Long Answer Questions-II (PYQ)

[6 Mark]

Q.1. Find the coordinates of the foot of perpendicular and the length of the perpendicular drawn from the point *P* (5, 4, 2) to the line $\vec{r} = -\hat{i} + 3\hat{j} + \hat{k} + \lambda(2\hat{i} + 3\hat{j} - \hat{k})$. **Also find the image of *P* in this line.**

Ans.

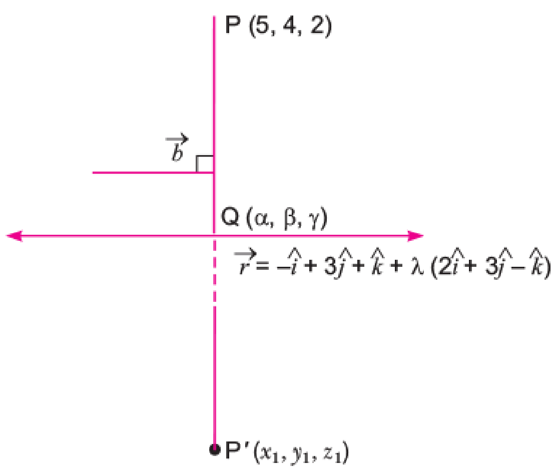
Given line is

$$\vec{r} = -\hat{i} + 3\hat{j} + \hat{k} + \lambda(2\hat{i} + 3\hat{j} - \hat{k})$$

It can be written in cartesian form as

$$\frac{x+1}{2} = \frac{y-3}{3} = \frac{z-1}{-1} \quad ...(i)$$

Let $Q(\alpha, \beta, \gamma)$ be the foot of perpendicular drawn from $P(5, 4, 2)$ to the line (i) and $P'(x_1, y_1, z_1)$ be the image of P on the line (i)



$\because$ $Q(\alpha, \beta, \gamma)$ lie on line (i)

$\therefore$ $\frac{\alpha+1}{2} = \frac{\beta-3}{3} = \frac{\gamma-1}{-1} = \lambda$ (say)

$\Rightarrow \alpha = 2\lambda - 1; \beta = 3\lambda + 3$ and $\gamma = -\lambda + 1$ $\quad ...(ii)$

Now, $\overrightarrow{PQ} = (\alpha - 5)\,\hat{i} + (\beta - 4)\,\hat{j} + (\gamma - 2)\hat{k}$

Parallel vector of line (i) $\vec{b} = 2\hat{i} + 3\hat{j} - \hat{k}$.

Obviously $\overrightarrow{PQ} \perp \vec{b}$ $\quad \Rightarrow \quad$ $\overrightarrow{PQ} \cdot \vec{b} = 0$

$2(\alpha - 5) + 3(\beta - 4) + (-1)(\gamma - 2) = 0$

$\Rightarrow 2\alpha - 10 + 3\beta - 12 - \gamma + 2 = 0$

$\Rightarrow 2\alpha + 3\beta - \gamma - 20 = 0$

$\Rightarrow 2(2\lambda - 1) + 3(3\lambda + 3) - (-\lambda + 1) - 20 = 0$ [Putting value of α, β, γ from (ii)]

$\Rightarrow 4\lambda - 2 + 9\lambda + 9 + \lambda - 1 - 20 = 0$

$\Rightarrow 14\lambda - 14 = 0 \Rightarrow \lambda = 1$

Hence the coordinates of foot of perpendicular Q are $(2 \times 1 - 1,\ 3 \times 1 + 3, -1 + 1)$, *i.e.*, $(1, 6, 0)$

$\therefore$ Length of perpendicular $= \sqrt{(5-1)^2 + (4-6)^2 + (2-0)^2}$

$= \sqrt{16+4+4} = \sqrt{24} = 2\sqrt{6}$ units.

Also, since Q is mid-point of PP'

$\therefore 1 = \frac{x_1+5}{2} \quad \Rightarrow \quad x_1 = -3$

$6 = \frac{y_1+4}{2} \Rightarrow \quad y_1 = 8$

$0 = \frac{z_1+2}{2} \Rightarrow \quad z_1 = -2$

Therefore required image is $(-3, 8, -2)$.

Q.2. Find the equation of the plane that contains the point (1, – 1, 2) and is perpendicular to both the planes 2*x* + 3*y* – 2*z* = 5 and *x* + 2*y* – 3*z* = 8. Hence find the distance of point *P*(–2, 5, 5) from the plane obtained above.

Ans.

Equation of plane containing the point $(1, -1, 2)$ is given by

$a(x-1)+b(y+1)+c(z-2)=0 \ldots(i)$

$\because$ (i) is perpendicular to plane $2x+3y-2z=5$

$\therefore$ $2a+3b-2c=0 \ldots(ii)$

Also, (i) is perpendicular to plane $x+2y-3z=8$

$a+2b-3c=0 \ldots(iii)$

From (ii) and (iii), we get

$\frac{a}{-9+4}=\frac{b}{-2+6}=\frac{c}{4-3}$

$\Rightarrow \frac{a}{-5}=\frac{b}{4}=\frac{c}{1}=\lambda \text{ (say)}$

$\Rightarrow$ $a=-5\lambda,\ b=4\lambda,\ c=1$

Putting these values in (i), we get

$-5\lambda(x-1)+4\lambda(y+1)+\lambda(z-2)=0$

$\Rightarrow -5(x-1)+4(y+1)+(z-2)=0$

$\Rightarrow -5x+5+4y+4+z-2=0 \Rightarrow -5x+4y+z+7=0$

$\Rightarrow 5x-4y-z-7=0 \ldots (iv)$ is the required equation of plane.

Again, if d be the distance of point $P(-2, 5, 5)$ to plane (iv), then

$$d = \left|\frac{5\times(-2)+(-4)\times5+(-1)\times5-7}{\sqrt{5^2+(-4)^2+(-1)^2}}\right|$$

$$= \left|\frac{-10-20-5-7}{\sqrt{25+16+1}}\right| = \frac{42}{\sqrt{42}} = \sqrt{42}\text{units}$$

Q.3. If the lines $\frac{x-1}{-3} = \frac{y-2}{-2k} = \frac{z-3}{2}$ and $\frac{x-1}{k} = \frac{y-2}{1} = \frac{z-3}{5}$ are perpendicular, then find the value of *k* and hence find the equation of plane containing these lines.

Ans.

Given lines are

$$\frac{x-1}{-3} = \frac{y-2}{-2k} = \frac{z-3}{2} \quad ...(i)$$

$$\frac{x-1}{k} = \frac{y-2}{1} = \frac{z-3}{5} \quad ...(ii)$$

Obviously, parallel vectors $\vec{b_1}$ and $\vec{b_2}$ of line (i) and (ii) respectively are:

$$\vec{b_1} = -3\hat{i} - 2k\hat{j} + 2\hat{k} \text{ and } \vec{b_2} = k\hat{i} + \hat{j} + 5\hat{k}$$

Lines $(i) \perp (ii) \Rightarrow \vec{b_1} \perp \vec{b_2}$

$$\Rightarrow \vec{b_1}.\vec{b_2} = 0 \Rightarrow -3k - 2k + 10 = 0$$

$$\Rightarrow -5k + 10 = 0 \Rightarrow k = \frac{-10}{-5} = 2$$

Putting $k = 2$ in (i) and (ii), we get

$$\frac{x-1}{-3} = \frac{y-2}{-4} = \frac{z-3}{2} \text{ and } \frac{x-1}{2} = \frac{y-2}{1} = \frac{z-3}{5}$$

Now, the equation of plane containing above two lines is

$$\begin{bmatrix} x-1 & y-2 & z-3 \\ -3 & -4 & 2 \\ 2 & 1 & 5 \end{bmatrix} = 0$$

$\Rightarrow (x-1)(-20-2) - (y-2)(-15-4) + (z-3)(-3+8) = 0$

$\Rightarrow -22(x-1) + 19(y-2) + 5(z-3) = 0$

$\Rightarrow -22x + 22 + 19y - 38 + 5z - 15 = 0$

$\Rightarrow -22x + 19y + 5z - 31 = 0 \Rightarrow \quad 22x - 19y - 5z + 31 = 0$

Note: Equation of plane containing lines $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and

$\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$, or $\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$

Q.4. Find the vector equation of the plane passing through the points (2, 1, –1) and (–1, 3, 4) and perpendicular to the plane *x* – 2*y* + 4*z* = 10. Also show that the plane thus obtained contains the line $\vec{r} = -\hat{i} + 3\hat{j} + 4\hat{k} + \lambda\,(3\hat{i} - 2\hat{j} - 5\hat{k})$.

Ans.

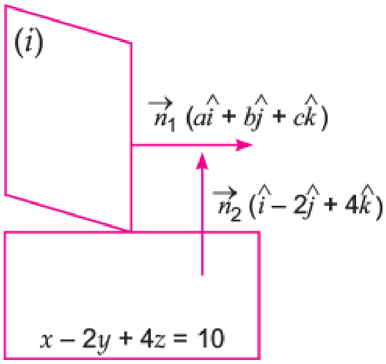
Let the equation of plane through $(2, 1, -1)$ be

$a(x-2) + b(y-1) + c(z+1) = 0 \;...(i)$

$\because \; (i)$ passes through $(-1, 3, 4)$

$\therefore \; a(-1-2) + b(3-1) + c(4+1) = 0$

$\Rightarrow \quad -3a + 2b + 5c = 0 \;...(ii)$



Also plane (i) is perpendicular to plane $x - 2y + 4z = 10$

$\Rightarrow \overrightarrow{n_1} \perp \overrightarrow{n_2} \Rightarrow \overrightarrow{n_1}.\overrightarrow{n_2} = 0$

$\therefore\ 1\,a - 2b + 4c = 0$ ...(iii)

From (ii) and (iii), we get

$\frac{a}{8+10} = \frac{b}{5+12} = \frac{c}{6-2} \qquad \Rightarrow \qquad \frac{a}{18} = \frac{b}{17} = \frac{c}{4} = \lambda$ (say)

$\Rightarrow a = 18\lambda,\ b = 17\lambda,\ c = 4\lambda$

Putting the value of a, b, c in (i), we get

$18\lambda(x-2) + 17\lambda(y-1) + 4\lambda(z+1) = 0$

$\Rightarrow 18x - 36 + 17y - 17 + 4z + 4 = 0$

$\Rightarrow 18x + 17y + 4z = 49$

$\therefore$ Required vector equation of plane is

$\vec{r}.(18\hat{i} + 17\hat{j} + 4\hat{k}) = 49$ ...(iv)

Obviously plane (iv) contains the line

$\vec{r} = (-\hat{i} + 3\hat{j} + 4\hat{k}) + \lambda(3\hat{i} - 2\hat{j} - 5\hat{k})$ ...(v)

Since, point $(-\hat{i}+3\hat{j}+4\hat{k})$ satisfy equation (iv) and vector $(18\hat{i}+17\hat{j}+4\hat{k})$ is perpendicular to, $(3\hat{i}-2\hat{j}+5\hat{k})$, as $(-\hat{i}+3\hat{j}+4\hat{k}).(18\hat{i}+17\hat{j}+4\hat{k}) = -18+51+16 = 49$

and $(18\hat{i}+17\hat{j}+4\hat{k}).(3\hat{i}-2\hat{j}-5\hat{k}) = 54-34-20 = 0$

Therefore, (iv) contains line (v).

Q.5. Let *P* (3, 2, 6) be a point in the space and *Q* be a point on the line $\vec{r}=(\hat{i}-\hat{j}+2\hat{k})+\mu(-3\hat{i}+\hat{j}+5\hat{k})$, then find the value of for which the vector μ is $\overrightarrow{PQ}$ parallel to the plane *x* – 4*y* + 3*z* = 1

Ans.

Let $P(3, 2, 6)$ be a point in the space and $Q(\alpha, \beta, \gamma)$ be a point on the given line represented in cartesian form as

$\frac{x-1}{-3}=\frac{y+1}{1}=\frac{z-2}{5}=\mu \quad ...(i)$

$Q(\alpha, \beta, \gamma)$ lie on line (i)

$\frac{\alpha-1}{-3}=\frac{\beta+1}{1}=\frac{\gamma-2}{5}=\mu$

$\alpha=-3\mu+1, \beta=\mu-1, \gamma=5\mu+2 \ ...(ii)$

Now, $\overrightarrow{PQ}=(\alpha-3)\hat{i}+(\beta-2)\hat{j}+(\gamma-6)\hat{k}$

Normal vector of plane, $\vec{n}=\hat{i}-4\hat{j}+3\hat{k}$

Obviously, $\overrightarrow{PQ}$ is perpendicular to $\vec{n}$.

$\therefore\ \overrightarrow{PQ}\ \vec{n} = 0$

$(\alpha - 3).1 + (\beta - 2).(-4) + (\gamma - 6).3 = 0$

$\Rightarrow\ \alpha - 3 - 4\beta + 8 + 3\gamma - 18 = 0 \qquad \Rightarrow \qquad \alpha - 4\beta + 3\gamma - 13 = 0$

Putting the value of $\alpha,\ \beta,\ \gamma$ from (ii), we get

$-\ 3\mu + 1 - 4(\mu - 1) + 3(5\mu + 2) - 13 = 0$

$\Rightarrow\ -3\mu + 1 - 4\mu + 4 + 15\mu + 6 - 13 = 0$

$\Rightarrow\ 8\mu - 2 = 0 \quad \Rightarrow \quad \mu = \frac{2}{8} = \frac{1}{4}$

Q.6. Find the vector and cartesian equations of the plane which bisects the line joining the points (3, –2, 1) and (1, 4, –3) at right angles.

Ans.

Let $P(3, -2, 1)$; $Q(1, 4, -3)$ be two points such that R (a point of plane) is mid point of $\overrightarrow{PQ}$ and $\overrightarrow{PQ}$ is perpendicular to required plane.

Now, coordinate of $R = \left(\frac{3+1}{2}, \frac{4-2}{2}, \frac{-3+1}{2}\right) = (2, 1, -1)$

Also, $\overrightarrow{PQ} = (1 - 3)\hat{i} + (4 + 2)\hat{j} + (-3 - 1)\hat{k} = -2\hat{i} + 6\hat{j} - 4\hat{k}$

Now, we have a normal vector $\overrightarrow{PQ}$ and a point $R(2, 1, -1)$ of required plane.

Therefore, vector equation of required plane is

$$(\vec{r} - (2\hat{i} + \hat{j} - \hat{k})).(-2\hat{i} + 6\hat{j} - 4\hat{k}) = 0$$

$$\{\vec{r} - (2\hat{i} + \hat{j} - \hat{k})\}.(\hat{i} - 3\hat{j} + 2\hat{k}) = 0$$

$$\Rightarrow \quad \vec{r}.(\hat{i} - 3\hat{j} + 2\hat{k}) - (2 - 3 - 2) = 0$$

$$\Rightarrow \quad \vec{r}.(\hat{i} - 3\hat{j} + 2\hat{k}) + 3 = 0$$

$$x - 3y + 2z + 3 = 0$$

Also, cartesian equation of required plane is

Q.7. Find the vector and Cartesian equations of a plane containing the two lines.

$$\vec{r} = (2\hat{i} + \hat{j} - 3\hat{k}) + \lambda(\hat{i} + 2\hat{j} + 5\hat{k}) \text{ and } \vec{r} = (3\hat{i} + 3\hat{j} + 2\hat{k}) + \mu(3\hat{i} - 2\hat{j} + 5\hat{k})$$

Ans.

Given lines are

$$\vec{r} = (2\hat{i} + \hat{j} - 3\hat{k}) + \lambda(\hat{i} + 2\hat{j} + 5\hat{k}) \quad ...(i)$$

$$\vec{r} = (3\hat{i} + 3\hat{j} + 2\hat{k}) + \mu(3\hat{i} - 2\hat{j} + 5\hat{k}) \quad ...(ii)$$

Here $\vec{a_1} = 2\hat{i} + \hat{j} + 3\hat{k}; \qquad \vec{a_2} = 3\hat{i} + 3\hat{j} + 2\hat{k}$

$$\vec{b_1} = \hat{i} + 2\hat{j} + 5\hat{k}; \qquad \vec{b_2} = 3\hat{i} - 2\hat{j} + 5\hat{k}$$

Now, $\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 5 \\ 3 & -2 & 5 \end{vmatrix}$

$= (10+10)\hat{i} - (5-15)\hat{j} + (-2-6)\hat{k} = 20\hat{i} + 10\hat{j} - 8\hat{k}$

Hence, vector equation of required plane is

$(\vec{r} - \vec{a_1}).(\vec{b_1} \times \vec{b_2}) = 0$

$\Rightarrow \quad \vec{r}.(\vec{b_1} \times \vec{b_2}) = \vec{a_1}.(\vec{b_1} \times \vec{b_2})$

$\Rightarrow \quad \vec{r}.\ (20\hat{i} + 10\hat{j} - 8\hat{k}) = (2\hat{i} + \hat{j} - 3\hat{k}).\ (20\hat{i} + 10\hat{j} - 8\hat{k})$

$\Rightarrow \quad \vec{r}.(20\hat{i} + 10\hat{j} - 8\hat{k}) = 40 + 10 + 24$

$\vec{r}.\ (20\hat{i} + 10\hat{j} - 8\hat{k}) = 74 \quad \Rightarrow \quad \vec{r}.(20\hat{i} + 10\hat{j} - 8\hat{k}) = 74 \quad \Rightarrow \quad \vec{r}.(10\hat{i} + 5\hat{j} - 4\hat{k}) = 37$

Therefore, Cartesian equation is $10x + 5y - 4z = 37$

Q.8. Find the distance of the point (–2, 3, –4) from the line $\frac{x+2}{3} = \frac{2y+3}{4} = \frac{3z+4}{5}$ measured parallel to the plane $4x + 12y - 3z + 1 = 0$.

Ans.

Let given point be $P(-2, 3, -4)$ and given line and plane be

$\frac{x+2}{3} = \frac{2y+3}{4} = \frac{3z+4}{5} \quad ...(i)$

$4x + 12y - 3z + 1 = 0 \quad ...(ii)$

Let $Q(\alpha, \beta, \gamma)$ be the point on line (i), such that

$\overrightarrow{PQ}$ parallel to plane (ii)

$\Rightarrow \overrightarrow{PQ} \perp \vec{n}$ [normal vector of (ii)]

Now, $\overrightarrow{PQ} = (\alpha+2)\hat{i} + (\beta-3)\hat{j} + (\gamma+4)\hat{k}$

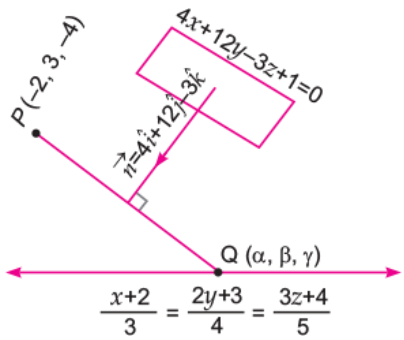
and $\vec{n} = 4\hat{i} + 12\hat{j} - 3\hat{k}$

$\therefore\ \overrightarrow{PQ}\cdot\vec{n}=0$

$\Rightarrow 4(\alpha+2)+12(\beta-3)-3(\gamma+4)=0$

$\Rightarrow 4\alpha+8+12\beta-36-3\gamma-12=0$

$\Rightarrow 4\alpha+12\beta-3\gamma=40 \qquad ...(iii)$



Also, $Q(\alpha,\beta,\gamma)$ lie on line (i)

$\frac{\alpha+2}{3}=\frac{2\beta+3}{4}=\frac{3\gamma+4}{5}=\lambda\ \text{(say)}$

$\alpha=3\lambda-2,\ \beta=\frac{4\lambda-3}{2},\ \gamma=\frac{5\lambda-4}{3}$

Putting the value of α,β,γ in (iii), we get

$4(3\lambda-2)+12\left(\frac{4\lambda-3}{2}\right)-3\left(\frac{5\lambda-4}{3}\right)=40$

$\Rightarrow 12\lambda-8+24\lambda-18-15\lambda+4=40$

$\Rightarrow 31\lambda-22=40$

$\Rightarrow 31\lambda=62$

$\Rightarrow \lambda=2$

$\therefore\ \alpha=3\times2-2=4,\ \beta=\frac{4\times2-3}{2}=\frac{5}{2},\qquad \gamma=\frac{5\times2-4}{3}=2$

Hence $Q\equiv\left(4,\frac{5}{2},2\right)$

$\therefore$ Distance, $PQ=\sqrt{(4+2)^2+\left(\frac{5}{2}-3)\right)^2+(2+4)^2}=\sqrt{36+36+\frac{1}{4}}=\sqrt{\frac{289}{4}}=\frac{17}{2}$ units.

Q.9. Find the equation of the plane passing through the point (–1, 3, 2) and perpendicular to each of the planes *x* + 2*y*+ 3*z* = 5 and 3*x* + 3*y* + *z* = 0.

Ans.

The equation of plane through $(-1, 3, 2)$ can be expressed as

$$A(x+1) + B(y-3) + C(z-2) = 0$$

As the required plane is perpendicular to $x + 2y + 3z = 5$ and $3x + 3y + z = 0$, we get

$$A + 2B + 3C = 0 \quad \text{and } 3A + 3B + C = 0$$

$$\Rightarrow \frac{A}{2-9} = \frac{B}{9-1} = \frac{C}{3-6} \qquad \Rightarrow \qquad \frac{A}{-7} = \frac{B}{8} = \frac{C}{-3}$$

$\therefore$ Direction ratios of normal to the required plane are $-7, 8, -3$.

Hence, equation of the plane will be

$$-7(x+1) + 8(y-3) - 3(z-2) = 0$$

$$\Rightarrow \qquad -7x - 7 + 8y - 24 - 3z + 6 = 0$$

or $\qquad 7x - 8y + 3z + 25 = 0$

Q.10. Find the distance of the point (2, 12, 5) from the point of intersection of the line

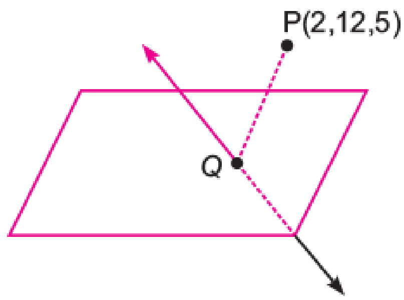
$\vec{r} = 2\hat{i} - 4\hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ **and the plane** $\vec{r}.(\hat{i} - 2\hat{j} + \hat{k}) = 0.$

Ans.

Given line and plane are

$$\vec{r} = (2\hat{i} - 4\hat{j} + 2\hat{k}) + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k}) \quad \dots (i)$$

and $\vec{r}.(\hat{i} - 2\hat{j} + \hat{k}) = 0 \quad \dots (ii)$



For intersection point Q, we solve equations (i) and (ii) by putting the value of $\vec{r}$ from (i) in (ii)

$$[(2\hat{i} - 4\hat{j} + 2\hat{k}) + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})].\ (\hat{i} - 2\hat{j} + \hat{k}) = 0$$

$$\Rightarrow \quad [(2 + 3\lambda)\hat{i} - (4 - 4\lambda)\hat{j} + (2 + 2\lambda)\hat{k}].(\hat{i} - 2\hat{j} + \hat{k}) = 0$$

$$\Rightarrow (2+3\lambda) + 2\ (4 - 4\lambda) + (2 + 2\lambda) = 0$$

$$\Rightarrow 2 + 3\lambda + 8 - 8\lambda + 2 + 2\lambda = 0 \Rightarrow 12 - 3\lambda = 0$$

$$\Rightarrow \lambda = 4$$

Hence position vector of intersecting point is $14\hat{i} + 12\hat{i} + 10\hat{k}$.

Co-ordinate of intersecting point, $Q \equiv (14, 12, 10)$

Required distance = $\sqrt{(14 - 2)^2 + (12 - 12)^2 + (10 - 5)^2}$

$= \sqrt{144 + 25} = \sqrt{169}$ units

$= 13$ units.

Q.11. The points *A* (4, 5, 10), *B* (2, 3, 4) and C (1, 2, –1) are three vertices of a parallelogram *ABCD*. Find the vector equations of the sides *AB* and *BC* and also find the coordinates of point *D*.

Ans.

The points $A\ (4, 5, 10)$, $B\ (2, 3, 4)$ and $C\ (1, 2, -1)$ are three vertices of parallelogram $ABCD$.

Let coordinates of D be (x, y, z)

Direction vector along AB is

$$\vec{a} = (2-4)\hat{i} + (3-5)\hat{j} + (4-10)\hat{k} =- 2\hat{i} - 2\hat{j} - 6\hat{k}$$

$\therefore$ Equation of line AB, is given by

$$\vec{b} = (4\hat{i} + 5\hat{j} + 10\hat{k}) + \lambda(2\hat{i} + 2\hat{j} + 6\hat{k})$$

Direction vector along BC is

$$\vec{c} = (1-2)\hat{i} + (2-3)\hat{j} + (-1-4)\hat{k} =- \hat{i} - \hat{j} - 5\hat{k}$$

$\therefore$ Equation of a line BC, is given by .

$$\vec{d} = (2\hat{i} + 3\hat{j} + 4\hat{k}) + \mu(\hat{i} + \hat{j} + 5\hat{k})$$

Since $ABCD$ is a parallelogram AC and BD bisect each other

$$\therefore \left[\frac{4+1}{2}, \frac{5+2}{2}, \frac{10-1}{2}\right] = \left[\frac{2+x}{2}, \frac{3+y}{2}, \frac{4+z}{2}\right]$$

$\Rightarrow 2 + x = 5,\ 3 + y = 7,\ 4 + z = 9$

$\Rightarrow x = 3,\ y = 4,\ z = 5$

Coordinates of D are $(3, 4, 5)$.

Q.12. Find the coordinates of the foot of the perpendicular and the perpendicular distance of the point *P* (3, 2, 1) from the plane $2x - y + z + 1 = 0$. Also find the image of the point in the plane.

Ans.

Let O (α, β, γ) be the image of the point P $(3, 2, 1)$ in the plane

$2x - y + z + 1 = 0$

PO is perpendicular to the plane and S is the mid-point of PO and the foot of the perpendicular.

Dr's of PS are $2, -1, 1$.

$\therefore$ Equation of PS are $\frac{x-3}{2} = \frac{y-2}{-1} = \frac{z-1}{1} = \mu$

$\therefore$ General point on line is S $(2\mu + 3, -\mu + 2, \mu + 1)$

If this point lies on plane, then

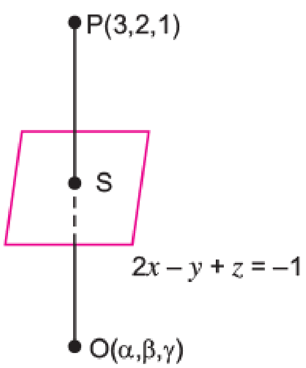
$2(2\mu + 3) - (-\mu + 2) + 1(\mu + 1) + 1 = 0$

$\Rightarrow \quad 6\mu + 6 = 0 \quad \Rightarrow \quad \mu = -1$

$\therefore$ Coordinates of S are $(1, 3, 0)$.

As S is the mid point of PO,

$\therefore$The coordinate of $S = \left(\frac{3+\alpha}{2}, \frac{2+\beta}{2}, \frac{1+\gamma}{2}\right) = (1, 3, 0)$



By comparing both sides, we get

$\frac{3+\alpha}{2} = 1 \quad \Rightarrow \quad \alpha = -1$

$\frac{2+\beta}{2} = 3 \quad \Rightarrow \quad \beta = 4$

$\frac{1+\gamma}{2} = 0 \quad \Rightarrow \quad \gamma = -1$

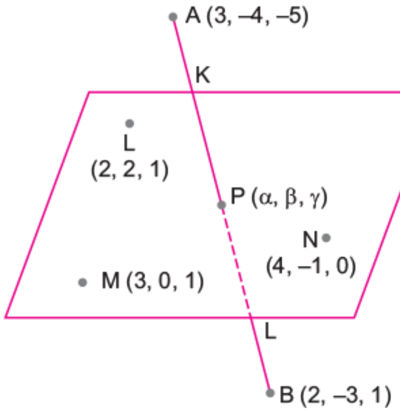
Image of point P is $(-1, 4, -1)$.

Q.13. Find the coordinate of the point *P* where the line through *A*(3, – 4, – 5) and *B* (2, – 3, 1) crosses the plane passing through three points *L*(2, 2, 1), *M*(3, 0, 1) and *N*(4, –1, 0). Also, find the ratio in which *P* divides the line segment AB.

Ans.

Let the coordinate of *P* be (α, β, γ).

Equation of plane passing through *L*(2, 2, 1), *M*(3, 0, 1) and *N*(4, –1, 0) is given by



$$\begin{vmatrix} x-2 & y-2 & z-1 \\ 3-2 & 0-2 & 1-1 \\ 4-2 & -1-2 & 0-1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-2 & y-2 & z-1 \\ 1 & -2 & 0 \\ 2 & -3 & -1 \end{vmatrix} = 0$$

$$\Rightarrow (x-2)(2-0) - (y-2)(-1-0) + (z-1)(-3+4) = 0$$

$$\Rightarrow 2(x-2) + (y-2) + z - 1 = 0$$

$$\Rightarrow 2x - 4 + y - 2 + z - 1 = 0$$

$$\Rightarrow 2x + y + z - 7 = 0 \;...(i)$$

Now, the equation of line passing through $A(3, -4, -5)$ and $B(2, -3, 1)$ is given by

$\Rightarrow \frac{x-3}{2-3} = \frac{y+4}{-3+4} = \frac{z+5}{1+5} \quad \frac{x-3}{-1} = \frac{y+4}{1} = \frac{z+5}{6} \quad ...(ii)$

$\because \; P(\alpha, \beta, \gamma)$ lie on line AB

$\Rightarrow \frac{\alpha-3}{-1} = \frac{\beta+4}{1} = \frac{\gamma+5}{6} = \lambda$ (say)

$\Rightarrow$ a = – λ + 3, b = λ – 4, g = 6λ – 5

Also $P(\alpha, \beta, \gamma)$ lie on plane (i)

$\Rightarrow 2\alpha + \beta + \gamma - 7 = 0$

$\Rightarrow 2(-\lambda + 3) + (\lambda - 4) + (6\lambda - 5) - 7 = 0$

$\Rightarrow -2\lambda + 6 + \lambda - 4 + 6\lambda - 5 - 7 = 0$

$\Rightarrow 5\lambda - 10 = 0$

$\Rightarrow \lambda = 2$

$\therefore \alpha = 1, \beta = -2, \gamma = 7$

$\therefore$ Co-ordinate of $P \equiv (1, -2, 7)$

Let P divides AB in the ratio $K : 1$.

$\therefore \; 1 = \frac{K \times 2 + 1 \times 3}{K+1}$

$\Rightarrow K + 1 = 2K + 3 \Rightarrow K = -2$

$\Rightarrow P$ divides AB externally in the ratio 2 : 1.

Q.14. Find the shortest distance between the lines $x + 1 = 2y = -12z$ and $x = y + 2 = 6z - 6$.

Ans.

Given lines are

$$x+1=2y=-12z \text{ and } x = y + 2 = 6z - 6$$

$$\Rightarrow \quad \frac{x-(-1)}{1}=\frac{y-0}{\frac{1}{2}}=\frac{z-0}{-\frac{1}{12}} \text{ and } \frac{x-0}{1}=\frac{y-(-2)}{1}=\frac{z-1}{\frac{1}{6}}$$

These lines may be written in vector form as

$$\vec{r}=(-\hat{i}+0\hat{j}+0\hat{k})+\lambda(\hat{i}+\tfrac{1}{2}\hat{j}-\tfrac{1}{12}\hat{k}) \quad \ldots(i)$$

and $\vec{r}=(0\hat{i}-2\hat{j}+\hat{k})+\lambda(\hat{i}+\hat{j}+\tfrac{1}{6}\hat{k}) \ldots(ii)$

We know that the shortest distance between

$\vec{r}=\vec{a_1}+\lambda\vec{b_1}$ and $\vec{r}=\vec{a_2}+\lambda\vec{b_2}$ is given by

$$\text{SD}=\left|\frac{(\vec{a_2}-\vec{a_1}).(\vec{b_1}\times\vec{b_2})}{|\vec{b_1}\times\vec{b_2}|}\right|$$

Here, $\vec{a_1}=-\hat{i}+0\hat{j}+0\hat{k},\qquad \vec{b_1}=\hat{i}+\frac{1}{2}\hat{j}-\frac{1}{12}\hat{k}$

$\vec{a_2}=0\hat{i}-2\hat{j}+\hat{k},\qquad \vec{b_2}=\hat{i}+\hat{j}+\frac{1}{6}\hat{k}$

Now, $\vec{a_2}-\vec{a_1}=(0\hat{i}-2\hat{j}+\hat{k})-(-\hat{i}+0\hat{j}+0\hat{k})=\hat{i}-2\hat{j}+\hat{k}$

$$\vec{b_1}\times\vec{b_2}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1/2 & -1/12\\ 1 & 1 & 1/6\end{vmatrix}=\left(\tfrac{1}{12}+\tfrac{1}{12}\right)\hat{i}-\left(\tfrac{1}{6}+\tfrac{1}{12}\right)\hat{j}+\left(1-\tfrac{1}{2}\right)\hat{k}$$

$$=\tfrac{1}{6}\hat{i}-\tfrac{1}{4}\hat{j}+\tfrac{1}{2}\hat{k}$$

$$\therefore \quad |\vec{b_1}\times\vec{b_2}|=\sqrt{\left(\tfrac{1}{6}\right)^2+\left(-\tfrac{1}{4}\right)^2+\left(\tfrac{1}{2}\right)^2}$$

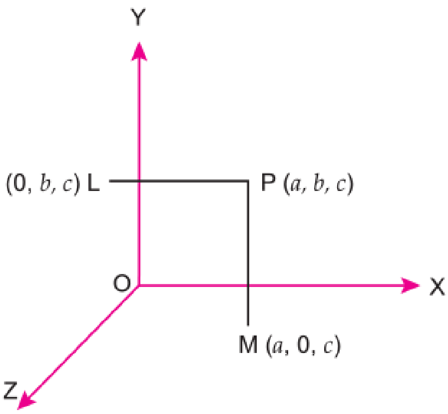
$$=\sqrt{\tfrac{1}{36}+\tfrac{1}{16}+\tfrac{1}{4}}=\sqrt{\tfrac{4+9+36}{144}}=\sqrt{\tfrac{49}{144}}=\tfrac{7}{12}$$

$$\therefore \text{ Required } S.D.=\left|\frac{(\hat{i}-2\hat{j}+\hat{k}).\left(\frac{1}{6}\hat{i}-\frac{1}{4}\hat{j}+\frac{1}{2}\hat{k}\right)}{\frac{7}{12}}\right|=\left|\frac{\frac{1}{6}+\frac{1}{2}+\frac{1}{2}}{\frac{7}{12}}\right|=\frac{\frac{7}{6}}{\frac{7}{12}}=\frac{7}{6}\times\frac{12}{7}=2 \text{ units}.$$

Q.15. From the point *P*(*a*, *b*, *c*), perpendiculars *PL* and *PM* are drawn to *YZ* and *ZX* planes respectively. Find the equation of the plane *OLM*.

Ans.

Obviously, the coordinates of *O*, *L* and *M* are (0, 0, 0), (0, *b*, *c*) and (*a*, 0, *c*).



Therefore, the equation of required plane is given by

$$\begin{vmatrix} x-0 & y-0 & z-0 \\ 0-0 & b-0 & c-0 \\ a-0 & 0-0 & c-0 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} x & y & z \\ 0 & b & c \\ a & 0 & c \end{vmatrix} = 0$$

$$\Rightarrow x(bc-0) - y(0-ac) + z(0-ab) = 0$$

$$\Rightarrow \quad bcx + acy - abz = 0$$

Q.16. Find the equation of the plane passing through the point (1, 1, –1) and perpendicular to the planes $x + 2y + 3z - 7 = 0$ and $2x - 3y + 4z = 0$.

Ans.

Let the equation of plane passing through point $(1, 1, -1)$ be

$a(x-1)+b(y-1)+c(z+1)=0 \quad \ldots(i)$

Since (i) is perpendicular to the plane $x+2y+3z-7=0$

$\therefore \quad 1.a+2.b+3.c=0 \quad \Rightarrow a+2b+3c=0 \quad \ldots(ii)$

Again plane (i) is perpendicular to the plane $2x-3y+4z=0$

$\therefore \quad 2.a-3.b+4.c=0 \quad \Rightarrow 2a-3b+4c=0 \quad \ldots(iii)$

From (ii) and (iii), we get

$$\frac{a}{8+9}=\frac{b}{6-4}=\frac{c}{-3-4} \quad \Rightarrow \quad \frac{a}{17}=\frac{b}{2}=\frac{c}{-7}=\lambda$$

$\Rightarrow \; a=17\lambda, b=2\lambda, c=-7\lambda$

Putting the value of a, b, c in (i), we get

$17\lambda(x-1)+2\lambda(y-1)-7\lambda(z+1)=0$

$\Rightarrow \; 17(x-1)+2(y-1)-7(z+1)=0$

$\Rightarrow \; 17x+2y-7z-17-2-7=0$

$\Rightarrow \; 17x+2y-7z-26=0$ is the required equation.

[**Note:** The equation of plane passing through (x_1, y_1, z_1) is given by $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$, where a, b, c are direction ratios of normal of plane.]

Q.17. Find the equation of the plane through the line of intersection of the planes $x+y+z=1$ and $2x+3y+4z=5$ which is perpendicular to the plane $x-y+z=0$. Also find the distance of the plane obtained above, from the origin.

Ans.

The equation of a plane passing through the intersection of the given planes is

$$(x + y + z - 1) + \lambda\,(2x + 3y + 4z - 5) = 0$$

$$\Rightarrow (1 + 2\lambda)\,x + (1 + 3\lambda)\,y + (1 + 4\lambda)\,z - (1+5\lambda) = 0 \quad \ldots (i)$$

Since, (i) is perpendicular to $x - y + z = 0$

$$\Rightarrow (1 + 2\lambda)\,1 + (1 + 3\lambda)\,(-1) + (1 + 4\lambda)\,1 = 0$$

$$\Rightarrow 1 + 2\lambda - 1 - 3\lambda + 1 + 4\lambda = 0$$

$$\Rightarrow 3\lambda + 1 = 0$$

$$\Rightarrow \lambda = -\frac{1}{3}$$

Putting the value of l in (i), we get

$$(1-\tfrac{2}{3})x + (1-1)y + (1-\tfrac{4}{3})z - (1-\tfrac{5}{3}) = 0 \quad \Rightarrow \quad \frac{x}{3} - \frac{z}{3} + \frac{2}{3} = 0$$

$\Rightarrow x - z + 2 = 0$, it is required plane.

Let d be the distance of this plane from origin.

$$\therefore\ d = \left|\frac{0.\,x + 0.\,y + 0.\,(-z) + 2}{\sqrt{1^2 + 0^2 + (-1)^2}}\right| = \left|\frac{2}{\sqrt{2}}\right| = \sqrt{2} \text{ units.}$$

[**Note:** The distance of the point (α, β, γ) to the plane $ax + by + cz + d = 0$ is given by $\left|\frac{a\alpha + b\beta + c\gamma + d}{\sqrt{a^2 + b^2 + c^2}}\right|$.

Q.18. Find the coordinates of the point where the line through the points *A* (3, 4, 1) and *B* (5, 1, 6) crosses the *XY*-plane.

Ans.

Let $P(\alpha, \beta, \gamma)$ be the point at which the given line crosses the XY plane

Now, the equation of given line is

$$\frac{x-3}{2} = \frac{y-4}{-3} = \frac{z-1}{5} \quad \ldots(i)$$

Since $P(\alpha, \beta, \gamma)$ lies on line (i)

$$\therefore \quad \frac{\alpha-3}{2} = \frac{\beta-4}{-3} = \frac{\gamma-1}{5} = \lambda \text{ (say)}$$

$\Rightarrow \alpha = 2\lambda + 3; \quad \beta = -3\lambda + 4$ and $\gamma = 5\lambda + 1$

Also $P(\alpha, \beta, \gamma)$ lie on given XY plane, *i.e.*, $z = 0$

$\therefore \; 0.\alpha + 0.\beta + \gamma = 0$

$\Rightarrow \; 5\lambda + 1 = 0$

$\Rightarrow \; \lambda = -\frac{1}{5}.$

Hence, the coordinates of required point is

$\alpha = 2 \times \left(-\frac{1}{5}\right) + 3 = \frac{13}{5}; \beta = -3 \times \left(-\frac{1}{5}\right) + 4 = \frac{23}{5}$ and $\gamma = 5 \times \left(-\frac{1}{5}\right) + 1 = 0$

i.e., required coordinates are $\left(\frac{13}{5}, \frac{23}{5}, 0\right)$.

Q.19. Find the vector equation of the plane passing through the points (3, 4, 2) and (7, 0, 6) and perpendicular to the plane 2*x* – 5*y* – 15 = 0. Also show that the plane thus obtained contains the line $\vec{r} = \hat{i} + 3\hat{j} - 2\hat{k} + \lambda(\hat{i} - \hat{j} + \hat{k}).$

Ans.

Let the equation of plane through $(3, 4, 2)$ be

$a(x-3)+b(y-4)+c(z-2)=0 \quad ...(i)$

$\because \quad (i)$ passes through $(7, 0, 6)$

$\therefore \quad a(7-3)+b(0-4)+c(6-2)=0 \Rightarrow 4a-4b+4c=0$

$\Rightarrow \quad a-b+c=0 \; ...(ii)$

Also, since plane (i) is perpendicular to plane $2x-5y-15=0$

$2a-5b+0c=0 \; ...(iii)$

From (ii) and (iii), we get

$\frac{a}{5}=\frac{b}{2}=\frac{c}{-3}=\lambda \quad (\text{say}) \Rightarrow a=5\lambda,\ b=2\lambda,\ c=-3\lambda.$

Putting the value of a, b, c in (i), we get

$5\lambda(x-3)+2\lambda(y-4)-3\lambda(z-2)=0$

$\Rightarrow \quad 5x-15+2y-8-3z+6=0$

$\Rightarrow \quad 5x+2y-3z=17$

$\therefore \quad$ Required vector equation of plane is $\vec{r}\,.\,(5\hat{i}+2\hat{j}-3\hat{k})=17 \quad ...(iv)$

Obviously, plane (iv) contains the line

$\vec{r}=(\hat{i}+3\hat{j}-2\hat{k})+\lambda(\hat{i}-\hat{j}+\hat{k}) \quad ...(v)$

Since, point $(\hat{i}+3\hat{j}-2\hat{k})$ satisfy the equation (iv) and vector is perpendicular to $(5\hat{i}+2\hat{j}-3\hat{k})$.

as $(\hat{i}+3\hat{j}-2\hat{k}).\,(5\hat{i}+2\hat{j}-3\hat{k})=5+6+6=17$ and $(5\hat{i}+2\hat{j}-3\hat{k}).(\hat{i}-\hat{j}+\hat{k})=5-2-3=0$

Therefore, (iv) contains line (v).

Q.20. Show that the lines

$$\overrightarrow{r} = 3\hat{i} + 2\hat{j} - 4\hat{k} + \boldsymbol{\lambda}(\hat{i} + 2\hat{j} + 2\hat{k});$$

$$\overrightarrow{r} = 5\hat{i} - 2\hat{j} + \mu(3\hat{i} + 2\hat{j} + 6\hat{k});$$

are intersecting. Hence find their point of intersection.

Ans.

Given lines are

$$\overrightarrow{r} = 3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k}) \text{ and } \overrightarrow{r} = 5\hat{i} + 2\hat{j} + \mu(3\hat{i} + 2\hat{j} + 6\hat{k})$$

Its corresponding cartesian forms are

$$\frac{x-3}{1} = \frac{y-2}{2} = \frac{z+4}{2} \quad ...(i)$$

$$\frac{x-5}{3} = \frac{y+2}{2} = \frac{z-0}{6} \quad ...(ii)$$

If two lines (i) and (ii) intersect, let interesting point be .

$\Rightarrow$ (α, β, γ) satisfy line (i)

$$\therefore \frac{\alpha-3}{1} = \frac{\beta-2}{2} = \frac{\gamma+4}{2} = \lambda = \text{(say)}$$

$\Rightarrow \alpha = \lambda + 3, \beta = 2\lambda + 2, \gamma = 2\lambda - 4$

Also, (α, β, γ) will satisfy line (ii)

$$\therefore \frac{\alpha-5}{3} = \frac{\beta+2}{2} = \frac{\gamma}{6}$$

$$\Rightarrow \frac{\lambda+3-5}{3} = \frac{2\lambda+2+2}{2} = \frac{2\lambda-4}{6}$$

$$\therefore \frac{\lambda-2}{3} = \frac{\lambda+2}{1} = \frac{\lambda-2}{3}$$

I II III

I and II $\Rightarrow$ $\frac{\lambda-2}{3}=\frac{\lambda+2}{1}$ $\Rightarrow$ $\lambda-2=3\lambda+6$ $\Rightarrow$ $\lambda=-4$

II and III $\Rightarrow$ $\frac{\lambda+2}{1}=\frac{\lambda-2}{3}$ $\Rightarrow$ $\lambda=-4$

$\therefore$ The value of l is same in both cases.

Hence, both lines intersect each other at point

$(\alpha, \beta, \gamma) \equiv (-4+3, 2\times(-4)+2, 2(-4)-4)$ $(-1, -6, -12)$

Q.21. Find the vector equation of the plane passing through three points with position vectors $\hat{i}+\hat{j}-2\hat{k}, 2\hat{i}-\hat{j}+\hat{k}$ **and** $\hat{i}+2\hat{j}+\hat{k}.$ **Also find the coordinates of the point of intersection of this plane and the line** $\vec{r}=3\hat{i}-\hat{j}-\hat{k}+\lambda(2\hat{i}-2\hat{j}+\hat{k}).$

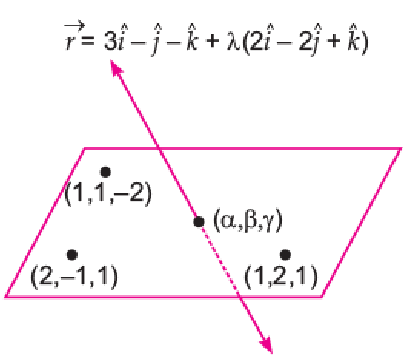
Ans.

The equation of plane passing through three points $\hat{i}+\hat{j}-2\hat{k}, 2\hat{i}-\hat{j}+\hat{k}$ and $\hat{i}+2\hat{j}+\hat{k}$

i.e., $(1, 1, -2), (2, -1, 1)$ and $(1, 2, 1)$ is

$$\begin{vmatrix} x-1 & y-1 & z+2 \\ 2-1 & -1-1 & 1+2 \\ 1-1 & 2-1 & 1+2 \end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix} x-1 & y-1 & z+2 \\ 1 & -2 & 3 \\ 0 & 1 & 3 \end{vmatrix}=0$$



$\Rightarrow (x-1)(-6-3)-(y-1)(3-0)+(z+2)(1+0)=0$

$\Rightarrow -9x+9-3y+3+z+2=0$

$\Rightarrow 9x+3y-z=14 \ldots(i)$

Its vector form is $\vec{r}.(9\hat{j}+3\hat{j}-\hat{k})=14$

The given line is $\vec{r}=(3\hat{i}-\hat{j}-\hat{k})+\lambda(2\hat{i}-2\hat{j}+\hat{k})$

Its cartesian form is

$\frac{x-3}{2}=\frac{y+1}{-2}=\frac{z+1}{1} \quad \ldots(ii)$

Let the line (ii) intersect plane (i) at (α, β, γ)

$\because (\alpha, \beta, \gamma)$ lie on (ii)

$\frac{\alpha-3}{2}=\frac{\beta+1}{-2}=\frac{\gamma+1}{1}=\lambda$ (say)

$\Rightarrow \alpha=2\lambda+3; \beta=-2\lambda-1; \gamma=\lambda-1$

Also, point (α, β, γ) lie on plane (i)

$\Rightarrow 9\text{ a}+3\text{b}-\text{g}=14$

$\Rightarrow 9(2\lambda+3)+3(-2\lambda-1)-(\lambda-1)=14$

$\Rightarrow 18\lambda+27-6\lambda-3-\lambda+1=14$

$\Rightarrow 11\lambda+25=14$

$\Rightarrow 11\lambda=14-25$

$\Rightarrow 11\lambda=-11$

$\Rightarrow \lambda=-1$

Therefore, point of intersection $\equiv (1, 1, -2)$.

Q.22. Find the coordinates of the point where the line through the points *A* (3, 4, 1) and *B* (5, 1, 6) crosses the plane determined by the points *P* (2, 1, 2), *Q* (3, 1, 0) and *R* (4, –2, 1).

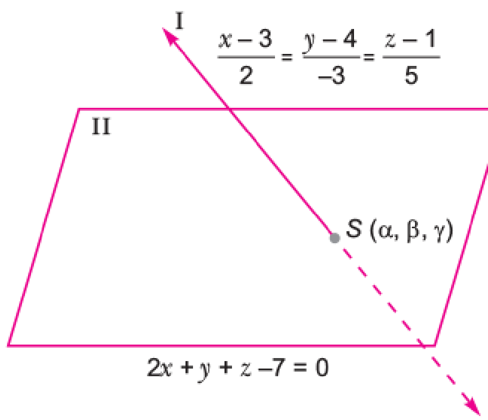
Ans.

The line through $A\ (3, 4, 1)$ and $B\ (5, 1, 6)$ is given by

$$\frac{x-3}{5-3} = \frac{y-4}{1-4} = \frac{z-1}{6-1} \quad \Rightarrow \quad \frac{x-3}{2} = \frac{y-4}{-3} = \frac{z-1}{5} \quad ...(i)$$

The equation of plane determined by the points $P\ (2, 1, 2)$, $Q\ (3, 1, 0)$ and $R\ (4, -2, 1)$ is given by,

$$\begin{vmatrix} x-2 & y-1 & z-2 \\ 3-2 & 1-1 & 0-2 \\ 4-2 & -2-1 & 1-2 \end{vmatrix} = 0 \quad \Rightarrow \quad \begin{vmatrix} x-2 & y-1 & z-2 \\ 1 & 0 & -2 \\ 2 & -3 & -1 \end{vmatrix} = 0$$



$$\Rightarrow (0-6)(x-2) - (-1+4)(y-1) + (-3-0)(z-2) = 0$$

$$\Rightarrow -6x + 12 - 3y + 3 - 3z + 6 = 0$$

$$\Rightarrow -6x - 3y - 3z + 21 = 0$$

$$\Rightarrow 2x + y + z - 7 = 0 \ ...(ii)$$

Let S (α, β, γ) be intersecting point of line (I) and plane (II)

$\because$ S (α, β, γ) lie on line (I)

$\frac{\alpha - 3}{2} = \frac{\beta - 4}{-3} = \frac{\gamma - 1}{5} = \lambda$

$\therefore$ $\alpha = 2\lambda + 3$, $\beta = -3\lambda + 4$, $\gamma = 5\lambda + 1$

$\because$ S (α, β, γ) also lie on plane (II)

$2a + b + g - 7 = 0$

$\Rightarrow 2\ (2\lambda + 3) + (-3\lambda + 4) + (5\lambda + 1) - 7 = 0$

$\Rightarrow 4\lambda + 6 - 3\lambda + 4 + 5\lambda + 1 - 7 = 0$

$\Rightarrow\ 6\lambda + 4 = 0 \quad \Rightarrow \quad \lambda = -\frac{4}{6} = -\frac{2}{3}$

$\therefore\ \alpha = 2 \times \left(\frac{-2}{3}\right) + 3 = -\frac{4}{3} + 3 - \frac{5}{3}$

$\beta = -3 \times \left(-\frac{2}{3}\right) + 4 = 2 + 4 = 6$ and $\gamma = 5 \times \left(\frac{-2}{3}\right) + 1 = \frac{-10}{3} + 1 = -\frac{7}{3}$

$\therefore$ Required point of intersection = $\left(\frac{5}{3}, 6, -\frac{7}{3}\right)$.

Q.23. Find the direction ratios of the normal to the plane, which passes through the points (1, 0, 0) and (0, 1, 0) and makes angle $\frac{\pi}{4}$ with the plane *x* + *y* = 3. Also find the equation of the plane.

Ans.

Let the equation of plane passing through the point $(1, 0, 0)$ be

$a(x-1)+b(y-0)+c(z-0)=0$

$\Rightarrow ax-a+by+cz=0$

$\Rightarrow ax+by+cz=a \ldots(i)$

Since, (i) also passes through $(0, 1, 0)$

$\Rightarrow 0+b+0=a$

$\Rightarrow b=a \ldots(ii)$

Given, the angle between plane (i) and plane $x+y=3$ is $\frac{\pi}{4}$.

$\therefore \cos\frac{\pi}{4}=\left|\frac{a.1+b.1+c.0}{\sqrt{a^2+b^2+c^2}\sqrt{1^2+1^2}}\right|$

$\Rightarrow \frac{1}{\sqrt{2}}=\left|\frac{a+b}{\sqrt{a^2+b^2+c^2}\sqrt{1+1}}\right|$

$\Rightarrow \frac{1}{\sqrt{2}}=\left|\frac{a+b}{\sqrt{a^2+b^2+c^2}\sqrt{2}}\right| \quad \Rightarrow \quad 1=\left|\frac{a+b}{\sqrt{a^2+b^2+c^2}}\right|$

$\Rightarrow \sqrt{a^2+b^2+c^2}=\pm(a+b) \quad \Rightarrow \quad a^2+b^2+c^2=(a+b)^2$

$\Rightarrow a^2+b^2+c^2=a^2+b^2+2ab$

$\Rightarrow c^2=2ab$

$\Rightarrow c^2=2a^2$ [From (ii)]

$\Rightarrow \sqrt{a^2+b^2+c^2}=\pm(a+b) \quad \Rightarrow \quad a^2+b^2+c^2=(a+b)^2$

Now, equation (i) becomes

$\mathrm{ax}+\mathrm{ay}\pm\sqrt{2}\mathrm{az}=a$

$\Rightarrow x+y\pm\sqrt{2}z=1,$ is the required equation of plane.

Therefore, required direction ratios are $1,\ 1,\ \pm\sqrt{2}$.

Q.24. Find the equation of the plane which contains the line of intersection of the planes

$$\vec{r}.\,(\hat{i} - 2\hat{j} + 3\hat{k}) - 4 = 0 \text{ and } \vec{r}.\,(-2\hat{i} + \hat{j} + \hat{k}) + 5 = 0$$

and whose intercept on *x*-axis is equal to that of on *y*-axis.

Ans.

Given planes are $\vec{r}.\,(\hat{i} - 2\hat{j} + 3\hat{k}) - 4 = 0$ and $\vec{r}.\,(-2\hat{i} + \hat{j} + \hat{k}) + 5 = 0$

These can be written in cartesian form as

$x - 2y + 3z - 4 = 0 \quad ...(i)$

and $-2x + y + z + 5 = 0 \quad ...(ii)$

Now the equation of plane containing the line of intersection of the planes (i) and (ii) is given by

$(x - 2y + 3z - 4) + \lambda(-2x + y + z + 5) = 0 \quad ...(iii)$

$\Rightarrow (1 - 2\lambda)\,x - (2 - \lambda)\,y + (3 + \lambda)\,z - 4 + 5\lambda = 0$

$\Rightarrow (1 - 2\lambda)\,x - (2 - \lambda)\,y + (3 + \lambda)\,z = 4 - 5\lambda$

$\Rightarrow \dfrac{x}{\frac{4-5\lambda}{1-2\lambda}} + \dfrac{y}{\frac{4-5\lambda}{-2+\lambda}} + \dfrac{z}{\frac{4-5\lambda}{3+\lambda}} = 1$

According to question $\dfrac{4-5\lambda}{1-2\lambda} = \dfrac{4-5\lambda}{-2+\lambda}$

$\Rightarrow 1 - 2\lambda = -2 + \lambda$

$\Rightarrow 3\lambda = 3$

$\Rightarrow \lambda = 1$

Putting the value of $\lambda = 1$ in (iii), we get

$(x - 2y + 3z - 4) + 1\ (-2x + y + z + 5) = 0$

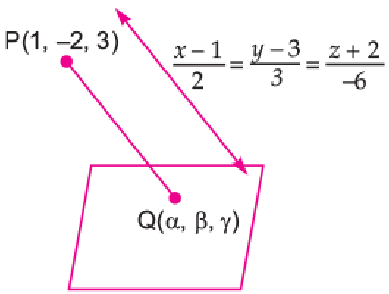
$-x - y + 4z + 1 = 0$ $\vec{r}.(\hat{i} + \hat{j} - 4\hat{k}) - 1 = 0$

$\Rightarrow\ x + y - 4z - 1 = 0$

$\Rightarrow$ Its vector form is

Q.25. Find the distance of the point (1, –2, 3) from the plane *x* – *y* + *z* = 5 measured parallel to the line whose direction cosines are proportional to 2, 3, –6.

Ans.



Let $Q\ (\alpha, \beta, \gamma)$ be the point on the given plane

$x - y + z = 5$ ...(i)

Since PQ is parallel to given line

$\frac{x-1}{2} = \frac{y-3}{3} = \frac{z+2}{-6}$ ...(ii) where $P\ (1, -2, 3)$ is the given point.

$\because$ PQ is parallel to given line (ii).

$\therefore \overrightarrow{PQ} \parallel \vec{b}$ (parallel vector of line).

$\Rightarrow \frac{\alpha - 1}{2} = \frac{\beta + 2}{3} = \frac{\gamma - 3}{-6} = \lambda$

$\Rightarrow \alpha = 2\lambda + 1, \beta = 3\lambda - 2, \gamma = -6\lambda + 3$

Now, $\because$ $Q(\alpha, \beta, \gamma)$ lie on plane (i)

$\alpha - \beta + \gamma = 5$

$2\lambda + 1 - 3\lambda + 2 - 6\lambda + 3 = 5$

$-7\lambda + 6 = 5 \qquad \Rightarrow -7\lambda = -1$

$\lambda = \frac{1}{7}$

$\alpha = 2 \times \frac{1}{7} + 1 = \frac{9}{7}; \beta = 3 \times \frac{1}{7} - 2 = -\frac{11}{7}$ and $\gamma = -6 \times \frac{1}{7} + 3 = \frac{15}{7}$

Therefore required distance

$$PQ = \sqrt{\left(\frac{9}{7} - 1\right)^2 + \left(-\frac{11}{7} + 2\right)^2 + \left(\frac{15}{7} - 3\right)^2} \text{ unit.}$$
$$= \sqrt{\frac{4}{49} + \frac{9}{49} \frac{36}{49}} = \sqrt{1} = 1$$

Q.26. Find the value of *p*, so that the lines $l_1 = \frac{1 - x}{3} = \frac{7y - 14}{p} = \frac{z - 3}{2}$ and $l_2 : \frac{7 - 7x}{3p} = \frac{y - 5}{1} = \frac{6 - z}{5}$ are perpendicular to each other. Also find the equations of a line passing through a point (3, 2, –4) and parallel to line l_1.

Ans.

Given line l_1 and l_2 are

$$l_1 \equiv \frac{1 - x}{3} = \frac{7y - 14}{p} = \frac{z - 3}{2}$$
$$\Rightarrow \quad \frac{x - 1}{-3} = \frac{y - 2}{\frac{p}{7}} = \frac{z - 3}{2}$$

$$l_2 \equiv \frac{7 - 7x}{3p} = \frac{y - 5}{1} = \frac{6 - z}{5}$$
$$\Rightarrow \quad \frac{x - 1}{\frac{-3p}{7}} = \frac{y - 5}{1} = \frac{z - 6}{-5}$$

Since $l_1 \perp l_2$

$$\Rightarrow (-3)\left(-\frac{3p}{7}\right)+\frac{p}{7}\times 1+2\times(-5)=0 \qquad \Rightarrow \frac{9p}{7}+\frac{p}{7}-10=0 \quad \Rightarrow \quad \frac{10p}{7}=10$$

$$\Rightarrow p=\frac{7\times 10}{10} \qquad \Rightarrow \qquad p=7$$

The equation of line passing through $(3, 2, -4)$ and parallel to l_1 is given by

$$\frac{x-3}{-3}=\frac{y-2}{\frac{p}{7}}=\frac{z+4}{2}$$

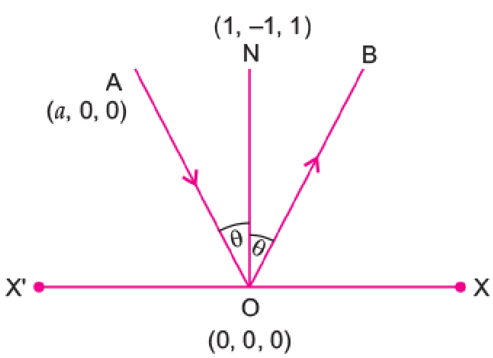
i.e., $\frac{x-3}{-3}=\frac{y-2}{1}=\frac{z+4}{2}$ $\qquad (\because\ p=7)$

Long Answer Questions-II (OIQ)

[6 Mark]

Q.1. A mirror and source of light are situated at the origin *O* and at a point on *OX* respectively. A ray of light from the source strike the mirror and is reflected. If the Dr's of the normal to the plane are 1, –1, 1, then find dc's of reflected ray.

Ans.



Let the source of light be situated at $(a, 0, 0)$ and AO and OB be incident and reflected rays. ON is the normal to the mirror at O.

Now Dr's at OA are $(a-0), (0-0), (0-0)$ *i.e.*, $a, 0, 0$

$\therefore$ Dc's of OA $\frac{a}{\sqrt{a^2+0^2+0^2}}, \frac{0}{\sqrt{a^2+0^2+0^2}}, \frac{0}{\sqrt{a^2+0^2+0^2}}$ $\qquad$ *i.e.*, 1, 0, 0.

Given, Dr's of ON are $1, -1, 1$

$\therefore$ Dc's of ON are $\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

Again let $\angle AON = \angle NOB$ = q [Law of reflection]

$\therefore \cos\theta = 1.\frac{1}{\sqrt{3}} + 0 + 0$ $\quad [\because \cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2]$

Let l, m, n be Dc's of reflected ray OB.

$\cos\theta = \frac{1}{\sqrt{3}}.l + \left(-\frac{1}{\sqrt{3}}\right)m + \frac{1}{\sqrt{3}}n$

$\Rightarrow \frac{1}{\sqrt{3}} = \frac{l}{\sqrt{3}} - \frac{m}{\sqrt{3}} + \frac{n}{\sqrt{3}} \quad \Rightarrow \quad l - m + n = 1$ $\quad \ldots (i)$

Also, $\cos 2\theta = 1.\, l + 0 \,.\, m + 0 \,.\, n$

$\Rightarrow 2\cos^2\theta - 1 = l$

$\Rightarrow 2 \times \frac{1}{3} - 1 = l \quad \Rightarrow \quad l = \frac{2-3}{3} = -\frac{1}{3}$

Putting in (i), we get $m - n = -\frac{4}{3}$ $\quad \ldots (ii)$

Also $l^2 + m^2 + n^2 = 1$

$m^2 + n^2 = 1 - \left(-\frac{1}{3}\right)^2 = \frac{8}{9}$ $\quad \ldots (iii)$

(ii) & $(iii) \Rightarrow m = -\frac{2}{3}$ and $n = \frac{2}{3}$, Hence, direction cosines of reflected ray are $-\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}$.

Q.2. Find the cartesian as well as vector equations of the planes through the intersection of planes $\vec{r}.\,(2\hat{i} + 6\hat{j}) + 12 = 0$ **and** $\vec{r}.(3\hat{i} - \hat{j} + 4\hat{k}) = 0$ **, which are at a unit distance from the origin.**

Ans.

The equation of the plane passing through the intersection of the planes

$\vec{r}.(2\hat{i}+6\hat{j})+12=0$ and $\vec{r}.(3\hat{i}-\hat{j}+4\hat{k})=0$ is

$\Rightarrow \vec{r}.\{(2+3\lambda)\hat{i}+(6-\lambda)\hat{j}+4\lambda\hat{k}\}+12=0$ ...(*i*)

The planes are at a unit distance from origin. Therefore, length of the perpendicular from the origin to the plane (*i*) = 1 unit.

$\therefore \frac{12}{\sqrt{(2+3\lambda)^2+(6-\lambda)^2+16\lambda^2}}=1$

$\Rightarrow 144=(2+3\lambda)^2+(6-\lambda)^2+16\lambda^2$ $\Rightarrow 144=40+26\lambda^2$ $\Rightarrow 26\lambda^2=104$

$\Rightarrow \lambda^2=4$ $\Rightarrow \lambda=\pm 2$

Putting the values of λ in equation (*i*), we get

$\vec{r}.(8\hat{i}+4\hat{j}+8\hat{k})+12=0$ and $\vec{r}.(-4\hat{i}+8\hat{j}-8\hat{k})+12=0$

which are the equations of the required planes. These equations can also be written as

$\vec{r}.(2\hat{i}+\hat{j}+2\hat{k})+3=0$ and $\vec{r}.(-\hat{i}+2\hat{j}-2\hat{k})+3=0$.

The above equations can be written in cartesian form as follows:

$2x+y+2z+3=0$ and $-x+2y-2z+3=0$

Q.3. A plane meets the coordinate axes in *A, B, C,* such that the centroid of the triangle *ABC* is the point (α, β, γ). Show that the equation of the plane is $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=3.$

Ans.

Let the equation of required plane be

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \qquad ...(i)$$

Then the coordinates of A, B, C are $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ respectively. So, the centroid of triangle ΔABC is $\left(\frac{a}{3},\frac{b}{3},\frac{c}{3}\right)$. But the coordinates of the centroid are (α, β, γ) as given in problem.

$$\alpha=\frac{a}{3},\ \beta=\frac{b}{3},\text{ and }\gamma=\frac{c}{3} \Rightarrow a=3\alpha,\ b=3\beta,\ c=3\gamma\ .$$

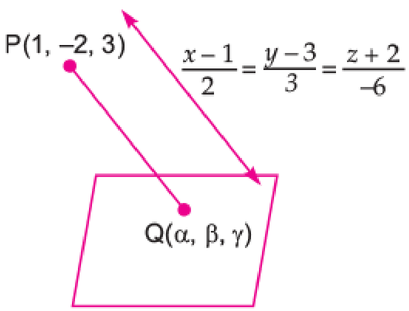
Substituting the values of a, b and c in equation (i), we get the required equation of the plane as follows

$$\frac{x}{3\alpha}+\frac{y}{3\beta}+\frac{z}{3\gamma}=1 \quad \Rightarrow \quad \frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=3.$$

Q.4. Find the distance of the point (1, –2, 3) from the plane $x - y + z = 5$, measured parallel to the line.

$$\frac{x-1}{2}=\frac{y-3}{3}=\frac{z+2}{-6}$$

Ans.



Let $Q(\alpha, \beta, \gamma)$ be the point on the given plane

$$x - y + z = 5 \qquad ...(i)$$

Since PQ is parallel to given line

$$\frac{x-1}{2}=\frac{y-3}{3}=\frac{z+2}{-6} \qquad ...(ii) \text{ where } P(1, -2, 3) \text{ is the given point.}$$

$\because$ PQ is parallel to given line (ii).

$\therefore \overrightarrow{PQ} \parallel \vec{b}$ (parallel vector of line).

$$\Rightarrow \frac{\alpha-1}{2}=\frac{\beta+2}{3}=\frac{\gamma-3}{-6}=\lambda$$

$\Rightarrow \alpha = 2\lambda + 1, \beta = 3\lambda - 2, \gamma = -6\lambda + 3$

Now, $\because Q(\alpha, \beta, \gamma)$ lie on plane (i)

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$2\lambda + 1 - 3\lambda + 2 - 6\lambda + 3 = 5$

$-7\lambda + 6 = 5 \qquad \Rightarrow -7\lambda = -1$

$\lambda = \frac{1}{7}$

$\alpha = 2 \times \frac{1}{7} + 1 = \frac{9}{7}; \beta = 3 \times \frac{1}{7} - 2 = -\frac{11}{7}$ and $\gamma = -6 \times \frac{1}{7} + 3 = \frac{15}{7}$

Therefore required distance

$$PQ = \sqrt{\left(\frac{9}{7} - 1\right)^2 + \left(-\frac{11}{7} + 2\right)^2 + \left(\frac{15}{7} - 3\right)^2} \text{ unit.}$$
$$= \sqrt{\frac{4}{49} + \frac{9}{49}\frac{36}{49}} = \sqrt{1} = 1$$