Exercise 13.4

Answer 1E.

(A)

We know the average velocity over a time interval of length h is given by

$$\frac{\vec{r}(t+h)-\vec{r}(t)}{h}$$

Now for the interval [0, 1]

$$\frac{\vec{r}(t+h)-\vec{r}(t)}{h}=\frac{\vec{r}(0+1)-\vec{r}(0)}{1}$$

$$=\frac{<4.5,\ 6,3>-<2.7,9.8,3.7>}{1}$$

$$=<1.8,\ -3.8,\ -0.7>$$

That is average velocity over [0, 1] is $1.8\hat{i}-3.8\hat{j}-0.7\hat{k}$

For the interval [0.5, 1]

$$\frac{\vec{r}(t+h)-\vec{r}(t)}{h}=\frac{\vec{r}(0.5+0.5)-\vec{r}(0.5)}{0.5}$$

$$=\frac{<4.5,\ 6.0,\ 3.0>-<3.5,\ 7.2,\ 3.3>}{0.5}$$

$$=<2.0,\ -2.4,\ -0.6>$$

That is average velocity over [0.5, 1] is $2\hat{i}-2.4\hat{j}-0.6\hat{k}$

For the interval [1, 2]

$$\frac{\vec{r}(t+h)-\vec{r}(t)}{h}=\frac{\vec{r}(1+1)-\vec{r}(1)}{1}$$

$$=\frac{<7.3,\ 7.8,\ 2.7>-<4.5,\ 6.0,\ 3.0>}{1}$$

$$=<2.8,\ 1.8,\ -0.3>$$

That is average velocity over [1, 2] is $2.8\hat{i}+1.8\hat{j}-0.3\hat{k}$

For the interval [1, 1.5]

$$\frac{\vec{r}(t+h)-\vec{r}(t)}{h}=\frac{\vec{r}(1+0.5)-\vec{r}(1)}{0.5}$$
$$=\frac{<5.9,\ 6.4,\ 2.8>-<4.5,\ 6.0,\ 3.0>}{0.5}$$
$$=<2.8,\ 0.8,\ -0.4>$$

That is average velocity over [1, 1.5] is $2.8\hat{i}+0.8\hat{j}-0.4\hat{k}$

(B)

The average velocity over interval [0.5, 1] is $<2,-2.4,\ -0.6>$

And over interval [1, 1.5] is $<2.8,\ 0.8,\ -0.4>$

Therefore the velocity of particle at $t=1$ is

$$\frac{<2,-2.4,\ -0.6>+<2.8,0.8,-0.4>}{2}$$
$$=<2.4,\ -0.8,\ -0.5>$$

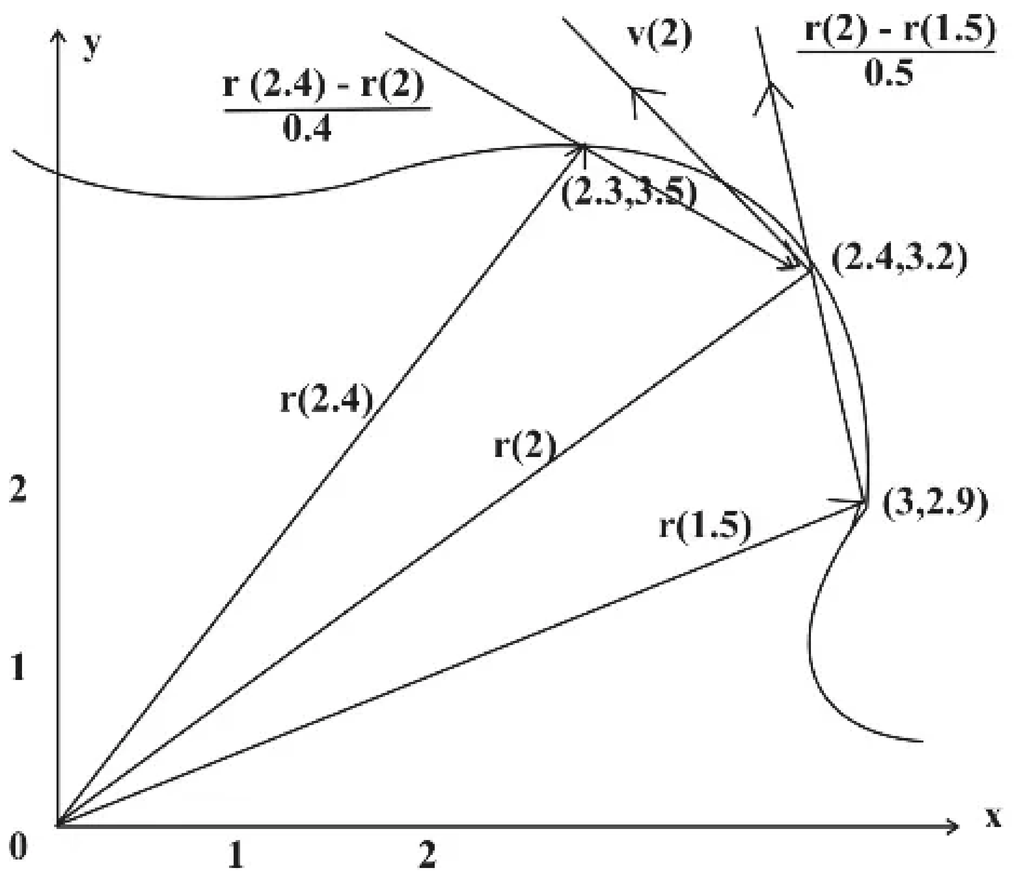
i.e. $\vec{v}(1)=2.4\hat{i}-0.8\hat{j}-0.5\hat{k}$

And speed is $\left|\vec{v}(1)\right|=\sqrt{(2.4)^2+(-0.8)^2+(-0.5)^2}$

$$=\boxed{2.58}$$

Answer 2E.

(A), (B)



(C)

$$\vec{v}(2)=\lim_{h\to 0}\frac{\vec{r}(2+h)-\vec{r}(2)}{h}$$

(D)

Speed of the particle at $t=2$ is $\left|\vec{v}(2)\right|$

$$\vec{v}(2)=\frac{1}{2}\left[\frac{\vec{r}(2.4)-\vec{r}(2)}{0.4}+\frac{\vec{r}(2)-\vec{r}(1.5)}{0.5}\right]$$

$$=\frac{1}{2}\left[\frac{<2.3,3.5>-<2.4,3.2>}{0.4}+\frac{<2.4,\ 3.2>-<3,\ 2.9>}{0.5}\right]$$

$$=\frac{1}{2}<-1.45,\ 1.35>$$

$$=<-0.725,\ 0.675>$$

Then $\left|\vec{v}(2)\right|=\sqrt{(-0.725)^2+(0.675)^2}$

$$=0.99\approx\boxed{1}$$

Answer 3E.

To find the velocity, acceleration and speed of the particle with the position function,

$$\mathbf{r}(t)=\left\langle -\frac{1}{2}t^2,t\right\rangle, t=2$$

Velocity of the particle,

$$\mathbf{v}(t)=\mathbf{r}'(t)$$

$$=\frac{d}{dt}\left\langle -\frac{1}{2}t^2,t\right\rangle$$

$$=\left\langle -\frac{1}{2}(2t),1\right\rangle$$

$$=\boxed{\langle -t,1\rangle}$$

And,

$$\mathbf{v}(2)=\langle -2,1\rangle$$

The acceleration of the particle is,

$$\begin{aligned}\mathbf{a}(t) &= \mathbf{r}''(t)\\ &= \frac{d}{dt}\mathbf{r}'(t)\\ &= \frac{d}{dt}\langle -t,1\rangle\\ &= \boxed{\langle -1,0\rangle}\end{aligned}$$

And,

$$\mathbf{a}(2) = \langle -1,0\rangle$$

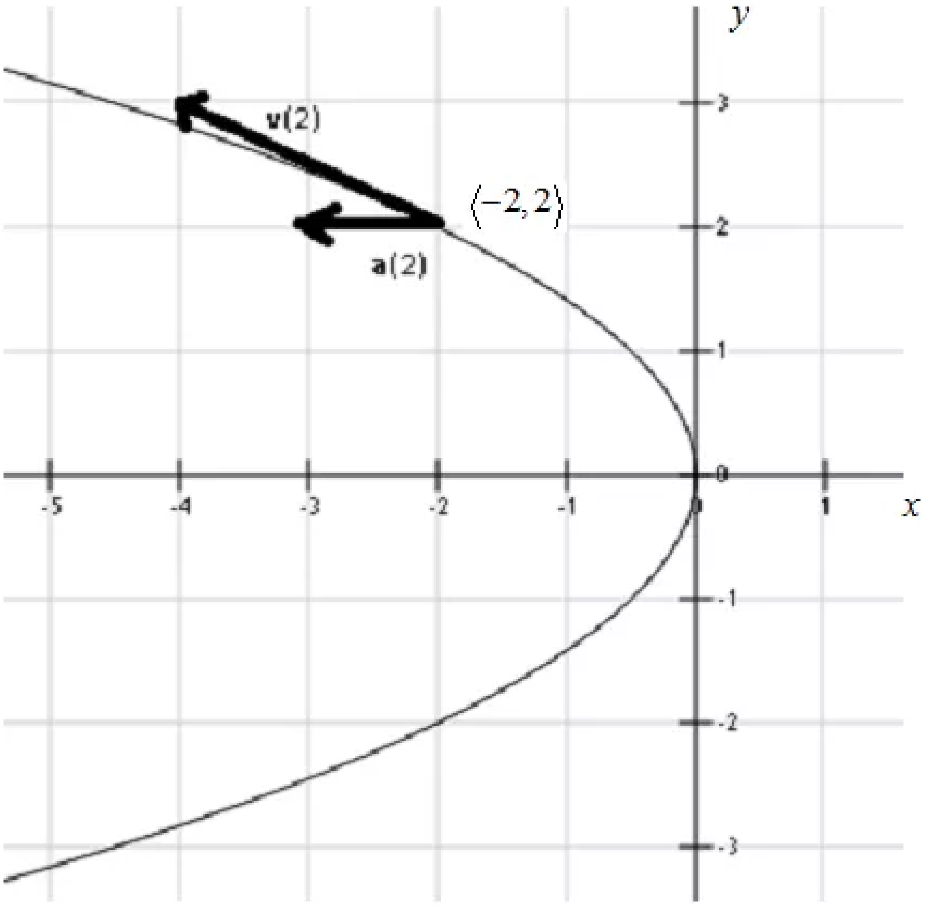
The speed of the particle is,

$$\begin{aligned}|\mathbf{v}(t)| &= |\langle -t,1\rangle|\\ &= \boxed{\sqrt{t^2+1}}\end{aligned}$$

At $t = 2$,

$$\begin{aligned}\mathbf{r}(2) &= \left\langle -\frac{1}{2}(2)^2, 2\right\rangle\\ &= \langle -2,2\rangle\end{aligned}$$

Finally, the sketch of the path of the particle, the velocity and acceleration vectors $\mathbf{v}(2)=\langle -2,1\rangle$, $\mathbf{a}(2)=\langle -1,0\rangle$ is shown below:



Answer 4E.

Consider the position vector,

$$r(t)=(2-t)\mathbf{i}+4\sqrt{t}\mathbf{j}\ ;a=1$$

The objective is to find the acceleration, speed and velocity of a particle with the given position vector.

The velocity of the particle is,

$$\begin{aligned} v(t) &= \frac{d}{dt}r(t) \\ &= \frac{d}{dt}\langle 2-t, 4\sqrt{t}\rangle \\ &= \left\langle -1, \frac{2}{\sqrt{t}} \right\rangle \end{aligned}$$

The velocity vector of the particle is $\boxed{v(t)=\left\langle -1,\frac{2}{\sqrt{t}}\right\rangle}$.

Find the velocity at $t=1$.

$$v(t)=\left\langle -1,\frac{2}{\sqrt{t}}\right\rangle$$

$$v(1)=\left\langle -1,\frac{2}{\sqrt{1}}\right\rangle$$
$$=\langle -1,2\rangle$$

Thus, the velocity at $t=1$ is $v(1)=\boxed{\langle -1,2\rangle}$.

The acceleration of the particle is derivative of the velocity vector.

The acceleration of the particle is,

$$a(t)=\frac{d}{dt}v(t)$$
$$=\frac{d}{dt}\left\langle -1,\frac{2}{\sqrt{t}}\right\rangle$$
$$=\frac{d}{dt}\left\langle -1,2t^{-\frac{1}{2}}\right\rangle$$
$$=\left\langle 0,2\left(-\frac{1}{2}t^{-\frac{1}{2}-1}\right)\right\rangle$$
$$=\left\langle 0,2\left(-\frac{1}{2}t^{-3/2}\right)\right\rangle$$
$$=\left\langle 0,-t^{-3/2}\right\rangle$$

The acceleration vector of the particle is $\boxed{a(t)=\left\langle 0,-t^{-3/2}\right\rangle}$.

Find the acceleration at $t=1$.

$$a(t)=\left\langle 0,-t^{-3/2}\right\rangle$$
$$a(1)=\langle 0,-1\rangle$$

Thus, the acceleration at $t=1$ is $a(1)=\boxed{\langle 0,-1\rangle}$.

The speed of the particle is,

$$|v(t)| = \left|\left\langle -1, \frac{2}{\sqrt{t}} \right\rangle\right|$$
$$= \sqrt{(-1)^2 + \left(\frac{2}{\sqrt{t}}\right)^2}$$
$$= \sqrt{1 + \frac{4}{t}}$$

Hence, the speed of the particle is $\boxed{|v(t)| = \sqrt{1 + \frac{4}{t}}}$.

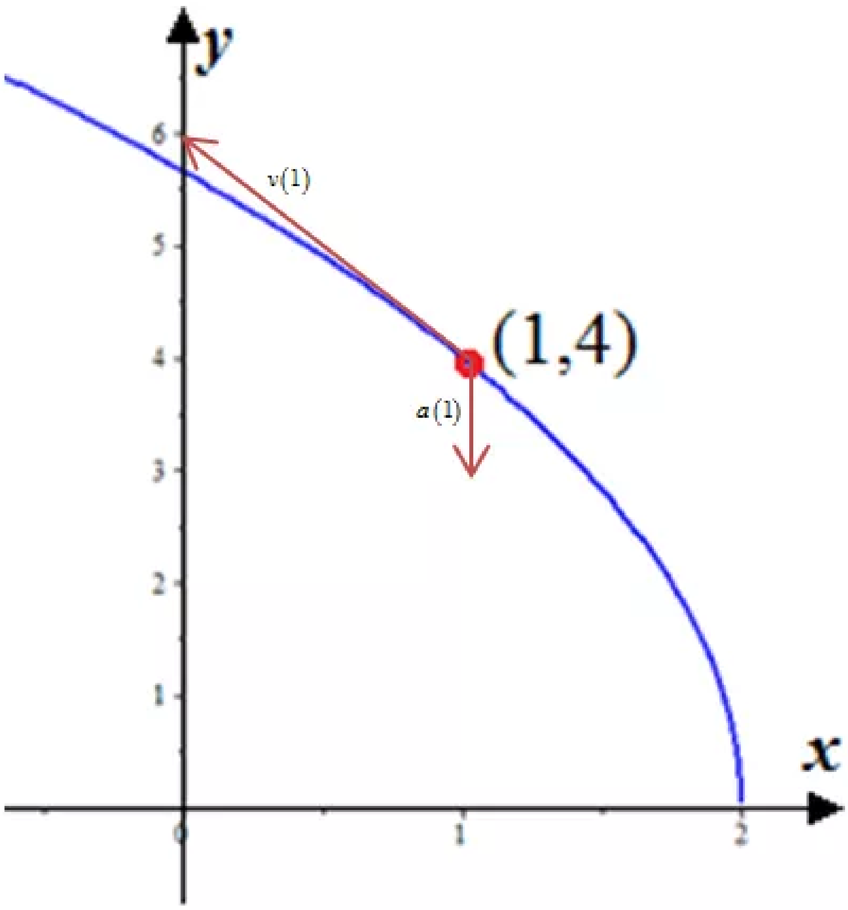
The speed of the particle at $t = 1$ is,

$$|\mathbf{v}(1)| = \sqrt{1 + \frac{4}{1}}$$
$$= \boxed{\sqrt{5}}$$

The position vector at $t = 1$ is,

$$r(t) = \langle 2 - t, 4\sqrt{t} \rangle$$
$$r(1) = \langle 2 - 1, 4\sqrt{1} \rangle$$
$$= \langle 1, 4 \rangle$$

Sketch the graph of the path of the particle and draw the velocity and acceleration vectors.



Answer 5E.

To find the velocity, acceleration and speed of the particle with the position function,

$$\mathbf{r}(t)=\langle 3\cos t, 2\sin t\rangle, t=\pi/3$$

Formulas for differentiation,

$$\frac{d}{dt}(\cos t)=-\sin t$$
$$\frac{d}{dt}(\sin t)=\cos t$$

Velocity of the particle,

$$\begin{aligned}\mathbf{v}(t)&=\mathbf{r}'(t)\\&=\frac{d}{dt}\langle 3\cos t, 2\sin t\rangle\\&=\boxed{\langle -3\sin t, 2\cos t\rangle}\end{aligned}$$

And,

$$\mathbf{v}(t)=\langle -3\sin t, 2\cos t\rangle$$

$$\begin{aligned}\mathbf{v}\left(\frac{\pi}{3}\right)&=\left\langle -3\sin\frac{\pi}{3}, 2\cos\frac{\pi}{3}\right\rangle\\&=\left\langle -3\left(\frac{\sqrt{3}}{2}\right), 2\left(\frac{1}{2}\right)\right\rangle\\&=\left\langle \frac{-3\sqrt{3}}{2}, 1\right\rangle\end{aligned}$$

The acceleration of the particle is,

$$\begin{aligned}\mathbf{a}(t) &= \mathbf{r}''(t)\\ &= \frac{d}{dt}\mathbf{r}'(t)\\ &= \frac{d}{dt}\langle -3\sin t, 2\cos t\rangle\\ &= \boxed{\langle -3\cos t, -2\sin t\rangle}\end{aligned}$$

And,

$$\begin{aligned}\mathbf{a}(t) &= \langle -3\cos t, -2\sin t\rangle\\ \mathbf{a}\left(\frac{\pi}{3}\right) &= \left\langle -3\cos\left(\frac{\pi}{3}\right), -2\sin\left(\frac{\pi}{3}\right)\right\rangle\\ &= \left\langle -3\left(\frac{1}{2}\right), -2\left\langle\frac{\sqrt{3}}{2}\right\rangle\right\rangle\\ &= \left\langle \frac{-3}{2}, -\sqrt{3}\right\rangle\end{aligned}$$

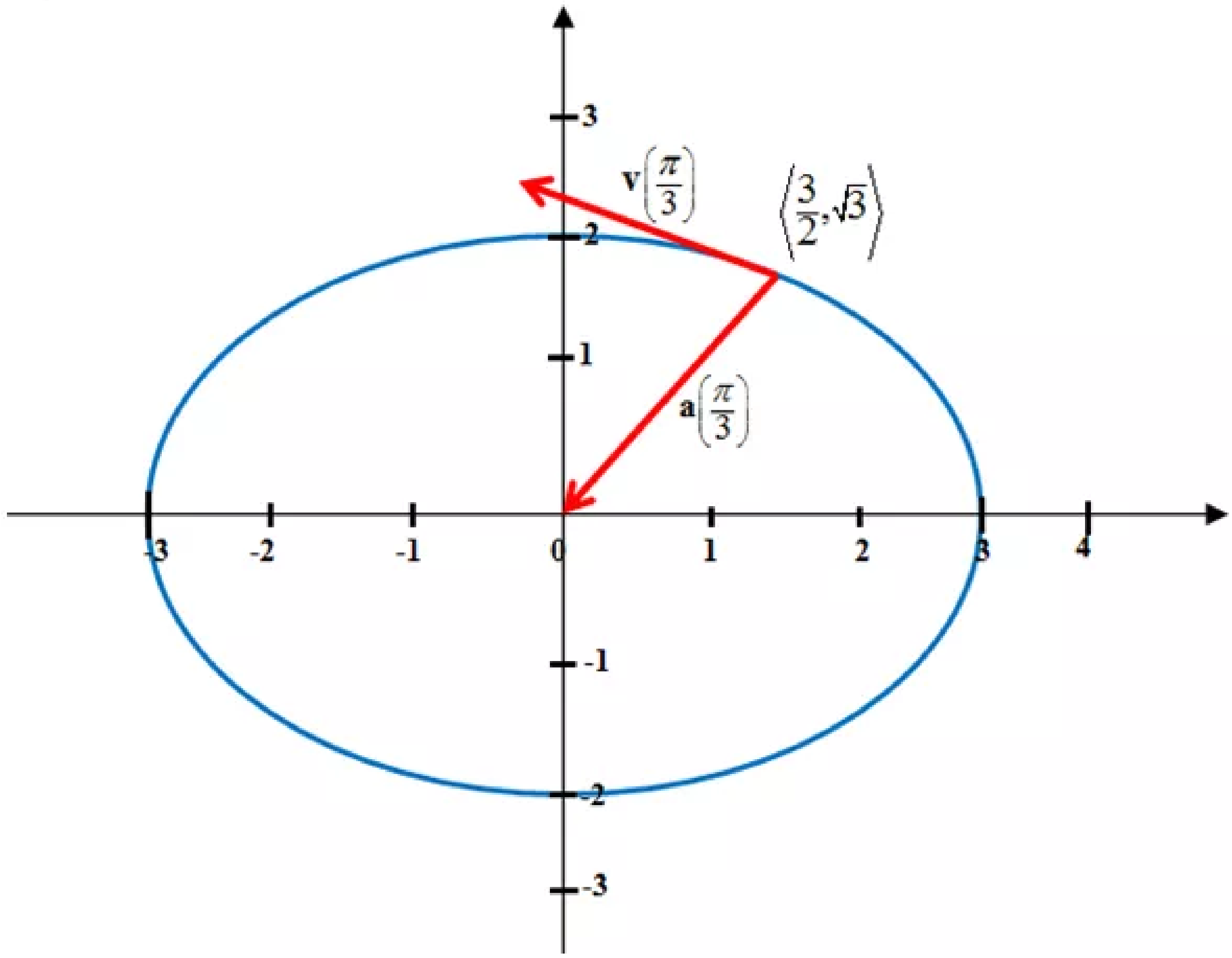
The speed of the particle is,

$$\begin{aligned}|\mathbf{v}(t)| &= |\langle -3\sin t, 2\cos t\rangle|\\ &= \sqrt{(-3\sin t)^2 + (2\cos t)^2}\\ &= \sqrt{9\sin^2 t + 4\cos^2 t}\\ &= \sqrt{5\sin^2 t + 4\sin^2 t + 4\cos^2 t}\\ &= \sqrt{5\sin^2 t + 4}\end{aligned}$$

At $t = \pi/3$,

$$\begin{aligned}\mathbf{r}(t) &= \langle 3\cos t, 2\sin t\rangle\\ \mathbf{r}\left(\frac{\pi}{3}\right) &= \left\langle 3\cos\left(\frac{\pi}{3}\right), 2\sin\left(\frac{\pi}{3}\right)\right\rangle\\ &= \left\langle 3\left(\frac{1}{2}\right), 2\left(\frac{\sqrt{3}}{2}\right)\right\rangle\\ &= \left\langle \frac{3}{2}, \sqrt{3}\right\rangle\end{aligned}$$

Finally, the sketch of the path of the particle, the velocity and acceleration vectors $\mathbf{v}\left(\frac{\pi}{3}\right)=\left\langle\frac{-3\sqrt{3}}{2},1\right\rangle$, $\mathbf{a}\left(\frac{\pi}{3}\right)=\left\langle\frac{-3}{2},-\sqrt{3}\right\rangle$ is shown below:



Answer 7E.

Find the velocity, acceleration and speed of the particle with the following position function:

$\mathbf{r}(t)=\langle t,t^2,2\rangle, t=1$

Velocity of the particle is as follows:

$$\begin{aligned}\mathbf{v}(t)&=\mathbf{r}'(t)\\&=\frac{d}{dt}\langle t,t^2,2\rangle\\&=\boxed{\langle 1,2t,0\rangle}\end{aligned}$$

Now, proceed as follows:

$\mathbf{v}(1)=\langle 1,2,0\rangle$

The acceleration of the particle is calculated as follows:

$$\begin{aligned}\mathbf{a}(t) &= \mathbf{r}''(t) \\ &= \frac{d}{dt}\mathbf{r}'(t) \\ &= \frac{d}{dt}\langle 1,2t,0\rangle \\ &= \boxed{\langle 0,2,0\rangle}\end{aligned}$$

Obtain the following result:

$$\mathbf{a}(1) = \langle 1,2,0\rangle$$

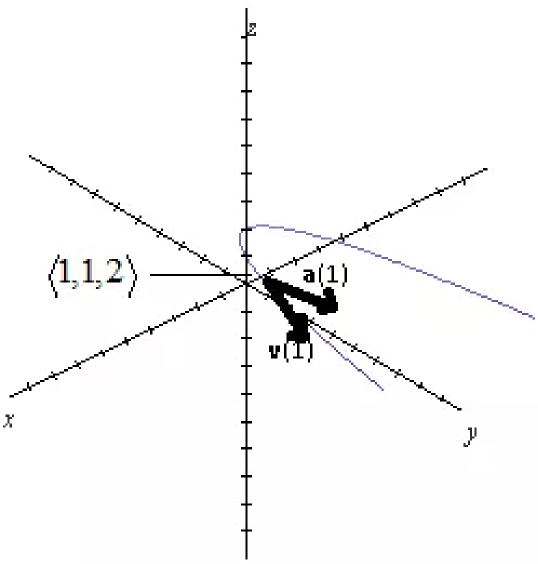
The speed of the particle is calculated as follows:

$$\begin{aligned}|\mathbf{v}(t)| &= |\langle 1,2t,0\rangle| \\ &= \sqrt{(1)^2+(2t)^2+0^2} \\ &= \boxed{\sqrt{1+4t^2}}\end{aligned}$$

At $t=1$, obtain the following values:

$$\mathbf{r}(t) = \langle t,t^2,2\rangle$$
$$\mathbf{r}(1) = \langle 1,1,2\rangle$$

Finally, the sketch of the path of the particle, the velocity and acceleration vectors $\mathbf{v}(1)=\langle 1,2,0\rangle$, $\mathbf{a}(1)=\langle 1,2,0\rangle$ is as follows:



Answer 8E.

The position function is $\mathbf{r}(t) = t\mathbf{i} + 2\cos t\mathbf{j} + \sin t\mathbf{k}$.

That is $\mathbf{r}(t) = \langle t, 2\cos t, \sin t\rangle$

The objective is to find velocity, acceleration and speed of a particle.

The point on the curve at $t = 0$ is $\langle 0, 2, 0\rangle$

Use the following formulas:

(i) The velocity of a particle at time t is $\mathbf{v}(t) = r'(t)$

(ii) The acceleration of a particle at time t is $\mathbf{a}(t) = r''(t)$.

First differentiate the position function, $\mathbf{r}(t) = \langle t, 2\cos t, \sin t\rangle$ with respect to t.

$\mathbf{r}'(t) =< 1, -2\sin t, \cos t >$

The velocity at $t = 0$ is $\mathbf{v}(0) =< 1, -2\sin 0, \cos 0 >= \boxed{\langle 1, 0, 1\rangle}$

Again differentiate the function $\mathbf{r}'(t) =< 1, -2\sin t, \cos t >$ with respect to t.

$$\begin{aligned}\mathbf{r}''(t) &= \frac{d}{dt} < 1, -2\sin t, \cos t > \\ &= \langle 0, -2\cos t, -\sin t\rangle\end{aligned}$$

The acceleration vector at $t = 0$ is,

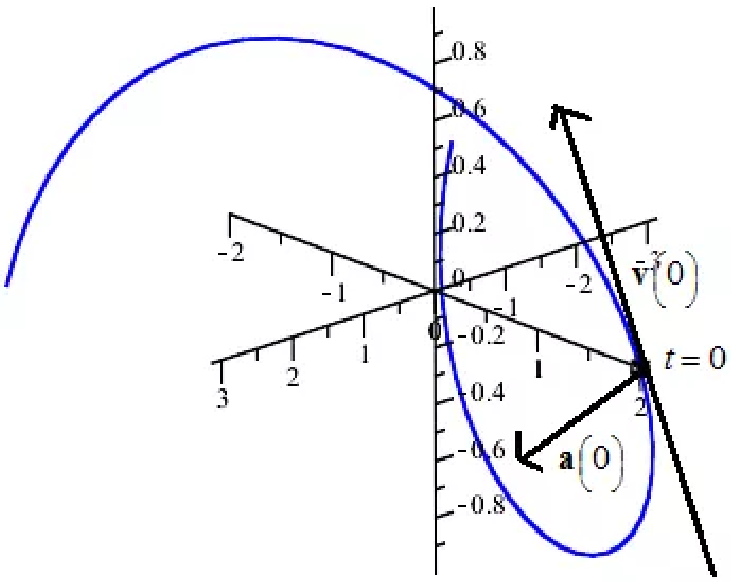
$$\begin{aligned}\mathbf{a}(0) &= \langle 0, -2\cos 0, -\sin 0\rangle \\ &= \boxed{\langle 0, -2, 0\rangle}\end{aligned}$$

The speed of a particle is $|\mathbf{v}(t)| = \sqrt{1+4\sin^2 t+\cos^2 t}$.

The speed of the particle at $t=0$ is,

$$|\mathbf{v}(0)| = \sqrt{1+4\sin^2 0+\cos^2 0} = \boxed{\sqrt{2}}$$

The velocity and acceleration of a particle at $t=0$ are shown below:



Answer 9E.

$$\vec{r}(t) = <t^2+1, t^3, t^2-1>$$

$$\text{Velocity } \vec{v}(t) = \vec{r}\,'(t) = <2t, 3t^2, 2t>$$

$$\text{Acceleration } \vec{a}(t) = \vec{r}\,''(t) = <2, 6t, 2>$$

$$\begin{aligned}\text{Speed } |\vec{v}(t)| &= \sqrt{(2t)^2+(3t^2)^2+(2t)^2}\\ &= \sqrt{4t^2+9t^4+4t^2}\\ &= \sqrt{9t^4+8t^2}\\ &= |t|\sqrt{9t^2+8}\end{aligned}$$

Answer 10E.

$$\vec{r}(t) = <2\cos t, 3t, 2\sin t>$$

Velocity $\vec{v}(t)=\vec{r}\,'(t)= \;<-2\sin t,\ 3,\ 2\cos t>$

Acceleration $\vec{a}(t)=\vec{r}\,''(t)=<-2\cos t,\ 0,-2\sin t>$

$$\begin{aligned}\text{Speed } \left|\vec{v}(t)\right| &= \sqrt{(-2\sin t)^2+3^2+(2\cos t)^2} \\ &= \sqrt{4\sin^2 t+9+4\cos^2 t} \\ &= \sqrt{4+9} \\ &= \sqrt{13}\end{aligned}$$

Answer 11E.

Consider the position function

$$\mathbf{r}(t)=\sqrt{2}t\,\mathbf{i}+e^t\,\mathbf{j}+e^{-t}\mathbf{k}$$

Velocity is the rate of change of position vector with respect to time *t*.

That is

$$\begin{aligned}\mathbf{v}(t) &= \mathbf{r}'(t) \\ &= \frac{d}{dt}\left(\sqrt{2}t\,\mathbf{i}+e^t\,\mathbf{j}+e^{-t}\mathbf{k}\right) \\ &= \boxed{\sqrt{2}\,\mathbf{i}+e^t\,\mathbf{j}-e^{-t}\mathbf{k}}\end{aligned}$$

And

The Acceleration is the rate of change of velocity with respect to time *t*.

Therefore

$$\begin{aligned}\mathbf{a}(t) &= \mathbf{v}'(t) \\ &= \mathbf{r}''(t) \\ &= \frac{d}{dt}\left(\sqrt{2}\,\mathbf{i}+e^t\,\mathbf{j}-e^{-t}\mathbf{k}\right) \\ &= 0\,\mathbf{i}+e^t\,\mathbf{j}+e^{-t}\mathbf{k}\end{aligned}$$

$$=\boxed{e^t\,\mathbf{j}+e^{-t}\mathbf{k}}$$

Now find the speed of a particle

The speed of the particle at time *t* is the magnitude of the velocity vector, that is, $|v(t)|$.

Since speed cannot be negative

Thus

Speed = $|v(t)|$

$$|\mathbf{v}(t)| = |\mathbf{r}'(t)|$$
$$= \frac{ds}{dt}$$

= rate of change of distance with respect to time.

Suppose

If $\mathbf{v}(t) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ then $|v(t)| = \sqrt{a^2 + b^2 + c^2}$

Here $\mathbf{v}(t) = \sqrt{2}\,\mathbf{i} + e^t\mathbf{j} - e^{-t}\mathbf{k}$

Then

$$|\mathbf{v}(t)| = \sqrt{\left(\sqrt{2}\right)^2 + \left(e^t\right)^2 + \left(-e^{-t}\right)^2}$$

$= \sqrt{2 + e^{2t} + e^{-2t}}$ Squaring

$= \dfrac{\sqrt{2e^{2t} + e^{4t} + 1}}{\sqrt{e^{2t}}}$ Multiply and divide by 1 or $\sqrt{e^{2t}}$

$= \sqrt{\dfrac{\left(e^{2t} + 1\right)^2}{e^{2t}}}$ Use the formula $(a+b)^2 = a^2 + b^2 + 2ab$

$= \dfrac{e^{2t} + 1}{e^t}$ Cancel the square root

$= \boxed{e^t + e^{-t}}$ Simplify

Thus the speed of a particle is $\boxed{e^t + e^{-t}}$

Answer 12E.

Consider the position function, $\mathbf{r}(t) = t^2\,\mathbf{i} + 2t\,\mathbf{j} + \ln t\,\mathbf{k}$.

The objective is to find the velocity, acceleration, and speed of a particle with the position function.

Find velocity function $\mathbf{v}(t)$ as follows:

The derivative of the position function is velocity function.

That is, $\mathbf{v}(t) = \mathbf{r}'(t)$.

$$\begin{aligned}
\mathbf{v}(t) &= \frac{d}{dt}(\mathbf{r}(t)) \\
&= \frac{d}{dt}\left(t^2\,\mathbf{i} + 2t\,\mathbf{j} + \ln t\,\mathbf{k}\right) \\
&= \frac{d}{dt}(t^2)\mathbf{i} + \frac{d}{dt}(2t)\mathbf{j} + \frac{d}{dt}(\ln t)\mathbf{k} \\
&= 2t\mathbf{i} + 2\mathbf{j} + \frac{1}{t}\mathbf{k}
\end{aligned}$$

Hence, the velocity function is, $\boxed{\mathbf{v}(t) = 2t\mathbf{i} + 2\mathbf{j} + \frac{1}{t}\mathbf{k}}$.

Find the acceleration $\mathbf{a}(t)$ as follows:

The derivative of the velocity function is the acceleration.

That is, $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.

$$\begin{aligned}
\mathbf{a}(t) &= \frac{d}{dt}(\mathbf{v}(t)) \\
&= \frac{d}{dt}\left(2t\mathbf{i} + 2\mathbf{j} + \frac{1}{t}\mathbf{k}\right) \\
&= \frac{d}{dt}(2t)\mathbf{i} + \frac{d}{dt}(2)\mathbf{j} + \frac{d}{dt}\left(\frac{1}{t}\right)\mathbf{k} \\
&= 2\mathbf{i} + 0\cdot\mathbf{j} + \left(\frac{-1}{t^2}\right)\mathbf{k}
\end{aligned}$$

Hence, the acceleration is, $\boxed{\mathbf{a}(t) = 2\mathbf{i} + 0\cdot\mathbf{j} - \frac{1}{t^2}\mathbf{k}}$.

Find the speed of a particle as follows:

The speed of the particle at time t is the magnitude of the velocity vector. That is, $|\mathbf{v}(t)|$.

$$|\mathbf{v}(t)| = \sqrt{(2t)^2 + 2^2 + \left(\frac{1}{t}\right)^2}$$
$$= \sqrt{4t^2 + 4 + \frac{1}{t^2}}$$
$$= \sqrt{\frac{4t^4 + 4t^2 + 1}{t^2}}$$
$$= \frac{1}{|t|}\sqrt{4t^4 + 4t^2 + 1}$$

Hence, the speed of the particle is, $\boxed{\frac{1}{|t|}\sqrt{4t^4 + 4t^2 + 1}}$.

Answer 13E.

$$\vec{r}(t) = \left(e^t \cos t\,\hat{i} + e^t \sin t\,\hat{j} + e^t t\,\hat{k}\right)$$

$$\text{Velocity } \vec{v}(t) = \vec{r}\,'(t) = \left(e^t \cos t - e^t \sin t\right)\hat{i} + \left(e^t \sin t + e^t \cos t\right)\hat{j} + \left(e^t.t + e^t.1\right)\hat{k}$$
$$= e^t\left[(\cos t - \sin t)\hat{i} + (\sin t + \cos t)\hat{j} + (t+1)\hat{k}\right]$$

$$\text{Acceleration } \vec{a}(t) = \vec{r}\,''(t) = \left(e^t \cos t - e^t \sin t - e^t \sin t - e^t \cos t\right)\hat{i}$$
$$+\left(e^t \sin t + e^t \cos t + e^t \cos t - e^t \sin t\right)\hat{j}$$
$$+\left(e^t t + e^t.1 + e^t\right)\hat{k}$$
$$= -2e^t \sin t\,\hat{i} + 2e^t \cos t\,\vec{j} + e^t(t+2)\,\hat{k}$$

$$\text{Speed } |\vec{v}(t)| = \sqrt{e^{2t}\left[(\cos t - \sin t)^2 + (\sin t + \cos t)^2 + (t+1)^2\right]}$$

$$= e^t\sqrt{\cos^2 t + \sin^2 t - 2\sin t\cos t + \sin^2 t + \cos^2 t + 2\sin t\cos t + t^2 + 2t + 1}$$
$$= e^t\sqrt{1 + 1 + t^2 + 2t + 1}$$
$$= \boxed{e^t\sqrt{t^2 + 2t + 3}}$$

Answer 14E.

If x and y are twice differentiable functions of t, and $\mathbf{r}$ is a vector-valued function given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, then the velocity vector is given by
$\mathbf{v}(t) = \mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$.

Evaluate $x'(t)$.

$$\begin{aligned} x'(t) &= \frac{d}{dt}(t^2) \\ &= 2t \end{aligned}$$

Now, find $y'(t)$.

$$\begin{aligned} y'(t) &= \frac{d}{dt}(\sin t - t\cos t) \\ &= t\sin t \end{aligned}$$

Find $z'(t)$.

$$\begin{aligned} z'(t) &= \frac{d}{dt}(\cos t + t\sin t) \\ &= t\cos t \end{aligned}$$

Replace the obtained values in $\mathbf{v}(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$.

$\mathbf{v}(t) = 2t\mathbf{i} + t\sin t\mathbf{j} + t\cos t\mathbf{k}$

Thus, we get the velocity function as $\mathbf{v}(t) = 2t\mathbf{i} + t\sin t\mathbf{j} + t\cos t\mathbf{k}$.

Now, determine the acceleration given by $\mathbf{a}(t) = \mathbf{r}''(t) = x''(t)\mathbf{i} + y''(t)\mathbf{j} + z''(t)\mathbf{k}$.

Find $x''(t)$, $y''(t)$ and $z''(t)$.

$$\begin{aligned} x''(t) &= \frac{d}{dt}(2t) = 2 \\ y''(t) &= \frac{d}{dt}(t\sin t) = \sin(t) + t\cos(t) \\ z''(t) &= \frac{d}{dt}(t\cos t) = \cos(t) - t\sin(t) \end{aligned}$$

Therefore, the acceleration of the object is obtained as

$\mathbf{v}(t) = 2\mathbf{i} + [\sin(t) + t\cos(t)]\mathbf{j} + [\cos(t) - t\sin(t)]\mathbf{k}$.

Replace the obtained values in $\mathbf{v}(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$.

$\mathbf{v}(t) = 2t\mathbf{i} + t\sin t\mathbf{j} + t\cos t\mathbf{k}$

Thus, we get the velocity function as $\mathbf{v}(t) = 2t\mathbf{i} + t\sin t\mathbf{j} + t\cos t\mathbf{k}$.

Now, determine the acceleration given by $\mathbf{a}(t) = \mathbf{r}''(t) = x''(t)\mathbf{i} + y''(t)\mathbf{j} + z''(t)\mathbf{k}$.

Find $x''(t)$, $y''(t)$ and $z''(t)$.

$$x''(t) = \frac{d}{dt}(2t) = 2$$

$$y''(t) = \frac{d}{dt}(t\sin t) = \sin(t) + t\cos(t)$$

$$z''(t) = \frac{d}{dt}(t\cos t) = \cos(t) - t\sin(t)$$

Therefore, the acceleration of the object is obtained as

$\mathbf{v}(t) = 2\mathbf{i} + [\sin(t) + t\cos(t)]\mathbf{j} + [\cos(t) - t\sin(t)]\mathbf{k}$.

$$\begin{aligned}\|\mathbf{v}(t)\| &= \sqrt{(2t)^2 + (t\sin t)^2 + (t\cos t)^2} \\ &= \sqrt{5t^2} \\ &= \sqrt{5}t\end{aligned}$$

The speed of the object is thus obtained as $\boxed{\sqrt{5}t}$.

Answer 15E.

To find the velocity and the position vectors of a particle that has the given acceleration, velocity, and the position,

$\mathbf{a}(t) = \mathbf{i} + 2\mathbf{j}$

$\mathbf{v}(0) = \mathbf{k}$

$\mathbf{r}(0) = \mathbf{i}$

A known result is,

$$\int t^n dt = \frac{t^{n+1}}{n+1}, n \neq -1$$

First, integrate acceleration,

$$\mathbf{v}(t) = \int \mathbf{a}(t)dt$$
$$= \int (\mathbf{i} + 2\mathbf{j})\, dt$$
$$= t\mathbf{i} + 2t\mathbf{j} + \mathbf{C}$$

$$\mathbf{v}(t) = t\mathbf{i} + 2t\mathbf{j} + \mathbf{C} \quad \ldots\ldots (1)$$

Now we can plug in the first initial condition $\mathbf{v}(0) = \mathbf{k}$,

$$\mathbf{v}(0) = (0)\mathbf{i} + 2(0)\mathbf{j} + \mathbf{C}$$
$$\mathbf{C} = \mathbf{k}$$

Then by (1), the velocity vector $\boxed{\mathbf{v}(t) = t\mathbf{i} + 2t\mathbf{j} + \mathbf{k}}$

Next we can solve for the position function.

$$\mathbf{r}(t) = \int \mathbf{v}(t)\, dt$$
$$= \int (t\mathbf{i} + 2t\mathbf{j} + \mathbf{k})\, dt$$
$$= \frac{t^2}{2}\mathbf{i} + t^2\mathbf{j} + t\mathbf{k} + \mathbf{C}'$$

$$\mathbf{r}(t) = \frac{t^2}{2}\mathbf{i} + t^2\mathbf{j} + t\mathbf{k} + \mathbf{C}' \quad \ldots\ldots (2)$$

Now we can plug in the final initial condition $\mathbf{r}(0) = \mathbf{i}$,

$$\mathbf{r}(0) = \frac{0^2}{2}\mathbf{i} + 0^2\mathbf{j} + 0\mathbf{k} + \mathbf{C}'$$
$$\mathbf{C}' = \mathbf{i}$$

Then by (2),

$$\mathbf{r}(t) = \frac{t^2}{2}\mathbf{i} + t^2\mathbf{j} + t\mathbf{k} + \mathbf{i}$$

Therefore, the position vector $\boxed{\mathbf{r}(t) = \left(\frac{t^2}{2} + 1\right)\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}}$

Answer 16E.

Find the velocity and the position vectors of a particle that has the given acceleration, velocity, and the position,

$$\mathbf{a}(t) = 2\mathbf{i} + 6t\,\mathbf{j} + 12t^2\,\mathbf{k}$$
$$\mathbf{v}(0) = \mathbf{i}$$
$$\mathbf{r}(0) = \mathbf{j} - \mathbf{k}$$

A known result is,

$$\int t^n dt = \frac{t^{n+1}}{n+1}, n \neq -1$$

First, integrate acceleration,

$$\begin{aligned}\mathbf{v}(t) &= \int \mathbf{a}(t)\,dt \\ &= \int \left(2\mathbf{i} + 6t\,\mathbf{j} + 12t^2\,\mathbf{k}\right)dt \\ &= 2t\,\mathbf{i} + 3t^2\,\mathbf{j} + 4t^3\,\mathbf{k} + \mathbf{C}\end{aligned}$$

Therefore, the velocity vector is,

$$\mathbf{v}(t) = 2t\,\mathbf{i} + 3t^2\,\mathbf{j} + 4t^3\,\mathbf{k} + \mathbf{C} \quad \cdots\cdots(1)$$

Plug in the first initial condition $\mathbf{v}(0) = \mathbf{i}$,

$$\begin{aligned}(0)\mathbf{i} + 2(0)\mathbf{j} + \mathbf{C} &= \mathbf{i} \\ \mathbf{C} &= \mathbf{i}\end{aligned}$$

Then by (1),

$$\begin{aligned}\mathbf{v}(t) &= 2t\,\mathbf{i} + 3t^2\,\mathbf{j} + 4t^3\,\mathbf{k} + \mathbf{i} \\ &= (2t+1)\mathbf{i} + 3t^2\,\mathbf{j} + 4t^3\,\mathbf{k}\end{aligned}$$

Therefore, the velocity vector is $\mathbf{v}(t) = \boxed{(2t+1)\mathbf{i} + 3t^2\,\mathbf{j} + 4t^3\,\mathbf{k}}$.

Solve for the position function by integrate the velocity vector,

$$\mathbf{r}(t)=\int \mathbf{v}(t)\,dt$$
$$=\int\left((2t+1)\mathbf{i}+3t^2\,\mathbf{j}+4t^3\,\mathbf{k}\right)dt$$
$$\mathbf{r}(t)=\left(t^2+t\right)\mathbf{i}+t^3\,\mathbf{j}+t^4\,\mathbf{k}+\mathbf{D}\quad\cdots\cdots(2)$$

Plug in the final initial condition $\mathbf{r}(0)=\mathbf{j}-\mathbf{k}$,

$$\left(0^2+0\right)\mathbf{i}+0^3\,\mathbf{j}+0^4\mathbf{k}+\mathbf{D}=\mathbf{j}-\mathbf{k}$$
$$\mathbf{D}=\mathbf{j}-\mathbf{k}$$

Then by (2),

$$\mathbf{r}(t)=\left(t^2+t\right)\mathbf{i}+t^3\,\mathbf{j}+t^4\,\mathbf{k}+\mathbf{j}-\mathbf{k}$$
$$=\left(t^2+t\right)\mathbf{i}+\left(t^3+1\right)\mathbf{j}+\left(t^4-1\right)\mathbf{k}$$

Therefore, the position vector is $\mathbf{r}(t)=\boxed{\left(t^2+t\right)\mathbf{i}+\left(t^3+1\right)\mathbf{j}+\left(t^4-1\right)\mathbf{k}}$.

Answer 17E.

A position vector is represented as $r(t)$ which shows the position of any point P in space with respect to the three coordinates and it is also known as location or radius vector.

(a)

Consider the acceleration of a particle

$$\mathbf{a}(t)=(2t)\mathbf{i}+(\sin t)\mathbf{j}+(\cos 2t)\mathbf{k}.$$

Since $\mathbf{a}(t)=\mathbf{v}'(t)$, find the velocity of the particle by integrating $\mathbf{a}(t)$ with respect to t.

$$\mathbf{v}(t)=\int \mathbf{a}(t)\,dt$$
$$=\int\left[(2t)\mathbf{i}+(\sin t)\mathbf{j}+(\cos 2t)\mathbf{k}\right]dt$$
$$=\left(\frac{2t^2}{2}\right)\mathbf{i}+(-\cos t)\mathbf{j}+\left(\frac{\sin 2t}{2}\right)\mathbf{k}+\mathbf{C}$$
$$\mathbf{v}(t)=\left(t^2\right)\mathbf{i}+(-\cos t)\mathbf{j}+\left(\frac{\sin 2t}{2}\right)\mathbf{k}+\mathbf{C}\ \ldots(1)$$

Given that the velocity of the particle at $t=0$ is $\mathbf{v}(0)=\mathbf{i}\ldots(2)$

To find the integration constant $\mathbf{C}$, substitute $t=0$ in the equation (1) and equate it to the equation (2).

$$\mathbf{v}(t)=(t^2)\mathbf{i}+(-\cos t)\mathbf{j}+\left(\frac{\sin 2t}{2}\right)\mathbf{k}+\mathbf{C}$$

$$\mathbf{v}(0)=(0^2)\mathbf{i}+(-\cos 0)\mathbf{j}+\left(\frac{\sin 2(0)}{2}\right)\mathbf{k}+\mathbf{C}$$

$$\mathbf{i}=(0)\mathbf{i}+(-1)\mathbf{j}+\left(\frac{\sin 0}{2}\right)\mathbf{k}+\mathbf{C}$$

$$\mathbf{i}=(-1)\mathbf{j}+\left(\frac{0}{2}\right)\mathbf{k}+\mathbf{C}$$

$$\mathbf{C}=\mathbf{i}+\mathbf{j}$$

Substitute $\mathbf{C}=\mathbf{i}+\mathbf{j}$ in the equation (1).

$$\mathbf{v}(t)=(t^2)\mathbf{i}+(-\cos t)\mathbf{j}+\left(\frac{\sin 2t}{2}\right)\mathbf{k}+\mathbf{C}$$

$$\mathbf{v}(t)=(t^2)\mathbf{i}+(-\cos t)\mathbf{j}+\left(\frac{\sin 2t}{2}\right)\mathbf{k}+\mathbf{i}+\mathbf{j}$$

$$\mathbf{v}(t)=(1+t^2)\mathbf{i}+(1-\cos t)\mathbf{j}+\left(\frac{\sin 2t}{2}\right)\mathbf{k}$$

The velocity of the particle is;

$$\mathbf{v}(t)=(1+t^2)\mathbf{i}+(1-\cos t)\mathbf{j}+\left(\frac{\sin 2t}{2}\right)\mathbf{k}.$$

Since $\mathbf{v}(t)=\mathbf{r}'(t)$, find the velocity of the particle by integrating $\mathbf{v}(t)$ with respect to t.

$$\begin{aligned}\mathbf{r}(t)&=\int \mathbf{v}(t)\,dt\\ &=\int\left[(1+t^2)\mathbf{i}+(1-\cos t)\mathbf{j}+\left(\frac{\sin 2t}{2}\right)\mathbf{k}\right]dt\\ &=\left(t+\frac{t^3}{3}\right)\mathbf{i}+(t-\sin t)\mathbf{j}+\left(-\frac{\cos 2t}{4}\right)\mathbf{k}+\mathbf{D}\end{aligned}$$

$$\mathbf{r}(t)=\left(t+\frac{t^3}{3}\right)\mathbf{i}+(t-\sin t)\mathbf{j}+\left(-\frac{\cos 2t}{4}\right)\mathbf{k}+\mathbf{D}\ldots(3)$$

Given that the position vector of the particle at $t=0$ is $\mathbf{r}(0)=\mathbf{j}\ldots(4)$

To find the integration constant $\mathbf{D}$, substitute $t=0$ in the equation (3) and equate it to the equation (4).

$$\mathbf{r}(t)=\left(t+\frac{t^3}{3}\right)\mathbf{i}+(t-\sin t)\mathbf{j}+\left(-\frac{\cos 2t}{4}\right)\mathbf{k}+\mathbf{D}$$

$$\mathbf{r}(0)=\left(0+\frac{0^3}{3}\right)\mathbf{i}+(0-\sin 0)\mathbf{j}+\left(-\frac{\cos 2(0)}{4}\right)\mathbf{k}+\mathbf{D}$$

$$\mathbf{j}=(0)\mathbf{i}+(0)\mathbf{j}+\left(-\frac{1}{4}\right)\mathbf{k}+\mathbf{D}$$

$$\mathbf{D}=\mathbf{j}+\frac{1}{4}\mathbf{j}$$

Substitute $\mathbf{D}=\mathbf{j}+\frac{1}{4}\mathbf{j}$ in the equation (3).

$$\mathbf{r}(t)=\left(t+\frac{t^3}{3}\right)\mathbf{i}+(t-\sin t)\mathbf{j}+\left(-\frac{\cos 2t}{4}\right)\mathbf{k}+\mathbf{D}$$

$$\mathbf{r}(t)=\left(t+\frac{t^3}{3}\right)\mathbf{i}+(t-\sin t)\mathbf{j}+\left(-\frac{\cos 2t}{4}\right)\mathbf{k}+\mathbf{j}+\frac{1}{4}\mathbf{j}$$

$$\mathbf{r}(t)=\left(t+\frac{t^3}{3}\right)\mathbf{i}+(1+t-\sin t)\mathbf{j}+\left(\frac{1}{4}-\frac{\cos 2t}{4}\right)\mathbf{k}$$

$$\mathbf{r}(t)=\left(t+\frac{t^3}{3}\right)\mathbf{i}+(1+t-\sin t)\mathbf{j}+\left(\frac{1-\cos 2t}{4}\right)\mathbf{k}$$

Therefore, the required position vector of the particle is;

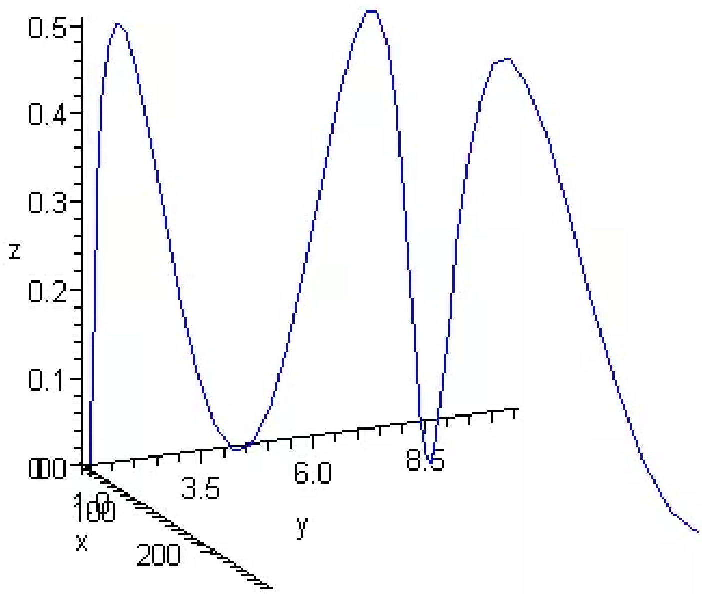
$$\boxed{\mathbf{r}(t)=\left(t+\frac{t^3}{3}\right)\mathbf{i}+(1+t-\sin t)\mathbf{j}+\left(\frac{1-\cos 2t}{4}\right)\mathbf{k}}$$

(b)

Graph the path of the particle using a computer whose position vector is given below;

$$\mathbf{r}(t)=\left(t+\frac{t^3}{3}\right)\mathbf{i}+(1+t-\sin t)\,\mathbf{j}+\left(\frac{1-\cos 2t}{4}\right)\mathbf{k}.$$

Use MAPLE to sketch the graph as shown below.



Answer 19E.

Consider the position function of particle $\mathbf{r}(t)=\langle t^2,5t,t^2-16t\rangle$.

To determine that, when the speed of a particle is minimum if the position function of the particle is given by, $\mathbf{r}(t)=\langle t^2,5t,t^2-16t\rangle$.

State the formulae for differentiation.

$$\frac{d}{dt}\left(t^n\right)=nt^{n-1}$$

$$\frac{d}{dt}\left(\sqrt{f(t)}\right)=\frac{1}{2\sqrt{f(t)}}f'(t)$$

The velocity function $\mathbf{v}(t)=\mathbf{r}'(t)$.

$$\mathbf{v}(t)=\langle 2t,5,2t-16\rangle$$

Find the speed of the particle, $|\mathbf{v}(t)|$.

$$
\begin{aligned}
|\mathbf{v}(t)| &= \sqrt{(2t)^2 + (5)^2 + (2t-16)^2} \\
&= \sqrt{4t^2 + 25 + (2t-16)^2} \\
&= \sqrt{4t^2 + 25 + 4t^2 - 64t + 256} \\
&= \sqrt{8t^2 - 64t + 281}
\end{aligned}
$$

Find the minimum speed by equating the derivative of the speed is equal to zero.

$$
\begin{aligned}
\frac{d}{dt}|\mathbf{v}(t)| &= 0 \\
\frac{1}{2\sqrt{8t^2 - 64t + 281}}(16t - 64) &= 0 \\
16t - 64 &= 0 \\
t &= 4
\end{aligned}
$$

So, the speed of the particle is minimum when $\boxed{t = 4\,\text{sec}}$.

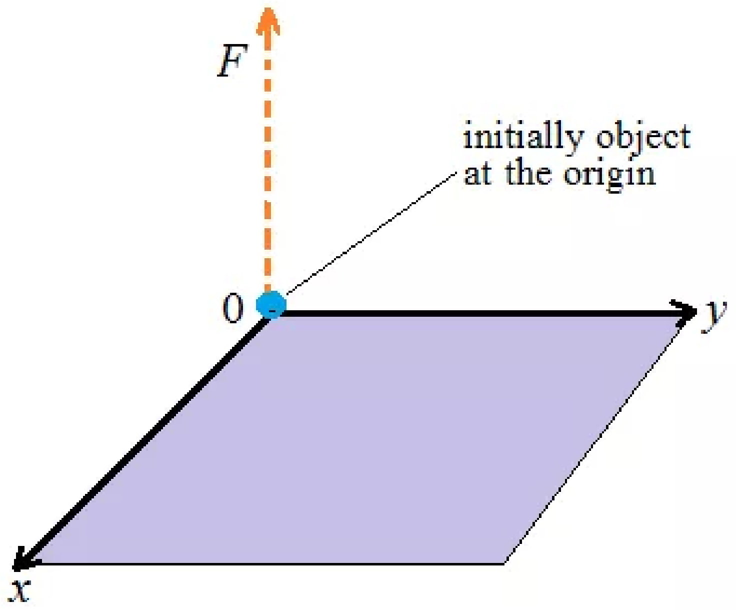
Answer 20E.

$$
\begin{aligned}
\vec{r}(t) &= t^3\hat{i} + t^2\hat{j} + t^3\hat{k} \\
\vec{v}(t) &= \vec{r}\,'(t) \\
&= 3t^2\hat{i} + 2t\hat{j} + 3t^2\hat{k} \\
\vec{a}(t) &= \vec{r}\,''(t) \\
&= 6t\hat{i} + 2\hat{j} + 6t\hat{k}
\end{aligned}
$$

$$
\begin{aligned}
\text{Therefore } \vec{F}(t) &= m\vec{a}(t) \\
&= m\left[6t\hat{i} + 2\hat{j} + 6t\hat{k}\right]
\end{aligned}
$$

Answer 21E.

Suppose that an object starts at the origin with initial velocity $\mathbf{v}(0)=\mathbf{i}-\mathbf{j}$ as shown below.



A force with magnitude 20 N acts directly upward from the $xy-$plane on an object with mass 4 kg. It is needed to find its position function and its speed at time t.

Acceleration of the object is defined as the derivative of the velocity:

$\vec{\mathbf{a}}=\vec{\mathbf{v}}'(t)$

Then $\vec{\mathbf{v}}(t)=\int \vec{\mathbf{a}}(t)\,dt$

$=\int(5\mathbf{k})\,dt$

$=5t\,\mathbf{k}+\vec{C}$

Where $\vec{C}$ is constant vector of integration

From the data, it is given that $\vec{\mathbf{v}}(0)=\mathbf{i}-\mathbf{j}$

Then $\mathbf{i}-\mathbf{j}=5(0)\mathbf{k}+\vec{C}$

$\vec{C}=\mathbf{i}-\mathbf{j}$

Therefore, $\vec{\mathbf{v}}(t)=\mathbf{i}-\mathbf{j}+5t\,\mathbf{k}$

Velocity of the object is defined as the derivative of the position vector:

$\vec{\mathbf{v}}(t)=\vec{\mathbf{r}}'(t)$

Then $\vec{\mathbf{r}}(t)=\int \vec{\mathbf{v}}(t)\,dt$

That is $\vec{\mathbf{r}}(t)=\int(\mathbf{i}-\mathbf{j}+5t\,\mathbf{k})\,dt$

$=t\,\mathbf{i}-t\,\mathbf{j}+\frac{5}{2}t^2\,\mathbf{k}+\vec{D}$

Where $\vec{D}$ is constant vector of integration.

Since the object starts at origin then

$\vec{\mathbf{r}}(0)=\vec{0}$

$\vec{0}=0\,\mathbf{i}-0\,\mathbf{j}+\frac{5}{2}(0^2)\mathbf{k}+\vec{D}$

$\vec{D}=\vec{0}$

Then $\vec{\mathbf{r}}(t)=t\,\mathbf{i}-t\,\mathbf{j}+\frac{5}{2}t^2\,\mathbf{k}$

Thus position function at time t is $\vec{\mathbf{r}}(t)=\boxed{t\,\mathbf{i}-t\,\mathbf{j}+\frac{5}{2}t^2\,\mathbf{k}}$

The speed of the object at time t is the magnitude of the velocity vector, that is $\left|\vec{\mathbf{v}}(t)\right|$

As $\vec{\mathbf{v}}(t)=\mathbf{i}-\mathbf{j}+5t\,\mathbf{k}$, speed of the object at time t is

$$\begin{aligned}\left|\vec{\mathbf{v}}(t)\right| &= \sqrt{(1)^2+(-1)^2+(5t)^2}\\ &= \sqrt{1+1+25t^2}\\ &= \sqrt{2+25t^2}\end{aligned}$$

Thus, speed of the object at time t is $\left|\vec{\mathbf{v}}(t)\right|=\boxed{\sqrt{2+25t^2}}$

Answer 22E.

Let the velocity of the particle is

$$\vec{v}(t) = \langle f(t), g(t), h(t) \rangle \quad \text{---------------} (1)$$

Then acceleration is

$$\vec{a}(t) = \langle f'(t), g'(t), h'(t) \rangle \quad \text{---------------} (2)$$

The speed of the particle is given by

$$\left|\vec{v}(t)\right| = \sqrt{f^2(t) + g^2(t) + h^2(t)}$$

It is given that $\left|\vec{v}(t)\right|$=constant

i.e. $\sqrt{f^2(t) + g^2(t) + h^2(t)}$=constant

i.e. $f^2(t) + g^2(t) + h^2(t) =$ constant

Differentiating both sides with respect to t

$$2f(t)f'(t) + 2g(t)g'(t) + 2h(t)h'(t) = 0$$

i.e. $f(t)f'(t) + g(t)g'(t) + h(t)h'(t) = 0$

i.e. $\langle f(t), g(t), h(t) \rangle . \langle f'(t), g'(t), h'(t) \rangle = 0$

i.e. $\vec{v}(t).\vec{a}(t) = 0$ (Because of (1) and (2))

i.e. $\vec{v}(t)$ and $\vec{a}(t)$ are orthogonal

That is if a particle moves with a constant speed then the velocity and acceleration vectors are orthogonal

Answer 23E.

The initial speed of a projectile is 200m/s.

The projectile is fired with an angle of elevation 60°.

(a)

The objective is to find the range of the projectile.

Find the range of the projectile, using formula for distance traveled.

$$d = \frac{v_0^2 \sin(2\alpha)}{g}$$

The initial velocity $v_0 = 200\text{m/s}$, $g = 9.8\text{m/s}^2$, and $\alpha = 60°$

Substitute the values v_0, g, and α in the above formula.

$$d = \frac{200^2 \times \sin(120°)}{9.8}$$
$$= \frac{200^2 \times \sin(90+30)}{9.8}$$
$$= \frac{40000 \times \cos(30°)}{9.8}$$
$$= \frac{40000 \times \frac{\sqrt{3}}{2}}{9.8}$$
$$= \frac{40000 \times \sqrt{3}}{2 \times 9.8}$$
$$= \frac{20000 \times \sqrt{3}}{9.8}$$
$$= \frac{34641.01615}{9.8}$$
$$= 3534.797$$
$$\approx 3535$$

Therefore, the range of the projectile is, $\boxed{3535\text{m}}$

(b)

Find the maximum height of the projectile, using parametric equations of the trajectory.

$$x = (v_0 \cos(\alpha))t,\ y = (v_0 \sin(\alpha))t - \frac{1}{2}gt^2$$

Substitute the values of $v_0, g,$ and α in the parametric equations.

$$\begin{aligned} x &= (200\cos(60°))t \\ &= 200 \cdot \frac{1}{2}t \\ &= 100t \end{aligned}$$

$$\begin{aligned} y &= (200\sin(60°))t - \frac{9.8}{2}t^2 \\ &= \frac{\sqrt{3}\times 200}{2}t - \frac{9.8}{2}t^2 \end{aligned}$$

Write the vector form.

$$\mathbf{r}(t) = 100t\mathbf{i} + \left(\frac{\sqrt{3}\times 200}{2}t - \frac{9.8}{2}t^2\right)\mathbf{j}$$

Find the maximum height of the projectile reached.

Observe that this height will be critical point for the graph of vertical motion.

So, need to derivative of y with respect to t.

$$\begin{aligned} \frac{dy}{dt} &= \frac{d}{dt}\left(\frac{\sqrt{3}\times 200}{2}t - \frac{9.8}{2}t^2\right) \\ &= \frac{\sqrt{3}\times 200}{2} - 9.8t \end{aligned}$$

Now find $\frac{dy}{dt} = 0$

$$\begin{aligned} \frac{\sqrt{3}\times 200}{2} - 9.8t &= 0 \\ 9.8t &= \frac{\sqrt{3}\times 200}{2} \\ t &= \frac{\sqrt{3}\times 200}{19.6} \end{aligned}$$

Substitute the value of t into the formula for vertical position.

$$y = \frac{\sqrt{3}\times 200}{2}t - \frac{9.8}{2}t^2$$
$$= \frac{\sqrt{3}\times 200}{2}\left(\frac{\sqrt{3}\times 200}{19.6}\right) - \frac{9.8}{2}\left(\frac{\sqrt{3}\times 200}{19.6}\right)^2$$
$$= \frac{3\times 40,000}{39.2} - \frac{9.8}{2}\left(\frac{3\times 40,000}{384.16}\right)$$
$$= \frac{3\times 40,000}{39.2} - 9.8\left(\frac{3\times 40,000}{768.32}\right)$$
$$= \frac{120,000}{39.2} - 9.8\left(\frac{120,000}{768.32}\right)$$
$$= 3061.2 - 1530.6$$
$$= 1530.612245$$
$$\approx 1531$$

Therefore, the maximum height of the projectile reached is, $\boxed{1531\text{m}}$.

(c)

Find the speed of the particle at impact.

First solve for the time at which the projectile impacts the ground.

The range of the projectile is 3535 meters.

Find that at time of impact:

$$x = 100t$$
$$3535 = 100t$$
$$35.35 \approx t$$

Solve for impact velocity:

The velocity of the particle is,

$$\mathbf{v}(t) = \mathbf{r}'(t)$$

The vector $\mathbf{r}(t)=100t\mathbf{i}+\left(\frac{\sqrt{3}\times 200}{2}t-\frac{9.8}{2}t^2\right)\mathbf{j}$

Differentiate the vector $\mathbf{r}(t)$ with respect to t.

$$\mathbf{v}(t)=100\mathbf{i}+\left(\frac{\sqrt{3}\times 200}{2}-9.8t\right)\mathbf{j}$$
$$=100\mathbf{i}+\left(\frac{\sqrt{3}\times 200}{2}-9.8(35.35)\right)\mathbf{j}$$
$$=100\mathbf{i}+\left(100\sqrt{3}-346.430\right)\mathbf{j}$$
$$=100\mathbf{i}+\left(173.21-346.430\right)\mathbf{j}$$
$$=100\mathbf{i}-173.220\mathbf{j}$$

Hence, the impact velocity is, $\mathbf{v}(t)\approx 100\mathbf{i}-173.220\mathbf{j}$

Find the speed of the particle, using the formula

Impact speed is just the magnitude of impact velocity,

$$|\mathbf{v}(t)|=\sqrt{(100)^2+(-173.200)^2}$$
$$=199.996$$
$$\approx 200$$

Therefore, the speed of the impact is, $\boxed{200\text{m/s}}$

Answer 24E.

(a)

Find the range of the projectile.

The parametric equations of the trajectory of a projectile are

$$x=(v_0\cos\alpha)t \quad y=(v_0\sin\alpha)t-\frac{1}{2}gt^2$$

Here v_0 is the initial speed of the projectile, is the angle of elevation, and $g\approx 9.8\ \text{m/s}^2$ is the acceleration due to gravity.

If we place the origin at the ground, then the initial position of $(0,100)$ and so we add 100 to the expression for y. Use $v_0 = 200$, and $\alpha = 60°$.

$$\begin{aligned} x &= (200\cos 60°)t \\ &= 100t \end{aligned}$$

And

$$\begin{aligned} y &= 100 + (200\sin 60°)t - \frac{1}{2}(9.8)t^2 \\ &= 100 + 100t\sqrt{3} - 4.9t^2 \end{aligned}$$

The range is the value of x when $y = 0$, that is, when $4.9t^2 - 100t\sqrt{3} - 100 = 0$.

Solving the quadratic equation, to get

$$\begin{aligned} t &= \frac{100\sqrt{3} + \sqrt{30,000 + 1960}}{9.8} \\ &\approx 35.92 \end{aligned}$$

Then

$$\begin{aligned} x &= 100(35.92) \\ &= 3592 \end{aligned}$$

Therefore, the range of the projectile is $\boxed{3592 \text{ m}}$.

(b)

Find the maximum height reached.

The velocity of the projectile is

$$\begin{aligned}\mathbf{v}(t) &= \mathbf{r}'(t) \\ &= 100\mathbf{i} + \left(100\sqrt{3} - 9.8t\right)\mathbf{j}\end{aligned}$$

The maximum height is reached when the y-coordinate of v is 0.

$$100\sqrt{3} - 9.8t = 0$$

Solving this equation for t, to get

$$\begin{aligned}t &= \frac{100\sqrt{3}}{9.8} \\ &\approx 17.67\end{aligned}$$

Then

$$\begin{aligned}y &= 100 + 100(17.67)\sqrt{3} - 4.9(17.67)^2 \\ &= 1631\end{aligned}$$

Therefore, the maximum height reached is $\boxed{1631\ \text{m}}$.

(c)

Find the speed at impact.

The speed of impact occurs when $t \approx 35.92$.

$$\begin{aligned}|\mathbf{v}(t)| &= \sqrt{(100)^2 + \left[100\sqrt{3} - 9.8(35.62)\right]^2} \\ &\approx 532\end{aligned}$$

Therefore, the speed at impact is $\boxed{532\ \text{m/s}}$.

Answer 25E.

If the ball is projected with speed v_0 making an angle α with ground, then we know the maximum range of the ball is given by

$$R = \frac{2v_0^2 \sin\alpha \cos\alpha}{g}$$

Here $v_0 = ?$, $R = 90$ m, $g = 9.8\, m/s^2$, $\alpha = 45°$

Then $$90 = \frac{2v_0^2 \sin 45° \cos 45°}{9.8}$$

i.e. $$v_0^2 = \frac{90 \times 9.8}{2 \sin 45° \cos 45°}$$

i.e. $v_0^2 = 882$

i.e. $v_0 = 29.69$

That is initial speed of the ball is

$$v_0 \approx 30\, m/s$$

Answer 26E.

Consider the following data:

A gun is fired with angle of elevation 30^0.

The maximum height of the shell is 500 m.

The objective is to find the muzzle speed.

The position vector of the projectile is,

$$\mathbf{r}(t) = (v_0 \cos\alpha)t\,\mathbf{i} + \left[(v_0 \sin\alpha)t - \frac{1}{2}gt^2\right]\mathbf{j}$$

And, the parametric equations of trajectory are,

$$x = (v_0 \cos\alpha)t$$
$$y = (v_0 \sin\alpha)t - \frac{1}{2}gt^2$$

Here,

$$\alpha = 30°$$
$$h_{max} = 500\,\text{m}$$
$$g = 9.8\,\text{m/s}^2$$

Substitute these values in the above parametric equations to get,

$$\begin{aligned} x &= v_0 \cos\alpha \\ &= v_0 \cos 30° \\ &= \frac{\sqrt{3}}{2} v_0 \end{aligned}$$

And,

$$\begin{aligned} y &= (v_0 \sin\alpha)t - \frac{1}{2}gt^2 \\ &= (v_0 \sin 30°)t - \frac{1}{2}(9.8)t^2 \\ &= \frac{1}{2}v_0 t - 4.9t^2 \end{aligned}$$

Differentiate y with respect to t to get,

$$\frac{dy}{dt} = \frac{1}{2}v_0 - 9.8t$$

For maxima/minima, set $\frac{dy}{dt} = 0.$

$$\frac{1}{2}v_0 - 9.8t = 0$$

$$9.8t = \frac{1}{2}v_0$$

$$t = \frac{v_0}{19.6}$$

Since $h_{max} = 500\,\text{m}$, then

$$\frac{1}{2}v_0 t - 4.9t^2 = 500$$

Substitute $t = \frac{v_0}{19.6}$ in the above equation to get,

$$\frac{1}{2}v_0\left(\frac{v_0}{19.6}\right) - 4.9\left(\frac{v_0}{19.6}\right)^2 = 500$$

$${v_0}^2\left(\frac{1}{2\times 19.6} - \frac{4.9}{(19.6)^2}\right) = 500$$

$${v_0}^2\left(\frac{5}{392}\right) = 500$$

$${v_0}^2 = \frac{500\times 392}{5}$$

$${v_0}^2 = 39200$$

$$v_0 = \sqrt{39200}$$

$$v_0 \approx 198$$

Thus, the muzzle speed is $v_0 \approx \boxed{198\ \text{m/s}}$.

Answer 27E.

We know if the projectile is projected with initial speed v_0 making an angle α with horizontal then the range is given by

$$R = \frac{v_0^2 \sin 2\alpha}{g}$$

Here $R = 800 \text{ m}$, $v_0 = 150 \, m/s$, $g = 9.8 \, m/s^2$, $\alpha = ?$

Then $$800 = \frac{(150)^2 \sin 2\alpha}{9.8}$$

i.e. $$\sin 2\alpha = \frac{800 \times 9.8}{(150)^2}$$

i.e. $\sin 2\alpha = 0.348$

i.e. $2\alpha = \sin^{-1}(0.348) = 20.39$

i.e. $\alpha = 10.19$

We know $\sin 2(90 - \alpha) = \sin(180° - 2\alpha)$

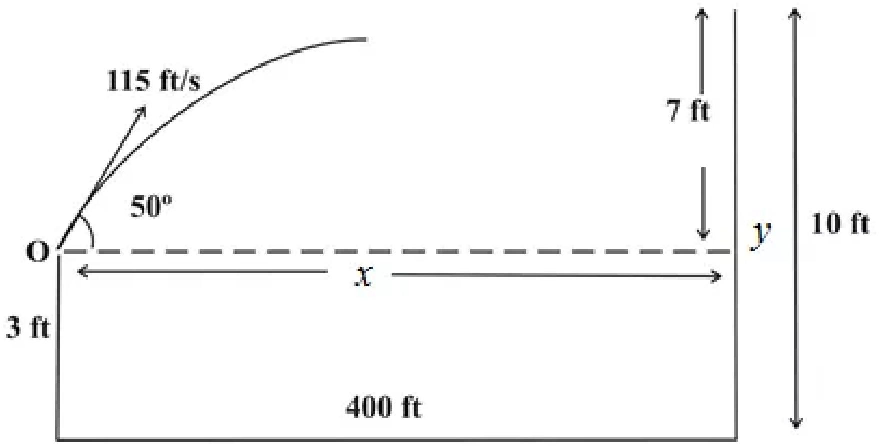
$= \sin 2\alpha$

Therefore the two required angles of elevation are 10.2° and 90° - 10.2°

That is 10.2° and 79.8°

Answer 28E.

Let the *x*-coordinate represent the distance from home plate, while the *y*-coordinate denotes height above the ground is shown in below figure



Use the foot as the unit of distance and the second as the unit of time. The initial position is then

$$\mathbf{r}_0 = x_0\mathbf{i} + y_0\mathbf{j}.$$

The initial velocity $\mathbf{v}_0$ is given in terms of the speed $|\mathbf{v}_0| = 115$ ft/s and the angle $\theta = 50°$ above the horizontal. Thus

$$\begin{aligned}\mathbf{v}_0 &= |\mathbf{v}_0|(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) \\ &= 115(\cos 50°\mathbf{i} + \sin 50°\mathbf{j}).\end{aligned}$$

The acceleration is $\mathbf{a} = -g\mathbf{j},$ where g is the strength of the gravitational field, measured in ft/s^2. In these units, $g \approx\sim 32.174.$

Using the formula for motion with constant acceleration:

$$\begin{aligned}\mathbf{r}(t) &= \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{a}t^2 \\ &= 3\mathbf{j} + 115(\cos 50°\mathbf{i} + \sin 50°\mathbf{j})t + \frac{1}{2}(-32.174\mathbf{j})t^2 \\ &= 3\mathbf{j} + 115t\cos 50°\mathbf{i} + 115t\sin 50°\mathbf{j} - 16.087t^2\mathbf{j} \\ &= (115t\cos 50°)\mathbf{i} + (115t\sin 50° - 16.087t^2 + 3)\mathbf{j}\end{aligned}$$

Where $x(t) = 115t\cos 50°$ and $y(t) = 115t\sin 50° - 16.087t^2 + 3$

The question of whether this trajectory clears the fence can be rephrased as the question of whether the ball is above the level $y = 10$ top of the fence when the ball reaches the fence at $x = 400.$

Solving the equation $x = 400$ yield the time when the ball reaches the fence:

$$\begin{aligned}x(t) &= 115t\cos 50° \\ t &= \frac{x(t)}{115\cos 50°} \\ t &= \frac{400}{115\cos 50°} \\ t &\approx 5.4\end{aligned}$$

Substituting this into the equation for the height $y(t)$:

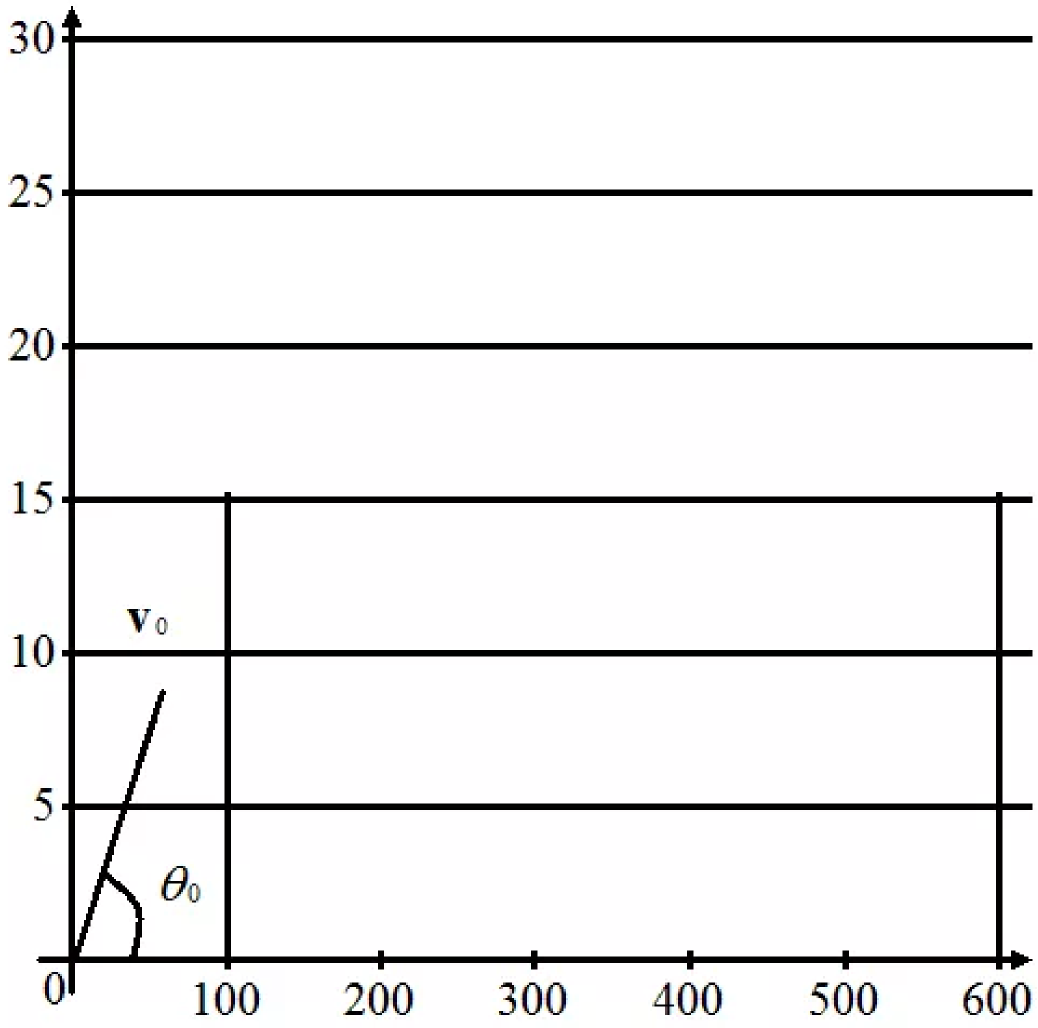
$$\begin{aligned} y(5.4) &= 115t\sin 50° - 16.087t^2 + 3 \\ &= 115(5.4)\sin 50° - 16.087(5.4)^2 + 3 \\ &= 115(5.4)(0.766) - 16.087(5.4)^2 + 3 \\ &\approx 9.62 \end{aligned}$$

Because $9.62 < 10$, the ball does not clear the fence .

Answer 29E.

Consider a projectile is fired with angle of elevation α_0 and initial velocity $\mathbf{v}_0$

First draw the picture of the situation and various different firing angles.



The equation of motion for this problem,

$$x = x_0 + v_0 t\cos(\theta_0)$$

$$y = y_0 + v_0 t\sin(\theta_0) - \frac{1}{2}gt^2$$

Where $x_0 = 0, y_0 = 0, \mathbf{v}_0 = 80\text{m/s}$, g is the gravitational acceleration is 9.8m/s^2

$$x = x_0 + v_0 t\cos(\theta_0)$$

$$x = 0 + v_0 t\cos(\theta_0)$$

$$t = \frac{x_0}{\mathbf{v}_0\cos(\theta_0)}$$

$$y = y_0 + v_0 t\sin(\theta_0) - \frac{1}{2}gt^2$$

$$y = 0 + v_0\left(\frac{x_0}{\mathbf{v}_0\cos(\theta_0)}\right)\sin(\theta_0) - \frac{1}{2}g\left(\frac{x_0}{\mathbf{v}_0\cos(\theta_0)}\right)^2$$

To solve the value of y

$$y = \frac{x_0\sin(\theta_0)}{\cos(\theta_0)} - \frac{gx^2}{2\mathbf{v}_0\cos^2(\theta_0)}$$

$$= x\tan(\theta_0) - \frac{gx^2}{2\mathbf{v}_0^{\ 2}}\sec^2(\theta_0)$$

$$= x\tan(\theta_0) - \frac{gx^2}{2\mathbf{v}_0^{\ 2}}\left(1 + \tan^2(\theta_0)\right)$$

$$= x\tan(\theta_0) - \frac{gx^2}{2\mathbf{v}_0^{\ 2}} - \frac{gx^2}{2\mathbf{v}_0^{\ 2}}\tan^2(\theta_0)$$

$$2\mathbf{v}_0^{\ 2}y = x2\mathbf{v}_0^{\ 2}\tan(\theta_0) - gx^2 - gx^2\tan^2(\theta_0)$$

$$2\mathbf{v}_0^{\ 2}y - x2\mathbf{v}_0^{\ 2}\tan(\theta_0) + gx^2 + gx^2\tan^2(\theta_0) = 0$$

$$\frac{2\mathbf{v}_0^{\ 2}y - x2\mathbf{v}_0^{\ 2}\tan(\theta_0) + gx^2 + gx^2\tan^2(\theta_0) = 0}{gx^2}$$

$$\frac{2\mathbf{v}_0^{\ 2}y}{gx^2} - \frac{x2\mathbf{v}_0^{\ 2}}{gx^2}\tan(\theta_0) + 1 + \tan^2(\theta_0) = 0$$

$$\tan^2(\theta_0) - \frac{x2\mathbf{v}_0^{\ 2}}{gx^2}\tan(\theta_0) + \left(1 + \frac{2\mathbf{v}_0^{\ 2}y}{gx^2}\right) = 0$$

By use the quadratic formula:

$$\tan(\theta_0) = \left(\frac{v^2_0}{gx}\right) \pm \frac{1}{2}\sqrt{\left(-\frac{2v^2_0}{gx}\right)^2 - 4\left(1+\frac{2v^2_0}{gx^2}\right)}$$

$$\theta_0 = \tan^{-1}\left[\left(\frac{v^2_0}{gx}\right) \pm \frac{1}{2}\sqrt{\left(-\frac{2v^2_0}{gx}\right)^2 - 4\left(1+\frac{2v^2_0}{gx^2}\right)}\right]$$

To set fire to the city by catapulting heated rocks over the wall, so the following (x, y) pairs $(100m, 15m)$ and the commander of an attacking army and the closets then get to the wall is 100m.

So the (x, y) pairs $((500+100)m, 0) = (600m, 0)$, and $v_0 = 80\text{m/s}$

Therefore,

$\theta_0 = \{12.99^0, 85.54^0\}$ at $(x, y) = (100m, 15m)$

$\theta_0 = \{37.37^0, 56.63^0\}$ at $(x, y) = (600m, 0m)$

So, the range of angles should $\boxed{12.99^0 \le \theta \le 37.37^0}$ or $\boxed{56.63^0 \le \theta_0 \le 85.54^0}$.

Answer 30E.

In the case of a projectile, the equation for y-direction motion are given by $y = y_0 + u_y t - \frac{1}{2}gt^2$ and $v_y = u\sin\theta - gt$.

At $t = 0$, we get $y = 0$ and $u_y = u\sin\theta$.

Then, we get $y = u\sin\theta - \frac{1}{2}gt^2$.

The maximum height reached is given by $H = \frac{u^2\sin^2\theta}{2g}$. But, at maximum height $v_y = 0$.

Then, we get $0 = u\sin\theta - gt$ or $t = \frac{u\sin\theta}{g}$. The time taken to reach the maximum height is $t = \frac{u\sin\theta}{g}$.

Now, we have to find the time taken when $H' = \frac{3}{4}H$. We then get the maximum height reached as $H' = \frac{3u^2\sin^2\theta}{8g}$. Thus, $\frac{3u^2\sin^2\theta}{8g} = u\sin\theta - \frac{1}{2}gt^2$.

Use the quadratic formula to find the roots of t.

$$t = \frac{u\sin\theta \pm \sqrt{u^2\sin^2\theta - \frac{3}{4}u^2\sin^2\theta}}{g}$$

$$= \frac{u\sin\theta \pm \frac{1}{2}u\sin\theta}{g}$$

$$= \frac{3u\sin\theta}{2g}, \frac{u\sin\theta}{2g}$$

Since $H' < H$, the only possible value for t is $\frac{u\sin\theta}{2g}$.

Therefore, we can say that the projectile needs only half the time to reach its maximum height.

Answer 31E.

Consider the acceleration vector:

$$\mathbf{a}(t) = -4\mathbf{j} - 32\mathbf{k}$$

Given that the initial velocity is $\mathbf{v}(0) = 50\mathbf{i} + 80\mathbf{k}$

To find the velocity, Integrate $\mathbf{a}(t)$ with respect to 't'we get

$$\begin{aligned}\mathbf{v}(t) &= \int \mathbf{a}(t)dt \\ &= \int(-4\mathbf{j} - 32\mathbf{k})dt \\ &= -4t\mathbf{j} - 32t\mathbf{k} + \mathbf{C}\end{aligned}$$

To find the constant **C**, use the fact that the initial velocity is $\mathbf{v}(0) = 50\mathbf{i} + 80\mathbf{k}$.

Then the equation becomes,

$$\begin{aligned}\mathbf{v}(0) &= \mathbf{C} \\ 50\mathbf{i} + 80\mathbf{k} &= \mathbf{C}\end{aligned}$$

Therefore, $\mathbf{C} = 50\mathbf{i} + 80\mathbf{k}$.

Now substitute $\mathbf{C} = 50\mathbf{i} + 80\mathbf{k}$ in $\mathbf{v}(t) = -4t\mathbf{j} - 32t\mathbf{k} + \mathbf{C}$

Then, the velocity vector is:

$$\begin{aligned}\mathbf{v}(t) &= -4t\mathbf{j} - 32t\mathbf{k} + 50\mathbf{i} + 80\mathbf{k} \\ &= 50\mathbf{i} - 4t\mathbf{j} + (-32t + 80)\mathbf{k}\end{aligned}$$

Since $\mathbf{v}(t) = \mathbf{r}'(t)$.

To find the position function $\mathbf{r}(t)$, integrate $\mathbf{v}(t)$ with respect to 't'

$$\begin{aligned}\mathbf{r}(t) &= \int \mathbf{v}(t)dt \\ &= \int \left[50\mathbf{i} - 4t\mathbf{j} + (-32t + 80)\mathbf{k}\right]dt \\ &= 50t\mathbf{i} - 2t^2\mathbf{j} + \left(-16t^2 + 80t\right)\mathbf{k} + \mathbf{D}\end{aligned}$$

The initial position of the ball is 0, that is, $\mathbf{r}(0) = 0$

So,

$$\begin{aligned}\mathbf{r}(0) &= 0 + \mathbf{D} \\ 0 &= \mathbf{D} \\ \mathbf{D} &= \mathbf{0}\end{aligned}$$

Thus, the position vector is $\mathbf{r}(t) = 50t\mathbf{i} - 2t^2\mathbf{j} + \left(-16t^2 + 80t\right)\mathbf{k}$.

The ball lands when the value of the *z*-component ($-16t^2 + 80t$) is 0, which occurs when $t = 5$. Thus, the ball lands at the position $\mathbf{r}(5)$.

$$\begin{aligned}\mathbf{r}(5) &= 50(5)\mathbf{i} - 2(5)^2\mathbf{j} + \left[-16(5)^2 + 80(5)\right]\mathbf{k} \\ &= 250\mathbf{i} - 50\mathbf{j}\end{aligned}$$

The ball lands at the point $\boxed{(250, -50, 0)}$

Now find the speed of the ball. That is $\left|\mathbf{v}(t)\right|_{t=5}$

$$\begin{aligned}\mathbf{v}(t) &= \left|50\mathbf{i} - 4t\mathbf{j} + (-32t + 80)\mathbf{k}\right| \\ \mathbf{v}(5) &= \left|50\mathbf{i} - 4(5)\mathbf{j} + (-32(5) + 80)\mathbf{k}\right| \\ &= \left|50\mathbf{i} - 20\mathbf{j} - 80\mathbf{k}\right| \\ &= \sqrt{2500 + 400 + 6400} \\ &= \sqrt{9300} \\ &= \boxed{10\sqrt{93}\text{ ft/s}}\end{aligned}$$

Thus, the ball lands at about the point $\boxed{(250, -50, 0)}$ with a speed of approximately $\boxed{10\sqrt{93}\text{ ft/s}}$.

Answer 33E.

(a)

Consider the function $f(x)=\frac{3}{400}x(40-x)$

Let the point A corresponds to the origin in R^2,and the position of the boat a time 0 is $\mathbf{r}(0)=0$.

The velocity of the boat at any time t.

Since the boat is proceeds at a constant speed of 5 m/s from a point A

This means that after t seconds the $x-$coordinate of the boat is $5t$

Therefore the time t seconds the velocity of the water at the point,

$$\begin{aligned}f(5t)&=\frac{3}{400}(5t)(40-5t)\\&=\frac{3}{2}t-\frac{3}{16}t^2\end{aligned}$$

And the velocity of the boat at time t is,

$$\mathbf{v}(t)=5\mathbf{i}+\left(\frac{3}{2}t-\frac{3}{16}t^2\right)\mathbf{j}$$

(a)

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And the velocity of the boat at time t is,

$$\mathbf{v}(t)=5\mathbf{i}+\left(\frac{3}{2}t-\frac{3}{16}t^2\right)\mathbf{j}$$

The position vector of the boat by integration.

$$\int \mathbf{v}(t) = \int\left[5\mathbf{i}+\left(\frac{3}{2}t-\frac{3}{16}t^2\right)\mathbf{j}\right]dt$$

$$\mathbf{r}(t) = 5t\mathbf{i}+\left(\frac{3}{4}t^2-\frac{1}{16}t^3\right)\mathbf{j}+C$$

But $C=0$ because $\mathbf{r}(0)=0$

The boat reaches the opposite shore when $x=40$ then $t=8$ $\left(\text{Since, } \frac{40}{5}\right)$

$$\mathbf{r}(8)=5(8)\mathbf{i}+\left(\frac{3}{4}(8)^2-\frac{1}{16}(8)^3\right)\mathbf{j}$$

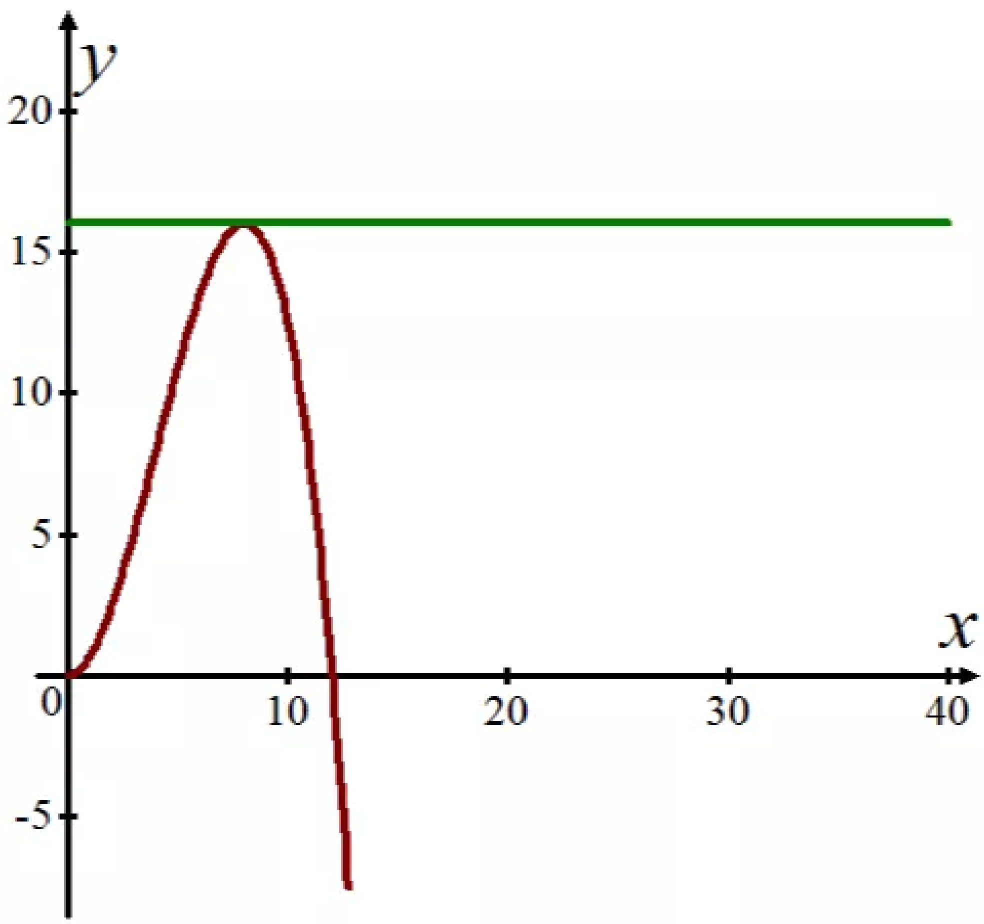
$$=40\mathbf{i}+16\mathbf{j}$$

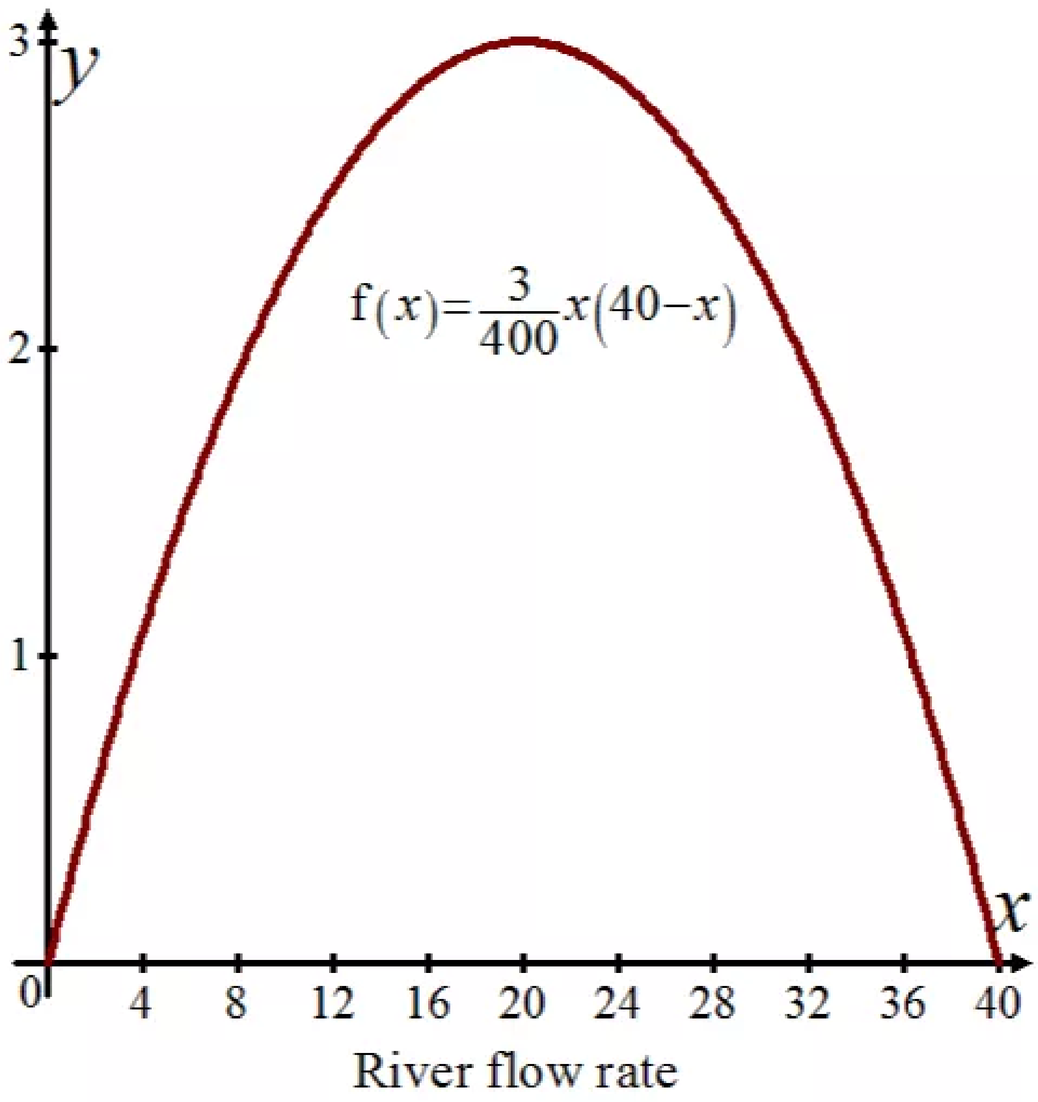
Therefore, the boat ends up 16 meter down the shore from its started.

The plot of,

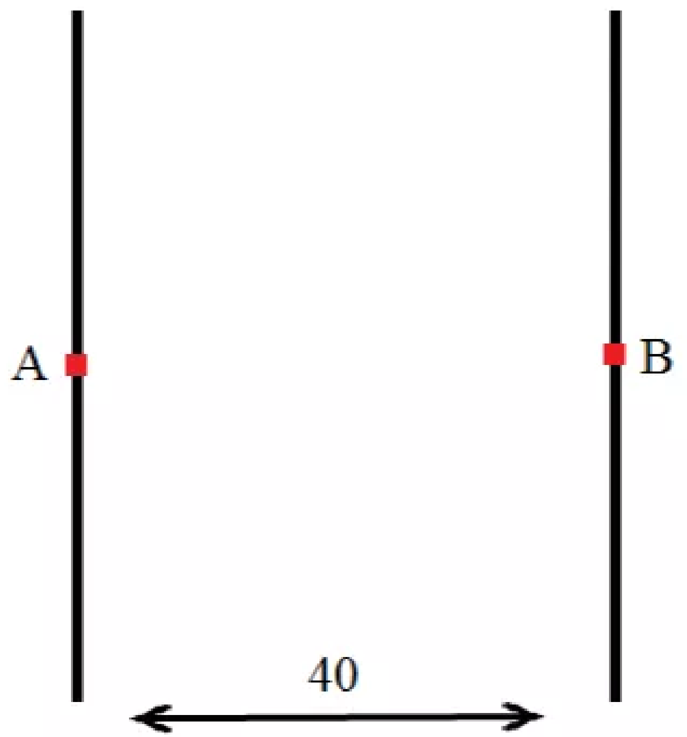
$$x=5t,\ y=\frac{3}{4}t^2-\frac{1}{16}t^3,\quad 0\le \text{t} \le 8$$

The path of motion of the boat is shown below and 16 m far down the river on the opposite bank will the boat touch shore.





y
3
2
1
0
$f(x)=\frac{3}{400}x(40-x)$
x
4
8
12
16
20
24
28
32
36
40
River flow rate



A
B
40

(b)

The pilot the boat to land at the point B on the east bank directly opposite A.

If maintain a constant speed of 5m/s.

The velocity supplied by the boat's engine is $5\cos(\alpha)\mathbf{i}+5\sin(\alpha)\mathbf{j}$

Where α is the direction and $-90^0<\alpha<0^0$, the end up at point B if $\alpha\geq 0^0$.

The $x-$component of the velocity is $5\cos(\alpha)$ meters per second, the $x-$coordinate of the boat at time t will be $5\cos(\alpha)t$.

Which means the velocity supplied by the river itself at this time,

$$\begin{aligned} f(5\cos(\alpha)t) &= \frac{3}{400}(5\cos(\alpha)t)(40-5\cos(\alpha)t) \\ &= \frac{3}{2}\cos(\alpha)t-\frac{3}{16}\cos^2(\alpha)t^2 \end{aligned}$$

The velocity vector of the boat at time t is,

$$\mathbf{v}(t)=5\cos(\alpha)\mathbf{i}+\left(5\sin(\alpha)+\frac{3}{2}\cos(\alpha)t-\frac{3}{16}\cos^2(\alpha)t^2\right)\mathbf{j}$$

The position vector of the boat by integration.

$$\mathbf{r}(t)=5\cos(\alpha)t\mathbf{i}+\left(5\sin(\alpha)t+\frac{3}{4}\cos(\alpha)t^2-\frac{1}{16}\cos^2(\alpha)t^3\right)\mathbf{j}+C$$

But $C=0$ because $\mathbf{r}(0)=0$.

The boat will reach point B when $x=40$ and $y=0$.

$$\begin{aligned} 5\cos(\alpha)t &= 40 \\ t &= \frac{8}{\cos(\alpha)} \end{aligned}$$

And

$$5\sin(\alpha)\left(\frac{8}{\cos(\alpha)}\right)+\frac{3}{4}\cos(\alpha)\left(\frac{8}{\cos(\alpha)}\right)^2-\frac{1}{16}\cos^2(\alpha)\left(\frac{8}{\cos(\alpha)}\right)^3=0$$

$$\begin{aligned} 640\sin(\alpha)+256 &= 0 \\ \sin(\alpha) &= -\frac{2}{5} \\ \alpha &\approx -23.58^0 \end{aligned}$$

The boat must maintain a upstream of about 23.58^0 at the point B

But, $\sin(\alpha) = -\frac{2}{5}$ so $\cos(\alpha) = \frac{\sqrt{21}}{5}$

But, $t = \frac{8}{\cos(\alpha)}$

$$t = \frac{8}{\left(\frac{\sqrt{25}}{5}\right)}$$

$$\approx 8.73 \text{ seconds}$$

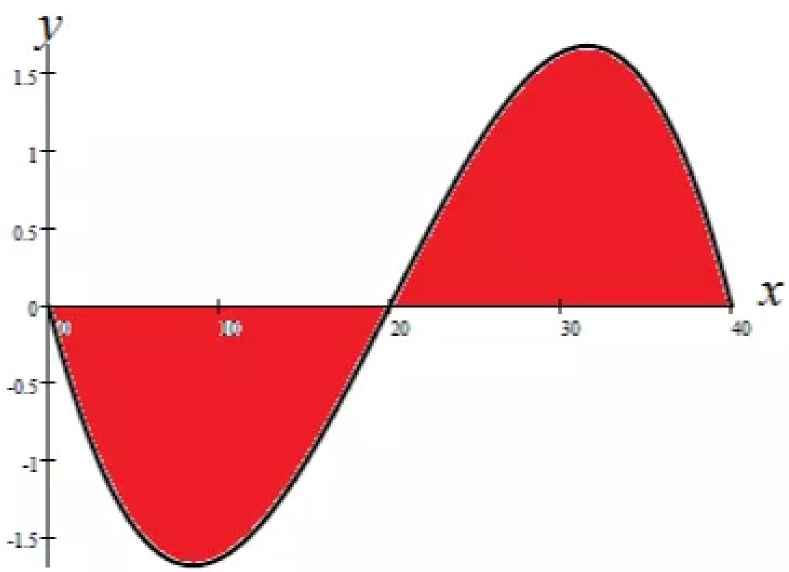
Therefore,

$$\mathbf{r}(t) = \sqrt{21}t\mathbf{i} + \left(-2t + \frac{3\sqrt{21}}{20}t^2 - \frac{21}{400}t^3\right)\mathbf{j}$$

The boat's path by the parametric curve,

$$x = \sqrt{21}t, y = \left(-2t + \frac{3\sqrt{21}}{20}t^2 - \frac{21}{400}t^3\right), 0 \le t \le 8.73$$

The path of motion is shown below.



Answer 34E.

Let α be the angle north of east that the boat heads, so the velocity of the boat in still water is given by $5(\cos\alpha)i + 5(\sin\alpha)j$. At t seconds, the boat is $5(\cos\alpha)t$ meters from the west bank, at which point the velocity of the water is

$$3\sin\left(\frac{\pi x}{40}\right)j = 3\sin\left[\pi\cdot\left(\frac{5(\cos\alpha)t}{40}\right)\right]j = 3\sin\left(\frac{\pi}{8}t\cos\alpha\right)j$$

The resultant velocity of the boat then is given by

$v(t) = 5(\cos\alpha)i + \left[5\sin\alpha + 3\sin\left(\frac{\pi}{8}t\cos\alpha\right)\right]j$. Integrating, we get

$$r(t) = (5t\cos\alpha)i + \left[5t\sin\alpha - \frac{24}{\pi\cos\alpha}\cos\left(\frac{\pi}{8}t\cos\alpha\right)\right]j + C$$

If we place the origin at A then

$$r(0) = 0 \Rightarrow -\frac{24}{\pi\cos\alpha}j + C = 0 \Rightarrow C = \frac{24}{\pi\cos\alpha}j \text{ and}$$

$$r(t) = (5t\cos\alpha)i + \left[5t\sin\alpha - \frac{24}{\pi\cos\alpha}\cos\left(\frac{\pi}{8}t\cos\alpha\right) + \frac{24}{\pi\cos\alpha}\right]j$$

The boat will reach the east bank when $5t\cos\alpha = 40 \Rightarrow t = \frac{8}{\cos\alpha}$. In order to land at point B(40,0) we need $5t\sin\alpha - \frac{24}{\pi\cos\alpha}\cos\left(\frac{\pi}{8}t\cos\alpha\right) + \frac{24}{\pi\cos\alpha} = 0$.

$$\Rightarrow 5\left(\frac{8}{\cos\alpha}\right)\sin\alpha - \frac{24}{\pi\cos\alpha}\cos\left[\frac{\pi}{8}\left(\frac{8}{\cos\alpha}\right)\cos\alpha\right] + \frac{24}{\pi\cos\alpha} = 0$$

$$\Rightarrow \frac{1}{\cos\alpha}\left(40\sin\alpha - \frac{24}{\pi}\cos\pi + \frac{24}{\pi}\right) = 0 \Rightarrow 40\sin\alpha + \frac{48}{\pi} = 0 \Rightarrow \sin\alpha = -\frac{6}{5\pi}$$

Therefore, $\alpha = \sin^{-1}\left(-\frac{6}{5\pi}\right) \approx -22.5^\circ$ south of east.

Answer 35E.

It is given that $\mathbf{r}'(t) = c\times\mathbf{r}(t)$. Then, we get $\mathbf{r}'(t)\cdot\mathbf{r}(t) = c\times\mathbf{r}(t)\cdot\mathbf{r}(t)$ or $\mathbf{r}'(t)\cdot\mathbf{r}(t) = 0$. We note that the dot product of $\mathbf{r}'(t)$ and $\mathbf{r}(t)$ is zero. Thus, we can say that position vector is perpendicular to the velocity vector. Also, we can say that $\mathbf{r}'(t)$ is perpendicular to the plane containing c and $\mathbf{r}(t)$.

Answer 36E.

(a) If a particle moves along a straight line, then **r**(*t*) is a linear function of *t*.

We know that **a**(*t*) is obtained by differentiating **r**(*t*) twice with respect of *t*.

Since **r**(*t*) is a linear function of *t*, we can say that $\mathbf{r}''(t)$ is zero.

Thus, the acceleration vector is zero.

(b) If the particle moves with a constant speed, then we cannot make any conclusions about the acceleration.

This is because the acceleration of a particle dependents on its velocity and not speed.

Answer 37E.

$$\vec{r}(t) = \left(3t - t^3\right)\hat{i} + 3t^2\hat{j}$$
$$\vec{r}\,'(t) = \left(3 - 3t^2\right)\hat{i} + 6t\,\hat{j}$$
$$\vec{r}\,''(t) = -6t\hat{i} + 6\hat{j}$$

$$\begin{aligned}\left|\vec{r}\,'(t)\right| &= \sqrt{\left(3-3t^2\right)^2 + (6t)^2} \\ &= \sqrt{9 + 9t^4 - 18t^2 + 36t^2} \\ &= \sqrt{9t^4 + 18t^2 + 9} \\ &= \sqrt{\left(3t^2+3\right)^2} = 3t^2 + 3\end{aligned}$$

The tangential component of acceleration vector is

$$\begin{aligned} a_T &= \frac{\vec{r}\,'(t)\vec{r}\,''(t)}{\left|\vec{r}\,'(t)\right|} \\ &= \frac{\left(3-3t^2\right)(-6t) + (6t)6}{3t^2+3} \\ &= \frac{-18t + 18t^3 + 36t}{3t^2+3} \\ &= \frac{18t^3 + 18t}{3t^2+3} \\ &= \frac{6t\left(3t^2+3\right)}{3t^2+3} \\ &= 6t \end{aligned}$$

Now $\vec{r}'(t)\times\vec{r}''(t)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3-3t^2 & 6t & 0\\ -6t & 6 & 0\end{vmatrix}$

$$=\left[\left(3-3t^2\right)6+36t^2\right]\hat{k}$$
$$=\left[18-18t^2+36t^2\right]\hat{k}$$
$$=\left(18t^2+18\right)\hat{k}$$

$$\left|\vec{r}'(t)\times\vec{r}''(t)\right|=\sqrt{\left(18t^2+18\right)^2}$$
$$=18t^2+18$$

The normal component of acceleration vector is

$$a_N=\frac{\left|\vec{r}'(t)\times\vec{r}''(t)\right|}{\left|\vec{r}'(t)\right|}$$
$$=\frac{18t^2+18}{3t^2+3}$$
$$=\frac{6\left(3t^2+3\right)}{3t^2+3}$$
$$=6$$

Answer 38E.

$$\vec{r}(t)=(1+t)\hat{i}+\left(t^2-2t\right)\hat{j}$$
$$\vec{r}'(t)=\hat{i}+(2t-2)\hat{j}$$
$$=\hat{i}+2(t-1)\hat{j}$$
$$\vec{r}''(t)=2\hat{j}$$

$$\left|\vec{r}'(t)\right|=\sqrt{1^2+\left[2(t-1)\right]^2}$$
$$=\sqrt{1+4\left(t^2-2t+1\right)}$$
$$=\sqrt{1+4t^2-8t+4}$$
$$=\sqrt{4t^2-8t+5}$$

The tangential component of acceleration vector is

$$a_T = \frac{\vec{r}\,'(t)\,\vec{r}\,''(t)}{\left|\vec{r}\,'(t)\right|}$$

$$= \frac{\left[2(t-1)\right]2}{\sqrt{4t^2-8t+5}}$$

$$= \frac{4t-4}{\sqrt{4t^2-8t+5}}$$

Now $\vec{r}\,'(t)\times\vec{r}\,''(t) = \left[\hat{i}+2(t-1)\hat{j}\right]\times 2\hat{j}$

$$= 2\hat{i}\times\hat{j}+2(t-1)\hat{j}\times 2\hat{j}$$

$$= 2\hat{k}+0$$

$$= 2\hat{k}$$

$$\left|\vec{r}\,'(t)\times\vec{r}\,''(t)\right| = \sqrt{2^2}$$

$$= 2$$

The normal component of acceleration vector is

$$a_N = \frac{\left|\vec{r}\,'(t)\times\vec{r}\,''(t)\right|}{\left|\vec{r}\,'(t)\right|}$$

$$= \frac{2}{\sqrt{4t^2-8t+5}}$$

Answer 39E.

Consider the following vector

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$$

Find the tangential and normal components of the acceleration vector:

Recall the tangential components of acceleration vector is

$$a_T = \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{\mathbf{r}'(t)}$$

Differentiating $\mathbf{r}(t)$ with respect to t is

$$\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}$$

Again differentiating $\mathbf{r}(t)$ with respect to t is

$$\mathbf{r}''(t) = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j}$$

$$|\mathbf{r}'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}$$

$$= \sqrt{\sin^2 t + \cos^2 t + 1}$$

$$= \sqrt{1+1}$$

$$= \sqrt{2}$$

The tangential components of acceleration vector is

$$a_T = \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{\mathbf{r}'(t)}$$

$$= \frac{(-\sin t)(-\cos t) + \cos t(-\sin t)}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}(\sin t\cos t - \cos t\sin t)$$

$$= 0$$

Therefore, the tangential component of acceleration vector is $\boxed{0}$.

Recall the normal component of acceleration vector is

$$a_N = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|}$$

Now $\mathbf{r}'(t)\times\mathbf{r}''(t)=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -\sin t & \cos t & 1\\ -\cos t & -\sin t & 0\end{vmatrix}$

$$=\sin t\,\mathbf{i}+(-\cos t)\,\mathbf{j}+\left(\sin^2 t+\cos^2 t\right)\mathbf{k}$$

$$=\sin t\mathbf{i}-\cos t\,\mathbf{j}+\mathbf{k}$$

$$\left|\mathbf{r}'(t)\times\mathbf{r}''(t)\right|=\sqrt{\sin^2 t+\cos^2 t+1}$$

$$=\sqrt{1+1}$$

$$=\sqrt{2}$$

The normal component of acceleration vector is

$$a_N=\frac{\left|\vec{r}'(t)\times\vec{r}''(t)\right|}{\left|\vec{r}'(t)\right|}$$

$$=\frac{\sqrt{2}}{\sqrt{2}}$$

$$=1$$

The normal component of acceleration vector is $\boxed{1}$.

Answer 40E.

A particle moves with position function $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + 3t\mathbf{k}$.

To find the tangential and normal components of the acceleration of this particle:

The tangential component of acceleration vector is

$$a_T = \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|} \quad \text{...... (1)}$$

Now,

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + 3t\mathbf{k}$$

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 3\mathbf{k}$$

$$\mathbf{r}''(t) = 2\mathbf{j}$$

And the magnitude of the vector $\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 3\mathbf{k}$ is,

$$|\mathbf{r}'(t)| = \sqrt{1^2 + (2t)^2 + 3^2}$$

$$= \sqrt{1 + 4t^2 + 9}$$

$$= \sqrt{4t^2 + 10}$$

Substitute the values of $\mathbf{r}'(t), \mathbf{r}''(t), |\mathbf{r}'(t)|$ in the equation (1) to get the tangential component of acceleration vector.

$$a_T = \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|}$$

$$= \frac{\langle 1, 2t, 3\rangle \cdot (0, 2, 0)}{\sqrt{4t^2 + 10}}$$

$$= \boxed{\frac{4t}{\sqrt{4t^2 + 10}}}$$

The normal component of the acceleration of the particle is,

$$a_N = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|} \quad \ldots\ldots (2)$$

Now the cross product of $\mathbf{r}'(t), \mathbf{r}''(t)$ is

$$\mathbf{r}'(t)\times\mathbf{r}''(t) = (\mathbf{i}+2t\mathbf{j}+3\mathbf{k})\times(2\mathbf{j})$$

$$= 2(\mathbf{i}\times\mathbf{j}) + 2(2t)(\mathbf{j}\times\mathbf{j}) + 6(\mathbf{k}\times\mathbf{j})$$

$= 2\mathbf{k} + 4t(\mathbf{0}) + 6(-\mathbf{i})$ Since $\mathbf{i}\times\mathbf{j}=\mathbf{k}, \mathbf{j}\times\mathbf{k}=\mathbf{i}, \mathbf{k}\times\mathbf{i}=\mathbf{j}$

$$= -6\mathbf{i}+2\mathbf{k}$$

So the magnitude of the vector $\mathbf{r}'(t)\times\mathbf{r}''(t)$ is,

$$|\mathbf{r}'(t)\times\mathbf{r}''(t)| = \sqrt{(-6)^2+2^2}$$

$$= \sqrt{36+4}$$

$$= \sqrt{40}$$

$$= 2\sqrt{10}$$

So by (2), the normal component of acceleration vector is,

$$a_N = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|}$$

$$= \boxed{\frac{2\sqrt{10}}{\sqrt{4t^2+10}}}$$

Answer 41E.

$$\vec{r}(t) = e^t\hat{i} + \sqrt{2}t\,\hat{j} + e^{-t}\hat{k}$$

$$\vec{r}\,'(t) = e^t\hat{i} + \sqrt{2}\,\hat{j} - e^{-t}\hat{k}$$

$$\vec{r}\,''(t) = e^t\hat{i} + e^{-t}\hat{k}$$

$$\left|\vec{r}\,'(t)\right| = \sqrt{\left(e^{t}\right)^{2} + \left(\sqrt{2}\right)^{2} + \left(-e^{-t}\right)^{2}}$$
$$= \sqrt{e^{2t} + 2 + e^{-2t}}$$
$$= \sqrt{\frac{e^{4t} + 2e^{2t} + 1}{e^{2t}}}$$
$$= \sqrt{\frac{\left(e^{2t} + 1\right)^{2}}{e^{2t}}}$$
$$= \frac{e^{2t} + 1}{e^{t}}$$
$$= e^{t} + e^{-t}$$

The tangential component of acceleration vector is

$$a_T = \frac{\vec{r}\,'(t) \cdot \vec{r}\,''(t)}{\left|\vec{r}\,'(t)\right|}$$
$$= \frac{e^{t}.e^{t} - \left(e^{-t}\right)\left(e^{-t}\right)}{e^{t} + e^{-t}}$$
$$= \frac{e^{2t} - e^{-2t}}{e^{t} + e^{-t}}$$
$$= \frac{\left(e^{t} + e^{-t}\right)\left(e^{t} - e^{-t}\right)}{e^{t} + e^{-t}}$$
$$= \boxed{e^{t} - e^{-t}}$$

Now $\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ e^{t} & \sqrt{2} & -e^{-t} \\ e^{t} & 0 & e^{-t} \end{vmatrix}$

$$= \sqrt{2}e^{-t}\hat{i} - \left(e^{t}.e^{-t} + e^{t}e^{-t}\right)\hat{j} - \sqrt{2}e^{t}\hat{k}$$
$$= \sqrt{2}e^{-t}\hat{i} - 2\hat{j} - \sqrt{2}e^{t}\hat{k}$$

$$\left|\vec{r}\,'(t) \times \vec{r}\,''(t)\right| = \sqrt{\left(\sqrt{2}e^{-t}\right)^{2} + 2^{2} + \left(\sqrt{2}e^{t}\right)^{2}}$$
$$= \sqrt{2e^{-2t} + 4 + 2e^{2t}}$$
$$= \sqrt{2}\sqrt{e^{-2t} + 2 + e^{2t}}$$
$$= \sqrt{2}\sqrt{\left(e^{t} + e^{-t}\right)^{2}}$$
$$= \sqrt{2}\left(e^{t} + e^{-t}\right)$$

The normal component is

$$a_N = \frac{\left|\vec{r}'(t)\times\vec{r}''(t)\right|}{\left|\vec{r}'(t)\right|}$$

$$= \frac{\sqrt{2}\left(e^t+e^{-t}\right)}{e^t+e^{-t}}$$

$$= \boxed{\sqrt{2}}$$

Answer 42E.

$$\vec{r}(t) = t\hat{i} + \cos^2 t\,\hat{j} + \sin^2 t\,\hat{k}$$

$$\vec{r}'(t) = \hat{i} + 2\cos t(-\sin t)\hat{j} + 2\sin t\cos t\hat{k}$$

$$= \hat{i} - \sin 2t\,\hat{j} + \sin 2t\,\hat{k}$$

$$\vec{r}''(t) = -\cos 2t.2\hat{j} + \cos 2t.2\hat{k}$$

$$= -2\cos 2t\,\hat{j} + 2\cos 2t\,\hat{k}$$

$$\left|\vec{r}'(t)\right| = \sqrt{1^2 + (-\sin 2t)^2 + (\sin 2t)^2}$$

$$= \sqrt{1+\sin^2 2t + \sin^2 2t}$$

$$= \sqrt{1+2\sin^2 2t}$$

The tangential component of acceleration vector is

$$a_T = \frac{\vec{r}'(t)\cdot\vec{r}''(t)}{\left|\vec{r}'(t)\right|}$$

$$= \frac{(-\sin 2t)(-2\cos 2t) + (\sin 2t)(2\cos 2t)}{\sqrt{1+2\sin^2 2t}}$$

$$= \frac{4\sin 2t\cos 2t}{\sqrt{1+2\sin^2 2t}}$$

$$= \boxed{\frac{2\sin 4t}{\sqrt{1+2\sin^2 2t}}}$$

Now $\vec{r}'(t)\times\vec{r}''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -\sin 2t & \sin 2t \\ 0 & -2\cos 2t & 2\cos 2t \end{vmatrix}$

$$= (-2\sin 2t\cos 2t + 2\cos 2t\sin 2t)\hat{i} + 2\cos 2t\,\hat{j} + (-2\cos 2t)\hat{k}$$

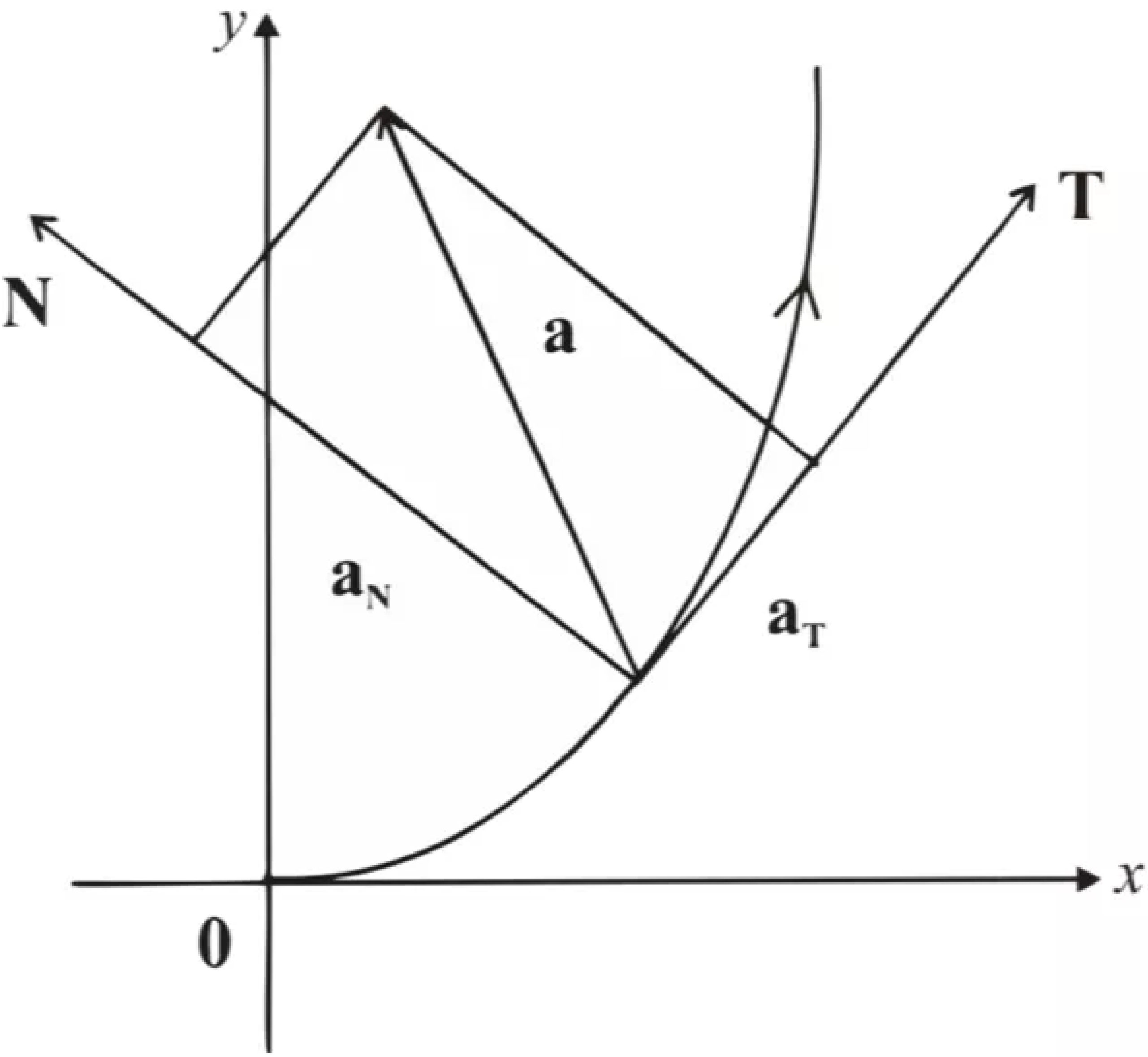
$$= 2\cos 2t\,\hat{j} - 2\cos 2t\hat{k}$$

$$\left|\vec{r}'(t)\times\vec{r}''(t)\right| = \sqrt{(2\cos 2t)^2 + (2\cos 2t)^2}$$
$$= \sqrt{4\cos^2 2t + 4\cos^2 2t}$$
$$= 2\sqrt{2\cos^2 2t}$$
$$= 2\sqrt{2}\cos 2t$$

The normal component of acceleration vector is

$$a_N = \frac{\left|\vec{r}'(t)\times\vec{r}''(t)\right|}{\left|\vec{r}'(t)\right|}$$
$$= \boxed{\frac{2\sqrt{2}\cos 2t}{\sqrt{1+2\sin^2 2t}}}$$

Answer 43E.



It is given that the magnitude of the acceleration vector **a** is:

$$|\vec{\mathbf{a}}| = 10\,\text{cm}/\text{s}^2$$

Now, from the figure, the tangential and normal components of **a** can be determined.

Since, it can be seen in the figure that the vector **a** makes the diagonal of the rectangle whose sides are given by the normal and tangential components, so, by Pythagoras theorem:

The tangential component is:

$$\boxed{a_T = 4.5\,\text{cm}/\text{s}^2}$$

And the normal component is:

$$\boxed{a_N = 9.0\,\text{cm}/\text{s}^2}$$

Verify:

$$\begin{aligned} a_T^2 + a_N^2 &= (4.5)^2 + (9.0)^2 \\ &\approx 100 \end{aligned}$$

Answer 44E.

It is given that

$$\vec{L}(t) = m\vec{r}(t) \times \vec{v}(t)$$

On differentiating both sides with respect to t

$$\vec{L}'(t) = \left[m\vec{r}'(t) \times \vec{v}(t) + m\vec{r}(t) \times \vec{v}'(t)\right]$$

Since $\vec{r}'(t) = \vec{v}(t)$

Then

$$\begin{aligned} \vec{L}'(t) &= \left[m\vec{v}(t) \times \vec{v}(t) + m\vec{r}(t) \times \vec{v}'(t)\right] \\ &= \left[0 + m\vec{r}(t) \times \vec{v}'(t)\right] \\ &= m\vec{r}(t) \times \vec{a}(t) \qquad (\text{As } \vec{v}'(t) = \vec{a}(t)) \end{aligned}$$

But $m\vec{r}(t) \times \vec{a}(t) = \vec{\tau}(t)$

Then $\vec{L}'(t) = \vec{\tau}(t)$

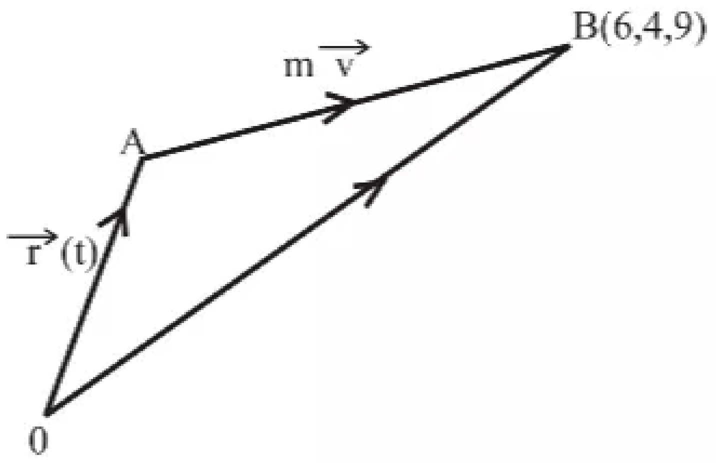
Now if $\vec{\tau}(t) = \vec{0}$ for all t

Then $\vec{L}'(t) = \vec{0}$

On integrating

$\vec{L}(t) = \text{constant vector}$

Answer 45E.



Let the engine is turned off when spaceship is at A. After A the spaceship will move along tangent to the curve at A. i.e. the spaceship will continue to move along it's velocity $\vec{v}(t)$ at A.

Let spaceship takes time m to reach at B from A. therefore total displacement of the spaceship in t + m time is $\overrightarrow{OB}$ and

$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$$
$$= \overrightarrow{OA} + m\vec{v}(t)$$

Now position function of spaceship is

$$\vec{r}(t)=(3+t)\hat{i}+(2+\ln t)\hat{j}+\left(7-\frac{4}{t^2+1}\right)\hat{k}$$

Differentiating with respect to t,

$$\vec{v}(t)=\vec{r}'(t)=\frac{d}{dt}\left[(3+t)\hat{i}+(2+\ln t)\hat{j}+\left(7-\frac{4}{t^2+1}\right)\hat{k}\right]$$

$$=\hat{i}+\frac{1}{t}\hat{j}-4\left(-\frac{1}{\left(t^2+1\right)^2}\right)(2t)\hat{k}$$

$$=\hat{i}+\frac{1}{t}\hat{j}+\frac{8t}{\left(t^2+1\right)^2}\hat{k}$$

Now, from $\overrightarrow{OA}+m\vec{v}(t)=\overrightarrow{OB}$, we have,

$$\vec{r}(t)+m\vec{v}(t)=6\hat{i}+4\hat{j}+9\hat{k}$$

$$\Rightarrow \quad (3+t)\hat{i}+(2+\ln t)\hat{j}+\left(7-\frac{4}{t^2+1}\right)\hat{k}+m\left[\hat{i}+\frac{1}{t}\hat{j}+\frac{8t}{\left(t^2+1\right)^2}\hat{k}\right]=6\hat{i}+4\hat{j}+9\hat{k}$$

$$\Rightarrow \quad (3+t+m)\hat{i}+\left(2+\ln t+\frac{m}{t}\right)\hat{j}+\left[7-\frac{4}{\left(t^2+1\right)}+\frac{8tm}{\left(t^2+1\right)^2}\right]\hat{k}=6\hat{i}+4\hat{j}+9\hat{k}$$

Equating coefficients of $\hat{i}$, $\hat{j}$, $\hat{k}$ we get,

$$3+t+m=6\Rightarrow m=3-t \qquad \text{----- (1)}$$

$$2+\ln t+\frac{m}{t}=4 \qquad \text{----- (2)}$$

And

$$7-\frac{4}{\left(t^2+1\right)}+\frac{8tm}{\left(t^2+1\right)^2}=9$$

$$\frac{7\left(t^2+1\right)^2-4\left(t^2+1\right)+8t(3-t)}{\left(t^2+1\right)^2}=9$$

$$\Rightarrow \quad 7\left(t^4+2t^2+1\right)-4t^2-4+24t-8t^2=9\left[t^4+2t^2+1\right]$$

$$\Rightarrow \quad 7t^4+14t^2+7-12t^2+24t-4=9t^4+18t^2+9$$

$$\Rightarrow \quad 7t^4+2t^2+24t+3=9t^4+18t^2+9$$

$$\Rightarrow \quad 2t^4+16t^2-24t+6=0$$

$$\Rightarrow \quad t^4+8t^2-12t+3=0$$

$$\Rightarrow \quad (t-1)\left(t^3+t^2+9t-3\right)=0$$

If $t-1=0 \Rightarrow t=1$

Therefore, $m=3-t=3-1=2$

Putting values of t and m in equation (2)

We get, $2+\ln 1+\frac{2}{1}=4$

$$2+0+2=4$$

$$\Rightarrow 4=4$$

i.e. values of m and t also satisfy equation (2)

Hence,

$$\boxed{\text{Engine should be turned off when t = 1}}$$

Answer 46E.

(A) Equation of motion of rocket is

$$m\frac{d\vec{v}}{dt}=\frac{dm}{dt}\vec{v}_e$$

Or, $$\frac{d\vec{v}}{dt}=\frac{1}{m}\frac{dm}{dt}\vec{v}_e$$

Integrating both sides with respect to t

From $t=0$ to $t=t$ we get,

$$\int_{t=0}^{t}\frac{d\vec{v}}{dt}=\vec{v}_e\int_{t=0}^{t}\frac{1}{m}\frac{dm}{dt}$$

$$\Rightarrow \int_{\vec{v}(0)}^{\vec{v}(t)}d\vec{v}=\vec{v}_e\int_{m(0)}^{m(t)}\frac{dm}{m}$$

$$\Rightarrow [\vec{v}]_{\vec{v}(0)}^{\vec{v}(t)}=\vec{v}_e[\ln m]_{m(0)}^{m(t)}$$

$$\Rightarrow \vec{v}(t)-\vec{v}(0)=\vec{v}_e\left[\ln m(t)-\ln m(0)\right]$$

$$\Rightarrow \vec{v}(t)=\vec{v}(0)+\vec{v}_e\ln\frac{m(t)}{m(0)}$$

Hence,

$$\boxed{\vec{v}(t)=\vec{v}(0)-\vec{v}_e\ln\frac{m(0)}{m(t)}}$$

(B) From part (a) we have

$$\vec{v}(t) = v(0) - \vec{v}_e \ln \frac{m(0)}{m(t)}$$

Given, rocket starts motion from rest.

Therefore, $v(0) = 0$ and thus,

$$\vec{v}(t) = -\vec{v}_e \ln \frac{m(0)}{m(t)}$$

Also, when speed of rocket is twice the speed of its own exhaust gases, we have

$$|\vec{v}(t)| = 2|\vec{v}_e|$$

Now, from $\vec{v}(t) = -\vec{v}_e \ln \frac{m(0)}{m(t)}$

$$|\vec{v}(t)| = |\vec{v}_e| \ln \frac{m(0)}{m(t)}$$

$$\Rightarrow \quad 2|\vec{v}_e| = |\vec{v}_e| \ln \frac{m(0)}{m(t)}$$

$$\Rightarrow \quad 2 = \ln \frac{m(0)}{m(t)}$$

$$\Rightarrow \quad \frac{m(0)}{m(t)} = e^2$$

$$\Rightarrow \quad m(t) = \frac{m(0)}{e^2} = e^{-2} m(0)$$

Therefore, mass of the fuel burnt

$$= m(0) - m(t)$$
$$= m(0) - e^{-2} m(0)$$
$$= m(0)\left[1 - e^{-2}\right]$$

Thus, the fraction of the initial mass which the rocket has to burn as fuel

$$= \frac{\text{mass of the fuel burnt}}{\text{Initial mass of the fuel}}$$
$$= \frac{m_0\left(1 - e^{-2}\right)}{m_0}$$
$$= 1 - e^{-2}$$

Hence,

The fraction of the initial mass that is burned as fuel $= 1 - e^{-2}$