Chapter
Properties of Triangles

Topic-1: Properties of Triangle, Solutions of Triangles, Inscribed & Circumscribed Circles, Regular Polygons

1 MCQs with One Correct Answer

1. In a triangle the sum of two sides is x and the product of the same sides is y. If $x^2 - c^2 = y$, where c is the third side of the triangle, then the ratio of the inradius to the circum-radius of the triangle is **[Adv. 2014]**

(a) $\frac{3y}{2x(x+c)}$ (b) $\frac{3y}{2c(x+c)}$

(c) $\frac{3y}{4x(x+c)}$ (d) $\frac{3y}{4c(x+c)}$

2. Let PQR be a triangle of area Δ with $a = 2$, $b = \frac{7}{2}$ and $c = \frac{5}{2}$; where a, b, and c are the lengths of the sides of the triangle opposite to the angles at P, Q and R respectively. Then $\frac{2\sin P - \sin 2P}{2\sin P + \sin 2P}$ equals. **[2012]**

(a) $\frac{3}{4\Delta}$ (b) $\frac{45}{4\Delta}$ (c) $\left(\frac{3}{4\Delta}\right)^2$ (d) $\left(\frac{45}{4\Delta}\right)^2$

3. If the angles A, B and C of a triangle are in an arithmetic progression and if a, b and c denote the lengths of the sides opposite to A, B and C respectively, then the value of the expression $\frac{a}{c}\sin 2C + \frac{c}{a}\sin 2A$ is **[2010]**

(a) $\frac{1}{2}$ (b) $\frac{\sqrt{3}}{2}$ (c) 1 (d) $\sqrt{3}$

4. Let $ABCD$ be a quadrilateral with area 18, with side AB parallel to the side CD and $2AB = CD$. Let AD be perpendicular to AB and CD. If a circle is drawn inside the quadrilateral $ABCD$ touching all the sides, then its radius is **[2007 - 3 Marks]**

(a) 3 (b) 2 (c) $\frac{3}{2}$ (d) 1

5. One angle of an isosceles Δ is 120° and radius of its incircle $= \sqrt{3}$. Then the area of the triangle in sq. units is **[2006 - 3M, –1]**

(a) $7+12\sqrt{3}$ (b) $12-7\sqrt{3}$

(c) $12+7\sqrt{3}$ (d) 4π

6. In a triangle ABC, a, b, c are the lengths of its sides and A, B, C are the angles of triangle ABC. The correct relation is given by **[2005S]**

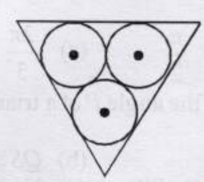
(a) $(b-c)\sin\left(\frac{B-C}{2}\right) = a\cos\frac{A}{2}$

(b) $(b-c)\cos\left(\frac{A}{2}\right) = a\sin\frac{B-C}{2}$

(c) $(b+c)\sin\left(\frac{B+C}{2}\right) = a\cos\frac{A}{2}$

(d) $(b-c)\cos\left(\frac{A}{2}\right) = 2a\sin\frac{B+C}{2}$

7. In an equilateral triangle, 3 coins of radii 1 unit each are kept so that they touch each other and also the sides of the triangle. Area of the triangle is **[2005S]**



(a) $4+2\sqrt{3}$ (b) $6+4\sqrt{3}$

(c) $12+\frac{7\sqrt{3}}{4}$ (d) $3+\frac{7\sqrt{3}}{4}$

8. The sides of a triangle are in the ratio $1 : \sqrt{3} : 2$, then the angles of the triangle are in the ratio [2004S]
(a) $1:3:5$ (b) $2:3:4$
(c) $3:2:1$ (d) $1:2:3$

9. If the angles of a triangle are in the ratio $4:1:1$, then the ratio of the longest side to the perimeter is [2003S]
(a) $\sqrt{3}:(2+\sqrt{3})$ (b) $1:6$
(c) $1:2+\sqrt{3}$ (d) $2:3$

10. Which of the following pieces of data does NOT uniquely determine an acute-angled triangle ABC (R being the radius of the circumcircle)? [2002S]
(a) $a, \sin A, \sin B$ (b) a, b, c
(c) $a, \sin B, R$ (d) $a, \sin A, R$

11. In a triangle ABC, let $\angle C = \frac{\pi}{2}$. If r is the inradius and R is the circumradius of the triangle, then $2(r+R)$ is equal to [2000S]
(a) $a+b$ (b) $b+c$
(c) $c+a$ (d) $a+b+c$

12. In a triangle ABC, $2ac\sin\frac{1}{2}(A-B+C) =$ [2000S]
(a) $a^2+b^2-c^2$ (b) $c^2+a^2-b^2$
(c) $b^2-c^2-a^2$ (d) $c^2-a^2-b^2$

13. In a triangle ABC, $\angle B = \frac{\pi}{3}$ and $\angle C = \frac{\pi}{4}$. Let D divide BC internally in the ratio $1:3$ then $\frac{\sin\angle BAD}{\sin\angle CAD}$ is equal to [1995S]
(a) $\frac{1}{\sqrt{6}}$ (b) $\frac{1}{3}$ (c) $\frac{1}{\sqrt{3}}$ (d) $\sqrt{\frac{2}{3}}$

14. If the lengths of the sides of triangle are 3, 5, 7 then the largest angle of the triangle is [1994]
(a) $\frac{\pi}{2}$ (b) $\frac{5\pi}{6}$ (c) $\frac{2\pi}{3}$ (d) $\frac{3\pi}{4}$

15. In a triangle ABC, angle A is greater than angle B. If the measures of angles A and B satisfy the equation $3\sin x - 4\sin^3 x - k = 0, 0 < k < 1$, then the measure of angle C is [1990 - 2 Marks]
(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{2\pi}{3}$ (d) $\frac{5\pi}{6}$

16. If the bisector of the angle P of a triangle PQR meets QR in S, then [1979]
(a) $QS = SR$ (b) $QS:SR = PR:PQ$
(c) $QS:SR = PQ:PR$ (d) None of these

2 Integer Value Answer/ Non-Negative Integer

17. Consider the hyperbola

$$\frac{x^2}{100}-\frac{y^2}{64}=1$$

with foci at S and S_1, where S lies on the positive x-axis. Let P be a point on the hyperbola, in the first quadrant. Let $\angle SPS_1 = \alpha$, with $\alpha < \frac{\pi}{2}$. The straight line passing through the point S and having the same slope as that of the tangent at P to the hyperbola, intersects the straight line S_1P at P_1. Let δ be the distance of P from the straight line SP_1, and $\beta = S_1P$. Then the greatest integer less than or equal to $\frac{\beta\delta}{9}\sin\frac{\alpha}{2}$ is ________. [Adv. 2022]

18. In a triangle ABC, let $AB = \sqrt{23}$, $BC = 3$ and $CA = 4$. Then the value of $\frac{\cot A + \cot C}{\cot B}$ is ______. [Adv. 2021]

19. Consider a triangle ABC and let a, b and c denote the lengths of the sides opposite to vertices A, B and C respectively. Suppose $a=6$, $b=10$ and the area of the triangle is $15\sqrt{3}$, if $\angle ACB$ is obtuse and if r denotes the radius of the incircle of the triangle, then r^2 is equal to [2010]

20. Let ABC and ABC' be two non-congruent triangles with sides $AB = 4$, $AC = AC' = 2\sqrt{2}$ and angle $B = 30°$. The absolute value of the difference between the areas of these triangles is [2009]

3 Numeric/ New Stem Based Questions

Consider an obtuse angled triangle ABC in which the difference between the largest and the smallest angle is $\frac{\pi}{2}$ and whose sides are in arithmetic progression. Suppose that the vertices of this triangle lie on a circle of radius 1.

21. Let a be the area of the triangle ABC. Then the value of $(64a)^2$ is [Adv. 2023]

22. Then the inradius of the triangle ABC is [Adv. 2023]

4 Fill in the Blanks

23. In a triangle ABC, $a:b:c = 4:5:6$. The ratio of the radius of the circumcircle to that of the incircle is .... [1996 - 1 Mark]

24. A circle is inscribed in an equilateral triangle of side a. The area of any square inscribed in this circle is .... [1994 - 2 Marks]

25. In a triangle ABC, AD is the altitude from A. Given $b > c$, $\angle C = 23°$ and $AD = \frac{abc}{b^2-c^2}$ then $\angle B =$ .... [1994 - 2 Marks]

26. If in a triangle ABC, $\frac{2\cos A}{a}+\frac{\cos B}{b}+\frac{2\cos C}{c}=\frac{a}{bc}+\frac{b}{ca}$, then the value of the angle A is .... degrees. [1993 - 2 Marks]

27. If the angles of a triangle are 30° and 45° and the included side is $(\sqrt{3}+1)$ cms, then the area of the triangle is .... [1988 - 2 Marks]

28. A polygon of nine sides, each of length 2, is inscribed in a circle. The radius of the circle is .... [1987 - 2 Marks]

29. In a triangle ABC, if $\cot A, \cot B, \cot C$ are in A.P., then a^2, b^2, c^2, are in .... progression. [1985 - 2 Marks]

30. The set of all real numbers a such that $a^2 + 2a, 2a + 3$ and $a^2 + 3a + 8$ are the sides of a triangle is ....
[1985 - 2 Marks]

31. ABC is a triangle with $\angle B$ greater than $\angle C$. D and E are points on BC such that AD is perpendicular to BC and AE is the bisector of angle A. Complete the relation [1980]

$$\angle DAE = \frac{1}{2}[(...) - \angle C]$$

32. ABC is a triangle, P is a point on AB, and Q is point on AC such that $\angle AQP = \angle ABC$. Complete the relation

$$\frac{\text{area of } \Delta APQ}{\text{area of } \Delta ABC} = \frac{(...)}{AC^2}$$ [1980]

33. In a ΔABC, $\angle A = 90°$ and AD is an altitude. Complete the relation $\frac{BD}{BA} = \frac{AB}{(....)}$ [1980]

6 MCQs with One or More than One Correct Answer

34. Consider a triangle PQR having sides of lengths p, q and r opposite to the angles P, Q and R, respectively. Then which of the following statements is (are) **TRUE**? [Adv. 2021]

(a) $\cos P \ge 1 - \frac{p^2}{2qr}$

(b) $\cos R \ge \left(\frac{q-r}{p+q}\right)\cos P + \left(\frac{p-r}{p+q}\right)\cos Q$

(c) $\frac{q+r}{p} < 2\frac{\sqrt{\sin Q \sin R}}{\sin P}$

(d) If $p < q$ and $p < r$, then $\cos Q > \frac{p}{r}$ and $\cos R > \frac{p}{q}$

35. Let x, y and z be positive real numbers. Suppose x, y and z are the lengths of the sides of a triangle opposite to its angles X, Y and Z, respectively. If [Adv. 2020]

$$\tan\frac{X}{2} + \tan\frac{Z}{2} = \frac{2y}{x+y+z},$$

then which of the following statements is/are TRUE?

(a) $2Y = X + Z$ (b) $Y = X + Z$

(c) $\tan\frac{X}{2} = \frac{x}{y+z}$ (d) $x^2 + z^2 - y^2 = xz$

36. In a triangle PQR, let $\angle PQR = 30°$ and the sides PQ and QR have lengths $10\sqrt{3}$ and 10, respectively. Then, which of the following statement(s) is (are) TRUE? [Adv. 2018]

(a) $\angle QPR = 45°$

(b) The area of the triangle PQR is $25\sqrt{3}$ and $\angle QRP = 120°$

(c) The radius of the incircle of the triangle PQR is $10\sqrt{3} - 15$

(d) The area of the circumcircle of the triangle PQR is 100π.

37. In a triangle XYZ, let x, y, z be the lengths of sides opposite to the angles X, Y, Z, respectively, and $2s = x + y + z$. If $\frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2}$ and area of incircle of the triangle XYZ is $\frac{8\pi}{3}$, then [Adv. 2016]

(a) area of the triangle XYZ is $6\sqrt{6}$

(b) the radius of circumcircle of the triangle XYZ is $\frac{35}{6}\sqrt{6}$

(c) $\sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2} = \frac{4}{35}$

(d) $\sin^2\left(\frac{X+Y}{2}\right) = \frac{3}{5}$

38. In a triangle PQR, P is the largest angle and $\cos P = \frac{1}{3}$. Further the incircle of the triangle touches the sides PQ, QR and RP at N, L and M respectively, such that the lengths of PN, QL and RM are consecutive even integers. Then possible length(s) of the side(s) of the triangle is (are) [Adv. 2013]

(a) 16 (b) 18 (c) 24 (d) 22

39. Let ABC be a triangle such that $\angle ACB = \frac{\pi}{6}$ and let a, b and c denote the lengths of the sides opposite to A, B and C respectively. The value(s) of x for which $a = x^2 + x + 1$, $b = x^2 - 1$ and $c = 2x + 1$ is (are) [2010]

(a) $-(2+\sqrt{3})$ (b) $1+\sqrt{3}$

(c) $2+\sqrt{3}$ (d) $4\sqrt{3}$

40. In ΔABC, internal angle bisector of $\angle A$ meets side BC in D. $DE \perp AD$ meets AC in E and AB in F. Then [2006 - 5M, –1]

(a) AE is HM of b & c (b) $AD = \frac{2bc}{b+c}\cos\frac{A}{2}$

(c) $EF = \frac{4bc}{b+c}\sin\frac{A}{2}$ (d) ΔAEF is isosceles

41. Let $A_0A_1A_2A_3A_4A_5$ be a regular hexagon inscribed in a circle of unit radius. Then the product of the lengths of the line segments A_0A_1, A_0A_2 and A_0A_4 is [1998 - 2 Marks]

(a) $\frac{3}{4}$ (b) $3\sqrt{3}$

(c) 3 (d) $\frac{3\sqrt{3}}{2}$

42. If in a triangle PQR, $\sin P, \sin Q, \sin R$ are in A.P., then [1998 - 2 Marks]

(a) the altitudes are in A.P.
(b) the altitudes are in H.P.
(c) the medians are in G.P.
(d) the medians are in A.P.

43. In a triangle, the lengths of the two larger sides are 10 and 9, respectively. If the angles are in A.P. Then the length of the third side can be [1987 - 2 Marks]

(a) $5-\sqrt{6}$ (b) $3\sqrt{3}$

(c) 5 (d) $5+\sqrt{6}$

(e) none

44. There exists a triangle ABC satisfying the conditions [1986 - 2 Marks]

(a) $b\sin A = a, A < \pi/2$ (b) $b\sin A > a, A > \pi/2$

(c) $b\sin A > a, A < \pi/2$ (d) $b\sin A < a, A < \pi/2, b > a$

(e) $b\sin A < a, A > \pi/2, b = a$

10 Subjective Problems

45. If I_n is the area of n sided regular polygon inscribed in a circle of unit radius and O_n be the area of the polygon circumscribing the given circle, prove that

$$I_n = \frac{O_n}{2}\left(1+\sqrt{1-\left(\frac{2I_n}{n}\right)^2}\right).$$ [2003 - 4 Marks]

46. If Δ is the area of a triangle with side lengths a, b, c, then show that $\Delta \le \frac{1}{4}\sqrt{(a+b+c)abc}$. Also show that the equality occurs in the above inequality if and only if $a = b = c$. [2001 - 6 Marks]

47. Let ABC be a triangle with incentre I and inradius r. Let D,E,F be the feet of the perpendiculars from I to the sides BC, CA and AB respectively. If r_1, r_2 and r_3 are the radii of circles inscribed in the quadrilaterals $AFIE$, $BDIF$ and $CEID$ respecitvely, prove that [2000 - 7 Marks]

$$\frac{r_1}{r-r_1}+\frac{r_2}{r-r_2}+\frac{r_3}{r-r_3}=\frac{r_1r_2r_3}{(r-r_1)(r-r_2)(r-r_3)}.$$

48. Let ABC be a triangle having O and I as its circumcenter and incentre respectively. If R and r are the circumradius and the inradius, respectively, then prove that $(IO)^2 = R^2 - 2Rr$. Further show that the triangle BIO is a right-angled triangle if and only if b is arithmetic mean of a and c. [1999 - 10 Marks]

49. Prove that a triangle ABC is equilateral if and only if $\tan A + \tan B + \tan C = 3\sqrt{3}$. [1998 - 8 Marks]

50. Consider the following statements concerning a triangle ABC [1994 - 5 Marks]

(i) The sides a, b, c and area Δ are rational.

(ii) $a, \tan\frac{B}{2}, \tan\frac{C}{2}$ are rational.

(iii) $a, \sin A, \sin B, \sin C$ are rational.

Prove that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i)

51. Let $A_1, A_2, \ldots\ldots, A_n$ be the vertices of an n-sided regular polygon such that $\frac{1}{A_1A_2}=\frac{1}{A_1A_3}+\frac{1}{A_1A_4}$, Find the value of n. [1994 - 4 Marks]

52. Three circles touch the one another externally. The tangent at their point of contact meet at a point whose distance from a point of contanct is 4. Find the ratio of the product of the radii to the sum of the radii of the circles. [1992 - 4 Marks]

53. In a triangle of base a, the ratio of the other two sides is $r (< 1)$. Show that the altitude of the triangle is less than or equal to $\frac{ar}{1-r^2}$ [1991 - 4 Marks]

54. The sides of a triangle are three consecutive natural numbers and its largest angle is twice the smallest one. Determine the sides of the triangle. [1991 - 4 Marks]

55. If in a triangle ABC, $\cos A\cos B + \sin A\sin B\sin C = 1$, Show that $a : b : c = 1 : 1 : \sqrt{2}$ [1986 - 5 Marks]

56. In a triangle ABC, the median to the side BC is of length $\frac{1}{\sqrt{11-6\sqrt{3}}}$ [1985 - 5 Marks] and it divides the angle A into angles 30° and 45°. Find the length of the side BC.

57. A ladder rests against a wall at an angle α to the horizontal. Its foot is pulled away from the wall through a distance a, so that it slides a distance b down the wall making an angle β with the horizontal. Show that $a = b\tan\frac{1}{2}(\alpha+\beta)$ [1985 - 5 Marks]

58. With usual notation, if in a triangle ABC;

$\frac{b+c}{11}=\frac{c+a}{12}=\frac{a+b}{13}$ then prove that

$\frac{\cos A}{7}=\frac{\cos B}{19}=\frac{\cos C}{25}$ [1984 - 4 Marks]

59. For a triangle ABC it is given that $\cos A + \cos B + \cos C = \frac{3}{2}$. Prove that the triangle is equilateral. [1984 - 4 Marks]

60. The ex-radii r_1, r_2, r_3 of ΔABC are in H.P. Show that its sides a, b, c are in A.P. [1983 - 3 Marks]

61. Let the angles A, B, C of a triangle ABC be in A.P. and let $b : c = \sqrt{3} : \sqrt{2}$. Find the angle A. [1981 - 2 Marks]

62. ABC is a triangle with $AB = AC$. D is any point on the side BC. E and F are points on the side AB and AC, respectively, such that DE is parallel to AC, and DF is parallel to AB. Prove that

$DF + FA + AE + ED = AB + AC$ [1980]

63. ABC is a triangle. D is the middle point of BC. If AD is perpendicular to AC, then prove that [1980]

$$\cos A\cos C = \frac{2(c^2-a^2)}{3ac}$$

64. (a) If a circle is inscribed in a right angled triangle ABC with the right angle at B, show that the diameter of the circle is equal to $AB + BC - AC$.

(b) If a triangle is inscribed in a circle, then the product of any two sides of the triangle is equal to the product of the diameter and the perpendicular distance of the third side from the opposite vertex. Prove the above statement. [1979]

65. A triangle ABC has sides $AB = AC = 5$ cm and $BC = 6$ cm Triangle $A'B'C'$ is the reflection of the triangle ABC in a line parallel to AB placed at a distance 2 cm from AB, outside the triangle ABC. Triangle $A''B''C''$ is the reflection of the triangle $A'B'C'$ in a line parallel to $B'C'$ placed at a distance of 2 cm from $B'C'$ outside the triangle $A'B'C'$. Find the distance between A and A''. [1978]

Topic-2: Heights & Distances

1 MCQs with One Correct Answer

1. A man from the top of a 100 metres high tower sees a car moving towards the tower at an angle of depression of 30°. After some time, the angle of depression becomes 60°. The distance [in metres] travelled by the car during this time is [2001S]

(a) $100\sqrt{3}$ (b) $200\sqrt{3}/3$

(c) $100\sqrt{3}/3$ (d) $200\sqrt{3}$

2. A pole stands vertically inside a triangular park ΔABC. If the angle of elevation of the top of the pole from each corner of the park is same, then in ΔABC the foot of the pole is at the [2000S]

(a) centroid (b) circumcentre

(c) incentre (d) orthocentre

3. From the top of a light-house 60 metres high with its base at the sea-level, the angle of depression of a boat is 15°. The distance of the boat from the foot of the light house is [1983 - 1 Mark]

(a) $\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)$ 60 metres (b) $\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)$ 60 metres

(c) $\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)^2$ metres (d) none of these

10 Subjective Problems

4. A bird flies in a circle on a horizontal plane. An observer stands at a point on the ground. Suppose 60° and 30° are the maximum and the minimum angles of elevation of the bird and that they occur when the bird is at the points P and Q respectively on its path. Let θ be the angle of elevation of the bird when it is a point on the arc of the circle exactly midway between P and Q. Find the numerical value of $\tan^2\theta$.[Assume that the observer is not inside the vertical projection of the path of the bird.] [1998 - 8 Marks]

5. A tower AB leans towards west making an angle α with the vertical. The angular elevation of B, the topmost point of the tower is β as observed from a point C due west of A at a distance d from A. If the angular elevation of B from a point D due east of C at a distance $2d$ from C is γ, then prove that $2\tan\alpha = -\cot\beta + \cot\gamma$. [1994 - 4 Marks]

6. An observer at O notices that the angle of elevation of the top of a tower is 30°. The line joining O to the base of the tower makes an angle of $\tan^{-1}(1/\sqrt{2})$ with the North and is inclined Eastwards. The observer travels a distance of 300 meters towards the North to a point A and finds the tower to his East. The angle of elevation of the top of the tower at A is ϕ, Find ϕ and the height of the tower [1993 - 5 Marks]

7. A man notices two objects in a straight line due west. After walking a distance c due north he orserves that the objects subtend an angle α at his eye; and, after walking a further distance $2c$ due north, an angle β. Show that the distance between the objects is $\frac{8c}{3\cot\beta - \cot\alpha}$; the height of the man is being ignored. [1991 - 4 Marks]

8. A vertical tower PQ stands at a point P. Points A and B are located to the South and East of P respectively. M is the mid point of AB. PAM is an equilateral triangle; and N is the foot of the perpendicular from P on AB. Let $AN = 20$ metres and the angle of elevation of the top of the tower at N is $\tan^{-1}(2)$. Determine the height of the tower and the angles of elevation of the top of the tower at A and B. [1990 - 4 Marks]

9. ABC is a triangular park with $AB = AC = 100$ m. A television tower stands at the midpoint of BC. The angles of elevation of the top of the tower at A, B, C are 45°, 60°, 60°, respectively. Find the height of the tower. [1989 - 5 Marks]

10. A sign-post in the form of an isosceles triangle ABC is mounted on a pole of height h fixed to the ground. The base BC of the triangle is parallel to the ground. A man standing on the ground at a distance d from the sign-post finds that the top vertex A of the triangle subtends an angle β and either of the other two vertices subtends the same angle α at his feet. Find the area of the triangle. [1988 - 5 Marks]

11. Four ships A, B, C and D are at sea in the following relative positions : B is on the straight line segment AC, B is due North of D and D is due west of C. The distance between B and D is 2 km. $\angle BDA = 40°$, $\angle BCD = 25°$. What is the distance between A and D? [Take $\sin 25° = 0.423$] [1983 - 3 Marks]

12. A vertical pole stands at a point Q on a horizontal ground. A and B are points on the ground, d meters apart. The pole subtends angles α and β at A and B respectively. AB subtends an angle γ at Q. Find the height of the pole. [1982 - 3 Marks]

13. (i) PQ is a vertical tower. P is the foot and Q is the top of the tower. A, B, C are three points in the horizontal plane through P. The angles of elevation of Q from A, B, C are equal, and each is equal to θ. The sides of the triangle ABC are a, b, c; and the area of the triangle ABC is Δ. Show that the height of the tower is $\frac{abc\tan\theta}{4\Delta}$.

(ii) AB is a vertical pole. The end A is on the level ground. C is the middle point of AB. P is a point on the level ground. The portion CB subtends an angle β at P. If $AP = n\,AB$, then show that $\tan\beta = \frac{n}{2n^2+1}$ [1980]

14. (a) A balloon is observed simultaneously from three points A, B and C on a straight road directly beneath it. The angular elevation at B is twice that at A and the angular elevation at C is thrice that at A. If the distance between A and B is a and the distance between B and C is b, find the height of the balloon in terms of a and b.

(b) Find the area of the smaller part of a disc of radius 10 cm, cut off by a chord AB which subtends an angle of $22\frac{1}{2}°$ at the circumference. [1979]

Answer Key

Topic-1 : Properties of Triangle, Solutions of Triangles, Inscribed & Circumscribed Circles, Regular Polygons

1. (b)	**2.** (c)	**3.** (d)	**4.** (b)	**5.** (c)	**6.** (b)	**7.** (b)	**8.** (d)	**9.** (a)	**10.** (d)
11. (a)	**12.** (b)	**13.** (a)	**14.** (c)	**15.** (c)	**16.** (c)	**17.** (7)	**18.** (2)	**19.** (3)	**20.** (4)
21. (1008.00)	**22.** (0.25)	**23.** (16 : 7)	**24.** $\frac{a^2}{6}$	**25.** (113°)	**26.** (90°)	**27.** $\frac{\sqrt{3}+1}{2}$ sq. units			
28. $\operatorname{cosec}\frac{\pi}{9}$	**29.** Arithmetic	**30.** $(5,\infty)$	**31.** $\angle B$	**32.** AP^2	**33.** BC	**34.** (a, b)	**35.** (b, c)		
36. (b, c, d)	**37.** (a, c, d)	**38.** (b, d)	**39.** (b)	**40.** (a,b,c,d)	**41.** (c)	**42.** (b)	**43.** (a, d)	**44.** (a, d)	

Topic-2 : Heights & Distances

1. (b)	**2.** (b)	**3.** (b)

Hints & Solutions

Topic-1: Properties of Triangle, Solutions of Triangles, Inscribed & Circumscribed Circles, Regular Polygons

1. **(b)** Let two sides of the triangle be a and b.

$\therefore$ $a+b=x$ and $ab=y$

Now $x^2-c^2=y$, where c is the third side of Δ.

$$\Rightarrow (a+b)^2-c^2=ab \Rightarrow \frac{a^2+b^2-c^2}{2ab}=-\frac{1}{2}$$

$$\Rightarrow \cos C=-\frac{1}{2} \Rightarrow C=120^\circ \quad \text{.... (i)}$$

$$\therefore \frac{r}{R}=\frac{\Delta}{s}\times\frac{4\Delta}{abc}, \text{ where } \Delta \text{ is area of the triangle}$$

$$\Rightarrow \frac{r}{R}=\frac{4\Delta^2}{\frac{(a+b+c)}{2}abc}=\frac{8\times\left(\frac{1}{2}ab\sin C\right)^2}{(a+b+c)abc}$$

$$=\frac{2a^2b^2\sin^2 120^\circ}{(a+b+c)abc}=\frac{2ab\times\frac{3}{4}}{(x+c)c}=\frac{3y}{2c(x+c)}$$

2. **(c)** $\dfrac{2\sin P-\sin 2P}{2\sin P+\sin 2P}=\dfrac{2\sin P-2\sin P\cos P}{2\sin P+2\sin P\cos P}=\dfrac{1-\cos P}{1-\sin P}$

$$=\frac{2\sin^2\frac{P}{2}}{2\cos^2\frac{P}{2}}=\tan^2\frac{P}{2}=\frac{(s-b)(s-c)}{s(s-a)} \quad \text{where } s=\frac{a+b+c}{2}$$

$$=\frac{(s-b)^2(s-c)^2}{s(s-a)(s-b)(s-c)}=\frac{(a+c-b)^2(a+b-c)^2}{16.\Delta^2}$$

$$=\frac{\left(2+\frac{5}{2}-\frac{7}{2}\right)^2\left(2+\frac{7}{2}-\frac{5}{2}\right)^2}{16\Delta^2}=\frac{1\times 9}{16\Delta^2}=\left(\frac{3}{4\Delta}\right)^2$$

3. **(d)** $\because$ A, B, C are in A.P., $\therefore$ $B=60^\circ$

Now $\dfrac{a}{c}\sin 2C+\dfrac{c}{a}\sin 2A$

$$=\frac{\sin A}{\sin C}.2\sin C\cos C+\frac{\sin C}{\sin A}.2\sin A\cos A$$

$$=2(\sin A\cos C+\cos A\sin C)=2\sin(A+C)$$

$$=2\sin(\pi-B)=2\sin B=2\times\frac{\sqrt{3}}{2}=\sqrt{3}$$

4. **(b)** Given : $AB \parallel CD$, AD perpendicular to both AB and CD, $CD=2AB$ Let $AB=a$ then $CD=2a$

Let radius of circle be r and circle touches AB at P, BC at Q, AD at R and CD at S.

Then $AR=AP=r$, $BP=BQ=a-r$

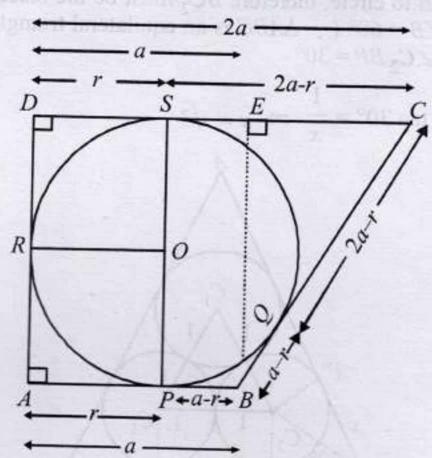
$DR=DS=r$ and $CQ=CS=2a-r$

Now in ΔBEC, $BC^2=BE^2+EC^2$

$$\Rightarrow (a-r+2a-r)^2=(2r)^2+(a)^2$$

$$\Rightarrow 9a^2+4r^2-12ar=4r^2+a^2 \Rightarrow a=\frac{3}{2}r$$

Now, area (quad. $ABCD$) = 18



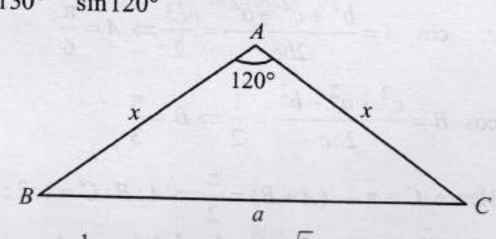
$$\Rightarrow \text{Area (quad. } ABED) + \text{Area } (\Delta BEC) = 18$$

$$\Rightarrow a\times 2r+\frac{1}{2}\times a\times 2r=18$$

$$\Rightarrow ar=6 \Rightarrow \frac{3r^2}{2}=6 \Rightarrow r=2 \quad \left[\because a=\frac{3}{2}r\right]$$

5. **(c)** By Sine law,

$$\frac{x}{\sin 30^\circ}=\frac{a}{\sin 120^\circ} \Rightarrow a=x\sqrt{3}$$



$$\therefore \Delta=\frac{1}{2}\times x\times x\sin 120^\circ=\frac{\sqrt{3}}{4}x^2 \quad \text{.... (i)}$$

Now, $r=\dfrac{\Delta}{s} \Rightarrow rs=\Delta \Rightarrow \dfrac{(2x+a)}{2}\sqrt{3}=\dfrac{\sqrt{3}}{4}x^2$

$$\Rightarrow x=2(2+\sqrt{3})$$

On putting the value of x in (i), we get

$$\Delta=\frac{\sqrt{3}}{4}\times 4(4+3+4\sqrt{3})=7\sqrt{3}+12 \text{ sq.units.}$$

6. **(b)** By Sine law, $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k$ (let)

$$\therefore\ \frac{b-c}{a}=\frac{\sin B-\sin C}{\sin A}=\frac{2\cos\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)}{2\sin A/2\cos A/2}$$

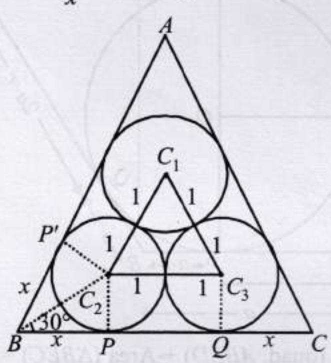
$$=\frac{\sin A/2\sin\left(\frac{B-C}{2}\right)}{\sin A/2\cos A/2}=\frac{\sin\left(\frac{B-C}{2}\right)}{\cos A/2}$$

$\therefore\quad (b-c)\cos A/2=a\sin\left(\frac{B-C}{2}\right)$

7. **(b)** Let C_1, C_2 and C_3 be the centres of the three circular coins. For circle with centre C_2, BP and BP' are two tangents from B to circle, therefore BC_2 must be the bisector of $\angle B$. But $\angle B=60°$ ($\because$ ΔABC is an equilateral triangle)

$\therefore\quad \angle C_2BP=30°$

$\Rightarrow\quad \tan 30°=\frac{1}{x}\Rightarrow x=\sqrt{3}$



$\therefore\quad BC=BP+PQ+QC=x+2+x=2+2\sqrt{3}$

$\therefore$ Area of $\Delta ABC=\frac{\sqrt{3}}{4}\times(2+2\sqrt{3})^2=4\sqrt{3}+6$ sq. units.

8. **(d)** Given : Sides are in the ratio $1:\sqrt{3}:2$

Let $a=k$, $b=\sqrt{3}\,k$ and $c=2k$

$\therefore\quad \cos A=\frac{b^2+c^2-a^2}{2bc}=\frac{\sqrt{3}}{2}\Rightarrow A=\frac{\pi}{6}$

$\cos B=\frac{c^2+a^2-b^2}{2ac}=\frac{1}{2}\Rightarrow B=\frac{\pi}{3}$

Hence $C=\pi-(A+B)=\frac{\pi}{2}\Rightarrow A:B:C=1:2:3$

9. **(a)** Given : $A:B:C=4:1:1$

Let $A=4x$, $B=x$, $C=x$ But $A+B+C=180°$

$\therefore\quad 4x+x+x=180°\Rightarrow x=30°$

$\therefore\quad A=120°$, $B=30°$ and $C=30°$

Now by sine law, $\frac{a}{\sin 120°}=\frac{b}{\sin 30°}=\frac{c}{\sin 30°}$

$\Rightarrow\quad \frac{a}{\sqrt{3}/2}=\frac{b}{1/2}=\frac{c}{1/2}\Rightarrow a:b:c=\sqrt{3}:1:1$

$\therefore\quad$ Ratio of longest side to the perimeter

$=\sqrt{3}:1+1+\sqrt{3}=\sqrt{3}:2+\sqrt{3}$

10. **(d)** We know by Sine law in ΔABC as

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin(\pi-A-B)}=2R$$

$$\Rightarrow\quad \frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin(A+B)}=2R$$

(a) By using the value of a, $\sin A$, $\sin B$; we can find the value of b, c, $\angle A$, $\angle B$ and $\angle C$.

(b) By using the value of a, b, c we can find the value of $\angle A$, $\angle B$ and $\angle C$ using cosine law.

(c) By using the value of a, $\sin B$, R we can find the value of $\sin A$, b and then $\sin(A+B)$ and hence $\angle C$ can be found.

(d) By using the value of a, $\sin A$, R then we can find only the ratio $\frac{b}{\sin B}=\frac{c}{\sin(A+B)}$. We can not find the values of b, c, $\sin B$, $\sin C$ separately.

$\therefore$ Δ can not be determined in this case.

11. **(a)** We know $2R=\frac{c}{\sin C}\Rightarrow c=2R\quad (\because \angle C=90°)$

Now, $\tan\frac{C}{2}=\frac{r}{s-c}$

$\Rightarrow r=s-c\quad (\because \angle C=90°)$

$\Rightarrow a+b-c=2r\Rightarrow 2r+c=a+b$

$\Rightarrow\quad 2r+2R=a+b\qquad (\because\ c=2R)$

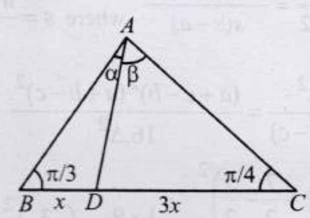
12. **(b)** In ΔABC, $A+B+C=180°$

$\Rightarrow\quad A+C-B=180°-2B$

Now $2ac\sin\left[\frac{1}{2}(A-B+C)\right]=2ac\sin(90°-B)$

$=2ac\cos B=\frac{2ac\,(a^2+c^2-b^2)}{2ac}=a^2+c^2-b^2$

13. **(a)**



In ΔABD, on applying Sine law we get

$$\frac{AD}{\sin\pi/3}=\frac{x}{\sin\alpha}\Rightarrow AD=\frac{\sqrt{3}x}{2\sin\alpha}\quad \text{....(i)}$$

In ΔACD, applying Sine law, we get

$$\frac{AD}{\sin\pi/4}=\frac{3x}{\sin\beta}\Rightarrow AD=\frac{3x}{\sqrt{2}\sin\beta}\quad \text{....(ii)}$$

From (i) and (ii), $\frac{\sqrt{3}x}{2\sin\alpha}=\frac{3x}{\sqrt{2}\sin\beta}\ \therefore\ \frac{\sin\alpha}{\sin\beta}=\frac{1}{\sqrt{6}}$

14. **(c)** Let $a=3$, $b=5$, $c=7$ then the largest angle is opposite to the longest side, i.e., $\angle C$

Now, $\cos C=\frac{a^2+b^2-c^2}{2ab}=\frac{9+25-49}{2\times3\times5}=\frac{-1}{2}$

$\therefore\quad C=2\pi/3$

15. **(c)** Given : $A > B$ and $3\sin x - 4\sin^3 x - k = 0$, $0 < k < 1$

$\Rightarrow \sin 3x = k$

Since A and B satisfy above equation.

$\therefore$ $\sin 3A = k$ and $\sin 3B = k \Rightarrow \sin 3A - \sin 3B = 0$

$$\Rightarrow 2\cos\frac{3A+3B}{2}\sin\frac{3A-3B}{2} = 0$$

$$\Rightarrow \cos\left(\frac{3A+3B}{2}\right) = 0 \text{ or } \sin\left(\frac{3A-3B}{2}\right) = 0$$

$$\Rightarrow \frac{3A+3B}{2} = \frac{\pi}{2} \text{ or } \frac{3A-3B}{2} = 0$$

$$\Rightarrow A + B = \frac{\pi}{3} \text{ or } A = B$$

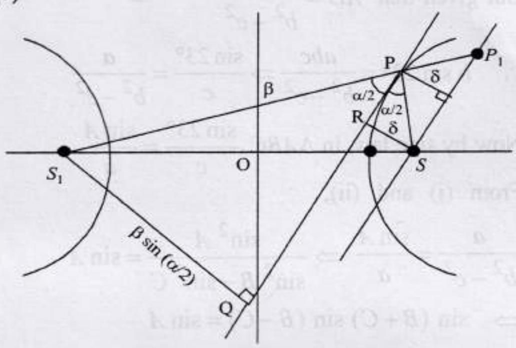
But given that $A > B$, $\therefore A \neq B$ Hence, $A + B = \frac{\pi}{3}$

But $A + B + C = \pi \Rightarrow C = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ $\therefore C = 2\pi/3$

16. **(c)** We know that bisector of an angle, in a triangle, divides the opposite side in the same ratio as the sides containing the angle.

$\therefore QS : SR = PQ : PR$

17. **(7)**



In ΔS_1QP, $\sin\frac{\alpha}{2} = \frac{S_1Q}{\beta} \Rightarrow S_1Q = \beta\sin\frac{\alpha}{2}$

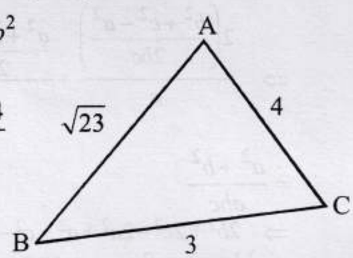
Product of distances of any tangent from two foci = b^2

$\delta . S_1Q = \delta \times \beta\sin\frac{\alpha}{2} = b^2$

So, $\frac{\beta\delta\sin\frac{\alpha}{2}}{9} = \frac{b^2}{9} = \frac{64}{9}$

$\therefore \left[\frac{64}{9}\right] = 7$

18. **(2)**



Given that c = $\sqrt{23}$; a = 3; b = 4

We have $\cot A = \frac{\cos A}{\sin A} = \frac{b^2 + c^2 - a^2}{2bc\sin A}$

$$= \frac{b^2 + c^2 - a^2}{2.2\Delta}\left\{\Delta = \frac{1}{2}bc\sin A\right\}$$

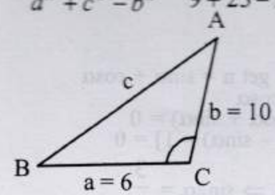
$$\therefore \cot A = \frac{b^2 + c^2 - a^2}{4\Delta}$$

Similarly, $\cot B = \frac{a^2 + c^2 - b^2}{4\Delta}$ & $\cot C = \frac{a^2 + b^2 - c^2}{4\Delta}$

Now, $\frac{\cot A + \cot C}{\cot B} = \frac{b^2 + c^2 - a^2 + a^2 + b^2 - c^2}{a^2 + c^2 - b^2}$

$$= \frac{2b^2}{a^2 + c^2 - b^2} = \frac{2(16)}{9 + 23 - 16} = \frac{32}{16} = 2$$

19. **(3)**



We know that area of triangle $= \frac{1}{2}ab\sin C$

$\Rightarrow \frac{1}{2} \times 6 \times 10 \times \sin C = 15\sqrt{3} \Rightarrow \sin C = \frac{\sqrt{3}}{2}$

$\Rightarrow C = 120°$ as C is obtuse angle.

Now $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$, [cosine rule]

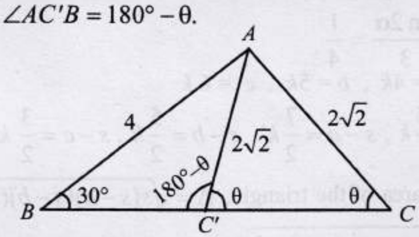
$\therefore \cos 120° = \frac{36 + 100 - c^2}{120}$

$\Rightarrow c^2 = 196 \Rightarrow c = 14$, $\therefore s = \frac{a+b+c}{2} = 15$

Hence radius of incircle, $r = \frac{\Delta}{s} = \frac{15\sqrt{3}}{15} = \sqrt{3}$

$\therefore r^2 = 3$

20. **(4)** Let $\angle ACC' = \theta$, then $\angle AC'C = \theta$ ($\because AC = AC'$) and $\angle AC'B = 180° - \theta$.



Applying sine law in $\Delta ABC'$, we get

$$\frac{4}{\sin(180 - \theta)} = \frac{2\sqrt{2}}{\sin 30°} \Rightarrow \sin\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = 45°$$

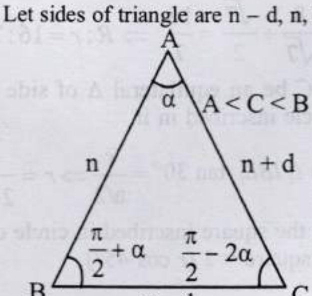
$\therefore \angle CAC' = 90°$

Now required area $= ar(\Delta ABC) - ar(\Delta ABC')$

$= ar(\Delta ACC') = \frac{1}{2} \times AC \times AC'$

$= \frac{1}{2} \times 2\sqrt{2} \times 2\sqrt{2} = 4$ sq. units.

21. **(1008.00)** Let sides of triangle are n – d, n, n + d.



$n - d = 2\sin\alpha$ ... (i)

$n + d = 2\sin\left(\frac{\pi}{2}+\alpha\right)$

$\Rightarrow n + d = 2\cos\alpha$ ... (ii)

$n = 2\sin\left(\frac{\pi}{2}-2\alpha\right)$

$\Rightarrow n = 2\cos 2\alpha$ ...(iii)

Adding (i) and (ii), we get $n = \sin\alpha + \cos\alpha$

$\Rightarrow 2\cos 2\alpha = \sin\alpha + \cos\alpha$

$2(\cos^2\alpha - \sin^2\alpha) - (\cos\alpha + \sin\alpha) = 0$

$(\cos\alpha + \sin\alpha)\,[2(\cos\alpha - \sin\alpha) - 1] = 0$

$\Rightarrow 2(\cos\alpha - \sin\alpha) = 1 \Rightarrow \sin 2\alpha = \frac{3}{4}$

Then, $a = \frac{1}{2}.n.(n + d).\sin\alpha = \frac{1}{2}.\,2\cos 2\alpha.\,2\cos\alpha.\,\sin\alpha$

$= \sin 2\alpha.\ \cos 2a$

$= \frac{3}{4}\times\frac{\sqrt{7}}{4} = \frac{3\sqrt{7}}{16}$

$(64a)^2 = \left(64\times\frac{3\sqrt{7}}{16}\right)^2 = 16\times 9\times 7 = 1008.$

22. **(0.25)** From solution 14

$r = \frac{\Delta}{s} = \frac{1}{2}\frac{n(n+d)\sin\alpha}{\left(\frac{3n}{2}\right)}$

$= \frac{(n+d).\sin\alpha}{3}$

$= \frac{2\cos\alpha.\sin\alpha}{3}(\text{from (ii)})$

$r = \frac{\sin 2\alpha}{3} = \frac{1}{4}$

23. Let $a = 4k$, $b = 5k$, $c = 6k$

$s = \frac{15}{2}k$, $s-a = \frac{7}{2}k$, $s-b = \frac{5}{2}k$, $s-c = \frac{3}{2}k$

Now, area of the triangle, $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{15\times7\times5\times3\left(\frac{k}{2}\right)^4} = 15\sqrt{7}\left(\frac{k}{2}\right)^2$

Radius of circle, $r = \frac{\Delta}{s} = 15\sqrt{7}\left(\frac{k}{2}\right)^2 \div \frac{15}{2}k = \sqrt{7}\frac{k}{2}$

Radius of circumcircle, $R_2 = \frac{abc}{4S}$

Radius of incircle $= \frac{4.5.6k^3}{4.15\sqrt{7}\,k^2/4} = \frac{8}{\sqrt{7}}k.$

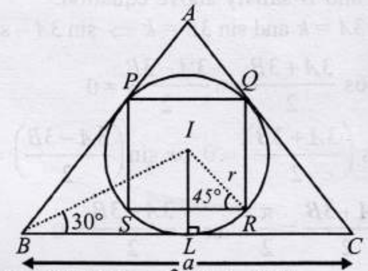
$\therefore\quad \frac{R}{r} = \frac{8}{\sqrt{7}} \div \frac{\sqrt{7}}{2} = \frac{16}{7} \Rightarrow R : r = 16 : 7$

24. Let that ABC be an equilateral Δ of side a and r be the radius of circle inscribed in it.

$\therefore$ In right ΔIBL, $\tan 30° = \frac{r}{a/2} \Rightarrow r = \frac{a}{2\sqrt{3}}$

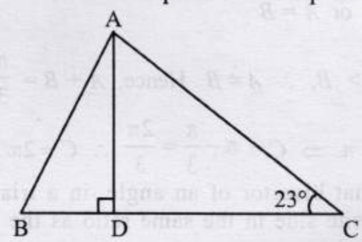
If $PQRS$ be the square inscribed in circle of radius r, then side of square = 2 ($r\cos 45°$)

$= \frac{2r}{\sqrt{2}} = \frac{2}{\sqrt{2}}\times\frac{a}{2\sqrt{3}} = \frac{a}{\sqrt{6}}$



$\therefore$ Area of square $= a^2/6$ sq. units.

25.



In right angle ΔADC, $AD = b\sin 23°$

But given that $AD = \frac{abc}{b^2-c^2}$

$\therefore\quad b\sin 23° = \frac{abc}{b^2-c^2} \Rightarrow \frac{\sin 23°}{c} = \frac{a}{b^2-c^2}$ ....(i)

Now by sine law, in ΔABC $\frac{\sin 23°}{c} = \frac{\sin A}{a}$ ....(ii)

From (i) and (ii),

$\frac{a}{b^2-c^2} = \frac{\sin A}{a} \Rightarrow \frac{\sin^2 A}{\sin^2 B - \sin^2 C} = \sin A$

$\Rightarrow\ \sin(B+C)\sin(B-C) = \sin A$

$\Rightarrow\ \sin(B-C) = 1 \Rightarrow B - C = 90° \Rightarrow B = 90° + 23° = 113°$

26. In ΔABC, $\frac{2\cos A}{a} + \frac{\cos B}{b} + \frac{2\cos C}{c} = \frac{a}{bc} + \frac{b}{ca}$

$\Rightarrow \frac{2\left(\frac{b^2+c^2-a^2}{2bc}\right)}{a} + \frac{\frac{a^2+c^2-b^2}{2ac}}{b} + \frac{2\left(\frac{b^2+a^2-c^2}{2ab}\right)}{c}$

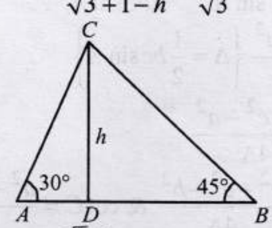
$= \frac{a^2+b^2}{abc}$

$\Rightarrow\ 2b^2 + 2c^2 - 2a^2 + a^2 + c^2 - b^2 + 2a^2 + 2b^2 - 2c^2 = 2a^2 + 2b^2$

$\Rightarrow b^2 + c^2 = a^2 \Rightarrow \Delta ABC$ is right angled at A $\therefore \angle A = 90°$

27. $BD = BC = h$

$\tan 30° = \frac{h}{\sqrt{3}+1-h} = \frac{1}{\sqrt{3}}$

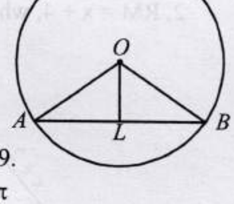


$\Rightarrow \quad h\sqrt{3}+h=\sqrt{3}+1 \Rightarrow h=1$

$\therefore \quad \text{Area} = \frac{1}{2}\times(\sqrt{3}+1)\times 1 = \frac{\sqrt{3}+1}{2}$ sq. units

28. Let $AB = 2$ units be one of the sides of the polygon.

$\therefore \quad \angle AOB = 2\pi/9$ where O is the centre of circle.



If $OL \perp AB$, then $AL = 1$ and $\angle AOL = \pi/9$

Now $OA = AL$ cosec $\pi/9$ = cosec $\pi/9$.

$\therefore \quad$ Radius of the circle = cosec $\frac{\pi}{9}$

29. $\because \quad \cot A, \ \cot B, \ \cot C$ are in A.P.

$\therefore \quad \cot B - \cot A = \cot C - \cot B$

$\Rightarrow \quad \frac{\sin(A-B)}{\sin A \sin B} = \frac{\sin(B-C)}{\sin B \sin C}$

$\Rightarrow \quad \sin(A-B)\sin(A+B) = \sin(B+C)\sin(B-C)$

$\Rightarrow \quad \sin^2 A - \sin^2 B = \sin^2 B - \sin^2 C$

$\Rightarrow \quad a^2 - b^2 = b^2 - c^2 \ \Rightarrow \ a^2, b^2, c^2$ are in A.P.

30. Let $x = a^2 + 2a$; $y = 2a + 3$ and $z = a^2 + 3a + 8$

If x, y and z are sides of a Δ, then $x + y > z$

$\Rightarrow a^2 + 4a + 3 > a^2 + 3a + 8 \Rightarrow a > 5$ .... (i)

$y + z > x \Rightarrow a^2 + 5a + 11 > a^2 + 2a \Rightarrow 3a > -11 \Rightarrow a > -11/3$ .... (ii)

and $z + x > y$

$\Rightarrow \ 2a^2 + 5a + 8 > 2a + 3 \ \Rightarrow \ 2a^2 + 3a + 5 > 0$

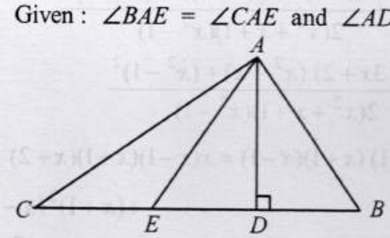
Here coeff. of $a^2 > 0$ and $D = 9 - 40 = -$ve

$\therefore$ It is true for all values of a. Hence an identity.

From (i) and (ii), we get $a > 5$.

$\therefore \quad a \in (5, \infty)$

31. Given : $\angle BAE = \angle CAE$ and $\angle ADB = \angle ADC = 90°$



Now $\quad \angle DAE = \angle BAE - \angle BAD$

$= \angle CAE - (90° - \angle B)$

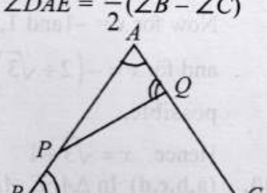
$= (\angle CAD - \angle DAE) - 90° + \angle B$

$= (90° - \angle C) - \angle DAE - 90° + \angle B$

$\Rightarrow 2\angle DAE = \angle B - \angle C \quad \Rightarrow \angle DAE = \frac{1}{2}(\angle B - \angle C)$

32. In ΔAPQ and ΔACB

$\angle A = \angle A$ (common)



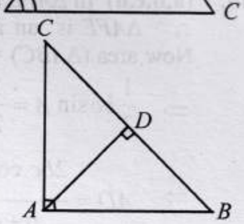
$\angle AQP = \angle ABC$ (given)

$\therefore \quad \Delta APQ \sim \Delta ACB$

(By AA similarity)

$\Rightarrow \quad \frac{Ar(\Delta APQ)}{Ar(\Delta ACB)} = \frac{AP^2}{AC^2}$

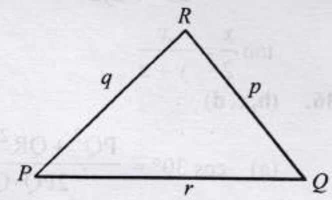
33. We know that altitude from right vertex to hypotenuse in right angled triangle divides it into two triangles each being similar to the original triangle.



$\therefore \quad \Delta BDA \sim \Delta BAC$

$\Rightarrow \quad \frac{BD}{BA} = \frac{AB}{BC}$

34. **(a, b)**



(a) Since $AM \ge G.M$

$\therefore \quad \frac{p+q}{2} \ge \sqrt{pq}$

$\Rightarrow \quad P^2 + q^2 \ge 2pq$

$\cos P = \frac{q^2 + r^2 - P^2}{2qr} = \frac{q^2 + r^2}{2pr} - \frac{P^2}{2qr} \ge 1 - \frac{P^2}{2qr}$

So (a) is correct.

(b) $(p+q)\cos R \ge (q-r)\cos P + (p-r)\cos Q$

$\Rightarrow (p\cos R + r\cos P) + (q\cos R + r\cos Q) \ge q\cos P + p\cos Q$

[By projection formula $p = q\cos R + r\cos Q$]

$\Rightarrow \quad q + p \ge r$

[In triangle sum of two sides greater than third sides]

So (b) is correct.

(c) By sine rule,

$P = K\sin P, q = K\sin Q, r = K\sin R$

$\frac{q+r}{P} = \frac{\sin Q + \sin R}{\sin P} \ge \frac{2\sqrt{\sin Q \times \sin R}}{\sin P}$

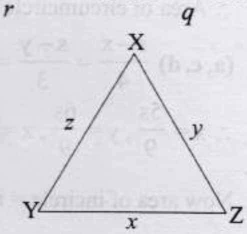
$[\because \ AM \ge GM]$

So (c) is incorrect.

(d) If $p < q$ and $q < r$ then p is smallest side, therefore one of Q or R be obtuse angle. So, one of $\cos Q$ or $\cos R$ can be negative. Therefore $\cos Q > \frac{p}{r}$ and $\cos R > \frac{p}{q}$ cannot hold always.

So (d) is incorrect.

35. **(b, c)** $\quad$ Let $x + y + z = 2s$



$\tan\frac{x}{2} + \tan\frac{z}{2} = \frac{2y}{x+y+z}$

$\Rightarrow \frac{\Delta}{s(s-x)} + \frac{\Delta}{s(s-z)} = \frac{2y}{2s}$

$\Rightarrow \frac{\Delta}{s}\left(\frac{2s-(x+z)}{(s-x)(s-z)}\right) = \frac{y}{s}$

$\Rightarrow \Delta^2 = (s-x)^2(s-z)^2 \Rightarrow s(s-y) = (s-x)(s-z)$

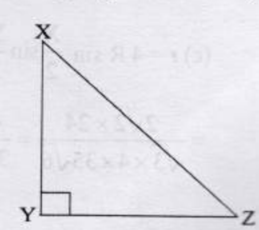
$\Rightarrow (x+y+z)(x+z-y) = (y+z-x)(x+y-z)$

$\Rightarrow (x+z)^2 - y^2 = y^2 - (z-x)^2$

$\Rightarrow (x+z)^2 + (x-z)^2 = 2y^2 \Rightarrow x^2 + z^2 = y^2 \Rightarrow \angle Y = \frac{\pi}{2}$

(By converse of Pythagoras theorem)

$\Rightarrow \angle Y = \angle X + \angle Z$



$\tan\frac{x}{2} = \frac{\Delta}{s(s-x)}$

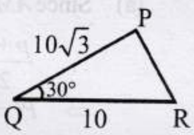
$\tan\frac{x}{2} = \frac{\frac{1}{2}xz}{\frac{(y+z)^2 - x^2}{4}}$

$\tan\frac{x}{2} = \frac{2xz}{2z^2 + 2yz}$ $\quad(\because y^2 = x^2 + z^2)$

$\tan\frac{x}{2} = \frac{x}{y+z}$

36. **(b, c, d)**

(a) $\cos 30° = \frac{PQ^2 + QR^2 - PR^2}{2PQ \cdot QR}$ [cosine law]



$\Rightarrow \frac{\sqrt{3}}{2} = \frac{(10\sqrt{3})^2 + 10^2 - PR^2}{2\times 10\sqrt{3}\times 10}$

$\Rightarrow PR^2 = 100 \Rightarrow PR = 10$

$\therefore \angle P = \angle Q = 30°$, $\therefore$ (a) is false.

(b) Area of $\Delta PQR = \frac{1}{2} PQ \times QR \times \sin 30°$

$= \frac{1}{2}\times 10\sqrt{3}\times 10\times\frac{1}{2} = 25\sqrt{3}$

Also $\angle R = 180° - 30° - 30° = 120°$, $\therefore$ (b) is true.

(c) $r = \frac{\Delta}{s} = \frac{25\sqrt{3}}{\left(\frac{10\sqrt{3}+10+10}{2}\right)} = \frac{5\sqrt{3}}{2+\sqrt{3}}$

$= 5\sqrt{3}(2-\sqrt{3}) = 10\sqrt{3} - 15$, $\therefore$ (c) is true.

(d) $R = \frac{abc}{4\Delta} = \frac{10\sqrt{3}\times 10\times 10}{4\times 25\sqrt{3}} = 10$

$\therefore$ Area of circumcircle $= \pi R^2 = 100\pi$, $\therefore$ (d) is true.

37. **(a, c, d)** $\frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2} = \frac{s-x+s-y+s-z}{4+3+2} = \frac{s}{9}$

$\therefore x = \frac{5s}{9}, y = \frac{6s}{9}, z = \frac{7s}{9}$

Now area of incircle $= \pi r^2 = \pi\frac{\Delta^2}{s^2} = \frac{8\pi}{3}$

$\Rightarrow \frac{s(s-x)(s-y)(s-z)}{s^2} = \frac{8}{3}$

$\Rightarrow \frac{4\times 3\times 2s^3}{9\times 9\times 9s} = \frac{8}{3} \Rightarrow s = 9 \therefore x = 5, y = 6, z = 7$

(a) Area $(\Delta XYZ) = \sqrt{s(s-x)(s-y)(s-z)}$

$= \sqrt{9\times 4\times 3\times 2} = 6\sqrt{6}$

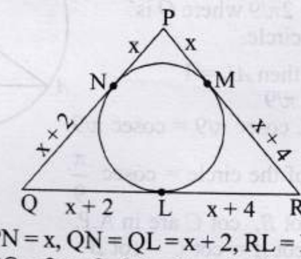
(b) Radius of circumcircle, $R = \frac{xyz}{4\Delta} = \frac{5\times 6\times 7}{4\times 6\sqrt{6}} = \frac{35\sqrt{6}}{24}$

(c) $r = 4R\sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2} \Rightarrow \sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2} = \frac{r}{4R}$

$= \frac{2\sqrt{2}\times 24}{\sqrt{3}\times 4\times 35\sqrt{6}} = \frac{4}{35}\left[\because r = \frac{2\sqrt{2}}{\sqrt{3}} \text{ and } R = \frac{35\sqrt{6}}{24}\right]$

(d) $\sin^2\frac{X+Y}{2} = \cos^2\frac{Z}{2} = \frac{s(s-z)}{xy} = \frac{9\times 2}{5\times 6} = \frac{3}{5}$

38. **(b, d)** Since length of PN, QL and RM are consecutive even integers, therefore we can suppose PN = x, QL = x + 2, RM = x + 4, where x is an even integer.



$\therefore$ PM = PN = x, QN = QL = x + 2, RL = RM = x + 4

Hence, PQ = 2x + 2, QR = 2x + 6, PR = 2x + 4

Now $\cos P = \frac{1}{3} \Rightarrow \frac{PQ^2 + PR^2 - QR^2}{2PQ.PR} = \frac{1}{3}$

$\Rightarrow \frac{(2x+2)^2 + (2x+4)^2 - (2x+6)^2}{2.(2x+2).(2x+4)} = \frac{1}{3}$

$\Rightarrow 3[(x+1)^2 + (x+2)^2 - (x+3)^2] = 2(x+1)(x+2)$

$\Rightarrow 3(x^2 - 4) = 2(x+1)(x+2) \Rightarrow x = 8$

$\therefore$ PQ = 18, QR = 22, PR = 20

39. **(b)** Given : $a = x^2 + x + 1, b = x^2 - 1, c = 2x + 1$ and $\angle C = \pi/6$

Now, $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$ [cosine rule]

$\Rightarrow \cos\frac{\pi}{6} = \frac{(x^2+x+1)^2 + (x^2-1)^2 - (2x+1)^2}{2(x^2+x+1)(x^2-1)}$

$\Rightarrow \frac{\sqrt{3}}{2} = \frac{(x^2+3x+2)(x^2-x) + (x^2-1)^2}{2(x^2+x+1)(x^2-1)}$

$\Rightarrow \sqrt{3}(x^2+x+1)(x+1)(x-1) = x(x-1)(x+1)(x+2) + (x+1)^2(x-1)^2$

$\Rightarrow (x+1)(x-1)\left[\sqrt{3}(x^2+x+1) - x(x+2) - (x+1)(x-1)\right] = 0$

$\Rightarrow (x+1)(x-1)\left[\left(\sqrt{3}-2\right)x^2 + \left(\sqrt{3}-2\right)x + \left(\sqrt{3}+1\right)\right] = 0$

$\therefore x = -1, 1, \left(\sqrt{3}+1\right), -\left(\sqrt{3}+2\right)$

Now for $x = -1$ and 1, $b = 0$ which is not possible and for $x = -\left(2+\sqrt{3}\right)$, $c = -4 - 2\sqrt{3} + 1 < 0$, which is not possible.

Hence, $x = \sqrt{3} + 1$

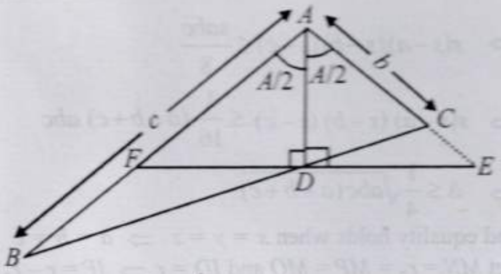
40. **(a,b,c,d)** In ΔAFE, $AF = AE$ [By simple geometry]

$\therefore$ ΔAFE is an isosceles triangle.

Now area (ΔABC) = area (ΔABD) + area (ΔADC)

$\Rightarrow \frac{1}{2}bc\sin A = \frac{1}{2}cAD\sin\frac{A}{2} + \frac{1}{2}bAD\sin\frac{A}{2}$

$\Rightarrow AD = \frac{2bc\cos\frac{A}{2}}{b+c}$ ....(i)

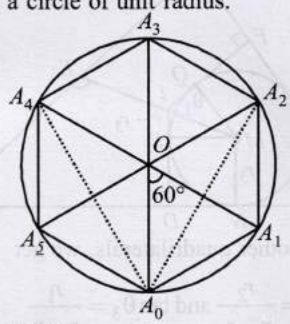


Also $AD = AE \cos \frac{A}{2}$ .... (ii)

From (i) and (ii), $AE = \frac{2bc}{b+c} =$ HM of b and c.

Now, $EF = 2DE = 2.\ AD \tan\frac{A}{2} = \frac{4bc\sin\frac{A}{2}}{b+c}$

41. **(c)** Given that $A_0A_1A_2A_3A_4A_5$ is a regular hexagon inscribed in a circle of unit radius.



$\angle A_0OA_1 = \frac{360°}{6} = 60°$

In ΔOA_0A_1, $OA_0 = OA_1 = 1$

$\therefore\ \angle OA_0A_1 = \angle OA_1A_0 = 60°$

$\Rightarrow\ \Delta OA_0A_1$ is an equilateral triangle.

$\therefore\ A_0A_1 = 1 = A_1A_2 = A_2A_3 = A_3A_4 = A_4A_5 = A_5A_0$

$\Rightarrow\ \angle A_0A_1A_2 = 60° + 60° = 120°$

In $\Delta A_0A_1A_2$, $\cos 120° = \frac{(A_0A_1)^2 + (A_1A_2)^2 - (A_0A_2)^2}{2(A_0A_1)\ (A_1A_2)}$

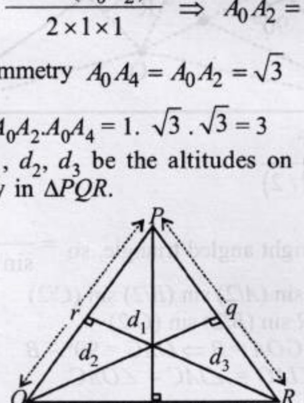
[using cosine law]

$\Rightarrow\ -\frac{1}{2} = \frac{1+1-(A_0A_2)^2}{2\times1\times1} \Rightarrow A_0A_2 = \sqrt{3}$

Now by symmetry $A_0A_4 = A_0A_2 = \sqrt{3}$

$\therefore\ A_0A_1.A_0A_2.A_0A_4 = 1.\ \sqrt{3}\ .\sqrt{3} = 3$

42. **(b)** Let d_1, d_2, d_3 be the altitudes on *QR, RP* and *PQ* respectively in ΔPQR.



$\therefore$ area $(\Delta PQR) = \Delta = \frac{1}{2}pd_1 = \frac{1}{2}qd_2 = \frac{1}{2}rd_3$

$\Rightarrow\ d_1 = \frac{2\Delta}{p}, d_2 = \frac{2\Delta}{q}, d_3 = \frac{2\Delta}{r}$

$\Rightarrow\ d_1 = \frac{2\Delta}{k\sin P}, d_2 = \frac{2\Delta}{k\sin Q}, d_3 = \frac{2\Delta}{k\sin R}$

[using sine law]

$\Rightarrow\ d_1, d_2, d_3$ are in H.P. (As given that $\sin P$, $\sin Q$, $\sin R$ are in A.P.)

43. **(a, d)** Since the angles of triangle are in A.P., Let $\angle A = x + d$, $\angle B = x$, $\angle C = x - d$

Then by $\angle$ sum property of triangle, we have

Now $\angle A + \angle B + \angle C = 180°$

$\therefore\ x + d + x + x - d = 180° \Rightarrow x = 60°\quad \therefore\ \angle B = 60°$

Now $\cos B = \frac{a^2 + c^2 - b^2}{2ac}$ (cosine formula)

$\Rightarrow\ \cos 60° = \frac{100 + c^2 - 81}{2\times10\times c} \Rightarrow c^2 - 10c + 19 = 0$

$\Rightarrow\ c = \frac{10\pm\sqrt{100-76}}{2} \Rightarrow c = 5\pm\sqrt{6}$

Given that $a = 10$, $b = 9$ are the longer sides

$\therefore\ c = 5 + \sqrt{6}$ or $5 - \sqrt{6}$, both are possible.

44. **(a, d)** In a ΔABC, $\frac{\sin A}{a} = \frac{\sin B}{b} \Rightarrow a\sin B = b\sin A$

(a) $b\sin A = a \Rightarrow a\sin B = a$

$\Rightarrow\ \sin B = 1 \Rightarrow B = \pi/2$

$\because\ A < \pi/2$, $\therefore\ \Delta ABC$ is possible.

(b) $b\sin A > a \Rightarrow a\sin B > a \Rightarrow \sin B > 1$, which is impossible. $\therefore\ \Delta ABC$ is not possible. (c) ΔABC is not possible as in (b)

(d) $b\sin A < a \Rightarrow a\sin B < a \Rightarrow \sin B < 1$

$\therefore$ value of $\angle B$ exists.

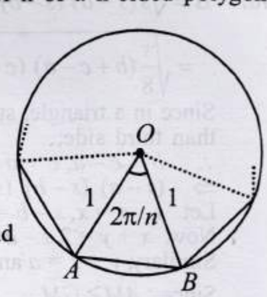
Now, $b > a \Rightarrow B > A$. Since $A < \pi/2$

The ΔABC is possible when either $B > \pi/2$ or $B < \pi/2$.

(e) $\because b = a$, $\therefore\ B = A$. But $A > \pi/2$

$\therefore\ B > \pi/2$. But it is not possible for any triangle.

45. Let *OAB* be one triangle out of n of a n sided polygon inscribed in a circle of radius 1.



Then $\angle AOB = \frac{2\pi}{n}$

$OA = OB = 1$

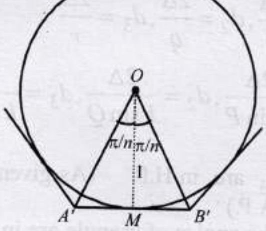
$\therefore$ using Area of isosceles triangle with vertical angle θ and equal sides as r

$= \frac{1}{2}r^2\sin\theta$

$\therefore\ $ area $(\Delta OAB) = \frac{1}{2}\sin\frac{2\pi}{n}$

$\therefore\ I_n = \frac{n}{2}\sin\frac{2\pi}{n}$ ...... (i)

Further consider the n sided polygon subscribing the circle.



$A'MB'$ is the tangent of the circle at $M \Rightarrow A'MB' \perp OM$

$\Rightarrow$ $A'MO$ is right angled triangle, right angle at M.

$\therefore \quad A'M = \tan\dfrac{\pi}{n}$

Now, area of $\Delta A'MO = \dfrac{1}{2} \times 1 \times \tan\dfrac{\pi}{n}$

$\therefore \quad$ Area of $\Delta A'B'O = \tan\dfrac{\pi}{n}$

Hence, $O_n = n\tan\dfrac{\pi}{n}$ ......(ii)

We have to prove $I_n = \dfrac{O_n}{2}\left(1+\sqrt{1-\left(\dfrac{2I_n}{n}\right)^2}\right)$

or $\dfrac{2I_n}{O_n} - 1 = \sqrt{1-\left(\dfrac{2I_n}{n}\right)^2}$

LHS $= \dfrac{2I_n}{O_n} - 1 = \dfrac{n\sin\dfrac{2\pi}{n}}{n\tan\dfrac{\pi}{n}} - 1$ [From (i) and (i)]

$= 2\cos^2\dfrac{\pi}{n} - 1 = \cos\dfrac{2\pi}{n}$

RHS $= \sqrt{1-\left(\dfrac{2I_n}{n}\right)^2} = \sqrt{1-\sin^2\dfrac{2\pi}{n}}$ [From (i)]

$= \cos\dfrac{2\pi}{n} \quad \therefore$ LHS = RHS

46. $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{\dfrac{s}{8}(b+c-a)(c+a-b)(a+b-c)}$

Since in a triangle, sum of two sides is always greater than third side;

$\therefore \quad b+c-a, c+a-b, a+b-c > 0$

$\Rightarrow (s-a)(s-b)(s-c) > 0$

Let $s-a = x, s-b = y, s-c = z$

Now, $x + y = 2s - a - b = c$

Similary, $y + z = a$ and $z + x = b$

Since $AM \geq GM$

$\Rightarrow \dfrac{x+y}{2} \geq \sqrt{xy} \Rightarrow 2\sqrt{xy} \leq c,$

$\dfrac{y+z}{2} \geq \sqrt{yz} \Rightarrow 2\sqrt{yz} \leq a$

and $\dfrac{z+x}{2} \geq \sqrt{zx} \Rightarrow 2\sqrt{zx} \leq b, \quad \therefore 8xyz \leq abc$

$\Rightarrow (s-a)(s-b)(s-c) \leq \dfrac{1}{8}abc$

$\Rightarrow s(s-a)(s-b)(s-c) \leq \dfrac{sabc}{8}$

$\Rightarrow s(s-a)(s-b)(s-c) \leq \dfrac{1}{16}(a+b+c)abc$

$\Rightarrow \Delta \leq \dfrac{1}{4}\sqrt{abc(a+b+c)}$

and equality holds when $x = y = z \Rightarrow a = b = c$

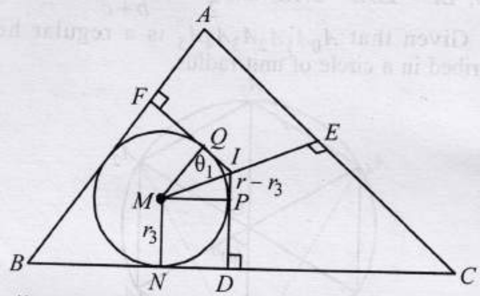
47. Let $MN = r_3 = MP = MQ$ and $ID = r \Rightarrow IP = r - r_3$

Clearly IP and IQ are tangents to circle with centre M.

$\therefore \quad IM$ must be the $\angle$ bisector of $\angle PIQ$

$\therefore \quad \angle PIM = \angle QIM = \theta_1$ (let)

Also from ΔIPM, $\tan\theta_1 = \dfrac{MP}{IP} = \dfrac{r_3}{r - r_3}$



Similarly, in other quadrilaterals, we get

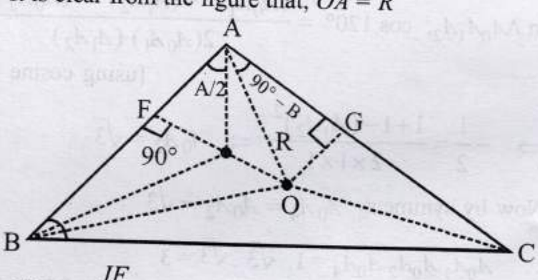
$\tan\theta_2 = \dfrac{r_2}{r - r_2}$ and $\tan\theta_3 = \dfrac{r_1}{r - r_1}$

Also $2\theta_1 + 2\theta_2 + 2\theta_3 = 2\pi \Rightarrow \theta_1 + \theta_2 + \theta_3 = \pi$

$\Rightarrow \tan\theta_1 + \tan\theta_2 + \tan\theta_3 = \tan\theta_1 \cdot \tan\theta_2 \cdot \tan\theta_3$

$\Rightarrow \dfrac{r_1}{r-r_1} + \dfrac{r_2}{r-r_2} + \dfrac{r_3}{r-r_3} = \dfrac{r_1 r_2 r_3}{(r-r_1)(r-r_2)(r-r_3)}$

48. It is clear from the figure that, $OA = R$



$AI = \dfrac{IF}{\sin(A/2)}$

$\therefore$ $DAIF$ is right angled triangle, so $= \dfrac{r}{\sin(A/2)}$

But $r = 4R\sin(A/2)\sin(B/2)\sin(C/2)$

$\therefore \quad AI = 4R\sin(B/2)\sin(C/2)$

Again, $\quad \angle GOA = B \Rightarrow OAG = 90° - B$

Therefore, $\angle IAO = \angle IAC - \angle OAC$

$= A/2 - (90° - B) = \dfrac{1}{2}(A + 2B - 180°)$

$= \dfrac{1}{2}(A + 2B - A - B - C) = \dfrac{1}{2}(B - C)$

In ΔOAI, $OI^2 = OA^2 + AI^2 - 2(OA)(AI)\cos(\angle IAO)$
$= R^2 + [4R\sin(B/2)\sin(C/2)]^2$

$$-2R\cdot[4R\sin(B/2)\sin(C/2)]\cos\left(\frac{B-C}{2}\right)$$

$= [R^2 + 16R^2\sin^2(B/2)\sin^2(C/2)$

$$-8R^2\sin(B/2)\sin^2(C/2)\cos\left(\frac{B-C}{2}\right)\Bigg]$$

$= R^2[1 + 16\sin^2(B/2)\sin^2(C/2)$

$$-8\sin(B/2)\sin(C/2)\cos\left(\frac{B-C}{2}\right)\Bigg]$$

$= R^2[1 + 8\sin(B/2)\sin(C/2)$

$$\left\{2\sin(B/2)\sin(C/2)-\cos\left(\frac{B-C}{2}\right)\right\}\Bigg]$$

$= R^2[1 + 8\sin(B/2)\sin(C/2)$

$$\left\{\cos\left(\frac{B-C}{2}\right)+\cos\left(\frac{B+C}{2}\right)-\cos\left(\frac{B-C}{2}\right)\right\}\Bigg]$$

$$= R^2\left[1-8\sin(B/2)\sin(C/2)\cos\left(\frac{B+C}{2}\right)\right]$$

$$= R^2\left[1-8\sin(B/2)\sin(C/2)\cos\left(\frac{\pi}{2}-\frac{A}{2}\right)\right]$$

$$\left[\because \frac{A}{2}+\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2}\right]$$

$= R^2[1 - 8\sin(A/2)\sin(B/2)\sin(C/2)]$

$$= R^2\left[1-8\left(\frac{r}{4R}\right)\right] = R^2 - 2Rr$$

Now, in right angled ΔBIO,

$OB^2 = BI^2 + IO^2$

$\Rightarrow R^2 = BI^2 + IO^2 \Rightarrow 2Rr = BI^2$

$\Rightarrow 2Rr = r^2/\sin^2(B/2)$

$\Rightarrow 2R = r/\sin^2(B/2)$

$\Rightarrow 2R\sin^2 B/2 = r$

$\Rightarrow R(1-\cos B) = r$

$$\Rightarrow \frac{abc}{4\Delta}(1-\cos B) = r$$

$$\Rightarrow abc(1-\cos B) = \frac{4\Delta^2}{s}$$

$$\Rightarrow abc\left[1-\frac{a^2+c^2-b^2}{2ac}\right] = \frac{4\Delta^2}{s}$$

$$\Rightarrow abc\left[\frac{2ac-a^2-c^2+b^2}{2ac}\right] = \frac{4\Delta^2}{s}$$

$$\Rightarrow b\left[b^2-(a-c)^2\right] = \frac{4\Delta^2}{s}$$

$$\Rightarrow b\left[b^2-(a-c)^2\right] = 8(s-a)(s-b)(s-c)$$

$$\Rightarrow b\left[\{b-(a-c)\}\{b+(a-c)\}\right] = 8(s-a)(s-b)(s-c)$$

$$\Rightarrow b\left[(b+c-a)(b+a-c)\right] = 8(s-a)(s-b)(s-c)$$

$$\Rightarrow b\left[2s-2a(2s-2c)\right] = 8(s-a)(s-b)(s-c)$$

$$\Rightarrow b\left[2.2(s-a)(s-c)\right] = 8(s-a)(s-b)(s-c)$$

$\Rightarrow b = 2s - 2b \Rightarrow 2b = a + c$

Which shows that b is arithmetic mean between and c.

49. Let ABC is an equilateral triangle, then

$A = B = C = 60°$

$\Rightarrow \tan A + \tan B + \tan C = 3\sqrt{3}$

Conversely, suppose

$\tan A + \tan B + \tan C = 3\sqrt{3}$

Now using A.M. $\geq$ G.M. (equality occurs when numbers are equal)

For $\tan A$, $\tan B$, $\tan C$, we get

$$\frac{\tan A+\tan B+\tan C}{3} \geq (\tan A\tan B\tan C)^{1/3}$$

But in any ΔABC, we know that

$\Rightarrow (\tan A + \tan B + \tan C)^{2/3} \geq 3$

$[\because \tan A + \tan B + \tan C = \tan A\tan B\tan C]$

$\Rightarrow \tan A + \tan B + \tan C \geq 3\sqrt{3}$,

where equality occurs when $\tan A$, $\tan B$, $\tan C$ are equal, i.e., $A = B = C$

$\Rightarrow \Delta ABC$ is equilateral.

50. (I) $a, b\ c$ and Δ are rational.

$$\Rightarrow s = \frac{a+b+c}{2} \text{ is also rational}$$

$$\Rightarrow \tan B/2 = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} = \frac{\Delta}{s(s-b)} \text{ is also rational}$$

$$\text{and } \tan C/2 = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \frac{\Delta}{s(s-c)} \text{ is also rational}$$

Therefore (I) $\Rightarrow$ (II).

(II) a, $\tan B/2$, $\tan C/2$ are rational.

$$\Rightarrow \sin B = \frac{2\tan B/2}{1+\tan^2 B/2}$$

$$\text{and } \sin C = \frac{2\tan C/2}{1+\tan^2 C/2} \text{ are rationals.}$$

$$\text{Now } \tan A/2 = \tan\left[90°-\left(\frac{B}{2}+\frac{C}{2}\right)\right] = \cot\left(\frac{B}{2}+\frac{C}{2}\right)$$

$$= \frac{1}{\tan\left(\frac{B}{2}+\frac{C}{2}\right)} = \frac{1-\tan B/2\tan C/2}{\tan B/2+\tan C/2} \text{ is rational}$$

$$\therefore \sin A = \frac{2\tan A/2}{1+\tan^2 A/2} \text{ is rational.}$$

Therefore (II) $\Rightarrow$ (III)

(III) a, $\sin A$, $\sin B$, $\sin C$ are rational.

$$\text{Now } \frac{a}{\sin A} = 2R \Rightarrow R \text{ is rational}$$

$\therefore b = 2R\sin B$, $c = 2R\sin C$ are rationals.

$$\therefore \Delta = \frac{1}{2}bc\sin A \text{ is rational}$$

Therefore (III) $\Rightarrow$ (I).

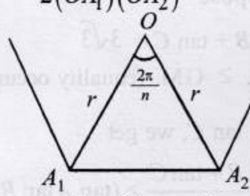
51. Let O be the centre and r be the radius of the circle passing through the vertices $A_1, A_2, \ldots\ldots, A_n$.

Then, $\angle A_1OA_2 = \dfrac{2\pi}{n}$

also, $OA_1 = OA_2 = r$

Again by cos formula, we know that,

$$\cos\left(\frac{2\pi}{n}\right) = \frac{OA_1^2 + OA_2^2 - A_1A_2^2}{2(OA_1)(OA_2)}$$



$$\Rightarrow \cos\left(\frac{2\pi}{n}\right) = \frac{r^2 + r^2 - A_1A_2^2}{2(r)(r)}$$

$$\Rightarrow 2r^2\cos\left(\frac{2\pi}{n}\right) = 2r^2 - A_1A_2^2$$

$$\Rightarrow A_1A_2^2 = 2r^2 - 2r^2\cos\left(\frac{2\pi}{n}\right)$$

$$\Rightarrow A_1A_2^2 = 2r^2\left[1 - \cos\left(\frac{2\pi}{n}\right)\right]$$

$$\Rightarrow A_1A_2^2 = 2r^2.2\sin^2\left(\frac{\pi}{n}\right) \Rightarrow A_1A_2^2 = 4r^2\sin^2\left(\frac{\pi}{n}\right)$$

$$\Rightarrow A_1A_2 = 2r\sin\left(\frac{\pi}{n}\right)$$

Similarly, $A_1A_2 = 2r\sin\left(\dfrac{2\pi}{n}\right)$

and $A_1A_4 = 2r\sin\left(\dfrac{3\pi}{n}\right)$

Since, $\dfrac{1}{A_1A_2} = \dfrac{1}{A_1A_3} + \dfrac{1}{A_1A_4}$ [given]

$$\Rightarrow \frac{1}{2r\sin(\pi/n)} = \frac{1}{2r\sin(2\pi/n)} + \frac{1}{2r\sin(3\pi/n)}$$

$$\Rightarrow \frac{1}{\sin(\pi/n)} = \frac{1}{\sin(2\pi/n)} + \frac{1}{2r\sin(3\pi/n)}$$

$$\Rightarrow \frac{1}{\sin(\pi/n)} = \frac{\sin\left(\frac{3\pi}{n}\right) + \sin\left(\frac{2\pi}{n}\right)}{\sin(2\pi/n)\sin(3\pi/n)}$$

$$\Rightarrow \sin\left(\frac{2\pi}{n}\right).\sin\left(\frac{3\pi}{n}\right) = \sin\left(\frac{\pi}{n}\right)\sin\left(\frac{3\pi}{n}\right) + \sin\left(\frac{\pi}{n}\right).\sin\left(\frac{2\pi}{n}\right)$$

$$\Rightarrow \sin\left(\frac{2\pi}{n}\right)\left[\sin\left(\frac{3\pi}{n}\right) - \sin\left(\frac{\pi}{n}\right)\right] = \sin\left(\frac{\pi}{n}\right)\sin\left(\frac{3\pi}{n}\right)$$

$$\Rightarrow \sin\left(\frac{2\pi}{n}\right)\left[\left\{2\cos\left(\frac{3\pi+\pi}{2n}\right)\sin\left(\frac{3\pi-\pi}{2n}\right)\right\}\right] = \sin\left(\frac{\pi}{n}\right).\sin\left(\frac{3\pi}{n}\right)$$

$$\Rightarrow 2\sin\left(\frac{2\pi}{n}\right).\cos\left(\frac{2\pi}{n}\right).\sin\left(\frac{\pi}{n}\right) = \sin\left(\frac{\pi}{n}\right)\sin\left(\frac{3\pi}{n}\right)$$

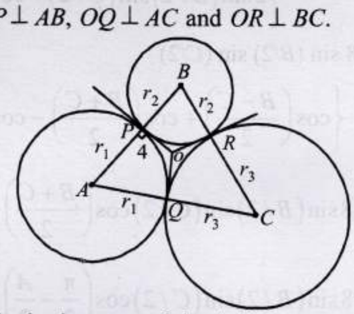
$$\Rightarrow 2\sin\left(\frac{2\pi}{n}\right)\cos\left(\frac{2\pi}{n}\right) = \sin\left(\frac{3\pi}{n}\right)$$

$$\Rightarrow \sin\left(\frac{4\pi}{n}\right) = \sin\left(\frac{3\pi}{n}\right)$$

$$\Rightarrow \frac{4\pi}{n} = \pi - \frac{3\pi}{n} \Rightarrow \frac{7\pi}{n} = \pi$$

$$\Rightarrow n = 7$$

52. Let us consider three circles with centres at A, B and C and with radii r_1, r_2 and r_3 respectively which touch each other externally at P, Q and R. Let the common tangents at P, Q and R meet each other at O. Then $OP = OQ = OR = 4$ [Lengths of tangents from an external point to a circle are equal]. Also $OP \perp AB$, $OQ \perp AC$ and $OR \perp BC$.



$\Rightarrow$ O is the incentre of the ΔABC.

Hence, for ΔABC, $s = \dfrac{(r_1 + r_2) + (r_2 + r_3) + (r_3 + r_1)}{2}$

$$\Rightarrow s = r_1 + r_2 + r_3$$

$\therefore$ $\Delta = \sqrt{(r_1 + r_2 + r_3)\cdot r_1 \cdot r_2 \cdot r_3}$ [Heron's formula]

Now $r = \dfrac{\Delta}{s}$

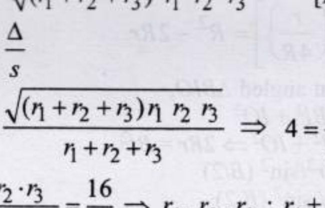
$$\Rightarrow 4 = \frac{\sqrt{(r_1 + r_2 + r_3)r_1\ r_2\ r_3}}{r_1 + r_2 + r_3} \Rightarrow 4 = \frac{\sqrt{r_1\ r_2\ r_3}}{\sqrt{r_1 + r_2 + r_3}}$$

$$\Rightarrow \frac{r_1\cdot r_2\cdot r_3}{r_1 + r_2 + r_3} = \frac{16}{1} \Rightarrow r_1 \cdot r_2 \cdot r_3 : r_1 + r_2 + r_3 = 16 : 1$$

53. In ΔABC, $BC = a$ and $\dfrac{c}{b} = r$

Let altitude $AD = h$

In ΔABD, $h = c\sin B$



$$= \frac{c\,a\sin B}{a}$$

$$= \frac{c\,k\sin A\sin B}{k\sin A} = \frac{c\sin A\sin B}{\sin(B+C)}$$

$$= \frac{c\sin A\sin B\sin(B-C)}{\sin(B+C)\sin(B-C)} = \frac{c\sin A\sin B\sin(B-C)}{\sin^2 B - \sin^2 C}$$

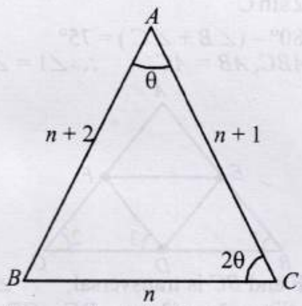
$$= \frac{c \cdot \frac{a}{k} \cdot \frac{b}{k} \cdot \sin(B-C)}{\frac{b^2}{k^2} - \frac{c^2}{k^2}} = \frac{abc \sin(B-C)}{b^2 - c^2}$$

$$= \frac{a\left(\frac{c}{b}\right)\sin(B-C)}{1-\left(\frac{c}{b}\right)^2} = \frac{ar\sin(B-C)}{1-r^2} \le \frac{ar}{1-r^2}$$

$[\because \sin(B-C) \le 1]$

$$\therefore\ h \le \frac{ar}{1-r^2}$$

54. Let the sides of ΔABC be n, $n+1$, $n+2$; where $n \in N$.
Let $a = n$, $b = n+1$, $c = n+2$



Let the smallest angle $\angle A = \theta$ then the greatest $\angle C = 2\theta$

In ΔABC, $\frac{\sin\theta}{n} = \frac{\sin 2\theta}{n+2} \Rightarrow \frac{\sin\theta}{n} = \frac{2\sin\theta\cos\theta}{n+2}$

$\because\ \sin\theta \ne 0$, $\therefore\ \frac{1}{n} = \frac{2\cos\theta}{n+2} \Rightarrow \cos\theta = \frac{n+2}{2n}$ ...(i)

In ΔABC, $\cos\theta = \frac{(n+1)^2 + (n+2)^2 - n^2}{2(n+1)(n+2)}$ [Cosine Law] ...(ii)

From (i) and (ii), $\frac{(n+1)^2 + (n+2)^2 - n^2}{2(n+1)(n+2)} = \frac{n+2}{2n}$

$\Rightarrow\ (n+2)^2\ (n+1) = n\,(n+2)^2 + n\,(n+1)^2 - n^3$

$\Rightarrow\ n\,(n+2)^2 + (n+2)^2 = n\,(n+2)^2 + n\,(n+1)^2 - n^3$

$\Rightarrow\ n^2 + 4n + 4 = n^3 + 2n^2 + n - n^3$

$\Rightarrow\ (n+1)(n-4) = 0 \Rightarrow n = 4$ (as $n \ne -1$)

$\therefore$ Sides of triangle are 4, 5 and 6.

55. Given : In ΔABC, $\cos A \cos B + \sin A \sin B \sin C = 1$

$\Rightarrow\ \sin C = \frac{1-\cos A\cos B}{\sin A \sin B}$

$\because\ \sin C \le 1$, $\therefore\ \frac{1-\cos A\cos B}{\sin A\sin B} \le 1$

$\Rightarrow\ 1 \le \cos A\cos B + \sin A \sin B \Rightarrow 1 \le \cos(A-B)$

$\because\ \cos(A-B) \le 1$, $\therefore\ \cos(A-B) = 1$

$\Rightarrow\ A - B = 0 \Rightarrow A = B$

$\therefore\ \cos A \cos A + \sin A \sin A \sin C = 1$ $[\because A = B]$

$\Rightarrow\ \sin^2 A \sin C = \sin^2 A \Rightarrow \sin^2 A\,(\sin C - 1) = 0$

$\Rightarrow\ \sin A = 0$ or $\sin C = 1$

The only possibility is $\sin C = 1 \Rightarrow C = \pi/2$

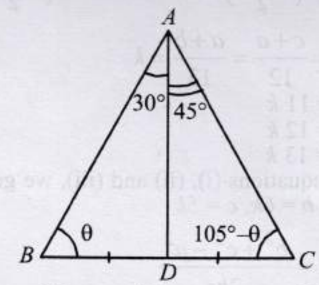
$\Rightarrow\ A + B = \pi/2$

But $A = B \Rightarrow A = B = \pi/4$

Now, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ [sine rule]

$\Rightarrow\ \frac{a}{\sin 45^\circ} = \frac{b}{\sin 45^\circ} = \frac{c}{\sin 90^\circ} \Rightarrow a : b : c = 1 : 1 : \sqrt{2}$

56. Let AD be the median in ΔABC.
Let $\angle B = \theta$ then $\angle C = 105° - \theta$



In ΔABD, $\frac{BD}{\sin 30^\circ} = \frac{AD}{\sin\theta} \Rightarrow BD = \frac{AD}{2\sin\theta}$

In ΔACD, $\frac{DC}{\sin 45^\circ} = \frac{AD}{\sin(105^\circ - \theta)}$

$\Rightarrow DC = \frac{AD}{\sqrt{2}\sin(105^\circ - \theta)}$

$\because\ BD = DC$, $\therefore\ \frac{AD}{2\sin\theta} = \frac{AD}{\sqrt{2}\sin(105^\circ-\theta)}$

$\Rightarrow\ \sin(90^\circ + 15^\circ - \theta) = \sqrt{2}\sin\theta$

$\Rightarrow\ \cos(15^\circ - \theta) = \sqrt{2}\sin\theta$

$\Rightarrow\ \cos 15^\circ \cos\theta + \sin 15^\circ \sin\theta = \sqrt{2}\sin\theta$

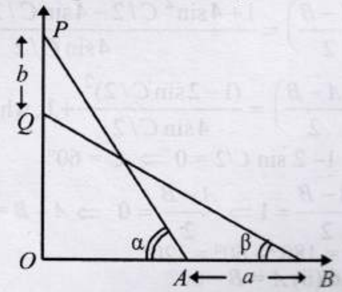
$\Rightarrow\ \cot\theta = \frac{\sqrt{2} - \sin 15^\circ}{\cos 15^\circ} = \frac{5-\sqrt{3}}{\sqrt{3}+1} = 3\sqrt{3} - 4$

$\Rightarrow\ \operatorname{cosec}\theta = \sqrt{1 + \cot^2\theta} = 2\sqrt{11 - 6\sqrt{3}}$

Now, $BD = \frac{AD}{2\sin\theta} = \frac{1}{\sqrt{11-6\sqrt{3}}} \times \frac{2\sqrt{11-6\sqrt{3}}}{2} = 1$

$\therefore\ BC = 2\ BD = 2$ units

57. Let ℓ be the length of the ladder, then



In ΔBOQ, $\cos\beta = \frac{OB}{BQ} \Rightarrow OB = \ell\cos\beta$

Similarly in ΔPOA, $\cos\alpha = \frac{OA}{PA} \Rightarrow OA = \ell\cos\alpha$

Now $a = OB - OA = \ell(\cos\beta - \cos\alpha)$

Also in ΔOAP, $OP = \ell\sin\alpha$

and in ΔOQB, $OQ = \ell\sin\beta$

$\therefore \quad b = OP - OQ = \ell(\sin\alpha - \sin\beta)$ ....(ii)

Dividing equation (i) by (ii), we get

$$\frac{a}{b} = \frac{\cos\beta - \cos\alpha}{\sin\alpha - \sin\beta} = \frac{2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}{2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}$$

$$\Rightarrow \frac{a}{b} = \tan\left(\frac{\alpha+\beta}{2}\right), \ \therefore a = b\tan\left(\frac{\alpha+\beta}{2}\right)$$

58. Let $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13} = k$

$\Rightarrow b + c = 11k$ ....(i)

$c + a = 12k$ ....(ii)

$a + b = 13k$ ....(iii)

On solving equations (i), (ii) and (iii), we get

$a = 7k, b = 6k, c = 5k$

Now, $\cos A = \frac{b^2+c^2-a^2}{2bc}$

$$= \frac{36k^2 + 25k^2 - 49k^2}{2\times 6k \times 5k} = \frac{1}{5}$$

$$\cos B = \frac{c^2+a^2-b^2}{2ca} = \frac{25k^2+49k^2-36k^2}{2\times 5k\times 7k} = \frac{19}{35}$$

$$\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{49k^2+36k^2-25k^2}{2\times 7k\times 6k} = \frac{5}{7}$$

$$\therefore \frac{\cos A}{1/5} = \frac{\cos B}{19/35} = \frac{\cos C}{5/7}$$

$$\Rightarrow \frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}$$

59. In ΔABC, $\cos A + \cos B + \cos C = \frac{3}{2}$

$$\Rightarrow 2\sin\frac{C}{2}\cos\frac{A-B}{2} = \frac{3-2\cos C}{2}$$

$$\Rightarrow 2\sin\frac{C}{2}\cos\frac{A-B}{2} = \frac{3-2(1-2\sin^2 C/2)}{2}$$

$$\Rightarrow \cos\left(\frac{A-B}{2}\right) = \frac{1+4\sin^2 C/2}{4\sin C/2}$$

$$\Rightarrow \cos\left(\frac{A-B}{2}\right) = \frac{1+4\sin^2 C/2 - 4\sin C/2 + 4\sin C/2}{4\sin C/2}$$

$$\Rightarrow \cos\left(\frac{A-B}{2}\right) = \frac{(1-2\sin C/2)^2}{4\sin C/2} + 1, \text{ which is possible}$$

only when $1 - 2\sin C/2 = 0 \Rightarrow C = 60°$

$\therefore \cos\frac{A-B}{2} = 1 \Rightarrow \frac{A-B}{2} = 0 \Rightarrow A - B = 0$ ....(i)

and $A + B = 180° - 60° = 120°$ ....(ii)

From (i) and (ii) $A = B = 60°$

$\Rightarrow A = B = C = 60°$, $\therefore \Delta ABC$ is an equilateral triangle.

60. Ex-radii of a ΔABC are $r_1 = \frac{\Delta}{s-a}, r_2 = \frac{\Delta}{s-b}, r_3 = \frac{\Delta}{s-c}$

Since r_1, r_2, r_3 are in HP., $\therefore \frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3}$ are in AP

$\Rightarrow \frac{s-a}{\Delta}, \frac{s-b}{\Delta}, \frac{s-c}{\Delta}$ are in AP

$\Rightarrow s-a, s-b, s-c$ are in AP

$\Rightarrow -a, -b, -c$ are in AP. $\Rightarrow a, b, c$ are in A.P.

61. As the angles A, B, C of ΔABC are in AP

$\therefore$ Let $A = x - d, B = x, C = x + d$

But $A + B + C = 180°$, $\therefore x - d + x + x + d = 180°$

$\Rightarrow x = 60° \therefore \angle B = 60°$

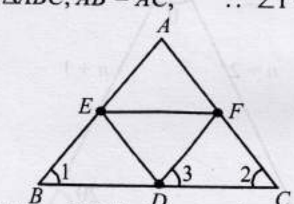
Now by sine law in ΔABC,

$$\frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{b}{c} = \frac{\sin B}{\sin C} \Rightarrow \frac{\sqrt{3}}{\sqrt{2}} = \frac{\sin 60°}{\sin C}$$

$$\Rightarrow \frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{3}}{2\sin C} \Rightarrow \sin C = \frac{1}{\sqrt{2}} = \sin 45°$$

$\therefore \angle A = 180° - (\angle B + \angle C) = 75°$

62. Given : In ΔABC, $AB = AC$, $\therefore \angle 1 = \angle 2$ ....(i)



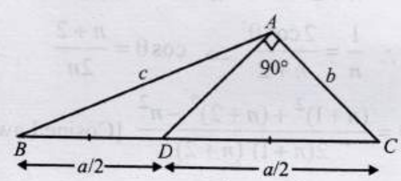
$\because$ $AB \parallel DF$ and BC is transversal, $\therefore \angle 1 = \angle 3$ ....(ii)

From (i) and (ii), $\angle 2 = \angle 3 \Rightarrow DF = CF$ ....(iii)

Similaraly we can prove $DE = BE$

Now, $DF + FA + AE + ED = CF + FA + AE + BE = AC + AB$ [using (iii) and (iv)]

63.



In ΔACD, $\cos C = \frac{b}{a/2} = \frac{2b}{a}$ ....(i)

In ΔABC, $\cos C = \frac{a^2+b^2-c^2}{2ab}$ [cosine law] ....(ii)

From (i) and (ii),

$$\frac{2b}{a} = \frac{a^2+b^2-c^2}{2ab} \Rightarrow b^2 = \frac{1}{3}(a^2-c^2) \quad ....(iii)$$

Also, $\cos A = \frac{b^2+c^2-a^2}{2bc}$

[cosine law]

$$\therefore \cos A \cos C = \frac{b^2+c^2-a^2}{2bc} \times \frac{2b}{a} = \frac{b^2+c^2-a^2}{ac}$$

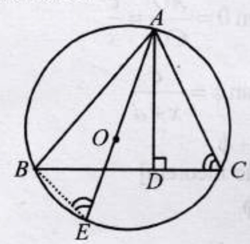
$$= \frac{\frac{1}{3}(a^2-c^2)+(c^2-a^2)}{ac} = \frac{2(c^2-a^2)}{3ac}$$

64. **(a)** Inradius of the circle is given by

$$r = (s-b)\tan\frac{B}{2} = \left(\frac{a+b+c}{2} - b\right)\tan\frac{\pi}{4} = \frac{a+c-b}{2}$$

$2r = a + c - b \Rightarrow$ Diameter $= BC + AB - AC$

(b) **Given :** ΔABC in which $AD \perp BC$, AE is diameter of circumcircle of ΔABC.



To prove :

$AB \times AC = AE \times AD$

Construction : Join BE

Proof : $\angle ABE = 90°$ [Angle in a semi circle]

In Δ's ABE and ADC

$\angle ABE = \angle ADC$ [= 90°]

$\angle AEB = \angle ACD$ [Angles in the same segment]

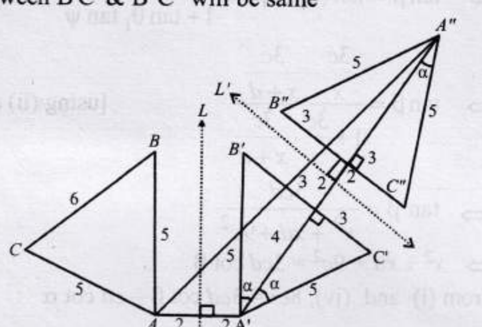
$\therefore$ $\Delta ABE \sim \Delta ADC$ [By AA similarity]

$$\Rightarrow \frac{AB}{AD} = \frac{AE}{AC} \Rightarrow AB \times AC = AD \times AE \quad \text{proved}$$

65. Let L be the line parallel to side AB of ΔABC, at a distance of 2 cm from AB, in which the first reflection $\Delta A'\ B'\ C'$ is obtained. Let L' be the second line parallel to $B'\ C'$, at a distance of 2 cm from $B'\ C'$, in which reflection of $\Delta A'\ B'\ C'$ is taken as $\Delta A''B''C''$.

In figure, size of $\Delta A''B''C''$ is same to the size of $\Delta A'B'C'$.

In the figure, distance between AB & A'B' and distance between B'C' & B''C'' will be same



From figure $AA' = 4$ cm and $A'A'' = 12$ cm. So to find AA'' it suffices to know $\angle AA'A''$, clearly

$$\angle AA'A'' = 90° + \alpha, \text{ where } \sin\alpha = \frac{3}{5}$$

$$\Rightarrow \cos\angle AA'A'' = \cos(90° + \alpha) = -\sin\alpha = -\frac{3}{5}$$

Now, in $\Delta AA'A''$, $\cos(90° + \alpha) = \frac{(AA')^2 + (A'A'')^2 - (AA'')^2}{2(AA')(A'A'')}$

[cosine law]

$$\therefore AA'' = \sqrt{(AA')^2 + (A'A'')^2 - 2AA' \times A'A''.\cos(90° + \alpha)}$$

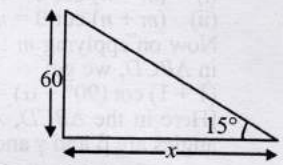
$$= \sqrt{16 + 144 + 96 \times \frac{3}{5}} = \sqrt{\frac{1088}{5}} = 8\sqrt{\frac{17}{5}}\text{cm}$$

Topic-2: Heights & Distances

1. (b)

2. (b) Since angle of elevation of the top of the pole from each corner of the triangular park ABC therefore foot of the pole will be equidistant from the each corner of the triangular park ABC. Hence foot of the pole will be at the point of circumcentre of the triangle ABC.

3. (b) $\tan 15° = \frac{60}{x}$



$\Rightarrow x = 60 \cot 15°$

$$= 60\left[\frac{\sqrt{3}+1}{\sqrt{3}-1}\right]$$

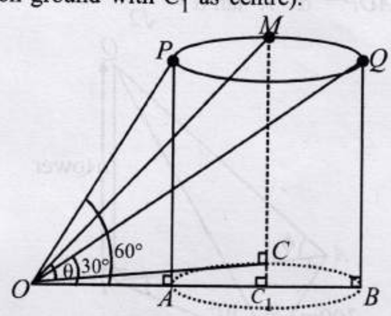
4. Let A, B and C be the projections of the points P, Q and M on the ground.

$\therefore \angle POA = 60°, \angle QOB = 30°, \angle MOC = \theta$

Let h be the height of circle from ground, then

$AP = CM = BQ = h$

Let $OA = x$ and $AB = d$ (diameter of the projection of the circle on ground with C_1 as centre).



Now in ΔPAO, $\tan 60° = \frac{h}{x} \Rightarrow x = \frac{h}{\sqrt{3}}$ ....(i)

In ΔQBO, $\tan 30° = \frac{h}{x+d} \Rightarrow x + d = h\sqrt{3}$

$$\Rightarrow \quad d = h\sqrt{3} - \frac{h}{\sqrt{3}} = \frac{2h}{\sqrt{3}} \quad ....(ii)$$

In ΔOCM, $\tan\theta = \frac{h}{OC}$

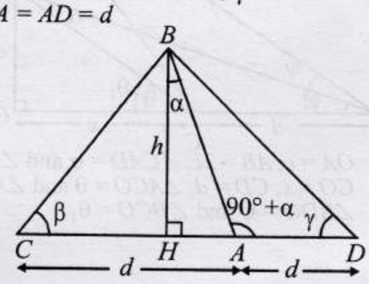
$$\Rightarrow \tan^2\theta = \frac{h^2}{OC^2} = \frac{h^2}{OC_1^2 + C_1C^2} = \frac{h^2}{\left(x + \frac{d}{2}\right)^2 + \left(\frac{d}{2}\right)^2}$$

$$= \frac{h^2}{\left(\frac{h}{\sqrt{3}} + \frac{h}{\sqrt{3}}\right)^2 + \left(\frac{h}{\sqrt{3}}\right)^2} \quad \text{[using (i) and (ii)]}$$

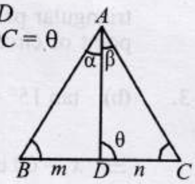
$$= \frac{h^2}{\frac{4h^2}{3} + \frac{h^2}{3}} = \frac{3}{5}$$

5. Let AB be the tower leaning towards west making an angle α with vertical. At C, angle of elevation of B is β and at D the angle of elevation of B is γ

$CA = AD = d$



***m : n* theorem:** In ΔABC where point D divides BC in the ratio $m : n$. and $\angle ADC = \theta$



(i) $(m+n)\cot\theta = n\cot B - m\cot C$
(ii) $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$

Now on applying $m : n$ theorem in ΔBCD, we get

$(1+1)\cot(90^\circ+\alpha) = 1.\cot\beta - 1.\cot\gamma$

[Here in the ΔBCD, A divides CD in the ratio 1 : 1, base angles are β and γ and $\angle BAD = 90^\circ + \alpha$]

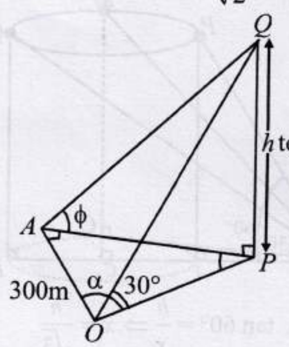
$\Rightarrow\ -2\tan\alpha = \cot\beta - \cot\gamma$

$\Rightarrow\ 2\tan\alpha = \cot\gamma - \cot\beta$

6. Let PQ be the tower of height h. A is in the north of O and P is towards east of A.

$\therefore\quad \angle OAP = 90^\circ;\ \angle QOP = 30^\circ;\ \angle QAP = \phi$

$\angle AOP = \alpha,\ \therefore\ \tan\alpha = \dfrac{1}{\sqrt{2}}$



Now in ΔOPQ, $\tan 30^\circ = \dfrac{h}{OP} \Rightarrow OP = h\sqrt{3}$ ....(i)

In ΔAPQ, $\tan\phi = \dfrac{h}{AP} \Rightarrow AP = h\cot\phi$ ....(ii)

$\because\quad \tan\alpha = \dfrac{1}{\sqrt{2}},\ \therefore\ \sin\alpha = \dfrac{\tan\alpha}{\sqrt{1+\tan^2\alpha}} = \dfrac{1}{3}$

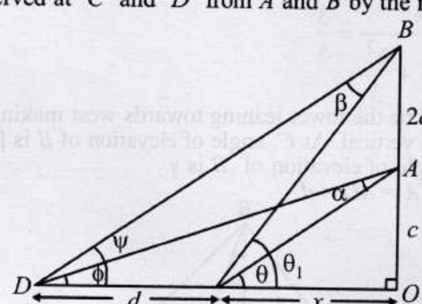
Now in ΔAOP, $\sin\alpha = \dfrac{AP}{OP} \Rightarrow \dfrac{1}{\sqrt{3}} = \dfrac{h\cot\phi}{h\sqrt{3}}$ [using (i) and (ii)]

$\Rightarrow\ \cot\phi = 1\ \Rightarrow\ \phi = 45^\circ$.

Now in right ΔOAP, $OP^2 = OA^2 + AP^2$

$\Rightarrow\ 3h^2 = 90000 + h^2\cot^2 45^\circ\ \Rightarrow\ h = 150\sqrt{2}$m

7. Let the man initially be standing at the position O after walking a distance 'c', the man reached at A and then after walking a distance '$2c$' reached at B. The two objects are observed at 'C' and 'D' from A and B by the man.



Now Let $OA = c$, $AB = 2c$, $\angle CAD = \alpha$ and $\angle CBD = \beta$
$CO = x$, $CD = d$, $\angle ACO = \theta$ and $\angle ADC = \phi$,
$\angle BDO = \psi$ and $\angle BCO = \theta_1$

In ΔACO, $\tan\theta = \dfrac{AO}{CO} = \dfrac{c}{x}$

In ΔADO, $\tan\phi = \dfrac{c}{x+d}$

Now, $\theta = \alpha + \phi$
[Exterior angle theorem]

$\Rightarrow\ \alpha = \theta - \phi$

$\Rightarrow\ \tan\alpha = \tan(\theta-\phi) = \dfrac{\tan\theta - \tan\phi}{1+\tan\theta\tan\phi} = \dfrac{\dfrac{c}{x} - \dfrac{c}{x+d}}{1+\dfrac{c}{x}\dfrac{c}{x+d}}$

$\Rightarrow\ \tan\alpha = \dfrac{cx+cd-cx}{x^2+dx+c^2}$,

$\Rightarrow x^2 + c^2 + xd = cd\cot\alpha$ ....(i)

Now in ΔBOD, $\tan\psi = \dfrac{3c}{x+d}$ ....(ii)

and in ΔBOC, $\tan\theta_1 = \dfrac{3c}{x}$ ....(iii)

But $\theta_1 = \psi + \beta$
[Exterior angle theorem]

$\Rightarrow\ \tan\beta = \tan(\theta_1 - \psi) = \dfrac{\tan\theta_1 - \tan\psi}{1+\tan\theta_1\tan\psi}$

$\Rightarrow\ \tan\beta = \dfrac{\dfrac{3c}{x} - \dfrac{3c}{x+d}}{1+\dfrac{3c}{x}\cdot\dfrac{3c}{x+d}}$ [using (ii) and (iii)]

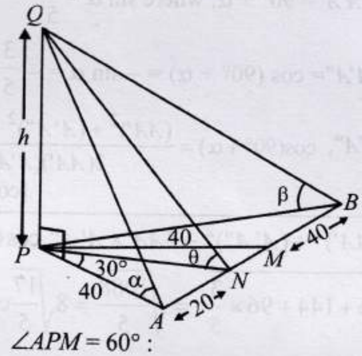
$\Rightarrow\ \tan\beta = \dfrac{3cd}{x^2+xd+9c^2}$

$\Rightarrow\ x^2 + xd + 9c^2 = 3cd\cot\beta$ ....(iv)

From (i) and (iv), $8c^2 = 3cd\cot\beta - cd\cot\alpha$

$\therefore\quad d = \dfrac{8c}{3\cot\beta - \cot\alpha}$

8. Since A and B are located to the south and east of P respectively, $\therefore\ \angle APB = 90^\circ$.



$\therefore\quad \angle APM = 60^\circ$:

Since $PN \perp AB$, therefore $AN = NM = 20$ m

$\because\quad PAM$ is an equilateralΔ, $\therefore\ AP = 40$ m

$\because\ M$ is mid point of AB, MB = 40 m

Let angles of elevation of top of the tower PQ of height h from A, N and B be α, θ and β respectively.

$\therefore \quad Q = \tan^{-1}2 \Rightarrow \tan Q = 2$

In ΔPQN, $\tan\theta = \frac{PQ}{PN} \Rightarrow 2 = \frac{h}{PN} \Rightarrow PN = h/2$ ....(i)

In equilateral ΔAPM, $\angle APM = 60°$ and PN is altitude, $\therefore \angle APN = 30°$

$\therefore$ In ΔAPN $\tan \angle APN = \frac{AN}{PN}$

$\Rightarrow \tan 30° = \frac{20}{h/2}$ [using (i)]

$\therefore \quad h = 40\sqrt{3}$ m

In ΔAPQ, $\tan \alpha = \frac{h}{AP} = \frac{40\sqrt{3}}{40} = \sqrt{3} \Rightarrow \alpha = 60°$

Also in ΔBPQ, $\tan\beta = h/PB$

$PB = \sqrt{PN^2 + NB^2} = \sqrt{(20\sqrt{3})^2 + (60)^2} = 40\sqrt{3}$ m

$\therefore \quad \tan\beta = \frac{40\sqrt{3}}{40\sqrt{3}} \Rightarrow \beta = 45°$

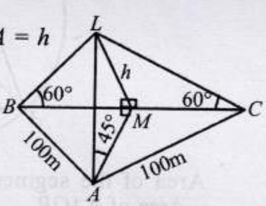
Thus height of the tower is $40\sqrt{3}m$ and angles of elevation are $60°, 45°$.

9. Let ABC be the triangular region with $AB = AC = 100\ m$ and M be the mid point of BC at which tower LM stands.

Since ΔABC is isosceles and M is mid point of BC.

$\therefore \quad AM \perp BC$

Let $LM = h$ be the height of the tower.



In ΔAML, $\tan 45° = \frac{h}{MA} \Rightarrow MA = h$

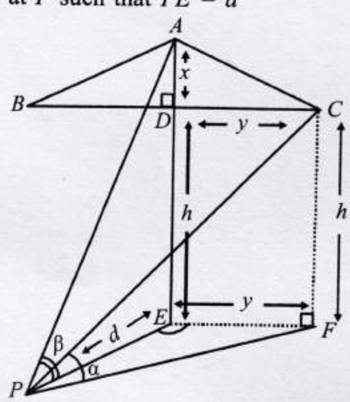
Also in ΔBML, $\tan 60° = \frac{h}{BM}$

$\Rightarrow BM = \frac{h}{\sqrt{3}}$

Now in right ΔAMB, $AB^2 = AM^2 + BM^2$

$\Rightarrow (100)^2 = h^2 + \frac{h^2}{3} \Rightarrow h = 50\sqrt{3}$ m

10. Let ABC be the isosceles triangular sign board with BC horizontal. DE be the pole of height h. Let the man be standing at P such that $PE = d$



Let $AD = x$ and $BC = 2y$

Since, ΔABC is isosceles with $AB = AC$

$\therefore \quad D$ is mid point of BC.

Now, $\angle APE = \beta$ and $\angle CPF = \alpha$

In ΔAEP, $\tan\beta = \frac{h+x}{d} \Rightarrow x = d\tan\beta - h$ ....(i)

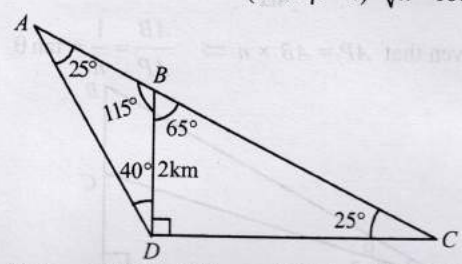
In ΔCFP, $\tan\alpha = \frac{h}{PF} \Rightarrow \tan\alpha = \frac{h}{\sqrt{d^2+y^2}}$

$\Rightarrow \quad y^2 + d^2 = h^2\cot^2\alpha \Rightarrow y = \sqrt{h^2\cot^2\alpha - d^2}$ ....(ii)

Now area of $\Delta ABC = \frac{1}{2} \times BC \times AD = \frac{1}{2} \times 2y \times x = xy$

$= (d\tan\beta - h)\sqrt{h^2\cot^2\alpha - d^2}$

11.



Here, $\angle ADC = 130°$, $\therefore \angle DAC = 180° - (25° + 130°) = 25°$

From the figure, in ΔABD, using sine law

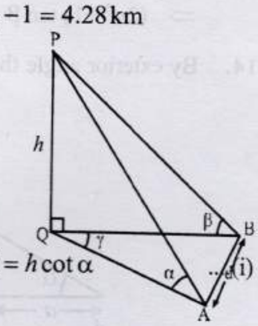
$\frac{AD}{\sin 115°} = \frac{BD}{\sin 25°} \Rightarrow$

$AD = \frac{2\sin(90°+25°)}{\sin 25°} = \frac{2\cos 25°}{\sin 25°}$

$\Rightarrow AD = 2\cot 25°$

$= 2\sqrt{\frac{1}{\sin^2 25°} - 1} = 2\sqrt{\frac{1}{(0.423)^2} - 1} = 4.28$ km

12. Let height of pole PQ be h.



In ΔAQP, $\tan\alpha = \frac{h}{AQ} \Rightarrow AQ = h\cot\alpha$ ....(i)

In ΔBQP, $\tan\beta = \frac{h}{BQ} \Rightarrow BQ = h\cot\beta$ ....(ii)

In ΔABQ, $\cos\gamma = \frac{AQ^2 + BQ^2 - AB^2}{2AQ \cdot BQ}$

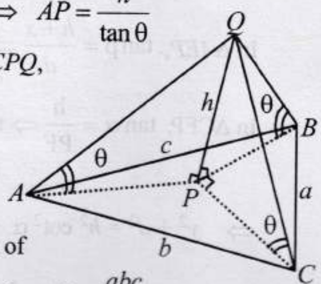
$\therefore \quad \cos\gamma = \frac{h^2\cot^2\alpha + h^2\cot^2\beta - d^2}{2h^2\cot\alpha\cot\beta}$

$\Rightarrow \quad h = \frac{d}{\sqrt{\cot^2\alpha + \cot^2\beta - 2\cot\alpha\cot\beta\cos\gamma}}$

13. (i) Let h be the height of tower PQ.

In ΔAPQ, $\tan\theta = \dfrac{h}{AP} \Rightarrow AP = \dfrac{h}{\tan\theta}$

Similarly in Δ's BPQ and CPQ,



$$BP = \frac{h}{\tan\theta} = CP$$

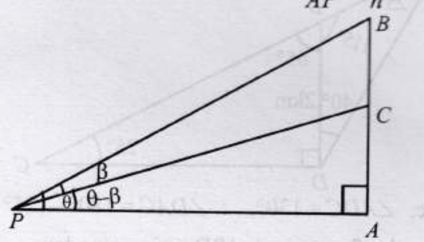
$\Rightarrow AP = BP = CP$

$\Rightarrow P$ is the circumcentre of

ΔABC with circum radius $R = AP = \dfrac{abc}{4\Delta}$

$\therefore \quad h = AP\tan\theta = \dfrac{abc\tan\theta}{4\Delta}$

(ii) Given that $AP = AB \times n \Rightarrow \dfrac{AB}{AP} = \dfrac{1}{n} = \tan\theta$

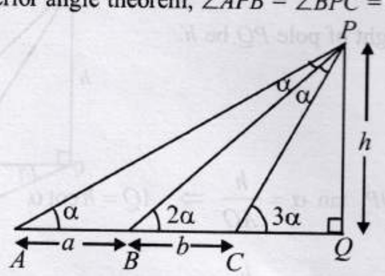


Also $\tan(\theta-\beta) = \dfrac{AC}{AP} = \dfrac{1}{2}\dfrac{AB}{AP} = \dfrac{1}{2n}$

$$\Rightarrow \frac{\tan\theta - \tan\beta}{1+\tan\theta\tan\beta} = \frac{1}{2n} \Rightarrow \frac{\frac{1}{n} - \tan\beta}{1+\frac{1}{n}\tan\beta} = \frac{1}{2n}$$

$$\Rightarrow (2n^2+1)\tan\beta = n \Rightarrow \tan\beta = \frac{n}{2n^2+1}$$

14. By exterior angle theorem, $\angle APB = \angle BPC = \alpha$



Also in ΔABP, $\angle BAP = \angle APB$, $\quad \therefore AB = PB = a$

In ΔPBC,

Now $\dfrac{a}{\sin(180° - 3\alpha)} = \dfrac{b}{\sin\alpha} = \dfrac{PC}{\sin 2\alpha}$ [sine law] ....(i)

$$\Rightarrow \frac{a}{3\sin\alpha - 4\sin^3\alpha} = \frac{b}{\sin\alpha} = \frac{PC}{2\sin\alpha\cos\alpha}$$

$$\Rightarrow \frac{a}{3-4\sin^2\alpha} = \frac{b}{1} = \frac{PC}{2\cos\alpha}$$

$$\Rightarrow \sin^2\alpha = \frac{3b-a}{4b} \Rightarrow \cos\alpha = \frac{1}{2}\sqrt{\frac{b+a}{b}}$$

Also $PC = 2b\cos\alpha = \sqrt{b(a+b)}$

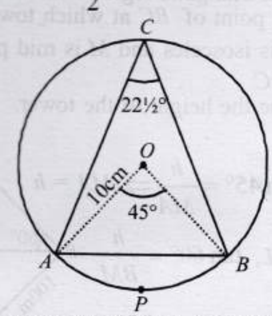
Now in ΔPCQ

$\sin 3\alpha = \dfrac{h}{PC} \Rightarrow h = PC.\left(\dfrac{a\sin\alpha}{b}\right)$ [using eqn. (i)]

$$\Rightarrow h = \sqrt{b(a+b)}.\frac{a}{b}.\sqrt{\frac{(3b-a)}{4b}}$$

$$\Rightarrow h = \frac{a}{2b}.\sqrt{(a+b)(3b-a)}$$

(b) $\because \angle ACB = 22\dfrac{1}{2}°, \quad \therefore \angle AOB = 45°$



Area of the segment APB = Area of the sector $APBO$ − Area of ΔAOB

$= \dfrac{1}{8}\pi r^2 - \dfrac{1}{2}\times 10\times 10\sin 45°$ $\quad [\because \Delta = \dfrac{1}{2}bc\sin A]$

$= \dfrac{3.14\times 100}{8} - \dfrac{50}{\sqrt{2}} = 3.91$ sq. cm.