le 0$

Sign scheme of $\frac{7x-12}{3x-5}$ is as follows:

$+$ | $5/3$ | $-$ | $12/7$ | $+$

Fig. 1.7

$\Rightarrow x\in(5/3,12/7]$

$x=5/3$ is not included in the solution as at $x=5/3$, denominator becomes zero.

Example 1.4 Solve $(x-1)^2(x+4)<0$.

Sol. $(x-1)^2(x+4)<0$ (1)

Sign scheme of $(x-1)^2(x+4)$ is as follows:

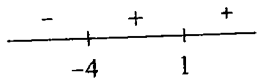


Fig. 1.8

Sign of expression does not change at $x = 1$ as $(x - 1)$ factor has even power.
Hence, solution of (1) is $x \in (-\infty, -4)$.

Example 1.5 Solve $x > \sqrt{(1-x)}$.

Sol. Given inequality can be solved by squaring both sides. But sometimes squaring gives extraneous solutions that do not satisfy the original inequality. Before squaring, we must restrict x for which terms in the given inequality are well-defined.

$x > \sqrt{(1-x)}$. Here x must be positive.

Here $\sqrt{1-x}$ is defined only when $1 - x \geq 0$ or $x \leq 1$ (1)

Squaring the given inequality we get $x^2 > 1 - x$

$$\Rightarrow x^2 + x - 1 > 0 \Rightarrow \left(x - \frac{-1-\sqrt{5}}{2}\right)\left(x - \frac{-1+\sqrt{5}}{2}\right) > 0$$

$$\Rightarrow x < \frac{-1-\sqrt{5}}{2} \text{ or } x > \frac{-1+\sqrt{5}}{2} \quad (2)$$

From (1) and (2), $x \in \left(\frac{\sqrt{5}-1}{2}, 1\right]$ (as x is +ve)

Example 1.6 Find the domain of $f(x) = \sqrt{1-\sqrt{1-\sqrt{1-x^2}}}$.

Sol. $f(x) = \sqrt{1-\sqrt{1-\sqrt{1-x^2}}}$

$$\Rightarrow 1 - \sqrt{1-\sqrt{1-x^2}} \geq 0$$

$$\Rightarrow \sqrt{1-\sqrt{1-x^2}} \leq 1$$

$$\Rightarrow 1 - \sqrt{1-x^2} \leq 1$$

$$\Rightarrow \sqrt{1-x^2} \geq 0$$

$$\Rightarrow 1 - x^2 \geq 0$$

$$\Rightarrow x^2 \leq 1 \Rightarrow x \in [-1, 1].$$

Sign Scheme of $F(x) = f_1(x) f_2(x) f_3(x) \ldots f_n(x)$

Put the values of x, which are roots of the equation, $f_1(x) = 0, f_2(x) = 0, \ldots, f_n(x) = 0$ on the number line and follow the same procedure explained in the above problems.

Example 1.7 Solve $(x-1)|x+1|\cos x > 0$, for $x \in [-\pi, \pi]$.

Sol. Let $f(x) = (x-1)|x+1|\cos x$

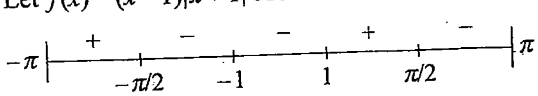


Fig. 1.9

$\cos x = 0 \Rightarrow x = \pm \pi/2$.
So, critical points are $-\pi/2, -1, 1, \pi/2$
For $x \in (\pi/2, \pi)$, $\cos x < 0 \Rightarrow f(x) < 0$
At $x = \pi/2$, and $x = 1$, $f(x)$ changes sign as shown in the sign scheme.
At $x = -1$, $f(x)$ does not change sign as $|x+1| > 0$ for all x.
Hence, $f(x) > 0 \Rightarrow x \in (-\pi, -\pi/2) \cup (1, \pi/2)$.

Example 1.8 Find the domain of

$$f(x) = \sqrt{x-4-2\sqrt{(x-5)}} - \sqrt{x-4+2\sqrt{(x-5)}}$$

Sol. $f(x) = \sqrt{x-4-2\sqrt{(x-5)}} - \sqrt{x-4+2\sqrt{(x-5)}}$

$$= \sqrt{x-5-2\sqrt{(x-5)}+1} - \sqrt{x-4+2\sqrt{(x-5)}+1}$$

$$= \sqrt{\left(\sqrt{(x-5)}-1\right)^2} - \sqrt{\left(\sqrt{x-5}+1\right)^2}$$

$$= \left|\sqrt{(x-5)}-1\right| - \left|\sqrt{(x-5)}+1\right|$$

Hence domain is $[5, \infty)$

Concept Application Exercise 1.1

Find the domain of the following functions:

1. $f(x) = \dfrac{x-3}{(x+3)\sqrt{x^2-4}}$
2. $f(x) = \sqrt{2-x} - \dfrac{1}{\sqrt{9-x^2}}$
3. $f(x) = \sqrt{\dfrac{x-2}{x+2}} + \sqrt{\dfrac{1-x}{1+x}}$
4. $f(x) = \sqrt{\left(\dfrac{2}{x^2-x+1} - \dfrac{1}{x+1} - \dfrac{2x-1}{x^3+1}\right)}$
5. $f(x) = \sqrt{x-\sqrt{1-x^2}}$
6. Find the range of $f(x) = \dfrac{x^2+1}{x^2+2}$
7. Solve $x(e^x-1)(x+2)(x-3)^2 \leq 0$.

FUNCTION

Roughly speaking, term function is used to define the dependence of one physical quantity on another, e.g., volume V of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$. This dependence of V on r would be denoted as $V = f(r)$ and we would simply say that V is a function of r. Here f is purely a symbol (for that matter, any other letter could have been used in place of f), and it is simply used to represent the dependence of one quantity on the other.

Definition of Function

Function can be easily defined with the help of the concept of mapping. Let A and B be any two non-empty sets. "A function from A and B is a rule or correspondence that assigns to each element of set A, one and only one element of set B". Let the correspondence be f. Then mathematically we write $f: A \to B$ where $y = f(x)$, $x \in A$ and $y \in B$. We say that y is the image of x under f (or x is the pre-image of y).

- A mapping $f: A \to B$ is said to be a function if each element in the set A has a image in set B. It is possible that a few elements in the set B are present which are not the images of any element in set A.
- Every element in set A should have one and only one image. That means it is impossible to have more than one image for a specific element in set A. Functions cannot be multi-valued (A mapping that is multi-valued is called a relation from A and B).

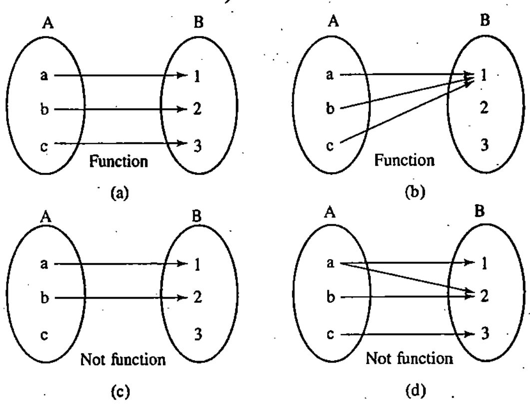


Fig. 1.10

Let us consider some other examples to make the above mentioned concepts clear.

a. Let $f: R^+ \to R$ where $y^2 = x$. This cannot be considered a function as each $x \in R^+$ would have two images namely $\pm\sqrt{x}$. Hence, it does not represent a function. Thus, it would be a relation.

b. Let $f: [-2, 2] \to R$, where $x^2 + y^2 = 4$. Here $y = \pm\sqrt{4 - x^2}$, that means for every $x \in [-2, 2]$ we would have two values of y (except when $x = \pm 2$). Hence, it does not represent a function.

c. Let $f: R \to R$ where $y = x^3$. Here for each $x \in R$ we would have a unique value of y in the set R (as cube of any two distinct real numbers are distinct). Hence, it would represent a function.

Function as a Set of Ordered Pairs

A function $f: A \to B$ can be expressed as a set of ordered pairs in which each ordered pair is such that its first element belongs to A and second element is the corresponding element of B.

As such a function $f: A \to B$ can be considered as a set of ordered pairs $(a, f(a))$ where $a \in A$ and $f(a) \in B$ which is the f image of a. Hence, f is a subset of $A \times B$.

As a particular type of relation, we can define a function as follows:

A relation R from a set A to a set B is called a function if

a. each element of A is associated with some element of B
b. each element of A has unique image in B

Thus, a function f from a set A to a set B is a subset of $A \times B$ in which each $a \in A$ appears in one and only one ordered pair belonging to f. Hence, a function f is a relation from A to B satisfying the following properties:

a. $f \subset A \times B$
b. $\forall\, a \in A \Rightarrow (a, f(a)) \in f$
c. $(a, b) \in f$ and $(a, c) \in f \Rightarrow b = c$

Thus, the ordered pairs of f must satisfy the property that each element of A appears in some ordered pair and no two ordered pairs have same first element.

Note:
Every function is a relation but every relation is not necessarily a function.

Distinction between a Relation and a Function by Graphs (Vertical Line Test)

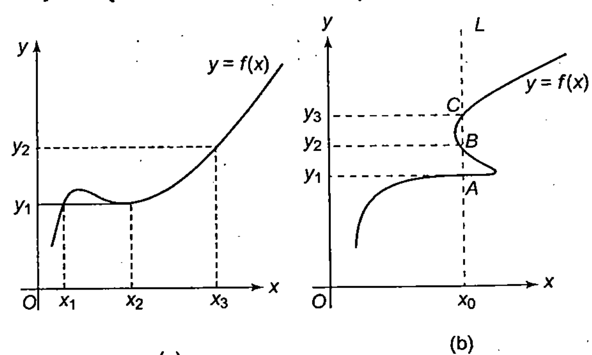


Fig. 1.11

The above figures show the graph of two arbitrary curves. In Fig. 1.11 (a), any line drawn parallel to y-axis would meet the curve at only one point. That means each element of A would have one and only one image. Thus, Fig. 1.11(a) represents the graph of a function.

In Fig. 1.11 (b), certain line parallel to y-axis, (e.g., line L) would meet the curve in more than one points (A, B and C). Thus, element x_0 of A would have three distinct images. Thus, this curve does not represent a function.

Hence, if $y = f(x)$ represents a function, lines drawn parallel to y-axis through different points corresponding to points of set X should meet the curve in one and only one point.

Consider the graph of following relations:

Equation of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is a relation, which is a combination of two functions $y = \pm b\sqrt{1 - \frac{x^2}{a^2}}$.

The upper branch represents function $y = b\sqrt{1 - \frac{x^2}{a^2}}$ and the lower branch represents the function $y = -b\sqrt{1 - \frac{x^2}{a^2}}$.

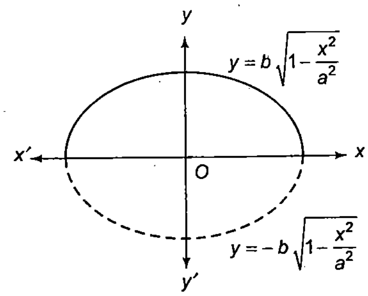


Fig. 1.12

Graph of a parabola $y^2 = x$

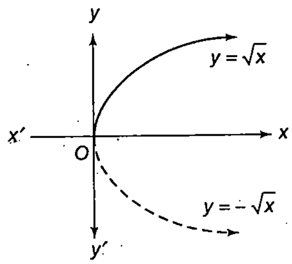


Fig. 1.13

Graph of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

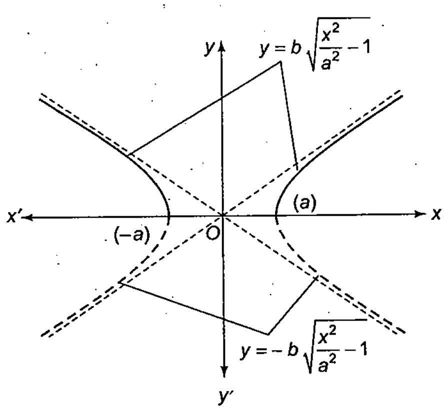


Fig. 1.14

Domain, Co-Domain, Range

Let $f: A \to B$ be a function. In general, sets A and B could be any arbitrary non-empty sets. But at this level, we would confine ourself only to real-valued functions. That means it would be invariably assumed that A and B are the subsets of real numbers.

Set A is called domain of the function f.

Set B is called co-domain of the function f.

The set of images of different elements of set A is called the range of the function f. It is obvious that a range would be a subset of co-domain as we may have few elements in co-domain which are not the images of any element of set A (of course, these elements of co-domain will not be included in the range). Range is also called domain of variation. Domain of function f is normally represented as Domain (f). Range is represented as Range (f). Note that sometimes domain of the function is not explicitly defined. In these cases, domain would mean the set of values of x for which $f(x)$ assumes real values. For example, if $y = f(x)$ then Domain $(f) = \{x : f(x) \text{ is a real number}\}$.

Rules for the Domain of a Function

a. Domain $(f(x) + g(x))$ = Domain $f(x) \cap$ Domain $g(x)$
b. Domain $(f(x) \times g(x))$ = Domain $f(x) \cap$ Domain $g(x)$
c. Domain $\left(\frac{f(x)}{g(x)}\right)$ = Domain $f(x) \cap$ Domain $g(x) \cap \{x : g(x) \neq 0\}$
d. Domain $\sqrt{f(x)}$ = Domain $f(x) \cap \{x : f(x) \geq 0\}$

Some Important Definitions

1. **Polynomial function:** If a function f is defined by $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$ where n is a non-negative integer and $a_0, a_1, a_2, \ldots, a_n$ are real numbers and $a_0 \neq 0$, then f is called a polynomial function of degree n.
2. **Algebraic function:** y is an algebraic function of x, if it is a function that satisfies an algebraic equation of the form $P_0(x)\,y^n + P_1(x)\,y^{n-1} + \cdots + P_{n-1}(x)\,y + P_n(x) = 0$ where n is a positive integer and $P_0(x)$, $P_1(x)$, $P_2(x)$ are polynomials in x. For example, $x^3 + y^3 - 3xy = 0$ or $y = |x|$ is an algebraic function, since it satisfies the equation $y^2 - x^2 = 0$.
   Note that all polynomial functions are algebraic but converse is not true.
   A function that is not algebraic is called *Transcedental function.*
3. **Rational function:** A function that can be written as the quotient of two polynomial function is said to be a rational function.
   If $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, and $Q(x) = b_0 + b_1x + b_2x^2 + \cdots + b_mx^m$ be two polynomial functions, then the function f is defined by $f(x) = \frac{P(x)}{Q(x)}$ is a rational function of x.
4. **Explicit function:** A function $y = f(x)$ is said to be an explicit function of x if the dependent variable y can be expressed in terms of independent variable x only. For example, (i) $y = x - \cos x$, (ii) $y = x + \log_e x - 2x^3$.
5. **Implicit function:** A function $y = f(x)$ is said to be an implicit function of x if y cannot be written in terms of x only. For example, (i) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, (ii) $xy = \sin(x + y)$.
6. **Bounded functions:** A function is said to be bounded if $|f(x)| \leq M$, where M is a finite positive real number.
7. **Identity function:** The function $f: R \to R$ is called an identity function if $f(x) = x \,\forall\, x \in R$.

DIFFERENT TYPES OF FUNCTIONS

Quadratic Function

Let $f(x) = ax^2 + bx + c$, where $a, b, c, \in R$ and $a \neq 0$.

We have $f(x) = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right]$

$$= a\left[x^2+\frac{b}{a}x+\frac{b^2}{4a^2}+\frac{c}{a}-\frac{b^2}{4a^2}\right]$$

$$= a\left[\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right]$$

$$= a\left[\left(x+\frac{b}{2a}\right)^2-\frac{D}{4a^2}\right] \Rightarrow \left(y+\frac{D}{4a}\right)=a\left(x+\frac{b}{2a}\right)^2.$$

Thus, $y = f(x)$ represents a parabola whose axis is parallel to y-axis and vertex $A\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$. For some values of $x, f(x)$ may be positive, negative or zero and for $a > 0$, the parabola opens upwards and for $a < 0$, the parabola opens downwards. This gives the following cases:

a. $a > 0$ and $D < 0$, so $f(x) > 0\ \forall\, x \in R$, i.e., $f(x)$ is positive for all values of x.

Range of function is $\left[-\frac{D}{4a}, \infty\right)$

$x = -\frac{b}{2a}$ is a point of minima.

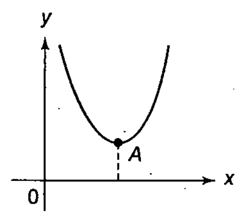


Fig. 1.15

b. $a < 0$ and $D < 0$ so $f(x) < 0\ \forall\ x \in R$, i.e., $f(x)$ is negative for all values of x.

Range of function is $\left(-\infty, -\frac{D}{4a}\right]$

$x = -\frac{b}{2a}$ is a point of maxima.

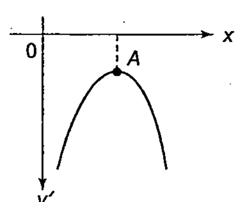


Fig. 1.16

c. $a > 0$ and $D = 0$, so $f(x) \geq 0\ \forall\ x \in R$, i.e., $f(x)$ is positive for all values of x except at vertex where $f(x) = 0$.

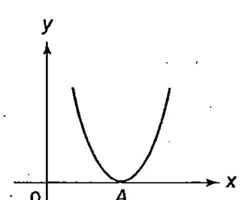


Fig. 1.17

d. $a > 0$ and $D > 0$

Let $f(x) = 0$ have two real roots α and were β (where $\alpha < \beta$) then .

$f(x) > 0\ \forall\ x \in (-\infty, \alpha) \cup (\beta, \infty)$ and $f(x) < 0\ \forall\ x \in (\alpha, \beta)$.

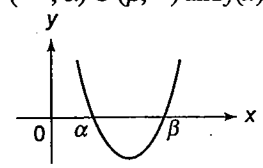


Fig. 1.18

e. $a < 0$ and $D = 0$

so $f(x) \leq 0\ \forall\ x \in R$, i.e., $f(x)$ is negative for all values of x except at vertex where $f(x) = 0$.

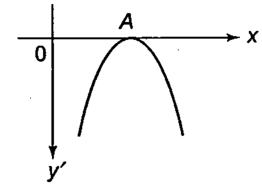


Fig. 1.19

f. $a < 0$ and $D > 0$

Let $f(x) = 0$ have two roots α and β (where $\alpha < \beta$) then $f(x) < 0\ \forall\ x \in (-\infty, \alpha) \cup (\beta, \infty)$ and $f(x) > 0$, $\forall\ x \in (\alpha, \beta)$

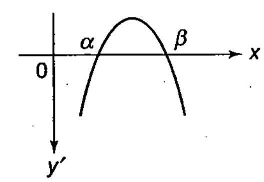


Fig. 1.20

Note: *If $f(x) \geq 0,\ \forall\ x \in R \Rightarrow a > 0$ and $D \leq 0$ and if $f(x) \leq 0\ \forall\ x \in R \Rightarrow a < 0$ and $D \leq 0$.*

Example 1.9 Find the range of $f(x) = x^2 - x - 3$.

Sol. $f(x) = x^2 - x - 3 = \left(x-\frac{1}{2}\right)^2 - \frac{1}{4} - 3 = \left(x-\frac{1}{2}\right)^2 - \frac{13}{4}$

Now $\left(x-\frac{1}{2}\right)^2 \geq 0\ \forall\, x \in R \Rightarrow \left(x-\frac{1}{2}\right)^2 - \frac{13}{4} \geq \frac{-13}{4}$, $\forall\ x \in R$

Hence, range is $\left[-\frac{13}{4}, \infty\right)$.

Example 1.10 Find the domain and range of

$$f(x) = \sqrt{x^2 - 3x + 2}.$$

Sol. For domain $x^2 - 3x + 2 \geq 0$

$\Rightarrow (x-1)(x-2) \geq 0$

$\Rightarrow x \in (-\infty, 1] \cup [2, \infty)$

Now, $f(x) = \sqrt{x^2 - 3x + 2}$

$$= \sqrt{\left(x-\frac{3}{2}\right)^2 + 2 - \frac{9}{4}}$$

$$= \sqrt{\left(x-\frac{3}{2}\right)^2 - \frac{1}{4}}$$

Now, the least permissible value of $\left(x-\frac{3}{2}\right)^2 - \frac{1}{4}$ is 0 when $\left(x-\frac{3}{2}\right) = \pm\frac{1}{2}$. Hence, the range is $[0, \infty)$.

Example 1.11 Find the range of the function $f(x) = 6^x + 3^x + 6^{-x} + 3^{-x} + 2$.

Sol. $f(x) = 6^x + 3^x + 6^{-x} + 3^{-x} + 2$

$$= \left(\sqrt{6^x} - \sqrt{6^{-x}}\right)^2 + \left(\sqrt{3^x} - \sqrt{3^{-x}}\right)^2 + 6 \geq 6.$$

Hence, the range is $[6, \infty)$.

Example 1.12 Find the domain and range of

$$f(x) = \sqrt{x^2 - 4x + 6}.$$

Sol. $x^2 - 4x + 6 = (x-2)^2 + 2$ which is always positive.

Hence, the domain is R.

Now, $f(x) = \sqrt{(x-2)^2 + 2}$

The least value of $f(x)$ is $\sqrt{2}$ when $x - 2 = 0$.

Hence, the range is $[\sqrt{2}, \infty)$.

Example 1.13 Find the range of $f(x) = \dfrac{x^2 - x + 1}{x^2 + x + 1}$.

Sol. Let $y = \dfrac{x^2 - x + 1}{x^2 + x + 1}$

$\Rightarrow (1-y)x^2 - (1+y)x + 1 - y = 0$

Now x is real, then $D \geq 0$

$\Rightarrow (1+y)^2 - 4(1-y)^2 \geq 0$

$\Rightarrow (1+y-2+2y)(1+y+2-2y) \geq 0$

$\Rightarrow (3y-1)(3-y) \geq 0$

$\Rightarrow 3\left(y-\frac{1}{3}\right)(y-3) \leq 0$

$\Rightarrow \frac{1}{3} \leq y \leq 3 \Rightarrow$ The range is $\left[\frac{1}{3}, 3\right]$.

Example 1.14 Find the complete set of values of a such that $\dfrac{x^2 - x}{1 - ax}$ attains all real values.

Sol. $y = \dfrac{x^2 - x}{1 - ax}$

$\Rightarrow x^2 - x = y - axy$

$\Rightarrow x^2 + x(ay - 1) - y = 0$

Since x is real $\Rightarrow (ay-1)^2 + 4y \geq 0$

$\Rightarrow a^2y^2 + 2y(2-a) + 1 \geq 0 \ \forall y \in R$

$\Rightarrow$ As $a^2 > 0, 4(2-a)^2 - 4a^2 \leq 0 \Rightarrow 4 - 4a \leq 0 \Rightarrow a \in [1, \infty)$.

Concept Application Exercise 1.2

1. Find the range of $f(x) = \dfrac{x^2 + 34x - 71}{x^2 + 2x - 7}$.
2. Find the range of $f(x) = \sqrt{x-1} + \sqrt{5-x}$.
3. If $f(x) = \sqrt{x^2 + ax + 4}$ is defined for all x, then find the values of a.
4. Find the domain and range of $f(x) = \sqrt{3 - 2x - x^2}$.

Modulus Function

$$y = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases} = \sqrt{x^2} = \max\{x, -x\}$$

Domain : R

Range : $[0, \infty)$

Nature : even function

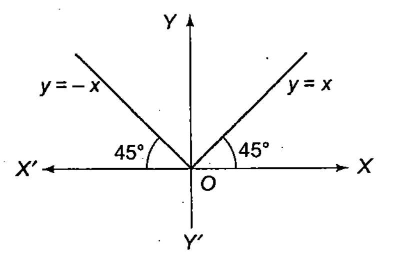


Fig. 1.21

$$y = |x - a| = \begin{cases} x - a, & x \geq a \\ a - x, & x < a \end{cases}, \text{ where } a > 0$$

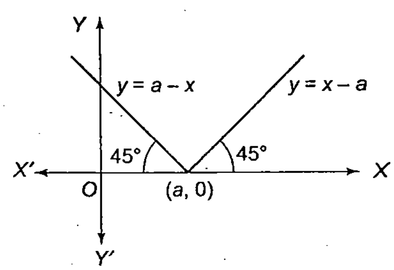
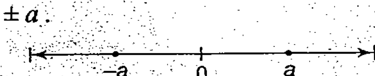


Fig. 1.21(a)

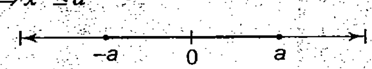
Properties of Modulus Function

a. $|x| = a \Rightarrow$ Points on the real number line whose distance from origin is a

$\Rightarrow x = \pm a$.



b. $|x| \leq a \Rightarrow x^2 \leq a^2$

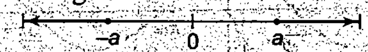


$\Rightarrow$ Points on the real number line whose distance from the origin is a or less than a

$\Rightarrow -a \le x \le a$; $(a \ge 0)$.

c. $|x| \ge a \Rightarrow x^2 \ge a^2$

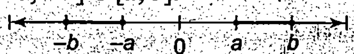
$\Rightarrow$ Points on the real number line whose distance from origin is a or greater than a



$\Rightarrow x \le -a$ or $x \ge a$; $(a \ge 0)$.

d. $a \le |x| \le b \Rightarrow a^2 \le x^2 \le b^2$

$\Rightarrow x \in [-b, -a] \cup [a, b]$.



e. $|x+y| = |x| + |y| \Leftrightarrow x$ and y have the same sign or at least one of x and y is zero or $xy \ge 0$.

f. $|x-y| = |x| - |y| \Rightarrow x \ge 0, y \ge 0$ and $|x| \ge |y|$ or $x \le 0, y \le 0$ and $|x| \ge |y|$.

g. $|x \pm y| \le |x| + |y|$.

h. $|x \pm y| \ge ||x| - |y||$.

Example 1.15 Solve $|3x-2| \le \dfrac{1}{2}$.

Sol. $|3x-2| \le \dfrac{1}{2}$

$\Rightarrow -\dfrac{1}{2} \le 3x - 2 \le \dfrac{1}{2}$

$\Rightarrow \dfrac{3}{2} \le 3x \le \dfrac{5}{2}$

$\Rightarrow \dfrac{1}{2} \le x \le \dfrac{5}{6}$

$\Rightarrow x \in [1/2, 5/6]$.

Example 1.16 Solve $||x-1|-5| \ge 2$.

Sol. $||x-1|-5| \ge 2$

$\Rightarrow |x-1| - 5 \le -2$ or $|x-1| - 5 \ge 2$

$\Rightarrow |x-1| \le 3$ or $|x-1| \ge 7$

$\Rightarrow -3 \le x-1 \le 3$ or $x - 1 \le -7$ or $x - 1 \ge 7$

$\Rightarrow -2 \le x \le 4$ or $x \le -6$ or $x \ge 8$

$\Rightarrow x \in (-\infty, -6] \cup [-2, 4] \cup [8, \infty)$.

Example 1.17 Solve $\dfrac{-1}{|x|-2} \ge 1$, where $x \in R, x \ne \pm 2$ or find the domain of $f(x) = \sqrt{\dfrac{1-|x|}{|x|-2}}$.

Sol. Given $\dfrac{-1}{|x|-2} \ge 1$

$\Rightarrow \dfrac{-1}{|x|-2} - 1 \ge 0$

$\Rightarrow \dfrac{-1-(|x|-2)}{|x|-2} \ge 0$

$\Rightarrow \dfrac{1-|x|}{|x|-2} \ge 0$

$\Rightarrow \dfrac{|x|-1}{|x|-2} \le 0$

$\Rightarrow \dfrac{y-1}{y-2} \le 0$, where $y = |x|$.

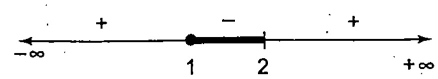


Fig. 1.22

$\Rightarrow 1 \le y < 2$

$\Rightarrow 1 \le |x| < 2$

$\Rightarrow x \in (-2, -1] \cup [1, 2)$.

Example 1.18 Solve $\dfrac{|x+3|+x}{x+2} > 1$.

Sol. We have $\dfrac{|x+3|+x}{x+2} > 1$

Clearly, L.H.S. of this inequation is meaningful for $x \ne -2$.

Given $\dfrac{|x+3|+x}{x+2} > 1$

$\Rightarrow \dfrac{|x+3|+x}{x+2} - 1 > 0$

$\Rightarrow \dfrac{|x+3|+x-x-2}{x+2} > 0$

$\Rightarrow \dfrac{|x+3|-2}{x+2} > 0$

If $|x+3| - 2 = 0 \Rightarrow x + 3 = \pm 2 \Rightarrow x = -5, -1$.

Hence, the sign scheme of the expression $\dfrac{|x+3|-2}{x+2}$ is as follows:

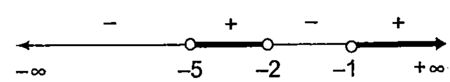


Fig. 1.23

From the above sign scheme, $x \in (-5, -2) \cup (-1, \infty)$.

Example 1.19 Solve $|x-1| + |x-2| \ge 4$.

Sol. Let $f(x) = |x-1| + |x-2|$

A.	B. $f(x)$	C. $f(x) \ge 4$	D. A $\cap$ C
$x < 1$	$1 - x + 2 - x$ $= 3 - 2x$	$3 - 2x \ge 4$ $\Rightarrow x \le -1/2$	$x \le -1/2$
$1 \le x \le 2$	$x - 1 + 2$ $- x = 1$	$1 \ge 4$, not possible	
$x > 2$	$x - 1 + x - 2$ $= 2x - 3$	$2x - 3 \ge 4$ $\Rightarrow x \ge 7/2$	$x \ge 7/2$

Hence, the solution is $x \in (-\infty, -1/2] \cup [7/2, \infty)$.

Example 1.20 Solve $|\sin x + \cos x| = |\sin x| + |\cos x|$, $x \in [0, 2\pi]$.

Sol. The given relation holds only when $\sin x$ and $\cos x$ have same sign or at least one of them is zero.

Hence, $x \in [0, \pi/2] \cup [\pi, 3\pi/2] \cup \{2\pi\}$.

Concept Application Exercise 1.3

1. Solve the following:
   a. $1 \le |x-2| \le 3$
   b. $0 < |x-3| \le 5$
   c. $|x-2|+|2x-3|=|x-1|$
   d. $\left|\dfrac{x-3}{x+1}\right| \le 1$
2. Find the domain of
   a. $f(x)=\dfrac{1}{\sqrt{x-|x|}}$
   b. $f(x)=\dfrac{1}{\sqrt{x+|x|}}$
3. Find the set of real value(s) of a for which the equation $|2x+3|+|2x-3|=ax+6$ has more than two solutions.
4. If $a<b<c$, then find the range of $f(x)=|x-a|+|x-b|+|x-c|$.
5. Find the range of $f(x)=\sqrt{1-\sqrt{x^2-6x+9}}$.

Trigonometric Functions

1. $y=f(x)=\sin x$
   Domain $\to R$, Range $\to [-1, 1]$
   Period $\to 2\pi$
   Nature $\to$ odd, many-one in its actual domain
   $\sin^2 x$, $|\sin x| \in [0, 1]$
   $\sin x = 0 \Rightarrow x = n\pi, n \in I$
   $\sin x = 1 \Rightarrow x = (4n+1)\pi/2, n \in I$
   $\sin x = -1 \Rightarrow x = (4n-1)\pi/2, n \in I$
   $\sin x = \sin\alpha \Rightarrow x = n\pi + (-1)^n\alpha, n \in I$
   $\sin x \ge 0 \Rightarrow x \in \bigcup_{n\in I} [2n\pi, \pi + 2n\pi]$

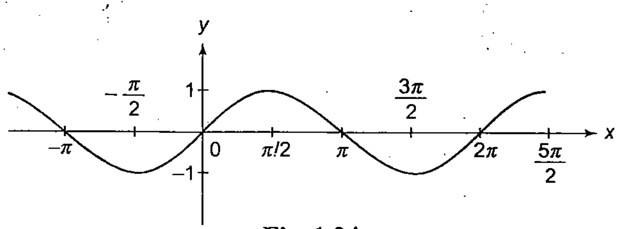


Fig. 1.24

2. $y=f(x)=\cos x$
   Domain $\to R$, Range $\to [-1, 1]$
   Period $\to 2\pi$
   Nature $\to$ even, many-one in its actual domain
   $\cos^2 x$, $|\cos x| \in [0, 1]$
   $\cos x = 0 \Rightarrow x = (2n+1)\pi/2, n \in I$
   $\cos x = 1 \Rightarrow x = 2n\pi, n \in I$
   $\cos x = -1 \Rightarrow x = (2n+1)\pi, n \in I$
   $\cos x = \cos\alpha \Rightarrow x = 2n\pi \pm \alpha, n \in I$
   $\cos x \ge 0 \Rightarrow x \in \bigcup_{n\in I} [2n\pi - \frac{\pi}{2}, 2n\pi + \pi/2]$

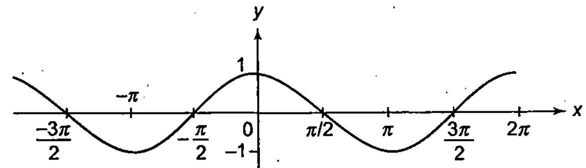


Fig. 1.25

3. $y=f(x)=\tan x$
   Domain $\to R \sim (2n+1)\pi/2, n \in I$;
   Range $\to (-\infty, \infty)$
   Period $\to \pi$.
   Nature $\to$ odd, many-one in its actual domain
   Discontinuous at $x=(2n+1)\pi/2, n \in I$
   $\tan^2 x$, $|\tan x| \in [0, \infty)$
   $\tan x = 0 \Rightarrow x = n\pi, n \in I$
   $\tan x = \tan\alpha \Rightarrow x = n\pi + \alpha, n \in I$

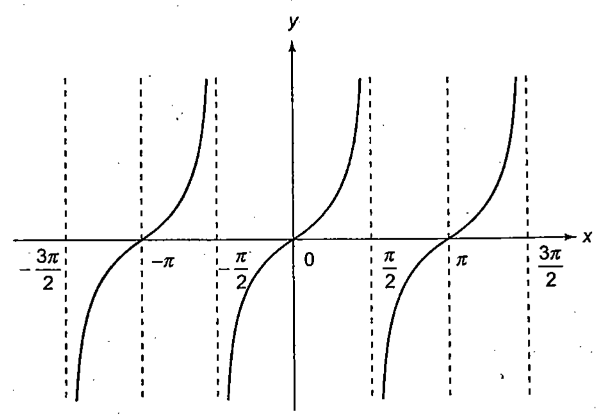


Fig. 1.26

4. $y=f(x)=\cot x$
   Domain $\to R \sim n\pi, n \in I$; Range $\to (-\infty, \infty)$;
   Period $\to \pi$;
   Nature $\to$ odd, many-one in its actual domain
   Discontinuous at $x = n\pi, n \in I$
   $\cot^2 x$, $|\cot x| \in [0, \infty)$
   $\cot x = 0 \Rightarrow x = (2n+1)\pi/2, n \in I$

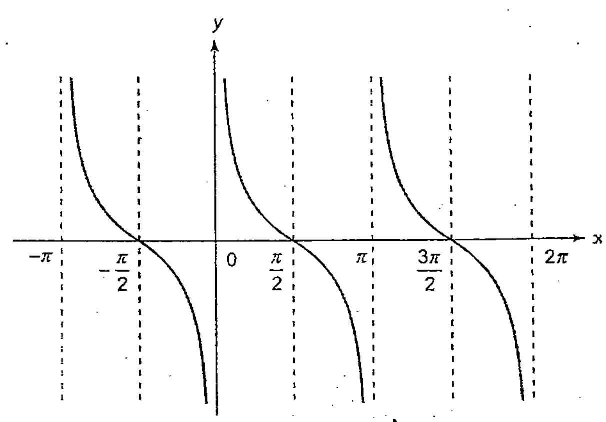


Fig. 1.27

5. $y = f(x) = \sec x$
Domain $\rightarrow R \sim (2n+1)\pi/2, n \in I$
Range $\rightarrow (-\infty, -1] \cup [1, \infty)$
Period $\rightarrow 2\pi$,
$\sec^2 x$, $|\sec x| \in [1, \infty)$
Nature $\rightarrow$ even, many-one in its actual domain

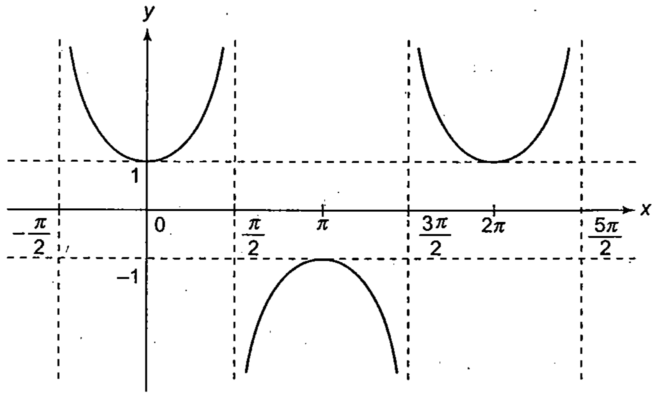


Fig. 1.28

6. $y = f(x) = \text{cosec } x$
Domain $\rightarrow R \sim n\pi, n \in I$
Range $\rightarrow (-\infty, -1] \cup [1, \infty)$
Period $\rightarrow 2\pi$;
$\text{cosec}^2 x$, $|\text{cosec} x| \in [1, \infty)$
Nature $\rightarrow$ odd, many-one in its actual domain

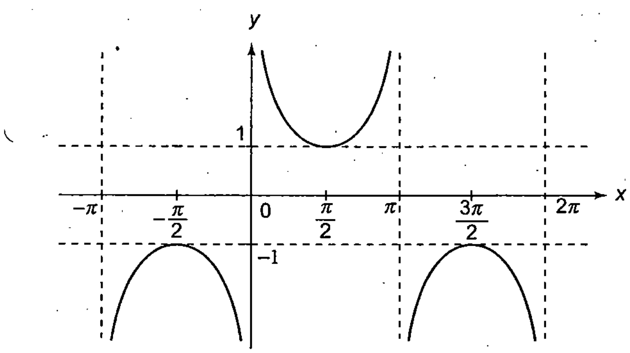


Fig. 1.29

Important Result

$$f(x) = a\cos x + b\sin x = \sqrt{a^2+b^2}\sin\left(x + \tan^{-1}\frac{a}{b}\right)$$
$$= \sqrt{a^2+b^2}\cos\left(x - \tan^{-1}\frac{b}{a}\right).$$

Proof: Let $a = r\sin\alpha$, $b = r\cos\alpha$

$\Rightarrow a^2 + b^2 = r^2$ and $\tan\alpha = \dfrac{a}{b}$

Now, $f(x) = r(\cos x\ \sin\alpha + \sin x\cos\alpha)$

$$= r\sin(x+\alpha) = \sqrt{a^2+b^2}\sin\left(x + \tan^{-1}\frac{a}{b}\right).$$

Since $-1 \le \sin\left(x + \tan^{-1}\dfrac{a}{b}\right) \le 1$

$$\Rightarrow -\sqrt{a^2+b^2} \le \sqrt{a^2+b^2}\sin\left(x + \tan^{-1}\frac{a}{b}\right) \le \sqrt{a^2+b^2}$$

$\Rightarrow$ The range of $f(x) = a\cos x + b\sin x$ is

$$\left[-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}\right].$$

Example 1.21 Find the domain of the functions.

$$f(x) = \frac{1}{1+2\sin x}.$$

Sol. To define $f(x)$, we must have $1 + 2\sin x \ne 0$

$$\Rightarrow \sin x \ne -\frac{1}{2} \Rightarrow x \ne n\pi + (-1)^n\frac{7\pi}{6}, n \in Z$$

Hence, the domain of the function is

$$R - \left\{n\pi + (-1)^n\frac{7\pi}{6}, n \in Z\right\}.$$

Example 1.22 Solve $\sin x > -\dfrac{1}{2}$ or find the domain of

$$f(x) = \frac{1}{\sqrt{1+2\sin x}}.$$

Sol. To define $f(x)$, we must have $1 + 2\sin x > 0$ or $\sin x > -\dfrac{1}{2}$.

The function $\sin x$ has the least positive period 2π. That is why it is sufficient to solve inequality of the form $\sin x > a$, $\sin x \ge a$, $\sin x < a$, $\sin x \le a$ first on the interval of length 2π, and then get the solution set by adding numbers of the form $2\pi n$, $n \in Z$, to each of the solutions obtained on that interval.

Thus, let us solve this inequality on the interval $\left[-\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right]$.

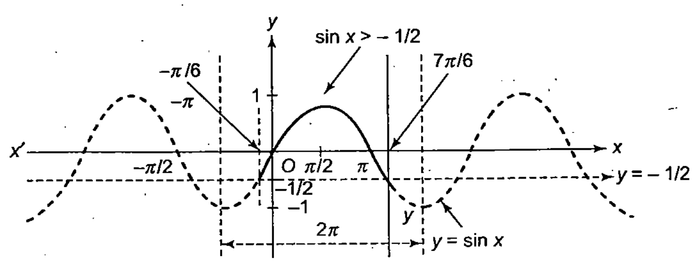


Fig. 1.30

From Fig. 1.30, $\sin x > -\dfrac{1}{2}$ when $-\dfrac{\pi}{6} < x < \dfrac{7\pi}{6}$

Thus, on generalizing the above solution, we get

$$2n\pi - \frac{\pi}{6} < x < 2n\pi + \frac{7\pi}{6}; n \in Z.$$

Example 1.23 Find the number of solutions of $\sin x = \dfrac{x}{10}$.

Sol. Here, let $f(x) = \sin x$ and $g(x) = \dfrac{x}{10}$. Also we know that

$$-1 \le \sin x \le 1 \Rightarrow -1 \le \frac{x}{10} \le 1 \Rightarrow -10 \le x \le 10.$$

Thus, sketch both the curves when $x \in [-10, 10]$.

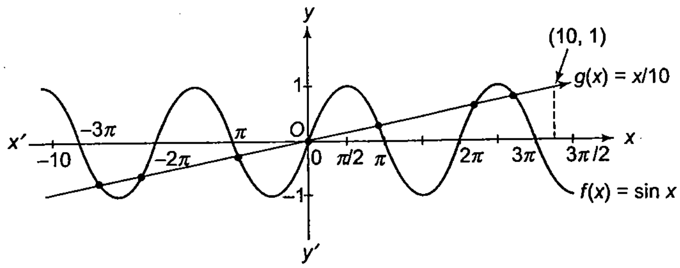


Fig. 1.31

From Fig. 1.31, $f(x) = \sin x$ and $g(x) = \dfrac{x}{10}$ intersect at 7 points. So, numbers of solutions are 7.

Example 1.24 Find the number of solutions of the equation $\sin x = x^2 + x + 1$.

Sol. Let $f(x) = \sin x$ and $g(x) = x^2 + x + 1 = \left(x + \dfrac{1}{2}\right)^2 + \dfrac{3}{4}$ as shown in Fig. 1.32, which do not intersect at any point, therefore, there is no solution.

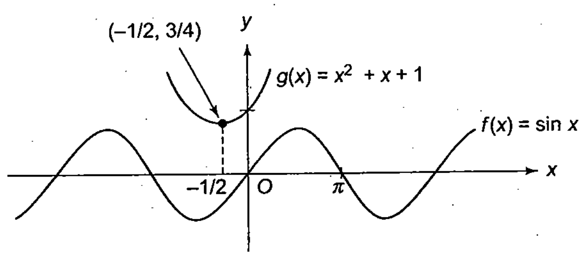


Fig. 1.32

Example 1.25 Find the range of $f(x) = \sin^2 x - \sin x + 1$

Sol. $f(x) = \sin^2 x - \sin x + 1 = \left(\sin x - \dfrac{1}{2}\right)^2 + \dfrac{3}{4}$

Now, $-1 \le \sin x \le 1 \Rightarrow -\dfrac{3}{2} \le \sin x - \dfrac{1}{2} \le \dfrac{1}{2}$

$\Rightarrow 0 \le \left(\sin x - \dfrac{1}{2}\right)^2 \le \dfrac{9}{4} \Rightarrow \dfrac{3}{4} \le \left(\sin x - \dfrac{1}{2}\right)^2 + \dfrac{3}{4} \le 3$

Hence, the range is $\left[\dfrac{3}{4}, 3\right]$.

Example 1.26 Find the range of $f(x) = \dfrac{1}{2\cos x - 1}$.

Sol. $-1 \le \cos x \le 1$

$\Rightarrow -2 \le 2\cos x \le 2$

$\Rightarrow -3 \le 2\cos x - 1 \le 1$

For $\dfrac{1}{2\cos x - 1}$, $-3 \le 2\cos x - 1 < 0$ or $0 < 2\cos x - 1 \le 1$

$\Rightarrow -\infty < \dfrac{1}{2\cos x - 1} \le \dfrac{-1}{3}$ or $1 \le \dfrac{1}{2\cos x - 1} < \infty$

Hence, the range is $\left(-\infty, -\dfrac{1}{3}\right] \cup [1, \infty)$.

Example 1.27 Find the domain for $f(x) = \sqrt{\cos(\sin x)}$.

Sol. $f(x) = \sqrt{\cos(\sin x)}$ is defined if

$\cos(\sin x) \ge 0$ (1)

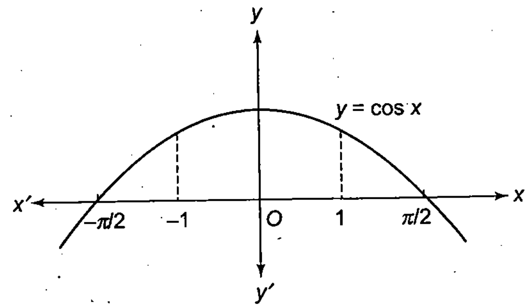


Fig. 1.33

As we know, $-1 \le \sin x \le 1$ for all x

$\cos\theta \ge 0$

{Here, $\theta = \sin x$ lies in the first and fourth quadrant}

i.e., $\cos(\sin x) \ge 0$ for all x

i.e., $x \in R$

Thus, the domain $f(x)$ is R.

Example 1.28 If $f(x) = \dfrac{\sin x}{\sqrt{1 + \tan^2 x}} - \dfrac{\cos x}{\sqrt{1 + \cot^2 x}}$, then find the range of $f(x)$.

Sol. $f(x) = \dfrac{\sin x}{|\sec x|} - \dfrac{\cos x}{|\operatorname{cosec} x|} = \sin x\,|\cos x| - \cos x\,|\sin x|$

Clearly, the domain of $f(x)$ is $R \sim \left\{n\pi, (2n+1)\dfrac{\pi}{2} / n \in I\right\}$

and period of $f(x)$ is 2π.

$$f(x) = \begin{cases} 0, & x \in (0, \pi/2) \\ -\sin 2x, & x \in (\pi/2, \pi) \\ 0, & x \in (\pi, 3\pi/2) \\ \sin 2x, & x \in (3\pi/2, 2\pi) \end{cases}$$

$\Rightarrow$ The range of $f(x)$ is $(-1, 1)$.

Example 1.29 Find the range of $f(x) = |\sin x| + |\cos x|, x \in R$.

Sol. $f(x) = |\sin x| + |\cos x| \ \forall x \in R$.

Clearly $f(x) > 0$.

Also, $f^2(x) = \sin^2 x + \cos^2 x + |2\sin x \cos x| = 1 + |\sin 2x|$

$\Rightarrow 1 \le f^2(x) \le 2$

$\Rightarrow 1 \le f(x) \le \sqrt{2}$.

Example 1.30 Find the range of $f(\theta) = 5\cos\theta + 3\cos\left(\theta + \dfrac{\pi}{3}\right) + 3$.

Sol. $f(\theta) = 5\cos\theta + 3\cos\left(\theta + \dfrac{\pi}{3}\right) + 3$

$= 5\cos\theta + \dfrac{3}{2}\cos\theta - \dfrac{3\sqrt{3}}{2}\sin\theta + 3$

$$= \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta + 3 = \sqrt{\left(\frac{169}{4}+\frac{27}{4}\right)}\ \sin(\theta-\alpha)+3$$

Thus, the range of $f(\theta)$ is $[-4, 10]$.

Concept Application Exercise 1.4

1. Find the domain of $f(x)=\sqrt{\sin x}+\sqrt{16-x^2}$.
2. Solve **(a)** $\tan x < 2$, **(b)** $\cos x \le -\frac{1}{2}$.
3. Prove that the least positive value of x, satisfying $\tan x = x+1$, lies in the interval $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.
4. Find the range of $f(x)=\sec\left(\frac{\pi}{4}\cos^2 x\right)$, where $-\infty < x < \infty$.
5. If $x \in [1, 2]$, then find the range of $f(x)=\tan x$.
6. Find the range of $f(x)=\dfrac{1}{1-3\sqrt{1-\sin^2 x}}$.

Inverse Trigonometric Functions

$f(x)=\sin^{-1}x$

Domain: $[-1, 1]$,

Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\sin^{-1}(\sin x)=x$, for all $x\in[-\pi/2, \pi/2]$

$\sin(\sin^{-1}x)=x$, for all $x\in[-1, 1]$

$\sin^{-1}(-x)=-\sin^{-1}(x)$, for all $x\in[-1, 1]$

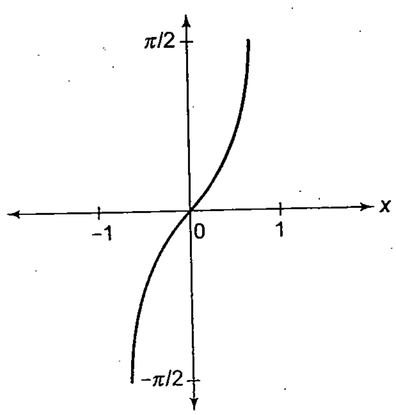


Fig. 1.34

$f(x)=\cos^{-1}x$

Domain: $[-1, 1]$

Range: $[0, \pi]$

$\cos^{-1}(\cos x)=x$, for all $x\in[0, \pi]$

$\cos(\cos^{-1}x)=x$, for all $x\in[-1, 1]$

$\cos^{-1}(-x)=\pi-\cos^{-1}x$, for all $x\in[-1, 1]$

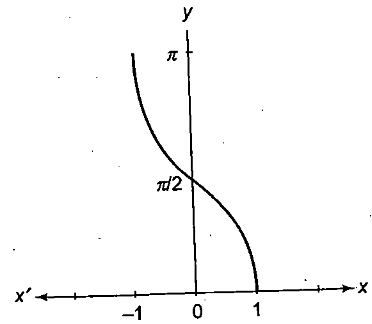


Fig. 1.35

$f(x)=\tan^{-1}x$

Domain: R

Range: $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$\tan^{-1}(\tan x)=x$, for all $x\in(-\pi/2, \pi/2)$

$\tan(\tan^{-1}x)=x$, for all $x\in R$

$\tan^{-1}(-x)=-\tan^{-1}x$, for all $x\in R$

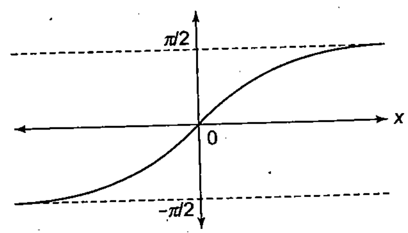


Fig. 1.36

$f(x)=\cot^{-1}x$

Domain: R

Range: $(0, \pi)$

$\cot^{-1}(\cot x)=x$, for all $x\in(0, \pi)$

$\cot(\cot^{-1}x)=x$, for all $x\in R$

$\cot^{-1}(-x)=\pi-\cot^{-1}x$, for all $x\in R$

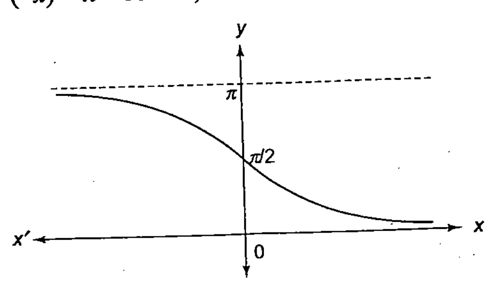


Fig. 1.37

$f(x)=\sec^{-1}x$

Domain: $(-\infty, -1]\cup[1, \infty)$

Range: $[0, \pi]-\{\pi/2\}$

$\sec^{-1}(\sec x)=x$, for all $x\in[0, \pi]-\{\pi/2\}$

$\sec(\sec^{-1}x)=x$, for all $x\in(-\infty, -1]\cup[1, \infty)$

$\sec^{-1}(-x)=\pi-\sec^{-1}x$, for all $x\in(-\infty, -1]\cup[1, \infty)$

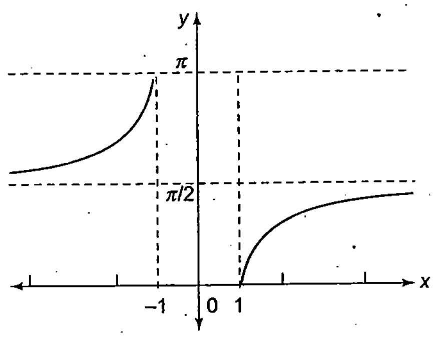


Fig. 1.38

$f(x) = \text{cosec}^{-1}x$

Domain: $(-\infty, -1] \cup [1, \infty)$

Range: $[-\pi/2, \pi/2] - \{0\}$

$\text{cosec}^{-1}(\text{cosec}\, x) = x$, for all $x \in [-\pi/2, \pi/2] - \{0\}$

$\text{cosec}(\text{cosec}^{-1} x) = x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

$\text{cosec}^{-1}(-x) = -\text{cosec}^{-1} x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

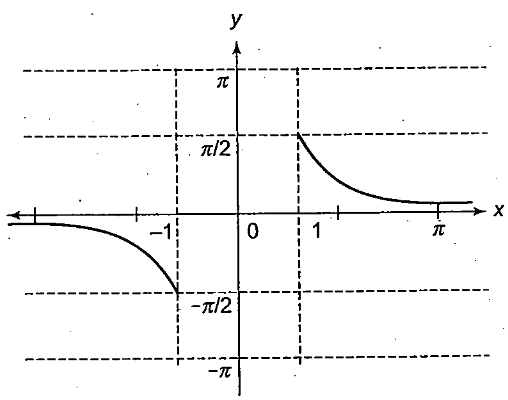


Fig. 1.39

Example 1.31 Find the domain of $f(x) = \sin^{-1}\left(\dfrac{x^2}{2}\right)$.

Sol. $f(x) = \sin^{-1}\left(\dfrac{x^2}{2}\right)$ is defined, if $-1 \le \dfrac{x^2}{2} \le 1$ or $-2 \le x^2 \le 2$

$\Rightarrow 0 \le x^2 \le 2$ (as x^2 cannot be negative)

$\Rightarrow -\sqrt{2} \le x \le \sqrt{2}$

Therefore, the domain of $f(x)$ is $[-\sqrt{2}, \sqrt{2}]$.

Example 1.32 Find the range of $f(x) = \sin^{-1} x + \tan^{-1} x + \cos^{-1} x$.

Sol. Clearly, the domain of the function is $[-1, 1]$.

Also $\tan^{-1} x \in \left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right]$ for $x \in [-1, 1]$.

Now, $\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}$ for $x \in [-1, 1]$.

Thus, $f(x) = \tan^{-1} x + \dfrac{\pi}{2}$, where $x \in [-1, 1]$.

Hence, the range is $\left[-\dfrac{\pi}{4} + \dfrac{\pi}{2}, \dfrac{\pi}{4} + \dfrac{\pi}{2}\right] = \left[\dfrac{\pi}{4}, \dfrac{3\pi}{4}\right]$.

Example 1.33 Find the domain of $f(x) = \sqrt{\cos^{-1} x - \sin^{-1} x}$.

Sol. We must have $\cos^{-1} x \ge \sin^{-1} x$

$\Rightarrow \dfrac{\pi}{2} - \sin^{-1} x \ge \sin^{-1} x$

$\Rightarrow \dfrac{\pi}{2} \ge 2\sin^{-1} x$

$\Rightarrow \sin^{-1} x \le \dfrac{\pi}{4}$, but $-\dfrac{\pi}{2} \le \sin^{-1} x$

$\Rightarrow -\dfrac{\pi}{2} \le \sin^{-1} x \le \dfrac{\pi}{4}$

$\Rightarrow \sin\left(-\dfrac{\pi}{2}\right) \le x \le \sin\dfrac{\pi}{4}$

$\left(\because \sin x \text{ is increasing function in } \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]\right)$

$\Rightarrow x \in \left[-1, \dfrac{1}{\sqrt{2}}\right]$.

Example 1.34 Find the range of $\tan^{-1}\left(\dfrac{2x}{1+x^2}\right)$.

Sol. First, we must get the range of $\dfrac{2x}{1+x^2} = y$

We have $yx^2 - 2x + y = 0$

Since x is real, $D \ge 0, \Rightarrow 4 - 4y^2 \ge 0 \Rightarrow -1 \le y \le 1$

$\Rightarrow \tan^{-1}(y) \in \left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right]$ (as $\tan x$ is an increasing function).

Example 1.35 Find the domain for $f(x) = \sin^{-1}\left(\dfrac{1+x^2}{2x}\right)$.

Sol. $f(x) = \sin^{-1}\left(\dfrac{1+x^2}{2x}\right)$ is defined for $-1 \le \dfrac{1+x^2}{2x} \le 1$, or

$\left|\dfrac{1+x^2}{2x}\right| \le 1$

$\Rightarrow |1 + x^2| \le |2x|$, for all x.

$\Rightarrow 1 + x^2 \le |2x|$, for all x (as $1 + x^2 > 0$)

$\Rightarrow x^2 - 2|x| + 1 \le 0$

$\Rightarrow |x|^2 - 2|x| + 1 \le 0$ (as $x^2 = |x^2|$)

$\Rightarrow (|x| - 1)^2 \le 0$

But $(|x| - 1)^2$ is always either positive or zero. Thus, $(|x| - 1)^2 = 0 \Rightarrow |x| = 1 \Rightarrow x = \pm 1$.

Thus, the domain for $f(x)$ is $\{-1, 1\}$.

Example 1.36 Find the range of $f(x) = \cot^{-1}(2x - x^2)$.

Sol. Let $\theta = \cot^{-1}(2x - x^2)$, where $\theta \in (0, \pi)$

$\Rightarrow \cot\theta = 2x - x^2$, where $\theta \in (0, \pi)$

$\Rightarrow \cot\theta = 1 - (1 - 2x + x^2)$, where $\theta \in (0, \pi)$

$\Rightarrow \cot\theta = 1 - (1 - x)^2$, where $\theta \in (0, \pi)$

$\Rightarrow \quad \cot\theta \le 1$, where $\theta \in (0, \pi)$

$\Rightarrow \quad \frac{\pi}{4} \le \theta < \pi.$

$\Rightarrow$ The range of $f(x) \in \left[\frac{\pi}{4}, \pi\right)$.

Concept Application Exercise 1.5

1. Find the domain of the following functions:

   a. $f(x) = \frac{\sin^{-1} x}{x}$

   b. $f(x) = \sin^{-1}(|x-1|-2)$

   c. $f(x) = \cos^{-1}(1+3x+2x^2)$

   d. $f(x) = \frac{\sin^{-1}(x-3)}{\sqrt{9-x^2}}$

   e. $f(x) = \cos^{-1}\left(\frac{6-3x}{4}\right) + \operatorname{cosec}^{-1}\left(\frac{x-1}{2}\right)$

   f. $f(x) = \sqrt{\sec^{-1}\left(\frac{2-|x|}{4}\right)}$

2. Find the range of $f(x) = \tan^{-1}\left(\sqrt{x^2-2x+2}\right)$.

3. Find the range of $f(x) = \sqrt{\cos^{-1}\sqrt{1-x^2} - \sin^{-1} x}$

4. Find the range of the function,

   $f(x) = \cot^{-1} \log_{0.5}(x^4 - 2x^2 + 3)$

Exponential and Logarithmic Functions

Exponential Function

$y = a^x, a > 1$
Domain $\rightarrow R$
Range $\rightarrow (0, \infty)$
Nature $\rightarrow$
Non-periodic
One – one
Neither odd nor even
Monotonically increasing, $(a > 1)$
Monotonically decreasing, $(0 < a < 1)$

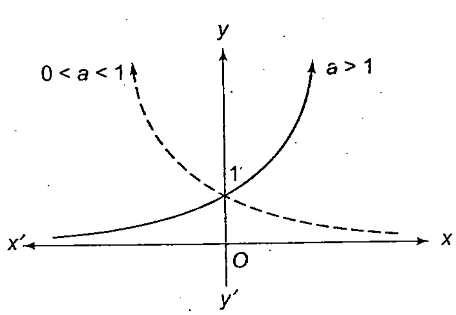


Fig. 1.40

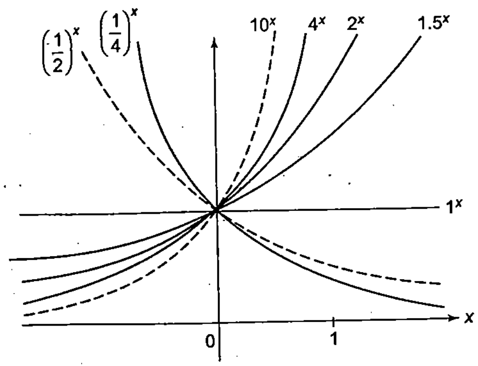


Fig. 1.41

Logarithmic Function

Logarithm function is the inverse of exponential function.

Hence, the domain and range of the logarithmic functions are range and domain of exponential function, respectively.

Also, the graph of function can be obtained by taking the mirror image of the graph of the exponential function in the line $y = x$.

$y = \log_a x, a > 0$ and $a \ne 1$
Domain $\rightarrow (0, \infty)$
Range $\rightarrow (-\infty, \infty)$
Period $\rightarrow$ non-periodic
Nature $\rightarrow$ neither odd nor even

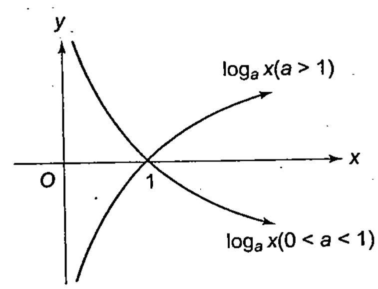


Fig. 1.42

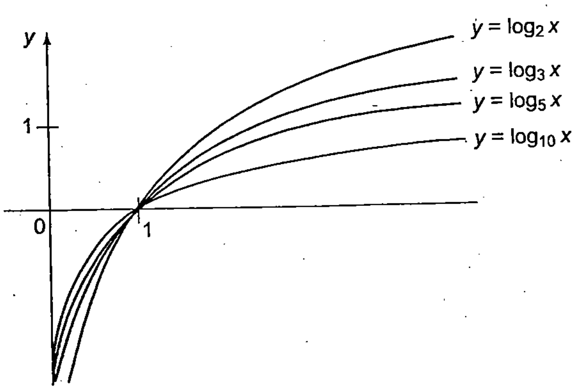


Fig. 1.43

Properties of Logarithmic Function

For $x, y > 0$ and $a > 0, a \ne 2$

1. $\log_a(xy) = \log_a x + \log_a y$
2. $\log_a(x/y) = \log_a x - \log_a y$
3. $\log_a(x^b) = b\log_a x$

4. $\log_{x^a} y^b = \frac{b}{a}\log_x y$

5. If $\log_a x > \log_a y \Rightarrow \begin{cases} x > y, & \text{if } a > 1 \\ x < y, & \text{if } 0 < a < 1 \end{cases}$

6. If $\log_a x = y \Rightarrow x = a^y$

7. If $\log_a x > y \Rightarrow \begin{cases} x > a^y, & \text{if } a > 1 \\ x < a^y, & \text{if } 0 < a < 1 \end{cases}$

8. $a^{\log_a x} = x$

9. $\log_y x = \dfrac{\log_a x}{\log_a y}$

10. $\log_a x > 0 \Rightarrow x > 1$ and $a > 1$ or $0 < x < 1$ and $0 < a < 1$

Example 1.37 Find the domain of $f(x) = \sqrt{\left(\dfrac{1-5^x}{7^{-x}-7}\right)}$.

Sol. We must have $g(x) = \left(\dfrac{1-5^x}{7^{-x}-7}\right) \geq 0 \Rightarrow \dfrac{5^x-1}{7^{-x}-7} \leq 0$

Now $5^x - 1 = 0 \Rightarrow x = 0$ and $7^{-x} - 7 = 0 \Rightarrow x = -1$

The sign scheme of $g(x)$ is

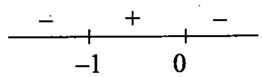


Fig. 1.44

Hence, from the sign scheme of $g(x)$

$x \in (-\infty, -1) \cup [0, \infty)$.

Example 1.38 Find the domain of

$$f(x) = \sqrt{(0.625)^{4-3x} - (1.6)^{x(x+8)}}.$$

Sol. Clearly, $(0.625)^{4-3x} \geq (1.6)^{x(x+8)}$

$\Rightarrow \left(\dfrac{5}{8}\right)^{4-3x} \geq \left(\dfrac{8}{5}\right)^{x(x+8)}$

$\Rightarrow \left(\dfrac{8}{5}\right)^{3x-4} \geq \left(\dfrac{8}{5}\right)^{x(x+8)}$

$\Rightarrow 3x - 4 \geq x^2 + 8x \Rightarrow x^2 + 5x + 4 \leq 0$

$\Rightarrow -4 \leq x \leq -1$

Hence, the domain of function $f(x)$ is $x \in [-4, -1]$.

Example 1.39 Find the range of

a. $f(x) = \log_e \sin x$

b. $f(x) = \log_3(5 - 4x - x^2)$

Sol. a. $f(x) = \log_e \sin x$ is defined if $\sin x \in (0, 1]$ for which $\log_e \sin x \in (-\infty, 0]$.

b. $f(x) = \log_3(5 - 4x - x^2)$

$= \log_3(9 - (x-2)^2)$

$f(x)$ is defined if $9 - (x-2)^2 > 0$, but $9 - (x-2)^2 \leq 9$

$\Rightarrow 0 < 9 - (x-2)^2 \leq 9$

$\Rightarrow -\infty < \log_3(9 - (x-2)^2) \leq \log_3 9$

Hence, the range is $(-\infty, 2]$.

Example 1.40 Find the domain of $f(x) = \log_{10}\log_2\log_{2/\pi}(\tan^{-1} x)^{-1}$.

Sol. We must have $\log_2 \log_{2/\pi}(\tan^{-1} x)^{-1} > 0$

$\Rightarrow \log_{2/\pi}(\tan^{-1} x)^{-1} > 1$

$\Rightarrow 0 < (\tan^{-1} x)^{-1} < 2/\pi$

$\Rightarrow \pi/2 < \tan^{-1} x < \infty$, which is not possible.

Hence, the domain is ϕ.

Example 1.41 Find the domain and range of $f(x) = \sqrt{\log_3(\cos(\sin x))}$.

Sol. $f(x) = \sqrt{\log_3(\cos(\sin x))}$.

$f(x)$ is defined only if $\log_3(\cos(\sin x)) \geq 0$

$\Rightarrow \cos(\sin x) \geq 1$

$\Rightarrow \cos(\sin x) = 1$ as $-1 \leq \cos\theta \leq 1$

$\Rightarrow \sin x = 0 \Rightarrow x = n\pi, n \in I$.

Hence, the domain consists of the multiples of π, i.e., Domain $= \{n\pi: n \in I\}$.

Also, the range is $\{0\}$.

Example 1.42 Solve $\log_x(x^2 - 1) \leq 0$.

Sol. Given $\log_x(x^2 - 1) \leq 0$

If $x > 1$, then

$\Rightarrow 0 < x^2 - 1 \leq 1$

$\Rightarrow 1 < x^2 \leq 2$

$\Rightarrow x \in [-\sqrt{2}, -1) \cup (1, \sqrt{2}]$

$\Rightarrow x \in (1, \sqrt{2}]$

If $0 < x < 1 \Rightarrow x^2 - 1 \geq 1 \Rightarrow x^2 \geq 2$

$\Rightarrow x \in (-\infty, -\sqrt{2}] \cup [\sqrt{2}, \infty)$

$\Rightarrow x = \phi$

Thus, $x \in (1, \sqrt{2}]$.

Example 1.43 Find the number of solutions of $2^x + 3^x + 4^x - 5^x = 0$.

Sol. $2^x + 3^x + 4^x - 5^x = 0$

$\Rightarrow 2^x + 3^x + 4^x = 5^x$

$\Rightarrow \left(\dfrac{2}{5}\right)^x + \left(\dfrac{3}{5}\right)^x + \left(\dfrac{4}{5}\right)^x = 1$

Now, the number of solution of the equation is equal to number of times

$y = \left(\dfrac{2}{5}\right)^x + \left(\dfrac{3}{5}\right)^x + \left(\dfrac{4}{5}\right)^x$ and $y = 1$ intersect.

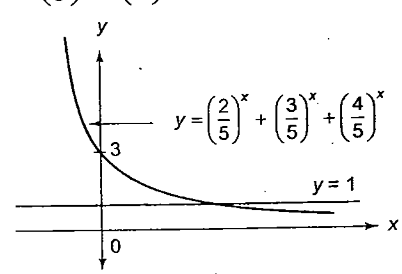


Fig. 1.45

From the graph, the equation has only one solution.

Example 1.44 Find the domain of $f(x) = \sin^{-1}(\log_9(x^2/4))$.

Sol. We have $f(x) = \sin^{-1}\left\{\log_9\left(\frac{x^2}{4}\right)\right\}$.

Since the domain of $\sin^{-1} x$ is $[-1\ 1]$.

Therefore, $f(x) = \sin^{-1}\left\{\log_9\left(\frac{x^2}{4}\right)\right\}$ is defined,

if $-1 \le \log_9\left(\frac{x^2}{4}\right) \le 1$

$\Rightarrow 9^{-1} \le \frac{x^2}{4} \le 9^1$

$\Rightarrow \frac{4}{9} \le x^2 \le 36$

$\Rightarrow \frac{2}{3} \le |x| \le 6$

$\Rightarrow x \in \left[-6, -\frac{2}{3}\right] \cup \left[\frac{2}{3}, 6\right]$ $(\because a \le |x| \le b \Leftrightarrow x \in [-b, -a] \cup [a, b])$

Hence, the domain of $f(x)$ is $\left[-6, -\frac{2}{3}\right] \cup \left[\frac{2}{3}, 6\right]$.

Example 1.45 Find the domain of function $f(x) = \log_4\{\log_5(\log_3(18x - x^2 - 77))\}$

Sol. We have $f(x) = \log_4\{\log_5(18x - x^2 - 77))\}$

Since $\log_a x$ is defined for all $x > 0$. Therefore, $f(x)$ is defined if

$\log_5\{\log_3(18x - x^2 - 77)\} > 0$ and $18x - x^2 - 77 > 0$

$\Rightarrow \log_3(18x - x^2 - 77) > 5^0$ and $x^2 - 18x + 77 < 0$

$\Rightarrow \log_3(18x - x^2 - 77) > 1$ and $(x - 11)(x - 7) < 0$

$\Rightarrow 18x - x^2 - 77 > 3^1$ and $7 < x < 11$

$\Rightarrow 18x - x^2 - 80 > 0$ and $7 < x < 11$

$\Rightarrow x^2 - 18x + 80 < 0$ and $7 < x < 11$

$\Rightarrow (x - 10)(x - 8) < 0$ and $7 < x < 11$

$\Rightarrow 8 < x < 10$ and $7 < x < 11$

$\Rightarrow 8 < x < 10$

$\Rightarrow x \in (8, 10)$.

Hence, the domain of $f(x)$ is $(8, 10)$.

Example 1.46 Find the domain of $f(x) = \sqrt{\log_{0.4}\left(\frac{x-1}{x+5}\right)}$.

Sol. $f(x) = \sqrt{\log_{0.4}\left(\frac{x-1}{x+5}\right)}$ exists if $\log_{0.4}\left(\frac{x-1}{x+5}\right) \ge 0$ and $\left(\frac{x-1}{x+5}\right) > 0$.

$\Rightarrow \frac{x-1}{x+5} \le (0.4)^0$ and $\frac{x-1}{x+5} > 0$

$\Rightarrow \frac{x-1}{x+5} \le 1$ and $\frac{x-1}{x+5} > 0$

$\Rightarrow \frac{x-1}{x+5} - 1 \le 0$ and $\frac{x-1}{x+5} > 0$

$\Rightarrow \frac{-6}{x+5} \le 0$ and $\frac{x-1}{x+5} > 0$

$\Rightarrow x + 5 > 0$ and $\frac{x-1}{x+5} > 0$

$\Rightarrow x > -5$ and $x - 1 > 0$ (as $x + 5 > 0$)

$\Rightarrow x > -5$ and $x > 1$

Thus, the domain $f(x) \in (1, \infty)$.

Concept Application Exercise 1.6

Find the domain of the following functions: (1 to 7)

1. $f(x) = \sqrt{4^x + 8^{\frac{2}{3}(x-2)} - 13 - 2^{2(x-1)}}$
2. $f(x) = \sin^{-1}(\log_2 x)$
3. $f(x) = \log_{(x-4)}(x^2 - 11x + 24)$
4. $f(x) = \frac{3}{4 - x^2} + \log_{10}(x^3 - x)$
5. $f(x) = \sqrt{\frac{\log_{0.3}|x-2|}{|x|}}$
6. $f(x) = \sqrt{\log_{10}\left\{\frac{\log_{10} x}{2(3 - \log_{10} x)}\right\}}$
7. $f(x) = \frac{1}{\sqrt{\log_{1/2}(x^2 - 7x + 13)}}$
8. Find the range of $f(x) = \log_2\left(\frac{\sin x - \cos x + 3\sqrt{2}}{\sqrt{2}}\right)$.

Greatest Integer and Fractional Part Function

Greatest Integer Function (Floor Value Function)

$y = f(x) = [x]$ (Greatest integer $\le x$)

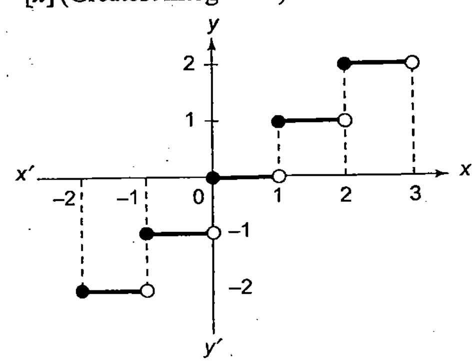


Fig. 1.46

Graph of $y = [x]$

Properties

- Domain $\to R$; Range $\to Z$;
- $[x] = n\ (n \in I) \Rightarrow x \in [n, n+1)$

- $x-1<[x]\le x$
- $[-x]+[x]=0$, if $x\in Z$
- $[-x]+[x]=-1$, if $x\notin Z$
- $[x]\ge n\Rightarrow x\ge n, n\in Z$
- $[x]\le n\Rightarrow x<n+1, n\in Z$
- $[x]>n\Rightarrow x\ge n+1, n\in Z$

e.g.,

$[x]\ge 2\Rightarrow x\in[2,\infty)$

$[x]>3\Rightarrow[x]\ge 4\Rightarrow x\in[4,\infty)$

$[x]\le 3\Rightarrow x\in(-\infty,4)$

- $\left[\frac{x}{n}\right]+\left[\frac{x+1}{n}\right]+\left[\frac{x+2}{n}\right]+\cdots+\left[\frac{x+n-1}{n}\right]=[x], n\in N$

or $[x]+\left[x+\frac{1}{n}\right]+\left[x+\frac{2}{n}\right]+\cdots+\left[x+\frac{n-1}{n}\right]=[nx]$.

Fractional Part Function

$y=f(x)=\{x\}=x-[x]$

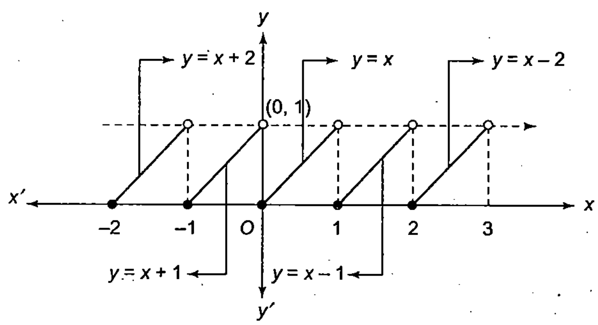


Fig. 1.47

Graph of $y=\{x\}$

Properties

- Domain $\to R$; Range $\to[0,1)$; Period $\to 1$.
- $[x+y]=[x]+[y]$, if $0\le\{x\}+\{y\}<1$
- $[x+y]=[x]+[y]+1$, $1\le\{x\}+\{y\}<2$
- $\{x\}+\{-x\}=0$ if $x\in I$
- $\{x\}+\{-x\}=1$ if $x\notin I$

Example 1.47 Find the domain of

$f(x)=\sqrt{([x]-1)}+\sqrt{(4-[x])}$, (where [ ] represents the greatest integer function).

Sol. Given $f(x)=\sqrt{([x]-1)}+\sqrt{(4-[x])}$

$\therefore f(x)$ is defined when $[x]-1\ge 0$ and $4-[x]\ge 0$

$\therefore 1\le[x]\le 4$ or $1\le x<5$.

Hence, the domain of $f(x)=D_f=[1,5)$.

Example 1.48 Find the domain and range of $f(x)=\sin^{-1}[x]$ (where [ ] represents the greatest integer function).

Sol. $f(x)=\sin^{-1}[x]$ is defined if $-1\le[x]\le 1$

$\Rightarrow[x]=-1,0,1$

$\Rightarrow x\in[-1,2)$

$\Rightarrow$ Range is $\{\sin^{-1}(-1),\sin^{-1}0,\sin^{-1}1\}\equiv\{-\pi/2,0,\pi/2\}$.

Example 1.49 Find the domain and range of $f(x)=\log\{x\}$, where { } represents the fractional part function).

Sol. We know that $0\le\{x\}<1\ \forall\ x\in R$

Now when $\{x\}=0$, $\log\{x\}$ is not defined. So x cannot be integer. Hence, the domain is $R-I$.

Now for $0<\{x\}<1$, $-\infty<\log\{x\}<0\Rightarrow$ Range is $(-\infty,0)$

Example 1.50 Find the range of $f(x)=[\sin\{x\}]$ where {.} represents the fractional part function, [.] represents greatest the integer function.

Sol. $f(x)=[\sin\{x\}]$

Here, $\{x\}$ can take all its possible values and sine function is defined for all real values.

Hence, $0\le\{x\}<1$

$\Rightarrow 0\le\sin\{x\}<\sin 1$

$\Rightarrow[\sin\{x\}]=0$.

Hence, the range is $\{0\}$.

Example 1.51 Solve $2[x]=x+\{x\}$, where [.] and {.} denote the greatest integer function and fractional part, respectively.

Sol. Given $2[x]=x+\{x\}$

$\Rightarrow 2[x]=[x]+2\{x\}$

$\Rightarrow\{x\}=\frac{[x]}{2}$

$\Rightarrow 0\le\frac{[x]}{2}<1$

$\Rightarrow 0\le[x]<2$

$\Rightarrow[x]=0,1$.

For $[x]=0$, we get $\{x\}=0\Rightarrow x=0$

For $[x]=1$, we get $\{x\}=\frac{1}{2}\Rightarrow x=\frac{3}{2}$.

Example 1.52 Find the range of $f(x)=\frac{x-[x]}{1-[x]+x}$, where [ ] represents the greatest integer function.

Sol. $f(x)=\frac{x-[x]}{1-[x]+x}=\frac{\{x\}}{1+\{x\}}=1-\frac{1}{1+\{x\}}$

Now, $0\le\{x\}<1$

$\Rightarrow 1\le\{x\}+1<2$

$\Rightarrow\frac{1}{2}<\frac{1}{1+\{x\}}\le 1$

$\Rightarrow -1\le-\frac{1}{1+\{x\}}<-\frac{1}{2}$

$\Rightarrow 0\le 1-\frac{1}{1+\{x\}}<\frac{1}{2}$

Example 1.53 Solve the system of equation in x, y, and z satisfying the following equations

$x+[y]+\{z\}=3.1$

$\{x\}+y+[z]=4.3$

$[x]+\{y\}+z=5.4$

(where [.] denotes the greatest integer function and {.} denotes fractional part.)

Sol. Adding all the three equations $2(x + y + z) = 12.8$ or
$x+y+z=6.4$ (1)
Adding first two equations, we get $x + y + z + [y] + \{x\}=7.4$ (2)
Adding 2nd and 3rd equations, we get $x + y + z + [z] + \{y\}=9.7$ (3)
Adding 1st and 4th equations, we get $x + y + z + [x] + \{z\}=8.5$ (4)
From (1) and (2), $[y]+\{x\}=1$
From (1) and (3), $[z]+\{y\}=3.3$
From (1) and (4), $[x]+\{z\}=2.1$
$\Rightarrow [x]=2, [y]=1, [z]=3, \{x\}=0, \{y\}=0.3$ and $\{z\}=0.1$
$\Rightarrow x=2, y=1.3, z=3.1.$

Example 1.54 Solve $x^2-4-[x]=0$ is (where [.] denotes the greatest integer function).

Sol. The best method to solve such system is graphical one.
Given equation is $x^2-4=[x]$
Then, the solutions of the equation are values of x where graph $y=x^2-4$ and $y=[x]$ intersect.

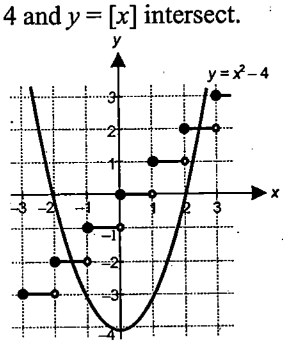


Fig. 1.48

From the graph, it is seen that graph intersects when $x^2-4=2$ and $x^2-4=-2$
$\Rightarrow x^2=6$ or $x^2=2$
$\Rightarrow x=\sqrt{6}$ or $-\sqrt{2}$

Example 1.55 If $f(x)=\begin{cases}[x], & 0\le\{x\}<0.5\\ [x]+1, & 0.5<\{x\}<1\end{cases}$ then prove that $f(x)=-f(-x)$ (where [.] and { } represent the greatest integer function and fractional part function).

Sol. $f(-x)=\begin{cases}[-x], & 0\le\{-x\}<0.5\\ [-x]+1, & 0.5<\{x\}<1\end{cases}$

$$=\begin{cases}[-x], & \{-x\}=0\\ [-x], & 0<\{-x\}<0.5\\ [-x]+1, & 0.5<\{-x\}<1\end{cases}$$

$$=\begin{cases}-[x], & \{x\}=0\\ -1-[x], & 0<1-\{x\}<0.5\\ -1-[x]+1, & 0.5<1-\{x\}<1\end{cases}$$

$$=\begin{cases}-[x], & \{x\}=0\\ -1-[x], & 0.5<\{x\}<1\\ -[x], & 0<\{x\}<0.5\end{cases}$$

$$=\begin{cases}-[x], & 0\le\{x\}<0.5\\ -1-[x], & 0.5<\{x\}<1\end{cases}$$

$$=\begin{cases}-[x], & 0\le\{x\}<0.5\\ 1+[x], & 0.5<\{x\}<1\end{cases}=-f(x)$$

Concept Application Exercise 1.7

In the following questions:
([x] and {x} represent the greatest integer function and fractional part function, respectively).

1. Solve $[x]^2-5[x]+6=0.$
2. If $y=3[x]+1=4[x-1]-10$, then find the value of $[x+2y]$.
3. Find the domain of
   a. $f(x)=\dfrac{1}{\sqrt{x-[x]}}$ **b.** $f(x)=\dfrac{1}{\log[x]}$ **c.** $f(x)=\log\{x\}.$
4. Find the domain of $f(x)=\dfrac{1}{\sqrt{|[|x|-1]|-5}}$.
5. Find the domain of $f(x)=\dfrac{\sqrt{(1-\sin x)}}{\log_5(1-4x^2)}+\cos^{-1}(1-\{x\}).$
6. Find the range of $f(x)=\cos(\log_e\{x\}).$
7. Find the domain and range of $f(x)=\cos^{-1}\sqrt{\log_{[x]}\left(\dfrac{|x|}{x}\right)}$.
8. Find the range of $f(x)=\log_{[x-1]}\sin x.$
9. Solve: $(x-2)[x]=\{x\}-1$, (where $[x]$ and $\{x\}$ denotes the greatest integer function less than or equal to x and fractional part function respectively).

Signum Function

$y=f(x)=\text{sgn}(x)$

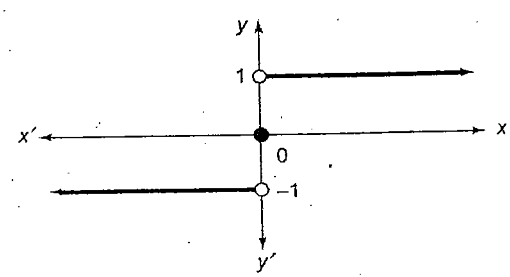


Fig. 1.49

$$\text{sgn}(x)=\begin{cases}\dfrac{|x|}{x}, & x\ne 0\\ 0, & x=0\end{cases}\text{ or}$$

$$\text{sgn}(x)=\begin{cases}-1, & x<0\\ 0, & x=0\\ 1, & x>0\end{cases}$$

Domain $\to R$;
Range $\to \{-1,0,1\}$;

Nature : Many one, odd function

$$\text{In general, } \operatorname{sgn}(f(x)) = \begin{cases} \dfrac{|f(x)|}{f(x)}, & f(x) \neq 0 \\ 0, & f(x) = 0 \end{cases}$$

$$\text{or } \operatorname{sgn}(f(x)) = \begin{cases} -1, & f(x) < 0 \\ 0, & f(x) = 0 \\ 1, & f(x) > 0 \end{cases}$$

Example 1.56 Write the equivalent (piecewise) definition of $f(x) = \operatorname{sgn}(\sin x)$.

Sol. $\operatorname{sgn}(\sin x) = \begin{cases} -1, & \sin x < 0 \\ 0, & \sin x = 0 \\ 1, & \sin x > 0 \end{cases}$

$$= \begin{cases} -1, & x \in ((2n+1)\pi, (2n+2)\pi), n \in Z \\ 0, & x = n\pi, n \in Z \\ 1, & x \in (2n\pi, (2n+1)\pi), n \in Z \end{cases}$$

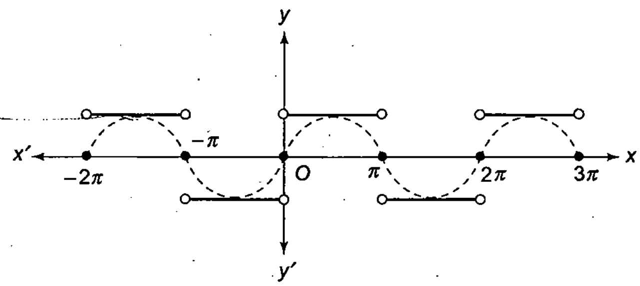


Fig. 1.50

Example 1.57 Find the range of $f(x) = \operatorname{sgn}(x^2 - 2x + 3)$.

Sol. $x^2 - 2x + 3 = (x-1)^2 + 1 > 0 \ \forall \ x \in R.$
$\Rightarrow f(x) = \operatorname{sgn}(x^2 - 2x + 3) = 1$
Hence, the range is $\{1\}$.

Functions of the Form $f(x) = \max.\{g_1(x), g_2(x), \ldots, g_n(x)\}$ or $f(x) = \min.\{g_1(x), g_2(x), \ldots, g_n(x)\}$

Let's consider the function $f(x) = \max.\{x, x^2\}$

To write the equivalent definition of the function, first draw the graph of $y = x$ and $y = x^2$.

Now from the graph, we can see that

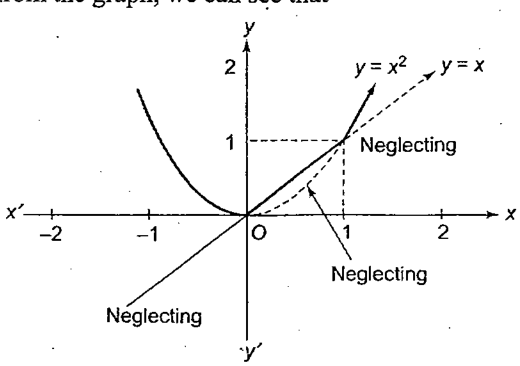


Fig. 1.51

For $x \in (-\infty, 0)$, the graph of $y = x^2$ lies above the graph of $y = x$, or $x^2 > x$.

For $x \in (0, 1)$, the graph of $y = x$ lies above the graph of $y = x^2$ or $x > x^2$.

For $x \in (1, \infty)$, the graph of $y = x^2$ lies above the graph of $y = x$ or $x^2 > x$.

$$\text{Hence, we have } f(x) = \begin{cases} x^2, & x < 0 \\ x, & 0 \le x \le 1 \\ x^2, & x > 0 \end{cases}$$

$$\text{For } f(x) = \min.\{x, x^2\}, \text{ we have } f(x) = \begin{cases} x, & x < 0 \\ x^2, & 0 \le x \le 1 \\ x, & x > 0 \end{cases}$$

Example 1.58 If $f: R \to R, g: R \to R$ be the two given functions, then prove that $2 \min \{f(x) - g(x), 0\} = f(x) - g(x) - |g(x) - f(x)|$.

Sol. $h(x) = 2 \min \{f(x) - g(x), 0\}$

$$= \begin{cases} 0 & f(x) > g(x) \\ 2(f(x) - g(x)), & f(x) \le g(x) \end{cases}$$

$$= \begin{cases} f(x) - g(x) - |f(x) - g(x)|, & f(x) > g(x) \\ f(x) - g(x) - |f(x) - g(x)|, & f(x) \le g(x) \end{cases}$$

$\therefore \quad h(x) = f(x) - g(x) - |g(x) - f(x)|.$

Example 1.59 Draw the graph of the function $f(x) = \max\{\sin x, \cos 2x\}$, $x \in [0, 2\pi]$. Write the equivalent definition of $f(x)$ and find the range of the function.

Sol. $\sin x = \cos 2x$
$\Rightarrow \sin x = 1 - 2\sin^2 x$
$\Rightarrow 2\sin^2 x + \sin x - 1 = 0$
$\Rightarrow (2\sin x - 1)(\sin x + 1) = 0$
$\Rightarrow \sin x = 1/2$ or $\sin x = -1$
$\Rightarrow x = \pi/6, 5\pi/6$ or $x = \pi$

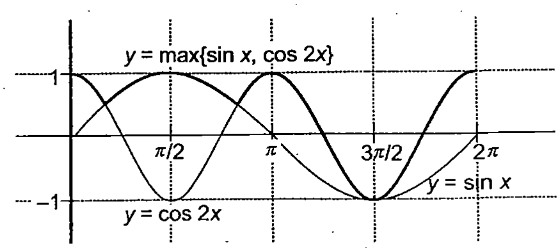


Fig. 1.52

$$\text{From the graph } f(x) = \begin{cases} \cos 2x, 0 \le x < \dfrac{\pi}{6} \\ \sin x, \dfrac{\pi}{6} \le x < \dfrac{5\pi}{6} \\ \cos 2x, \dfrac{5\pi}{6} < x \le 2\pi \end{cases}$$

Also range of the function is $[-1, 1]$.

Concept Application Exercise 1.8

1. Consider the function $f(x) = \max.\ \{1, |x-1|, \min\{4, |3x-1|\}\}\ \forall\ x \in R$, then find the value of $f(3)$.
2. Find the equivalent definition of $f(x) = \max.\ \{x^2, (1-x)^2, 2x(1-x)\}$ where $0 \le x \le 1$.
3. Write the equivalent definition and draw the graphs of the following function.
   a. $f(x) = \text{sgn}\ (\log_e |x|)$
   b. $f(x) = \text{sgn}\ (x^3 - x)$

DIFFERENT TYPES OF MAPPINGS(FUNCTIONS)

One-One and Many-One Functions

If each element in the domain of a function has a distinct image in the co-domain, the function is said to be one-one. One-one functions are also called **injective** functions.

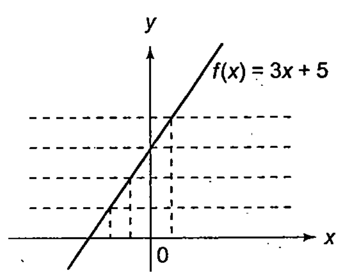
For example, $f: R \to R$ given by $f(x) = 3x + 5$ is one-one.

On the other hand, if there are at least two elements in the domain whose images are the same, the function is known as many-one.

For example, $f: R \to R$ given by $f(x) = x^2 + 1$ is many-one.

Note that a function will be either one-one or many-one.

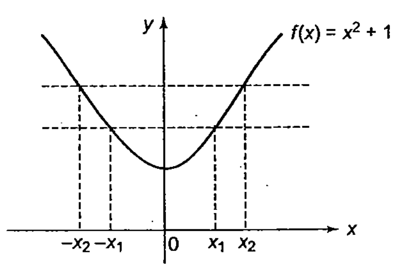
Lines drawn parallel to the x-axis from the each corresponding image point should intersect the graph of $y = f(x)$ at one (and only one) point if $f(x)$ is one-one and there will be at least one line parallel to x-axis that will cut the graph at least at two different points if $f(x)$ is many-one and vice versa.



Graph of $f(x) = 3x + 5$ (one-one function)

Fig. 1.53

Note that a many-one function can be made one-one by restricting the domain of the original function.



Graph of $f(x) = x^2 + 1$ (many-one function)

Fig. 1.54

Methods to Determine One-One and Many-One

a. Let $x_1, x_2 \in$ domain of f and if $x_1 \ne x_2 \Rightarrow f(x_1) \ne f(x_2)$ for every x_1, x_2 in the domain, then f is one-one else many-one.

b. Conversely, if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ for every x_1, x_2 in the domain, then f is one-one else many-one.

c. If the function is entirely increasing or decreasing in the domain, then f is one-one else many-one.

d. Any continuous function $f(x)$ that has at least one local maxima or local minima is many-one.

e. All even functions are many-one.

f. All polynomials of even degree defined in R have at least one local maxima or minima and hence are many-one in the domain R. Polynomials of odd degree can be one-one or many-one.

g. If f is a rational function, then $f(x_1) = f(x_2)$ will always be satisfied when $x_1 = x_2$ in the domain. Hence, we can write $f(x_1) - f(x_2) = (x_1 - x_2)\, g(x_1, x_2)$ where $g(x_1, x_2)$ is some function in x_1 and x_2. Now, if $g(x_1, x_2) = 0$ gives some solution which is different from $x_1 = x_2$ and lies in the domain, then f is many-one else one-one.

h. Draw the graph of $y = f(x)$ and determine whether $f(x)$ is one-one or many-one.

Example 1.60 Let $f: R \to R$ where $f(x) = \dfrac{x^2+4x+7}{x^2+x+1}$. Is $f(x)$ one-one?

Sol. Let $f(x_1) = f(x_2)$

$$\Rightarrow \frac{x_1^2+4x_1+7}{x_1^2+x_1+1} = \frac{x_2^2+4x_2+7}{x_2^2+x_2+1}$$

$\Rightarrow (x_1 - x_2)(2x_1 + 2x_2 + x_1x_2 + 1) = 0$

One solution is obviously $x_1 = x_2$

Let us consider $2x_1 + 2x_1 + x_1x_2 + 1 = 0$

Here, we have got a relation between x_1 and x_2 and for each value of x_1 in the domain we get a corresponding value of x_2 which may or may not be same as x_1. Let us check this out:

If $x_1 = 0$, we get $x_2 = -1/2 \ne x_1$ and both lie in the domain of f. Hence, we have two different values $x_1 = 0$ and $x_2 = -1/2$ for which $f(x)$ has the same value.

That is, $f(0) = f(-1/2) = 7$ and hence f is many-one.

Example 1.61 If $f : X \to [1, \infty)$ be a function defined as $f(x) = 1 + 3x^3$. Find the super set of all the sets X such that $f(x)$ is one-one.

Sol. Note that $f(x) \ge 1$

$\Rightarrow 1 + 3x^3 \ge 1$

$\Rightarrow x^3 \ge 0$

$\Rightarrow x \in [0, \infty)$.

Moreover, for $x_1, x_2 \in [0, \infty), x_1 \neq x_2$

$\Rightarrow 1+3x_1^3 \neq 1+3x_2^3$

$\Rightarrow f(x_1) \neq f(x_2)$

Thus, $f: [1, \infty)$ is one-one for $x \in [0, \infty)$.

Onto and Into Functions

Let $f: X \rightarrow Y$ be a function. If each element in the co-domain Y has at least one pre-image in the domain X, that is, for every $y \in Y$ there exists at least one element $x \in X$ such that $f(x) = y$, then f is onto. In other words, the range of $f = Y$ for onto functions.

On the other hand, if there exists at least one element in the co-domain Y which is not an image of any element in the domain X, then f is into.

Onto function is also called *surjective function* and a function which is both one-one and onto is called *bijective function*.

For example, $f: R \rightarrow R$ where $f(x) = \sin x$ is into.

$f: R \rightarrow R$ where $f(x) = ax^3 + b$ is onto where $a \neq 0, b \in R$.

Note that a function will be either onto or into.

Methods to Determine Whether a Function is Onto or Into

a. If range = co-domain, then f is onto. If range is a proper subset of co-domain, then f is into.
b. Solve $f(x) = y$ for x, say $x = g(y)$.

Now if $g(y)$ is defined for each $y \in$ co-domain and $g(y) \in$ domain of f for all $y \in$ co-domain, then $f(x)$ is onto. If this requirement is not met by at least one value of y in the co-domain, then $f(x)$ is into.

Remark

a. An into function can be made onto by redefining the co-domain as the range of the original function.
b. Any polynomial function $f: R \rightarrow R$ is onto if degree is odd; into if degree of f is even.

Number of Functions (Mappings)

Consider set A has n different elements and set B has r different elements and function $f: A \rightarrow B$

Description	Equivalent to number of ways in which n different balls can be distributed among r persons if	Number of functions
1. Total number of functions	Any one can get any number of objects	r^n
2. Total number of one-to-one function	Each gets exactly 1 object or permutation of n different objects taken r at a time	$\begin{cases} {}^{r}C_n \cdot n!, & r \geq n \\ 0, & r < n \end{cases}$
3. Total number of many-one functions	At least one gets more than one ball	$\begin{cases} r^n - {}^{n}C_n \cdot n!, & r \geq n \\ r^n, & r < n \end{cases}$
4. Total number of onto functions	Each gets at least one ball	$\begin{cases} r^n - {}^{r}C_1(r-1)^n + {}^{r}C_2(r-2)^n - {}^{r}C_3(r-3)^n + \cdots, & r < n \\ r!, & r = n \\ 0, & r > n \end{cases}$
5. Total number of into function		$\begin{cases} {}^{r}C_1(r-1)^n - {}^{n}C_2(r-2)^n + {}^{r}C_3(r-3)^n - \cdots, & r \leq n \\ r^n, & r > n \end{cases}$
6. Total number of constant functions	All the balls are received by any one person	r

Example 1.62 Let $f: R \rightarrow R$ where $f(x) = \sin x$. Show that f is into.

Sol. Since the co-domain of f is the set R, whereas the range of f is the interval $[-1, 1]$, hence f is into.

Can you make it onto?

The answer is 'yes', if you redefine the co-domain.

Let f be defined from R to another set $Y = [-1, 1]$, i.e., $f: R \rightarrow Y$ where $f(x) = \sin x$, then f is onto as range $f(x) = [-1, 1] = Y$.

Example 1.63 Let $f: N \rightarrow Z$ be a function defined as $f(x) = x - 1000$. Show that f is an into function.

Sol. Let $f(x) = y = x - 1000$

$\Rightarrow x = y + 1000 = g(y)$ (say)

Here, $g(y)$ is defined for each $y \in I$, but $g(y) \notin N$ for $y \leq -1000$. Hence, f is into.

Example 1.64 Let $A = \{x: -1 \le x \le 1\} = B$ be a mapping $f: A \to B$.

Then, match the following columns:

Column I (Function)	Column II (Type of mapping)
p. $f(x)=\|x\|$	a. one-one
q. $f(x)=x\|x\|$	b. many-one
r. $f(x)=x3$	c. onto
s. $f(x)=[x]$ where $[\cdot]$ represents greatest integer function	d. into
t. $f(x)=\sin\dfrac{\pi x}{2}$	

p−(b, d.) q−(a, c.) r−(a, c.) s−(b, d.) t.−(a, c.)

Sol. p. $f(x)=|x|$

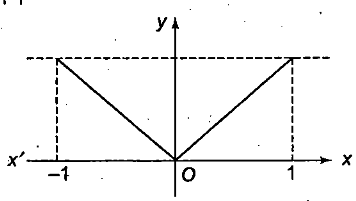


Fig. 1.55

The graph shows that $f(x)$ is many-one, as the straight line parallel to x-axis and cuts at two points. Here, the range for $f(x) \in [0, 1]$ which is clearly a subset of co-domain, i.e., $[0, 1] \subset [-1, 1]$. Thus, into.

Hence, function is many-one-into, therefore neither injective nor surjective.

q. $f(x)=x|x|=\begin{cases} x^2, & x \ge 0 \\ -x^2, & x < 0 \end{cases}$

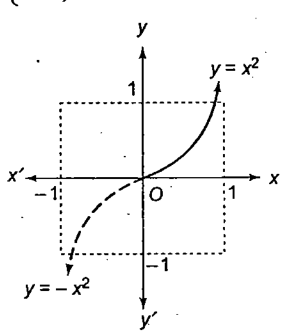


Fig. 1.56

The graph shows that $f(x)$ is one-one, as the straight line parallel to x-axis cuts only at one point.

Here, the range $f(x) \in [-1, 1]$.

Thus, range = co-domain.

Hence, onto.

Therefore, $f(x)$ is one-one and onto or (bijective).

r. $f(x)=x^3$,

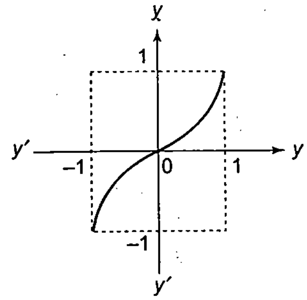


Fig. 1.57

The graph shows that $f(x)$ is one-one onto (i.e., bijective) (as explained in the above example).

s. $f(x)=[x]$,

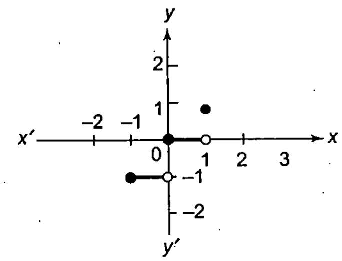


Fig. 1.58

The graph shows that $f(x)$ is many-one, as the straight line parallel to x-axis meets at more than one point.

Here, the range $f(x) \in \{-1, 0, 1\}$, which shows into as the range $\subset$ co-domain.

Hence, many-one-into.

t. $f(x)=\sin\dfrac{\pi x}{2}$

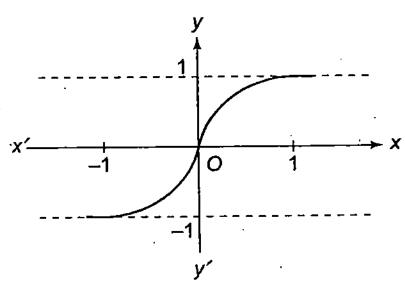


Fig. 1.59

The graph shows that $f(x)$ is one-one and onto as range = co-domain.

Therefore, $f(x)$ is bijective.

Example 1.65 Show $f: R \to R$ defined by $f(x)=(x-1)(x-2)(x-3)$ is surjective but not injective.

Sol.

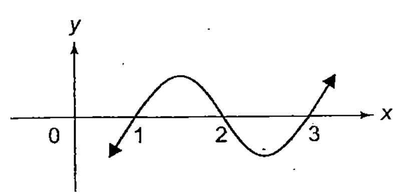


Fig. 1.60

Graphically, $y=f(x)=(x-1)(x-2)(x-3)$, which is clearly many-one and onto.

Example 1.66 If the function $f: R \to A$ given by $f(x)=\dfrac{x^2}{x^2+1}$ is surjection, then find A.

Sol. The domain of $f(x)$ is all real numbers.
Since, $f: R \to A$ is surjective, therefore A must be the range of $f(x)$.
Let $f(x)=y$

$$\Rightarrow \quad y=\frac{x^2}{x^2+1}$$

$$\Rightarrow \quad x^2y+y=x^2$$

$$\Rightarrow \quad x=\sqrt{\frac{y}{1-y}} \text{ exists, if } \frac{y}{1-y}\ge 0$$

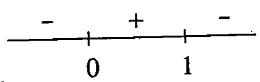


Fig. 1.61

$\Rightarrow \quad 0 \le y < 1$. Hence, $A \in [0, 1)$.

Example 1.67 If $f: R \to R$ be a function such that $f(x)=x^3+x^2+3x+\sin x$. Then identify the type of function.

Sol. $f(x)=x^3+x^2+3x+\sin x$
$\Rightarrow f'(x)$
$=3x^2+2x+3+\cos x$

$$=3\left[\left(x+\frac{1}{3}\right)^2+\frac{8}{9}\right]-(-\cos x)>0 \text{ as } 3\left[\left(x+\frac{1}{3}\right)^2+\frac{8}{9}\right]_{\min}$$

$=\dfrac{8}{3}$ and $-\cos x$ has a maximum value 1.
$\Rightarrow f(x)$ is strictly increasing and hence one-one.
Also $\lim_{x\to\infty} f(x)=\infty$ and $\lim_{x\to-\infty} f(x)=-\infty$. Thus, the range of $f(x)$ is R, hence onto.

Example 1.68 If $f: R \to R, f(x)=\begin{cases} x\,|x|-4, & x\in Q \\ x\,|x|-\sqrt{3}, & x\notin Q\end{cases}$, then identify the type of function.

Sol. $f(2)=f(3^{1/4}) \Rightarrow$ many-to-one function
and $f(x)\ne\sqrt{3}\ \forall\, x\in R \Rightarrow$ into function.

Concept Application Exercise 1.9

1. Which of the following functions from Z to itself are bijections?
   a. $f(x)=x^3$
   b. $f(x)=x+2$
   c. $f(x)=2x+1$
   d. $f(x)=x^2+x$
2. If $f: N\to Z\ f(n)=\begin{cases}\dfrac{n-1}{2}, & \text{when } n \text{ is odd}\\ -\dfrac{n}{2}, & \text{when } n \text{ is even}\end{cases}$ identify the type of function.
3. If $f: R\to R$ given by $f(x)=\dfrac{x^2-4}{x^2+1}$, identify the type of function.
4. If $f: R\to S$, defined by $f(x)=\sin x-\sqrt{3}\cos x+1$, is onto, then find the set S.
5. Let $f: (-1, 1)\to B$ be a function defined by $f(x)=\tan^{-1}\dfrac{2x}{1-x^2}$, then f is both one-one and onto when B is the interval.
   a. $\left[0, \dfrac{\pi}{2}\right)$
   b. $\left(0, \dfrac{\pi}{2}\right)$
   c. $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$
   d. $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$
6. Let $g: R\to\left(0, \dfrac{\pi}{3}\right]$ is defined by $g(x)=\cos^{-1}\left(\dfrac{x^2-k}{1+x^2}\right)$. Then find the possible values of 'k' for which g is subjective function.

EVEN AND ODD FUNCTIONS

Even Function

A function $y=f(x)$ is said to be an even function if $f(-x)=f(x)\ \forall\, x\in D_f$.
Graph of an even function $y=f(x)$ is symmetrical about the y-axis, i.e., if point (x, y) lies on the graph then $(-x, y)$ also lies on the graph.

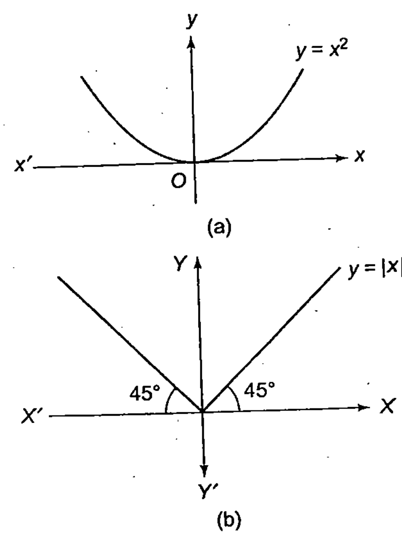


Fig. 1.62

Odd Function

A function $y=f(x)$ is said to be an odd function if $f(-x)=-f(x)\ \forall\, x\in D_f$.
Graph of an odd function $y=f(x)$ is symmetrical in opposite quadrants, i.e., if point (x, y) lies on the graph then $(-x, -y)$ also lies on the graph.

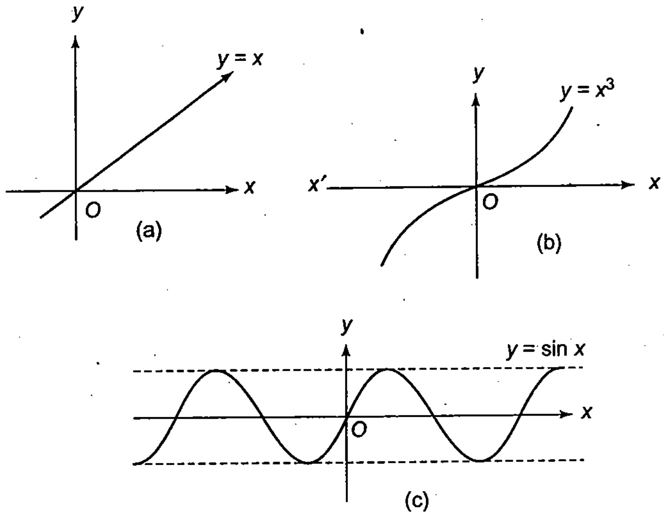


Fig. 1.63

Properties of Odd and Even Functions

- Sometimes, it is easy to prove that $f(x) - f(-x) = 0$ for even functions and $f(x) + f(-x) = 0$ for odd functions.
- A function can be either even or odd or neither.
- Function (not necessarily even or odd) can be expressed as a sum of an even and an odd function, i.e.,

$$f(x) = \left(\frac{f(x)+f(-x)}{2}\right) + \left(\frac{f(x)-f(-x)}{2}\right)$$

Let $h(x) = \left(\frac{f(x)+f(-x)}{2}\right)$ and $g(x) = \left(\frac{f(x)-f(-x)}{2}\right)$. It can now easily be shown that $h(x)$ is even and $g(x)$ is odd.

- The first derivative of an even function is an odd function and vice versa.
- If $x = 0 \in$ domain of f, then for odd function $f(x)$ which is continuous at $x = 0$, $f(0) = 0$, i.e., if for a function, $f(0) \neq 0$, then that function cannot be odd. It follows that for a differentiable even function $f'(0) = 0$, i.e., if for a differentiable function $f'(0) \neq 0$ then the function f cannot be even.
- $f(x) = 0$ is the only function which is defined on the entire number line is even and odd at the same time.
- Every even function $y = f(x)$ is many-one $\forall\, x \in D_f$.

$f(x)$	$g(x)$	$f(x) + g(x)$	$f(x) - g(x)$	$f(x)g(x)$	$f(x)/g(x)$	fog(x)
Even	Even	Even	Even	Even	Even	Even
Even	Odd	Neither even nor odd	Neither even nor odd	Odd	Odd	Even
Odd	Even	Neither even nor odd	Neither even nor odd	Odd	Odd	Even
Odd	Odd	Odd	Odd	Even	Even	Odd

Example 1.69 Which of the following functions is (are) even, odd or neither

a. $f(x) = x^2 \sin x$.

b. $f(x) = \sqrt{1+x+x^2} - \sqrt{1-x+x^2}$.

c. $f(x) = \log\left(\frac{1-x}{1+x}\right)$.

d. $f(x) = \log\left(x+\sqrt{1+x^2}\right)$.

e. $f(x) = \sin x - \cos x$.

f. $f(x) = \frac{e^x + e^{-x}}{2}$.

Sol. **a.** $f(-x) = (-x)^2 \sin(-x) = -x^2 \sin x = -f(x)$, hence $f(x)$ is odd.

b. $f(-x) = \sqrt{1+(-x)+(-x)^2} - \sqrt{1-(-x)+(-x)^2}$

$= \sqrt{1-x+x^2} - \sqrt{1+x+x^2}$

$= -f(x)$, hence $f(x)$ is odd.

c. $f(-x) = \log\left(\frac{1-(-x)}{1+(-x)}\right) = \log\left(\frac{1+x}{1-x}\right)$

$= -f(x)$, hence $f(x)$ is odd.

d. $f(-x) = \log\left(-x+\sqrt{1+(-x)^2}\right)$

$$= \log\left(\frac{(-x+\sqrt{1+x^2})(x+\sqrt{1+x^2})}{(x+\sqrt{1+x^2})}\right)$$

$$= \log\left(\frac{1}{x+\sqrt{1+x^2}}\right) = -f(x), \text{ hence } f(x) \text{ is odd.}$$

e. $f(-x) = \sin(-x) - \cos(-x) = -\sin x - \cos x$. Hence $f(x)$ is neither even nor odd.

f. $f(-x) = \frac{e^{-x} + e^{-(-x)}}{2} = \frac{e^{-x} + e^x}{2} = f(x)$, hence $f(x)$ is even.

Example 1.70 Prove that $f(x)$ given by $f(x+y) = f(x) + f(y)$ $\forall\, x \in R$ is an odd function.

Sol. Given $f(x+y) = f(x) + f(y)$ $\forall\, x \in R$ (1)

Replace y by $-x$, we have $f(x-x) = f(x) + f(-x)$

$\Rightarrow f(x) + f(-x) = f(0)$ (2)

Now put $x = y = 0$ in (1), we have $f(0+0) = f(0) + f(0)$

$\Rightarrow f(0) = 0$.

Then from (2), $f(x)+f(-x)=0$. Hence, $f(x)$ is an odd function.

Extension of Domain

Let a function be defined on a certain domain which is entirely non-negative (or non-positive). The domain of $f(x)$ can be extended to the set $X=\{-x : x \in \text{domain of } f(x)\}$ in two ways:

a. **Even extension:** The even extension is obtained by defining a new function $f(-x)$ for $x \in X$, such that $f(-x) = f(x)$.

b. **Odd extension:** The odd extension is obtained by defining a new function $f(-x)$ for $x \in X$, such that $f(-x) = -f(x)$.

Example 1.71 If $f(x)=\begin{cases} x^3+x^2 & \text{for } 0 \le x \le 2 \\ x+2 & \text{for } 2 < x \le 4 \end{cases}$. Then find the even and odd extension of $f(x)$.

Sol. For even extension $f(x)=f(-x)$

$$f(x)=f(-x)=\begin{cases} (-x)^3+(-x)^2, & 0 \le -x \le 2 \\ -x+2, & 2<-x\le 4 \end{cases}$$

$$=\begin{cases} -x+2, & -4 \le x < -2 \\ -x^3+x^2, & -2 \le x \le 0 \end{cases}$$

The odd extension of $f(x)$ is as follows:

$$h(x)=\begin{cases} x-2, & -4 \le x < -2 \\ x^3-x^2, & -2 \le x \le 0. \end{cases}$$

Example 1.72 Let the function $f(x) = x^2 + x + \sin x - \cos x + \log(1 + |x|)$ be defined on the interval [0, 1]. Define functions $g(x)$ and $h(x)$ in [–1, 0] satisfying $g(-x) = -f(x)$ and $h(-x) = f(x)$ $\forall\, x \in [0, 1]$.

Sol. Clearly, $g(x)$ is the odd extension of the function $f(x)$ and $h(x)$ is the even extension.

Since x^2, $\cos x$, $\log(1+|x|)$ are even functions and x, $\sin x$ are odd functions.

$g(x)=-x^2+x+\sin x+\cos x-\log(1+|x|)$

and $h(x)=x^2-x-\sin x-\cos x+\log(1+|x|)$

Clearly, this function satisfies the restriction of the problem.

Concept Application Exercise 1.10

Identify the following functions whether odd or even or neither

1. $f(x)=(g(x)-g(-x))^3$
2. $f(x)=\log\left(\dfrac{x^4+x^2+1}{x^2+x+1}\right)$
3. $f(x)=xg(x)g(-x)+\tan(\sin x)$
4. $f(x)=\cos|x|+\left[\left|\dfrac{\sin x}{2}\right|\right]$

where [.] denotes the greatest integer function.

5. $f(x)=\log\left(x+\sqrt{x^2+1}\right)$
6. $f(x)=\begin{cases} x|x|, & x \le -1 \\ [x+1]+[1-x], & -1<x<1, \\ -x|x|, & x \ge 1 \end{cases}$

where [ ] represents the greatest integer function.

PERIODIC FUNCTIONS

A function $f: X \to Y$ is said to be a periodic function if there exists a positive real number T such that $f(x+T) = f(x)$, for all $x \in X$. The least of all such positive numbers T is called the principal period or simply period of f. All periodic functions can be analyzed over an interval of one period within the domain as the same pattern shall be repetitive over the entire domain.

In other words, a function is said to be periodic function if its each value is repeated after a definite interval.

Here, the least positive value of T is called the fundamental period of the function. Clearly, $f(x) = f(x+T) = f(x+2T) = f(x+3T) = \ldots$

Important Facts about Periodic Functions

- If $f(x)$ is periodic with period T, then $af(x \pm b) \pm c$ where $a, b, c \in R$ $(a \ne 0)$ is also periodic with period T.
- If $f(x)$ is periodic with period T, then $f(ax+b)$, where $a, b, \in R$ $(a \ne 0)$ is also period with period $\dfrac{T}{|a|}$.

Proof: Consider $a > 0$

Let $f(x+T)=f(x)$ and $f[a(x+T')+b]=f(ax+b)$

$\Rightarrow f(ax+b+aT')=f(ax+b)$

$\Rightarrow f(y+aT')=f(y)=f(y+T)$

$\Rightarrow T=|aT'| \Rightarrow T'=T/|a|$ ($\because$ period is always positive).

- Let $f(x)$ has period $p=m/n$ ($m, n \in N$ and co-prime) and $g(x)$ has period $q=r/s$ ($r, s \in N$ and co-prime) and let t be the LCM of p and q, i.e., $t=\dfrac{\text{LCM of }(m, r)}{\text{HCF of }(n, s)}$.

  Then t will be the period of $f+g$, provided there does not exist a positive number k $(<t)$ for which $f(x+k)+g(x+k)=f(x)+g(x)$, else k will be the period. The same rule is applicable for any other algebraic combination of $f(x)$ and $g(x)$.

  LCM of p and q exists if p and q are rational quantities. If p and q are irrational, then LCM of p and q does not exist unless they have same irrational surd. LCM of rational and irrational is not possible.
- $\sin^n x$, $\cos^n x$, $\operatorname{cosec}^n x$ and $\sec^n x$ have period 2π if n is odd and π if n is even.
- $\tan^n x$ and $\cot^n x$ have period π whether n is odd or even.
- A constant function is periodic but does not have a well-defined period.

- If g is periodic, then fog will always be a periodic function. Period of fog may or may not be the period of g.
- A continuous periodic function is bounded.
- If $f(x)$, $g(x)$ are periodic functions with periods T_1, T_2, respectively, then, we have $h(x) = f(x) + g(x)$ has period as
  a. LCM of $\{T_1, T_2\}$; if $f(x)$ and $g(x)$ cannot be interchanged by adding a least positive number less than the LCM of $\{T_1, T_2\}$.
  b. k; if $f(x)$ and $g(x)$ can be interchanged by adding a least positive number k ($k <$ LCM of $\{T_1, T_2\}$).

  For example, consider the function $f(x) = |\sin x| + |\cos x|$, $|\sin x|$ and $|\cos x|$ have period π, hence according to the rule of LCM period of $f(x)$ is π.

  But $f\left(x+\frac{\pi}{2}\right) = \left|\sin\left(x+\frac{\pi}{2}\right)\right| + \left|\cos\left(x+\frac{\pi}{2}\right)\right|$

  $= |\cos x| + |\sin x|$. Hence, period of $f(x)$ is $\pi/2$.

Example 1.73 Find the periods (if periodic) of the following functions, [.] denotes the greatest integer function.

a. $f(x) = e^{\log(\sin x)} + \tan^3 x - \operatorname{cosec}(3x-5)$

b. $f(x) = x - [x-b], b \in R$

c. $f(x) = \dfrac{|\sin x + \cos x|}{|\sin x| + |\cos x|}$

d. $f(x) = \tan\dfrac{\pi}{2}[x]$

Sol. a. $f(x) = e^{\log(\sin x)} + \tan^3 x - \operatorname{cosec}(3x-5)$

Period of $e^{\log(\sin x)}$ is 2π, $\tan^3 x$ is π, $\operatorname{cosec}(3x-5)$ is $\dfrac{2\pi}{3}$

$\therefore$ period = L.C.M of $\left\{2\pi, \pi, \dfrac{2\pi}{3}\right\} = 2\pi.$

b. $f(x) = x - [x-b] = b + \{x-b\}$, ($\because$ period of $\{.\}$ is 1)
$\Rightarrow f(x)$ has period 1.

c. $f(x) = \dfrac{|\sin x + \cos x|}{|\sin x| + |\cos x|}$

Since period of $|\sin x + \cos x| = \pi$ and period of $|\sin x| + |\cos x|$ is $\pi/2$.

Hence, period of $f(x) =$ L.C.M. of $\left\{\dfrac{\pi}{2}, \pi\right\} = \pi$

d. $f(x) = \tan\dfrac{\pi}{2}[x] \Rightarrow \tan\dfrac{\pi}{2}[x+T] = \tan\dfrac{\pi}{2}[x]$

$\Rightarrow \dfrac{\pi}{2}[x+T] = n\pi + \dfrac{\pi}{2}[x]$

$\Rightarrow$ Period = 2 (least positive value).

Example 1.74 Find the period if $f(x) = \sin x + \{x\}$, where $\{x\}$ is the fractional part of x.

Sol. Here $\sin x$ is periodic with period 2π, $\{x\}$ is periodic with 1. LCM of 2π (irrational) and 1 (rational) does not exist. Thus, $f(x)$ is not periodic.

Example 1.75 If $f(x) = \sin x + \cos ax$ is a periodic function, show that a is a rational number.

Sol. Period of $\sin x = 2\pi = \dfrac{2\pi}{1}$ and period of $\cos ax = \dfrac{2\pi}{|a|}$

$\therefore$ period of $\sin x + \cos ax$ = LCM of $\dfrac{2\pi}{1}$ and $\dfrac{2\pi}{|a|}$

$= \dfrac{\text{LCM of } 2\pi \text{ and } 2\pi}{\text{HCF of 1 and } a} = \dfrac{2\pi}{\lambda}$ where λ is the HCF of 1 and a.

Since λ is the HCF of 1 and a. $\dfrac{1}{\lambda}$ and $\dfrac{|a|}{\lambda}$ should be both integers

Suppose $\dfrac{1}{\lambda} = p$ and $\dfrac{|a|}{\lambda} = q$, then $\dfrac{\frac{|a|}{\lambda}}{\frac{1}{\lambda}} = \dfrac{q}{p}$, where $p, q \in Z$

i.e., $|a| = \dfrac{q}{p}$.

Hence, a is the rational number.

Example 1.76 Discuss whether the function $f(x) = \sin(\cos x + x)$ is perodic or not, if yes then what is its period.

Sol. Clearly, $f(x+2\pi) = \sin(\cos(2\pi + x) + 2\pi + x)$
$= \sin(2\pi + (x + \cos x))$
$= \sin(x + \cos x)$

Hence, period is 2π.

Example 1.77 Find the period of $\cos(\cos x) + \cos(\sin x)$.

Sol. Clearly, the domain of the function is R
Let $f(x) = f(x+T)$, for all x
$\Rightarrow f(0) = f(T)$
$\Rightarrow \cos 1 + 1 = \cos(\cos T) + \cos(\sin T)$

Clearly, $T = \dfrac{\pi}{2}$ satisfies the equation, hence period is $\dfrac{\pi}{2}$.

Example 1.78 Find the period of the function satisfying the relation $f(x) + f(x+3) = 0\ \forall x \in R$.

Sol. Given $f(x) + f(x+3) = 0$ (1)
Replace x by $x+3$,
We have $f(x+3) + f(x+6) = 0$ (2)
From (1) and (2), $f(x) = f(x+6)$.
Hence, the function has period 6.

Concept Application Exercise 1.11

1. Match the column

Column I (Function)	Column II (Period)
p. $f(x) = \sin^3 x + \cos^4 x$	a. $\pi/2$
q. $f(x) = \cos^4 x + \sin^4 x$	b. π
r. $f(x) = \sin^3 x + \cos^3 x$	c. 2π
s. $f(x) = \cos^4 x - \sin^4 x$	

2. Which of the following functions is not periodic?
   a. $|\sin 3x| + \sin^2 x$ b. $\cos \sqrt{x} + \cos^2 x$
   c. $\cos 4x + \tan^2 x$ d. $\cos 2x + \sin x$
3. Let $[x]$ denote the greatest integer less than or equal to x. If the function $f(x) = \tan(\sqrt{[n]}\, x)$ has period $\frac{\pi}{3}$, then find the values of n.
4. Find the period of
   a. $\frac{|\sin 4x| + |\cos 4x|}{|\sin 4x - \cos 4x| + |\sin 4x + \cos 4x|}$
   b. $f(x) = \sin \frac{\pi x}{n!} - \cos \frac{\pi x}{(n+1)!}$
   c. $f(x) = \sin x + \tan \frac{x}{2} + \sin \frac{x}{2^2} + \tan \frac{x}{2^3} + \cdots + \sin \frac{x}{2^{n-1}} + \tan \frac{x}{2^n}$
5. If $f(x) = \lambda|\sin x| + \lambda^2 |\cos x| + g(\lambda)$ has the period equal to $\pi/2$, then find the value of λ.
6. If $f(x)$ satisfying the relation $f(x) + f(x+4) = f(x+2) + f(x+6)$ for all x, then prove that $f(x)$ is periodic and find its period.

COMPOSITE FUNCTION

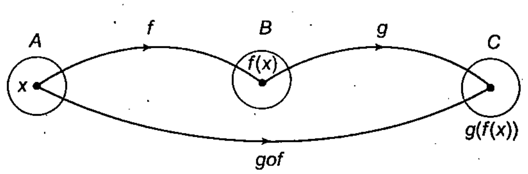


Fig. 1.64

Let A, B and C be three non-empty sets.

Let $f: A \to B$ and $g: B \to C$ be two functions, then $gof: A \to C$. This function is called composition of f and g, given by

$$gof(x) = g(f(x)) \ \forall\, x \in A.$$

Thus, the image of every $x \in A$ under the function gof is the g-image of the f-image of x.

The gof is defined only if $\forall\, x \in A, f(x)$ is an element of the domain of g so that we can take its g-image.

The range of f must be a subset of the domain of g in gof.

Properties of Composite Functions

a. The composition of functions is not commutative in general, i.e., $fog \neq gof$.

b. The composition of functions is associative i.e., if $h: A \to B, g: B \to C$ and $f: C \to D$ be three functions, then $(fog)oh = fo(goh)$.

c. The composition of any function with the identity function is the function itself, i.e., $f: A \to B$ then $foI_A = I_B of = f$ where I_A and I_B are the identity functions of A and B, respectively.

d. If $f: A \to B$ and $g: B \to C$ are one-one, then $gof: A \to C$ is also one-one.

Proof:
Suppose $gof(x_1) = gof(x_2)$
$\Rightarrow g(f(x_1)) = g(f(x_2))$
$\Rightarrow f(x_1) = f(x_2)$, as g is one-one
$\Rightarrow x_1 = x_2$, as f is one-one
Hence, gof is one-one.

e. If $f: A \to B$ and $g: B \to C$ are onto, then $gof: A \to C$ is also onto.

Proof:
Given an arbitrary element $z \in C$, there exists a pre-image y of z under g such that $g(y) = z$, since g is onto. Further, for $y \in B$, there exists an element x in A with $f(x) = y$, since f is onto.
Therefore, $gof(x) = g(f(x)) = g(y) = z$, showing that gof is onto.

f. If $gof(x)$ is one-one, then $f(x)$ is necessarily one-one but $g(x)$ may not be one-one.
Consider the function $f(x)$ and $g(x)$ as shown in the following figure.

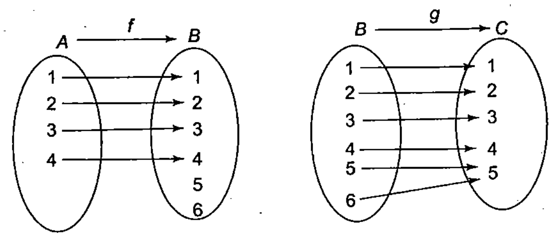


Fig. 1.65

Here f is one-one, but g is many-one. But $g(f(x))$: {(1, 1), (2, 2), (3, 3), (4, 4)} is one-one.

g. If $gof(x)$ is onto, then $g(x)$ is necessarily onto but $f(x)$ may not be onto.

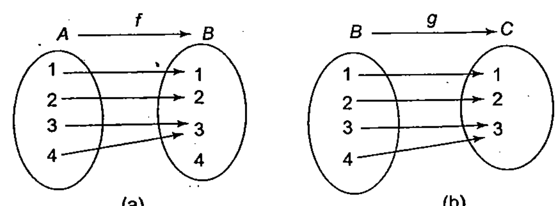


Fig. 1.66

Here, f is into and g is onto. But $gof(x)$: {(1, 1), (2, 2), (3, 3), (4, 3)} is onto.
Thus, it can be verified in general that gof is one-one implies that f is one-one. Similarly, gof is onto implies that g is onto.

Example 1.79 Let f: {2, 3, 4, 5} → {3, 4, 5, 9} and g: {3, 4, 5, 9} → {7, 11, 15} be functions defined as $f(2) = 3$, $f(3) = 4$, $f(4) = f(5) = 5$ and $g(3) = g(4) = 7$ and $g(5) = g(9) = 11$. Find gof.

Sol. We have $gof(2) = g(f(2)) = g(3) = 7$, $gof(3) = g(f(3)) = g(4) = 7$, $gof(4) = g(f(4)) = g(5) = 11$ and $gof(5) = g(5) = 11$.

Example 1.80 Let $f(x)$ and $g(x)$ be bijective functions where f: {a, b, c, d} → {1, 2, 3, 4} and g: {3, 4, 5, 6} → {w, x, y, z}, respectively. Then, find the number of elements in the range set of $g(f(x))$.

Sol. The range of $f(x)$ for which $g(f(x))$ is defined is $\{3, 4\}$.
Hence, the domain of $g\{f(x)\}$ has two elements.
$\therefore$ The range of $g(f(x))$ also has two elements.

Example 1.81 Let $f(x) = ax + b$ and $g(x) = cx + d$, $a \neq 0$, $c \neq 0$. Assume $a = 1, b = 2$. If $(fog)(x) = (gof)(x)$ for all x, what can you say about c and d?

Sol. $(fog)(x) = f(g(x)) = a(cx + d) + b$
and $(gof)(x) = g(f(x)) = c(ax + b) + d$
Given that $(fog)(x) = (gof)(x)$ and at $a = 1, b = 2$
$\Rightarrow \quad cx + d + 2 = cx + 2c + d \Rightarrow c = 1$ and d is arbitrary.

Example 1.82 Suppose that $g(x) = 1 + \sqrt{x}$ and $f(g(x)) = 3 + 2\sqrt{x} + x$, then find the function $f(x)$.

Sol. $g(x) = 1 + \sqrt{x}$ and $f(g(x)) = 3 + 2\sqrt{x} + x$ (1)
$\Rightarrow \quad f(1 + \sqrt{x}) = 3 + 2\sqrt{x} + x$
Put $1 + \sqrt{x} = y \Rightarrow x = (y - 1)^2$
Then $f(y) = 3 + 2(y - 1) + (y - 1)^2 = 2 + y^2$.
Therefore, $f(x) = 2 + x^2$.

Example 1.83 The function $f(x)$ is defined in [0, 1]. Find the domain of $f(\tan x)$.

Sol. Here, $f(x)$ is defined in [0, 1].
$\Rightarrow \quad x \in [0, 1]$, i.e., the only value of x that we can substitute lies between [0, 1]
For $f(\tan x)$ to be defined, we must have $0 \leq \tan x \leq 1$ [as x replaced by $\tan x$]

i.e., $n\pi \leq x \leq n\pi + \dfrac{\pi}{4}, n \in Z$ [in general]

Thus, the domain for $f(\tan x) \in \left[n\pi, n\pi + \dfrac{\pi}{4}\right], n \in Z$.

Example 1.84 $f(x) = \begin{cases} x + 1, & x < 0 \\ x^2, & x \geq 0 \end{cases}$ and $g(x) = \begin{cases} x^3, & x < 1 \\ 2x - 1, & x \geq 1 \end{cases}$, then find $f(g(x))$ and find its domain and range.

Sol. $f(g(x)) = \begin{cases} g(x) + 1, & g(x) < 0 \\ (g(x))^2, & g(x) \geq 0 \end{cases}$

$$= \begin{cases} x^3 + 1, & x^3 < 0,\ x < 1 \\ 2x - 1 + 1, & 2x - 1 < 0,\ x \geq 1 \\ (x^3)^2, & x^3 \geq 0,\ x < 1 \\ (2x - 1)^2, & 2x - 1 \geq 0,\ x \geq 1 \end{cases} = \begin{cases} x^3 + 1, & x < 0 \\ x^6, & 0 \leq x < 1 \\ (2x - 1)^2, & x \geq 1 \end{cases}$$

For $x < 0, x^3 + 1 \in (-\infty, 1)$
For $0 \leq x < 1, x^6 \in [0, 1)$
For $x \geq 1, (2x - 1)^2 \in [1, \infty)$
Hence, the range is R and function is many-one.

Concept Application Exercise 1.12

1. If f be the greatest integer function and g be the modulus function, then find the value of $(gof)\left(-\dfrac{5}{3}\right) - (fog)\left(-\dfrac{5}{3}\right)$.
2. Let $f(x) = \begin{cases} 1 + |x|, & x < -1 \\ [x], & x \geq -1 \end{cases}$, where [.] denotes the greatest integer function. Then find the value of $f\{f(-2.3)\}$.
3. If $f(x) = \log\left[\dfrac{1 + x}{1 - x}\right]$, then prove that $f\left[\dfrac{2x}{1 + x^2}\right] = 2f(x)$.
4. If the domain for $y = f(x)$ is $[-3, 2]$, find the domain of $g(x) = f(|[x]|)$. where [ ] denotes the greatest integer function).
5. Let f be a function defined on [0, 2], then the domain of function $g(x) = f(9x^2 - 1)$.
6. $f(x) = \begin{cases} \log_e x, & 0 < x < 1 \\ x^2 - 1, & x \geq 1 \end{cases}$ and $g(x) = \begin{cases} x + 1, & x < 2 \\ x^2 - 1, & x \geq 2 \end{cases}$, then find $g(f(x))$.
7. Let $f(x) = \tan x$, $g(f(x)) = f\left(x - \dfrac{\pi}{4}\right)$ wher $f(x)$ and $g(x)$ are real-valued functions. Prove that $f(g(x)) = \tan\left(\dfrac{x - 1}{x + 1}\right)$.
8. A function f has domain $[-1, 2]$ and range $[0, 1]$. Find the domain and range of the function g defined by $g(x) = 1 - f(x + 1)$.

INVERSE FUNCTIONS

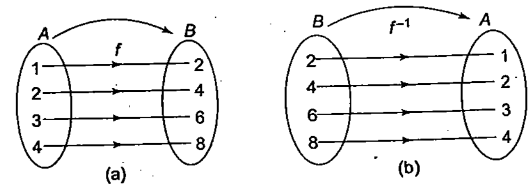


Fig. 1.67

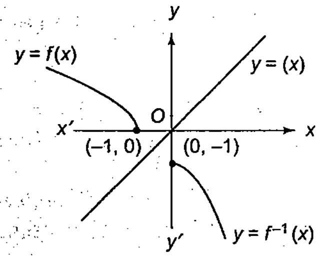
If $f: A \to B$ be a function defined by $y = f(x)$ such that f is both one-one and onto, then there exists a unique function $g: B \to A$ such that for each $y \in B$, $g(y) = x$ if and only if $y = f(x)$. The function g so defined is called the inverse of f and denoted by f^{-1}. Also if g is the inverse of f, then f is the inverse of g and the two functions f and g are said to be inverses of each other.

The condition for the existence of inverse of a function is that the function must be one-one and onto. Whenever an inverse function is defined, the range of the original function becomes the domain of the inverse function and the domain of the original function becomes the range of the inverse function.

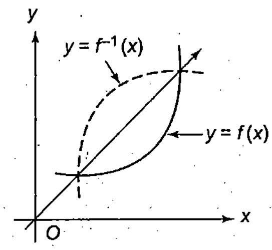
Properties of Inverse Functions

- The inverse of bijective function is unique and bijective.
- Let $f: A \to B$ be a function such that f is bijective and $g: B \to A$ is inverse of f, then $fog = I_B$ = identity function of set B. Then $gof = I_A$ = identity function of set A.
- If $fog = gof$ then either $f^{-1} = g$ or $g^{-1} = f$ and $fog(x) = gof(x) = x$.

- If f and g are two bijective functions such that $f: A \to B$ and $g: B \to C$, then $gof: A \to C$ is bijective.
  Also $(gof)^{-1} = f^{-1}og^{-1}$.
- Graphs of $y = f(x)$ and $y = f^{-1}(x)$ are symmetrical about $y = x$ line and intersect on line $y = x$ or $f(x) = f^{-1}(x) = x$ whenever graphs intersect.



(a)



(b)

Fig. 1.68

But in the case of the function $f(x) = \begin{cases} x+4, & x \in [1, 2] \\ -x+7, & x \in [5, 6] \end{cases}$,

$$f^{-1}(x) = \begin{cases} x-4, & x \in [5, 6] \\ 7-x, & x \in [1, 2] \end{cases}$$

$y = f(x)$ and $y = f^{-1}(x)$ intersect at (3/2, 11/2) and (11/2, 3/2) which do not lie on the line $y = x$.

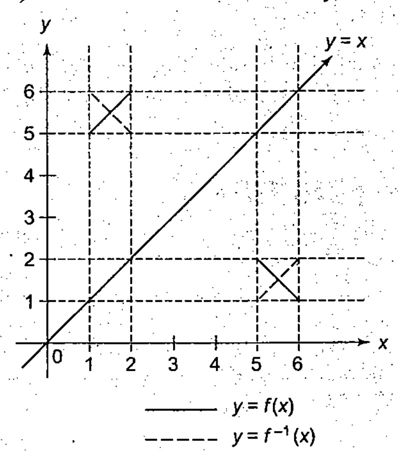


Fig. 1.69

Example 1.87 Which of the following functions has inverse function?

a. $f: Z \to Z$ defined by $f(x) = x + 2$
b. $f: Z \to Z$ defined by $f(x) = 2x$
c. $f: Z \to Z$ defined by $f(x) = x$
d. $f: Z \to Z$ defined by $f(x) = |x|$

Sol. Functions in options **a** and **c** are both one-one and have range Z, i.e., onto, hence invertible. $f: Z \to Z$ defined by $f(x) = 2x$ is one-one but has only even integers in the range, hence not onto.
$f: Z \to Z$ defined by $f(x) = |x|$ is many-one and has range $N \cup \{0\}$.
Thus, both the functions are not invertible.

Example 1.88 Let $A = R - \{3\}$, $B = R - \{1\}$ and let $f: A \to B$ defined by $f(x) = \dfrac{x-2}{x-3}$. Is f invertible? Explain.

Sol. Let $x_1, x_2 \in AC$
and let $f(x_1) = f(x_2)$

$$\Rightarrow \frac{x_1 - 2}{x_1 - 3} = \frac{x_2 - 2}{x_2 - 3}$$

$$\Rightarrow x_1x_2 - 3x_1 - 2x_2 + 6 = x_1x_2 - 3x_2 - 3x_2 - 2x_1 + 6$$

$$\Rightarrow x_1 = x_2$$

$\Rightarrow$ f is one-one.

To find whether f is onto or not, first let us find the range of f.

Let $y = f(x) = \dfrac{x-2}{x-3}$

$$\Rightarrow xy - 3y = x - 2$$

$$\Rightarrow x(y-1) = 3y - 2$$

$$\Rightarrow x = \frac{3y-2}{y-1}$$

x is defined if $y \neq 1$, i.e., the range of f is $R - \{1\}$ which is also the co-domain of f.

Also, for no value of y, x can be 3, i.e., if we put

$$3 = x = \frac{3y-2}{y-1}$$

$\Rightarrow$ $3y - 3 = 3y - 2 \Rightarrow -3 = -2$ not possible. Hence, f is onto.

Example 1.89 Let $f: R \to R$ be defined by $f(x) = (e^x - e^{-x})/2$. Is $f(x)$ invertible? If so, find its inverse.

Sol. Let us check for inevitability of $f(x)$

(a) One-one

Let $x_1, x_2 \in R$ and $x_1 < x_2$

$$\Rightarrow e^{x_1} < e^{x_2} \quad (\because e > 1) \qquad (1)$$

Also $x_1 < x_2 \Rightarrow -x_2 < -x_1$

$$\Rightarrow e^{-x_2} < e^{-x_1} \quad (\because e > 1) \qquad (2)$$

$$(1) + (2) \Rightarrow \frac{1}{2}(e^{x_1} - e^{-x_1}) < \frac{1}{2}(e^{x_2} - e^{-x_2}) \Rightarrow f(x_1) < f(x_2).$$

i.e. f is one-one.

(b) Onto

As $x \to \infty, f(x) \to \infty$

Similarly, as $x \to -\infty, f(x) \to -\infty$, i.e., $-\infty < f(x) < \infty$ so long as $x \in (-\infty, \infty)$.

Hence, the range of f is same as the set R. Therefore, $f(x)$ is onto.

Since $f(x)$ is both one-one and onto, $f(x)$ is invertible.

(c) To find f^{-1}

$y = f(x) = (e^x - e^{-x})/2$

$\Rightarrow e^x - e^{-x} = 2y$

$\Rightarrow e^{2x} - 2ye^x - 1 = 0$

$\Rightarrow e^x = \dfrac{2y \pm \sqrt{4y^2+4}}{2} = y \pm \sqrt{y^2+1}$

$\Rightarrow e^x = y + \sqrt{y^2+1}$

(as $y - \sqrt{y^2+1} < 0$ for all y and e^x is always positive)

$\Rightarrow x = \log_e\left(y + \sqrt{y^2+1}\right)$

$\Rightarrow f^{-1}(x) = \log_e\left(x + \sqrt{x^2+1}\right)$

Example 1.90 If $f(x) = (ax^2+b)^3$, then find the function g such that $f(g(x)) = g(f(x))$.

Sol. $f(g(x)) = g(f(x))$

$f(x) = (ax^2+b)^3$

If $g(x) = f^{-1}(x)$

$y = (ax^2+b)^3 \Rightarrow \sqrt{\dfrac{y^{1/3}-b}{a}} = x$

$\Rightarrow g(x) = \sqrt{\dfrac{x^{1/3}-b}{a}}$

Example 1.91 If $f(x) = 3x - 2$ and $(gof)^{-1}(x) = x - 2$, then find the function $g(x)$.

Sol. $f(x) = 3x - 2$

$\Rightarrow f^{-1}(x) = \dfrac{x+2}{3}$

Now $(gof)^{-1}(x) = x - 2$

$\Rightarrow f^{-1}og^{-1}(x) = x - 2$

$\Rightarrow f^{-1}(g^{-1}(x)) = x - 2$

$\Rightarrow \dfrac{g^{-1}(x)+2}{3} = x - 2$

$\Rightarrow g^{-1}(x) = 3x - 8$

$\Rightarrow g(x) = \dfrac{x+8}{3}$

Example 1.92 Find the inverse of $f(x) = \begin{cases} x, & x<1 \\ x^2, & 1 \le x \le 4 \\ 8\sqrt{x}, & x>4 \end{cases}$

Sol. Given $f(x) = \begin{cases} x, & x<1 \\ x^2, & 1 \le x \le 4 \\ 8\sqrt{x}, & x>4 \end{cases}$

Let $f(x) = y$

$\Rightarrow x = f^{-1}(y)$ (1)

$\therefore x = \begin{cases} y, & y<1 \\ \sqrt{y}, & 1 \le \sqrt{y} \le 4 \\ y^2/64, & y^2/64 > 4 \end{cases} = \begin{cases} y, & y<1 \\ \sqrt{y}, & 1 \le y \le 16 \\ y^2/64, & y>16 \end{cases}$

$f^{-1}(y) = \begin{cases} y, & y<1 \\ \sqrt{y}, & 1 \le y \le 16 \\ y^2/64, & y>16 \end{cases}$ [From (1)]

Hence, $f^{-1}(x) = \begin{cases} x, & x<1 \\ \sqrt{x}, & 1 \le x \le 16 \\ y^2/64, & x>16 \end{cases}$

Example 1.93 Solve the equation $x^2 - x + 1 = \dfrac{1}{2} + \sqrt{x - \dfrac{3}{4}}$, where $x \ge \dfrac{3}{4}$.

Sol. $f(x) = x^2 - x + 1$

and $g(x) = \dfrac{1}{2} + \sqrt{x - \dfrac{3}{4}}$

are inverse of one another so $f(x) = g(x)$.

When $f(x) = x$

$\Rightarrow x^2 - x + 1 = x$

$\Rightarrow x = 1$

Concept Application Exercise 1.13

Find the inverse of the following functions:

1. $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} + 2$
2. $f: R \to (-\infty, 1)$ is given by $f(x) = 1 - 2^{-x}$
3. Let $f: (2, 3) \to (0, 1)$ be defined by $f(x) = x - [x]$, where [.] represents greatest integer function.
4. $f: Z \to Z$ be defined by $f(x) = [x+1]$, where [.] denotes the greatest integer function.
5. $f(x) = \begin{cases} x^3 - 1, & x<2 \\ x^2 + 3, & x \ge 2 \end{cases}$
6. $f: [-1, 1] \to [-1, 1]$ defined by $f(x) = x|x|$
7. $f: (-\infty, 1] \to \left[\dfrac{1}{2}, \infty\right)$, where $f(x) = 2^{x(x-2)}$

IDENTICAL FUNCTION

Two functions f and g are said to be identical if

a. The domain of f = the domain of g, i.e., $D_f = D_g$.

b. The range of f = the range of g.

c. $f(x) = g(x)$, $\forall\ x \in D_f$ or $x \in D_g$, e.g., $f(x) = x$ and $g(x) = \sqrt{x^2}$ are not identical functions as $D_f = D_g$ but $R_f = R$, $R_g = [0, \infty)$.

Example 1.94 Find the values of x for which the following functions are identical.

a. $f(x) = x$ and $g(x) = \dfrac{1}{1/x}$

b. $f(x) = \cos x$ and $g(x) = \dfrac{1}{\sqrt{1+\tan^2 x}}$

c. $f(x) = \dfrac{\sqrt{9-x^2}}{\sqrt{x-2}}$ and $g(x) = \sqrt{\dfrac{9-x^2}{x-2}}$

d. $f(x) = \tan^{-1}x + \tan^{-1}\dfrac{1}{x}$ and $g(x) = \sin^{-1}x + \cos^{-1}x$

Sol. a. $f(x) = x$ is defined for all x.

But $g(x) = \dfrac{1}{1/x} = x$ is not defined for $x = 0$ as $1/x$ is not defined at $x = 0$.

Hence, both the functions are identical for $x \in R - \{0\}$.

b. $f(x) = \cos x$ has the domain R and the range $[-1, 1]$.

But $g(x) = \dfrac{1}{\sqrt{1+\tan^2 x}} = \dfrac{1}{\sqrt{\sec^2 x}} = |\cos x|$, has domain $R - \{(2n+1)\pi/2, n \in Z\}$ as $\tan x$ is not defined for $x = (2n+1)\pi/2, n \in Z$.

Also, the range of $g(x) = |\cos x|$ is $[0, 1]$.

Hence, $f(x)$ and $g(x)$ are identical if x lies in 1st and 4th quadrant.

$$\Rightarrow x \in \left(-\frac{\pi}{2} + 2n\pi, \frac{\pi}{2} + 2n\pi\right),\ n \in Z.$$

c. $f(x) = \dfrac{\sqrt{9-x^2}}{\sqrt{x-2}}$ is defined if $9 - x^2 \ge 0$ and $x - 2 > 0$

$\Rightarrow x \in [-3, 3]$ and $x > 2 \Rightarrow x \in (2, 3]$

$g(x) = \sqrt{\dfrac{9-x^2}{x-2}}$ is defined if $\dfrac{9-x^2}{x-2} \ge 0$

$$\Rightarrow \frac{x^2-9}{x-2} \le 0$$

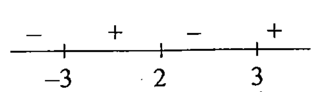


Fig. 1.70

From the sign scheme, $x \in (-\infty, -3] \cup (2, 3]$.

Hence, $f(x)$ and $g(x)$ are identical if $x \in (2, 3]$.

d. $f(x) = \tan^{-1}x + \tan^{-1}\dfrac{1}{x} = \begin{cases} \dfrac{\pi}{2}, & x > 0 \\ -\dfrac{\pi}{2}, & x < 0 \end{cases}$

and $g(x) = \sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}$ for $x \in [-1, 1]$.

Hence, the functions are identical if $x \in (0, 1]$

TRANSFORMATION OF GRAPHS

a. $f(x)$ transforms to $f(x) \pm a$ (where a is +ve)

That is, $f(x) \to f(x) + a$ shift the given graph of $f(x)$ upward through a units.

$f(x) \to f(x) - a$, shift the given graph of $f(x)$ downward through a units.

Graphically, it could be stated as

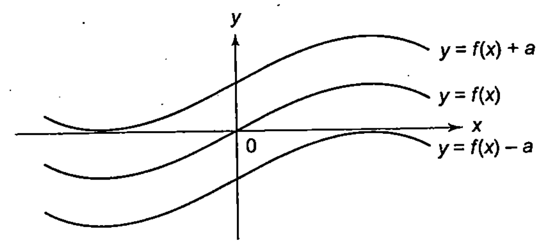


Fig. 1.71

b. $f(x)$ transforms to $f(x - a)$

That is, $f(x) \to f(x - a)$; a is positive. Shift the graph of $f(x)$ through a unit towards right.

That is, $f(x) \to f(x + a)$; a is positive. Shift the graph of $f(x)$ through a units towards left.

Graphically, it could be stated as

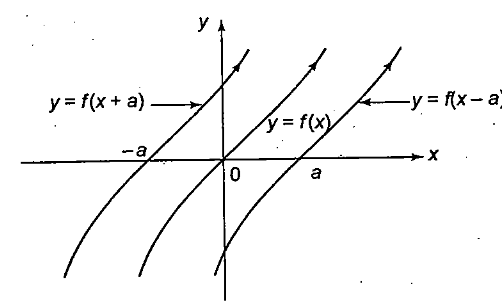


Fig. 1.72

Example 1.95 Plot $y = |x|$, $y = |x - 2|$ and $y = |x + 2|$.

Sol. As discussed $f(x) \to f(x - a)$; shift towards right.

$\Rightarrow y = |x - 2|$ is shifted 2 units towards right.

Also $y = |x + 2|$ is shifted 2 units towards left.

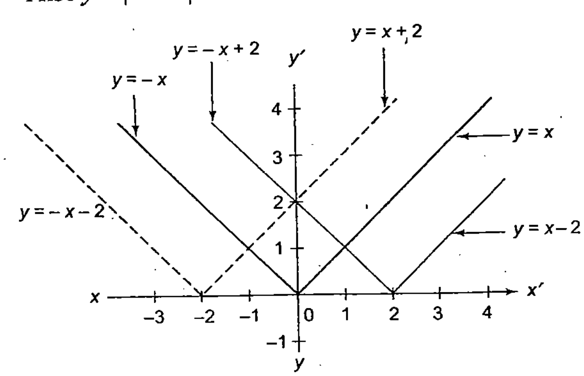


Fig. 1.73

c. $f(x)$ transforms to $f(ax)$

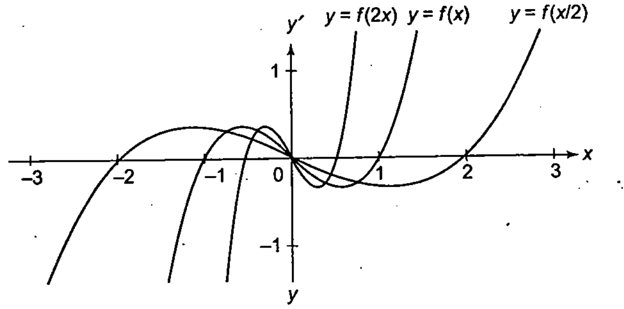


Fig. 1.74

That is, $f(x) \to f(ax); a > 1$.
Shrink (or contract) the graph of $f(x)$ a times along the x-axis.

Again $f(x) \to f\left(\frac{1}{a}x\right)$; $a > 1$, stretch (or expand) the graph of $f(x)$ a times along the x-axis.

Graphically, it could be stated as shown in Fig.1.74.

Example 1.96 Plot $y = \sin x$ and $y = \sin 2x$.

Sol. Here $y = \sin 2x$, shrink (or contract) the graph of $\sin x$ by factor of 2 along the x-axis.

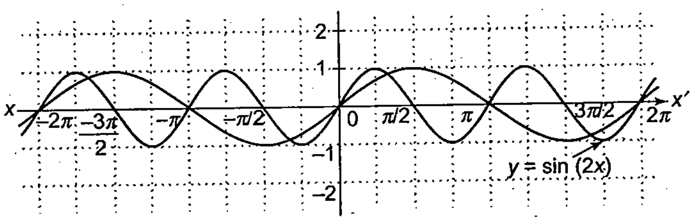


Fig. 1.75

From Fig. 1.75, $\sin x$ is periodic with period 2π and $\sin 2x$ with period π.

Example 1.97 Plot $y = \sin x$ and $y = \sin \frac{x}{2}$.

Sol. Here $y = \sin\left(\frac{x}{2}\right)$; stretch (or expand) the graph of $\sin x$, 2 times along the x-axis.

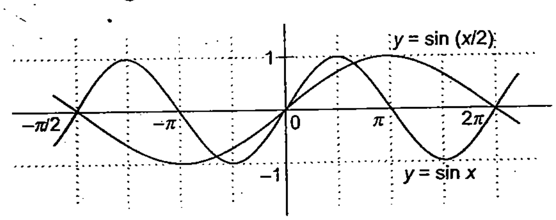


Fig. 1.76

From Fig. 1.76, $\sin x$ is periodic with period 2π and $\sin\left(\frac{x}{2}\right)$ is periodic with period 4π.

d. $f(x)$ transforms to $y = af(x)$

It is clear that the corresponding points (points with same x co-ordinates) would have their ordinates in the ratio of 1 : a.

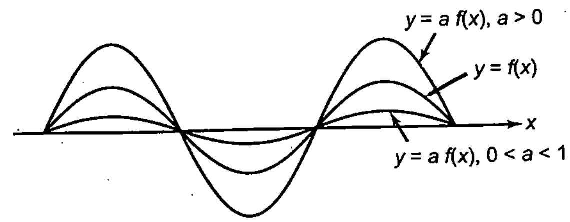


Fig. 1.77

Example 1.98 Consider the function $f(x) = \begin{cases} 2x+3, & x \le 1 \\ -x^2+6, & x > 1 \end{cases}$

Then draw the graph of the function $y = f(x)$, $y = f(|x|)$, $y = |f(x)|$ and $y = |f(|x|)|$.

Sol.

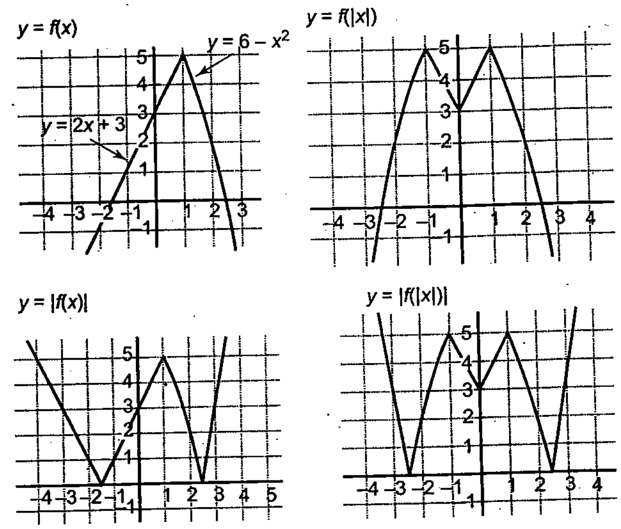


Fig. 1.78

Example 1.99 Plot $y = \sin x$ and $y = 2 \sin x$.

Sol. We know $y = \sin x$ and $f(x) \to a f(x)$

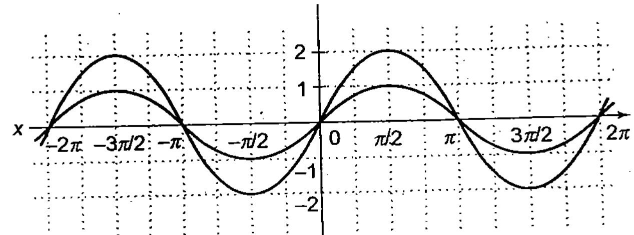


Fig. 1.79

⇒ Stretch the graph of $f(x)$ a times along the y-axis.
$\therefore y = 2 \sin x$.
⇒ Stretch the graph of $\sin x$, 2 times along y-axis.

e. $f(x)$ transforms to $f(-x)$

That is, $f(x) \to f(-x)$
To draw $y = f(-x)$, take the image of the curve $y = f(x)$ in the y-axis as plane mirror.
Or
Turn the graph of $f(x)$ by 180° about the y-axis.
Graphically, it is shown as

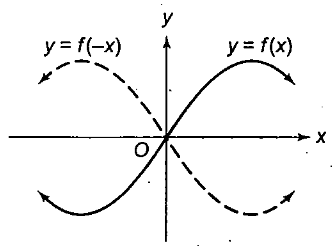


Fig. 1.80

Example 1.100 Plot the curve $y = \log_e(-x)$.

Sol. Here $y = \log_e(-x)$; take mirror image of $y = \log_e x$ about y-axis. Graphically, it is shown as

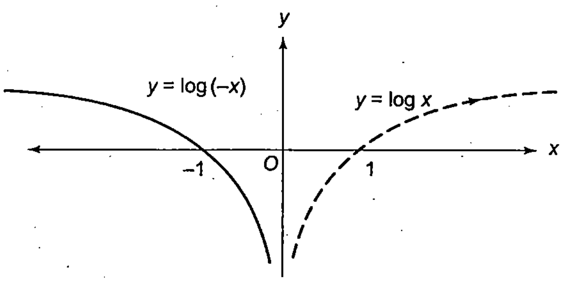


Fig. 1.81

f. $f(x)$ transforms to $-f(x)$

That is, $f(x) \to -f(x)$
To draw $y = -f(x)$, take image of $y = f(x)$ in the x-axis as plane mirror.
Or
Turn the graph of $f(x)$ by 180° about x-axis.

g. $f(x)$ transforms to $-f(-x)$

That is, $f(x) \to -f(-x)$
To draw $y = -f(-x)$, take image of $f(x)$ about y-axis to obtain $f(-x)$ and then the image of $f(-x)$ about x-axis to obtain $-f(-x)$.
$\therefore\ f(x) \to -f(-x)$
$\Rightarrow$ **a.** Image about y-axis
b. Image about x-axis
Graphically, it is shown as

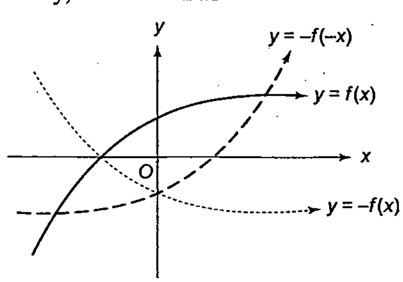


Fig. 1.82

h. $f(x)$ transform to $y = |f(x)|$

$|f(x)| = f(x)$ if $f(x) \geq 0$ and $|f(x)| = -f(x)$ if $f(x) < 0$.
It means that the graph of $f(x)$ and $|f(x)|$ would coincide if $f(x) \geq 0$ and the parts where $f(x) < 0$ would get inverted in the upward direction.
Figure 1.82 would make the procedure clear.

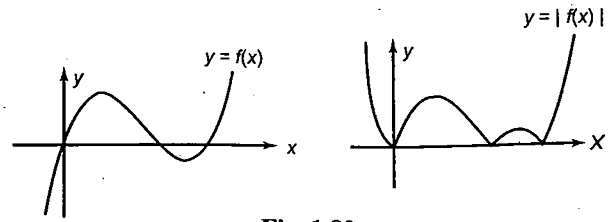


Fig. 1.83

Example 1.101 Draw the graph for $y = |\log x|$.

Sol. To draw graph for $y = |\log x|$, we have to follow two steps:
a. Leave the (+ve) part of $y = \log x$ as it is.
b. Take images of (–ve) part of $y = \log x$, i.e., the part below x-axis in the x-axis as plane mirror. Graphically, it is shown as

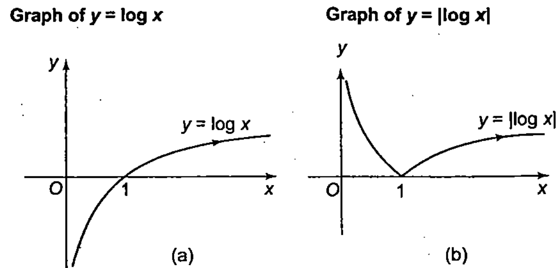


Fig. 1.84

$y = \log_e x$ is differentiable for all $x \in (0, \infty)$ (Fig. 1.84(a))
$y = |\log_e x|$ is clearly differentiable for all $x \in (0, \infty) - \{1\}$ as at $x = 1$ there is a sharp edge (Fig. 1.84(b)).

Example 1.102 Sketch the graph for $y = |\sin x|$.

Sol. Here $y = \sin x$ is known.
$\therefore$ To draw $y = |\sin x|$, we take the mirror image (in x-axis) of the part of the graph of $\sin x$ which lies below x-axis.

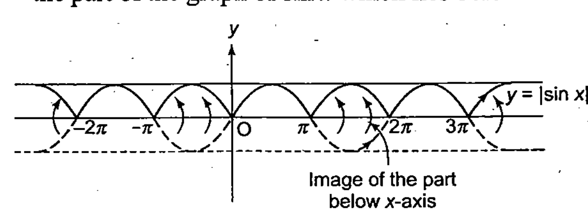


Fig. 1.85

From the above figure, it is clear
$y = |\sin x|$ is differentiable for all $x \in R - \{n\pi, n \in \text{integer}\}$.

i. $f(x)$ transforms to $f(|x|)$

That is, $f(x) \to f(|x|)$
If we know $y = f(x)$, then to plot $y = f(|x|)$, we would follow two steps:
a. Leave the graph lying right side of the y-axis as it is.
b. Take the image of $f(x)$ in the right of y-axis with y-axis as the plane mirror and the graph of $f(x)$ lying left-side of the y-axis (if it exists) is omitted.
Or
Neglect the curve for $x < 0$ and take the images of curves for $x \geq 0$ about y-axis.

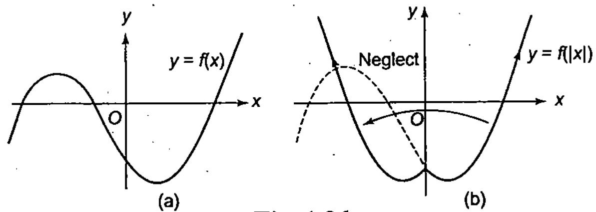


Fig. 1.86

Example 1.103 Sketch the curve $y = \log |x|$.

Sol. As we know, the curve $y = \log x$.

$\therefore y = \log |x|$ could be drawn in two steps:

a. Leave the graph lying right side of y-axis as it is.

b. Take the image of $f(x)$ in the y-axis as plane mirror.

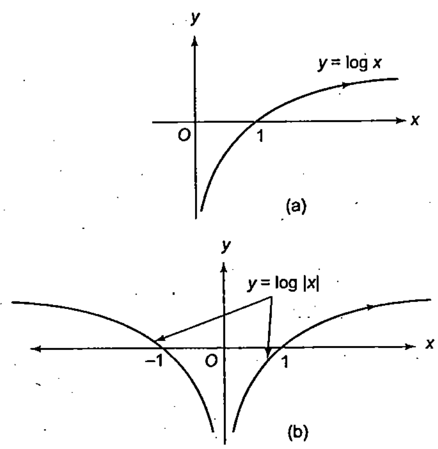


Fig. 1.87

j. Drawing the graph of $|y| = f(x)$ from the known graph of $y = f(x)$

Clearly, $|y| \geq 0$, if $f(x) < 0$, graph of $|y| = f(x)$ would not exist. And if $f(x) \geq 0$, $|y| = f(x)$ would give $y = \pm f(x)$. Hence, the graph of $|y| = f(x)$ would exist only in the regions where $f(x)$ is non-negative and will be reflected about x-axis only in those regions. Regions where $f(x) < 0$ will be neglected.

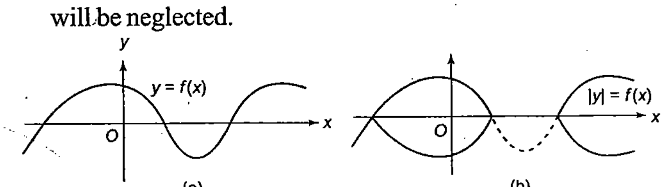


Fig. 1.88

Example 1.104 Sketch the curve $|y| = (x-1)(x-2)$.

Sol.

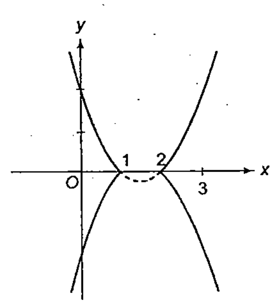


Fig. 1.89

k. Drawing the graph of $y = [f(x)]$ from the known graph of $y = f(x)$

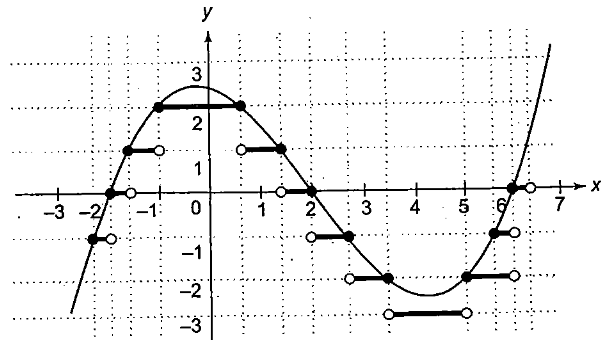


Fig. 1.90

It is clear that if $n \leq f(x) < n+1, n \in I$ then $[f(x)] = n$. Thus, we would draw lines parallel to the x-axis passing through different integral points. Hence, the values of x can be obtained so that $f(x)$ lies between two successive integers.

This procedure can be clearly understood from Fig. 1.90.

Concept Application Exercise 1.14

Draw the graph of the following functions: (1 to 5)

1. $f(x) = \sin |x|$
2. $f(x) = ||x-2|-3|$
3. $|f(x)| = \tan x$
4. $f(x) = |x^2 - 3|x| + 2|$
5. $f(x) = -|x-1|^{1/2}$
6. Find the total number of solutions of $\sin \pi x = |\ln |x||$.
7. Solve $\left|\dfrac{x^2}{x-1}\right| \leq 1$ using the graphical method.
8. Which of the following pair(s) of function have same graphs?

(a) $f(x) = \dfrac{\sec x}{\cos x} - \dfrac{\tan x}{\cot x}$, $g(x) = \dfrac{\cos x}{\sec x} + \dfrac{\sin x}{\operatorname{cosec} x}$

(b) $f(x) = \operatorname{sgn}(x^2 - 6x + 10)$,
$g(x) = \operatorname{sgn}\left(\cos^2 x + \sin^2\left(x + \dfrac{\pi}{3}\right)\right)$, where sgn denotes signum function.

(c) $f(x) = e^{\ln(x^2+3x+3)}$, $g(x) = x^2 + 3x + 3$

(d) $f(x) = \dfrac{\sin x}{\sec x} + \dfrac{\cos x}{\operatorname{cosec} x}$, $g(x) = \dfrac{2\cos^2 x}{\cot x}$

MISCELLANEOUS SOLVED PROBLEMS

1. Let $f: X \to Y$ be a function defined by $f(x) = a \sin\left(x + \dfrac{\pi}{4}\right) + b \cos x + c$. If f is both one-one and onto, find sets X and Y.

Sol. $f(x) = a \sin\left(x + \dfrac{\pi}{4}\right) + b \cos x + c$

$\Rightarrow f(x)=a\left\{\sin x\cos\frac{\pi}{4}+\cos x\sin\frac{\pi}{4}\right\}+b\cos x+c$

$\Rightarrow f(x)=\frac{a}{\sqrt{2}}\sin x+\left(\frac{a}{\sqrt{2}}+b\right)\cos x+c$ (1)

Let $\left(\frac{a}{\sqrt{2}}\right)=r\cos\alpha,\ \left(\frac{a}{\sqrt{2}}+b\right)=r\sin\alpha$

$\Rightarrow f(x)=r\,[\cos\alpha\sin x+\sin\alpha\cos x]+c$

$\Rightarrow f(x)=r\,[\sin(x+\alpha)]+c$

where $r=\sqrt{a^2+\sqrt{2}ab+b^2}$

and $\alpha=\tan^{-1}\left(\frac{a+b\sqrt{2}}{a}\right)$. (2)

For f to be one-one, we must have $-\pi/2\le x+\alpha\le\pi/2$. Thus,

domain $\in\left[\frac{-\pi}{2}-\alpha,\frac{\pi}{2}-\alpha\right]$ and range $\in[c-r,c+r]$.

Or $X=\left[\frac{-\pi}{2}-\alpha,\frac{\pi}{2}-\alpha\right]$ and $Y=[c-r,c+r]$.

2. Find the set of all solutions of the equation $2^{|y|}-|2^{y-1}-1|=2^{y-1}+1$.

Sol. Here, $2^{|y|}-|2^{y-1}-1|=2^{y-1}+1$.

Case I: $y<0$

$\Rightarrow 2^{-y}+(2^{y-1}-1)=2^{y-1}+1;\ \begin{cases}\text{as when } y<0\\ |y|=-y\\ \text{and } |2^{y-1}-1|=-(2^{y-1}-1)\end{cases}$.

$\Rightarrow 2^{-y}=2^1$

Hence, $y=-1$, which is true when $y<0$. (1)

Case II: $0\le y<1$

$\Rightarrow 2^{y}+(2^{y-1}-1)=2^{y-1}+1;\ \begin{cases}\text{as when } 0\le y<1\\ |y|=y\\ \text{and } |2^{y-1}-1|=-(2^{y-1}-1)\end{cases}$

$\Rightarrow 2^y=2$

$\Rightarrow y=1$, which shows no solution as $0\le y<1$. (2)

Case III: $y\ge 1$

$\Rightarrow 2^y-(2^{y-1}-1)=2^{y-1}+1$

$\Rightarrow 2^y=2^{y-1}+2^{y-1};\ \begin{cases}\text{as when } y\ge 0\\ |y|=y\\ \text{and } |2^{y-1}-1|=(2^{y-1}-1)\end{cases}$

$\Rightarrow 2^y=2^y$, which is an identity, therefore it is true $\forall\ y\ge 1$ (3)

Hence, from (1), (2) and (3) the solution of the set is $\{y: y\ge 1\cup y=-1\}$.

3. Let $x\in\left(0,\frac{\pi}{2}\right)$, then find the domain of the function

$$f(x)=\frac{1}{\sqrt{-\log_{\sin x}\tan x}}.$$

Sol. Here $x\in\left(0,\frac{\pi}{2}\right)$

$\Rightarrow\ 0<\sin x<1$ (1)

and we know $\begin{cases}\log_a x<b\ \Rightarrow & x>a^b, & \text{if } 0<a<1\\ & x<a^b, & \text{if } a>1\end{cases}$ (2)

Thus, $f(x)=\frac{1}{\sqrt{-\log_{\sin x}\tan x}}$ exists, if $-\log_{\sin x}(\tan x)>0$

$\Rightarrow\ \log_{\sin x}\tan x<0$

[as inequality sign changes on multiplying by –ve]

$\Rightarrow\ \tan x>(\sin x)^0$ [using (1) and (2)]

$\Rightarrow\ \tan x>1$

$\Rightarrow\ x\in\left(\frac{\pi}{4},\frac{\pi}{2}\right)$. [as $x\in(0,\pi/2)$]

4. Find whether the given function is even or odd function, where $f(x)=\frac{x(\sin x+\tan x)}{\left[\frac{x+\pi}{\pi}\right]-\frac{1}{2}}$, where [ ] denotes the greatest integer function.

Sol. $f(x)=\frac{x(\sin x+\tan x)}{\left[\frac{x+\pi}{\pi}\right]-\frac{1}{2}}=\frac{x(\sin x+\tan x)}{\left[\frac{x}{\pi}\right]+1-\frac{1}{2}}$

$=\frac{x(\sin x+\tan x)}{\left[\frac{x}{\pi}\right]+0.5}$

$\Rightarrow\quad f(-x)=\frac{-x(\sin(-x)+\tan(-x))}{\left[-\frac{x}{\pi}\right]+0.5}$

$\Rightarrow\quad f(-x)=\begin{cases}\dfrac{x(\sin x+\tan x)}{-1-\left[\frac{x}{\pi}\right]+0.5}, & x\ne n\pi\\ 0, & x=n\pi\end{cases}$

Hence, $f(-x)=-\left(\dfrac{x(\sin x+\tan x)}{\left[\frac{x}{\pi}\right]+0.5}\right)$ and $f(-x)=0$.

$f(-x)=-f(x)$. Hence, $f(x)$ is an odd function if $x \neq n\pi$ and $f(x)=0$ if $x=n\pi$ is both even and odd function.

5. Let $f(x)$ be periodic and k be a positive real number such that $f(x+k)+f(x)=0$ for all $x \in R$. Prove that $f(x)$ is a periodic with period $2k$.

Sol. We have $f(x+k)+f(x)=0, \forall x \in R$

$\Rightarrow f(x+k)=-f(x), \forall x \in R$. Put $x=x+k$

$\Rightarrow f(x+2k)=-f(x+k), \forall x \in R$ (as $f(x+k)=-f(x)$)

$\Rightarrow f(x+2k)=f(x), \forall x \in R$

which clearly shows that $f(x)$ is periodic with period $2k$.

6. If $f(x)$ be a polynomial function satisfying $f(x)\, f\left(\frac{1}{x}\right)$ $= f(x)+f\left(\frac{1}{x}\right)$ and $f(4)=65$. Then find $f(6)$.

Sol. Let $f(x)=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_{n-1}x+a_n$

Then, $f(x)\, f\left(\frac{1}{x}\right)=f(x)+f\left(\frac{1}{x}\right)$

$$\Rightarrow (a_0x^n+a_1x^{n-1}+\cdots+a_n)\left(\frac{a_0}{x^n}+\frac{a_1}{x^{n-1}}+\cdots+a_n\right)$$

$$= (a_0x^n+a_1x^{n-1}+\cdots+a_n)+\left(\frac{a_0}{x^n}+\frac{a_1}{x^{n-1}}+\cdots+a_n\right)$$

On comparing the coefficient of x^n, we have $a_0\, a_n=a_0$

$\Rightarrow a_n=1$ (as $a_0 \neq 0$)

On comparing the coefficient of x^{n-1}, we have $a_0a_{n-1}+a_na_1=a_1$.

$\Rightarrow a_0a_{n-1}+a_1=a_1$ (as $a_n=1$)

$\Rightarrow a_0a_{n-1}=0$

$\Rightarrow a_{n-1}=0$ (as $a_0 \neq 0$)

Similarly, $a_{n-1}=a_{n-2}=\cdots=a_1=0$

and $a_0=\pm 1$

$\therefore f(x)=\pm x^n+1, f(4)=\pm 4^n+1$

$\Rightarrow 4^n+1=65$ (as $f(4)=65$)

$\Rightarrow 4^n=64$

$\Rightarrow n=3$.

So, $f(x)=x^3+1$. Hence, $f(6)=6^3+1=217$.

7. Consider a real-valued function $f(x)$ satisfying $2f(xy)=(f(x))^y+(f(y))^x \;\forall\, x,y \in R$ and $f(1)=a$ where $a \neq 1$. Prove that $(a-1)\sum_{i=1}^{n} f(i)=a^{n+1}-a$.

Sol. We have $2f(xy)=(f(x))^y+(f(y))^x$

Replacing y by 1, we get $2f(x)=f(x)+(f(1))^x \Rightarrow f(x)=a^x$

$$\Rightarrow \sum_{i=1}^{n} f(i)=a+a^2+\cdots+a^n=\frac{a^{n+1}-a}{a-1}.$$

$$\Rightarrow (a-1)\sum_{i=1}^{n} f(i)=a^{n+1}-a.$$

8. If $f(x+y+1)=(\sqrt{f(x)}+\sqrt{f(y)})^2$ and $f(0)=1, \forall x,y \in R$. Determine $f(n), n \in N$.

Sol. Given $f(x+y+1)=(\sqrt{f(x)}+\sqrt{f(y)})^2$

Putting $x=y=0$,

then $f(1)=(\sqrt{f(0)}+\sqrt{f(0)})^2=(1+1)^2=2^2$.

Again putting $x=0, y=1$

Then $f(2)=(\sqrt{f(0)}+\sqrt{f(1)})^2=(1+2)^2=3^2$

and for $x=1, y=1$

$f(3)=(\sqrt{f(1)}+\sqrt{f(1)})^2=(2+2)^2=4^2$.

Hence, $f(n)=(n+1)^2$.

9. Check whether the function defined by $f(x+\lambda)=1+\sqrt{2f(x)-f^2(x)}\;\forall\, x \in R$ is periodic or not. If yes, then find its period. ($\lambda>0$)

Sol. For the function to be true, $2f(x)-f^2(x) \geq 0$

$\Rightarrow f(x)\,[f(x)-2] \leq 0 \Rightarrow 0 \leq f(x) \leq 2$ (1)

and from the given function, $f(x+\lambda) \geq 1 \Rightarrow f(x) \geq 1$ (2)

From (1) and (2), we have $1 \leq f(x) \leq 2$

Again, we have $\{f(x+\lambda)-1\}^2=2f(x)-f^2(x)$

$\Rightarrow \{f(x+\lambda)-1\}^2=1+\{2f(x)-f^2(x)-1\}$

$\Rightarrow \{f(x+\lambda)-1\}^2=1-\{f(x)-1\}^2$ (3)

Replacing x by $x+\lambda$, we get

$\{f(x+2\lambda)-1\}^2=1-\{f(x+\lambda)-1\}^2$ (4)

Subtracting (3) from (4), we get

$\{f(x+2\lambda)-1\}^2=\{f(x)-1\}^2$

$\Rightarrow |f(x+2\lambda)-1|=|f(x)-1|$ ($\because 1 \leq f(x) \leq 2$)

$\Rightarrow f$ is periodic with period 2λ.

10. If for all real values of u and v, $2f(u)\cos v=f(u+v)+f(u-v)$, prove that for all real values of x

a. $f(x)+f(-x)=2a\cos x$.

b. $f(\pi-x)+f(-x)=0$.

c. $f(\pi-x)+f(x)=2b\sin x$.

Deduce that $f(x)=a\cos x+b\sin x$, where a, b are arbitrary constants.

Sol. Given $2f(u)\cos v=f(u+v)+f(u-v)$ (1)

Putting $u=0$ and $v=x$ in (1), we get

$f(x)+f(-x)=2f(0)\cos x=2a\cos x$ (2)

a is an arbitrary constant.

Now putting $u=\frac{\pi}{2}-x$ and $v=\frac{\pi}{2}$ in (1), we get

$f(\pi-x)+f(-x)=0$ (3)

Again putting $u=\pi/2$ and $v=\pi/2-x$ in (1), we get

$f(\pi-x)+f(x)=2f(\pi/2)\sin x=2b\sin x$ (4)

b is an arbitrary constant.

Adding (2) and (4), we get

$2f(x)+f(\pi-x)+f(-x)=2a\cos x+2b\sin x$

$\Rightarrow\quad 2f(x)+0=2a\cos x+2b\sin x$ [From (3)]

$\therefore\quad f(x)=a\cos x+b\sin x.$

11. Let $f(x)=\dfrac{9^x}{9^x+3}$. Show $f(x)+f(1-x)=1$, and hence evaluate

$$f\left(\frac{1}{1996}\right)+f\left(\frac{2}{1996}\right)+f\left(\frac{3}{1996}\right)+\cdots+f\left(\frac{1995}{1996}\right).$$

Sol. $f(x)=\dfrac{9^x}{9^x+3}$ (1)

and $f(1-x)=\dfrac{9^{1-x}}{9^{1-x}+3}$

$$\Rightarrow\quad f(1-x)=\frac{\frac{9}{9^x}}{\frac{9}{9^x}+3}=\frac{9}{9+3.9^x}$$

$$\Rightarrow\quad f(1-x)=\frac{3}{(3+9^x)} \qquad (2)$$

Adding (1) and (2), we get $f(x)+f(1-x)$

$$=\frac{9^x}{9^x+3}+\frac{3}{(3+9^x)}=1$$

$$\Rightarrow\quad f(x)+f(1-x)=1 \qquad (3)$$

Now, putting $x=\dfrac{1}{1996},\ \dfrac{2}{1996},\ \dfrac{3}{1996},\ \cdots,\ \dfrac{998}{1996}$ in (3), we get

$$f\left(\frac{1}{1996}\right)+f\left(\frac{1995}{1996}\right)=1,\ f\left(\frac{2}{1996}\right)+f\left(\frac{1994}{1996}\right)=1,$$

$$f\left(\frac{3}{1996}\right)+f\left(\frac{1993}{1996}\right)=1$$

$\vdots$

$$f\left(\frac{997}{1996}\right)+f\left(\frac{999}{1996}\right)=1\ ,f\left(\frac{998}{1996}\right)+f\left(\frac{998}{1996}\right)=1$$

or $f\left(\dfrac{998}{1996}\right)=\dfrac{1}{2}$

Adding all the above expression, we get

$$f\left(\frac{1}{1996}\right)+f\left(\frac{2}{1996}\right)+\cdots+f\left(\frac{1995}{1996}\right)$$

$$=(1+1+1+\cdots+997)+\frac{1}{2}=997+\frac{1}{2}=997.5$$

12. Let $f(x)$ be defined on $[-2, 2]$ and is given by

$$f(x)=\begin{cases}-1, & -2\le x\le 0\\ x-1, & 0<x\le 2\end{cases},\ \text{and } g(x)=f(|x|)+|f(x)|.$$

Then find $g(x)$.

Sol. We have

$$f(x)=\begin{cases}-1, & -2\le x\le 0\\ x-1, & 0<x\le 2\end{cases}$$

$$\Rightarrow\quad f(|x|)=\begin{cases}-1, & -2\le |x|\le 0\\ |x|-1, & 0\le |x|\le 2\end{cases}$$

$$\Rightarrow\quad f(|x|)=|x|-1,\ 0\le |x|\le 2$$

(as $-2\le |x|<0$ is not possible)

$$\Rightarrow\quad f(|x|)=\begin{cases}-x-1, & -2\le x\le 0\\ x-1, & 0<x\le 2\end{cases} \qquad (1)$$

again, $f(x)=\begin{cases}-1, & -2\le x\le 0\\ x-1, & 0\le x\le 2\end{cases}$

$$\Rightarrow\quad |f(x)|=\begin{cases}|-1|, & -2\le x\le 0\\ |x-1|, & 0<x\le 2\end{cases}$$

$$\Rightarrow\quad |f(x)|=\begin{cases}1, & -2\le x\le 0\\ -(x-1), & 0<x\le 1\\ +(x-1), & 1<x\le 2\end{cases} \qquad (2)$$

$\therefore\ g(x)=f(|x|)+|f(x)|$ can be expressed as

$$g(x)=\begin{cases}(-x-1)+1, & -2\le x\le 0\\ (x-1)+(1-x), & 0\le x\le 1\\ (x-1)+(x-1), & 1\le x\le 2\end{cases}$$

[using (1) and (2)]

$$\Rightarrow\quad g(x)=\begin{cases}-x, & -2\le x\le 0\\ 0, & 0<x\le 1\\ 2(x-1), & 1<x\le 2\end{cases}$$

13. For what integral value of n, is 3π period of the function $\cos(nx)\sin\left(\dfrac{5x}{n}\right)$?

Sol. Let $f(x)=\cos nx \sin\left(\frac{5x}{n}\right)$

$f(x)$ is periodic

$\Rightarrow$ $f(x+\lambda)=f(x)$ where λ is period.

$$\Rightarrow \cos(nx+n\lambda)\sin\left(\frac{5x+5\lambda}{n}\right)=\cos nx \sin\left(\frac{5x}{n}\right)$$

at $x=0$, $\cos n\lambda \sin\left(\frac{5\lambda}{n}\right)=0$

If $\cos n\lambda=0$

$$\Rightarrow n\lambda=r\pi+\frac{\pi}{2}, r\in I$$

$$\Rightarrow n(3\pi)=r\pi+\frac{\pi}{2} \qquad (\because \lambda=3\pi)$$

$$\Rightarrow 3n-r=\frac{1}{2} \text{ (Impossible).}$$

Again, let $\sin\left(\frac{5\lambda}{n}\right)=0$

$$\therefore \frac{5\lambda}{n}=p\pi \qquad (p\in I)$$

$$\Rightarrow \frac{5(3\pi)}{n}=p\pi \qquad (\because \lambda=3\pi)$$

$$\Rightarrow n=\frac{15}{p}$$

For $p=\pm1, \pm3, \pm5, \pm15$

$\therefore n=\pm15, \pm5, \pm3, \pm1$ $\qquad (\because n\in I)$

14. Let $f: R\to R$ and $g: R\to R$ be functions defined by $f(x)=\begin{cases}2x, & x<1,\\ 2x^2-1, & x\geq1,\end{cases}$ $g(x)=\begin{cases}x+2, & x<0\\ 2x, & x\geq0.\end{cases}$

Find (a) $f+g$, (b) fg.

Sol. $(f+g): R\to R$ and $(fg): R\to R$ are functions defined by $(f+g)(x)=f(x)+g(x)$ and $(fg)(x)=f(x)\,g(x)$. To find $(f+g)(x)$ and $(fg)(x)$, we rewrite $f(x)$ and $g(x)$ as:

$$f(x)=\begin{cases}2x, & x<0\\ 2x, & 0\leq x<1\\ 2x^2-1, & x\geq1\end{cases} \quad g(x)=\begin{cases}x+2, & x<0\\ 2x, & 0\leq x<1\\ 2x, & x\geq1\end{cases}$$

$$\text{Hence, a. } (f+g)(x)=\begin{cases}3x+2, & x<0\\ 4x, & 0\leq x<1\\ 2x^2+2x-1, & x\geq1\end{cases}$$

$$\text{b. } (fg)(x)=\begin{cases}2x^2+4x, & x<0\\ 4x^2, & 0\leq x<1\\ 4x^3-2x, & x\geq1\end{cases}$$

15. If $f(x)=-1+|x-2|, 0\leq x\leq4$ and $g(x)=2-|x|, -1\leq x\leq3$. Then find $(fog)(x)$ and $(gof)(x)$.

Sol. We have

$$f(x)=\begin{cases}1-x, & 0\leq x\leq2\\ x-3, & 2<x\leq4\end{cases}$$

$$\text{and } g(x)=\begin{cases}2+x, & -1\leq x\leq0\\ 2-x, & 0<x\leq3\end{cases}$$

$$\therefore (fog)x=f\{g(x)\}=\begin{cases}1-g(x), & 0\leq g(x)\leq2\\ g(x)-3, & 2<g(x)\leq4\end{cases}$$

$$=\begin{cases}1-(2+x), & 0\leq2+x\leq2 \text{ and } -1\leq x\leq0\\ 2+x-3, & 2<2+x\leq4 \text{ and } -1\leq x\leq0\\ 1-(2-x), & 0\leq2-x\leq2 \text{ and } 0<x\leq3\\ 2-x-3, & 2<2-x\leq4 \text{ and } 0<x\leq3\end{cases}$$

$$=\begin{cases}-1-x, & -2\leq x\leq0 \text{ and } -1\leq x\leq0\\ x-1, & 0<x\leq2 \text{ and } -1\leq x\leq0\\ -1+x, & 0\leq x\leq2 \text{ and } 0<x\leq3\\ -x-1, & -2\leq x<0 \text{ and } 0<x\leq3\end{cases}$$

$$=\begin{cases}-1-x, & -1\leq x\leq0\\ x-1, & x\in\phi\\ -1+x, & 0<x\leq2\\ -x-1, & x\in\phi\end{cases}$$

$$=\begin{cases}-1-x, & -1\leq x\leq0\\ x-1, & 0<x\leq2\end{cases}$$

$$\text{and } (gof)x=g\{f(x)\}=\begin{cases}2+f(x), & -1\leq f(x)\leq0\\ 2-f(x), & 0<f(x)\leq3\end{cases}$$

$$=\begin{cases}2+1-x, & -1\leq1-x\leq0 \text{ and } 0\leq x\leq2\\ 2-(1-x), & 0<1-x\leq3 \text{ and } 0\leq x\leq2\\ 2+x-3, & -1\leq x-3\leq0 \text{ and } 2<x\leq4\\ 2-(x-3), & 0<x-3\leq3 \text{ and } 2<x\leq4\end{cases}$$

$$=\begin{cases}3-x, & 1\leq x\leq2 \text{ and } 0\leq x\leq2\\ 1+x, & -2\leq x<1 \text{ and } 0\leq x\leq2\\ x-1, & 2\leq x\leq3 \text{ and } 2<x\leq4\\ -x+5, & 3<x\leq6 \text{ and } 2<x\leq4\end{cases}$$

$$=\begin{cases}3-x, & 1\leq x\leq2\\ 1+x, & 0\leq x<1\\ x-1, & 2<x\leq3\\ 5-x, & 3<x\leq4\end{cases}=\begin{cases}1+x, & 0\leq x<1\\ 3-x, & 1\leq x\leq2\\ x-1, & 2<x\leq3\\ 5-x, & 3<x\leq4\end{cases}$$

EXERCISES

Subjective Type

Solutions on page 1.57

1. Write explicit functions of y defined by the following equations and also find domains of definitions of the given implicit functions:
   a. $x+|y|=2y$ b. $e^y-e^{-y}=2x$
   c. $10^x+10^y=10$ d. $x^2-\sin^{-1}y=\frac{\pi}{2}$
2. Let $g(x)=\sqrt{x-2k}$, $\forall\ 2k\le x<2(k+1)$, where $k\in$ integer, check whether $g(x)$ is periodic or not.
3. Let $f(x)=x^2-2x$, $x\in R$ and $g(x)=f(f(x)-1)+f(5-f(x))$. Show that $g(x)\ge 0\ \forall\ x\in R$.
4. If f and g are two distinct linear functions defined on R such that they map $[-1, 1]$ onto $[0, 2]$ and $h: R-\{-1, 0, 1\}\to R$ defined by $h(x)=\frac{f(x)}{g(x)}$, then show that $|h(h(x))+h(h(1/x))|>2$.
5. Let $f(x)=\begin{cases}x^2-4x+3, & x<3\\ x-4, & x\ge 3\end{cases}$ and $g(x)=\begin{cases}x-3, & x<4\\ x^2+2x+2, & x\ge 4\end{cases}$. Describe the function f/g and find its domain.
6. Let $f(x)=\log_2\log_3\log_4\log_5(\sin x+a^2)$. Find the set of values of a for which domain of $f(x)$ is R.
7. A certain polynomial $P(x)$, $x\in R$ when divided by $x-a$, $x-b$, $x-c$ leaves remainders a, b, c, respectively. Then find the remainder when $P(x)$ is divided by $(x-a)(x-b)(x-c)$ (a, b, c are distinct).
8. Let $R=\{(x, y): x, y\in R, x^2+y^2\le 25\}$ and $R'=\left\{(x,y): x, y\in R, y\ge \frac{4}{9}x^2\right\}$, then find the domain and range of $R\cap R'$.
9. If f is a polynomial function satisfying $2+f(x)f(y)=f(x)+f(y)+f(xy)$, $\forall\ x, y\in R$ and if $f(2)=5$, then find the value of $f(f(2))$.
10. If $f(a-x)=f(a+x)$ and $f(b-x)=f(b+x)$ for all real x, where a, b ($a>b$) are constants, then prove that $f(x)$ is a periodic function.
11. If p, q are positive integers, f is a function defined for positive numbers and attains only positive values, such that $f(xf(y))=x^p y^q$, then prove that $p^2=q$.
12. If $f: R\to[0,\infty)$ is a function such that $f(x-1)+f(x+1)=\sqrt{3}\ f(x)$, then prove that $f(x)$ is periodic and find its period.
13. If a, b be two fixed positive integers such that $f(a+x)=b+[b^3+1-3b^2f(x)+3b\{f(x)\}^2-\{f(x)\}^3]^{1/3}$ for all real x, then prove that $f(x)$ is a periodic and find its period.
14. Let $f(x, y)$ be a periodic function, satisfying the condition $f(x, y)=f(2x+2y, 2y-2x)\ \forall\ x, y\in R$ and let $g(x)$ be a function defined as $g(x)=f(2^x, 0)$. Prove that $g(x)$ is periodic function and find its period.
15. Let $f: R\to R$, $f(x)=\frac{x-a}{(x-b)(x-c)}$, $b>c$. If f is onto, then prove that $a\in(b, c)$.
16. Show that there exists no polynomial $f(x)$ with integral coefficients which satisfy $f(a)=b$, $f(b)=c$, $f(c)=a$, where a, b, c are distinct integers.
17. Consider the function $f(x)=\begin{cases}x-[x]-\frac{1}{2}, & \text{If } x\notin I\\ 0, & \text{If } x\in I\end{cases}$ where [.] denotes the fractional integral function and I is the set of integers. Then find $g(x)=\max.\{x^2, f(x), |x|\}$; $-2\le x\le 2$.
18. Determine all functions $f: R\to R$ such that $f(x-f(y))=f(f(y))+xf(y)+f(x)-1\ \forall\ x, y\in R$.
19. Let $f(x)=(2\cos x-1)(2\cos 2x-1)(2\cos 2^2x-1)\ldots(2\cos 2^{n-1}x-1)$ (where $n\ge 1$). Then prove that $f\left(\frac{2\pi k}{2^n\pm 1}\right)=1\ \forall\ k\in I$.
20. If $f(x)=\frac{a^x}{a^x+\sqrt{a}}$ $(a>0)$, then find the value of $\sum_{r=1}^{2n-1}2f\left(\frac{r}{2n}\right)$.

Objective Type

Solutions on page 1.60

Each question has four choices a, b, c, and d, out of which *only one* is correct.

1. The function $f: N\to N$ (N is the set of natural numbers) defined by $f(n)=2n+3$ is
   a. surjective only b. injective only
   c. bijective d. None of these
2. The function $f(x)=\sin\left(\log(x+\sqrt{1+x^2})\right)$ is
   a. even function b. odd function
   c. neither even nor odd d. periodic function
3. If x is real, then the value of the expression $\frac{x^2+14x+9}{x^2+2x+3}$ lies between
   a. 5 and 4 b. 5 and -4
   c. -5 and 4 d. None of these
4. The function $f: R\to R$ is defined by $f(x)=\cos^2x+\sin^4x$ for $x\in R$, then the range of $f(x)$ is
   a. $\left(\frac{3}{4}, 1\right]$ b. $\left[\frac{3}{4}, 1\right)$
   c. $\left[\frac{3}{4}, 1\right]$ d. $\left(\frac{3}{4}, 1\right)$
5. The domain of the function $f(x)=\log_{3+x}(x^2-1)$ is
   a. $(-3, -1)\cup(1,\infty)$
   b. $[-3, -1)\cup[1,\infty)$

c. $(-3,-2)\cup(-2,-1)\cup(1,\infty)$
d. $[-3,-2)\cup(-2,-1)\cup[1,\infty)$

6. The domain of the function $f(x)=\left[\log_{10}\left(\dfrac{5x-x^2}{4}\right)\right]^{1/2}$ is
a. $-\infty<x<\infty$ b. $1\le x\le 4$
c. $4\le x\le 16$ d. $-1\le x\le 1$

7. The domain of the function $f(x)=\dfrac{\sin^{-1}(3-x)}{\ln(|x|-2)}$ is
a. $[2,4]$ b. $(2,3)\cup(3,4]$
c. $[2,\infty)$ d. $(-\infty,-3)\cup[2,\infty)$

8. The domain of $f(x)=\log|\log x|$ is
a. $(0,\infty)$ b. $(1,\infty)$
c. $(0,1)\cup(1,\infty)$ d. $(-\infty,1)$

9. The domain of $f(x)=\dfrac{\log_2(x+3)}{x^2+3x+2}$ is
a. $R-\{-1,-2\}$ b. $(-2,\infty)$
c. $R-\{-1,-2,-3\}$ d. $(-3,\infty)-\{-1,-2\}$

10. Let $f:\left[-\dfrac{\pi}{3},\dfrac{2\pi}{3}\right]\to[0,4]$ be a function defined as $f(x)=\sqrt{3}\sin x-\cos x+2$. Then $f^{-1}(x)$ is given by
a. $\sin^{-1}\left(\dfrac{x-2}{2}\right)-\dfrac{\pi}{6}$ b. $\sin^{-1}\left(\dfrac{x-2}{2}\right)+\dfrac{\pi}{6}$
c. $\dfrac{2\pi}{3}+\cos^{-1}\left(\dfrac{x-2}{2}\right)$ d. None of these

11. If $F(n+1)=\dfrac{2F(n)+1}{2}$ $n=1,2,\ldots$ and $F(1)=2$, then $F(101)$ equals
a. 52 b. 49 c. 48 d. 51

12. The domain of the function $f(x)=\dfrac{1}{\sqrt{{}^{10}C_{x-1}-3\times{}^{10}C_x}}$ contains the points
a. 9, 10, 11 b. 9, 10, 12
c. all natural numbers d. None of these

13. The domain of the function $f(x)=\dfrac{x}{\sqrt{\sin(\ln x)-\cos(\ln x)}}$ $(n\in Z)$ is
a. $(e^{2n\pi},e^{(3n+1/2)\pi})$ b. $(e^{(2n+1/4)\pi},e^{(2n+5/4)\pi})$
c. $(e^{2n+1/4)\pi},e^{(3n-3/4)\pi})$ d. None of these

14. If f is a function such that $f(0)=2$, $f(1)=3$ and $f(x+2)=2f(x)-f(x+1)$ for every real x, then $f(5)$ is
a. 7 b. 13
c. 1 d. 5

15. The range of $f(x)=\sin^{-1}\left(\dfrac{x^2+1}{x^2+2}\right)$ is
a. $[0,\pi/2]$ b. $(0,\pi/6)$
c. $[\pi/6,\pi/2)$ d. None of these

16. The function $f(x)=\dfrac{\sec^{-1}x}{\sqrt{x-[x]}}$, where $[x]$ denotes the greatest integer less than or equal to x, is defined for all $x\in$
a. R b. $R-\{(-1,1)\cup\{n\mid n\in Z\}\}$
c. $R^+-(0,1)$ d. $R^+-\{n\mid n\in N\}$

17. The domain of $f(x)=\cos^{-1}\left(\dfrac{2-|x|}{4}\right)+[\log(3-x)]^{-1}$ is
a. $[-2,6]$ b. $[-6,2)\cup(2,3)$
c. $[-6,2]$ d. $[-2,2]\cup(2,3)$

18. The domain of the function $f(x)=\sqrt{\log\left(\dfrac{1}{|\sin x|}\right)}$ is
a. $R-\{-\pi,\pi\}$ b. $R-\{n\pi\mid n\in Z\}$
c. $R-\{2n\pi\mid n\in z\}$ d. $(-\infty,\infty)$

19. The domain of the function $f(x)=\log_2\left(-\log_{1/2}\left(1+\dfrac{1}{x^{1/4}}\right)-1\right)$ is
a. $(0,1)$ b. $(0,1]$
c. $[1,\infty)$ d. $(1,\infty)$

20. The range of $f(x)=\sin^{-1}(\sqrt{x^2+x+1})$ is
a. $\left(0,\dfrac{\pi}{2}\right]$ b. $\left(0,\dfrac{\pi}{3}\right]$
c. $\left[\dfrac{\pi}{3},\dfrac{\pi}{2}\right]$ d. $\left[\dfrac{\pi}{6},\dfrac{\pi}{3}\right]$

21. If $f(x)=\text{maximum}\left\{x^3,x^2,\dfrac{1}{64}\right\}\ \forall\ x\in[0,\infty)$, then
a. $f(x)=\begin{cases}x^2, & 0\le x\le 1\\ x^3, & x>1\end{cases}$ b. $f(x)=\begin{cases}\frac{1}{64}, & 0\le x\le\frac{1}{4}\\ x^2, & \frac{1}{4}<x\le 1\\ x^3, & x>1\end{cases}$
c. $f(x)=\begin{cases}\frac{1}{64}, & 0\le x\le\frac{1}{8}\\ x^2, & \frac{1}{8}<x\le 1\\ x^3, & x>1\end{cases}$ d. $f(x)=\begin{cases}\frac{1}{64}, & 0\le x\le\frac{1}{8}\\ x^3, & x>1/8\end{cases}$

22. If the period of $\dfrac{\cos(\sin(nx))}{\tan(x/n)}$, $n\in N$, is 6π, then n is equal to
a. 3 b. 2
c. 6 d. 1

23. The number of real solutions of the equation $\log_{0.5}|x|=2|x|$ is
a. 1 b. 2
c. 0 d. None of these

24. The period of the function $\left|\sin^3\dfrac{x}{2}\right|+\left|\cos^5\dfrac{x}{5}\right|$ is
a. 2π b. 10π
c. 8π d. 5π

25. If $f(x)=\sqrt[n]{x^m}$, $n\in N$, is an even function, then m is
a. even integer b. odd integer
c. any integer d. $f(x)$-even is not possible

26. If f is periodic, g is polynomial function and $f(g(x))$ is periodic and $g(2)=3$, $g(4)=7$ then $g(6)$ is
 a. 13 b. 15
 c. 11 d. None of these

27. The period of function $2^{\{x\}}+\sin \pi x+3^{\{x/2\}}+\cos 2\pi x$ (where $\{x\}$ denotes the fractional part of x) is
 a. 2 b. 1
 c. 3 d. None of these

28. The equation $||x-2|+a|=4$ can have four distinct real solutions for x if a belongs to the interval
 a. $(-\infty,-4)$ b. $(-\infty,0]$
 c. $[4,\infty)$ d. None of these

29. Given the function $f(x)=\dfrac{a^x+a^{-x}}{2}$ (where $a>2$). Then $f(x+y)+f(x-y)=$
 a. $2f(x).f(y)$ b. $f(x).f(y)$
 c. $\dfrac{f(x)}{f(y)}$ d. None of these

30. If $\log_3(x^2-6x+11)\le 1$, then exhaustive range of values of x is
 a. $(-\infty,2)\cup(4,\infty)$
 b. $(2,4)$
 c. $(-\infty,1)\cup(1,3)\cup(4,\infty)$
 d. None of these

31. The domain of the function $f(x)=\sqrt{x^2-[x]^2}$, where $[x]$ = the greatest integer less than or equal to x, is
 a. R b. $[0,+\infty)$
 c. $(-\infty,0]$ d. None of these

32. The range of the function $f(x)=|x-1|+|x-2|$, $-1\le x\le 3$, is
 a. $[1,3]$ b. $[1,5]$
 c. $[3,5]$ d. None of these

33. Which of the following functions is inverse to itself?
 a. $f(x)=\dfrac{1-x}{1+x}$ b. $f(x)=5^{\log x}$
 c. $f(x)=2^{x(x-1)}$ d. None of these

34. A function $F(x)$ satisfies the functional equation $x^2F(x)+F(1-x)=2x-x^4$ for all real x. $F(x)$ must be
 a. x^2 b. $1-x^2$
 c. $1+x^2$ d. x^2+x+1

35. If $f(x)=\begin{cases} x^2\sin\dfrac{\pi x}{2}, & |x|<1 \\ x|x|, & |x|\ge 1 \end{cases}$ then $f(x)$ is
 a. an even function b. an odd function
 c. a periodic function d. None of these

36. Function $f:(-\infty,-1)\to(0,e^5]$ defined by $f(x)=e^{x^3-3x+2}$ is
 a. many-one and onto b. many-one and into
 c. one-one and onto d. one-one and into

37. If $f(x)=\dfrac{1}{x}$, $g(x)=\dfrac{1}{x^2}$ and $h(x)=x^2$
 a. $fog(x)=x^2$ $x\ne 0$, $h(g(x))=\dfrac{1}{x^2}$
 b. $h(g(x))=\dfrac{1}{x^2}$ $x\ne 0$, $fog(x)=x^2$
 c. $fog(x)=x^2, x\ne 0$, $h(g(x))=(g(x))^2, x\ne 0$
 d. None of these

38. If $[x]$ and $\{x\}$ represent the integral and fractional parts of x, respectively, then the value of $\sum_{r=1}^{2000}\dfrac{\{x+r\}}{2000}$ is
 a. x b. $[x]$
 c. $\{x\}$ d. $x+2001$

39. If $f(x)$ is a polynomial satisfying $f(x)f(1/x)=f(x)+f(1/x)$ and $f(3)=28$, then $f(4)$ is equal to
 a. 63 b. 65
 c. 17 d. None of these

40. The values of b and c for which the identity $f(x+1)-f(x)=8x+3$ is satisfied, where $f(x)=bx^2+cx+d$, are
 a. $b=2, c=1$ b. $b=4, c=-1$
 c. $b=-1, c=4$ d. $b=-1, c=1$

41. Let $f:R\to R$, $g:R\to R$ be two given functions such that f is injective and g is surjective, then which of the following is injective?
 a. g_0f b. f_0g
 c. g_0g d. None of these

42. $f:N\to N$ where $f(x)=x-(-1)^x$ then f is
 a. one-one and into
 b. many-one and into
 c. one-one and onto
 d. many-one and onto

43. If $g(x)=x^2+x-2$ and $\dfrac{1}{2}gof(x)=2x^2-5x+2$, then which is not a possible $f(x)$?
 a. $2x-3$ b. $-2x+2$
 c. $x-3$ d. None of these

44. If $f:R\to R$ is an invertible function such that $f(x)$ and $f^{-1}(x)$ are symmetric about the line $y=-x$, then
 a. $f(x)$ is odd
 b. $f(x)$ and $f^{-1}(x)$ may not be symmetric about the line $y=x$
 c. $f(x)$ may not be odd
 d. None of these

45. Let $f:N\to N$ defined by $f(x)=x^2+x+1$, $x\in N$, then f is
 a. One-one onto
 b. Many-one onto
 c. One-one but not onto
 d. None of these

46. Let $f:X\to Y$ $f(x)=\sin x+\cos x+2\sqrt{2}$ is invertible. Then which $X\to Y$ is not possible?
 a. $\left[\dfrac{\pi}{4},\dfrac{5\pi}{4}\right]\to[\sqrt{2},3\sqrt{2}]$
 b. $\left[-\dfrac{3\pi}{4},\dfrac{\pi}{4}\right]\to[\sqrt{2},3\sqrt{2}]$
 c. $\left[-\dfrac{3\pi}{4},\dfrac{3\pi}{4}\right]\to[\sqrt{2},3\sqrt{2}]$
 d. None of these

47. If $f(x) = ax^7 + bx^3 + cx - 5$, a, b, c are real constants and $f(-7) = 7$, then the range of $f(7) + 17\cos x$ is
  a. $[-34, 0]$ b. $[0, 34]$
  c. $[-34, 34]$ d. None of these

48. If $f(x) = \dfrac{\sin([x]\pi)}{x^2+x+1}$, where [.] denotes the greatest integer function, then
  a. f is one-one
  b. f is not one-one and non-constant
  c. f is a constant function
  d. None of these

49. Let S be the set of all triangles and R^+ be the set of positive real numbers. Then the function $f: S \to R^+, f(\Delta) =$ area of Δ, where $\Delta \in S$ is
  a. injective but not surjective
  b. surjective but not injective
  c. injective as well as surjective
  d. neither injective nor surjective

50. The graph of $(y - x)$ against $(y + x)$ is shown

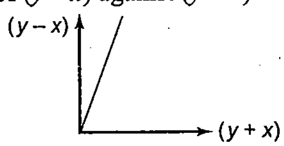
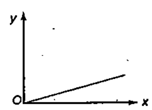
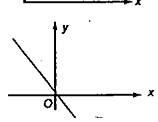


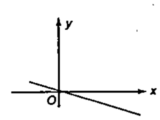
Fig. 1.91

Which one of the following shows the graph of y against x?

a. 

b. 

c. 

d. 

51. If $g: [-2, 2] \to R$ where $f(x) = x^3 + \tan x + \left[\dfrac{x^2+1}{P}\right]$ is a odd function, then the value of parametric P where [.] denotes the greatest integer function is
  a. $-5 < P < 5$ b. $P < 5$
  c. $P > 5$ d. None of these

52. If $f\left(2x + \dfrac{y}{8}, 2x - \dfrac{y}{8}\right) = xy$, then $f(m, n) + f(n, m) = 0$
  a. only when $m = n$ b. only when $m \neq n$
  c. only when $m = -n$ d. for all m and n

53. If $f(x + y) = f(x) + f(y) - xy - 1 \ \forall\, x, y \in R$ and $f(1) = 1$, then the number of solutions of $f(n) = n$, $n \in N$ is
  a. 0 b. 1
  c. 2 d. more than 2

54. The range of the function $f(x) = \dfrac{e^x - e^{|x|}}{e^x + e^{|x|}}$
  a. $(-\infty, \infty)$ b. $[0, 1)$
  c. $(-1, 0]$ d. $(-1, 1)$

55. If $f: R \to R$ is a function satisfying the property $f(2x + 3) + f(2x + 7) = 2, \forall\, x \in R$, then the fundamental period of $f(x)$ is
  a. 2 b. 4 c. 8 d. 12

56. Let $f: R \to \left[0, \dfrac{\pi}{2}\right)$ defined by $f(x) = \tan^{-1}(x^2 + x + a)$, then the set of values of a for which f is onto is
  a. $[0, \infty)$ b. $[2, 1]$
  c. $\left[\dfrac{1}{4}, \infty\right)$ d. None of these

57. The domain of the function $f(x) = \dfrac{1}{\sqrt{\{\sin x\} + \{\sin(\pi + x)\}}}$, where $\{\cdot\}$ denotes the fractional part, is
  a. $[0, \pi]$ b. $(2n + 1)\,\pi/2, n \in Z$
  c. $(0, \pi)$ d. None of these

58. $f(x) = \dfrac{\cos x}{\left[\dfrac{2x}{\pi}\right] + \dfrac{1}{2}}$, where x is not an integral multiple of π and [.] denotes the greatest integer function is
  a. An odd function b. Even function
  c. Neither odd nor even d. None of these

59. Let $f(x) = ([a]^2 - 5[a] + 4)x^3 - (6\{a\}^2 - 5\{a\} + 1)x - (\tan x) \times \operatorname{sgn} x$ be an even function for all $x \in R$, then the sum of all possible values of 'a' is (where $[\cdot]$ and $\{\cdot\}$ denote greatest integer function and fractional part functions, respectively)
  a. $\dfrac{17}{6}$ b. $\dfrac{53}{6}$ c. $\dfrac{31}{3}$ d. $\dfrac{35}{3}$

60. Let $f: [-10, 10] \to R$, where $f(x) = \sin x + [x^2/a]$ be an odd function. Then the set of values of parameter a is/are
  a. $(-10, 10) \sim \{0\}$ b. $(0, 10)$
  c. $[100, \infty)$ d. $(100, \infty)$

61. The function f satisfies the functional equation $3f(x) + 2f\left(\dfrac{x+59}{x-1}\right) = 10x + 30$ for all real $x \neq 1$. The value of $f(7)$ is
  a. 8 b. 4
  c. -8 d. 11

62. The period of the function $f(x) = [6x + 7] + \cos \pi x - 6x$, where [.] denotes the greatest integer function, is
  a. 3 b. 2π
  c. 2 d. None of these

63. If the graph of the function $f(x) = \dfrac{a^x - 1}{x^n(a^x + 1)}$ is symmetrical about y-axis, then n equals
  a. 2 b. $\dfrac{2}{3}$ c. $\dfrac{1}{4}$ d. $-\dfrac{1}{3}$

64. If $f(x)$ is an even function and satisfies the relation $x^2 f(x) - 2f\left(\dfrac{1}{x}\right) = g(x)$ where $g(x)$ is an odd function, then $f(5)$ equals
  a. 0 b. $\dfrac{50}{75}$ c. $\dfrac{49}{75}$ d. None of these

65. If $f(x + y) = f(x).f(y)$ for all real x, y and $f(0) \neq 0$, then the function $g(x) = \dfrac{f(x)}{1 + \{f(x)\}^2}$ is
  a. Even function b. Odd function
  c. Odd if $f(x) > 0$ d. Neither even nor odd

66. Possible values of a such that the equation $x^2 + 2ax + a = \sqrt{a^2 + x - \frac{1}{16}} - \frac{1}{16}$, $x \geq -a$, has two distinct real roots are given by
  a. $[0, 1]$ b. $[-\infty, 0)$
  c. $[0, \infty)$ d. $\left(\frac{3}{4}, \infty\right)$

67. Let $g(x) = f(x) - 1$. If $f(x) + f(1-x) = 2\ \forall\ x \in R$, then $g(x)$ is symmetrical about
  a. Origin b. The line $x = \frac{1}{2}$
  c. The point $(1, 0)$ d. The point $\left(\frac{1}{2}, 0\right)$

68. Domain (D) and range (R) of $f(x) = \sin^{-1}(\cos^{-1}[x])$ where [.] denotes the greatest integer function is
  a. $D \equiv x \in [1, 2), R \in \{0\}$
  b. $D \equiv x \in [0, 1], R \equiv \{-1, 0, 1\}$
  c. $D \equiv x \in [-1, 1], R \equiv \left\{0, \sin^{-1}\left(\frac{\pi}{2}\right), \sin^{-1}(\pi)\right\}$
  d. $D \equiv x \in [-1, 1], R \equiv \left\{-\frac{\pi}{2}, 0, \frac{\pi}{2}\right\}$

69. If $f(x+1) + f(x-1) = 2f(x)$ and $f(0) = 0$, then $f(n)$, $n \in N$, is
  a. $nf(1)$ b. $\{f(1)\}^n$
  c. 0 d. None of these

70. The range of the function f defined by $f(x) = \left[\frac{1}{\sin\{x\}}\right]$ (where [.] and {.} respectively denote the greatest integer and the fractional part functions) is
  a. I, the set of integers
  b. N, the set of natural numbers
  c. W, the set of whole numbers
  d. $\{1, 2, 3, 4, \ldots\}$

71. If $[\cos^{-1} x] + [\cot^{-1} x] = 0$, where [.] denotes the greatest integer function, then the complete set of values of x is
  a. $(\cos 1, 1]$ b. $(\cos 1, \cot 1)$
  c. $(\cot 1, 1]$ d. $[0, \cot 1)$

72. If $f(x)$ and $g(x)$ are periodic functions with period 7 and 11, respectively. Then the period of $F(x) = f(x)\, g\left(\frac{x}{5}\right) - g(x)\, f\left(\frac{x}{3}\right)$ is
  a. 177 b. 222
  c. 433 d. 1155

73. The period of the function
$$f(x) = c^{\sin^2 x + \sin^2\left(x + \frac{\pi}{3}\right) + \cos x \cos\left(x + \frac{\pi}{3}\right)}$$
is (where c is constant)
  a. 1 b. $\frac{\pi}{2}$
  c. π d. Cannot be determined

74. If $f(x + f(y)) = f(x) + y\ \forall\ x, y \in R$ and $f(0) = 1$, then the value of $f(7)$ is
  a. 1 b. 7
  c. 6 d. 8

75. Let $f(x) = \sqrt{|x| - \{x\}}$ (where {.} denotes the fractional part of x) and X, Y are its domain and range, respectively, then
  a. $x \in \left(-\infty, \frac{1}{2}\right]$ and $Y \in \left[\frac{1}{2}, \infty\right)$
  b. $x \in \left(-\infty, -\frac{1}{2}\right] \cup [0, \infty)$ and $Y \in \left[\frac{1}{2}, \infty\right)$
  c. $X \in \left(-\infty, -\frac{1}{2}\right] \cup [0, \infty)$ and $Y \in [0, \infty)$
  d. None of these

76. Let f be a function satisfying of x then $f(xy) = \frac{f(x)}{y}$ for all positive real numbers x and y if $f(30) = 20$, then the value of $f(40)$ is
  a. 15 b. 20
  c. 40 d. 60

77. The domain of the function $f(x) = \sqrt{\ln_{(|x|-1)}(x^2 + 4x + 4)}$ is
  a. $[-3, -1] \cup [1, 2]$
  b. $(-2, -1) \cup [2, \infty)$
  c. $(-\infty, -3] \cup (-2, -1) \cup (2, \infty)$
  d. None of these

78. The range of $f(x) = [1 + \sin x] + \left[2 + \sin\frac{x}{2}\right] + \left[3 + \sin\frac{x}{3}\right] + \cdots + \left[n + \sin\frac{x}{n}\right]$, $\forall\, x \in [0, \pi]$, where [.] denotes the greatest integer function, is
  a. $\left\{\frac{n^2 + n - 2}{2}, \frac{n(n+1)}{2}\right\}$
  b. $\left\{\frac{n(n+1)}{2}\right\}$
  c. $\left\{\frac{n^2 + n - 2}{2}, \frac{n(n+1)}{2}, \frac{n^2 + n + 2}{2}\right\}$
  d. $\left\{\frac{n(n+1)}{2}, \frac{n^2 + n + 2}{2}\right\}$

79. The total number of solutions of $[x]^2 = x + 2\{x\}$, where [.] and {.} denote the greatest integer function and fractional part, respectively, is equal to
  a. 2 b. 4
  c. 6 d. None of these

80. The domain of $f(x) = \sqrt{2\{x\}^2 - 3\{x\} + 1}$, where {.} denotes the fractional part in $[-1, 1]$, is
  a. $[-1, 1] \sim \left(\frac{1}{2}, 1\right)$

b. $\left[-1, -\frac{1}{2}\right] \cup \left[0, \frac{1}{2}\right] \cup \{1\}$

c. $\left[-1, \frac{1}{2}\right]$

d. $\left[-\frac{1}{2}, 1\right]$

81. The range of $\sin^{-1}\left[x^2+\frac{1}{2}\right]+\cos^{-1}\left[x^2-\frac{1}{2}\right]$, where [.] denotes the greatest integer function, is

a. $\left\{\frac{\pi}{2}, \pi\right\}$ b. $\{\pi\}$
c. $\left\{\frac{\pi}{2}\right\}$ d. None of these

82. If the period of $\frac{\cos(\sin(nx))}{\tan\left(\frac{x}{n}\right)}$, $n \in N$ is 6π then $n =$

a. 3 b. 2
c. 6 d. 1

83. The domain of $f(x) = \ln(ax^3 + (a+b)x^2 + (b+c)x + c)$, where $a > 0$, $b^2 - 4ac = 0$, is (where [·] represents greatest integer function).

a. $(-1, \infty) \sim \left\{-\frac{b}{2a}\right\}$ b. $(1, \infty) \sim \left\{-\frac{b}{2a}\right\}$
c. $(-1, 1) \sim \left\{-\frac{b}{2a}\right\}$ d. None of these

84. The period of $f(x) = [x] + [2x] + [3x] + [4x] + \cdots [nx] - \frac{n(n+1)}{2}x$, where $n \in N$, is (where [·] represents greatest integer function)

a. n b. 1
c. $\frac{1}{n}$ d. None of these

85. If $f\left(x+\frac{1}{2}\right) + f\left(x-\frac{1}{2}\right) = f(x)$ for all $x \in R$, then the period of $f(x)$ is

a. 1 b. 2
c. 3 d. 4

86. If $f: R^+ \to R$, $f(x) + 3xf\left(\frac{1}{x}\right) = 2(x+1)$, then $f(99)$ is equal to

a. 40 b. 30
c. 50 d. 60

87. If $f: X \to Y$, where X and Y are sets containing natural numbers, $f(x) = \frac{x+5}{x+2}$ then the number of elements in the domain and range of $f(x)$ are respectively

a. 1 and 1 b. 2 and 1
c. 2 and 2 d. 1 and 2

88. If $f(x) = \begin{cases} x^2 & \text{for } x \geq 0 \\ x & \text{for } x < 0 \end{cases}$ then $fof(x)$ is given by

a. x^2 for $x \geq 0$, x for $x < 0$
b. x^4 for $x \geq 0$, x^2 for $x < 0$
c. x^4 for $x \geq 0$, $-x^2$ for $x < 0$
d. x^4 for $x \geq 0$, x for $x < 0$

89. If the graph of $y = f(x)$ is symmetrical about lines $x = 1$ and $x = 2$, then which of the following is true?

a. $f(x+1) = f(x)$ b. $f(x+3) = f(x)$
c. $f(x+2) = f(x)$ d. None of these

90. Let $f(x) = x + 2|x+1| + 2|x-1|$. If $f(x) = k$ has exactly one real solution, then the value of k is

a. 3 b. 0
c. 1 d. 2

91. The domain of $f(x) = \sin^{-1}[2x^2 - 3]$, where [.] denotes the greatest integer function, is

a. $\left(-\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}\right)$

b. $\left(-\sqrt{\frac{3}{2}}, -1\right] \cup \left(-\sqrt{\frac{5}{2}}, \sqrt{\frac{5}{2}}\right)$

c. $\left(-\sqrt{\frac{5}{2}}, \sqrt{\frac{5}{2}}\right)$

d. $\left(-\sqrt{\frac{5}{2}}, -1\right] \cup \left[1, \sqrt{\frac{5}{2}}\right)$

92. The range of $f(x) = \cos^{-1}\left(\frac{1+x^2}{2x}\right) + \sqrt{2-x^2}$ is

a. $\left\{0, 1+\frac{\pi}{2}\right\}$ b. $\{0, 1+\pi\}$
c. $\left\{1, 1+\frac{\pi}{2}\right\}$ d. $\{1, 1+\pi\}$

93. If $f(x) = \begin{cases} x, & x \text{ is rational} \\ 1-x, & x \text{ is irrational} \end{cases}$ then $f(f(x))$ is

a. $x \; \forall \; x \in R$ b. $= \begin{cases} x, & x \text{ is irrational} \\ 1-x, & x \text{ is rational} \end{cases}$
c. $\begin{cases} x, & x \text{ is rational} \\ 1-x, & x \text{ is irrational} \end{cases}$ d. None of these

94. The range of $f(x) = [\,|\sin x| + |\cos x|\,]$, where [.] denotes the greatest integer function, is

a. $\{0\}$ b. $\{0, 1\}$
c. $\{1\}$ d. None of these

95. If $f(x) = \log_e\left(\frac{x^2+e}{x^2+1}\right)$, then the range of $f(x)$ is

a. $(0, 1)$ b. $[0, 1]$
c. $[0, 1)$ d. $(0, 1]$

96. The domain of the function $f(x)=\dfrac{1}{\sqrt{4x-|x^2-10x+9|}}$ is

a. $(7-\sqrt{40}, 7+\sqrt{40})$ b. $(0, 7+\sqrt{40})$

c. $(7-\sqrt{40}, \infty)$ d. None of these

97. If the function $f:[1,\infty)\to[1,\infty)$ is defined by $f(x)=2^{x(x-1)}$, then $f^{-1}(x)$ is

a. $\left(\dfrac{1}{2}\right)^{x(x-1)}$ b. $\dfrac{1}{2}(1+\sqrt{1+4\log_2 x})$

c. $\dfrac{1}{2}(1-\sqrt{1+4\log_2 x})$ d. Not defined

98. The number of roots of the equation $x\sin x=1$, $x\in[-2\pi,0)\cup(0,2\pi]$, is

a. 2 b. 3

c. 4 d. 0

99. The number of solutions of $2\cos x=|\sin x|$, $0\le x\le 4\pi$, is

a. 0 b. 2

c. 4 d. Infinite

100. If $af(x+1)+bf\left(\dfrac{1}{x+1}\right)=x, x\ne-1, a\ne b$, then $f(2)$ is equal to

a. $\dfrac{2a+b}{2(a^2-b^2)}$ b. $\dfrac{a}{a^2-b^2}$

c. $\dfrac{a+2b}{a^2-b^2}$ d. None of these

101. The number of solutions of $\tan x-mx=0$, $m>1$ in $\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$ is

a. 1 b. 2

c. 3 d. m

102. The range of $f(x)=[\sin x+[\cos x+[\tan x+[\sec x]]]]$, $x\in(0,\pi/4)$, where [.] denotes the greatest integer function $\le x$, is

a. $\{0, 1\}$ b. $\{-1, 0, 1\}$

c. $\{1\}$ d. None of these

103. If $f(3x+2)+f(3x+29)=0\ \forall\ x\in R$, then the period of $f(x)$ is

a. 7 b. 8

c. 10 d. None of these

104. Let $f(x)=\begin{cases}\sin x+\cos x, & 0<x<\dfrac{\pi}{2}\\ a, & x=\pi/2\\ \tan^2 x+\operatorname{cosec} x, & \pi/2<x<\pi\end{cases}$

then its odd extension is

a. $\begin{cases}-\tan^2 x-\operatorname{cosec} x, & -\pi<x<-\dfrac{\pi}{2}\\ -a, & x=-\dfrac{\pi}{2}\\ -\sin x+\cos x, & -\dfrac{\pi}{2}<x<0\end{cases}$

b. $\begin{cases}-\tan^2 x+\operatorname{cosec} x, & -\pi<x<-\dfrac{\pi}{2}\\ -a, & x=-\dfrac{\pi}{2}\\ \sin x-\cos x, & -\dfrac{\pi}{2}<x<0\end{cases}$

c. $\begin{cases}-\tan^2 x+\operatorname{cosec} x, & -\pi<x<-\dfrac{\pi}{2}\\ a, & x=-\dfrac{\pi}{2}\\ \sin x-\cos x, & -\dfrac{\pi}{2}<x<0\end{cases}$

d. $\begin{cases}\tan^2 x+\cos x, & -\pi<x<-\dfrac{\pi}{2}\\ -a, & x=-\dfrac{\pi}{2}\\ \sin x+\cos x, & -\dfrac{\pi}{2}<x<0\end{cases}$

105. If f and g are one-one function, then

a. $f+g$ is one-one b. fg is one-one

c. fog is one-one d. None of these

106. The domain of $f(x)$ is (0, 1), then, domain of $f(e^x)+f(\ln|x|)$ is

a. $(-1, e)$ b. $(1, e)$

c. $(-e, -1)$ d. $(-e, 1)$

107. The domain of $f(x)=\dfrac{1}{\sqrt{|\cos x|+\cos x}}$ is

a. $[-2n\pi, 2n\pi]$, $n\in Z$

b. $(2n\pi, \overline{2n+1}\pi)$, $n\in Z$

c. $\left(\dfrac{(4n+1)\pi}{2}, \dfrac{(4n+3)\pi}{2}\right)$, $n\in Z$

d. $\left(\dfrac{(4n-1)\pi}{2}, \dfrac{(4n+1)\pi}{2}\right)$, $n\in Z$

108. If $f(2x+3y, 2x-7y)=20x$, then $f(x, y)$ equals

a. $7x-3y$ b. $7x+3y$

c. $3x-7y$ d. $x-ky$

109. Let $X=\{a_1, a_2, \dots, a_6\}$ and $Y=\{b_1, b_2, b_3\}$. The number of functions f from x to y such that it is onto and there are exactly three elements x in X such that $f(x)=b_1$ is

a. 75 b. 90

c. 100 d. 120

110. Let $f: R\to R$ and $g: R\to R$ be two one-one and onto functions such that they are the mirror images of each other about the line $y=a$. If $h(x)=f(x)+g(x)$, then $h(x)$ is

a. One-one and onto.

b. Only one-one and not onto.

c. Only onto but not one-one.

d. Neither one-one nor onto.

111. If $f(x)=(-1)^{\left[\frac{2x}{\pi}\right]}$, $g(x)=|\sin x|-|\cos x|$ and $\phi(x)=f(x)g(x)$ (where [.] denotes the greatest integer function) then the respective fundamental periods of $f(x)$, $g(x)$ and $\phi(x)$ are

a. π, π, π b. $\pi, 2\pi, \pi$
c. $\pi, \pi, \frac{\pi}{2}$ d. $\pi, \frac{\pi}{2}, \pi$

112. Let $f(n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, then $f(1)+f(2)+f(3)+\cdots+f(n)$ is equal to
a. $nf(n)-1$ b. $(n+1)f(n)-n$
c. $(n+1)f(n)+n$ d. $nf(n)+n$

113. Let $f(x) = e^{\{e^{|x|}\operatorname{sgn} x\}}$ and $g(x) = e^{[e^{|x|}\operatorname{sgn} x]}$, $x \in R$ where { } and [ ] denotes the fractional and integral part functions, respectively. Also $h(x) = \log(f(x)) + \log(g(x))$ then for real x, $h(x)$ is
a. An odd function.
b. An even function.
c. Neither an odd nor an even function.
d. Both odd as well as even function.

114. Let $f_1(x) = \begin{cases} x, & 0 \le x \le 1 \\ 1, & x > 1 \\ 0, & \text{otherwise} \end{cases}$
and $f_2(x) = f_1(-x)$ for all x
$f_3(x) = -f_2(x)$ for all x
$f_4(x) = f_3(-x)$ for all x
Which of the following is necessarily true?
a. $f_4(x) = f_1(x)$ for all x b. $f_1(x) = -f_3(-x)$ for all x
c. $f_2(-x) = f_4(x)$ for all x d. $f_1(x) + f_3(x) = 0$ for all x

115. The number of solutions of the equation $[y + [y]] = 2\cos x$, where $y = \frac{1}{3}[\sin x + [\sin x + [\sin x]]]$ (where [.] denotes the greatest integer function) is
a. 4 b. 2
c. 3 d. 53

116. The sum of roots of the equation $\cos^{-1}(\cos x) = [x]$, [.] denotes the greatest integer function is
a. $2\pi+3$ b. $\pi+3$
c. $\pi-3$ d. $2\pi-3$

117. The range of
$$f(x) = \sqrt{(1-\cos x)\sqrt{(1-\cos x)\sqrt{(1-\cos x)\sqrt{\ldots\infty}}}}$$
is
a. $[0, 1]$ b. $[0, 1/2]$
c. $[0, 2]$ d. None of these

118. Let $h(x) = |kx + 5|$, the domain of $f(x)$ is $[-5, 7]$, the domain of $f(h(x))$ is $[-6, 1]$ and the range of $h(x)$ is the same as the domain of $f(x)$, then the value of k is
a. 1 b. 2
c. 3 d. 4

119. The range of $f(x) = (x+1)(x+2)(x+3)(x+4)+5$ for $x \in [-6, 6]$ is
a. $[4, 5045]$ b. $[0, 5045]$
c. $[-20, 5045]$ d. None of these

120. The exhaustive domain of
$$f(x) = \sqrt{x^{12} - x^9 + x^4 - x + 1}$$
is
a. $[0, 1]$ b. $[1, \infty)$
c. $(-\infty, 1]$ d. R

121. The range of $f(x) = \sec^{-1}(\log_3 \tan x + \log_{\tan x} 3)$ is
a. $\left[\frac{\pi}{3}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{2\pi}{3}\right]$ b. $\left[0, \frac{\pi}{2}\right)$
c. $\left(\frac{2\pi}{3}, \pi\right]$ d. None of these

122. The range of the function $f(x) = {}^{7-x}P_{x-3}$ is
a. $\{1, 2, 3\}$ b. $\{1, 2, 3, 4, 5, 6\}$
c. $\{1, 2, 3, 4\}$ d. $\{1, 2, 3, 4, 5\}$

123. A real-valued function $f(x)$ satisfies the functional equation $f(x-y) = f(x)f(y) - f(a-x)f(a+y)$, where a is a given constant and $f(0) = 1$. $f(2a-x)$ is equal to
a. $f(x)$ b. $-f(x)$
c. $f(-x)$ d. $f(a)+f(a-x)$

Multiple Correct Answers Type

Solutions on page 1.71

Each question has four choices a, b, c, and d, out of which *one or more* answers are correct.

1. Let $f(x) = \max\{1 + \sin x, 1, 1 - \cos x\}$, $x \in [0, 2\pi]$ and $g(x) = \max\{1, |x-1|\}$ $x \in R$, then
a. $g(f(0)) = 1$ b. $g(f(1)) = 1$
c. $f(f(1)) = 1$ d. $f(g(0)) = 1 + \sin 1$

2. Which of the following functions are identical?
a. $f(x) = \ln x^2$ and $g(x) = 2\ln x$
b. $f(x) = \log_x e$ and $g(x) = \frac{1}{\log_e x}$
c. $f(x) = \sin(\cos^{-1} x)$ and $g(x) = \cos(\sin^{-1} x)$
d. None of these

3. Which of the following function/functions have the graph symmetrical about the origin?
a. $f(x)$ given by $f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right)$
b. $f(x)$ given by $f(x) + f(y) = f\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$
c. $f(x)$ given by $f(x+y) = f(x) + f(y)\ \forall\ x, y \in R$
d. None of these

4. If the function f satisfies the relation $f(x+y) + f(x-y) = 2f(x)f(y)\ \forall\ x, y \in R$ and $f(0) \ne 0$, then
a. $f(x)$ is an even function
b. $f(x)$ is an odd function
c. If $f(2) = a$ then $f(-2) = a$
d. If $f(4) = b$ then $f(-4) = -b$

5. Consider the function $y = f(x)$ satisfying the condition $f\left(x + \frac{1}{x}\right) = x^2 + \frac{1}{x^2}$ $(x \ne 0)$, then
a. domain of $f(x)$ is R
b. domain of $f(x)$ is $R - (-2, 2)$

c. range of $f(x)$ is $[-2, \infty)$
d. range of $f(x)$ is $[2, \infty)$

6. Let $f(x) + f(y) = f(x\sqrt{1-y^2} + y\sqrt{1-x^2})$ ($f(x)$ is not identically zero). Then
a. $f(4x^3-3x)+3f(x)=0$
b. $f(4x^3-3x)=3f(x)$
c. $f(2x\sqrt{1-x^2})+2f(x)=0$
d. $f(2x\sqrt{1-x^2})=2f(x)$

7. Consider the real-valued function satisfying $2f(\sin x) + f(\cos x) = x$. Then
a. domain of $f(x)$ is R
b. domain of $f(x)$ is $[-1, 1]$
c. range of $f(x)$ is $\left[-\frac{2\pi}{3}, \frac{\pi}{3}\right]$
d. range of $f(x)$ is R

8. If $f(x)$ satisfies the relation $f(x+y)=f(x)+f(y)$ for all $x, y \in R$ and $f(1) = 5$, then
a. $f(x)$ is an odd function b. $f(x)$ is an even function
c. $\sum_{r=1}^{m} f(r) = 5\ {}^{m+1}C_2$ d. $\sum_{r=1}^{m} f(r) = \frac{5m(m+2)}{3}$

9. Let $f(x) = \begin{cases} x^2-4x+3, & x<3 \\ x-4, & x \geq 3 \end{cases}$ and

$g(x) = \begin{cases} x-3, & x<4 \\ x^2+2x+2, & x \geq 4 \end{cases}$

then, which of the following is/are true?
a. $(f+g)(3.5)=0$ b. $f(g(3))=3$
c. $(fg)(2)=1$ d. $(f-g)(4)=0$

10. $f(x)=x^2-2ax+a(a+1), f: [a, \infty) \to [a, \infty)$. If one of the solutions of the equation $f(x)=f^{-1}(x)$ is 5049, then the other may be
a. 5051 b. 5048
c. 5052 d. 5050

11. Which of the following function is/are periodic
a. $f(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$
b. $f(x) = \begin{cases} x-[x]; & 2n \leq x < 2n+1 \\ \frac{1}{2}; & 2n+1 \leq x < 2n+2 \end{cases}$, where [.] denotes the greatest integer function, $n \in Z$
c. $f(x) = (-1)^{\left[\frac{2x}{\pi}\right]}$, where [.] denotes the greatest integer function
d. $f(x) = x - [x+3] + \tan\left(\frac{\pi x}{2}\right)$, where [.] denotes the greatest integer function, and a is a rational number

12. If $f: R^+ \to R^+$ is a polynomial function satisfying the functional equation $f(f(x)) = 6x - f(x)$, then $f(17)$ is equal to
a. 17 b. −51
c. 34 d. −34

13. Let $f: R \to R$ be a function defined by $f(x+1) = \frac{f(x)-5}{f(x)-3}$ $\forall x \in R$. Then which of the following statement(s) is/are true
a. $f(2008)=f(2004)$ b. $f(2006)=f(2010)$
c. $f(2006)=f(2002)$ d. $f(2006)=f(2018)$

14. Let $f(x)=\sec^{-1}[1+\cos^2 x]$ where [.] denotes the greatest integer function. Then
a. the domain of f is R
b. the domain of f is $[1, 2]$
c. the domain of f is $[1, 2]$
d. the range of f is $\{\sec^{-1} 1, \sec^{-1} 2\}$

15. Which of the following pairs of functions is/are identical?
a. $f(x)=\tan(\tan^{-1} x)$ and $g(x)=\cot(\cot^{-1} x)$
b. $f(x)=\text{sgn}(x)$ and $g(x)=\text{sgn}(\text{sgn}(x))$
c. $f(x)=\cot^2 x \cdot \cos^2 x$ and $g(x)=\cot^2 x - \cos^2 x$
d. $f(x)=e^{\ln \sec^{-1} x}$ and $g(x)=\sec^{-1} x$

16. $f: R \to [-1, \infty)$ and $f(x)=\ln([|\sin 2x|+|\cos 2x|])$ (where [.] is the greatest integer function).
a. $f(x)$ has range Z
b. $f(x)$ is periodic with fundamental period $\pi/4$
c. $f(x)$ is invertible in $\left[0, \frac{\pi}{4}\right]$
d. $f(x)$ is into function

17. Which of the following is/are not a function ([.] and {.} denotes the greatest integer and fractional part functions respectively)?
a. $\frac{1}{\ln[1-|x|]}$ b. $\frac{x!}{\{x\}}$
c. $x!\ \{x\}$ d. $\frac{\ln(x-1)}{\sqrt{(1-x^2)}}$

18. If the following functions are defined from $[-1, 1]$ to $[-1, 1]$, select those which are not objective
a. $\sin(\sin^{-1} x)$ b. $\frac{2}{\pi}\sin^{-1}(\sin x)$
c. $(\text{sgn}(x)) \ln(e^x)$ d. $x^3(\text{sgn}(x))$

19. If $f: R \to N \cup \{0\}$, where f(area of triangle joining points $P(5, 0)$, $Q(8, 4)$ and $R(x, y)$ such that the angle PRQ is a right) = number of triangle. Then, which of the following is true ?
a. $f(5)=4$ b. $f(7)=0$
c. $f(6.25)=2$ d. $f(x)$ is into

20. If $f(x)$ is a polynomial of degree n such that $f(0)=0$, $f(1)=\frac{1}{2}, \ldots, f(n)=\frac{n}{n+1}$, then the value of $f(n+1)$ is

a. 1 when n is odd
b. $\frac{n}{n+2}$ when n is even
c. $-\frac{n}{n+1}$ when n is odd
d. -1 when n is even

21. Let $f(x)=\frac{3}{4}x+1$, and $f^n(x)$ be defined as $f^2(x)=f(f(x))$, and for $n\geq 2$, $f^{n+1}(x)=f(f^n(x))$. If $\lambda=\lim_{n\to\infty} f^n(x)$, then

a. λ is independent of x
b. λ is a linear polynomial in x
c. the line $y=\lambda$ has slope 0
d. the line $4y=\lambda$ touches the unit circle with centre at the origin.

22. The domain of the function

$$f(x)=\log_e\left\{\log_{|\sin x|}(x^2-8x+23)-\frac{3}{\log_2|\sin x|}\right\}$$

contains which of the following interval / intervals.

a. $(3,\pi)$
b. $\left(\pi,\frac{3}{2}\right)$
c. $\left(\frac{3\pi}{2},5\right)$
d. None of these

23. Let $f(x)=\text{sgn}(\cot^{-1}x)+\tan\left(\frac{\pi}{2}[x]\right)$, where $[x]$ is the greatest integer function less than or equal to x. Then which of the following alternatives is/are true?

a. $f(x)$ is many one but not even function
b. $f(x)$ is periodic function
c. $f(x)$ is bounded function
d. Graph of $f(x)$ remains above the x-axis

Reasoning Type

Solutions on page 1.75

Each question has four choices a, b, c and d, out of which *only one* is correct. Each question contains STATEMENT 1 and STATEMENT 2.

a. if both the statements are TRUE and STATEMENT 2 is the correct explanation of STATEMENT 1.
b. if both the statements are TRUE but STATEMENT 2 is NOT the correct explanation of STATEMENT 1.
c. if STATEMENT 1 is TRUE and STATEMENT 2 is FALSE.
d. if STATEMENT 1 is FALSE and STATEMENT 2 is TRUE.

1. **Statement 1:** $f(x)=\log_e x$ cannot be expressed as a sum of odd and even function.
Statement 2: $f(x)=\log_e x$ is neither odd nor even function.

2. **Statement 1:** If $g(x)=f(x)-1$. If $f(x)+f(1-x)=2\ \forall\ x\in R$, then $g(x)$ is symmetrical about the point $(1/2, 0)$.
Statement 2: If $g(a-x)=-g(a+x)\ \forall\ x\in R$, then $g(x)$ is symmetrical about the point $(a, 0)$.

3. Consider the function satisfying the relation

$$\text{if } f\left(\frac{2\tan x}{1+\tan^2 x}\right)=\frac{(1+\cos 2x)(\sec^2 x+2\tan x)}{2}$$

Statement 1: Range of $y=f(x)$ is R.
Statement 2: Linear function has range R if domain is R.

4. Consider the function if $f(x)=\sin(kx)+\{x\}$, where $\{x\}$ represents the fractional part function.
Statement 1: $f(x)$ is periodic for $k=m\pi$ where m is a rational number.
Statement 2: The sum of two periodic functions is always periodic.

5. **Statement 1:** Function $f(x)=x^2+\tan^{-1}x$ is a non-periodic function.
Statement 2: The sum of two non-periodic function is always non-periodic.

6. **Statement 1:** If $x\in[1,\sqrt{3}]$, then the range of $f(x)=\tan^{-1}x$ is $[\pi/4, \pi/3]$.
Statement 2: If $x\in[a, b]$, then the range of $f(x)$ is $[f(a), f(b)]$.

7. **Statement 1:** $f: N\to R$, $f(x)=\sin x$ is a one-one function.
Statement 2: The period of $\sin x$ is 2π and 2π is an irrational number.

8. **Statement 1:** A continuous surjective function $f: R\to R$, $f(x)$ can never be a periodic function.
Statement 2: For a surjective function $f: R\to R$, $f(x)$ to be periodic, it should necessarily be a discontinuous function.

9. **Statement 1:** The solution of equation $||x^2-5x+4|-|2x-3||=|x^2-3x+1|$ is $x\in(-\infty,1]\cup\left[\frac{3}{2},4\right]$.
Statement 2: If $|x+y|=|x|+|y|$, then $x.y\geq 0$.

10. Consider f and g be real-valued functions such that $f(x+y)+f(x-y)=2f(x).g(y)\ \forall\ x, y\in R$.
Statement 1: If $f(x)$ is not identically zero and $|f(x)|\leq 1\ \forall\ x\in R$, then $|g(y)|\leq 1\ \forall\ y\in R$.
Statement 2: For any two real numbers x and y, $|x+y|\leq |x|+|y|$.

11. **Statement 1:** $f(x)=\cos(x^2-\tan x)$ is a non-periodic function.
Statement 2: $x^2-\tan x$ is a non-periodic function.

12. **Statement 1:** The period of function $f(x)=\sin\{x\}$ is 1, where $\{.\}$ represents fractional part function.
Statement 2: $g(x)=\{x\}$ has period 1.

13. **Statement 1:** If $f: R\to R$, $y=f(x)$ is periodic and continuous function, then $y=f(x)$ cannot be onto.
Statement 2: A continuous periodic function is bounded.

14. Consider the functions $f(x)=\log_e x$ and $g(x)=2x+3$.
Statement 1: $f(g(x))$ is a one-one function.
Statement 2: $g(x)$ is a one-one function.

15. Consider the functions $f: R\to R$, $f(x)=x^3$ and $g: R\to R$, $g(x)=3x+4$.
Statement 1: $f(g(x))$ is an onto an function.
Statement 2: $g(x)$ is an onto function.

16. **Statement 1:** $f(x)=\sin x$ and $g(x)=\cos x$ are identical functions.
Statement 2: Both the functions have the same domain and range.

17. **Statement 1:** The period of $f(x)=\sin x$ is $2\pi \Rightarrow$ the period of $g(x)=|\sin x|$ is π.
Statement 2: The period of $f(x)=\cos x$ is $2\pi \Rightarrow$ the period of $g(x)=|\cos x|$ is π.

18. **Statement 1:** $f(x)=\sqrt{ax^2+bx+c}$ has a range $[0,\infty)$ if $b^2-4ac>0$.
Statement 2: $ax^2+bx+c=0$ has real roots if $b^2-4ac=0$.
19. **Statement 1:** If $f(x)=\cos x$ and $g(x)=x^2$, then $f(g(x))$ is an even function.
Statement 2: If $f(g(x))$ is an even function, then both $f(x)$ and $g(x)$ must be even function.
20. **Statement 1:** The graph of $y=\sec^2 x$ is symmetrical about y-axis.
Statement 2: The graph of $y=\tan x$ is symmetrical about origin.

Linked Comprehension Type

Solutions on page 1.76

Based upon each paragraph, three multiple choice questions have to be answered. Each question has four choices a, b, c and d, out of which *only one* is correct.

For Problems 1–3

Consider the functions

$$f(x)=\begin{cases}x+1, & x\le 1\\ 2x+1, & 1<x\le 2\end{cases} \text{ and } g(x)=\begin{cases}x^2, & -1\le x<2\\ x+2, & 2\le x\le 3\end{cases}$$

1. The domain of the function $f(g(x))$ is
   a. $[0,\sqrt{2}]$ b. $[-1,2]$
   c. $[-1,\sqrt{2}]$ d. None of these
2. The range of the function $f(g(x))$ is
   a. $[1,5]$ b. $[2,3]$
   c. $[1,2]\cup(3,5]$ d. None of these
3. The number of roots of the equation $f(g(x))=2$ is
   a. 1 b. 2
   c. 4 d. None of these

For Problems 4–6

Consider the function $f(x)$ satisfying the identity $f(x)+f\left(\dfrac{x-1}{x}\right)=1+x, \forall\ x\in R-\{0,1\}$ and $g(x)=2f(x)-x+1$.

4. The domain of $y=\sqrt{g(x)}$ is
   a. $\left(-\infty,\dfrac{1-\sqrt{5}}{2}\right]\cup\left[1,\dfrac{1+\sqrt{5}}{2}\right]$
   b. $\left(-\infty,\dfrac{1-\sqrt{5}}{2}\right]\cup(0,1)\cup\left[\dfrac{1+\sqrt{5}}{2},\infty\right)$
   c. $\left[\dfrac{-1-\sqrt{5}}{2},0\right]\cup\left[\dfrac{-1+\sqrt{5}}{2},1\right]$
   d. None of these
5. The range of $y=g(x)$ is
   a. $(-\infty,5]$ b. $[1,\infty)$
   c. $(-\infty,1]\cup[5,\infty)$ d. None of these
6. The number of roots of the equation $g(x)=1$ is
   a. 2 b. 1 c. 3 d. 0

For Problems 7–9

Let $f: N\to R$ be a function satisfying the following conditions, $f(1)=1/2$ and $f(1)+2,f(2)+3,f(3)+\cdots+nf(n)=n(n+1)$, $f(n)$ for $n\ge 2$.

7. The value of $f(1003)=\dfrac{1}{K}$, where K equals
   a. 1003 b. 2003 c. 2005 d. 2006
8. The value of $f(999)$ is $\dfrac{1}{K}$, where K equals
   a. 999 b. 1000 c. 1998 d. 2000
9. $f(1), f(2), f(3), f(4), \ldots$ represents a series of
   a. an A.P b. a G.P.
   c. a H.P. d. An arithmetico-geometric

For Problems 10–12

If $(f(x))^2\times f\left(\dfrac{1-x}{1+x}\right)=64x,\ \forall x\in Df$, then

10. $f(x)$ is equal to
    a. $4x^{2/3}\left(\dfrac{1+x}{1-x}\right)^{1/3}$ b. $x^{1/3}\left(\dfrac{1-x}{1+x}\right)^{1/3}$
    c. $x^{2/3}\left(\dfrac{1-x}{1+x}\right)^{1/3}$ d. $x\left(\dfrac{1+x}{1-x}\right)^{1/3}$
11. The domain of $f(x)$ is
    a. $[0,\infty)$ b. $R-\{1\}$
    c. $(-\infty,\infty)$ d. None of these
12. The value of $f(9/7)$ is
    a. $8(7/9)^{2/3}$ b. $4(9/7)^{1/3}$
    c. $-8(9/7)^{2/3}$ d. None of these

For Problems 13–15

$f(x)=\begin{cases}x-1, & -1\le x\le 0\\ x^2, & 0\le x\le 1\end{cases}$ and $g(x)=\sin x$. Consider the functions $h_1(x)=f(|g(x)|)$ and $h_2(x)=|f(g(x))|$.

13. Which of the following is not true about $h_1(x)$?
    a. It is periodic function with period π
    b. Range is $[0, 1]$
    c. Domain is R
    d. None of these
14. Which of the following is not true about $h_2(x)$?
    a. Domain is R
    b. It periodic function with period 2π
    c. Range is $[0, 1]$
    d. None of these
15. For $h_1(x)$ and $h_2(x)$ are identical function, then which of the following is not true ?
    a. Domain of $h_1(x)$ and $h_2(x)$, $x\in[2n\pi\ (2n+1)\pi], n\in Z$
    b. Range of $h_1(x)$ and $h_2(x)$ is $[0, 1]$
    c. Period of $h_1(x)$ and $h_2(x)$ is π
    d. None of these

For Problems 16–18

If $a_0 = x$, $a_{n+1} = f(a_n)$, where $n = 0, 1, 2, \ldots$, then answer the following questions.

16. If $f(x) = \sqrt[m]{(a - x^m)}, x > 0, m \geq 2, m \in N$. Then
   a. $a_n = x, n = 2k + 1$, where k is integer
   b. $a_n = f(x)$ if $n = 2k$, where k is integer
   c. Inverse of a_n exists for any value of n and m
   d. None of these

17. If $f(x) = \dfrac{1}{1-x}$, then which of the following is not true?
   a. $a_n = \dfrac{1}{1-x}$ if $n = 3k + 1$
   b. $a_n = \dfrac{x-1}{x}$ if $n = 3k + 2$
   c. $a_n = x$ if $n = 3k$
   d. None of these

18. If $f: R \to R$ be given by $f(x) = 3 + 4x$ and $a_n = A + Bx$, then which of the following is not true?
   a. $A + B + 1 = 2^{2n+1}$ b. $|A - B| = 1$
   c. $\lim\limits_{n\to\infty} \dfrac{A}{B} = -1$ d. None of these

For Problems 19–21

Let $f(x) = f_1(x) - 2f_2(x)$,

where $f_1(x) = \begin{cases} \min\{x^2, |x|\}, & |x| \leq 1 \\ \max\{x^2, |x|\}, & |x| > 1 \end{cases}$

and $f_2(x) = \begin{cases} \min\{x^2, |x|\}, & |x| > 1 \\ \max\{x^2, |x|\}, & |x| \leq 1 \end{cases}$

and $g(x) = \begin{cases} \min\{f(t) : -3 \leq t \leq x, & -3 \leq x < 0\} \\ \max\{f(t) : 0 \leq t \leq x, & 0 \leq x \leq 3\}. \end{cases}$

19. For $-3 \leq x \leq -1$, the range of $g(x)$ is
   a. $[-1, 3]$ b. $[-1, -15]$
   c. $[-1, 9]$ d. None of these

20. For $x \in (-1, 0)$, $f(x) + g(x)$ is
   a. $x^2 - 2x + 1$ b. $x^2 + 2x - 1$
   c. $x^2 + 2x + 1$ d. $x^2 - 2x - 1$

21. The graph of $y = g(x)$ in its domain is broken at
   a. 1 point b. 2 points
   c. 3 points d. None of these

For Problems 22–24

Let $f(x) = \begin{cases} 2x + a, & x \geq -1 \\ bx^2 + 3, & x < -1 \end{cases}$

and $g(x) = \begin{cases} x + 4, & 0 \leq x \leq 4 \\ -3x - 2, & -2 < x < 0 \end{cases}$

22. $g(f(x))$ is not defined if
   a. $a \in (10, \infty), b \in (5, \infty)$ b. $a \in (4, 10), b \in (5, \infty)$
   c. $a \in (10, \infty), b \in (0, 1)$ d. $a \in (4, 10), b \in (1, 5)$

23. If the domain of $g(f(x))$ is $[-1, 4]$, then
   a. $a = 1, b > 5$ b. $a = 2, b > 7$
   c. $a = 2, b > 10$ d. $a = 0, b \in R$

24. If $a = 2$ and $b = 3$, then the range of $g(f(x))$ is
   a. $(-2, 8]$ b. $(0, 8]$
   c. $[4, 8]$ d. $[-1, 8]$

For Problems 25–27

Let $f: R \to R$ is a function satisfying $f(2 - x) = f(2 + x)$ and $f(20 - x) = f(x)$, $\forall\, x \in R$. For this function f, answer the following.

25. If $f(0) = 5$, then the minimum possible number of values of x satisfying $f(x) = 5$, for $x \in [0, 170]$, is
   a. 21 b. 12
   c. 11 d. 22

26. The graph of $y = f(x)$ is not symmetrial about
   a. symmetrical about $x = 2$
   b. symmetrical about $x = 10$
   c. symmetrical about $x = 8$
   d. None of these

27. If $f(2) \neq f(6)$, then the
   a. fundamental period of $f(x)$ is 1
   b. fundamental period of $f(x)$ may be 1
   c. period of $f(x)$ cannot be 1
   d. fundamental period of $f(x)$ is 8

For Problems 28–30

Consider two functions $f(x) = \begin{cases} [x], & -2 \leq x \leq -1 \\ |x| + 1, & -1 < x \leq 2 \end{cases}$ and

$g(x) = \begin{cases} [x], & -\pi \leq x < 0 \\ \sin x, & 0 \leq x \leq \pi \end{cases}$, where [.] denotes the greatest integer function.

28. The exhaustive domain of $g(f(x))$ is
   a. $[0, 2]$ b. $[-2, 0]$
   c. $[-2, 2]$ d. $[-1, 2]$

29. The range of $g(f(x))$ is
   a. $[\sin 3, \sin 1]$ b. $[\sin 3, 1] \cup \{-2, -1, 0\}$
   c. $[\sin 1, 1] \cup \{-2, -1\}$ d. $[\sin 1, 1]$

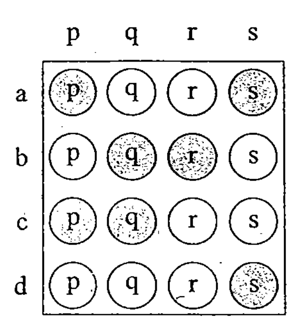
30. The number of integral points in the range of $g(f(x))$ is
   a. 2 b. 4
   c. 3 d. 5

Matrix-Match Type

Solutions on page 1.79

Each question contains statements given in two columns which have to be matched.

Statements a, b, c, d in column I have to be matched with statements p, q, r, s in column II. If the correct match is a-p, a-s, b-q, b-r, c-p, c-q and d-s, then the correctly bubbled 4 × 4 matrix should be as follows:



1. The function $f(x)$ is defined on the interval [0, 1].
Then match the following columns

Column I: Function	Column II: Domain
a. $f(\tan x)$	p. $\left[2n\pi-\frac{\pi}{2}, 2n\pi+\frac{\pi}{2}\right], n \in Z$
b. $f(\sin x)$	q. $\left[2n\pi, 2n\pi+\frac{\pi}{6}\right]\cup\left[2n\pi+\frac{5\pi}{6}, (2n+1)\pi\right], n\in Z$
c. $f(\cos x)$	r. $[2n\pi, (2n+1)\pi], n \in Z$
d. $f(2\sin x)$	s. $\left[n\pi, n\pi+\frac{\pi}{4}\right], n \in Z$

2.

Column I: Function	Column II: Type of function
a. $f(x)=\{(\operatorname{sgn} x)^{\operatorname{sgn} x}\}^n; x \neq 0$, n is an odd integer	p. odd function
b. $f(x)=\frac{x}{e^x-1}+\frac{x}{2}+1$	q. even function
c. $f(x)=\begin{cases}0, & \text{If } x \text{ is rational}\\ 1, & \text{If } x \text{ is irrational}\end{cases}$	r. neither odd nor even function
d. $f(x)=\max\{\tan x, \cot x\}$	s. periodic

3.

Column I: Functions	Column II: Values of x for which both the functions in any option of the column I are identical
a. $f(x)=\tan^{-1}\left(\frac{2x}{1-x^2}\right), g(x)=2\tan^{-1}x$	p. $x\in\{-1, 1\}$
b. $f(x)=\sin^{-1}(\sin x)$ and $g(x)=\sin(\sin^{-1}x)$	q. $x\in[-1, 1]$
c. $f(x)=\log_{x^2} 25$ and $g(x)=\log_x 5$	r. $x\in(-1, 1)$
d. $f(x)=\sec^{-1}x+\operatorname{cosec}^{-1}x, g(x)=\sin^{-1}x+\cos^{-1}x$	s. $x\in(0, 1)$

4.

Column I	Column II
a. $f: R\to\left[\frac{3\pi}{4},\pi\right)$ and $f(x)=\cot^{-1}(2x-x^2-2)$, then $f(x)$ is	p. one-one
b. $f: R\to R$ and $f(x)=e^{px}\sin qx$ where $p, q\in R^+$, then $f(x)$ is	q. into
c. $f: R^+\to[4, \infty]$ and $f(x)=4+3x^2$, then $f(x)$ is'	r. many-one
d. $f: X\to X$ and $f(f(x))=x\ \forall\ x\in X$, then $f(x)$ is	s. onto

5. Let $f: R\to R$ and $g: R\to R$ be functions such that $f(g(x))$ is a one-one function.

Column I	Column II
a. Then $g(x)$	p. must be one-one
b. Then $f(x)$	q. may not be one-one
c. If $g(x)$ is onto then $f(x)$	r. may be many-one
d. If $g(x)$ is into then $f(x)$	s. must be many-one

6.

Column I: Function	Column II: Period
a. $f(x)=\cos(\lvert\sin x\rvert-\lvert\cos x\rvert)$	p. π
b. $f(x)=\cos(\tan x+\cot x)\cdot\cos(\tan x-\cot x)$	q. $\pi/2$
c. $f(x)=\sin^{-1}(\sin x)+e^{\tan x}$	r. 4π
d. $f(x)=\sin^3 x\sin 3x$	s. 2π

7. {.} denotes the fractional part function and [.] denotes the greatest integer function:

Column I: (Function)	Column II: (Period)
a. $f(x)=e^{\cos^4\pi x+x-[x]+\cos^2\pi x}$	p. 1/3
b. $f(x)=\cos 2\pi\{2x\}+\sin 2\pi\{2x\}$	q. 1/4
c. $f(x)=\sin 3\pi\{x\}+\tan\pi[x]$	r. 1/2
d. $f(x)=3x-[3x+a]-b$, where $a, b\in R^+$	s. 1

8.

Column I: (Function)	Column II: (Range)
a. $f(x)=\log_3(5+4x-x^2)$	p. function not defined
b. $f(x)=\log_3(x^2-4x-5)$	q. $[0,\infty)$
c. $f(x)=\log_3(x^2-4x+5)$	r. $(-\infty, 2]$
d. $f(x)=\log_3(4x-5-x^2)$	s. R

9.

Column I: Equation	Column II: Number of roots
a. $x^2\tan x=1, x\in[0,2\pi]$	p. 5
b. $2^{\cos x}=\lvert\sin x\rvert, x\in[0,2\pi]$	q. 2
c. If $f(x)$ is a polynomial of degree 5 with real coefficients such that $f(\lvert x\rvert)=0$ has 8 real roots, then the number of roots of $f(x)=0$	r. 3
d. $7^{\lvert x\rvert}(\lvert 5-\lvert x\rvert\rvert)=1$	s. 4

Integer Type

Solutions on page 1.81

1. Let f be a real-valued invertible function such that $f\left(\frac{2x-3}{x-2}\right)=5x-2,\ x\neq 2$. Then the value of $f^{-1}(13)$ is

2. Number of values of x for which $\left\lvert\left\lvert\lvert x^2-x+4\rvert-2\right\rvert-3\right\rvert=x^2+x-12$ is

3. Let $f(x)=3x^2-7x+c$, where 'c' is a variable coefficient and $x>\frac{7}{6}$. Then the value of $[c]$ such that $f(x)$ touches $f^{-1}(x)$ is (where [·] represents greatest integer function)

4. Number of integral values of x for which
$$\frac{\left(2^{\frac{\pi}{\tan^{-1}x}}-4\right)(x-4)(x-10)}{x!-(x-1)!}<0 \text{ is}$$

5. Let $f: R^+\to R$ be a function which satisfies $f(x)\cdot f(y)=f(xy)+2\left(\frac{1}{x}+\frac{1}{y}+1\right)$ for $x, y>0$, then possible value of $f(1/2)$ is

6. A continuous function $f(x)$ on $R\to R$ satisfies the relation $f(x)+f(2x+y)+5xy=f(3x-y)+2x^2+1$ for $\forall x, y\in R$, then the value of $\lvert f(4)\rvert$ is

7. Let $a>2$ be a constant. If there are just 18 positive integers satisfying the inequality $(x-a)(x-2a)(x-a^2)<0$, then the value of a is

8. Number of integers in the domain of function, satisfying $f(x)+f(x^{-1})=\frac{x^3+1}{x}$, is

9. $f: R\to R$ $f(x^2+x+3)+2f(x^2-3x+5)=6x^2-10x+17$ $\forall x\in R$, then the value of $f(5)$ is

10. If $f(x)$ is an odd function and $f(1)=3$, and $f(x+2)=f(x)+f(2)$, then the value of $f(3)$ is

11. Let $f: R\to R$ be a continuous onto function satisfying $f(x)+f(-x)=0,\ \forall x\in R$.
If $f(-3)=2$ and $f(5)=4$ in $[-5, 5]$, then the minimum number of roots of the equation $f(x)=0$ is

12. Number of integral values of x for which the function $\sqrt{\sin x+\cos x}+\sqrt{7x-x^2-6}$ is defined is

13. Suppose that f is an even, periodic function with period 2, and that $f(x) = x$ for all x in the interval [0, 1]. The value of $[10\, f(3.14)]$ is (where [·] represents the greatest integer function)

14. If $f(x) = \sqrt{4-x^2} + \sqrt{x^2-1}$, then the maximum value of $(f(x))^2$ is

15. The function $f(x) = \dfrac{x+1}{x^3+1}$ can be written as the sum of an even function $g(x)$ and an odd function $h(x)$. Then the value of $|g(0)|$ is

16. If T is the period of the function $f(x) = [8x + 7] + |\tan 2\pi x + \cot 2\pi x| - 8x$ (where [·] denotes the greatest integer function), then the value of $1/T$ is

17. If a, b and c are non-zero rational numbers, then the sum of all the possible values of $\dfrac{|a|}{a} + \dfrac{|b|}{b} + \dfrac{|c|}{c}$ is

18. An even polynomial function $f(x)$ satisfies a relation $f(2x)\left(1 - f\left(\dfrac{1}{2x}\right)\right) + f(16x^2y) = f(-2) - f(4xy)\ \forall\ x, y \in R - \{0\}$ and $f(4) = -255$, $f(0) = 1$, then the value of $|(f(2)+1)/2|$ is

19. If $f(x) = \sin^2 x + \sin^2\left(x + \dfrac{\pi}{3}\right) + \cos x \cos\left(x + \dfrac{\pi}{3}\right)$ and $g\left(\dfrac{5}{4}\right) = 1$ then $(gof)(x)$ is

20. Let $E = \{1, 2, 3, 4\}$ and $F = \{1, 2\}$. If N is number of onto functions from E to F, then the value of $N/2$ is

21. The function f is continuous and has the property $f(f(x)) = 1 - x$, then the value of $f\left(\dfrac{1}{4}\right) + f\left(\dfrac{3}{4}\right)$ is

22. Number of integral values of x satisfying the inequality $\left(\dfrac{3}{4}\right)^{6x+10-x^2} < \dfrac{27}{64}$

23. A function f from integers to integers is defined as $f(x) = \begin{cases} n+3, & n \in \text{odd} \\ n/2, & n \in \text{even} \end{cases}$. Suppose $k \in$ odd and $f(f(f(k))) = 27$, then the sum of digits of k is

24. If θ be the fundamental period of function $f(x) = \sin^{99}x + \sin^{99}\left(x + \dfrac{2\pi}{3}\right) + \sin^{99}\left(x + \dfrac{4\pi}{3}\right)$, then complex number $z = |z|(\cos\theta + i\sin\theta)$ lies in the quadrant number.

25. If $x = \dfrac{4}{9}$ satisfy the equation $\log_a(x^2 - x + 2) > \log_a(-x^2 + 2x + 3)$, then sum of all possible distinct values of $[x]$ is (where [·] represesnts greatest integer function)

26. If $4^x - 2^{x+2} + 5 + ||b-1|-3| = |\sin y|$, $x, y, b \in R$, then the possible value of b is

27. If $f: N \to N$, and $x_2 > x_1 \Rightarrow f(x_2) > f(x_1)$, $\forall\ x_1, x_2 \in N$ and $f(f(n)) = 3n$, $\forall\ n \in N$, then $f(2) =$

28. Number of integral values of a for which $f(x) = \log(\log_{1/3}(\log_7(\sin x + a)))$ be defined for every real value of x

29. Let $f(x) = \sin^{23}x - \cos^{22}x$ and $g(x) = 1 + \dfrac{1}{2}\tan^{-1}|x|$, then the number of values of x in interval $[-10\pi, 8\pi]$ satisfying the equation $f(x) = \text{sgn}(g(x))$ is

30. Suppose that $f(x)$ is a function of the form $f(x) = \dfrac{ax^8 + bx^6 + cx^4 + dx^2 + 15x + 1}{x}$ $(x \neq 0)$. If $f(5) = 2$, then the value of $|f(-5)|/4$ is

Archives

Solutions on page 1.84

Subjective

1. Find the domain and range of the function $f(x) = \dfrac{x^2}{1+x^2}$. Is the function one-to-one? **(IIT-JEE, 1978)**

2. Draw the graph of $y = |x|^{1/2}$ for $-1 \le x \le 1$ **(IIT-JEE, 1978)**

3. If $f(x) = x^9 - 6x^8 - 2x^7 + 12x^6 + x^4 - 7x^3 + 6x^2 + x - 3$, find $f(6)$. **(IIT-JEE, 1979)**

4. Let f be a one-one function with domain $\{x, y, z\}$ and range $\{1, 2, 3\}$. It is given that exactly one of the following statements is true and the remaining two are false $f(x) = 1$, $f(y) \neq 1$, $f(z) \neq 2$ determine $f^{-1}(1)$. **(IIT-JEE, 1982)**

5. Find the natural number a for which $\sum_{k=1}^{n} f(a + k) = 16(2^n - 1)$, where the function f satisfies the relation $f(x + y) = f(x)\, f(y)$ for all natural numbers x, y and further $f(1) = 2$. **(IIT-JEE, 1992)**

6. Let $\{x\}$ and $[x]$ denote the fractional and integral part of a real number x, respectively. Solve $4\{x\} = x + [x]$. **(IIT-JEE, 1992)**

7. A function $f: R \to R$ is defined by $f(x) = \dfrac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$. Find the interval of values of α for which f is onto. Is the function one-to-one for $\alpha = 3$? Justify your answer. **(IIT-JEE, 1996)**

Objective

Fill in the blanks

1. The values of $f(x) = 3\sin\left(\sqrt{\dfrac{\pi^2}{16} - x^2}\right)$ lie in the interval ________ **(IIT-JEE, 1983)**

2. The domain of the function $f(x) = \sin^{-1}\left(\log_2 \dfrac{x^2}{2}\right)$ is given by ________. **(IIT-JEE, 1984)**

3. Let A be a set of n distinct elements. Then the total number of distinct functions from A to A is ________ and out of these ________ are onto functions. **(IIT-JEE, 1985)**

4. If $f(x) = \sin \log_e\left(\dfrac{\sqrt{4-x^2}}{1-x}\right)$, then the domain of $f(x)$ is ________ and its range is ________. **(IIT-JEE, 1985)**

5. There are exactly two distinct linear functions, ______, and ______ which map $[-1, 1]$ onto $[0, 2]$.

6. If f is an even function defined on the interval $(-5, 5)$, then four real values of x satisfying the equation $f(x) = f\left(\frac{x+1}{x+2}\right)$ are ______, ______, ______, and ______. **(IIT-JEE, 1985)**

7. If $f(x) = \sin^2 x + \sin^2\left(x + \frac{\pi}{3}\right) + \cos x \cos\left(x + \frac{\pi}{3}\right)$ and $g\left(\frac{5}{4}\right) = 1$, then $(gof)(x) =$ ______. **(IIT-JEE, 1996)**

8. The domain of the function $f(x) = \sin^{-1}\left(\frac{8(3)^{x-2}}{1-3^{2(x-1)}}\right)$ is **(IIT-JEE, 2011)**

True or false

1. If $f(x) = (a - x^n)^{1/n}$ where $a > 0$ and n is a positive integer, then $f[f(x)] = x$. **(IIT-JEE, 1983)**

2. The function $f(x) = \frac{x^2 + 4x + 30}{x^2 - 8x + 18}$ is not onto. **(IIT-JEE, 1983)**

3. If $f_1(x)$ and $f_2(x)$ are defined on the domain D_1 and D_2 respectively, then $f_1(x) + f_2(x)$ is defined on $D_1 \cup D_2$. **(IIT-JEE, 1988)**

Multiple choice questions with one correct answer

1. Let R be the set of real numbers. If $f : R \to R$ is a function defined by $f(x) = x^2$, then f is
   a. Injective but not surjective
   b. Surjective but not injective
   c. Bijective
   d. None of these **(IIT-JEE, 1979)**

2. The entire graph of the equation $y = x^2 + kx - x + 9$ is strictly above the x-axis if and only if
   a. $k < 7$ b. $-5 < k < 7$
   c. $k > -5$ d. None of these **(IIT-JEE, 1979)**

3. Let $f(x) = |x - 1|$. Then
   a. $f(x^2) = (f(x))^2$ b. $f(x+y) = f(x) + f(y)$
   c. $f(|x|) = |f(x)|$ d. None of these **(IIT-JEE, 1983)**

4. If x satisfies $|x-1| + |x-2| + |x-3| \ge 6$, then
   a. $0 \le x \le 4$ b. $x \le -2$ or $x \ge 4$
   c. $x \le 0$ or $x \ge 4$ d. None of these **(IIT-JEE, 1983)**

5. If $f(x) = \cos(\log_e x)$, then $f(x)f(y) - \frac{1}{2}\left[f\left(\frac{x}{y}\right) + f(xy)\right]$ has the value
   a. -1 b. $1/2$
   c. -2 d. None of these **(IIT-JEE, 1983)**

6. The domain of definition of the function $y = \frac{1}{\log_{10}(1-x)} + \sqrt{x+2} =$
   a. $(-3, -2)$ excluding -2.5 b. $[0, 1]$ excluding 0.5
   c. $[-2, 1)$ excluding 0 d. None of these **(IIT-JEE, 1983)**

7. Which of the following functions is periodic?
   a. $f(x) = x - [x]$ where $[x]$ denotes the largest integer less than or equal to the real number x
   b. $f(x) = \sin\frac{1}{x}$ for $x \ne 0$, $f(0) = 0$
   c. $f(x) = x \cos x$
   d. None of these **(IIT-JEE, 1983)**

8. If the function $f : [1, \infty) \to [1, \infty)$ is defined by $f(x) = 2^{x(x-1)}$, then $f^{-1}(x)$ is
   a. $\left(\frac{1}{2}\right)^{x(x-1)}$ b. $\frac{1}{2}\left(1 + \sqrt{1 + 4\log_2 x}\right)$
   c. $\frac{1}{2}\left(1 - \sqrt{1 + 4\log_2 x}\right)$ d. Not defined **(IIT-JEE, 1992)**

9. Let $f(x) = \sin x$ and $g(x) = \log_e|x|$. If the ranges of the composition function fog and gof are R_1 and R_2, respectively, then
   a. $R_1 = \{u : -1 \le u < 1\}, R_2 = \{v : -\infty < v < 0\}$
   b. $R_1 = \{u : -\infty < u < 0\}, R_2 = \{v : -\infty < v < 0\}$
   c. $R_1 = \{u : -1 < u < 1\}, R_2 = \{v : -\infty < v < 0\}$
   d. $R_1 = \{u : -1 \le u \le 1\}, R_2 = \{v : -\infty < v \le 0\}$ **(IIT-JEE, 1994)**

10. Let $f(x) = (x+1)^2 - 1, x \ge -1$. Then the set $\{x : f(x) = f^{-1}(x)\}$ is
   a. $\left\{0, -1, \frac{-3 + i\sqrt{3}}{2}, \frac{-3 - i\sqrt{3}}{2}\right\}$
   b. $\{0, 1, -1\}$
   c. $\{0, -1\}$
   d. empty **(IIT-JEE, 1995)**

11. Let $f(x)$ be defined for all $x > 0$ and be continuous. Let $f(x)$ satisfy $f\left(\frac{x}{y}\right) = f(x) - f(y)$ for all x, y and $f(e) = 1$. Then
   a. $f(x)$ is bounded b. $f\left(\frac{1}{x}\right) \to 0$ as $x \to 0$
   c. $x f(x) \to 1$ as $x \to 0$ d. $f(x) = \log_e x$ **(IIT-JEE, 1995)**

12. The domain of definition of the function $f(x)$ given by the equation $2^x + 2^y = 2$ is
   a. $0 < x \le 1$ b. $0 \le x \le 1$
   c. $-\infty < x \le 0$ d. $-\infty < x < 1$ **(IIT-JEE, 2000)**

13. Let $g(x) = 1 + x - [x]$ and $f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$. Then for all x, $f(g(x))$ is equal to (where $[\cdot]$ represents greatest integer function)
   a. x b. 1
   c. $f(x)$ d. $g(x)$ **(IIT-JEE, 2001)**

14. If $f : [1, \infty) \to [2, \infty)$ is given by $f(x) = x + \frac{1}{x}$, then $f^{-1}(x)$ equals **(IIT-JEE, 2001)**

a. $\frac{\left(x+\sqrt{x^2-4}\right)}{2}$ b. $\frac{x}{1+x^2}$

c. $\frac{\left(x-\sqrt{x^2-4}\right)}{2}$ d. $1+\sqrt{x^2-4}$

15. The domain of definition of $f(x)=\frac{\log_2(x+3)}{x^2+3x+2}$ is

a. $R-\{-1,-2\}$ b. $(-2,\infty)$

c. $R-\{-1,-2,-3\}$ d. $(-3,\infty)-\{-1,-2\}$

(IIT-JEE, 2001)

16. Let $E=\{1,2,3,4\}$ and $F=\{1,2\}$. Then the number of onto functions from E to F is

a. 14 b. 16 c. 12 d. 8

(IIT-JEE, 2001)

17. Let $f(x)=\frac{\alpha x}{x+1}, x\neq -1$. Then for what value of α is $f(f(x))=x$?

a. $\sqrt{2}$ b. $-\sqrt{2}$

c. 1 d. -1 (IIT-JEE, 2001)

18. Suppose $f(x)=(x+1)^2$ for $x\geq -1$. If $g(x)$ is the function whose graph is the reflection of the graph of $f(x)$ with respect to the line $y=x$, then $g(x)$ equals

a. $-\sqrt{x}-1, x\geq 0$ b. $\frac{1}{(x+1)^2}, x>-1$

c. $\sqrt{x+1}, x\geq -1$ d. $\sqrt{x}-1, x\geq 0$

(IIT-JEE, 2002)

19. Let function $f: R\rightarrow R$ be defined by $f(x)=2x+\sin x$ for $x\in R$, then f is

a. one-to-one and onto

b. one to one but NOT onto

c. onto but NOT one-to-one

d. neither one-to-one nor onto (IIT-JEE, 2002)

20. If $f:[0,\infty)\rightarrow[0,\infty)$, and $f(x)=\frac{x}{1+x}$, then f is

a. one-one and onto b. one-one but not onto

c. onto but not one-one d. neither one-one nor onto

(IIT-JEE, 2003)

21. The domain of definition of the function

$$f(x)=\sqrt{\sin^{-1}(2x)+\frac{\pi}{6}}$$

for real-valued x is

a. $\left[-\frac{1}{4},\frac{1}{2}\right]$ b. $\left[-\frac{1}{2},\frac{1}{2}\right]$

c. $\left(-\frac{1}{2},\frac{1}{9}\right)$ d. $\left[-\frac{1}{4},\frac{1}{4}\right]$

(IIT-JEE, 2003)

22. The range of the function $f(x)=\frac{x^2+x+2}{x^2+x+1}, x\in R$, is

a. $(1,\infty)$ b. $(1, 11/7)$

c. $(1, 7/3]$ d. $(1, 7/5)$ (IIT-JEE, 2003)

23. If $f(x)=\sin x+\cos x$, $g(x)=x^2-1$, then $g(f(x))$ is invertible in the domain

a. $\left[0,\frac{\pi}{2}\right]$ b. $\left[-\frac{\pi}{4},\frac{\pi}{4}\right]$

c. $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ d. $[0,\pi]$ (IIT-JEE, 2004)

24. If the functions $f(x)$ and $g(x)$ are defined on $R\rightarrow R$ such that $f(x)=\begin{cases}0, & x\in \text{rational}\\ x, & x\in\text{irrational}\end{cases}$

and $g(x)=\begin{cases}0, & x\in\text{irrational}\\ x, & x\in\text{rational}\end{cases}$ then $(f-g)(x)$ is

a. one-one and onto b. neither one-one nor onto

c. one-one but not onto d. onto but not one-one

(IIT-JEE, 2005)

25. X and Y are two sets and $f: X\rightarrow Y$. If $\{f(c)=y; c\subset X, y\subset Y\}$ and $f^{-1}(d)=x; d\subset Y, x\subset X\}$, then the true statement is

a. $f(f^{-1}(b))=b$ b. $f^{-1}(f(a))=a$

c. $f(f^{-1}(b))=b, b\subset y$ d. $f^{-1}(f(a))=a, a\subset x$

(IIT-JEE, 2005)

Multiple choice questions with one or more than one correct answer

1. If $y=f(x)=\frac{x+2}{x-1}$ then

a. $x=f(y)$

b. $f(1)=3$

c. y increases with x for $x<1$

d. f is a rational function of x (IIT-JEE, 1984)

2. Let $g(x)$ be a function defined on $[-1, 1]$. If the area of the equilateral triangle with two of its vertices at (0, 0) and $(x, g(x))$ is $\sqrt{3}/4$ then the function $g(x)$ is

a. $g(x)=\pm\sqrt{1-x^2}$ b. $g(x)=\sqrt{1-x^2}$

c. $g(x)=-\sqrt{1-x^2}$ d. $g(x)=\sqrt{1+x^2}$

(IIT-JEE, 1989)

3. If $f(x)=\cos[\pi^2]x+\cos[-\pi^2]x$, where $[x]$ stands for the greatest integer function, then

a. $f\left(\frac{\pi}{2}\right)=-1$ b. $f(\pi)=1$

c. $f(-\pi)=0$ d. $f\left(\frac{\pi}{4}\right)=1$

(IIT-JEE, 1991)

4. If $f(x)=3x-5$, then $f^{-1}(x)$

a. is given by $\frac{1}{3x-5}$

b. is given by $\frac{x+5}{3}$

c. does not exist because f is not one-one

d. does not exist because f is not onto

(IIT-JEE, 1998)

5. If $g(f(x)) = |\sin x|$ and $f(g(x)) = (\sin\sqrt{x})^2$, then

a. $f(x) = \sin^2 x, g(x) = \sqrt{x}$

b. $f(x) = \sin x, g(x) = |x|$

c. $f(x) = x^2, g(x) = \sin\sqrt{x}$

d. f and g cannot be determined **(IIT-JEE, 1998)**

Match the following type

This question contains statements given in two columns which have to be matched. Statements **a, b, c, d** in Column I have to be matched with statements **p, q, r, s** in Column II. The answers to these questions have to be appropriately bubbled as illustrated in the following example. If the correct match is **a–p, a–s, b–q, b–r, c–q** and **d–s**, then the correctly bubbled 4 × 4 matrix should be as follows:

	p	**q**	**r**	**s**
a	p	q	r	s
b	p	q	r	s
c	p	q	r	s
d	p	q	r	s

1. Let $f(x) = \dfrac{x^2 - 6x + 5}{x^2 - 5x + 6}$

Match of expressions/statements in Column I with expressions/statements in Column II

Column I	**Column II**
a. If $-1 < x < 1$, then $f(x)$ satisfies	p. $0 < f(x) < 1$
b. If $1 < x < 2$, then $f(x)$ satisfies	q. $f(x) < 0$
c. If $3 < x < 5$, then $f(x)$ satisfies	r. $f(x) > 0$
d. If $x > 5$, then $f(x)$ satisfies	s. $f(x) < 1$

(IIT-JEE, 2007)

ANSWERS AND SOLUTIONS

Subjective Type

1. a. $x + |y| = 2y$

If $y \ge 0$, we have $x + y = 2y \Rightarrow y = x$

$\Rightarrow \; y = x, x \ge 0$

If $y < 0 \quad x - y = 2y \Rightarrow y = \dfrac{x}{3}$

$\Rightarrow \; y = \dfrac{x}{3}; \quad x < 0$

$$\Rightarrow \; y = \begin{cases} \dfrac{x}{3}, & x < 0 \\ x, & x \ge 0 \end{cases} \quad D_f \equiv R.$$

b. $e^y - e^{-y} = 2x$

$\Rightarrow \; e^{2y} - 2xe^y - 1 = 0$ (Multiplying by e^y)

$\Rightarrow \; e^y = \dfrac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}$

$\Rightarrow \; e^y = x + \sqrt{x^2 + 1}$ (as $\sqrt{x^2 + 1} > x$, then $x - \sqrt{x^2 + 1} < 0$, which is not possible)

$\Rightarrow \; y = \ln(x + \sqrt{x^2 + 1})$

$D_f = R$

c. $10^x + 10^y = 10$

$\Rightarrow \; 10^y = 10 - 10^x$

$\Rightarrow \; y = \log_{10}(10 - 10^x)$

For domain $10 - 10^x > 0 \Rightarrow 10^x < 10 \Rightarrow x < 1$

$\Rightarrow \; D_f \equiv (-\infty, 1)$

d. $x^2 - \sin^{-1} y = \dfrac{\pi}{2}$

$\Rightarrow \; \sin^{-1} y = x^2 - \pi/2$

$\Rightarrow \; y = \sin(x^2 - \pi/2)$

$D_f \equiv R$

2. $g(x) = \sqrt{x - 2k}, \forall\, 2k \le x < 2(k+1)$, where $k \in$, integer

$$\Rightarrow \; g(x) = \begin{cases} \cdots \\ \cdots \\ \sqrt{x+2}, & -2 \le x < 0 \\ \sqrt{x}, & 0 \le x < 2 \\ \sqrt{x-2}, & 2 \le x < 4 \\ \sqrt{x-4}, & 4 \le x < 6 \\ \cdots \\ \cdots \end{cases}$$

$\Rightarrow \; g$ is periodic with period = 2

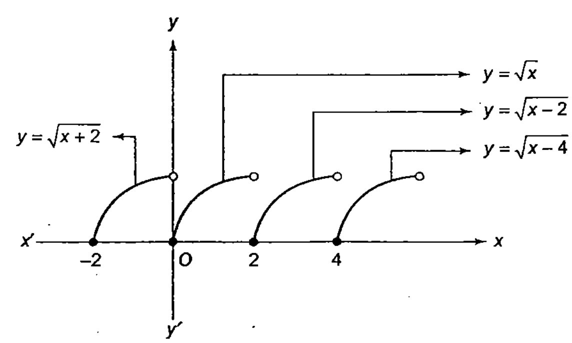


Fig. 1.92

3. Given $f(x) = x^2 - 2x = (x-1)^2 - 1$

$\Rightarrow \quad g(x) = f(f(x) - 1) + f(5 - f(x))$

$= f[(x-1)^2 - 2] + f[6 - (x-1)^2]$

$= [(x-1)^2 - 2 - 1]^2 - 1 + [6 - (x-1)^2 - 1]^2 - 1$

$= (x-1)^4 - 6(x-1)^2 + 9 - 1 + (x-1)^4$

$- 10(x-1)^2 + 25 - 1$

$= 2(x-1)^4 - 16(x-1)^2 + 32$
$= 2[(x-1)^4 - 8(x-1)^2 + 16]$
$= 2[(x-1)^2 - 4]^2 \geq 0 \ \forall x \in R$

4. Let two linear functions be $f(x) = ax + b$ and $g(x) = cx + d$
They map $[-1, 1] \to [0, 2]$ and mapping is onto
$\Rightarrow f(-1) = 0$ and $f(1) = 2$ and $g(-1) = 2$ and $g(1) = 0$
$\Rightarrow -a + b = 0$ and $a + b = 2$ (1)
and $-c + d = 2$ and $c + d = 0$ (2)
$\Rightarrow a = b = 1$ and $c = -1, d = 1$
$\Rightarrow f(x) = x + 1$ and $g(x) = -x + 1$

$$\Rightarrow h(x) = \frac{x+1}{1-x} \Rightarrow h(h(x)) = \frac{\frac{x+1}{1-x}+1}{\frac{x+1}{1-x}-1} = \frac{1}{x}$$

$\Rightarrow h(h(1/x)) = x$
$\Rightarrow |h(h(x)) + h(h(1/x))| = |x + 1/x| > 2.$

5. $$f(x) = \begin{cases} x^2 - 4x + 3, & x < 3 \\ x - 4, & x \geq 3 \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} x^2 - 4x + 3, & x < 3 \\ x - 4, & 3 \leq x < 4 \\ x - 4, & x \geq 4 \end{cases} \quad (1)$$

$$g(x) = \begin{cases} x - 3, & x < 4 \\ x^2 + 2x + 2, & x \geq 4 \end{cases}$$

$$\Rightarrow g(x) = \begin{cases} x - 3, & x < 3 \\ x - 3, & 3 \leq x < 4 \\ x^2 + 2x + 2, & x \geq 4 \end{cases} \quad (2)$$

From (1) and (2), we have

$$\frac{f(x)}{g(x)} = \begin{cases} \frac{x^2 - 4x + 3}{x - 3}, & x < 3 \\ \frac{x-4}{x-3} & 3 < x < 4 \\ \frac{x-4}{x^2 + 2x + 2}, & x \geq 4 \end{cases}$$

Clearly, $f(x)/g(x)$ is not defined at $x = 3$, hence the domain is $R - \{3\}$.

6. Given $f(x) = \log_2 \log_3 \log_4 \log_5 (\sin x + a^2)$
$f(x)$ is defined only if $\log_3 \log_4 \log_5 (\sin x + a^2) > 0, \forall x \in R$
$\Rightarrow \log_4 \log_5 (\sin x + a^2) > 1, \forall x \in R$
$\Rightarrow \log_5 (\sin x + a^2) > 4, \forall x \in R$
$\Rightarrow (\sin x + a^2) > 5^4, \forall x \in R$
$\Rightarrow a^2 > 625 - \sin x, \forall x \in R$
$\Rightarrow a^2$ must be greater than maximum value of $625 - \sin x$ which is 626 (when $\sin x = -1$)
$\Rightarrow a^2 > 626$
$\Rightarrow a \in \left(-\infty, -\sqrt{626}\right) \cup \left(\sqrt{626}, \infty\right)$

7. By remainder theorem, $P(a) = a, P(b) = b$ and $P(c) = c$.
Let the required remainder be $R(x)$, then $P(x) = (x-a)(x-b)(x-c)Q(x) + R(x)$, where $R(x)$ is a polynomial of degree at most 2.
We get $R(a) = a, R(b) = b$ and $R(c) = c$.
So, the equation $R(x) - x = 0$ has three roots a, b and c.
But its degree is at most 2, So, $R(x) - x$ must be zero polynomial (or identity). Hence, $R(x) = x$.

8.

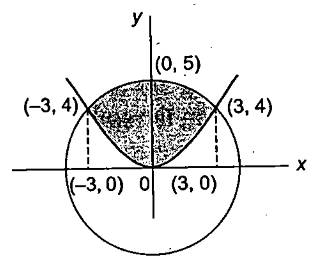


Fig. 1.93

The equation $x^2 + y^2 = 25$ represents a circle with centre (0, 0) and radius 5 and the equation $y = \frac{4}{9}x^2$ represents a parabola with vertex (0, 0). Hence, $R \cap R'$ is the set of points indicated in the figure $= \{(x, y): -3 \leq x \leq 3, 0 \leq y \leq 5\}$. Thus, the domain $R \cap R' = [-3, 3]$ and the range $R \cap R' = [0, 5]$.

9. Put $y = \frac{1}{x}$
$\Rightarrow 2 + f(x)\, f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right) + f(1)$ (1)
Now put $x = 1$
$\Rightarrow 2 + (f(1))^2 = 3f(1)$
$\Rightarrow f(1) = 1$ or 2
But $f(1) \neq 1$, otherwise from the given relation $2 + f(x) f(1) = f(x) + f(1) + f(x)$ or $f(x) = 1$, which is not possible as given that $f(2) = 5$.
Hence, $f(1) = 2$.
$\Rightarrow$ From (1), we have $f(x)\, f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$
$\Rightarrow f(x) = \pm x^n + 1$
$\Rightarrow f(2) = \pm 2^n + 1 = 5$
$\Rightarrow 2^n = 4 \Rightarrow n = 2$
$\Rightarrow f(x) = x^2 + 1$
$\Rightarrow f(f(2)) = f(5) = 26$

10. $f(x) = f(b + (x - b))$
$= f(b - (x - b))$
$= f(2b - x)$
$= f(a + (2b - x - a))$
$= f(a - (2b - x - a))$
$= f(2a - 2b + x)$
Hence, $f(x)$ is periodic with period $2a - 2b$.

11. Given $f(xf(y)) = x^p y^q$

$$\Rightarrow \quad x = \frac{\{f(xf(y))\}^{1/p}}{y^{q/p}} \qquad (1)$$

Let $xf(y) = 1 \Rightarrow x = \dfrac{1}{f(y)}$, then from (1)

$$f(y) = \frac{y^{q/p}}{\{f(1)\}^{1/p}}$$

$$\Rightarrow \quad f(1) = \frac{1}{\{f(1)\}^{1/p}}$$

$\Rightarrow \quad f(1) = 1$

$\Rightarrow \quad f(y) = y^{q/p} \qquad (2)$

Now, $f(x\,y^{q/p}) = x^p y^q$. Put $y^{q/p} = z$, we get

$f(xz) = (xz)^p$

$\Rightarrow \quad f(x) = x^p \qquad (3)$

From (2) and (3) $x^p = x^{q/p} \Rightarrow p^2 = q$.

12. $f(x-1) + f(x+1) = \sqrt{3} f(x) \qquad (1)$

Putting $x+2$ for x in relation (1) we get $f(x+1) + f(x+3) =$
$\sqrt{3} f(x+2) \qquad (2)$

From (1) and (2), we get

$$f(x-1) + 2f(x+1) + f(x+3) = \sqrt{3}\,(f(x) + f(x+2))$$
$$= \sqrt{3}\left(\sqrt{3}\, f(x+1)\right)$$
$$= 3f(x+1)$$

$\Rightarrow \quad f(x-1) + f(x+3) = f(x+1) \qquad (3)$

Putting $x+2$ for x in relation (3), we get

$f(x+1) + f(x+5) = f(x+3) \qquad (4)$

Adding (3) and (4) in $f(x-1) = -f(x+5)$

Now, put $x+1$ for x, $f(x) = -f(x+6) \qquad (5)$

Put $x+6$ in place of x in (5), we get $f(x+6) = -f(x+12)$

$\therefore$ from (5) again, $f(x) = -[-f(x+12)] = f(x+12)$

$\therefore$ the period of $f(x)$ is 12.

13. $f(a+x) = b + [1 + b^3 - 3b^2 f(x) + 3b\,\{f(x)\}^2 - \{f(x)\}^3]^{1/3}$

$= b + [1 + \{b - f(x)\}^3]^{1/3}$

$\Rightarrow \quad f(a+x) - b = [1 - \{f(x) - b\}^3]^{1/3}$

$\Rightarrow \quad \phi(a+x) = [1 - \{\phi(x)^3\}]^{1/3} \qquad (1)$

where $\phi(x) = f(x) - b$

$\Rightarrow \quad \phi(2a+x) = [1 - \{\phi(x+a)\}^3]^{1/3} = \phi(x)$ from (1)

$\Rightarrow \quad f(x+2a) - b = f(x) - b$

$\Rightarrow \quad f(x+2a) = f(x)$

$\therefore \quad f(x)$ is periodic with period $2a$.

14. $f(x, y) = f(2x+2y, 2y-2x)$

(Replacing x by $2x+2y$ and y by $2y-2x$)

$= f(2(2x+2y) + 2(2y-2x),$
$2(2y-2x) - 2(2x+2y))$

$f(x, y) = f(8x, -8y) = f(8(-8x), -8(8y))$

$= f(-64x, -64y)$

$= f(-64(-64x), -64y(-64y)) = f(2^{12}x, 2^{12}y)$

$f(x, 0) = f(2^{12}x, 0)$

$f(2^y, 0) = f(2^{12} . 2^y, 0) = f(2^{12+y}, 0)$

$\Rightarrow \quad g(y) = g(y+12)$

Hence, $g(x)$ is periodic and its period is 12.

15. $y = \dfrac{x-a}{(x-b)(x-c)} \Rightarrow yx^2 - [(b+c)y + 1]x + bcy + a = 0$

Now x is real, $\Rightarrow D \geq 0$

$\Rightarrow \quad [(b+c)y - 1]^2 - 4y(bcy - a) \geq 0, \forall\, y \in R,$

(as given that $f(x)$ is an onto function)

$\Rightarrow \quad (b-c)^2 y^2 - 2(b+c-2a)y + 1 \geq 0, \forall\, y \in R$

$D \leq 0$

$\Rightarrow \quad 4(b+c-2a)^2 - 4(b-c)^2 \leq 0$

$\Rightarrow \quad (b+c-2a-b+c)(b+c-2a+b-c) \leq 0$

$\Rightarrow \quad (c-a)(b-a) \leq 0$

$\Rightarrow \quad c \leq a$ and $b \geq a$ or $b > c$ and $c \geq a$ and $b \leq a$

$\Rightarrow \quad c \leq a \leq b$ (as $b > c$)

$\Rightarrow \quad a \in (b, c)$

16. Let $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, $a_i \in I\,(i = 0, 1, 2, \ldots, n)$

Now, $f(a) = a_0 + a_1 a + a_2 a^2 + \cdots + a_n a^n = b$

$f(b) = a_0 + a_1 b + a_2 b^2 + \cdots + a_n b^n = c$

$f(x) = a_0 + a_1 c + a_2 c^2 + \cdots + a_n c^n = a$

$\therefore f(a) - f(b) = (a-b) f_1(a, b) = b - c,$

where $f_1(a, b)$ is an integer

Similarly, $(b-c) f_1(b, c) = c - a$

and $(c-a) f_1(c, a) = a - b$

Multiplying all these, we get $f_1(a, b) f_1(b, c) f_1(c, a) = 1$

$\Rightarrow \quad f_1(a, b) = f_1(b, c) = f_1(c, a) = 1$

$\Rightarrow \quad a - b = b - c$, $c - a = a - b$ and $c - a = a - b$

$\Rightarrow \quad a = b = c$ which is a contradiction.

Hence, no such polynomial exists.

17. Clearly, from graph $g(x) = \begin{cases} x^2, & -2 \leq x \leq -1 \\ 1-x, & -1 < x \leq -1/4 \\ \frac{1}{2} + x, & -\frac{1}{4} < x < 0 \\ 1+x, & 0 \leq x < 1 \\ x^2, & 1 \leq x \leq 2 \end{cases}$

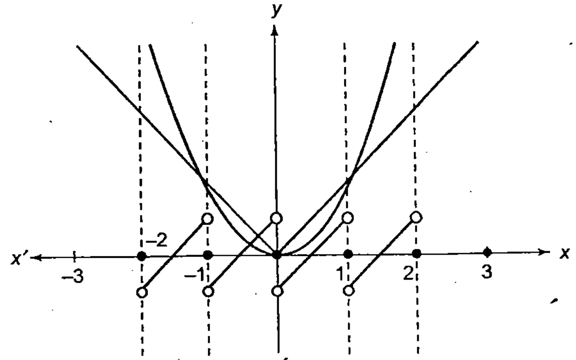


Fig. 1.94

18. Given $f(x-f(y))=f(f(y))+xf(y)+f(x)-1$ (1)

Putting $x=f(y)=0$, then $f(0)=f(0)+0+f(0)-1$

$\therefore \quad f(0)=1$ (2)

Again putting $x=f(y)=\lambda$ in (1)

Then $f(0)=f(\lambda)+\lambda^2+f(\lambda)-1$

$\Rightarrow \quad 1=2f(\lambda)+\lambda^2-1$ {from (2)}

$\therefore \quad f(\lambda)=\dfrac{2-\lambda^2}{2}=1-\dfrac{\lambda^2}{2}$

Hence, $f(x)=1-\dfrac{x^2}{2}$ is the unique function.

19. Since $f(x)=(2\cos x-1)(2\cos 2x-1)(2\cos 2^2x-1)\ldots(2\cos 2^{n-1}x-1)$

$$\therefore\ f(x)=\frac{(2\cos x+1)(2\cos x-1)(2\cos 2x-1)\times(2\cos 2^2x-1)\ldots(2\cos 2^{n-1}x-1)}{(2\cos x+1)}$$

$$=\frac{(4\cos^2 x-1)(2\cos 2x-1)(2\cos 2^2x-1)\ldots(2\cos 2^{n-1}x-1)}{(2\cos x+1)}$$

$$=\frac{(2\cos 2x+1)(2\cos 2x-1)(2\cos 2^2x-1)\ldots(2\cos 2^{n-1}x-1)}{(2\cos x+1)}$$

$$=\frac{(4\cos^2 2x-1)(2\cos 2^2x-1)\ldots(2\cos 2^{n-1}x-1)}{(2\cos x+1)}$$

$$=\frac{(2\cos 2^2x+1)(2\cos 2^2x-1)\ldots(2\cos 2^{n-1}x-1)}{(2\cos x+1)}$$

Proceeding in similarly way

$$f(x)=\frac{(2\cos 2^{n-1}x+1)(2\cos 2^{n-1}x-1)}{(2\cos x+1)}$$

$$=\frac{(4\cos^2 2^{n-1}x-1)}{(2\cos x+1)}=\frac{(2\cos 2^n x+1)}{(2\cos x+1)}$$

$$\Rightarrow\ f\left(\frac{2\pi k}{2^n\pm1}\right)=\frac{2\cos\left(\dfrac{2^{n+1}\pi k}{2^n\pm1}\right)+1}{2\cos\left(\dfrac{2\pi k}{2^n\pm1}\right)+1}$$

$$=\frac{2\cos\left(2\pi k\mp\dfrac{2\pi k}{2^n\pm1}\right)+1}{2\cos\left(\dfrac{2\pi k}{2^n\pm1}\right)+1}$$

$$=\frac{2\cos\left(\dfrac{2\pi k}{2^n\pm1}\right)+1}{2\cos\left(\dfrac{2\pi k}{2^n\pm1}\right)+1}=1$$

20. $f(x)=\dfrac{a^x}{a^x+\sqrt{a}}$

$\Rightarrow\ f(1-x)=\dfrac{a^{1-x}}{a^{1-x}+\sqrt{a}}=\dfrac{a^1}{a^1+\sqrt{a}a^x}=\dfrac{\sqrt{a}}{\sqrt{a}+a^x}$

$\Rightarrow\ f(x)+f(1-x)=1$

Also, $f\left(\dfrac{1}{2}\right)=\dfrac{1}{2}$

$\Rightarrow\ \sum_{r=1}^{2n-1} 2f\left(\dfrac{r}{2n}\right)$

$$=2\left[\begin{array}{l}f\left(\frac{1}{2n}\right)+f\left(\frac{2}{2n}\right)+\cdots+f\left(\frac{n-1}{2n}\right)\\+f\left(\frac{n}{2n}\right)+f\left(\frac{n+1}{2n}\right)+\cdots\\+f\left(\frac{2n-1}{2n}\right)\end{array}\right]$$

$$=2\left\{\begin{array}{l}\left[f\left(\frac{1}{2n}\right)+f\left(\frac{2n-1}{2n}\right)\right]+\left[f\left(\frac{2}{2n}\right)+f\left(\frac{2n-2}{2n}\right)\right]\\+\cdots+\left[f\left(\frac{n-1}{2n}\right)+f\left(\frac{n+1}{2n}\right)\right]+f\left(\frac{1}{2}\right)\end{array}\right\}$$

$$=2\left\{\begin{array}{l}\left[f\left(\frac{1}{2n}\right)+f\left(1-\frac{1}{2n}\right)\right]+\left[f\left(\frac{2}{2n}\right)+f\left(1-\frac{2}{2n}\right)\right]\\+\cdots+\left[f\left(\frac{n-1}{2n}\right)+f\left(1-\frac{n-1}{2n}\right)\right]+\frac{1}{2}\end{array}\right\}$$

$=2[1+1+1+\cdots+(n-1)\text{ times}]+1$

$=2n-1$

Objective Type

1. b. $f: N\to N, f(n)=2n+3$.

Here, the range of the function is $\{5, 6, 7, \ldots\}$ or $N-\{1, 2, 3, 4\}$

which is a subset of N (co-domain).

Hence, function is into.

Also, it is clear that $f(n)$ is one-one or injective.

Hence, $f(n)$ is injective only.

2. b. $f(x)=\sin\left(\log(x+\sqrt{1+x^2})\right)$

$\Rightarrow\ f(-x)=\sin[\log(-x+\sqrt{1+x^2})]$

$\Rightarrow\ f(-x)=\sin\log\left((\sqrt{1+x^2}-x)\dfrac{(\sqrt{1+x^2}+x)}{(\sqrt{1+x^2}+x)}\right)$

$\Rightarrow\ f(-x)=\sin\log\left[\dfrac{1}{(x+\sqrt{1+x^2})}\right]$

$\Rightarrow \quad f(-x)= \sin\left[-\log(x+\sqrt{1+x^2})\right]$

$\Rightarrow \quad f(-x)=-\sin\left[\log(x+\sqrt{1+x^2})\right]$

$\Rightarrow \quad f(-x)=-f(x)$

$\Rightarrow$ $f(x)$ is an odd function.

3. c. $\dfrac{x^2+14x+9}{x^2+2x+3}=y$

$\Rightarrow \quad x^2+14x+9=x^2y+2xy+3y$

$\Rightarrow \quad x^2(y-1)+2x(y-7)+(3y-9)=0$

Since x is real,

$\therefore \quad 4(y-7)^2-4(3y-9)(y-1)>0$

$\Rightarrow \quad 4(y^2+49-14y)-4(3y^2+9-12y)>0$

$\Rightarrow \quad (y+5)(y-4)<0;$

$\therefore$ y lies between -5 and 4.

4. c. $y=f(x)=\cos^2 x+\sin^4 x$

$\Rightarrow \quad y=f(x)=\cos^2 x+\sin^2 x(1-\cos^2 x)$

$\Rightarrow \quad y=\cos^2 x+\sin^2 x-\sin^2 x\cos^2 x$

$\Rightarrow \quad y=1-\sin^2 x\cos^2 x$

$\Rightarrow \quad y=1-\dfrac{1}{4}\sin^2 2x$

$\therefore \quad \dfrac{3}{4}\le f(x)\le 1 \qquad (\because 0\le\sin^2 2x\le 1)$

$\Rightarrow \quad f(x)\in[3/4,1]$

5. c. $f(x)$ is to be defined when $x^2-1>0$ and $3+x>0$ and $3+x\ne 1$

$\Rightarrow \quad x^2>1$ and $x>-3$ and $x\ne-2$

$\Rightarrow \quad x<-1$ or $x>1$ and $x>-3$ and $x\ne-2$

$\therefore \quad D_f=(-3,-2)\cup(-2,-1)\cup(1,\infty)$

6. b. We have $f(x)=\left[\log_{10}\left(\dfrac{5x-x^2}{4}\right)\right]^{1/2}$ (1)

From (1), clearly $f(x)$ is defined for those values of x for which $\log_{10}\left[\dfrac{5x-x^2}{4}\right]\ge 0$

$\Rightarrow \quad \left(\dfrac{5x-x^2}{4}\right)\ge 10^0$

$\Rightarrow \quad \left(\dfrac{5x-x^2}{4}\right)\ge 1$

$\Rightarrow \quad x^2-5x+4\le 0$

$\Rightarrow \quad (x-1)(x-4)\le 0$

Hence, the domain of the function is $[1,4]$.

7. b. $f(x)=\dfrac{\sin^{-1}(3-x)}{\log(|x|-2)}$

Let $g(x)=\sin^{-1}(3-x)$

$\Rightarrow \quad -1\le 3-x\le 1$

The domain of $g(x)$ is $[2,4]$

and let $h(x)=\log(|x|-2)$

$\Rightarrow \quad |x|-2>0$ or $|x|>2$

$\Rightarrow \quad x<-2$ or $x>2$

$\Rightarrow \quad (-\infty,-2)\cup(2,\infty)$

We know that

$(f/g)(x)=\dfrac{f(x)}{g(x)}\ \forall x\in D_1\cap D_2-\{x\in R: g(x)=0\}$

$\therefore$ the domain of $f(x)=(2,4]-\{3\}=(2,3)\cup(3,4]$.

8. c. $f(x)=\log|\log x|$, $f(x)$ is defined if $|\log x|>0$ and $x>0$, i.e., if $x>0$ and $x\ne 1$ $\qquad (\because |\log x|>0$ if $x\ne 1)$

$\Rightarrow \quad x\in(0,1)\cup(1,\infty)$.

9. d. Here $x+3>0$ and $x^2+3x+2\ne 0$

$\therefore \quad x>-3$ and $(x+1)(x+2)\ne 0$, i.e., $x\ne-1,-2$

$\therefore$ The domain $=(-3,\infty)-\{-1,-2\}$.

10. b. $y=f(x)=\sqrt{3}\sin x-\cos x+2=2\sin\left(x-\dfrac{\pi}{6}\right)+2$ (1)

Since $f(x)$ is one-one and onto, f is invertible.

From (1) $\sin\left(x-\dfrac{\pi}{6}\right)=\dfrac{y-2}{2}$

$\Rightarrow \quad x=\sin^{-1}\dfrac{y-2}{2}+\dfrac{\pi}{6}$

$\Rightarrow \quad f^{-1}(x)=\sin^{-1}\left(\dfrac{x-2}{2}\right)+\dfrac{\pi}{6}$

11. a. $F(n+1)=\dfrac{2F(n)+1}{2}\Rightarrow F(n+1)-F(n)=\dfrac{1}{2}$

Put $n=1,2,3,\ldots,100$ and add, we get

$F(101)-F(1)=100\times\dfrac{1}{2}$

$\Rightarrow \quad F(101)=52 \qquad [\because F(1)=2]$

12. d. Given function is defined if ${}^{10}C_{x-1}>3\ {}^{10}C_x$

$\Rightarrow \quad \dfrac{1}{11-x}>\dfrac{3}{x} \quad \Rightarrow 4x>33$

$\Rightarrow \quad x\ge 9$ but $x\le 10 \quad \Rightarrow x=9, 10.$

13. b. For the domain $\sin(\ln x)>\cos(\ln x)$ and $x>0$

$2n\pi+\dfrac{\pi}{4}<\ln x<2n\pi+\dfrac{5\pi}{4}, n\in N\cup\{0\}$

14. b. Put $x=0 \Rightarrow f(2)=2f(0)-f(1)=2\times 2-3=1$

Put $x=1 \Rightarrow f(3)=6-1=5$

Put $x=2 \Rightarrow f(4)=2f(2)-f(3)=2\times 1-5=-3$

Put $x=3 \Rightarrow f(5)=2f(3)-f(4)=2(5)-(-3)=13.$

15. c. Here, $\dfrac{x^2+1}{x^2+2}=1-\dfrac{1}{x^2+2}$

Now, $2 \le x^2 + 2 < \infty$ for all $x \in R$

$\Rightarrow \quad \frac{1}{2} \ge \frac{1}{x^2+2} > 0$

$\Rightarrow \quad -\frac{1}{2} \le \frac{-1}{x^2+2} < 0$

$\Rightarrow \quad \frac{1}{2} \le 1 - \frac{1}{x^2+2} < 1$

$\Rightarrow \quad \frac{\pi}{6} \le \sin^{-1}\left(1 - \frac{1}{x^2+2}\right) < \frac{\pi}{2}$

16. b. The function $\sec^{-1} x$ is defined for all $x \in R-(-1, 1)$ and the function $\frac{1}{\sqrt{x-[x]}}$ is defined for all $x \in R-Z$

So the given function is defined for all $x \in R - \{(-1, 1) \cup \{n \mid n \in Z\}\}$.

17. b. $\cos^{-1}\left(\frac{2-|x|}{4}\right)$ exists if $-1 \le \frac{2-|x|}{4} \le 1$

$\Rightarrow \quad -6 \le -|x| \le 2$

$\Rightarrow \quad -2 \le |x| \le 6$

$\Rightarrow \quad |x| \le 6$

$\Rightarrow \quad -6 \le x \le 6$

The function $[\log(3-x)]^{-1} = \frac{1}{\log(3-x)}$ is defined if $3 - x > 0$ and $x \ne 2$, i.e., if $x \ne 2$ and $x < 3$.

Thus, the domain of the given function is

$\{x \mid -6 \le x \le 6\} \cap \{x \mid x \ne 2, x < 3\} = [-6, 2) \cup (2, 3)$.

18. b. $f(x)$ is defined for $\log\left(\frac{1}{|\sin x|}\right) \ge 0$

$\Rightarrow \quad \frac{1}{|\sin x|} \ge 1$ and $|\sin x| \ne 0$.

$\Rightarrow \quad |\sin x| \ne 0 \qquad \left[\because \frac{1}{|\sin x|} \ge 1 \text{ for all } x\right]$

$\Rightarrow \quad x \ne n\pi, n \in Z$

Hence, the domain of $f(x) = R - \{n\pi : n \in Z\}$.

19. a. $f(x)$ is defined if $-\log_{1/2}\left(1+\frac{1}{x^{1/4}}\right) - 1 > 0$

$\Rightarrow \quad \log_{1/2}\left(1+\frac{1}{x^{1/4}}\right) < -1$

$\Rightarrow \quad 1 + \frac{1}{x^{1/4}} > \left(\frac{1}{2}\right)^{-1}$

$\Rightarrow \quad \frac{1}{x^{1/4}} > 1$

$\Rightarrow \quad 0 < x < 1$

20. c. For the function to get defined $0 \le x^2 + x + 1 \le 1$,

but $x^2 + x + 1 \ge \frac{3}{4} \Rightarrow \frac{\sqrt{3}}{2} \le \sqrt{x^2+x+1} \le 1$

$\Rightarrow \quad \frac{\pi}{3} \le \sin^{-1}(\sqrt{x^2+x+1}) \le \frac{\pi}{2}$.

21. c.

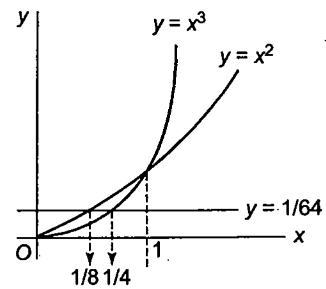


Fig. 1.95

Clearly, from the graph $f(x) = \begin{cases} \frac{1}{64}, & 0 \le x \le \frac{1}{8} \\ x^2, & \frac{1}{8} < x \le 1 \\ x^3, & x > 1 \end{cases}$

22. c. The period of $\cos(\sin nx)$ is $\frac{\pi}{n}$ and the period of $\tan\left(\frac{x}{n}\right)$ is πn.

Thus, $6\pi = \text{LCM}\left(\frac{\pi}{n}, \pi n\right)$.

By checking for the different values of n, $n = 6$.

23. b. Draw the graph of $y = \log_{0.5}|x|$ and $y = 2|x|$

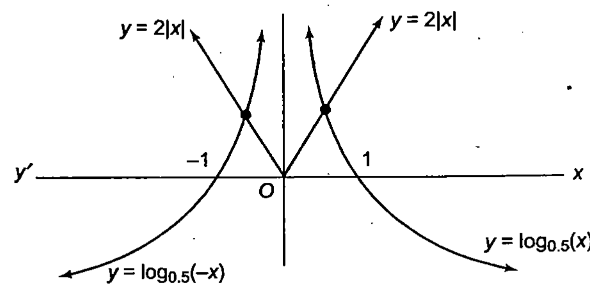


Fig. 1.96

Clearly, from the graph, there are two solutions.

24. b. $f(x) = \left|\sin^3 \frac{x}{2}\right| + \left|\cos^5 \frac{x}{5}\right|$

The period of $\sin^3 x$ is 2π

$\Rightarrow \quad$ The period of $\sin^3 \frac{x}{2}$ is $\frac{2\pi}{1/2} = 4\pi$

$\Rightarrow \quad$ The period of $\left|\sin^3 \frac{x}{2}\right|$ is 2π

The period of $\cos^5 x$ is 2π

$\Rightarrow \quad$ The period of $\cos^5 \frac{x}{5}$ is $\frac{2\pi}{\left(\frac{1}{5}\right)} = 10\pi$

$\Rightarrow \quad$ The period of $\left|\cos^5 \frac{x}{2}\right|$ is 5π

Now the period of $f(x)$ = LCM of $\{2\pi, 10\pi\} = 10\pi$.

25. a. Given $f(x) = \sqrt[n]{x^m}$, $n \in N$ is an even function where $m \in I$.

$\Rightarrow f(x) = f(-x)$

$\Rightarrow \sqrt[n]{x^m} = \sqrt[n]{(-x)^m}$

$\Rightarrow x^m = (-x)^m$

$\Rightarrow$ m is an even integer

$\Rightarrow m = 2k, k \in I$

26. c. From the given data $g(x)$ must be linear function

Hence, $g(x) = ax + b$

Also $g(2) = 2a + b = 3$ and $g(4) = 4a + b = 7$

Solving, we get $a = 2$ and $b = -1$

Hence, $g(x) = 2x - 1$

Then, $g(6) = 11$.

27. a. The period of $\sin \pi x$ and $\cos 2\pi x$ is 2 and 1, respectively

The period of $2^{\{x\}}$ is 1

The period of $3^{\{x/2\}}$ is 2

Hence, the period of $f(x)$ is LCM of 1 and 2 = 2.

28. a. $|x - 2| + a = \pm 4$

$\Rightarrow |x - 2| = \pm 4 - a$

for 4 real roots, $4 - a > 0$ and $-4 - a > 0$

$\Rightarrow a \in (-\infty, -4)$

29. a. We have $f(x + y) + f(x - y)$

$$= \frac{1}{2}[a^{x+y} + a^{-x-y} + a^{x-y} + a^{-x+y}]$$

$$= \frac{1}{2}[a^x(a^y + a^{-y}) + a^{-x}(a^y + a^{-y})]$$

$$= \frac{1}{2}(a^x + a^{-x})(a^y + a^{-y}) = 2f(x)f(y)$$

30. d. $\log_3(x^2 - 6x + 11) \le 1$

$\Rightarrow 0 < x^2 - 6x + 11 \le 3$

$\Rightarrow x \in [2, 4]$

31. d. $x^2 - [x]^2 \ge 0 \Rightarrow x^2 \ge [x]^2$

This is true for all positive values of x and all negative integer x.

32. b.

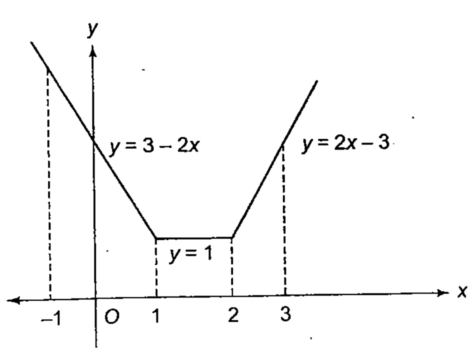


Fig. 1.97

Clearly, from the graph, the range is $[1, f(-1)] \equiv [1, 5]$

If $x < 1, f(x) = -(x - 1) - (x - 2) = -2x + 3$.

In this interval, $f(x)$ is decreasing.

If $1 \le x < 2, f(x) = x - 1 - (x - 2) = 1$

In this interval, $f(x)$ is constant.

If $2 \le x \le 3, f(x) = x - 1 + x - 2 = 2x - 3$

In this interval, $f(x)$ is increasing.

$\therefore \max f(x)$ = the greatest among $f(-1)$ and $f(3) = 5$, $\min f(x) = f(1) = 1$

So, the range = $[1, 5]$.

33. a. By checking for different function, we find that for $f(x) = \dfrac{1-x}{1+x}, f^{-1}(x) = f(x)$.

34. b. $x^2F(x) + F(1 - x) = 2x - x^4$ (1)

Replacing x by $1 - x$, we get

$\Rightarrow (1 - x)^2 F(1 - x) + F(x) = 2(1 - x) - (1 - x)^4$ (2)

Eliminating $F(1 - x)$ from (1) and (2), we get $F(x) = 1 - x^2$.

35. b.
$$f(-x) = \begin{cases} (-x)^2 \sin \dfrac{\pi(-x)}{2}, & |-x| < 1 \\ (-x)|-x|, & |-x| \ge 1 \end{cases}$$
$$= \begin{cases} -x^2 \sin \dfrac{\pi x}{2}, & |x| < 1 \\ -x|x|, & |x| \ge 1 \end{cases}$$
$$= -f(x).$$

36. d. $f(x) = e^{x^3 - 3x + 2}$

Let $g(x) = x^3 - 3x + 2$; $g'(x) = 3x^2 - 3 = 3(x^2 - 1)$

$g'(x) \ge 0$ for $x \in (-\infty, -1]$

$\therefore f(x)$ is increasing function $\therefore f(x)$ is one-one

Now, the range of $f(x) = (0, e^4]$

But co-domain is $(0, e^5]$. $\therefore f(x)$ is an into function.

37. c. $f(x) = \dfrac{1}{x}, g(x) = \dfrac{1}{x^2}$ and $h(x) = x^2$

$f(g(x)) = x^2, x \ne 0$

$h(g(x)) = \dfrac{1}{x^4} = (g(x))^2, x \ne 0$

38. c. $\displaystyle\sum_{r=1}^{2000} \frac{\{x + r\}}{2000} = \sum_{r=1}^{2000} \frac{\{x\}}{2000} = 2000 \frac{\{x\}}{2000} = \{x\}$.

39. b. $f(x) = x^n + 1$

$\Rightarrow f(3) = 3^n + 1 = 28$

$\Rightarrow 3^n = 27$

$\therefore n = 3$

$\Rightarrow f(4) = 4^3 + 1 = 65$.

40. b. $\because f(x + 1) - f(x) = 8x + 3$

$\Rightarrow \{b(x + 1)^2 + c(x + 1) + d\} - \{bx^2 + cx + d\} = 8x + 3$

$\Rightarrow b\{(x + 1)^2 - x^2\} + c = 8x + 3$

$\Rightarrow b(2x + 1) + c = 8x + 3$

On comparing co-efficient of x and constant term, we get $2b = 8$ and $b + c = 3$

then $b = 4$ and $c = -1$.

41. d. If f is injective and g is surjective

$\Rightarrow$ fog is injective

$\Rightarrow$ fof is injective.

42. c. $f(x)=\begin{cases} x-1, & x \text{ is even} \\ x+1, & x \text{ is odd} \end{cases}$, which is clearly are one-one and onto.

43. c. $\frac{1}{2}(gof)(x)=2x^2-5x+2$ or $\frac{1}{2}g[f(x)]=2x^2-5x+2$

$\therefore [\{f(x)\}^2+\{f(x)\}-2]=2[2x^2-5x+2]$

$\Rightarrow f(x)^2+f(x)-(4x^2-10x+6)=0$

$\therefore f(x)=\dfrac{-1\pm\sqrt{1+4(4x^2-10x+6)}}{2}$

$=\dfrac{-1\pm\sqrt{(16x^2-40x+25}}{2}=\dfrac{-1\pm(4x-5)}{2}=2x-3$ or $-2x+2$

44. a. Since $f(x)$ and $f^{-1}(x)$ are symmetric about the line $y=-x$.

If (α,β) lies on $y=f(x)$ then $(-\beta,-\alpha)$ on $y=f^{-1}(x)$

$\Rightarrow$ $(-\alpha,-\beta)$ lies on $y=f(x)$

$\Rightarrow$ $y=f(x)$ is odd.

45. c. Let $x, y\in N$ such that $f(x)=f(y)$

Then $f(x)=f(y)$

$\Rightarrow x^2+x+1=y^2+y+1$

$\Rightarrow (x-y)(x+y+1)=0$

$\Rightarrow x=y$ or $x=(-y-1)\notin N$

$\therefore$ f is one-one.

Also, $f(x)$ does not take all positive integral values. Hence f is into.

46. c. $f(x)=\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)+2\sqrt{2}$

or, $f(x)=\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)+2\sqrt{2}$

$\Rightarrow Y=\left[\sqrt{2}, 3\sqrt{2}\right]$ and $X=\left[-\frac{3\pi}{4},\frac{\pi}{4}\right]$ or $\left[\frac{\pi}{4},\frac{5\pi}{4}\right]$

47. a. $f(7)+f(-7)=-10$

$\Rightarrow f(7)=-17$

$\Rightarrow f(7)+17\cos x=-17+17\cos x$ which has the range $[-34, 0]$.

48. c. $f(x)=\dfrac{\sin[x]\pi}{x^2+x+1}$

Let $[x]=n\in$ integer

$\Rightarrow \sin[x]\pi=0$

$\Rightarrow f(x)=0$

$\Rightarrow f(x)$ is constant function.

49. b. Two triangles may have equal areas

$\therefore$ f is not one-one.

Since each positive real number can represent area of a triangle.

$\therefore$ f is onto.

50. c. $\dfrac{y-x}{y+x}=k(k>1)$; $y-x=k(y+x)$

$\Rightarrow y(1+k)=x(1+k)$

$\Rightarrow y=\left(\dfrac{1+k}{1-k}\right)x$, where $\dfrac{1+k}{1-k}<-1$

51. c. $g(x)=x^3+\tan x+\left[\dfrac{x^2+1}{P}\right]$

$\Rightarrow g(-x)=(-x)^3+\tan(-x)+\left[\dfrac{(-x)^2+1}{P}\right]$

$\Rightarrow g(-x)=-x^3-\tan x+\left[\dfrac{x^2+1}{P}\right]$

$\Rightarrow g(x)+g(-x)=0$

because $g(x)$ is a odd function

$\therefore \left(-x^3-\tan x+\left[\dfrac{x^2+1}{P}\right]\right)+\left(\begin{array}{c}-x^3-\tan x \\ +\left[\dfrac{x^2+1}{P}\right]\end{array}\right)=0$

$\Rightarrow 2\left[\dfrac{(x^2+1)}{P}\right]=0\Rightarrow 0\le\dfrac{x^2+1}{P}<1$

Now $x\in[-2,2]$

$\Rightarrow 0\le\dfrac{5}{P}<1\Rightarrow P>5$

52. d. Let $2x+\frac{y}{8}=\alpha$ and $2x-\frac{y}{8}=\beta$, then $x=\dfrac{\alpha+\beta}{4}$ and $y=4(\alpha-\beta)$.

Given, $f\left(2x+\frac{y}{8}, 2x-\frac{y}{8}\right)=xy$

$\Rightarrow f(\alpha,\beta)=\alpha^2-\beta^2$

$\Rightarrow f(m,n)+f(n,m)=m^2-n^2+n^2-m^2=0$ for all m, n

53. b. Given $f(x+y)=f(x)+f(y)-xy-1\ \forall x, y\in R$

$f(1)=1$

$f(2)=f(1+1)=f(1)+f(1)-1-1=0$

$f(3)=f(2+1)=f(2)+f(1)-2.\,1-1=-2$

$f(n+1)=f(n)+f(1)-n-1=f(n)-n<f(n)$

Thus, $f(1)>f(2)>f(3)>\ldots$ and $f(1)=1$.

$\therefore$ $f(1)=1$ and $f(n)<1$, for $n>1$

Hence, $f(n)=n$, $n\in N$ has only one solution $n=1$.

54. c. $f(x)=\dfrac{e^x-e^{|x|}}{e^x+e^{|x|}}=\begin{cases} 0, & x\ge 0 \\ \dfrac{e^x-e^{-x}}{e^x+e^{-x}}, & x<0 \end{cases}$

Clearly, $f(x)$ is identically zero if $x\ge 0$ (1)

If $x<0$, let $y=f(x)=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}\Rightarrow e^{2x}=\dfrac{1+y}{1-y}$

$\because$ $x<0\Rightarrow e^{2x}<1\Rightarrow 0<e^{2x}<1$

$\therefore$ $0<\dfrac{1+y}{1-y}<1$

$\Rightarrow \dfrac{1+y}{1-y}>0$ and $\dfrac{1+y}{1-y}<1$

$\Rightarrow (y+1)(y-1)<0$ and $\dfrac{2y}{1-y}<0$

$\Rightarrow -1<y<1$ and $y<0$ or $y>1$

$\Rightarrow -1<y<0$ (2)

Combining (1) and (2), we get $-1<y\le 0 \Rightarrow$ Range $=(-1,0]$.

55. c. $f(2x+3)+f(2x+7)=2$ (1)

Replace x by $x+2$, $f(2x+7)+f(2x+11)=2$ (2)

from (1) – (2) we get $f(2x+3)-f(2x+11)=0$

$\Rightarrow f(2x+3)=f(2x+11)$

$\Rightarrow f(2x+3)=f(2(x+4)+3)$

$\Rightarrow$ Period of $f(x)$ is 8

56. c. Since co-domain $=\left[0,\dfrac{\pi}{2}\right)$

$\therefore$ for f to be onto, the range $=\left[0,\dfrac{\pi}{2}\right)$

This is possible only when $x^2+x+a\ge 0 \quad \forall x\in R$

$\therefore 1^2-4a\le 0 \Rightarrow a\ge \dfrac{1}{4}$

57. d. $f(x)=\dfrac{1}{\sqrt{\{\sin x\}+\{\sin(\pi+x)\}}}=\dfrac{1}{\sqrt{\{\sin x\}+\{-\sin x\}}}$

Now $\{\sin x\}+\{-\sin x\}=\begin{cases}0, & \sin x \text{ is an integer}\\ 1, & \sin x \text{ is not an integer}\end{cases}$

For $f(x)$ to get defined $\{\sin x\}+\{-\sin x\}\ne 0$

$\Rightarrow \sin x \ne$ integer

$\Rightarrow \sin x\ne \pm 1, 0$

$\Rightarrow x\ne \dfrac{n\pi}{2}, n\in I$

Hence, the domain is $R-\left\{\dfrac{n\pi}{2}/n\in I\right\}$.

58. a. $f(-x)=\dfrac{\cos(-x)}{\left[-\dfrac{2x}{\pi}\right]+\dfrac{1}{2}}=\dfrac{\cos x}{-1-\left[\dfrac{2x}{\pi}\right]+\dfrac{1}{2}}$

(as x is not an integral multiple of π)

$\Rightarrow f(-x)=-\dfrac{\cos x}{\left[\dfrac{2x}{\pi}\right]+\dfrac{1}{2}}=-f(x)$

$\Rightarrow f(x)$ is an odd function.

59. d. $f(x)=\alpha x^3-\beta x-(\tan x)\operatorname{sgn} x$

$f(-x)=f(x)$

$\Rightarrow -\alpha x^3+\beta x-\tan x \operatorname{sgn} x=\alpha x^3-\beta x-(\tan x)(\operatorname{sgn} x)$

$\Rightarrow 2(-\alpha x^2-\beta)x=0 \ \forall\ x\in R$

$\Rightarrow \alpha=0$ and $\beta=0$

$\therefore [a]^2-5[a]+4=0$ and $6\{a\}^2-5\{a\}+1=0$

$\Rightarrow (3\{x\}-1)(2\{x\}-1)=0$

$\therefore a=1+\dfrac{1}{3}, 1+\dfrac{1}{2}, 4+\dfrac{1}{3}, 4+\dfrac{1}{2}$

Sum of values of $a=\dfrac{35}{3}$

60. d. Since $f(x)$ is an odd function, $\left[\dfrac{x^2}{a}\right]=0$ for all $x\in$ $[-10,10]$

$\Rightarrow 0\le \dfrac{x^2}{a}<1$ for all $x\in[-10,10]\Rightarrow a>100$.

61. b. $3f(x)+2f\left(\dfrac{x+59}{x-1}\right)=10x+30$

For $x=7$, $3f(7)+2f(11)=70+30=100$.

For $x=11$, $3f(11)+2f(7)=140$.

$\dfrac{f(7)}{-20}=\dfrac{f(11)}{-220}=\dfrac{-1}{9-4}\Rightarrow f(7)=4$.

62. c. $f(x)=[6x+7]+\cos\pi x-6x$
$=[6x]+7+\cos\pi x-6x$
$=7+\cos\pi x-\{6x\}$

$\{6x\}$ has the period 1/6 and $\cos\pi x$ has the period 2, then the period of $f(x)=$ LCM of 2 and 1/6 which is 2. Hence, the period is 2.

63. d. $f(x)=\dfrac{a^x-1}{x^n(a^x+1)}$

$f(x)$ is symmetrical about y-axis

$\Rightarrow f(x)=f(-x)$

$\Rightarrow \dfrac{a^x-1}{x^n(a^x+1)}=\dfrac{a^{-x}-1}{(-x)^n(a^{-x}+1)}$

$\Rightarrow \dfrac{a^x-1}{x^n(a^x+1)}=\dfrac{1-a^x}{(-x)^n(1+a^x)} \Rightarrow x^n=-(-x)^n$

$\Rightarrow$ the value of n which satisfy this relation is $-\dfrac{1}{3}$.

64. a. $x^2f(x)-2f\left(\dfrac{1}{x}\right)=g(x)$ and $2f\left(\dfrac{1}{x}\right)-4x^2f(x)=2x^2g\left(\dfrac{1}{x}\right)$

(Replacing x by $\dfrac{1}{x}$)

$\Rightarrow -3x^2f(x)=g(x)+2x^2g\left(\dfrac{1}{x}\right)$

(Eliminating $f\left(\dfrac{1}{x}\right)$)

$\Rightarrow f(x)=-\left(\dfrac{g(x)+2x^2g\left(\dfrac{1}{x}\right)}{3x^2}\right)$

$\because g(x)$ and x^2 are odd and even functions, respectively. So, $f(x)$ is an odd function. But $f(x)$ is given even

$\Rightarrow f(x)=0\ \forall x$. Hence, $f(5)=0$.

65. a. Given $f(x+y)=f(x)f(y)$. Put $x=y=0$, then $f(0)=1$.

Put $y=-x$, then $f(0)=f(x)f(-x)\Rightarrow f(-x)=\dfrac{1}{f(x)}$

Now, $g(x)=\dfrac{f(x)}{1+\{f(x)\}^2}$

$$\Rightarrow \quad g(-x)=\frac{f(-x)}{1+\{f(-x)\}^2}=\frac{\frac{1}{f(x)}}{1+\frac{1}{\{f(x)\}^2}}$$

$$=\frac{f(x)}{1+\{f(x)\}^2}=g(x)$$

66. d. The equation is $x^2+2ax+\frac{1}{16}=-a+\sqrt{a^2+x-\frac{1}{16}}$,

$\Rightarrow f(x)=f^{-1}(x)$

which has the solution if $x^2+2ax+\frac{1}{16}=x$

$\Rightarrow x^2+(2a-1)x+\frac{1}{16}=0$

For real and distinct roots $(2a-1)^2-4\frac{1}{16}\geq 0$

$\Rightarrow 2a-1\leq\frac{-1}{2}$ or $2a-1\geq\frac{1}{2}\Rightarrow a\leq\frac{1}{4}$ or $a\geq\frac{3}{4}$.

67. d. $f(x)-1+f(1-x)-1=0$; so $g(x)+g(1-x)=0$

Replacing x by $x+\frac{1}{2}$, we get $g\left(\frac{1}{2}+x\right)+g\left(\frac{1}{2}-x\right)=0$.

So, it is symmetrical about $\left(\frac{1}{2},0\right)$.

68. a. When $[x]=0$ we have $\sin^{-1}(\cos^{-1}0)=\sin^{-1}(\pi/2)$, not defined.

When $[x]=-1$ we have $\sin^{-1}(\cos^{-1}-1)=\sin^{-1}(\pi)$, not defined.

When $[x]=1$ we have $\sin^{-1}(\cos^{-1}1)=\sin^{-1}(0)=0$.

Hence, $x\in[1,2)$ and the range of function is $\{0\}$.

69. a. Putting $x=1, f(2)+f(0)=2f(1)\Rightarrow f(2)=2f(1)$

Putting $x=2, f(3)+f(1)=2f(2)$

$\Rightarrow f(3)=2\times 2f(1)-f(1)=3f(1)$, and so on.

$\therefore f(n)=nf(1)$, for $n=1,2,\ldots,n$

$f(n+1)+f(n-1)=2f(n)$

$\Rightarrow f(n+1)+(n-1)f(1)=2nf(1)$

$\Rightarrow f(n+1)=(n+1)f(1)$

70. d. $\because \{x\}\in[0,1)$

$\sin\{x\}\in(0,\sin 1)$ as $f(x)$ is defined if $\sin\{x\}\neq 0$

$\Rightarrow \frac{1}{\sin\{x\}}\in\left(\frac{1}{\sin 1},\infty\right) \Rightarrow \left[\frac{1}{\sin\{x\}}\right]\in\{1,2,3,\ldots\}$

Note that $1<\frac{\pi}{3}\Rightarrow \sin 1<\sin\frac{\pi}{3}=0.866\Rightarrow\frac{1}{\sin 1}>1.155$.

71. c. We have $[\cos^{-1}x]\geq 0\ \forall x\in[-1,1]$

and $[\cot^{-1}x]\geq 0\ \forall x\in R$

Hence, $[\cot^{-1}x]+[\cot^{-1}x]=0$

$\Rightarrow [\cot^{-1}x]=[\cot^{-1}x]=0$

If $[\cos^{-1}x]=0\Rightarrow x\in(\cos 1,1]$

If $[\cot^{-1}x]=0\Rightarrow x\in(\cot 1,\infty)$

$\Rightarrow x\in(\cot 1,1]$

72. d. The period of $f(x)$ is $7\Rightarrow$ The period of $f\left(\frac{x}{3}\right)$ is $\frac{7}{1/3}=21$

The period of $g(x)$ is $11\Rightarrow$ The period of $g\left(\frac{x}{5}\right)$ is $\frac{11}{1/5}=55$

Hence, T_1 = period of $f(x)g\left(\frac{x}{5}\right)=7\times 55=385$ and

T_2 = period of $g(x)f\left(\frac{x}{3}\right)=11\times 21=231$.

$\therefore$ Period of $F(x)=\text{LCM}\{T_1,T_2\}$

$=\text{LCM}\{385,231\}$

$=7\times 11\times 3\times 5$

$=1155$.

73. d. $\sin^2 x+\sin^2\left(x+\frac{\pi}{3}\right)+\cos x\cos\left(x+\frac{\pi}{3}\right)$

$$=\sin^2 x+\left(\frac{\sin x}{2}+\frac{\sqrt{3}\cos x}{2}\right)^2+\cos x\left(\frac{\cos x}{2}-\frac{\sqrt{3}\sin x}{2}\right)$$

$$=\sin^2 x+\frac{\sin^2 x}{4}+\frac{3\cos^2 x}{4}+\frac{\cos^2 x}{2}$$

$$=\frac{5\sin^2 x}{4}+\frac{5\cos^2 x}{4}=5/4.$$

Hence, $f(x)=c^{5/4}$ = constant, which is periodic whose period cannot be determined.

74. a. $f(x+f(y))=f(x)+y, f(0)=1$

Putting $y=0$, we get $f(x+f(0))=f(x)+0$

$\Rightarrow f(x+1)=f(x)\ \forall x\in R$

Thus, $f(x)$ is the period with 1 as one of its period.

$\Rightarrow f(7)=f(6)=f(5)=\cdots=f(1)=(0)=1$.

75. c. $f(x)=\sqrt{|x|-\{x\}}$ is defined if $|x|\geq\{x\}$

$\Rightarrow x\in\left(-\infty-\frac{1}{2}\right]\cup[0,\infty)\Rightarrow Y\in[0,\infty)$.

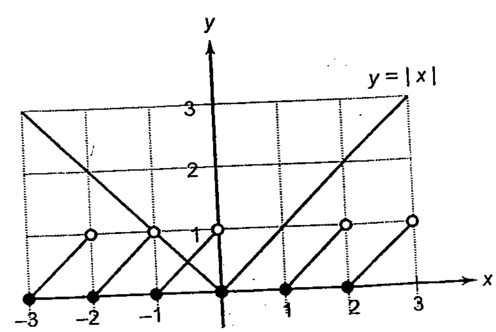


Fig. 1.98

76. a. $f(xy)=\dfrac{f(x)}{y}$

$\Rightarrow \quad f(y)=\dfrac{f(1)}{y}$ (putting $x=1$)

$\Rightarrow \quad f(30)=\dfrac{f(1)}{30}$ or $f(1)=30\times f(30)=30\times 20=600$.

Now, $f(40)=\dfrac{f(1)}{40}=\dfrac{600}{40}=15$.

77.c. **Case I**

$0<|x|-1<1 \Rightarrow 1<|x|<2$, then

$x^2+4x+4\le 1$

$\Rightarrow \; x^2+4x+3\le 0$

$\Rightarrow \; -3\le x\le -1$

So $x\in(-2,-1)$ (1)

Case II

$|x|-1>1 \Rightarrow |x|>2$, then $x^2+4x+4\ge 1$

$\Rightarrow \; x^2+4x+3\ge 0$

$\Rightarrow \; x\ge -1$ or $x\le -3$

So, $x\in(-\infty,-3]\cup(2,\infty)$ (2)

From (1) and (2), $x\in(-\infty,-3]\cup(-2,-1)\cup(2,\infty)$.

78.d. $f(x)=\dfrac{n(n+1)}{2}+[\sin x]+\left[\sin\dfrac{x}{2}\right]+\cdots+\left[\sin\dfrac{x}{n}\right]$

Thus, the range of $f(x)=\left\{\dfrac{n(n+1)}{2},\dfrac{n(n+1)}{2}+1\right\}$ as $x\in[0,\pi]$.

79.b. $[x]^2=x+2\{x\}$

$\Rightarrow [x]^2=[x]+3\{x\}$

$\Rightarrow \{x\}=\dfrac{[x]^2-[x]}{3}$

$\Rightarrow \quad 0\le\dfrac{[x]^2-[x]}{3}<1$

$\Rightarrow \quad 0\le [x]^2-[x]<3$

$\Rightarrow \; [x]\in\left(\dfrac{1-\sqrt{3}}{2},0\right]\cup\left[1,\dfrac{1+\sqrt{3}}{2}\right)$

$\Rightarrow \; [x]=-1,0,1,2$

$\Rightarrow \; \{x\}=\dfrac{2}{3},0,0,\dfrac{2}{3}$, (respectively)

$\Rightarrow \quad x=-\dfrac{1}{3},0,1,\dfrac{8}{3}$

80.b. We must have

$2\{x\}^2-3\{x\}+1\ge 0\Rightarrow \{x\}\ge 1$ or $\{x\}\le 1/2$.

Thus, we have $0\le\{x\}\le 1/2\Rightarrow x\in\left[n,n+\dfrac{1}{2}\right], n\in I$.

81.b. $\left[x^2+\dfrac{1}{2}\right]=\left[x^2-\dfrac{1}{2}+1\right]=1+\left[x^2-\dfrac{1}{2}\right]$.

Thus, from domain point of view,

$\left[x^2-\dfrac{1}{2}\right]=0,-1\Rightarrow\left[x^2+\dfrac{1}{2}\right]=1,0$

$\Rightarrow \; f(x)=\sin^{-1}(1)+\cos^{-1}(0)$ or $\sin^{-1}(0)+\cos^{-1}(-1)$

$\Rightarrow \; f(x)=\{\pi\}$

82.c. The period of $\cos(\sin nx)$ is $\dfrac{\pi}{n}$ and the period of $\tan\left(\dfrac{x}{n}\right)$ is πn.

Thus, $6\pi=\text{LCM}\left(\dfrac{\pi}{n},\pi n\right)$

$\Rightarrow \; 6\pi=\dfrac{\pi}{n}\lambda_1\Rightarrow n=\dfrac{\lambda_1}{6}$, and $6\pi=\lambda_2\,\pi n\Rightarrow n=\dfrac{6}{\lambda_2}$, λ_1, $\lambda_2\in I^+$

From $n=\dfrac{6}{\lambda_2}\Rightarrow n=6,3,2,1$.

Clearly, for $n=6$, we get the period of $f(x)$ to be 6π.

83.a. We must have $ax^3+(a+b)x^2+(b+c)x+c>0$

$\Rightarrow \; ax^2(x+1)+bx(x+1)+c(x+1)>0$

$\Rightarrow \; (x+1)(ax^2+bx+c)>0$

$\Rightarrow \; a(x+1)\left(x+\dfrac{b}{2a}\right)^2>0$ as $b^2=4ac$

$\Rightarrow \; x>-1$ and $\ne-\dfrac{b}{2a}$

84.b. $f(x)=[x]+[2x]+[3x]+\cdots+[nx]-(x+2x+3x+\cdots+nx)$

$=-(\{x\}+\{2x\}+\{3x\}+\cdots+\{nx\})$

The period of $f(x)=\text{LCM}\left(1,\dfrac{1}{2},\dfrac{1}{3},\ldots,\dfrac{1}{n}\right)=1$.

85.c. $f\left(x+\dfrac{1}{2}\right)+f\left(x-\dfrac{1}{2}\right)=f(x)$

$\Rightarrow \; f(x+1)+f(x)=f\left(x+\dfrac{1}{2}\right)$

$\Rightarrow \; f(x+1)+f\left(x-\dfrac{1}{2}\right)=0$

$\Rightarrow \; f\left(x+\dfrac{3}{2}\right)=-f(x)$

$\Rightarrow \; f(x+3)=-f\left(x+\dfrac{3}{2}\right)=f(x)$

$\therefore \; f(x)$ is periodic with period 3.

86.c. $f(x)+3xf\left(\dfrac{1}{x}\right)=2(x+1)$ (1)

Replacing x by $\dfrac{1}{x}$, we get

$f\left(\dfrac{1}{x}\right)+3\dfrac{1}{x}f(x)=2\left(\dfrac{1}{x}+1\right)$

$\Rightarrow \; xf\left(\dfrac{1}{x}\right)+3f(x)=2(x+1)$ (2)

From (1) and (2), we have $f(x)=\dfrac{x+1}{2}$

$\Rightarrow f(99)=50$

87.a. Let $y=\dfrac{x+5}{x+2}=1+\dfrac{3}{x+2}\Rightarrow x=1$

Also, $y-1=\dfrac{3}{x+2} \quad\Rightarrow\quad x+2=\dfrac{3}{y-1}$

$\Rightarrow\ x=-2+\dfrac{3}{y-1}$

$\Rightarrow\ y=2$ only as x and y are natural numbers.

88.d. $f(f(x))=\begin{cases}(f(x))^2, & \text{for } f(x)\geq 0\\ f(x), & \text{for } f(x)<0\end{cases}$

$=\begin{cases}(x^2)^2, & x^2\geq 0, x\geq 0\\ x^2, & x\geq 0, x<0\\ x^2, & x^2<0, x\geq 0\\ x, & x<0, x<0\end{cases} = \begin{cases}x^4, & x\geq 0\\ x, & x<0\end{cases}$

89.c. From the given data

$f(1-x)=f(1+x) \qquad (1)$

and $f(2-x)=f(2+x) \qquad (2)$

In (2) replacing x by $1+x$, we have

$f(1-x)=f(3+x)$

$\Rightarrow\ f(1+x)=f(3+x) \qquad$ [from (1)]

$\Rightarrow\ f(x)=f(2+x)$

90.a. Let $f(x)=x+2|x+1|+2|x-1|$

$\Rightarrow f(x)=\begin{cases}x-2(x+1)-2(x-1), & x<-1\\ x+2(x+1)-2(x-1), & -1\leq x\leq 1\\ x+2(x+1)+2(x-1), & x>1\end{cases}$

or $f(x)=\begin{cases}-3x, & x<-1\\ x+4, & -1\leq x\leq 1\\ 5x, & x>1\end{cases}$

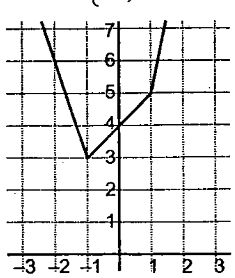


Fig. 1.99

Graph of $y=f(x)$ is as shown. Clearly $y=k$ can intersect $y=f(x)$ at exactly one point only if $k=3$

91.d. We must have $-1\leq[2x^2-3]\leq 1$

$\Rightarrow\ -1\leq 2x^2-3<2\Rightarrow 1\leq x^2<\dfrac{5}{2}$

$\Rightarrow\ x\in\left(-\sqrt{\dfrac{5}{2}},-1\right]\cup\left[1,\sqrt{\dfrac{5}{2}}\right)$

92.c. $\cos^{-1}\left(\dfrac{1+x^2}{2x}\right)$ is defined if $\left|\dfrac{1+x^2}{2x}\right|\leq 1$ and $x\neq 0$

$\Rightarrow\ 1+x^2-2|x|\leq 0$

$\Rightarrow\ (|x|-1)^2\leq 0$

$\Rightarrow\ x=1,-1$

Thus, the domain of $f(x)$ is $\{1,-1\}$. Hence, the range is $\{1,1+\pi\}$.

93.a. $f(f(x))=\begin{cases}f(x), & f(x) \text{ is rational}\\ 1-f(x), & f(x) \text{ is irrational}\end{cases}$

$\Rightarrow\ f(f(x))=\begin{cases}x, & x \text{ is rational}\\ 1-(1-x)=x, & x \text{ is irrational.}\end{cases}$

94.c. $y=|\sin x|+|\cos x|$

$\Rightarrow\ y^2=1+|\sin 2x|$

$\Rightarrow\ 1\leq y^2\leq 2$

$\Rightarrow\ y\in[1,\sqrt{2}]$

$\Rightarrow\ f(x)=1\ \forall x\in R$

95.d. $f(x)=\ln\left(\dfrac{x^2+e}{x^2+1}\right)=\ln\left(\dfrac{x^2+1+e-1}{x^2+1}\right)=\ln\left(1+\dfrac{e-1}{x^2+1}\right)$

Now, $1\leq x^2+1<\infty$

$\Rightarrow\ 0<\dfrac{1}{x^2+1}\leq 1 \quad\Rightarrow\quad 0<\dfrac{e-1}{x^2+1}\leq e-1$

$\Rightarrow\ 1<1+\dfrac{e-1}{x^2+1}\leq e \quad\Rightarrow\quad 0<\ln\left(1+\dfrac{e-1}{x^2+1}\right)\leq 1$

Hence, the range is (0, 1].

96.d. $f(x)=\dfrac{1}{\sqrt{4x-|x^2-10x+9|}}$

For $f(x)$ to be defined $|x^2-10x+9|<4x$

$\Rightarrow\ x^2-10x+9<4x$ and $x^2-10x+9>-4x$

$\Rightarrow\ x^2-14x+9<0$ and $x^2-6x+9>0$

$\Rightarrow\ x\in\left(7-\sqrt{40},7+\sqrt{40}\right)$ and $x\in R-\{-3\}$

$\Rightarrow\ x\in(7-\sqrt{40},-3)\cup(-3,7+\sqrt{40})$

97.b. Given $y=2^{x(x-1)}$

$\Rightarrow\ x(x-1)=\log_2 y$

$\Rightarrow\ x^2-x-\log_2 y=0$

$\Rightarrow\ x=\dfrac{1\pm\sqrt{1+4\log_2 y}}{2}$

Only $x=\dfrac{1+\sqrt{1+4\log_2 y}}{2}$ lies in the domain.

$\Rightarrow\ f^{-1}(x)=\dfrac{1}{2}[1+\sqrt{1+4\log_2 x}]$

98.c. $x\sin x=1 \qquad (1)$

$\Rightarrow\ y=\sin x=\dfrac{1}{x}$

Root of equation (1) will be given by the point(s) of intersection of the graphs $y=\sin x$ and $y=\dfrac{1}{x}$. Graphically, it is clear that we get four roots.

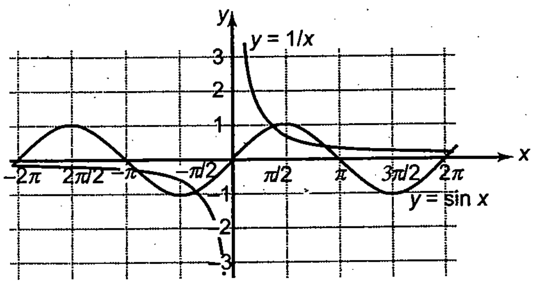


Fig. 1.100

99.c. See the graph of $y = 2\cos x$ and $y = |\sin x|$. Their points of intersection represent the solution of the given equation.

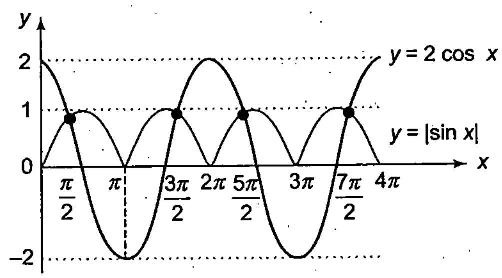


Fig. 1.101

We find that the graphs intersect at four points. Hence, the equation has four solutions.

100.a. $af(x+1)+bf\left(\frac{1}{x+1}\right)=(x+1)-1$ (1)

Replacing $x+1$ by $\frac{1}{x+1}$, we get

$\therefore af\left(\frac{1}{x+1}\right)+bf(x+1)=\frac{1}{x+1}-1$ (2)

(1) × a – (2) × b ⇒ $(a^2-b^2)\,f(x+1)=a(x+1)-a-\frac{b}{x+1}+b$

Putting $x=1$, $(a^2-b^2)f(2)=2a-a-\frac{b}{2}+b=a+\frac{b}{2}$

$=\frac{2a+b}{2}$

101.c.

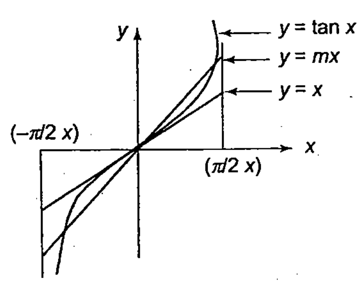


Fig. 1.102

In $\left(-\frac{\pi}{2},0\right)$, the graph of $y=\tan x$ lies below the line $y=x$ which is the tangent at $x=0$ and in $\left(0,\frac{\pi}{2}\right)$ it lies above the lies $y=x$.

For $m>1$, the line $y=mx$ lies below $y=x$ in $\left(-\frac{\pi}{2},0\right)$ and above $y=x$ in $\left(0,\frac{\pi}{2}\right)$. Thus graphs of $y=\tan x$ and $y=mx$, $m>1$, meet at three points including $x=0$ in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ independent of m.

102.c. Given $f(x)=[\sin x+[\cos x+[\tan x+[\sec x]]]]$

$=[\sin+p]$, where $p=[\cos x+[\tan x+[\sec x]]]$

$=[\sin x]+p$, (as p is an integer)

$=[\sin x]+[\cos x+[\tan x+[\sec x]]]$

$=[\sin x]+[\cos x]+[\tan x]+[\sec x]$

Now, for $x\in(0,\pi/4)$, $\sin x\in\left(0,\frac{1}{\sqrt{2}}\right)$, $\cos x\in\left(\frac{1}{\sqrt{2}},1\right)$,

$\tan x\in(0,1)$, $\sec x\in\left(1,\sqrt{2}\right)$

⇒ $[\sin x]=0$, $[\cos x]=0$, $[\tan x]=0$ and $[\sec x]=1$

⇒ The range of $f(x)$ is 1.

103.d. $f(3x+2)+f(3x+29)=0$ (1)

Replacing x by $x+9$, we get

$f(3(x+9)+2)+f(3(x+9)+29)=0$

⇒ $f(3x+29)+f(3x+56)=0$ (2)

From (1) and (2), we get

$f(3x+2)=f(3x+56)$

⇒ $f(3x+2)=f(3(x+18)+2)$

⇒ $f(x)$ is periodic with period 54.

104.b. For odd function

$f(x)=-f(-x)$

$$=-\begin{cases}\sin(-x)+\cos(-x) & 0\le -x<\pi/2\\ a, & -x=\pi/2\\ \tan^2(-x)+\operatorname{cosec}(-x), & \pi/2<-x<\pi\end{cases}$$

$$=\begin{cases}\sin x-\cos x, & -\pi/2<x\le 0\\ -a, & x=-\pi/2\\ \tan^2 x+\operatorname{cosec}x, & -\pi<x<-\pi/2\end{cases}$$

105.c. (a) $f(x)=\sin x$ and $g(x)=\cos x$, $x\in[0,\pi/2]$

Here, both $f(x)$ and $g(x)$ are one-one functions, but $h(x)=f(x)+g(x)=\sin x+\cos x$ is many-one as $h(0)=h(\pi/2)=1$.

(b) $h(x)=f(x)\,g(x)=\sin x\cos x=\frac{\sin 2x}{2}$ is many-one, as $h(0)=h(\pi/2)=0$.

(c) It is a fundamental property.

106.c. $f(x)$ is defined for $x \in (0, 1)$

$\Rightarrow$ $f(e^x) + f(\ln|x|)$ is defined for,

$0 < e^x < 1$ and $0 < \ln|x| < 1$

$\Rightarrow$ $-\infty < x < 0$ and $1 < |x| < e$

$\Rightarrow$ $x \in (-\infty, 0)$ and $x \in (-e, -1) \cup (1, e)$

$\Rightarrow$ $x \in (-e, -1)$

107.d. $|\cos x| + \cos x = \begin{cases} 0, & \cos x \le 0 \\ 2\cos x, & \cos x > 0 \end{cases}$

For $f(x)$ to defined $\cos x > 0$

$\Rightarrow x \in \left(\frac{(4n-1)\pi}{2}, \frac{(4n+1)\pi}{2}\right) n \in Z$ (1st and 4th quadrant).

108.b. Let $2x + 3y = m$ and $2x - 7y = n$

$\Rightarrow$ $y = \frac{m-n}{10}$ and $x = \frac{7m+3n}{20}$

$\Rightarrow$ $f(m, n) = 7m + 3n$

$\Rightarrow$ $f(x, y) = 7x + 3y$

109.d. Image b_1 is assigned to any three of the six pre-images in ${}^6C_3$ ways.

Rest two images can be assigned to remaining three pre-images in $2^3 - 2$ ways (as function is onto).

Hence number of functions are ${}^6C_3 \times (2^3 - 2) = 20 \times 6 = 120$

110.d. $y = f(x)$ and $y = g(x)$ are mirror image of each other about line $y = a$

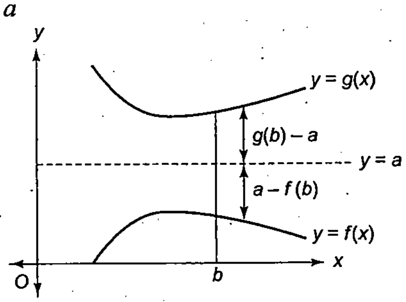


Fig. 1.103

$\Rightarrow$ for some $x = b$, $g(b) - a = a - f(b)$

$\Rightarrow$ $f(b) + g(b) = 2a$

$\Rightarrow$ $h(b) = f(b) + g(b) = 2a$ (constant)

Hence $h(x)$ is constant function. Thus it is neither one-one nor onto.

111.c. Clearly $f(x + \pi) = f(x)$, $g(x + \pi) = g(x)$ and $\phi\left(x + \frac{\pi}{2}\right)$

$= \{(-1) f(x)\} \{(-1) g(x)\} = \phi(x)$.

112.b. In the sum, $f(1) + f(2) + f(3) + \cdots + f(n)$, 1 occurs n times, $\frac{1}{2}$ occurs $(n-1)$ times, $\frac{1}{3}$ occurs $(n-2)$ times and so on

$\therefore f(1) + f(2) + f(3) + \cdots + f(n)$

$= n \cdot 1 + (n-1) \cdot \frac{1}{2} + (n-2) \cdot \frac{1}{3} + \cdots + 1 \cdot \frac{1}{n}$

$= n\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) - \left(\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{n-1}{n}\right)$

$= nf(n) - \left[\left(1 - \frac{1}{2}\right) + \left(1 - \frac{1}{3}\right) + \left(1 - \frac{1}{4}\right) + \cdots + \left(1 - \frac{1}{n}\right)\right]$

$= nf(n) - [n - f(n)]$

$= (n+1) f(n) - n$

113.a. $h(x) = \log(f(x) . g(x)) = \log e^{\{y\}+[y]} = \{y\} + [y] = e^{|x|} \operatorname{sgn} x$

$\therefore h(x) = e^{|x|} \operatorname{sgn} x = \begin{cases} e^x, & x > 0 \\ 0, & x = 0 \\ -e^{-x}, & x < 0 \end{cases}$

$\Rightarrow$ $h(-x) = \begin{cases} e^{-x}, & x < 0 \\ 0, & x = 0 \\ -e^x, & x > 0 \end{cases}$ $\Rightarrow$ $h(x) + h(-x) = 0$ for all x.

114.b.

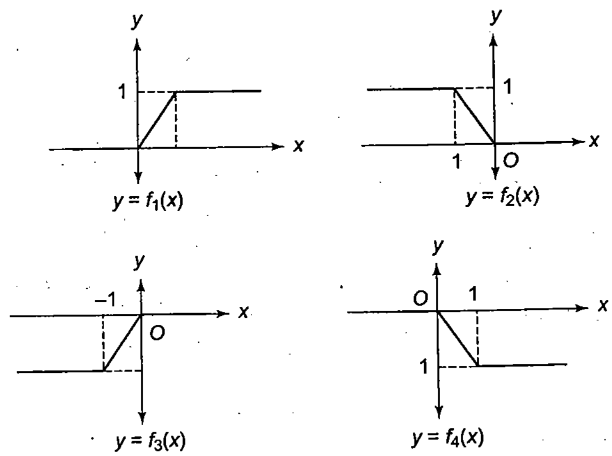


Fig. 1.104

115.d. $[y + [y]] = 2\cos x$

$\Rightarrow$ $[y] + [y] = 2\cos x$ $\quad$ ($\because [x + n] = [x] + n$ if $n \in I$)

$\Rightarrow$ $2[y] = 2\cos x \Rightarrow [y] = \cos x$ $\quad$ (1)

Also $y = \frac{1}{3}[\sin x + [\sin x + [\sin x]]]$

$= \frac{1}{3}(3[\sin x])$

$= [\sin x]$ $\quad$ (2)

From (1) and (2)

$[[\sin x]] = \cos x$

$\Rightarrow$ $[\sin x] = \cos x$

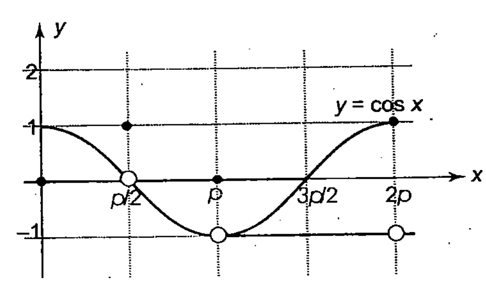


Fig. 1.105

The number of solutions is 0.

116.a. $\cos^{-1}(\cos x) = [x]$

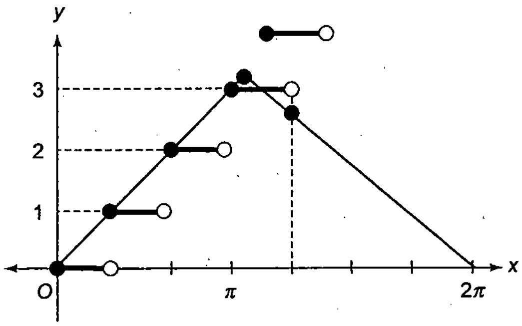


Fig. 1.106

The solutions are clearly 0, 1, 2, 3 and $3 = 2\pi - x$ or $x = 2\pi - 3$.

117.c. Given $f(x) = \sqrt{(1-\cos x)\sqrt{(1-\cos x)\sqrt{(1-\cos x)\sqrt{\ldots\infty}}}}$

$\Rightarrow f(x) = (1-\cos x)^{\frac{1}{2}}(1-\cos x)^{\frac{1}{4}}(1-\cos x)^{\frac{1}{8}}\ldots\infty$

$\Rightarrow f(x) = (1-\cos x)^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\infty}$

$\Rightarrow f(x) = (1-\cos x)^{\frac{\frac{1}{2}}{1-\frac{1}{2}}}$

$\Rightarrow f(x) = 1 - \cos x$

$\Rightarrow$ The range of $f(x)$ is [0, 2).

118.b. $-5 \le |kx+5| \le 7$

$\Rightarrow -12 \le kx \le 2$ where $-6 \le x \le 1$

$\Rightarrow -6 \le \frac{k}{2}x \le 1$ where $-6 \le x \le 1$

$\therefore k = 2.$ [$\because$ the range of $h(x)$ = the domain of $f(x)$]

119.a. Let $g(x) = (x+1)(x+2)(x+3)(x+4)$

The rough graph of $g(x)$ is given as

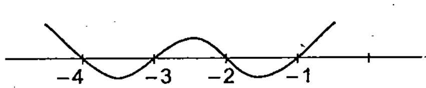


Fig. 1.107

$\therefore g(x) = (x+1)(x+2)(x+3)(x+4)$
$= (x+1)(x+4)(x+2)(x+3)$
$= (x^2+5x+4)(x^2+5x+6)$
$= t(t+2) = (t+1)^2 - 1,$

where $t = x^2 + 5x$

Now $g_{min} = -1$, for which $x^2 + 5x = -1$ has real roots in [– 6, 6]

Also $g(6) = 7 \times 8 \times 9 \times 10 = 5040$

Hence, the range of $g(x)$ is [– 1, 5040] for $x \in [-6, 6]$.

Then, the range of $f(x)$ is [4, 5045].

120.d. $f(x) = \sqrt{x^{12} - x^9 + x^4 - x + 1}$

We must have $x^{12} - x^9 + x^4 - x + 1 \ge 0$ (1)

Obviously (1) is satisfied by $x \in (-\infty, 0]$

Also, $x^9(x^3-1) + x(x^3-1) + 1 \ge 0 \ \forall x \in [1, \infty)$

Further, $x^{12} - x^9 + x^4 - x + 1 = (1-x) + x^4(1-x^5) + x^{12}$ is also satisfied by $x \in (0, 1)$.

Hence, the domain is R.

121.a. $f(x) = \sec^{-1}(\log_3 \tan x + \log_{\tan x} 3)$.

$f(x) = \sec^{-1}\left(\log_3 \tan x + \frac{1}{\log_3 \tan x}\right)$

Now for $\log_3 \tan x$ to get defined, $\tan x \in (0, \infty)$

$\Rightarrow \log_3 \tan x \in (-\infty, \infty)$ or $\log_3 \tan x \in R$

Also $x + \frac{1}{x} \le -2$ or $x + \frac{1}{x} \ge 2$

$\Rightarrow \log_3 \tan x + \frac{1}{\log_3 \tan x} \le -2$ or

$\log_3 \tan x + \frac{1}{\log_3 \tan x} \ge 2$

$\Rightarrow \sec^{-1}\left(\log_3 \tan x + \frac{1}{\log_3 \tan x}\right) \le \sec^{-1}(-2)$ or

$\sec^{-1}\left(\log_3 \tan x + \frac{1}{\log_3 \tan x}\right) \ge \sec^{-1} 2$

$\Rightarrow f(x) \le \frac{2\pi}{3}$ or $f(x) \ge \frac{\pi}{3}$

$\Rightarrow f(x) \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{2\pi}{3}\right]$

122.a. We have $f(x) = {}^{7-x}P_{x-3} = \frac{(7-x)!}{(10-2x)!}$

We must have $7 - x > 0, x \ge 3$ and $7 - x \ge x - 3$

$\Rightarrow x < 7, x \ge 3$ and $x \le 5$

$\Rightarrow 3 \le x \le 5$

$\Rightarrow x = 3, 4, 5$

Now, $f(3) = \frac{4!}{4!} = 1, f(4) = \frac{3!}{2!} = 3, f(5) = \frac{2!}{0!} = 2.$

Hence, $R_f = \{1, 2, 3\}$.

123.b. We have $f(x-y) = f(x)f(y) - f(a-x)f(a+y)$

Putting $x = a$ and $y = a - x$, we get

$f(a-(x-a)) = f(a)f(x-a) - f(0)f(x)$ (1)

Putting $x = 0, y = 0$, we get

$f(0) = f(0)(f(0)) - f(a)f(a)$

$\Rightarrow f(0) = (f(0))^2 - (f(a))^2$

$\Rightarrow 1 = (1)^2 - (f(a))^2$

$\Rightarrow f(a) = 0$

$\Rightarrow f(2a-x) = -f(x)$

Multiple Correct Answers Type

1. a, b, d.

$f(0) = \max\{1 + \sin 0, 1, 1 - \cos 0\} = 1$

$g(0) = \max\{1, |0-1|\} = 1$

$f(1) = \max\{1 + \sin 1, 1, 1 - \cos 1\} = 1 + \sin 1$
$g(f(0)) = g(1) = \max\{1, |1 - 1|\} = 1$
$f(g(0)) = f(1) = 1 + \sin 1$
$g(f(1)) = g(1 + \sin 1) = \max\{1, |1 + \sin 1 - 1|\} = 1$

2. b, c.

(a) For $f(x) = \log x^2, x^2 > 0 \Rightarrow x \in R - \{0\}$
For $g(x) = 2 \log x, x > 0$
Hence, $f(x)$ and $g(x)$ are not identical.

(b) $f(x) = \log_x e = \dfrac{1}{\log_e x} = g(x)$
Hence, the functions are identical.

(c) $f(x) = \sin(\cos^{-1} x) = \sin\left(\dfrac{\pi}{2} - \sin^{-1} x\right) = \cos\left(\sin^{-1} x\right)$
$= g(x)$
Hence, the functions are identical.

3. a, b, c.

$f(x) + f(y) = f\left(\dfrac{x+y}{1-xy}\right)$
Replace y by $-x \Rightarrow f(x) + f(-x) = f(0)$ (1)
Put $x = y = 0 \Rightarrow f(0) + f(0) = f(0) \Rightarrow f(0) = 0$
$\Rightarrow f(x) + f(-x) = 0$ (from (1))
Hence, $f(x)$ is an odd function.

$f(x) + f(y) = f\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$
Replace y by $-x \Rightarrow f(x) + f(-x) = f(0)$ (2)
Put $x = y = 0 \Rightarrow f(0) + f(0) = f(0)$
$\Rightarrow f(0) = 0 \Rightarrow f(x) + f(-x) = 0$ (from (2))
Hence, $f(x)$ is an odd function.

$f(x+y) = f(x) + f(y)$
Replace y by $-x \Rightarrow f(0) = f(x) + f(-x)$ (3)
Put $x = y = 0 \Rightarrow f(0+0) = f(0) + f(0) \Rightarrow f(0) = 0 \Rightarrow f(x) + f(-x) = 0$ (from (3))
Hence, $f(x)$ is an odd function.

4. a, c.

$f(x+y) + f(x-y) = 2f(x) \cdot f(y)$ (1)
Put $x = 0 \Rightarrow f(y) + f(-y) = 2f(0)f(y)$ (2)
Put $x = y = 0 \Rightarrow f(0) + f(0) = 2f(0)f(0)$
$\Rightarrow f(0) = 1$ (as $f(0) \neq 0$)
$\Rightarrow f(-y) = f(y)$ (from (2))
Hence, the function is even. Then $f(-2) = f(2) = a$.

5. b, d.

$$f\left(x + \frac{1}{x}\right) = x^2 + \frac{1}{x^2}$$

$$\Rightarrow f\left(x + \frac{1}{x}\right) = x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2$$

$$\Rightarrow f(y) = y^2 - 2$$

Now $y = x + \dfrac{1}{x} \geq 2$ or ≤ -2
Hence, the domain of the function is $(-\infty, -2] \cup [2, \infty)$
Also for these values of y, $y^2 \geq 4 \Rightarrow y^2 - 2 \geq 2$.
Hence, the range of the function is $[2, \infty)$.

6. a, d.

Given $f(x) + f(y) = \left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$ (1)

Replace y by $x \Rightarrow 2f(x) = f\left(2x\sqrt{1-x^2}\right)$

$3f(x) = f(x) + 2f(x)$
$= f(x) + f\left(2x\sqrt{1-x^2}\right)$
$= f\left(x\sqrt{1 - 4x^2(1-x^2)} + 2x\sqrt{1-x^2}\sqrt{1-x^2}\right)$
$= f\left(x\sqrt{(2x^2-1)^2} + 2x(1-x^2)\right)$
$= f\left(x|2x^2 - 1| + 2x - 2x^3\right)$
$= f\left(2x^3 - x + 2x - 2x^3\right)$ or $f\left(x - 2x^3 + 2x - 2x^3\right)$
$= f(x)$ or $f(3x - 4x^3)$
$\Rightarrow f(x) = 0$ or $3f(x) = f(3x - 4x^3)$
Consider $3f(x) = f(3x - 4x^3)$
Replace x by $-x$, we get
$3f(-x) = f(4x^3 - 3x)$ (2)
Also from (1), $f(x) + f(-x) = f(0)$
Put $x = y = 0$ in (1), we have $f(0) = 0 \Rightarrow f(x) + f(-x) = 0$, thus $f(x)$ is an odd function.
Now from (2) $-3f(x) = f(4x^3 - 3x)$
$\Rightarrow f(4x^3 - 3x) + 3f(x) = 0$

7. b, c.

Given $2f(\sin x) + f(\cos x) = x$ (1)

Replace x by $\dfrac{\pi}{2} - x$

$\Rightarrow 2f(\cos x) + f(\sin x) = \dfrac{\pi}{2} - x$ (2)

Eliminating $f(\cos x)$ from (1) and (2), we get

$\Rightarrow 3f(\sin x) = 3x - \dfrac{\pi}{2}$

$\Rightarrow f(\sin x) = x - \dfrac{\pi}{6}$

$\Rightarrow f(x) = \sin^{-1} x - \dfrac{\pi}{6}$

$f(x)$ has the domain $[-1, 1]$

Also, $\sin^{-1} x \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right] \Rightarrow \sin^{-1} x - \dfrac{\pi}{6} \in \left[-\dfrac{2\pi}{3}, \dfrac{\pi}{3}\right]$.

8. a, c.

$f(2) = f(1+1) = 2f(1) = 10$
$f(3) = f(2+1) = f(2) + f(1) = 10 + 5 = 15$
Then, $f(n) = 5n$

$$\Rightarrow \sum_{r=1}^{m} f(r) = 5\sum_{r=1}^{m} r = \frac{5m(m+1)}{2}$$

Replace y by $-x$, $\Rightarrow f(0) = f(x) + f(-x)$
Also put $x = y = 0 \Rightarrow f(0) = f(0) + f(0) \Rightarrow f(0) = 0$
$\Rightarrow f(x) + f(-x) = 0$, hence, the function is odd.

9. a, b, c.

$$(f+g)(3.5)=f(3.5)+g(3.5)=(-0.5)+0.5=0$$
$$f(g(3))=f(0)=3$$
$$(fg)(2)=f(2)g(2)=(-1)\times(-1)=1$$
$$(f-g)(4)=f(4)-g(4)=0-26=-26$$

10. b, d.

$$f(x)=x^2-2ax+a(a+1)$$
$$f(x)=(x-a)^2+a,\ x\in[a,\infty)$$

Let $y=(x-a)^2+a$ clearly $y\ge a$

$\Rightarrow\quad (x-a)^2=y-a$

$\Rightarrow\quad x=a+\sqrt{y-a}$

$\therefore\quad f^{-1}(x)=a+\sqrt{x-a}$

Now $f(x)=f^{-1}(x)$

$\Rightarrow\quad (x-a)^2+a=a+\sqrt{x-a}$

$(x-a)^2=\sqrt{x-a}$

$\Rightarrow\quad (x-a)^4=(x-a)$

$\Rightarrow\quad x=a$ or $(x-a)^3=1$

$\Rightarrow\quad x=a$ or $a+1$

If $a=5049$, then $a+1=5050$

If $a+1=5049$, then $a=5048$.

11. a, b, c, d.

$$f(x)=\begin{cases}1, & x \text{ is rational}\\ 0, & x \text{ is irrational}\end{cases}$$

$$\Rightarrow\quad f(x+k)=\begin{cases}1, & x+k \text{ is rational}\\ 0, & x+k \text{ is irrational}\end{cases}$$

where k is any rational number

$$\Rightarrow\quad f(x+k)=\begin{cases}1, & x \text{ is rational}\\ 0, & x \text{ is irrational}\end{cases}$$

$\Rightarrow\quad f(x+k)=f(x)$

$\Rightarrow\quad f(x)$ is periodic function, but its fundamental period cannot be determined

$$f(x)=\begin{cases}x-[x], & 2n\le x<2n+1\\ 1/2, & 2n+1\le x<2n+2\end{cases}$$

Draw the graph from which it can be verified that period is 2.

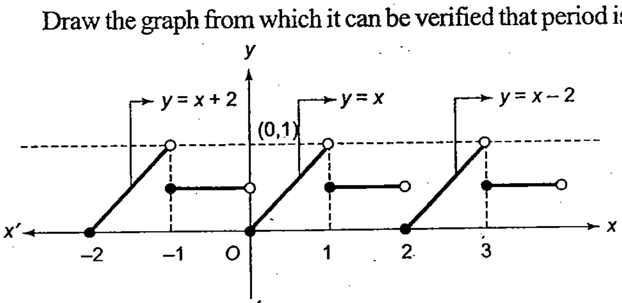


Fig. 1.108

$$f(x)=(-1)^{\left[\frac{2x}{\pi}\right]}$$

$$\Rightarrow f(x+\pi)=(-1)^{\left[\frac{2(\pi+x)}{\pi}\right]}=(-1)^{\left[\frac{2x}{\pi}\right]+2}=(-1)^{\left[\frac{2x}{\pi}\right]}$$

Hence, the period is π.

$$f(x)=x-[x+3]+\tan\left(\frac{\pi x}{2}\right)=\{x\}-3+\tan\left(\frac{\pi x}{2}\right)$$

$\{x\}$ is periodic with period 1, $\tan\left(\frac{\pi x}{2}\right)$ x is periodic with period 2.

Now, the LCM of 1 and 2 is 2. Hence, the period of $f(x)$ is 2.

12. b, c.

$f(x)$ must be a linear function, let $f(x)=ax+b$

$\Rightarrow\quad f(ax+b)=6x-ax-b$

$\Rightarrow\quad a(ax+b)+b=6x-ax-b$

$\Rightarrow\quad a^2=6-a$ and $ab+b=-b$

$\Rightarrow\quad a=2$ or $-3\Rightarrow b=0$

$\Rightarrow\quad f(x)=2x$ or $-3x\Rightarrow f(17)=34$ or -51

13. a, b, c, d.

$$f(x+1)=\frac{f(x)-5}{f(x)-3}\qquad(1)$$

$$\Rightarrow f(x)f(x+1)-3f(x+1)=f(x)-5$$

$$\Rightarrow f(x)=\frac{3f(x+1)-5}{f(x+1)-1}$$

Replacing x by $(x-1)$, we get

$$f(x-1)=\frac{3f(x)-5}{f(x)-1}\qquad(2)$$

Using (1), $$f(x+2)=\frac{f(x+1)-5}{f(x+1)-3}=\frac{\frac{f(x)-5}{f(x)-3}-5}{\frac{f(x)-5}{f(x)-3}-3}$$

$$=\frac{2f(x)-5}{f(x)-2}\qquad(3)$$

Using (2), $$f(x-2)=\frac{3f(x-1)-5}{f(x-1)-1}=\frac{3\left(\frac{3f(x)-5}{f(x)-1}\right)-5}{\frac{3f(x)-5}{f(x)-1}-1}$$

$$=\frac{2f(x)-5}{f(x)-2}\qquad(4)$$

Using (3) and (4), we have $f(x+2)=f(x-2)$

$\Rightarrow\quad f(x+4)=f(x)\Rightarrow f(x)$ is periodic with period 4.

14. a, d.

$f(x)=\sec^{-1}[1+\cos^2x]$

$f(x)$ is defined if $[1+\cos^2x]\le-1$ or $[1+\cos^2x]\ge1$

$\Rightarrow\quad [\cos^2x]\le-2$ (not possible) or $[\cos^2x]\ge0$

$\Rightarrow\quad \cos^2\ge0\Rightarrow x\in R$

Now $0\le\cos^2x\le1\Rightarrow1\le1+\cos^2x\le2$

$\Rightarrow\quad [1+\cos^2x]=1,2$

$\Rightarrow \quad \sec^{-1}[1+\cos^2 x] = \sec^{-1}1, \sec^{-1}2$

Hence, the range is $\{\sec^{-1}1, \sec^{-1}2\}$.

15. a, b, c.

$f(x) = \tan(\tan^{-1}x) = x$ for all x and $g(x) = \cot(\cot^{-1}x) = x$ for all x

Hence, this pair is identical functions.

$f(x) = \text{sgn}(x)$ and $g(x) = \text{sgn}(\text{sgn}(x))$ have domain R

$f(x)$ has range $\{-1, 0, 1\}$ and $g(x) = \text{sgn}(\text{sgn}(x))$ has range $\{-1, 0, 1\}$

Also $f(x) = g(x)$ for any x, then this pair is identical functions

$g(x) = \cot^2 x - \cos^2 x = \cos^2 x(\text{cosec}^2 x - 1) = \cos^2 x \cot^2 x = f(x)$

$f(x) = e^{\log_e \sec^{-1}x}$ has the domain $[1, \infty)$, whereas $g(x) = \sec^{-1}x$ has the domain $(-\infty, -1] \cup [1, \infty)$.

Hence, this pair is not identical functions.

16. b, d.

The period of $f(x) = |\sin 2x| + |\cos 2x|$ is $\pi/4$

$\Rightarrow \quad [f(x)]$ is also periodic with period $\pi/4$.

Also $1 \le f(x) \le \sqrt{2}$

$\Rightarrow \quad [f(x)] = 1$ $f(x)$ is a many-one and into function.

17. a, b, d.

$f(x) = \dfrac{1}{\ln[1-|x|]}$ is defined if $[1-|x|] > 0$ and $1 - [x] \ne 1$

$\Rightarrow \quad [1-|x|] \ge 2 \Rightarrow 1 - |x| \ge 2 \Rightarrow |x| \le -1$ which is not possible.

$f(x) = \dfrac{x!}{\{x\}}$. Here $x!$ is defined only when x is natural number, but $\{x\}$ becomes zero for these values of x. Hence, $f(x)$ is not defined in this case.

$f(x) = x!\ \{x\}$ is defined for x being a natural number. Hence, $f(x)$ is a function whose domain $x \in N$.

$f(x) = \dfrac{\ln(x-1)}{\sqrt{(1-x^2)}}$. Here $\ln(x-1)$ is defined only when $x - 1 > 0 \Rightarrow x > 1$. Also $1 - x^2 > 0$ for denominator, i.e., $-1 < x < 1$. Hence, $f(x)$ is not defined for any value of x.

18. b, c, d.

$f(x) = \sin(\sin^{-1}x) = x \ \forall\ x \in [-1, 1]$ which is one-one and onto.

$f(x) = \dfrac{2}{\pi}\sin^{-1}(\sin x) = \dfrac{2}{\pi}x$

The range of the function for $x \in [-1, 1]$ is $\left[-\dfrac{2}{\pi}, \dfrac{2}{\pi}\right]$ which is a subset of $[-1, 1]$.

Hence, the function is one-one but not onto, hence not bijective.

$$f(x) = (\text{sgn}(x))\ln(e^x) = (\text{sgn}(x))x = \begin{cases} x, & x > 0 \\ -x, & x < 0 \\ 0, & x = 0 \end{cases}.$$

This function has the range $[0, 1]$ which is a subset of $[-1, 1]$. Hence, the function is into. Also, the function is many-one.

$$f(x) = x^3\,\text{sgn}(x) = \begin{cases} x^3, & x > 0 \\ -x^3, & x < 0 \\ 0, & x = 0 \end{cases}$$

which is many-one and into.

19. a, b, c, d.

Since $\angle PRQ = \pi/2$ and points P, Q, R lie on the circle with PQ as diameter.

Also $PQ = 5$

Now, the maximum area of the triangle is $\Delta_{max} = \dfrac{1}{2}(5)\left(\dfrac{5}{2}\right) = 6.25$

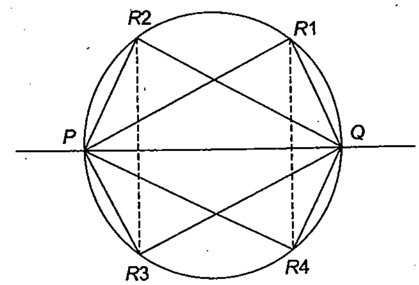


Fig. 1.109

For area = 5, we have four symmetrical positions of point R (shown as R_1, R_2, R_3, R_4)

For area = 6.25 we have exactly two points.

For area = 7, no such points exist.

20. a, b.

$(x+1)f(x) - x$ is a polynomial degree $n + 1$

$\Rightarrow \quad (x+1)f(x) - x = k(x)[x-1][x-2]\ldots[x-n]$ (i)

$\Rightarrow \quad [n+2]f(n+1) - (n+1) = k[(n+1)!]$

Also, $1 = k(-1)(-2)\ldots((-n-1))$ (Putting $x = -1$ in (i)]

$\Rightarrow \quad 1 = k(-1)^{n+1}(n+1)!$

$\Rightarrow \quad (n+2)f(n+1) - (n+1) = (-1)^{n+1}$

$\Rightarrow \quad f(n+1) = 1$, if n is odd and $\dfrac{n}{n+2}$, if n is even.

21. a, c, d.

$$f^2(x) = f\left(\frac{3}{4}x + 1\right) = \frac{3}{4}\left(\frac{3}{4}x + 1\right) + 1 = \left(\frac{3}{4}\right)^2 x + \frac{3}{4} + 1 \quad (1)$$

$$f^3(x) = f\{f^2(x)\} = \frac{3}{4}\{f^2(x) + 1\}$$

$$= \frac{3}{4}\left\{\left(\frac{3}{4}\right)^2 x + \frac{3}{4} + 1\right\} + 1$$

$$= \left(\frac{3}{4}\right)^3 x + \left(\frac{3}{4}\right)^2 + \frac{3}{4} + 1$$

$$\therefore\ f^n(x) = \left(\frac{3}{4}\right)^n x + \left(\frac{3}{4}\right)^{n-1} + \left(\frac{3}{4}\right)^{n-2} + \cdots + \left(\frac{3}{4}\right) + 1$$

$$= \left(\frac{3}{4}\right)^n x + \frac{1 - \left(\frac{3}{4}\right)^n}{1 - \frac{3}{4}}$$

$$\therefore\ \lambda = \lim_{n \to \infty} f^n(x) = 0 + 4 = 4$$

22.a, b, c.

$f(x)$ is defined if $\log_{|\sin x|}(x^2-8x+23)-\dfrac{3}{\log_2|\sin x|}>0$

$\Rightarrow \log_{|\sin x|}\left(\dfrac{x^2-8x+23}{8}\right)>0$

This is true if $|\sin x|\neq 0$, 1 and $\dfrac{x^2-8x+23}{8}<1$

Now, $\dfrac{x^2-8x+23}{8}<1 \Rightarrow x^2-8x+15<0$

$\Rightarrow x\in(3,5)-\left\{\pi, \dfrac{3\pi}{2}\right\}$

Domain $=(3,\pi)\cup\left(\pi,\dfrac{3}{2}\right)\cup\left(\dfrac{3\pi}{2},5\right)$.

23.a, b, c, d.

$f(x)=\text{sgn}(\cot^{-1}x)+\tan\left(\dfrac{\pi}{2}[x]\right)$

$\text{sgn}(\cot^{-1}x)$ is defined when $\cot^{-1}x$ is defined, which is for $\forall x\in R$.

$\tan\left(\dfrac{\pi}{2}[x]\right)$ is defined when $\dfrac{\pi}{2}[x]\neq\dfrac{(2n+1)}{2}\pi$, where $n\in Z$

$\Rightarrow [x]\neq 2n+1 \Rightarrow x\notin[2n+1, 2n+2)$

Hence domain of $f(x)$ is $\bigcup_{n\in Z}[2n, 2n+1)$

Also $\cot^{-1}x>0$, $\forall x\in R$,

Then $f(x)=1+\tan\left(\dfrac{\pi}{2}[x]\right)=1$

$\Rightarrow f(x)=1, x\in D_f$

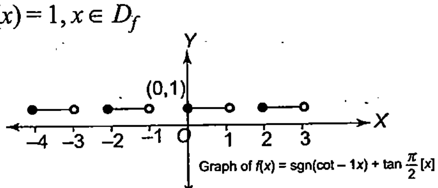


Fig. 1.110

From graph $f(x)$ is periodic with period 2.

Reasoning Type

1.b. A function which can be expressed as a sum of odd and even function need not to be odd or even.
But $f(x)=\log e^x$ is not defined for $x<0$, hence statement 2 is true but not correct explanation of statement 1.

2.a. $f(x)-1+f(1-x)-1=0$; so $g(x)+g(1-x)=0$

Replacing x by $x+\dfrac{1}{2}$, we get $g\left(\dfrac{1}{2}+x\right)+g\left(\dfrac{1}{2}-x\right)=0$.

So it is symmetrical about $\left(\dfrac{1}{2},0\right)$.

3.d. $f\left(\dfrac{2\tan x}{1+\tan^2 x}\right)=\dfrac{(1+\cos 2x)(\sec^2 x+2\tan x)}{2}$

$\Rightarrow f(\tan 2x)=\dfrac{2\cos^2 x(\sec^2 x+2\tan x)}{2}$

$=1+2\sin x\cos x=1+\sin 2x$

$\Rightarrow f(y)=1+y$ where $y=\sin 2x$, now $\sin 2x\in[-1,1]$

$\Rightarrow f(y)\in[0,2]$

Hence, statement 1 is false but statement 2 is true.

4. c. $\sin(kx)$ has period $\dfrac{\pi}{k}$ and period of $\{x\}$ is 1

Now LCM of $\dfrac{\pi}{k}$ and 1 exists only if k is a rational multiple of π (as LCM of rational and irrational number does not exist). Hence, statement 1 is true.
But statement 2 is false as sum of two periodic function is not necessarily periodic. Consider $f(x)=\sin x+\{x\}$.

5. c. Obviously, $f(x)=x^2+\tan^{-1}x$ is non-periodic, but sum of two non-periodic function is not always non-periodic, as $f(x)=x$ and $g(x)=-[x]$, where [.] represents the greatest integer function.
$f(x)+g(x)=x-[x]=\{x\}$ is a periodic function ({.} represents the fractional part function).

6. c. $f(x)=\tan^{-1}x$ is an increasing function, then the range of function is $[\tan^{-1}1, \tan^{-1}\sqrt{3}]\equiv[\pi/4, \pi/3]$.
Hence, statement 1 is true. But statement 2 is not true in general. For non-monotonic function, statement 2 is false.

7. a. For any integer k, we have $f(k)=f(2n\pi+k)$ where $n\in Z$, but $2n\pi+k$ is not integer, hence $f(x)$ is one-one.

8. a. Consider $f(x)=\tan x$, which is surjective, periodic but discontinuous.

9. b. $||x^2-5x+4|-|2x-3||=|x^2-3x+1|$

$\Rightarrow ||x^2-5x+4|-|2x-3||=|(x^2-5x+4)+(2x-3)|$

$\Rightarrow (x^2-5x+4)(2x-3)\leq 0$

$\Rightarrow (x-1)(2x-3)(x-4)\leq 0$

$\Rightarrow x\in(-\infty,1]\cup\left[\dfrac{3}{2},4\right]$

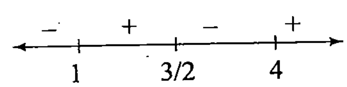


Fig. 1.111

Hence, statement 1 is true.
Statement 2 is true as it is the property of modulus function but is not a correct explanation of statement 1.

10. a. Let $\max|f(x)|=M$ where $0<M\leq 1$ (since f is not identically zero and $|f(x)|\leq 1\ \forall x\in R$)

Now, $f(x+y)+f(x-y)=2f(x).g(y)$

$\Rightarrow |2f(x).g(y)|=|f(x+y)+f(x-y)|$

$\Rightarrow 2|f(x)||g(y)|\leq|f(x+y)|+|f(x-y)|\leq M+M$

$\Rightarrow |g(y)|\leq 1$ for $y\in R$.

11. b. Obviously, both the statements are true but statement 2 is not a correct explanation of statement 1, as function $f(x)=\cos(2x+3)$ which is periodic though $g(x)=2x+3$ is non-periodic.

12. b. Obviously, both the statements are true but statement 2 is not a correct explanation of statement 1, as for $f(x)=\cos(\sin x)$ the period is π, where $\sin x$ has period 2π. Thus, the period of $f((g(x))$ is not always same as that of $g(x)$.

13. a. It is a fundamental concept.

14. b. Both the statements are true, but statement 2 is not a correct explanation of statement 1 as $f(g(x))$ is one-one when $g(x)$ is one-one and $f(x)$ is many-one.

15. b. Both the statements are true, but statement 2 is not a correct explanation of statement 1, as for $f(g(x))$ is onto it is necessary that $f(x)$ is onto, but there is no restriction on $g(x)$.

16. d. Statement 1 is false, though $f(x) = \sin x$ and $g(x) = \cos x$ have same domain and range, $\cos x = \sin x$ does not hold for all $x \in R$.
However, the statement 2 is true.

17. a.

18. d. If $b^2 - 4ax > 0$ then $ax^2 + bx + c = 0$ has real distinct roots α, β.

If $a > 0$, then for $f(x) = \sqrt{ax^2 + bx + c}$ to get defined, $ax^2 + bx + c \ge 0$, then the range of $f(x)$ is $[0, \infty)$ (as $b^2 - 4ac > 0$)

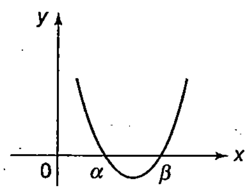


Fig. 1.112

If $a < 0$, then for $f(x)$ to get defined, $ax^2 + bx + c \ge 0$, then the range of $f(x)$ is $\left[0, -\frac{b}{2a}\right]$. (as $b^2 - 4ac > 0$)

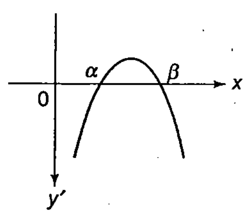


Fig. 1.113

Hence, statement 1 is false, but statement 2 is true.

19 .c. $fog(x)$ can be even also when one of them is even and other is odd.

20. a. Obviously, the graph of $y = \tan x$ is symmetrical about origin, as it is an odd function.
Also derivative of an odd function is an even function, and $\sec^2 x$ is derivative of $\tan x$, hence both the statements are true, and statement 2 is a correct explanation of statement 1.

Linked Comprehension Type

For Problems 1–3

1.c, 2.c, 3.b

Sol. $$f(x) = \begin{cases} x+1, & x \le 1 \\ 2x+1, & 1 < x \le 2 \end{cases}$$

$$g(x) = \begin{cases} x^2, & -1 \le x < 2 \\ x+2, & 2 \le x \le 3 \end{cases}$$

$$\Rightarrow \quad f(x) = \begin{cases} g(x)+1, & g(x) \le 1 \\ 2g(x)+1, & 1 < g(x) \le 2 \end{cases}$$

$$\Rightarrow \quad f(g(x)) = \begin{cases} x^2+1, & x^2 \le 1, \ -1 \le x < 2 \\ x+2+1, & x+2 \le 1, \ 2 \le x \le 3 \\ 2x^2+1, & 1 < x^2 \le 2, \ -1 \le x < 2 \\ 2(x+2)+1, & 1 < x+2 \le 2, \ 2 \le x \le 3 \end{cases}$$

$$\Rightarrow \quad f(g(x)) = \begin{cases} x^2+1, & -1 \le x \le 1 \\ 2x^2+1, & 1 < x \le \sqrt{2} \end{cases}$$

1. c. Hence, the domain of $f(x)$ is $[-1, \sqrt{2}]$.

2. c. For $-1 \le x \le 1$, we have $x^2 \in [0, 1] \Rightarrow x^2 + 1 \in [1, 2]$
For $1 < x \le \sqrt{2}$, we have $x^2 \in (1, 2] \Rightarrow 2x^2 + 1 \in (3, 5]$
Hence, the range is $[1, 2] \cup (3, 5]$.

3. b. For $f(g(x)) = 2 \Rightarrow x^2 + 1 = 2$ and $2x^2 + 1 = 2 \Rightarrow x = \pm 1$ or $x = \pm \frac{1}{\sqrt{2}}$

$\Rightarrow \quad x = \pm 1$ only. Hence, 2 roots

For Problems 4–6

4.b, 5.c, 6.d

Sol. $$f(x) + f\left(\frac{x-1}{x}\right) = 1 + x \qquad (1)$$

In (1) replace x by $\frac{x-1}{x}$, we have $f\left(\frac{x-1}{x}\right) + f\left(\frac{\frac{x-1}{x} - 1}{\frac{x-1}{x}}\right)$

$= 1 + \frac{x-1}{x}$

$$\Rightarrow \quad f\left(\frac{x-1}{x}\right) + f\left(\frac{1}{1-x}\right) = 1 + \frac{x-1}{x} \qquad (2)$$

Now from (1) – (2), we have $f(x) - f\left(\frac{1}{1-x}\right) = x - \frac{x-1}{x}$

(3)

In (3) replace x by $\frac{1}{1-x}$, we have $f\left(\frac{1}{1-x}\right) - f\left(\frac{x-1}{x}\right)$

$$= \frac{1}{1-x} - \frac{\frac{1}{1-x} - 1}{\frac{1}{1-x}}$$

$$\text{or } f\left(\frac{1}{1-x}\right) - f\left(\frac{x-1}{x}\right) = \frac{1}{1-x} - x \qquad (4)$$

Now from (1) + (3) + (4), we have $2f(x) = 1 + x + x - \frac{x-1}{x}$

$+ \frac{1}{1-x} - x$

$\Rightarrow f(x) = \frac{x^3 - x^2 - 1}{2x(x-1)}$

$f(x) = \frac{x^3 - x^2 - 1}{2x(x-1)}$

4. b. $\Rightarrow \quad g(x) = \frac{x^3 - x^2 - 1}{x(x-1)} - x + 1$

$= \frac{x^2 - x - 1}{x(x-1)}$

Now for $y = \sqrt{g(x)}$, we must have $\frac{x^2 - x - 1}{x(x-1)} \geq 0$ or

$$\frac{\left(x - \frac{1-\sqrt{5}}{2}\right)\left(x - \frac{1+\sqrt{5}}{2}\right)}{x(x-1)} \geq 0$$

$\Rightarrow x \in \left(-\infty, \frac{1-\sqrt{5}}{2}\right] \cup (0, 1) \cup \left[\frac{1+\sqrt{5}}{2}, \infty\right)$

5.c. $y = g(x) = \frac{x^2 - x - 1}{x(x-1)} \Rightarrow (y-1)x^2 + (1-y)x + 1 = 0$

Now x is real, $\Rightarrow D \geq 0 \Rightarrow (1-y)^2 - 4(y-1) \geq 0$

$\Rightarrow (y-1)(y-5) \geq 0$

$\Rightarrow y \in (-\infty, 1] \cup [5, \infty)$

6.d. $g(x) = 1 \Rightarrow \frac{x^2 - x - 1}{x(x-1)} = 1 \Rightarrow -x - 1 = -x$, which has no solutions.

For Problems 7–9

7.d, 8.c, 9.c

Sol.

Here,

$f(1) + 2f(2) + 3f(3) + \cdots + nf(n) = n(n+1) f(n)$, for $n \geq 2$ (1)

Replacing n by $n + 1$, we get

$f(1) + 2f(2) + 3f(3) + \cdots + (n+1)f(n+1)$

$= (n+1)(n+2)f(n+1)$ (2)

From (2) – (1), we get

$(n+1)f(n+1) = (n+1)\{(n+2)f(n+1) - nf(n)\}$

$\Rightarrow f(n+1) = (n+2)f(n+1) - nf(n)$

$\Rightarrow nf(n) = (n+2)f(n+1) - f(n+1)$

$\Rightarrow nf(n) = (n+1)f(n+1)$

Putting $n = 2, 3, 4, \ldots$, we get

$2f(2) = 3f(3) = 4f(4) = \ldots = nf(n)$

From (1), $f(1) + 2f(2) + 3f(3) + \cdots + nf(n) = n(n+1)f(n)$

$\Rightarrow f(1) + (n-1) \cdot nf(n) = n(n+1)f(n)$

$\Rightarrow f(1) = 2nf(n)$

$\Rightarrow f(n) = \frac{f(1)}{2n} = \frac{1}{2n}$

7. d. $f(1003) = \frac{1}{2(1003)} = \frac{1}{2006}$

8. c. $f(999) = \frac{1}{2(999)} = \frac{1}{1998}$

9. c. $f(1), f(2), f(3), \ldots$ are in H.P.

$\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots$ are in H.P.

For Problems 10–11

10. a, 11.b, 12.c

Sol. $(f(x))^2 f\left(\frac{1-x}{1+x}\right) = 64x$ (1)

Putting $\frac{1-x}{1+x} = y$, or $x = \frac{1-y}{1+y}$, we get

$\left\{f\left(\frac{1-y}{1+y}\right)\right\}^2 . f(y) = 64\left(\frac{1-y}{1+y}\right)$

$\Rightarrow f(x) . \left\{f\left(\frac{1-x}{1+x}\right)\right\}^2 = 64\left(\frac{1-x}{1+x}\right)$ (2)

From $(1)^2/(2)$, we get

$$\frac{f(x)^4 \left\{f\left(\frac{1-x}{1+x}\right)\right\}^2}{f(x)\left\{f\left(\frac{1-x}{1+x}\right)\right\}^2} = \frac{(64x)^2}{64\left(\frac{1-x}{1+x}\right)}$$

$\Rightarrow \{f(x)\}^3 = 64x^2\left(\frac{1+x}{1-x}\right)$

$\Rightarrow f(x) = 4x^{2/3}\left(\frac{1+x}{1-x}\right)^{1/3}$

$x = f(9/7) = -4(9/7)^{2/3}(2)$

For Problems 13–15

13.d, 14.c, 15.c

Sol. $|g(x)| = |\sin x|, x \in R$

$$f(|g(x)|) = \begin{cases} |\sin x| - 1, & -1 \leq |\sin x| < 0 \\ (|\sin x|)^2, & 0 \leq (|\sin x|) \leq 1 \end{cases} = \sin^2 x, x \in R$$

$$f(g(x)) = \begin{cases} \sin x - 1, & -1 \leq \sin x < 0 \\ \sin^2 x, & 0 \leq \sin x \leq 1 \end{cases}$$

$$= \begin{cases} \sin x - 1, & (2n+1)\pi < x < 2n\pi \\ \sin^2 x, & 2n\pi \leq x \leq (2n+1)\pi \end{cases}, n \in Z.$$

$$\Rightarrow \quad |f(g(x))| = \begin{cases} 1-\sin x, & (2n+1)\pi < x < 2n\pi \\ \sin^2 x, & 2n\pi \le x \le (2n+1)\pi \end{cases}, n \in Z$$

13. d. Clearly $h_1(x) = f(|g(x)|) = \sin^2 x$ has period π, range [0, 1] and domain R.

14. c. $h_2(x) = |f(g(x))|$ has domain R.

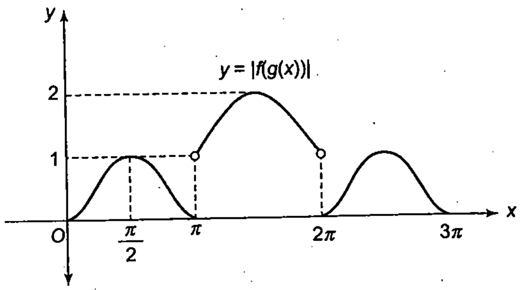


Fig. 1.114

Also from the graph, it is periodic with period 2π and has range [0, 2].

15. c. For $h_1(x) \equiv h_2(x) = \sin^2 x$, $x \in [2n\pi, (2n+1)\pi]$, $n \in Z$ and has range [0, 1] for the common domain.

Also, the period is 2π (from the graph).

For Problems 16–18

16.d, 17.d, 18.c

Sol. Given $a_{n+1} = f(a_n)$

Now $a_1 = f(a_0) = f(x)$

$\Rightarrow \quad a_2 = f(a_1) = f(f(a_0)) = fof(x)$

$$\Rightarrow a_n = \underbrace{fofofof \ldots f(x)}_{n \text{ times}}$$

16. d. $a_1 = f(x) = (a - x^m)^{1/m}$

$\Rightarrow \quad a_2 = f(f(x)) = [a - \{(a - x^m)^{1/m}\}^m]^{1/m} = x$

$\Rightarrow \quad a_3 = f(f(f(x))) = f(x)$

Obviously, the inverse does not exist when m is even and n is odd.

17. d.

$$\text{Now if } f(x) = \frac{1}{1-x} \Rightarrow fof(x) = \frac{1}{1-\dfrac{1}{1-x}} = \frac{x-1}{x}$$

$$\Rightarrow fofof(x) = \frac{\dfrac{1}{1-x} - 1}{\dfrac{1}{1-x}} = x$$

$$\Rightarrow \quad a_n = \underbrace{fofof \ldots of(x)}_{n \text{ times}} = \frac{1}{1-x} \text{ if } n = 3k+1$$

$$= \frac{x-1}{x} \text{ if } n = 3k+2$$

$= x$ if $n = 3k$

18. c. Since $a_1 = g(x) = 3 + 4x$

$\therefore a_2 = g\{g(x)\} = g(3+4x) = 3 + 4(3+4x) = (4^2 - 1) + 4^2 x$

$a_3 = g\{g^2(x)\} = g(15 + 4^2 x) = 3 + 4(15 + 4^2 x) = 63 + 4^3 x = (4^3 - 1) + 4^3 x$

Similarly, we get $a_n = (4^n - 1) + 4^n x$

$\Rightarrow \quad A = 4^n - 1$ and $B = 4^n$

$\Rightarrow \quad A + B + 1 = 2^{2n+1}$, $|A - B| = 1$ and $\lim_{n\to\infty} \frac{4^n - 1}{4^n}$

$$= \lim_{n\to\infty}\left(1 - \frac{1}{4^n}\right) = 1$$

For Problems 19–21

19.a, 20.b, 21.a

Sol. 19. a $\quad f_1(x) = x^2$ and $f_2(x) = |x|$

$\Rightarrow f(x) = f_1(x) - 2f_2(x) = x^2 - 2|x|$

Graph of $f(x)$

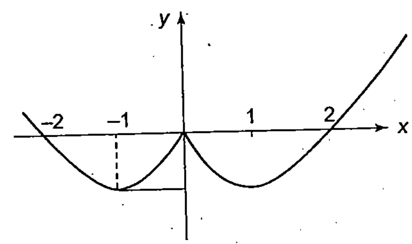


Fig. 1.115

$$g(x) = \begin{cases} f(x), & -3 \le x < -1 \\ -1, & -1 \le x < 0 \\ 0, & 0 \le x \le 2 \\ f(x), & 2 < x \le 3 \end{cases}$$

$$= \begin{cases} x^2 + 2x, & -3 \le x < -1 \\ -1, & -1 \le x < 0 \\ 0, & 0 \le x \le 2 \\ x^2 - 2x, & 2 < x \le 3 \end{cases}$$

The range of $g(x)$ for $[-3, -1]$ is $[-1, 3]$.

20. b. For $x \in (-1, 0)$, $f(x) + g(x) = x^2 + 2x - 1$.

21. a. Obviously, the graph is broken at $x = 0$.

For Problems 22–24

22.a, 23.d, 24.c

Sol. 22. a.

$g(f(x))$ is not defined if

(i) $-2 + a > 8$ and (ii) $b + 3 > 8$

$a > 10$ and $b > 5$

23. d.

$x \in [-1, 2]$

$\Rightarrow \quad -1 \le x \le 2$

$\Rightarrow \quad -2 \le 2x \le 4$

$\Rightarrow \quad -2 + a \le 2x + a \le 4 + a$

$\Rightarrow \quad -2 + a \le -2$ and $4 + a \le 4$, i.e., $a = 0$

b can take any value.

24. c.

If $a=2, b=3$

$$f(x)=\begin{cases}2x+2 & : x\ge -1\\ 3x^2+3 & : x<-1\end{cases}$$

The range of $f(x)$ is $[0, \infty)$.

For Problems 25–27

25.c, 26.c, 27.c

Sol. $f(2-x)=f(2+x)$ (1)

Replace x by $2-x$, $\Rightarrow f(x)=f(4-x)$ (2)

Also given $f(20-x)=f(x)$ (3)

From (1) and (2), $f(4-x)=f(20-x)$

Replace x by $4-x$, $\Rightarrow f(x)=f(x+16)$

Hence, the period of $f(x)$ is 16.

25. c. Given $f(0)=5$.

26. c. $f(2-x)=f(2+x)$

$\Rightarrow y=f(x)$ is symmetrical about $x=2$

Also $f(20-x)=f(x)$

$\Rightarrow f(20-(10+x))=f(10+x)$

$\Rightarrow f(10-x)=f(10+x)$

$\Rightarrow y=f(x)$ is symmetrical about $x=10$

27. c. If 1 is a period, then $f(x)=f(x+1)$, $\forall\ x\in R$

$\Rightarrow f(2)=f(3)=f(4)=f(5)=f(6)$

which contradicts the given hypotheses that $f(2)\,{}^1 f(6)$

$\therefore$ 1 cannot be period of $f(x)$.

For Problems 28–30

28. c, 29. c, 30. b

Sol. $$g(f(x))=\begin{cases}[f(x)] & -\pi\le f(x)<0\\ \sin f(x), & 0\le f(x)\le\pi\end{cases}$$

$$=\begin{cases}[[x]], & -\pi\le[x]<0, \quad -2\le x\le -1\\ [|x|+1], & -\pi\le|x|+1<0, \ -1<x\le 2\\ \sin[x], & 0\le[x]\le\pi, \quad -2\le x\le -1\\ \sin(|x|+1), & 0\le|x|+1\le\pi, \ -1<x\le 2\end{cases}$$

$$=\begin{cases}[x], & -2\le x\le -1\\ \sin(|x|+1), & -1<x\le 2\end{cases}$$

Hence, the domain is $[-2, 2]$.

Also for $-2\le x\le -1$, $[x]=-2,-1$

and for $-1<x\le 2$, $|x|+1\in[1,3]$

$\Rightarrow \sin(|x|+1)\in[\sin 3, 1]$

Hence, the range is $\{-2,-1\}\cup[\sin 3, 1]$

Also for $y\in[\sin 3, 1]$, $[y]=0, 1$

Hence, the number of integral points in the range is 4.

Matrix-Match Type

1. a $\to$ s; b $\to$ r; c $\to$ p; d $\to$ q.

$f(\tan x)$ is defined if $0\le\tan x\le 1$

$$\Rightarrow\ x\in\left[n\pi,\ n\pi+\frac{\pi}{4}\right], n\in I$$

$f(\sin x)$ if defined if $0\le\sin x\le 1$

$$\Rightarrow\ x\in[2n\pi,\ (2n+1)\pi], n\in I$$

$f(\cos x)$ is defined if $0\le\cos x\le 1$

$$\Rightarrow\ x\in\left[2n\pi-\frac{\pi}{2},\ 2n\pi+\frac{\pi}{2}\right], n\in I$$

$f(2\sin x)$ is defined if $0\le 2\sin x\le 1\Rightarrow 0\le\sin x\le 1/2$

$$\Rightarrow\ \left[2n\pi,\ 2n\pi+\frac{\pi}{6}\right]\cup\left[2n\pi+\frac{5\pi}{6},\ (2n+1)\pi\right], n\in I.$$

2. a $\to$ p; b $\to$ q; c $\to$ q, s; d $\to$ p, r

a. $$f(x)=\{(\operatorname{sgn} x)^{\operatorname{sgn} x}\}^n=\begin{cases}\left[(1)^1\right]^n, & x>0\\ \left[(-1)^{-1}\right]^n, & x<0\end{cases}$$

$$=\begin{cases}1, & x>0\\ -1, & x<0\end{cases}$$

Hence, $f(x)$ is an odd function.

b. $$f(x)=\frac{x}{e^x-1}+\frac{x}{2}+1$$

$$\Rightarrow\ f(-x)=\frac{-x}{e^{-x}-1}-\frac{x}{2}+1=\frac{xe^x}{e^x-1}-\frac{x}{2}+1$$

$$=\frac{xe^x-x+x}{e^x-1}-\frac{x}{2}+1$$

$$=x+\frac{x}{e^x-1}-\frac{x}{2}+1=\frac{x}{e^x-1}+\frac{x}{2}+1$$

$$=f(x)$$

c. $$f(x)=\begin{cases}0, & \text{If } x \text{ is rational}\\ 1, & \text{If } x \text{ is irrational}\end{cases}$$

$$\Rightarrow\ f(-x)=\begin{cases}0, & \text{If } -x \text{ is rational}\\ 1, & \text{If } -x \text{ is irrational}\end{cases}$$

$$=\begin{cases}0, & \text{If } x \text{ is rational}\\ 1, & \text{If } x \text{ is irrational}\end{cases}=f(x)$$

d. $f(x)=\max\{\tan x, \cot x\}$

From the graph of function it can be verified that $f(x)$ is neither odd nor even

Hence, $f(x)$ is an odd function.

Also $f(x+\pi)=\max\{\tan(x+\pi), \cot(x+\pi)\}$

$=\max\{\tan x, \cot x\}$

Hence, $f(x)$ is periodic with period π.

3. a $\to$ r, s; b $\to$ p, q, r, s; c $\to$ s. d $\to$ p.

a. $$\tan^{-1}\left(\frac{2x}{1-x^2}\right)\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$$

$$\Rightarrow 2\tan^{-1}x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$$

$\Rightarrow \tan^{-1} x \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \Rightarrow \tan^{-1} x \in (-1, 1)$.

b. $f(x) = \sin^{-1}(\sin x)$ and $g(x) = \sin(\sin^{-1} x)$
$f(x)$ is defined if $\sin x \in [-1, 1]$ which is true for all $x \in R$.
But $g(x)$ is defined for only $x \in [-1, 1]$
Hence, $f(x)$ and $g(x)$ are identical if $x \in [-1, 1]$.

c. $f(x) = \log_{x^2} 25$ and $g(x) = \log_x 5$
$f(x)$ is defined for $\forall\, x \in R - \{0, 1\}$ and $g(x)$ is defined for $(0, \infty) - \{1\}$.
Hence, $f(x)$ and $g(x)$ are identical if $x \in (0, 1) \cup (1, \infty)$.

d. $f(x) = \sec^{-1} x + \text{cosec}^{-1} x$, $g(x) = \sin^{-1} x + \cos^{-1} x$
$f(x)$ has domain $R - (-1, 1)$ and $g(x)$ has domain $[-1, 1]$
Hence, both the functions are identical only if $x = -1, 1$.

4. a → r, s; b → r, s; c → p, q; d → p, s.

a. $f(x) = \cot^{-1}(2x - x^2 - 2)$
$= \cot^{-1}(-1 - (x-1)^2)$
$-1 - (x-1)^2 \le -1$
$\Rightarrow f(0) = f(2)$. Hence $f(x)$ is many-one.
$\Rightarrow \cot^{-1}(2x - x^2 - 2) \in \left[\frac{3\pi}{4}, \pi\right)$
Hence, $f(x)$ is onto.
Also $f(3) = f(-1)$, hence function is many-one.
$-1 - (x-1)^2 = -5$.

b.

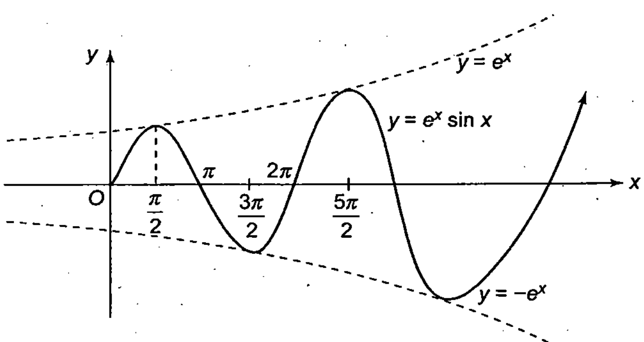


Fig. 1.116

Clearly, from the graph that $f(x)$ is many-one and onto.

c.

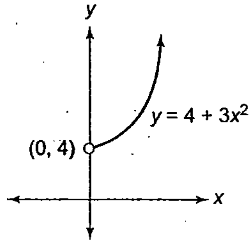


Fig. 1.117

d. Let $X = \{x_1, x_2, \ldots, x_n\}$
Let $f(x_1) = x_2$
$\Rightarrow f(f(x_1)) = f(x_2) \Rightarrow x_1$
Thus $f(x)$ is one-one and onto.

5. a → p; b → q, r; c → p; d → q, r.
Since $f(g(x))$ is a one-one function
$\Rightarrow f(g(x_1)) \ne f(g(x_2))$ whenever $g(x_1) = g(x_2)$
$\Rightarrow (g(x_1)) \ne (g(x_2))$ whenever $x_1 \ne x_2$
$\Rightarrow g(x)$ is one-one.
If $f(x)$ is not one-one, then $f(x) = y$ is satisfied by $x = x_1, x_2$
$\Rightarrow f(x_1) = f(x_2) = y$ also if $g(x)$ is onto, then
let $g(x_1) = x_1$ and $g(x_2) = x_2$
$\Rightarrow f(g(x_1)) = f(g(x_2))$
$\Rightarrow f(g(x))$ cannot be one-one.

6. a → q; b → q; c → s; d → p.

a. $f(x + \pi/2) = \cos(|\sin(x + \pi/2)| - |\cos(x + \pi/2)|)$
$= \cos(|\cos x| - |-\sin x|)$
$= \cos(|\cos x| - |\sin x|)$
$= \cos(|\sin x| - |\cos x|)$
$= f(x)$

b. $f(x + \pi/2) = \cos[\tan(x + \pi/2) + \cot(x + \pi/2)] \,.$
$\cos[\tan(x + \pi/2) - \cot(x + \pi/2)]$
$= \cos[-\cot x - \tan x] \,.\, \cos[-\cot x + \tan x]$
$= \cos(\tan x + \cot x) \,.\, \cos(\tan x - \cot x)$
$= f(x)$

c. The period of $\sin^{-1}(\sin x)$ is 2π. The period of $e^{\tan x}$ is π.
Thus, the period of $f(x) = \text{LCM}(2\pi, \pi) = 2\pi$

d. The given function is $f(x) = \sin^3 x \sin 3x$
$\Rightarrow f(x) = \left(\frac{3\sin x - \sin 3x}{4}\right)\sin 3x$
$\Rightarrow f(x) = \frac{3}{8}(\cos 2x - \cos 4x) - \frac{1}{8}(1 - \cos 6x)$
$\Rightarrow$ The period of $f(x)$ is π.

7. a → s; b → r; c → s; d → p.

a. $f(x) = e^{\cos^4 \pi x + x - [x] + \cos^2 \pi x}$
$\cos^2 \pi x$ and $\cos^4 \pi x$ has period 1
$x - [x] = \{x\}$ has period 1
Then the period of $f(x)$ is 1.

b. $f(x) = \cos 2\pi\{2x\} + \sin 2\pi\{2x\}$
The period $\{2x\}$ is 1/2, then the period of $f(x)$ is 1/2.

c. Clearly, $\tan \pi[x] = 0\ \forall\ x \in R$ and the period of $\sin 3\pi\{x\}$ is equal to 1.

d. $f(x) = 3x - [3x + a] - b = 3x + a - [3x + a] - (a + b)$
$= \{3x + a\} - (a + b)$
Thus, the period of $f(x)$ is 1.

8. a → r; b → s; c → q; d → p.

a. $f(x) = \log_3(5 + 4x - x^2)$
$= \log_3(9 - (x-2)^2)$

Now $-\infty < 9-(x-2)^2 \le 9$

But for $f(x)$ to get defined, $0 < 9-(x-2)^2 \le 9$

$\Rightarrow -\infty < \log_3(9-(x-2)^2) \le \log_3 9$

$\Rightarrow -\infty < \log_3(9-(x-2)^2) \le 2$

Hence, the range is $(-\infty, 2]$.

b. $f(x) = \log_3(x^2-4x-5)$
$= \log((x-2)^2-9)$

For $f(x)$ to get defined, $0 < (x-2)^2 - 9 < \infty$

$\Rightarrow \lim_{x\to 0} \log x < \log_e(x-2)^2 - 9 < \lim_{x\to\infty} \log x$

$\Rightarrow -\infty < f(x) < \infty$

Hence, the range is R.

c. $f(x) = \log_3(x^2-4x+5)$
$= \log_3((x-2)^2+1)$

$(x-2)^2 + 1 \in [1, \infty)$

$\Rightarrow \log_3(x^2-4x+5) \in [0, \infty)$.

d. $(x) = \log_3(4x-5-x^2)$
$= \log_3(-5-(x^2-4x))$
$= \log_3(-1-(x-2)^2)$

Now, $-1-(x-2)^2 < 0$ for all x

Hence, the function is not defined.

9. a → q. b → s. c → p. d → s.

p. $y = \tan x = \dfrac{1}{x^2}$

From the graph, it is clear that it will have two real roots

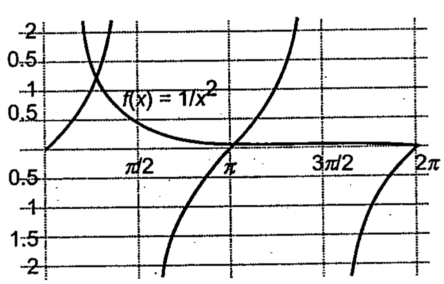


Fig. 1.118

q. See the graphs of $y = 2^{\cos x}$ and $y = |\sin x|$. Two curves meet at four points for $x \in [0, 2\pi]$.

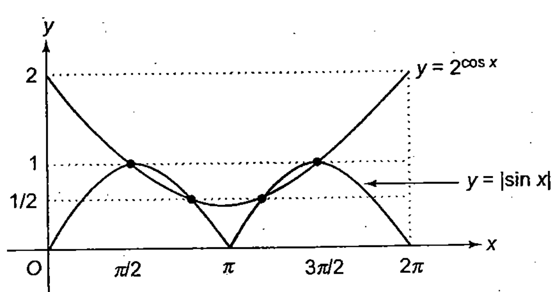


Fig. 1.119

So, the equation $2^{\cos x} = |\sin x|$ has four solutions.

r. Given that $f(|x|) = 0$ has 8 real roots $\Rightarrow f(x) = 0$ has 4 positive roots.

Since $f(x)$ is a polynomial of degree 5, $f(x)$ cannot have even number of real roots.

$\Rightarrow f(x)$ has all the five roots real and one root is negative.

s. $7^{|x|}(|5-|x||) = 1$.

$\Rightarrow |5-|x|| = 7^{-|x|}$

Draw the graph of $y = 7^{-|x|}$ and $y = |5-|x||$

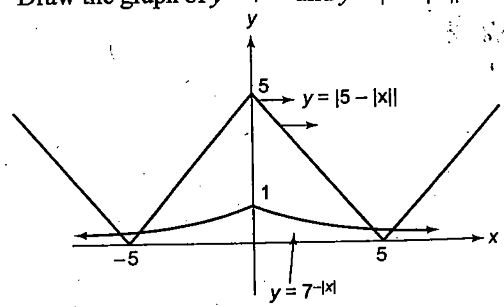


Fig. 1.120

From the graph, the number of roots is 4.

Integer Type

1.(3) We have $f\left(\dfrac{2x-3}{x-2}\right) = 5x-2 \Rightarrow f^{-1}(5x-2) = \dfrac{2x-3}{x-2}$

Let $5x-2 = 13$, then $x = 3$

Hence $f^{-1}(13) = \dfrac{2(3)-3}{3-2} = 3$

2.(1) $\left|\left||x^2-x+4| - 2\right| - 3\right| = x^2+x-12$

$\Rightarrow \left|\left|x^2-x+2\right| - 3\right| = x^2+x-12$

$\Rightarrow \left|x^2-x-1\right| = x^2+x-12$

$\Rightarrow 2x = 11$

$\Rightarrow x = 11/2$

3.(5) $f(x)$ and $f^{-1}(x)$ can only intersect on the line $y = x$ and therefore $y = x$ must be tangent at the common point of tangency

$\therefore 3x^2 - 7x + c = x$

$\Rightarrow 3x^2 - 8x + c = 0$ ...(1)

This equation must have equal roots

$\Rightarrow 64 - 12c = 0$

$\Rightarrow c = \dfrac{64}{12} = \dfrac{16}{3}$

4.(5) $x! - (x-1)! \ne 0 \Rightarrow x \in I^+ - \{1\}$

$2^{\frac{\pi}{\tan^{-1} x}} > 4$ as $\tan^{-1} x < \dfrac{\pi}{2}$

$\Rightarrow \dfrac{(x-4)(x-10)}{(x-1)!(x-1)} < 0$

$\Rightarrow x \in \{5, 6, \dots, 9\}$

5.(4) Put $x=1$ and $y=1$,

$f^2(1)-f(1)-6=0$

$\Rightarrow f(1)=3$ or $f(1)=-2$

Now put $y=1$

$\Rightarrow f(x)\cdot f(1)=f(x)+2\left(\frac{1}{x}+2\right)=f(x)+2\left(\frac{2x+1}{x}\right)$

$\Rightarrow f(x)[f(1)-1]=\frac{2(2x+1)}{2}$

$\Rightarrow f(x)=\frac{2(2x+1)}{x[f(1)-1]}$

For $f(1)=3, f(x)=\frac{2x+1}{x}$ (1)

and for $x=-2, f(x)=\frac{2(2x+1)}{-3x}$ (2)

$\Rightarrow f(1/2)=4$

6.(7) Let $2x+y=3x-y \Rightarrow 2y=x \Rightarrow y=\frac{x}{2}$

$\therefore$ Put $y=\frac{x}{2}$

$\Rightarrow f(x)+f\left(\frac{5x}{2}\right)+\frac{5x^2}{2}=f\left(\frac{5x}{2}\right)+2x^2+1$

$\Rightarrow f(x)=1-\frac{x^2}{2}$

$\Rightarrow f(4)=-7$

7.(5) As $a>2$, hence

$a^2>2a>a>2$

now $(x-a)(x-2a)(x-a^2)<0$

$\Rightarrow$ the solution set is as shown

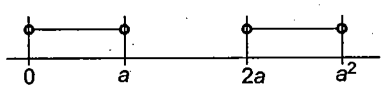


Fig. 1.121

between $(0, a)$ there are $(a-1)$ positive integers and between $(2a, a^2)$ there are a^2-2a-1 integer

$\therefore a^2-2a-1+a-1=18 \Rightarrow a^2-a-20=0$

$(a-5)(a+4)=0$

$\therefore a=5$

8.(2) $f(x)+f\left(\frac{1}{x}\right)=x^2+\frac{1}{x}$

replacing $x\to\frac{1}{x}$; $f\left(\frac{1}{x}\right)+f(x)=\frac{1}{x^2}+x$

$\Rightarrow \frac{1}{x^2}+x=x^2+\frac{1}{x}$

$\Rightarrow x-\frac{1}{x}=x^2-\frac{1}{x^2}$

$\Rightarrow \left(x-\frac{1}{x}\right)=\left(x-\frac{1}{x}\right)\left(x+\frac{1}{x}\right)$

$\Rightarrow \left(x-\frac{1}{x}\right)\left(x+\frac{1}{x}-1\right)=0$

$x=\frac{1}{x}$; $x+\frac{1}{x}=1$ (rejected)

Hence $x=1$ or -1

9.(7) Obviously f is a linear polynomial

Let $f(x)=ax+b$ hence $f(x^2+x+3)+2f(x^2-3x+5)\equiv 6x^2-10x+17$

$\Rightarrow [a(x^2+x+3)+b]+2[a(x^2-3x+5)+b]\equiv 6x^2-10x+17$

$\Rightarrow a+2a=6$ (1)

$\Rightarrow a-6a=-10$ (2)

(comparing coeff. of x^2 and coeff. of x on both sides)

$a\Rightarrow 2$

Again, $3a+b+10a+2b=17$ (Comparing constant term)

$\Rightarrow 6+b+20+2b=17$

$\therefore f(x)=2x-3$

$\Rightarrow f(5)=7$

10.(9) Given $f(x+2)=f(x)+f(2)$

Put $x=-1$, we have $f(1)=f(-1)+f(2)$

$\Rightarrow f(1)=-f(1)+f(2)$ (as $f(x)$ is an odd function)

$\Rightarrow f(2)=2f(1)=6$

Now put $x=1$,

We have $f(3)=f(1)+f(2)=3+6=9$

11.(3) $f(x)+f(-x)=0$

$\Rightarrow f(x)$ is an odd function.

Since points $(-3, 2)$ and $(5, 4)$ lie on the curve, therefore $(3, -2)$ and $(-5, -4)$ will also lie on the curve.

For minimum number of roots, graph of continuous function $f(x)$ is as follows:

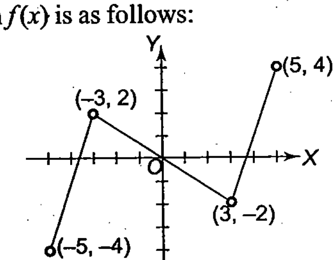


Fig. 1.122

From the above graph of $f(x)$, it is clear that equation $f(x)=0$ has at least three real roots.

12.(3) $f(x)=\sqrt{\sin x+\cos x}+\sqrt{7x-x^2-6}$

$=\sqrt{\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)}+\sqrt{(x-6)(1-x)}$

Now $f(x)$ is defined if $\sin\left(x+\frac{\pi}{4}\right)\geq 0$ and $(x-6)(1-x)\geq 0$

$\Rightarrow 0\leq x+\frac{\pi}{4}\leq\pi$ or $2\pi\leq x+\frac{\pi}{4}\leq 3\pi$ and $1\leq x\leq 6$

$\Rightarrow -\frac{\pi}{4}\leq x\leq\frac{3\pi}{4}$ or $\frac{7\pi}{4}\leq x\leq\frac{11\pi}{4}$ and $1\leq x\leq 6$

$\Rightarrow x\in\left[1,\frac{3\pi}{4}\right]\cup\left[\frac{7\pi}{4},6\right]$

Integral values of x are $x=1, 2$ and 6

13.(8) Since f is periodic with period 2 and $f(x)=x \ \forall \ x\in[0, 1]$

also $f(x)$ is even

$\Rightarrow$ symmetry about y-axis

$\therefore$ graph of $f(x)$ is as shown

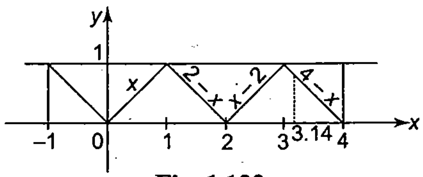


Fig. 1.123

$f(3.14) = 4 - 3.14 = 0.86$

14.(6) Let $x^2 = 4\cos^2\theta + \sin^2\theta$

then $(4 - x^2) = 3\sin^2\theta$ and $(x^2 - 1) = 3\cos^2\theta$

$\therefore\ f(x) = \sqrt{3}|\sin\theta| + \sqrt{3}|\cos\theta|$

$\Rightarrow\ y_{\min} = \sqrt{3}$ and

$y_{\max} = \sqrt{3}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = \sqrt{6}$

Hence range of $f(x)$ is $\left[\sqrt{3}, \sqrt{6}\right]$

Hence maximum value of $(f(x))^2$ is 6

15.(0) $g(x) = \dfrac{f(x) + f(-x)}{2}$

$= \frac{1}{2}\left[\frac{x+1}{x^3+1} + \frac{1-x}{1-x^3}\right]$

$= \frac{1}{2}\left[\frac{1}{x^2-x+1} + \frac{1}{1+x+x^2}\right]$

$= \frac{1}{2}\left[\frac{2(x^2+1)}{(x^2+1)^2 - x^2}\right]$

$= \frac{x^2+1}{x^4+x^2+1}$

$= \frac{x^4-1}{x^6+1} \Rightarrow g(0) = 1$

16.(4) $f(x) = [8x + 7] + |\tan 2\pi x + \cot 2\pi x| - 8x$

$= [8x] - 8x - 7 + |\tan 2\pi x + \cot 2\pi x|$

$= -\{8x\} + |\tan 2\pi x + \cot 2\pi x| + 7$

Period of $\{8x\}$ is 1/8

Also, $|\tan 2\pi x + \cot 2\pi x|$

$= \left|\frac{\sin 2\pi x}{\cos 2\pi x} + \frac{\cos 2\pi x}{\sin 2\pi x}\right| = \left|\frac{1}{\sin 2\pi x \cos 2\pi x}\right| = |2\operatorname{cosec} 4\pi x|$

Now period of $2\operatorname{cosec} 4\pi x$ is 1/2, then period of $|2\operatorname{cosec} 4\pi x|$ is 1/4

$\therefore$ Period is L.C.M. of $\frac{1}{8}$ and $\frac{1}{4}$ which is $\frac{1}{4}$

17.(0) Let $x = \frac{|a|}{a} + \frac{|b|}{b} + \frac{|c|}{c}$

If exactly one –ve, then $x = 1$

Exactly two –ve, then $x = -1$

All three – ve, then $x = -3$

All three +ve, then $x = 3$

Then the required sum is 0

18.(7) We have $f(2x) - f(2x)\, f\left(\frac{1}{2x}\right) + f(16x^2y) = f(-2) - f(4xy)$

Replacing y by $\frac{1}{8x^2}$, we get

$f(2x) - f(2x)\left(\frac{1}{2x}\right) + f(2) = f(-2) - f\left(\frac{1}{2x}\right)$

$\therefore\quad f(2x) + f\left(\frac{1}{2x}\right) = f(2x)\, f\left(\frac{1}{2x}\right)$ [As $f(x)$ is even]

$\therefore\quad f(2x) = 1 \pm (2x)^n$

$\Rightarrow\quad f(x) = 1 \pm x^n$

Now $f(4) = 1 \pm 4^n = -255$ (Given)

Taking negative sign, we get $256 = 4^n \Rightarrow n = 4$

Hence $f(x) = 1 - x^4$, which is an even function.

$\Rightarrow f(2) = -15$

19.(1) $f(x) = \sin^2 x + \sin^2\left(x + \frac{\pi}{3}\right) + \cos x \cos\left(x + \frac{\pi}{3}\right)$

$= \sin^2 x + \frac{1}{4}\left(\sin x + \sqrt{3}\cos x\right)^2 + \frac{1}{2}\cos x\left(\cos x - \sqrt{3}\sin x\right)$

$= \frac{5}{4}\left(\sin^2 x + \cos^2 x\right) = \frac{5}{4}$

$(gof)x = g[f(x)] = g(5/4) = 1$

20.(7) From E to F we can define, in all, $2 \times 2 \times 2 \times 2 = 16$ functions (2 options for each elements of E) out of which 2 are into, when all the elements of E either map to 1 or to 2

$\therefore$ Number of onto function $= 16 - 2 = 14$

21.(1) Given $f(f(x)) = -x + 1$

replacing $x \to f(x)$

$f(f(f(x))) = -f(x) + 1$

$f(1 - x) = -f(x) + 1$

$f(x) + f(1 - x) = 1$

$\Rightarrow\ f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right) = 1$

22. (7) $\left(\frac{3}{4}\right)^{6x+10-x^2} < \frac{27}{64}$

$\Rightarrow\ 6x + 10 - x^2 > 3$

$\therefore\ x^2 - 6x - 7 < 0$

$\therefore\ (x + 1)(x - 7) < 0$

$\Rightarrow\ 0, 1, 2, 3, 4, 5, 6$

23.(6) $\because\ k \in$ odd

$f(k) = k + 3$

$f(f(k)) = \frac{k+3}{2}$

If $\frac{k+3}{2}$ is odd $\Rightarrow 27 = \frac{k+3}{2} + 3 \Rightarrow k = 45$ not possible

$\Rightarrow\ \frac{k+3}{2}$ is even

$\therefore\ 27 = f(f(f(k))) = f\left(\frac{k+3}{2}\right) = \frac{k+3}{4}$

$\therefore\ k = 105$

verifying $f(f(f(105)))=f(f(108))=f(54)=27$

$\therefore\ k=105$

24.(3) Clearly fundamental period is $\frac{4\pi}{3}$, then z lies in the third quadrant.

25.(1) $\log_a(x^2-x+2)>\log_a(-x^2+2x+3)$

Put $x=\frac{4}{9}$, $\log_a\left(\frac{142}{81}\right)>\log_a\left(\frac{299}{81}\right)$

$\because\ \frac{142}{81}<\frac{299}{81}\Rightarrow 0<a<1$

$\Rightarrow\ \log_a(x^2-x+2)>\log_2(-x^2+2x+3)$

gives $0<x^2-x+2<-x^2+2x+3$

$x^2-x+2>0$ and $2x^2-3x-1<0$

$\Rightarrow\ \frac{3-\sqrt{17}}{4}<x<\frac{3+\sqrt{17}}{4}$

26.(4) $(2^{2x}-4\cdot 2^x+4)+1+||b-1|-3|=|\sin y|$

$\Rightarrow\ (2^x-2)^2+1+||b-1|-3|=|\sin y|$

$\Rightarrow\ (2^x-2)^2+1+||b-1|-3|=|\sin y|$

LHS ≥ 1 and RHS ≤ 1

$\therefore\ 2^x=2, |b-1|-3=0$

$\Rightarrow\ (b-1)=\pm 3$

$\Rightarrow\ b=4,-2$

27.(3) $f(3n)=f(f(f(n)))=3f(n),\ \forall\ n\in N$

Put $n=1, f(3)=3f(1)$

If $f(1)=1$, then $f(f(1))=f(1)=1$, but $f(f(n))=3n$

$\Rightarrow\ f(f(1))=3$ giving $1=3$ which is absurd.

$\therefore\quad f(1)\neq 1$

$\therefore\quad 3=f(f(1))>f(1)>1$

So $f(1)=2$

$f(2)=f(f(1))=3$

28.(3) $\log_{1/3}\log_7(\sin x+a)>0$

$\Rightarrow\ 0<\log_7(\sin x+a)<1$

$1<(\sin x+a)<7\quad \forall\, x\in R$ ['a' should be less than the minimum value of $7-\sin x$ and 'a' must be greater than maximum value of $1-\sin x$]

$\Rightarrow\ 1-\sin x<a<7-\sin x\ \forall\ x\in R$

$2<a<6$

29.(9) $g(x)=\frac{1}{2}\tan^{-1}|x|+1\quad\Rightarrow\ \text{sgn}(g(x))=1$

$\Rightarrow\ \sin^{23}x-\cos^{22}x=1$

$\Rightarrow\ \sin^{23}x=1+\cos^{22}x$ which is possible if $\sin x=1$ and $\cos x=0$

$\Rightarrow\ \sin x=1, x=2n\pi+\frac{\pi}{2}$

hence $-10\pi\leq 2n\pi+\frac{\pi}{2}\leq 8\pi\Rightarrow -\frac{21}{4}\leq n\leq\frac{15}{4}$

$\Rightarrow\ -5\leq n\leq 3$

Hence, number of values of $x=9$.

30.(7) $f(x)=\frac{ax^8+bx^6+cx^4+dx^2+15x+1}{x}$

$=\underbrace{ax^7+bx^5+cx^3+dx+\frac{1}{x}}_{\text{odd function}}+15$

Now $f(x)+f(-x)=30$

$\Rightarrow\ f(-5)=30-f(5)=28$

Archives

Subjective

1. Since $f(x)$ is defined and real for all real values of x, therefore domain of f is R.

Also, $\frac{x^2}{1+x^2}\geq 0$, for all $x\in R$

and $\frac{x^2}{1+x^2}<1\ (\because x^2<1+x^2)$ for all $x\in R$

$\therefore\quad 0\leq\frac{x^2}{1+x^2}<1\ \Rightarrow\ 0\leq f(x)<1$

$\Rightarrow$ The range of $f=[0,1)$.

Also, since $f(1)=f(-1)=1/2$

$\therefore\ f$ is not one-to-one.

2. $y=|x|^{1/2}, -1\leq x\leq 1$

$=\begin{cases}\sqrt{-x}, & x<0\\ \sqrt{x}, & x\geq 0\end{cases}$

$\Rightarrow\ y^2=-x$ if $-1\leq x\leq 0$ and $y^2=x$ if $0\leq x\leq 1$

[Here y should be always taken +ve, as by definition, y is a +ve square root]

Clearly, $y^2=-x$ represents upper half of left-hand parabola (upper half as y is +ve) and $y^2=x$ represents upper half of right-hand parabola.

Therefore, the resulting graph is shown as

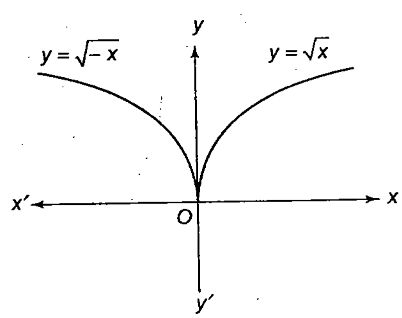


Fig. 1.124

3. $f(x)=x^9-6x^8-2x^7+12x^6+x^4-7x^3+6x^2+x-3$

Then $f(6)=6^9-6\times 6^8-2\times 6^7+12\times 6^6+6^4-7\times 6^3+6\times 6^2+6-3$

$= 6^9 - 6^9 - 2\times 6^7 + 2\times 6^7 + 6^4 - 7\times 6^3 + 6^3 + 6 - 3$

$= -6^3 + 6^3 + 6 - 3$

$= 3$

4. **Case I** $f(x)\neq 2$ is true, $f(y)=2$ and $f(z)\neq 1$ are false, then $f(x) = 1$ or 3, $f(y) = 1$ or 3 and $f(z)=1$

$\Rightarrow$ f is not one-one

Case II $f(x)\neq 2$ is false, $f(y)=2$ is true, $f(z)\neq 1$ is false, then $f(x)=2, f(y)=2, f(z)=1$

$\Rightarrow$ not possible

Case III $f(x)\neq 2$ is false, $f(y)=2$ is false, $f(z)\neq 1$ is true, then $f(x)=2$, $f(y)=1$ or 3, $f(z)=2$ or 3

$\Rightarrow$ $f(x)=2, f(z)=3, f(y)=1.$

5. Given that $f(x+y)=f(x)f(y)\ \forall\ x, y\in N$ and $f(1)=2$

$f(2)=f(1+1)=f(1)f(1)=2^2$

$\Rightarrow$ $f(3)=f(2+1)=f(2)f(1)=2^2\times 2=2^3$

Similarly $f(4)=2^4, \cdots, f(n)=2^n$

$$\sum_{k=1}^{n} f(a+k)= \sum_{k=1}^{n} f(a)f(k)$$

$$= f(a)\sum_{k=1}^{n} f(k)$$

$$= f(a)[f(1)+f(2)+\cdots+f(n)]$$

$$= f(a)[2+2^2+\cdots+2^n]$$

$$= f(a)\left[2\left(\frac{2^n-1}{2-1}\right)\right]$$

From $\sum_{k=1}^{n} f(a+k)=16(2^n-1)$, $f(a)=8=2^3 \Rightarrow a=3.$

6. Given that $4\{x\}=x+[x]$

where $[x]$ = greatest integer $\leq x$ and $\{x\}$ = fractional part of x

We know that $x=[x]+\{x\}$, for any $x\in R$

$\therefore$ Given equation becomes

$4\{x\}=[x]+\{x\}+[x]$

$\Rightarrow$ $3\{x\}=2[x]$

$\Rightarrow$ $[x]=\frac{3}{2}\{x\}$ (1)

Now $0\leq\{x\}<1$

$\Rightarrow$ $0\leq\frac{3}{2}\{x\}<\frac{3}{2}$

$\Rightarrow$ $0\leq[x]<\frac{3}{2}$ [Using equation (1)]

$\Rightarrow$ $[x]=0, 1$

If $[x]=0$

$\Rightarrow$ $\frac{3}{2}\{x\}=0$ [Using equation (1)]

$\Rightarrow$ $\{x\}=0$ $\therefore x=0+0=0$

If $[x]=1$, then

$\frac{3}{2}\{x\}=1$ [Using equation (1)]

$\Rightarrow$ $\{x\}=2/3$

$\Rightarrow$ $x=1+2/3=5/3$

Thus, $x=0, 5/3$

7. Let us put $y=\dfrac{\alpha x^2+6x-8}{\alpha+6x-8x^2}$

$\Rightarrow$ $(\alpha+6x-8x^2)y=\alpha x^2+6x-8$

$\Rightarrow$ $(\alpha+8y)x^2+6(1-y)x-(8+\alpha y)=0$

Since x is real,

$36(1-y)^2+4(\alpha+8y)(8+\alpha y)\geq 0$

$\Rightarrow$ $9(1-y)^2+(\alpha+8y)(8+\alpha y)\geq 0$

$\Rightarrow$ $y^2(9+8\alpha)+y(46+\alpha^2)+(9+8\alpha)\geq 0$ (1)

For (1) to hold for each $y\in R$, $9+8\alpha>0$ and $(46+\alpha^2)^2-4(9+8\alpha)^2\leq 0$

$\Rightarrow$ $\alpha>-\frac{9}{8}$ and $[46+\alpha^2-2(9+8\alpha)]\,[46+\alpha^2+2(9+8\alpha)]\leq 0$

$\Rightarrow$ $\alpha>-\frac{9}{8}$ and $(\alpha^2-16\alpha+28)(\alpha^2+16\alpha+64)\leq 0$

$\Rightarrow$ $\alpha>-\frac{9}{8}$ and $(\alpha-2)(\alpha-14)(\alpha+8)^2\leq 0$

[$\because (\alpha+8)^2\geq 0$]

$\Rightarrow$ $\alpha>-\frac{9}{8}$ and $2\leq\alpha\leq 14$

$\Rightarrow$ $2\leq\alpha\leq 14$

Thus, $f(x)=\dfrac{\alpha x^2+6x-8}{\alpha+6x-8x^2}$ will be onto if $2\leq\alpha\leq 14$

When $\alpha=3$, $f(x)=\dfrac{3x^2+6x-8}{3+6x-8x^2}$, in this case $f(x)=0$ implies $3x^2+6x-8=0$

$$\Rightarrow\ x=\frac{-6\pm\sqrt{36+96}}{6}=\frac{-6\pm\sqrt{132}}{6}=\frac{-6\pm 2\sqrt{33}}{6}$$

$$=\frac{1}{3}\left(-3\pm\sqrt{33}\right)$$

Thus, $f\left[\frac{1}{3}\left(-3+\sqrt{33}\right)\right]=f\left[\frac{1}{3}\left(-3-\sqrt{33}\right)\right]=0.$

Therefore, f is not one-to-one.

Objective

Fill in the blanks

1. For the given function to be defined $\frac{\pi^2}{16}-x^2\geq 0$

$\Rightarrow$ $-\pi/4\leq x\leq\pi/4$

$\therefore\ D_f = [-\pi/4, \pi/4]$

Now for $x \in [-\pi/4, \pi/4]$, $\sqrt{\pi^2/16 - x^2} \in [0, \pi/4]$ and sine function increases on $[0, \pi/4]$

$$\therefore\ \sin 0 \le \sin\sqrt{\frac{\pi^2}{16} - x^2} \le \sin \pi/4 = \frac{1}{\sqrt{2}}$$

$$\Rightarrow\ 0 \le 3\sin\sqrt{\frac{\pi^2}{16} - x^2} \le \frac{3}{\sqrt{2}}$$

$$\therefore\ f(x) \in \left[0, \frac{3}{\sqrt{2}}\right].$$

2. For $f(x)$ to be defined, we must have $-1 \le \log_2\left(\frac{x^2}{2}\right) \le 1$

$\Rightarrow\ 2^{-1} \le \frac{x^2}{2} \le 2^1 \Rightarrow 1 \le x^2 \le 4$

$\Rightarrow\ x \in [-2, -1] \cup [1, 2]$.

3. Set A has n distinct elements.
Then to define a function from A to A, we need to associate each element of set A to any one of the n elements of set A. So, the total number of functions from set A to set A is equal to the number of ways of doing n job where each job can be done in n ways. The total number of such ways is $n \times n \times n \times \ldots \times n$ n-times.
Hence, the total number of functions from A to A n^n
Now for an onto function from A to A, we need to associate each element of A to one and only one element of A. So, the total number of onto functions from set A to A is equal to the number of ways of arranging n elements in the range (set A) keeping n elements fixed in domain (set A). n elements can be arranged in n ! ways.
Hence, the total number of onto functions from A to A is $n!$

4. The given function is, $f(x) = \sin\left[\ln\left(\frac{\sqrt{4-x^2}}{1-x}\right)\right]$

Sine function is defined for all real numbers.
But logarithmic function is defined only for positive values.

Then $\frac{\sqrt{4-x^2}}{1-x} > 0$

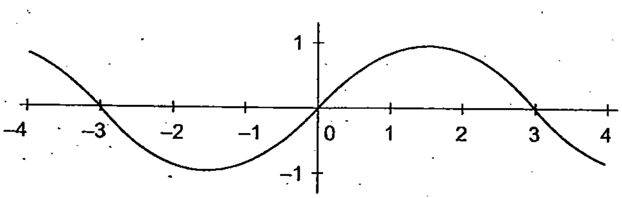


Fig. 1.125

$\Rightarrow\ 1 - x > 0$ also $4 - x^2 > 0$

$\Rightarrow\ x < 1$ and $-2 < x < 2$

$\Rightarrow\ x \in (-2, 1)$

$\Rightarrow$ Domain of f is $(-2, 1)$.

Also for $x \in (-2, 1)$, $\sin x \in (-1, \sin 1)$ as shown in graph.

5. According to the given data, there can be two possible linear functions, one is increasing and other is decreasing. Clearly, from the diagram, the functions are $f(x) = x + 1$ or $f(x) = -x + 1$.

6. Given that $f(x) = f\left(\frac{x+1}{x+2}\right)$ and f is an even function

$$\therefore\ f(x) = f(-x) = f\left(\frac{-x+1}{-x+2}\right)$$

$$\Rightarrow\ x = \frac{-x+1}{-x+2} \Rightarrow x^2 - 3x + 1 = 0$$

$$\Rightarrow\ x = \frac{3 \pm \sqrt{5}}{2}.$$

Also $f(-x) = f\left(\frac{x+1}{x+2}\right)$

$$\Rightarrow\ \frac{x+1}{x+2} = -x \Rightarrow x^2 + 3x + 1 = 0$$

$$\Rightarrow\ x = \frac{-3 \pm \sqrt{5}}{2}.$$

$\therefore$ Four values of x are $\frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}, \frac{-3+\sqrt{5}}{2}, \frac{-3-\sqrt{5}}{2}$.

7. $f(x) = \sin^2 x + \sin^2\left(x + \frac{\pi}{3}\right) + \cos x \cos\left(x + \frac{\pi}{3}\right)$

$$= \sin^2 x + \frac{1}{4}\left(\sin x + \sqrt{3}\cos x\right)^2 + \frac{1}{2}\cos x\left(\cos x - \sqrt{3}\sin x\right)$$

$$= \frac{5}{4}\left(\sin^2 x + \cos^2 x\right) = \frac{5}{4}$$

$(g\text{ of})x = g[f(x)] = g(5/4) = 1$

8. For domain

$$-1 \le \frac{8.3^{x-2}}{1 - 3^{2(x-1)}} \le 1$$

$$\Rightarrow\ -1 \le \frac{3^x - 3^{x-2}}{1 - 3^{2x-2}} \le 1$$

$$\Rightarrow \text{ For } \frac{3^x - 3^{x-2}}{1 - 3^{2x-2}} - 1 \le 0$$

$$\Rightarrow\ \frac{(3^x - 1)(3^{x-2} - 1)}{(3^{2x-2} - 1)} \ge 0$$

$$\Rightarrow\ x \in (-\infty, 0] \cup (1, \infty).$$

For $\dfrac{3^x - 3^{x-2}}{1 - 3^{2x-2}} + 1 \ge 0$

$$\Rightarrow \frac{(3^{x-2} - 1)(3^x + 1)}{(3^x \cdot 3^{x-2} - 1)} \ge 0$$

$\Rightarrow x \in (-\infty, 1) \cup [2, \infty)$

So, $x \in (-\infty, 0] \cup [2, \infty)$

True or false

1. $f(x) = (a - x^n)^{1/n}, a > 0, n$ is + ve integer

$$\Rightarrow f(f(x)) = f\left[\left(a - x^n\right)^{1/n}\right]$$

$$= \left[a - \left\{\left(a - x^n\right)^{1/n}\right\}^n\right]^{1/n}$$

$$= \left(a - a + x^n\right)^{1/n} = x$$

$\therefore$ Statement is true.

2. $f(x) = \dfrac{x^2 + 4x + 30}{x^2 - 8x + 18} = \dfrac{(x+2)^2 + 26}{(x-4)^2 + 2} = y$

For $y = 0$, there is no pre-image $x \in R$

$\therefore$ f is not onto.

$\therefore$ Statement is True.

3. We know that the sum of any two functions is defined only on the points where both f_1 and f_2 are defined, that is, $f_1 + f_2$ is defined on $D_1 \cap D_2$.

$\therefore$ The given statement is false.

Multiple choice questions with one correct answer

1. d. $f(x) = x^2$ is many-one as $f(1) = f(-1) = 1$.

Also f is into, as the range of function is $[0, \infty)$ which is subset of R (co-domain).

$\therefore$ f is neither injective nor surjective.

2. b. $y = x^2 + (k-1)x + 9 = \left(x + \dfrac{k-1}{2}\right)^2 + 9 - \left(\dfrac{k-1}{2}\right)^2$

For entire graph to be above x-axis, we should have

$$9 - \left(\frac{k-1}{2}\right)^2 > 0$$

$\Rightarrow k^2 - 2k - 35 < 0 \Rightarrow (k-7)(k+5) < 0$

$\Rightarrow -5 < k < 7$

3. d. $f(x) = |x - 1|$

$\Rightarrow f(x^2) = |x^2 - 1|$ and $(f(x))^2 = |x-1|^2 = x^2 - 2x + 1$

$\Rightarrow f(x^2) \ne (f(x))^2$

Hence, option **a** is not true.

$f(x+y) = f(x) + f(y) \Rightarrow |x + y - 1| = |x - 1| + |y - 1|$, which is absurd. Put $x = 2, y = 3$ and verify.

Hence, option **b** is not true.

Consider $f(|x|) = |f(x)|$

Put $x = -5$, then $f(|-5|) = f(5) = 4$ and $|f(-5)| = |-5 - 1| = 6$.

$\therefore$ **c** is not correct.

4. c. Let $|x-1| + |x-2| + |x-3| < 6$

$\Rightarrow |(x-1) + (x-2) + (x-3)| < |x-1| + |x-2| + |x-3| < 6$

$\Rightarrow |3x - 6| < 6$

$\Rightarrow |x - 2| < 2$

$\Rightarrow -2 < x - 2 < 2$

$\Rightarrow 0 < x < 4$

Hence, for $|x-1| + |x-2| + |x-3| \ge 6, x \le 0$ or $x \ge 4$.

5. d. $f(x) = \cos(\log x)$

$$\Rightarrow f(x)f(y) - \frac{1}{2}\left[f\left(\frac{x}{y}\right) + f(xy)\right]$$

$$= \cos(\log x)\cos(\log y) - \frac{1}{2}\left[\cos(\log x - \log y)\right.$$
$$\left. + \cos(\log x + \log y)\right]$$

$$= \cos(\log x)\cos(\log y) - \frac{1}{2}[2\cos(\log x)\cos(\log y)]$$

$$= 0$$

6. c. $y = \dfrac{1}{\log_{10}(1-x)} + \sqrt{x-2}$

$y = f(x) + g(x)$

Then, the domain of given function is $D_f \cap D_g$.

Now, for the domain of $f(x) = \dfrac{1}{\log_{10}(1-x)}$,

we know it is defined only when $1 - x > 0$ and $1 - x \ne 1$

$\Rightarrow x < 1$ and $x \ne 0 \therefore D_f = (-\infty, 1) - \{0\}$

For the domain of $g(x) = \sqrt{x+2}$

$x + 2 \ge 0 \Rightarrow x \ge -2$

$\therefore D_g = [-2, \infty)$

$\therefore$ common domain is $[-2, 1) - \{0\}$.

7. a. $f(x) = \{x\}$ is periodic with period 1

$f(x) = \sin\dfrac{1}{x}$ for $x \ne 0$, $f(0) = 0$ is non-periodic as

$g(x) = \dfrac{1}{x}$ is non-periodic

Also $f(x) = x\cos x$ is non-periodic as $g(x) = x$ is non-periodic.

8. b. $y = 2^{x(x-1)} \Rightarrow x^2 - x - \log_2 y = 0;$

$$\Rightarrow x = \frac{1}{2}\left(1 \pm \sqrt{1 + 4\log_2 y}\right)$$

Since $x \in [1, \infty)$, we choose $x = \dfrac{1}{2}\left(1 + \sqrt{1 + 4\log_2 y}\right)$

or $f^{-1}(x) = \dfrac{1}{2}\left(1 + \sqrt{1 + 4\log_2 x}\right)$.

9. d. We have $fog(x) = f(g(x)) = \sin(\log_e |x|)$
$\log_e |x|$ has range R, for which $\sin(\log_e|x|) \in [-1, 1]$
$\therefore \quad R_1 = \{u : -1 \le u \le 1\}$
Also $gof(x) = g(f(x)) = \log_e |\sin x|$
$\because 0 \le |\sin x| \le 1$
$\Rightarrow \quad -\infty < \log_e |\sin x| \le 0$
$\Rightarrow \quad R_2 = \{v : -\infty < v \le 0\}$.

10. c. Since $f(x) = (x+1)^2 - 1$ is continuous function, solution of $f(x) = f^{-1}(x)$ lies on the line $y = x$
$\Rightarrow \quad f(x) = f^{-1}(x) = x$
$\Rightarrow \quad (x+1)^2 - 1 = x$
$\Rightarrow \quad x^2 + x = 0$
$\Rightarrow \quad x = 0$ or -1
$\Rightarrow$ The required set is $\{0, -1\}$.

11. d. $f(x)$ is continuous for all $x > 0$ and $f\left(\frac{x}{y}\right) = f(x) - f(y)$
Also $f(e) = 1$
$\Rightarrow$ Clearly, $f(x) = \log_e x$ satisfies all these properties.
$\therefore \quad f(x) = \log_e x$, which is an unbounded function.

12. d. It is given that $2^x + 2^y = 2 \ \forall x, y \in R$
$\Rightarrow \quad 2^y = 2 - 2^x$
$\Rightarrow \quad y = \log_2(2 - 2^x)$
$\Rightarrow$ Function is defined only when $2 - 2^x > 0$ or $2^x < 2$ or $x < 1$

13. b. $g(x) = 1 + \{x\}, f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$ where $\{x\}$ represents the fractional part function.

$$\Rightarrow \quad f(g(x)) = \begin{cases} -1, & 1+\{x\} < 0 \\ 0, & 1+\{x\} = 0 \\ 1, & 1+\{x\} > 0 \end{cases}$$

$\Rightarrow \quad f(g(x)) = 1, 1 + \{x\} > 0 \ (\because 0 \le \{x\} < 1)$
$\Rightarrow \quad f(g(x)) = 1 \ \forall \ x \in R$

14. a. $f: [1, \infty) \to [2, \infty)$

$$f(x) = x + \frac{1}{x} = y$$

$$\Rightarrow \quad x^2 - yx + 1 = 0$$

$$\Rightarrow \quad x = \frac{y \pm \sqrt{y^2 - 4}}{2}$$

But given $f: [1, \infty) \to [2, \infty)$

$$\therefore \quad x = \frac{y + \sqrt{y^2 - 4}}{2}$$

$$\Rightarrow \quad f^{-1}(x) = \frac{x + \sqrt{x^2 - 4}}{2}$$

15. d. For domain of $f(x) = \dfrac{\log_2(x+3)}{x^2 + 3x + 2}$
$x^2 + 3x + 2 \ne 0$ and $x + 3 > 0$
$\Rightarrow \quad x \ne -1, -2$ and $x > -3$
$\therefore \quad D_f = (-3, \infty) - \{-1, -2\}$.

16. a. From E to F we can define, in all, $2 \times 2 \times 2 \times 2 = 16$ functions (2 options for each elements of E) out of which 2 are into, when all the elements of E map to either 1 or 2
$\therefore$ No. of onto function $= 16 - 2 = 14$.

17. d.
$$f(x) = \frac{\alpha x}{x+1}, \ x \ne -1$$

$$f(f(x)) = x \Rightarrow \frac{\alpha\left(\frac{\alpha x}{x+1}\right)}{\frac{\alpha x}{x+1} + 1} = x$$

$$\Rightarrow \quad \frac{\alpha^2 x}{(\alpha+1)x + 1} = x$$

$$\Rightarrow \quad (\alpha+1)x^2 + (1 - \alpha^2)x = 0 \qquad (1)$$

$\Rightarrow \quad \alpha + 1 = 0$ and $1 - \alpha^2 = 0$
[As true $\forall x \ne -1 \therefore$ Eq. (1) is an identity]
$\Rightarrow \quad \alpha = -1$.

18. d. Given that $f(x) = (x+1)^2, x \ge -1$
Now if $g(x)$ is the reflection of $f(x)$ in the line $y = x$, then $g(x)$ is an inverse function of $y = f(x)$.
Given $y = (x+1)^2 \ (x \ge -1$ and $y \ge 0)$
$\Rightarrow \quad x = \pm\sqrt{y} - 1$
$\Rightarrow \quad g(x) = f^{-1}(x) = \sqrt{x} - 1, x \ge 0$

19. a. Given that, $f(x) = 2x + \sin x, x \in R$
$\Rightarrow \quad f'(x) = 2 + \cos x$
but $-1 \le \cos x \le 1$
$\Rightarrow \quad 1 \le 2 + \cos x \le 3$
$\therefore \quad f'(x) > 0, \ \forall \ x \in R$
$\Rightarrow \quad f(x)$ is strictly increasing and hence one-one.
Also as $x \to \infty, f(x) \to \infty$ and $x \to -\infty, f(x) \to -\infty$
$\therefore$ Range of $f(x) = R =$ co-domain of $f(x)$
$\Rightarrow \quad f(x)$ is onto
Thus, $f(x)$ is one-one and onto.

20. b. Given that $f: [0, \infty) \to [0, \infty), f(x) = \dfrac{x}{x+1}$

$$\Rightarrow \quad f'(x) = \frac{1 + x - x}{(1+x)^2} = \frac{1}{(1+x)^2} > 0, \ \forall \ x$$

$\therefore \quad f$ is an increasing function $\Rightarrow f$ is one-one
Also $D_f = [0, \infty)$.

And for range let $\dfrac{x}{1+x} = y \Rightarrow x = \dfrac{y}{1-y}$

$\because x \ge 0 \Rightarrow 0 \le y < 1 \ \therefore R_f = [0, 1), \ne$ co-domain
$\therefore \quad f$ is not onto
Hence, $D_f \ne R_f \Rightarrow f$ is not onto.

21. a. For $f(x) = \sqrt{\sin^{-1}(2x) + \dfrac{\pi}{6}}$ to be defined and real
$\sin^{-1} 2x + \pi/6 \ge 0$

$\Rightarrow \quad \sin^{-1} 2x \geq -\dfrac{\pi}{6}$ (1)

But we know that $-\pi/2 \leq \sin^{-1} 2x \leq \pi/2$ (2)

Combining (1) and (2), $-\pi/6 \leq \sin^{-1} 2x \leq \pi/2$

$\Rightarrow \sin(-\pi/6) \leq 2x \leq \sin(\pi/2)$

$\Rightarrow -1/2 \leq 2x \leq 1$

$\Rightarrow -1/4 \leq x \leq 1/2$

$\therefore \quad D_f = \left[-\dfrac{1}{4}, \dfrac{1}{2}\right]$

22. c. We have $f(x) = \dfrac{x^2+x+2}{x^2+x+1} = \dfrac{(x^2+x+1)+1}{x^2+x+1}$

$$= 1 + \frac{1}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}$$

We can see here that as $x \to \infty$, $f(x) \to 1$ which is the minimum value of $f(x)$.

Also $f(x)$ is max when $\left(x+\dfrac{1}{2}\right)^2 + \dfrac{3}{4}$ is minimum which is so when $x = -1/2$.

$\therefore \quad f_{\max} = 1 + \dfrac{1}{3/4} = 7/3$

$\therefore \quad R_f = (1, 7/3]$

23. b. $f(x) = \sin x + \cos x$, $g(x) = x^2 - 1$

$\Rightarrow \quad g(f(x)) = (\sin x + \cos x)^2 - 1 = \sin 2x$

Clearly, $g(f(x))$ is invertible in $-\dfrac{\pi}{2} \leq 2x \leq \dfrac{\pi}{2}$

($\because \sin\theta$ is invertible when $-\pi/2 \leq \theta \leq \pi/2$)

$\Rightarrow \quad -\dfrac{\pi}{4} \leq x \leq \dfrac{\pi}{4}$.

24. a. We are given that

$$f: R \to R, f(x) = \begin{cases} 0, & x \in \text{rational} \\ x, & x \in \text{irrational} \end{cases}$$

$$g: R \to R, g(x) = \begin{cases} 0, & x \in \text{irrational} \\ x, & x \in \text{rational} \end{cases}$$

$\therefore \quad (f-g): R \to R$ such that

$$(f-g)(x) = \begin{cases} -x & \text{if } x \in \text{rational} \\ x & \text{if } x \in \text{irrational} \end{cases}$$

Since $f-g: R \to R$ for any x there is only one value of $(f(x) - g(x))$ whether x is rational or irrational. Moreover, as $x \in R$, $f(x) - g(x)$ also belongs to R. Therefore, $(f-g)$ is one-one and onto.

25. d. Given that X and Y are two sets and $f: X \to Y$. $\{f(c) = y; c \subset X, y \subset Y\}$ and $\{f^{-1} d = x; d \subset Y, x \subset X\}$.

The pictorial representation of given information is as shown

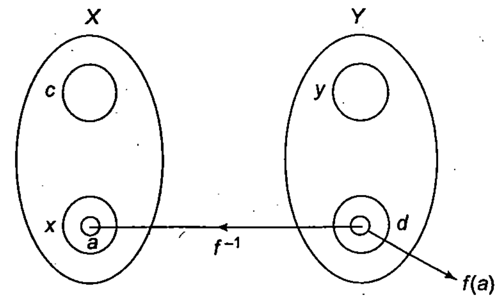


Fig. 1.126

Since $f^{-1} d = x$

$\Rightarrow \quad f(x) = d$

Now if $a \subset x \Rightarrow f(a) \subset f(x) = d$

$\Rightarrow \quad f^{-1}[fa] = a$

$\therefore \quad f^{-1}(f(a)) = a, a \subset x$ is the correct option.

Multiple choice question with more than one answer

1. a, d. Given that $f(x) = y = \dfrac{x+2}{x-1}$

a. Let $f(x) = \dfrac{x+2}{x-1} = y \Rightarrow x + 2 = xy - y$

$\Rightarrow \quad x = \dfrac{2+y}{y-1} \Rightarrow x = f(y)$

$\therefore$ **a** is correct.

b. $f(1) \neq 3$ $\therefore$ **b** is not correct.

c. $f'(x) = \dfrac{x-1-x-2}{(x-1)^2} = \dfrac{-3}{(x-1)^2} < 0$ for $\forall x \in R - \{1\}$

$\Rightarrow \quad f(x)$ is decreasing $\forall x \neq 1$

$\therefore$ **c** is not correct

d. $f(x) = \dfrac{x+2}{x-1}$ is a rational function of x

$\therefore$ **d.** is the correct answer

Thus, we get that **a,** and **d** are correct answer.

2. b., c. As $(0, 0)$ and $(x, g(x))$ are two vertices of an equilateral triangle; therefore, length of the side of Δ is

$$= \sqrt{(x-0)^2 + (g(x)-0)^2} = \sqrt{x^2 + (g(x))^2}$$

$\therefore$ The area of equilateral $\Delta = \dfrac{\sqrt{3}}{4}\left(x^2 + (g(x))^2\right)$

$= \dfrac{\sqrt{3}}{4}$

$\Rightarrow \quad g(x)^2 = 1 - x^2$

$\Rightarrow \quad g(x) = \pm\sqrt{1-x^2}$

$\therefore$ **b, c** are the correct answers as **a** is not a function ($\because$ image of x is not unique).

3. a, c. $f(x) = \cos[\pi^2]x + \cos[-\pi^2]x$

We know $9 < \pi^2 < 10$ and $-10 < -\pi^2 < -9$

$\Rightarrow \quad [\pi^2] = 9$ and $[-\pi^2] = -10$

$\Rightarrow f(x) = \cos 9x + \cos(-10x)$

$\Rightarrow f(x) = \cos 9x + \cos 10x$

a. $f\left(\frac{\pi}{2}\right) = \cos\frac{9\pi}{2} + \cos 5\pi = -1$ (true)

b. $f(\pi) = \cos 9\pi + \cos 10\pi = -1 + 1 = 0$ (false).

c. $f(-\pi) = \cos(-9\pi) + \cos(-10\pi) = \cos 9\pi + \cos 10\pi$
$= -1 + 1 = 0$ (true)

d. $f\left(\frac{\pi}{4}\right) = \cos\frac{9\pi}{4} + \cos\frac{5\pi}{2} = \cos\left(2\pi + \frac{\pi}{4}\right) + 0$ (false)

Thus, **a** and **c.** are correct options.

4. b. $f(x) = 3x - 5$ (given)

Let $y = f(x) = 3x - 5$

$\Rightarrow y + 5 = 3x \Rightarrow x = \frac{y+5}{3}$

$\Rightarrow f^{-1}(x) = \frac{x+5}{3}.$

5. a. If $f(x) = \sin^2 x$ and $g(x) = \sqrt{x}$

Now, $fog = f(g(x)) = f(\sqrt{x}) = \sin^2\sqrt{x}$

and $gof(x) = g(f(x)) = g(\sin^2 x) = \sqrt{\sin^2 x} = |\sin x|$

again if $f(x) = \sin x$, $g(x) = |x|$

$fog(x) = f(g(x)) = f(|x|) = \sin|x| \neq (\sin\sqrt{x})^2$

When $f(x) = x^2$, $g(x) = \sin\sqrt{x}$

$fog(x) = f[g(x)] = f(\sin\sqrt{x}) = (\sin\sqrt{x})^2$

and $(gof)(x) = g[f(x)] = g(x^2) = \sin\sqrt{x^2} = \sin|x|$
$\neq |\sin x|$

$\therefore$ **a** is the correct option.

Match the following type

1. We have $k f(x) = \frac{x^2 - 6x + 5}{x^2 - 5x + 6} = \frac{(x-5)(x-1)}{(x-2)(x-3)}$

a → p, r, s.

If $-1 < x < 1$, then $f(x) = \frac{(-\text{ve})(-\text{ve})}{(-\text{ve})(-\text{ve})} = +\text{ve} \therefore f(x) > 0$

Also $f(x) - 1 = \frac{-x-1}{x^2 - 5x + 6} = -\frac{(x+1)}{(x-2)(x-3)}$

for $-1 < x < 1$, $f(x) - 1 = \frac{-(+\text{ve})}{(-\text{ve})(-\text{ve})} = -\text{ve}$

$\Rightarrow f(x) - 1 < 0 \Rightarrow f(x) < 1$

$\therefore 0 < f(x) < 1$

b → q, s.

if $1 < x < 2$ then $f(x) = \frac{(-\text{ve})(+\text{ve})}{(-\text{ve})(-\text{ve})} = -\text{ve}$

$\therefore f(x) < 0$ and so $f(x) < 1$

c → q, s.

If $3 < x < 5$ then

$f(x) = \frac{(-\text{ve})(+\text{ve})}{(+\text{ve})(+\text{ve})} = -\text{ve}$

$\therefore f(x) < 0$ and so $f(x) < 1$

d → p, r, s.

For $x > 5$, $f(x) > 0$

Also $f(x) - 1 = \frac{-(x+1)}{(x-2)(x-5)} < 0$ for $x > 5$

$\Rightarrow f(x) < 1$, $\therefore 0 < f(x) < 1$