

निश्चित समाकल

Ex 10.1

योगफल की सीमा के रूप में (प्रथम सिद्धान्त से) निम्न निश्चित समाकलों के मान ज्ञात कीजिए :

प्रश्न 1.

$$\int_3^5 (x-2) dx$$

हल :

$$\int_3^5 (x-2) dx$$

परिभाषानुसार,

$$\int_a^b f(x) dx = \lim_{h \rightarrow 0} h [f(a+h) + f(a+2h) + f(a+3h) + \dots + f(a+nh)]$$

यहाँ $a = 3, b = 5, f(x) = x - 2, nh = 5 - 3 = 2$

$$\therefore \int_3^5 (x-2) dx$$

$$= \lim_{h \rightarrow 0} h [f(3+h) + f(3+2h) + f(3+3h) + \dots + f(3+nh)]$$

$$= \lim_{h \rightarrow 0} h [(3+h-2) + (3+2h-2) + (3+3h-2) + \dots + (3+nh-2)]$$

$$= \lim_{h \rightarrow 0} h [1+h+1+2h+1+3h+\dots+1+nh]$$

$$= \lim_{h \rightarrow 0} h [h(1+2+3+\dots+n) + (1+1+1+\dots+n \text{ बार})]$$

$$= \lim_{h \rightarrow 0} h \left[\frac{n(n+1)}{2} + n \right]$$

$$= \lim_{h \rightarrow 0} \frac{h(hn^2 + h + 2n)}{2}$$

$$= \lim_{h \rightarrow 0} \frac{h^2 n^2 + h^2 + 2nh}{2}$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \frac{(nh)^2 + 2(nh) + h^2 n}{2} \\
&= \frac{(2)^2 + 2 \times 2 + 0}{2} \\
&= \frac{4 + 4}{2} = \frac{8}{2} = 4
\end{aligned}$$

प्रश्न 2.

$$\int_a^b x^2 dx$$

हल :

$$\int_a^b x^2 dx$$

परिभाषानुसार,

$$\int_a^b f(x) dx = \lim_{h \rightarrow 0} h [f(a+h) + f(a+2h) + \dots + f(a+nh)]$$

यहाँ $a = a$, $b = b$, $nh = b - a$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} h [f(a+h) + f(a+2h) + \dots + f(a+nh)] \\
&= \lim_{h \rightarrow 0} h [(a+h)^2 + (a+2h)^2 + \dots + (a+nh)^2] \\
&= \lim_{h \rightarrow 0} h [(a^2 + h^2 + 2ah) + (a^2 + 2^2h^2 + 2 \cdot a^2h) + \dots + \\
&\quad (a^2 + n^2h^2 + 2anh)] \\
&= \lim_{h \rightarrow 0} h [(a^2 + a^2 + \dots + a^2) + h^2(1 + 2^2 + \dots + n^2) + \dots + \\
&\quad 2ah(1 + 2 + \dots + n)] \\
&= \lim_{h \rightarrow 0} h \left[a^2 \times n + h^2 \times \frac{n(2n+1)(n+1)}{6} + 2ah \times \frac{n(n+1)}{2} \right] \\
&= \lim_{h \rightarrow 0} \left[a^2 nh + h^3 \frac{(n)(n+1)(2n+1)}{6} + ah^2 \cdot n(n+1) \right] \\
&= \lim_{h \rightarrow 0} \left[a^2 (b-a) + \frac{(nh)(nh+h)(2nh+h)}{6} + anh(nh+h) \right]
\end{aligned}$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \left[a^2(b-a) + \frac{(b-a)(b-a+h)\{2(b-a)+h\}}{6} \right. \\
&\quad \left. + a(b-a)(b-a+h) \right] \\
&= a^2(b-a) + a(b-a)(b-a) + \frac{1}{3}(b-a)(b-a)(b-a) \\
&= (b-a)a^2 + a(b-a)^2 + \frac{1}{3}(b-a)^3 \\
&= \frac{b-a}{3} \left[3a^2 + 3a(b-a) + (b-a)^2 \right] \\
&= \frac{b-a}{3} \left[3a^2 + 3ab - 3a^2 + b^2 - 2ab + a^2 \right] \\
&= \frac{b-a}{3} [b^2 + ab + a^2] \\
&= \frac{b^3 - a^3}{3}
\end{aligned}$$

प्रश्न 3.

$$\int_1^3 (x^2 + 5x) dx$$

हल :

$$\int_1^3 (x^2 + 5x) dx$$

यहाँ $f(x) = x^2 + 5x$, $a = 1, b = 3$ $nh = 3 - 1 = 2$

$$\therefore \int_1^3 (x^2 + 5x) dx$$

$$= \lim_{h \rightarrow 0} h [f(1) + f(1+h) + \dots + f(1+(n-1)h)]$$

$$= \lim_{h \rightarrow 0} h [(1^2 + 5 \times 1) + \{(1+h)^2 + 5(1+h)\} + \dots +$$

$$\{1 + (n-1)h^2 + 5\{1 + (n-1)h\}$$

$$= \int_1^3 x^2 dx + \int_1^3 5x dx$$

$$= I_1 + I_2$$

...(i)

$$I_1 = \int_1^3 x^2 dx$$

समी. (i) में 11 व 12 का मान रखने पर,

$$\int_0^3 (x^2 + 5x) dx = \frac{26}{3} + 20$$
$$= \frac{26 + 60}{3} = \frac{86}{3}$$

प्रश्न 4.

$$\int_a^b e^{-x} dx$$

हल :

यहाँ $f(x) = e^{-x}$, 'a' = a, 'b' = b तथा $nh = b - a$

$$\therefore \int_a^b e^{-x} dx = \lim_{h \rightarrow 0} h[f(a+h) + f(a+2h) + \dots + f(a+nh)]$$

$$= \lim_{h \rightarrow 0} h[e^{-(a+h)} + e^{-(a+2h)} + \dots + e^{-(a+nh)}]$$

$$= \lim_{h \rightarrow 0} h e^{-a} [e^{-h} + e^{-2h} + \dots + e^{-nh}]$$

$$= \lim_{h \rightarrow 0} h e^{-a} \left[\frac{e^{-h} \{ (e^{-h})^n - 1 \}}{e^{-h} - 1} \right]$$

$$= h e^{-a} \lim_{h \rightarrow 0} e^{-1} \left[\frac{e^{-nh} - 1}{e^{-h} - 1} \right]$$

$$= e^{-a} e^{-1} \frac{h[e^{-(b-a)} - 1]}{e^{-h} - 1} \quad (\because nh = b - a)$$

$$= e^{-a} e^0 (e^{a-b} - 1) \cdot \frac{1}{\lim_{h \rightarrow 0} \frac{e^h - 1}{e^h}}$$

$$= e^{-a} (e^{a-b} - 1) \cdot \frac{1}{-1} \quad \left(\because \lim_{h \rightarrow 0} \frac{e^h - 1}{e^h} = -1 \right)$$

$$= -(e^{-b} - e^{-a})$$

$$= e^{-a} - e^{-b}$$

प्रश्न 5.

$$\int_0^2 (x+4) dx$$

हल :

$$\int_0^2 (x+4) dx$$

यहाँ $f(x) = x + 4$, $a = 0$, $b = 2$

तथा $nh = b - a = 2 - 0 = 2$

$$\therefore \int_0^2 (x+4) dx$$

$$= \lim_{h \rightarrow 0} h [f(0+h) + f(0+2h) + \dots + f(0+nh)]$$

$$= \lim_{h \rightarrow 0} h [f(h) + f(2h) + \dots + f(nh)]$$

$$= \lim_{h \rightarrow 0} h [f(h+4) + f(2h+4) + \dots + f(nh+4)]$$

$$= \lim_{h \rightarrow 0} h [h(1+2+\dots+n) + (4+4+\dots+4)]$$

$$= \lim_{h \rightarrow 0} h \left[h \left(\frac{n(n+1)}{2} \right) + 4n \right]$$

$$= \lim_{h \rightarrow 0} \left[\frac{nh(nh+h)}{2} + 4nh \right]$$

$$= \lim_{h \rightarrow 0} \left[\frac{2(2+h)}{2} + 4 \times 2 \right]$$

$$= (2+0)+8 = 10$$

प्रश्न 6.

$$\int_1^3 (2x^2 + 5) dx$$

हल :

$$\int_1^3 (2x^2 + 5) dx$$

यहाँ $f(x) = 2x^2 + 5$, $a = 1$, $b = 3$

तथा $nh = b - a = 3 - 1 = 2$

$$\begin{aligned}
&\therefore \int_1^3 (2x^2 + 5) dx \\
&= \lim_{h \rightarrow 0} h [f(1+h) + f(1+2h) + \dots + f(1+nh)] \\
&= \lim_{h \rightarrow 0} h [2(1+h)^2 + 5 + 2(1+2h)^2 + 5 + \dots + \\
&\qquad\qquad\qquad 2(1+nh)^2 + 5] \\
&= \lim_{h \rightarrow 0} h [2((1+h)^2 + (1+2h)^2 + \dots + (1+nh)^2) + \\
&\qquad\qquad\qquad (5 + 5 + \dots + 5)] \\
&= \lim_{h \rightarrow 0} h [2\{1 + h^2 + 2h + 1 + 2^2h^2 + 2 \cdot 2h + \dots + 1 + n^2h^2 \\
&\qquad\qquad\qquad + 2 \cdot nh\} + 5n] \\
&= \lim_{h \rightarrow 0} 2h [(1 + 1 + \dots + 1) + h^2(1 + 2^2 + \dots + n^2) \\
&\qquad\qquad\qquad + 2h(1 + 2 + \dots + n)] + \lim_{h \rightarrow 0} 5nh \\
&= \lim_{h \rightarrow 0} 2h \left[n + h^2 \frac{n(n+1)(2n+1)}{6} + \frac{2hn(n+1)}{2} \right] \\
&\qquad\qquad\qquad + \lim_{h \rightarrow 0} 5 \times 2 \\
&= \lim_{h \rightarrow 0} 2 \left[nh + \frac{nh(nh+h)(2nh+h)}{6} + nh(nh+h) \right] + 10 \\
&= \lim_{h \rightarrow 0} 2 \left[2 + \frac{2(2+h)(2 \times 2 + h)}{6} + 2(2+h) \right] + 10 \\
&= 2 \left[2 + \frac{2 \times 2 \times 4}{6} + 2 \times 2 \right] + 10 \\
&= 2 \left[2 + \frac{8}{3} + 4 \right] + 10 \\
&= 2 \left[6 + \frac{8}{3} \right] + 10 \\
&= 2 \times \frac{26}{3} + 10 \\
&= \frac{52 + 30}{3} = \frac{82}{3}
\end{aligned}$$

Ex 10.2

निम्न समाकलों के मान ज्ञात कीजिए

प्रश्न 1.

$$\int_1^3 (2x+1)^3 dx$$

हल :

$$\int_1^3 (2x+1)^3 dx$$

माना $2x + 1 = t$

$2dx = dt$

जब $x = 1$ तो $t = 3$

$2dx = dt$

जब $x = 3$ तो $t = 7$

$$\begin{aligned}\therefore \int_1^3 (2x+1)^3 dx &= \int_3^7 t^3 \frac{dt}{2} \\ &= \frac{1}{2} \left(\frac{t^4}{4} \right)_3^7 = \frac{1}{8} (7^4 - 3^4) \\ &= \frac{2320}{8} = 290\end{aligned}$$

प्रश्न 2.

$$\int_0^{\pi/2} \frac{\sin x}{1 + \cos^2 x} dx$$

हल :

$$\int_0^{\pi/2} \frac{\sin x}{1 + \cos^2 x} dx$$

माना $\cos x = t$

तो $-\sin x dx = dt$

या $\sin x dx = -dt$

जब $x = 0$

तौ $t = \cos 0 = 1$

जब $x = \frac{\pi}{2}$

तो $t = \cos \frac{\pi}{2} = 0$

$$\begin{aligned}\therefore \int_0^{\pi/2} \frac{\sin x}{1 + \cos^2 x} dx &= \int_1^0 \frac{-dt}{1+t^2} \\ &= -(\tan^{-1} t)_1^0 \\ &= -(\tan^{-1} 0 - \tan^{-1} 1) \\ &= -\left(0 - \frac{\pi}{4}\right) = \frac{\pi}{4}\end{aligned}$$

प्रश्न 3.

$$\int_1^3 \frac{\cos(\log x)}{x} dx$$

हल :

$$\int_1^3 \frac{\cos(\log x)}{x} dx$$

माना $\log x = t$

तौ $\frac{1}{x} dx = dt$

जब $x = 1$, तौ $t = \log 1 = 0$

जब $x = 3$, तौ $t = \log 3$

$$\begin{aligned}\therefore \int_1^3 \frac{\cos(\log 3)}{x} dx &= \int_0^{\log 3} \cos t dt \\ &= (\sin t)_0^{\log 3}\end{aligned}$$

$= \sin(\log 3)$

प्रश्न 4.

$$\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$$

हल :

$$\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$$

माना $\sqrt{x} = t$

$$\frac{1}{2\sqrt{x}} dx = dt$$

$$\frac{1}{\sqrt{x}} dx = 2dt$$

जब $x = 0$, तौ $t = \sqrt{0} = 0$

जब $x = 1$, तौ $t = \sqrt{1} = 1$

$$\begin{aligned} \int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx &= \int_0^1 e^t 2dt \\ &= 2[e^t]_0^1 \\ &= 2[e^1 - e^0] = 2(e - 1) \end{aligned}$$

प्रश्न 5.

$$\int_0^{\pi/2} \sqrt{1 + \sin x}$$

हल :

$$\begin{aligned} &\int_0^{\pi/2} \sqrt{1 + \sin x} dx \\ &= \int_0^{\pi/2} \sqrt{\left(\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2}\right)} dx \\ &= \int_0^{\pi/2} \sqrt{\left(\sin \frac{x}{2} + \cos \frac{x}{2}\right)^2} dx \\ &= \int_0^{\pi/2} \left(\sin \frac{x}{2} + \cos \frac{x}{2}\right) dx \\ &= \int_0^{\pi/2} \sin \frac{x}{2} dx + \int_0^{\pi/2} \cos \frac{x}{2} dx \quad \dots(1) \end{aligned}$$

माना $\frac{x}{2} = t$

तो $dx = 2dt$

जब $x = 0$ तो $t = \frac{0}{2} = 0$

जब $x = \frac{\pi}{2}$ तो $t = \frac{\pi}{2 \times 2} = \frac{\pi}{4}$

$\therefore \int_0^{\pi/2} \sqrt{1 + \sin x} dx = \int_0^{\pi/2} \sin \frac{x}{2} dx + \int_0^{\pi/2} \cos \frac{x}{2} dx$

समी. (1) से,

$$= \int_0^{\pi/4} \sin t \cdot 2dt + \int_0^{\pi/4} \cos t \cdot 2dt$$

$$= -2(\cos t)_0^{\pi/4} + 2(\sin t)_0^{\pi/4}$$

$$= -2 \left[\cos \frac{\pi}{4} - \cos 0 \right] + 2 \left[\sin \frac{\pi}{4} - \sin 0 \right]$$

$$= -2 \left[\frac{1}{\sqrt{2}} - 1 \right] + 2 \left[\frac{1}{\sqrt{2}} - 0 \right]$$

$$= -\frac{2}{\sqrt{2}} + 2 + \frac{2}{\sqrt{2}}$$

$$= 2$$

प्रश्न 6.

$$\int_0^c \frac{y}{\sqrt{y+c}} dy$$

हल :

$$\int_0^c \frac{y}{\sqrt{y+c}} dy$$

माना $y+c = t$

तो $y+c = t$

$dy = dt$

जय $y = 0$ तो $t = 0 + c = c$

जब $y = c$ तो $t = c + c = 2c$

$$\begin{aligned}
\therefore \int_0^c \frac{y}{y+c} dy &= \int_c^{2c} \frac{t-c}{\sqrt{t}} dt \\
&= \int_c^{2c} \left(\sqrt{t} - \frac{c}{\sqrt{t}} \right) dt \\
&= \int_c^{2c} t^{\frac{1}{2}} dt - c \int_c^{2c} t^{-\frac{1}{2}} dt \\
&= \left[\frac{2}{3} t^{\frac{3}{2}} \right]_c^{2c} - c \left[\frac{1}{\frac{1}{2}} t^{\frac{1}{2}} \right]_c^{2c} \\
&= \frac{2}{3} \left[(2c)^{\frac{3}{2}} - c^{\frac{3}{2}} \right] - 2c \left[(2c)^{\frac{1}{2}} - c^{\frac{1}{2}} \right] \\
&= \frac{2}{3} \left[c^{\frac{3}{2}} (2\sqrt{2} - 1) \right] - 2c \left[c^{\frac{1}{2}} (\sqrt{2} - 1) \right] \\
&= \frac{2}{3} c^{\frac{3}{2}} (2\sqrt{2} - 1) - 2c^{\frac{3}{2}} (\sqrt{2} - 1) \\
&= \frac{2}{3} c^{\frac{3}{2}} [2\sqrt{2} - 1 - 3\sqrt{2} + 3] \\
&= \frac{2}{3} [2 - \sqrt{2}] c^{\frac{3}{2}}
\end{aligned}$$

प्रश्न 7.

$$\int_0^\infty \frac{e^{\tan^{-1} x}}{1+x^2} dx$$

हल :

$$\int_0^\infty \frac{e^{\tan^{-1} x}}{1+x^2} dx$$

माना $\tan^{-1} x = t$

तो $\frac{1}{1+x^2} dx = dt$

जब $x = 0$, तो $t = \tan^{-1} 0 = 0$

जब $x = \infty$, तो $t = \tan^{-1}\infty = \frac{\pi}{2}$

$$\begin{aligned}\therefore \int_0^{\infty} \frac{e^{\tan^{-1}x}}{1+x^2} dx &= \int_0^{\frac{\pi}{2}} e^t dt \\ &= \left[e^t \right]_0^{\frac{\pi}{2}} \\ &= e^{\frac{\pi}{2}} - e^0 = e^{\frac{\pi}{2}} - 1\end{aligned}$$

प्रश्न 8.

$$\int_1^2 \frac{(1 + \log x)^2}{x} dx$$

हल :

$$\int_1^2 \frac{(1 + \log x)^2}{x} dx$$

माना $1 + \log x = t$

तौ $\frac{1}{x} dx = dt$

जब $x = 1$, तौ $t = 1 + \log 1 = 1$

जब $x = 2$, तौ $t = 1 + \log 2$

$$\begin{aligned}\therefore \int_1^2 \frac{(1 + \log x)^2}{x} dx &= \int_1^{1+\log 2} t^2 dt \\ &= \left[\frac{t^3}{3} \right]_1^{1+\log 2} \\ &= \frac{1}{3} \left[(1 + \log 2)^3 - 1^3 \right] \\ &= \frac{(1 + \log 2)^3}{3} - \frac{1}{3}\end{aligned}$$

प्रश्न 9.

$$\int_{\alpha}^{\beta} \frac{dx}{(x-\alpha)(\beta-x)}, \beta > \alpha$$

हल :

$$\int_{\alpha}^{\beta} \frac{dx}{(x-\alpha)(\beta-x)}$$

माना $I = \int_{\alpha}^{\beta} \frac{dx}{(x-\alpha)(\beta-x)}$

$$I = \int_{\alpha}^{\beta} \left(\frac{A}{x-\alpha} + \frac{B}{\beta-x} \right) dx$$

हल करने पर, $A = B = \frac{1}{\beta-\alpha}$

$$\therefore I = \frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} \left(\frac{1}{x-\alpha} + \frac{1}{\beta-x} \right) dx$$

$$= \frac{1}{\beta-\alpha} \left[\log(x-\alpha) + \log(\beta-x) \right]_{\alpha}^{\beta}$$

$$= \frac{1}{\beta-\alpha}$$

$$\left[\log(\beta-\alpha) + \log(\beta-\beta) - \log(\alpha-\alpha) + \log(\beta-\alpha) \right]$$

$$= \frac{1}{\beta-\alpha} \left[\log(\beta-\alpha) + 0 - 0 + \log(\beta-\alpha) \right]$$

$$= \frac{2\log(\beta-\alpha)}{\beta-\alpha}$$

प्रश्न 10.

$$\int_0^{\pi/4} \frac{(\sin x + \cos x)}{9 + 16 \sin 2x}$$

हल :

$$\int_0^{\pi/4} \frac{(\sin x + \cos x)}{9 + 16 \sin 2x}$$

माना $\sin x - \cos x = t$

$$\Rightarrow (\cos x + \sin x) dx = dt$$

$$\Rightarrow 1 - 2 \sin x \cos x = t^2$$

$$\Rightarrow \sin 2x = 1 - t^2$$

$$\text{जब } x = \frac{\pi}{4}, \text{ तो } t = \sin \frac{\pi}{4} - \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$$

$$\text{जब } x = 0, \text{ तो } t = \sin 0 - \cos 0 = -1$$

$$\begin{aligned} \therefore \int_0^{\frac{\pi}{4}} \frac{(\sin x + \cos x)}{9 + 16 \sin 2x} &= \int_{-1}^0 -1 \frac{dt}{9 + 16(1-t^2)} \\ &= \int_{-1}^0 -1 \frac{1}{25 - 16t^2} \\ &= \frac{1}{16} \int_{-1}^0 \frac{dt}{(5/4)^2 - t^2} \\ &= \frac{1}{16} \cdot \frac{1}{2 \times \frac{5}{4}} \left[\log \frac{\frac{5}{4} + t}{\frac{5}{4} - t} \right]_{-1}^0 \end{aligned}$$

$$= \frac{1}{40} [\log 1 - (\log 1 - \log 9)]$$

$$= \frac{1}{40} [\log 9]$$

$$= \frac{1}{40} \times 2 \times \log 3$$

$$= \frac{1}{20} \log 3$$

प्रश्न 11.

$$\int_{1/e}^e \frac{dx}{x(\log x)^{1/3}}$$

हल :

$$\int_{1/e}^e \frac{dx}{x(\log x)^{1/3}}$$

$$\text{माना } \log x = t$$

$$\text{तो } \frac{1}{x} dx = dt$$

$$\text{जब } x = \frac{1}{e}, \text{ तो } t = \log \frac{1}{e}$$

जब $x = e$, तो $t = \log e$

$$\begin{aligned}\therefore \int_{1/e}^e \frac{dx}{x(\log x)^{1/3}} &= \int_{\log 1/e}^{\log e} \frac{dt}{t^{1/3}} \\ &= \left[\frac{t^{-1/3+1}}{-1/3+1} \right]_{\log \frac{1}{e}}^{\log e} = \frac{3}{2} \left[\frac{2}{t^{2/3}} \right]_{\log \frac{1}{e}}^{\log e} \\ &= \frac{3}{2} \left[(\log e)^{3/2} - \left(\log \frac{1}{e} \right)^{3/2} \right] \\ &= \frac{3}{2} \left[1^{2/3} - 1^{2/3} \right] = 0\end{aligned}$$

प्रश्न 12.

$$\int_0^{\pi/4} \sin 2x \cos 3x \, dx$$

हल :

$$\begin{aligned}& \int_0^{\pi/4} \sin 2x \cos 3x \, dx \\ &= \frac{1}{2} \int_0^{\pi/4} 2 \sin 2x \cos 3x \, dx \\ &= \frac{1}{2} \int_0^{\pi/4} [\sin(2x+3x) + \sin(2x-3x)] \, dx \\ &= \frac{1}{2} \int_0^{\pi/4} [\sin 5x + \sin(-x)] \, dx \\ &= \frac{1}{2} \int_0^{\pi/4} [\sin 5x - \sin x] \, dx \\ &= \frac{1}{2} \left[\left(-\frac{\cos 5x}{5} \right)_0^{\pi/4} - (-\cos x)_0^{\pi/4} \right] \\ &= \frac{1}{2} \left[-\frac{1}{5}(\cos 5x)_0^{\pi/4} + (\cos x)_0^{\pi/4} \right]\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \left[-\frac{1}{5} \left(\cos \frac{5\pi}{4} - \cos 0 \right) + \left(\cos \frac{\pi}{4} - \cos 0 \right) \right] \\
&= \frac{1}{2} \left[-\frac{1}{5} \left(-\frac{1}{\sqrt{2}} - 1 \right) + \left(\frac{1}{\sqrt{2}} - 1 \right) \right] \\
&= \frac{1}{2} \left[\frac{1}{5\sqrt{2}} + \frac{1}{5} + \frac{1}{\sqrt{2}} - 1 \right] \\
&= \frac{1}{2} \left(\frac{3\sqrt{2}}{5} - \frac{4}{5} \right) \\
&= \frac{1}{10} (3\sqrt{2} - 4)
\end{aligned}$$

प्रश्न 13.

$$\int_e^{e^2} \left[\frac{1}{\log x} - \frac{1}{(\log x)^2} \right] dx$$

हल :

$$\int_e^{e^2} \left[\frac{1}{\log x} - \frac{1}{(\log x)^2} \right] dx$$

माना $\log x = t$
 $x = e^t$
 $dx = e^t dt$

$$\begin{aligned}
\int_e^{e^2} \left[\frac{1}{\log x} - \frac{1}{(\log x)^2} \right] dx &= \int_e^{e^2} \left[\frac{1}{t} - \frac{1}{t^2} \right] e^t dt \\
&= \int_e^{e^2} \frac{d}{dt} \left(e^t \cdot \frac{1}{t} \right) dt \\
&= \left(\frac{e^t}{t} \right)_e^{e^2} = \left(\frac{x}{\log x} \right)_e^{e^2} \\
&= \frac{e^2}{\log e^2} - \frac{e}{\log e}
\end{aligned}$$

$$= \frac{e^2}{2} - \frac{e}{1} = \frac{e^2}{2} - e$$

प्रश्न 14.

$$\int_0^1 \frac{x^3}{\sqrt{1-x^2}} dx$$

हल :

$$\int_0^1 \frac{x^3}{\sqrt{1-x^2}} dx$$

$$\text{माना } 1 - x^2 = t$$

$$\Rightarrow -2x dx = dt$$

$$\Rightarrow x dx = -\frac{1}{2} dt$$

$$\Rightarrow \sin 2x = 1 - t^2$$

$$\text{जब } x = 0 \text{ तौ } t = 1 - 0 = 1$$

$$\text{जब } x = 1 \text{ तौ } t = 1 - 1 = 0$$

$$\begin{aligned} \therefore \int_0^1 \frac{x^3}{\sqrt{1-x^2}} dx &= \int_0^1 \frac{x^2 \cdot x}{\sqrt{1-x^2}} dx \\ &= \int_1^0 \frac{1-t}{\sqrt{t}} \frac{dt}{-2} \\ &= -\frac{1}{2} \int_1^0 (t^{1/2} + t^{1/2}) dt \\ &= -\frac{1}{2} \left[\left(2t^{1/2}\right)_1^0 - \left(\frac{2t^{3/2}}{3}\right)_1^0 \right] \\ &= -\frac{1}{3} (t^{3/2})_1^0 - (t^{1/2})_1^0 \\ &= \frac{1}{3} (0 - 1) - (0 - 1) \\ &= \frac{1}{3} \times (-1) + 1 = 1 - \frac{1}{3} = \frac{2}{3} \end{aligned}$$

प्रश्न 15.

$$\int_{\pi/2}^{\pi} \frac{1 - \sin x}{1 - \cos x} dx$$

हल :

$$\begin{aligned} & \int_{\pi/2}^{\pi} \frac{1 - \sin x}{1 - \cos x} dx \\ &= \int_{\pi/2}^{\pi} \frac{1 - 2 \sin \frac{x}{2} \cos \frac{x}{2}}{2 \sin^2 \frac{x}{2}} dx \\ &= \int_{\pi/2}^{\pi} \left[\frac{1}{2} \operatorname{cosec}^2 \frac{x}{2} - \cot \frac{x}{2} \right] dx \\ &= -\frac{1}{2} [\cot x]_{\pi/2}^{\pi} - 2 \left[\log \left(\sin \frac{x}{2} \right) \right]_{\pi/2}^{\pi} \\ &= -\frac{1}{2} \left[\cot x - \cot \frac{\pi}{2} \right] - 2 \left[\log \left(\sin \frac{\pi}{2} \right) - \log \left(\sin \frac{\pi}{4} \right) \right] \\ &= -\frac{1}{2} (\infty - 0) - 2 \left(0 - \log \frac{1}{\sqrt{2}} \right) \\ &= 2 \log \frac{1}{\sqrt{2}} = \log \left(\frac{1}{\sqrt{2}} \right)^2 \\ &= \log \frac{1}{2} = \log \frac{e}{2} \end{aligned}$$

प्रश्न 16.

$$\int_0^{\pi/4} \frac{dx}{4 \sin^2 x + 5 \cos^2 x}$$

हल :

$$\int_0^{\pi/4} \frac{dx}{4 \sin^2 x + 5 \cos^2 x}$$

अंश व हर में $\cos^2 x$ से भाग देने पर

$$= \int_0^{\pi/4} \frac{\sec^2 x}{5 + 4 \tan^2 x} dx$$

माना $2 \tan x = t$

$$\Rightarrow 2 \sec^2 x \, dx = dt$$

$$\Rightarrow \sec^2 x \, dx = \frac{dt}{2}$$

जब $x = 0$ तौ $t = 0$

$$\text{जब } x = \frac{\pi}{4}, \text{ तो } t = 2 \tan \frac{\pi}{4} = 2 \times 1 = 2$$

$$\begin{aligned} \therefore \int_0^{\frac{\pi}{4}} \frac{dx}{4 \sin^2 x + 5 \cos^2 x} &= \frac{1}{2} \int_0^2 \frac{1}{(\sqrt{5})^2 + t^2} dt \\ &= \frac{1}{2} \left[\frac{1}{\sqrt{5}} \tan^{-1} \frac{t}{\sqrt{5}} \right]_0^2 \\ &= \frac{1}{2\sqrt{5}} \left[\tan^{-1} \frac{2}{\sqrt{5}} - \tan^{-1} 0 \right] \\ &= \frac{1}{2\sqrt{5}} \tan^{-1} \left(\frac{2}{\sqrt{5}} \right) \end{aligned}$$

प्रश्न 17.

$$\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx$$

हल :

$$\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx$$

$$\text{माना } I = \int_0^{\pi/2} \frac{\sin x}{(\sin x + \cos x)} dx \quad \dots(i)$$

$$I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\left[\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)\right]} dx$$

$$\text{या } I = \int_0^{\pi/2} \frac{\cos x}{(\cos x + \sin x)} dx \quad \dots(ii)$$

समी (i) तथा (ii) को जोड़ने पर

$$\begin{aligned}2I &= \int_0^{\pi/2} \frac{\sin x}{(\sin x + \cos x)} + \int_0^{\pi/2} \frac{\cos x}{(\cos x + \sin x)} dx \\&= \int_0^{\pi/2} \left[\frac{\sin x + \cos x}{\sin x + \cos x} \right] dx \\&= \int_0^{\pi/2} dx \\&= [x]_0^{\pi/2} = \frac{\pi}{2} \\I &= \frac{\pi}{4}\end{aligned}$$

प्रश्न 18.

$$\int_{-1}^1 x \tan^{-1} x dx$$

हल :

$$\begin{aligned}& \int_{-1}^1 x \tan^{-1} x dx \\&= 2 \int_0^1 x \tan^{-1} x dx \quad (\because x \tan^{-1} x \text{ सम फलन है।}) \\&= \left[2 \cdot \frac{x^2}{2} \cdot \tan^{-1} x \right]_0^1 - 2 \int_0^1 \frac{1}{2} \frac{x^2}{1+x^2} dx \\&= \left[x^2 \tan^{-1} x \right]_0^1 - \int_0^1 \frac{x^2 + 1 - 1}{1+x^2} dx \\&= \left[x^2 \tan^{-1} x \right]_0^1 - [x]_0^1 + \left[\tan^{-1} x \right]_0^1 \\&= \frac{\pi}{4} - 1 + \frac{\pi}{4} = \frac{\pi}{2} - 1 = \frac{\pi - 2}{2}\end{aligned}$$

प्रश्न 19.

$$\int_0^1 \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$$

हल :

$$\int_0^1 \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$$

माना $\sin^{-1} x = t,$

तो $x = \sin t$

$$\frac{1}{\sqrt{1-x^2}} dx = dt$$

जब $x = 0$ तौ $t = 0$

जब $x = 1$ तौ $t = \frac{\pi}{2}$

$$\therefore \int_0^1 \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$$

$$= \int_0^{\pi/2} \sin t \cdot t dt$$

II I

$$= t \int_0^{\pi/2} \sin t dt - \int_0^{\pi/2} \left[\frac{d}{dt}(t) \int \sin t dt \right] dt$$

$$= t[-\cos t]_0^{\pi/2} - \int_0^{\pi/2} 1 \cdot (-\cos t) dt$$

$$= -t[\cos t]_0^{\pi/2} + \int_0^{\pi/2} \cos t dt$$

$$= -t[\cos t]_0^{\pi/2} + [\sin t]_0^{\pi/2}$$

$$= \left(\sin \frac{\pi}{2} - \sin 0 \right) - \left(\frac{\pi}{2} \cos \frac{\pi}{2} - 0 \times \cos 0 \right)$$

$$= (1-0)-(0-0)$$

$$= 1$$

प्रश्न 20.

$$\int_0^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx$$

हल :

$$\int_0^{\infty} \frac{x^2}{(x^2 + a^2)(x^2 + b^2)} dx$$

माना $x^2 = y$

$$\text{तो } \frac{y}{(y + a^2)(y + b^2)} = \frac{A}{(y + a^2)} + \frac{B}{(y + b^2)}$$

तुलना करके हल करने पर,

$$A = \frac{a^2}{a^2 - b^2}$$

तथा $B = \frac{b^2}{b^2 - a^2}$

$$\begin{aligned} & \int_0^{\infty} \frac{x^2}{(x^2 + a^2)(x^2 + b^2)} dx \\ &= \frac{a^2}{a^2 - b^2} \int_0^{\infty} \frac{1}{a^2 + x^2} dx + \frac{b^2}{b^2 - a^2} \int_0^{\infty} \frac{1}{b^2 + x^2} dx \\ &= \frac{a^2}{a^2 - b^2} \cdot \frac{1}{a} \cdot \left(\tan^{-1} \frac{x}{a} \right)_0^{\infty} + \frac{b^2}{b^2 - a^2} \cdot \frac{1}{b} \cdot \left(\tan^{-1} \frac{x}{b} \right)_0^{\infty} \\ &= \frac{a}{a^2 - b^2} \cdot (\tan^{-1} \infty - \tan^{-1} 0) + \frac{b^2}{b^2 - a^2} (\tan^{-1} \infty - \tan^{-1} 0) \\ &= \frac{a^2}{a^2 - b^2} \left(\frac{\pi}{2} - 0 \right) + \frac{b^2}{b^2 - a^2} \left(\frac{\pi}{2} - 0 \right) \\ &= \frac{a^2}{a^2 - b^2} \times \frac{\pi}{2} + \frac{b^2}{b^2 - a^2} \times \frac{\pi}{2} \\ &= \frac{\pi}{2(a^2 - b^2)} (a - b) \\ &= \frac{\pi}{2(a + b)} \end{aligned}$$

प्रश्न 21.

$$\int_1^2 \log x dx$$

हल :

$$\begin{aligned}\int_1^2 \log x \, dx &= \int_1^2 \log x \cdot 1 \, dx \\ &= \left[\log x \int 1 \, dx \right]_1^2 - \int \left[\frac{d}{dx}(\log x) \cdot \int 1 \, dx \right] dx \\ &= [x \log x]_1^2 - \int_1^2 \frac{1}{x} \cdot x \, dx \\ &= (x \log x)_1^2 - \int_1^2 1 \, dx \\ &= (x \log x)_1^2 - (x)^2 \\ &= (2 \log 2 - 1 \log 1) - (2 - 1) \\ &= \log 2^2 - 0 - 1 \\ &= \log 4 - 1 \\ &= \log 4 - \log e \\ &= \log \frac{4}{e}\end{aligned}$$

प्रश्न 22.

$$\int_{4/\pi}^{2/\pi} \left(-\frac{1}{x^3} \right) \cos \left(\frac{1}{x} \right) dx$$

हल :

$$\int_{4/\pi}^{2/\pi} \left(-\frac{1}{x^3} \right) \cos \left(\frac{1}{x} \right) dx$$

माना $\frac{1}{x} = t$

$$-\frac{1}{x^2} dx = dt$$

जब $x = \frac{4}{\pi}$, तो $t = \frac{\pi}{4}$

जब $x = \frac{2}{\pi}$, तो $t = \frac{\pi}{2}$

$$\therefore \int_{4/\pi}^{2/\pi} \frac{1}{x} \cdot \cos \frac{1}{x} \cdot \left(-\frac{1}{x^2} \right) dx$$

$$\begin{aligned}
&= \int_{4/\pi}^{2/\pi} t \cos t \, dt \\
&= [t \sin t]_{\pi/2}^{\pi/4} - \int_{\pi/4}^{\pi/2} \sin t \, dt \\
&= \left[\frac{\pi}{2} \sin \frac{\pi}{2} - \frac{\pi}{4} \sin \frac{\pi}{4} \right] + [\cos t]_{\pi/4}^{\pi/2} \\
&= \left[\frac{\pi}{2} \times 1 - \frac{\pi}{4} \times \frac{1}{\sqrt{2}} \right] + \left[\cos \frac{\pi}{2} - \cos \frac{\pi}{4} \right] \\
&= \frac{\pi}{2} - \frac{\pi}{4\sqrt{2}} + \left(0 - \frac{1}{\sqrt{2}} \right) \\
&= \frac{\pi}{2} - \frac{\pi}{4\sqrt{2}} - \frac{1}{\sqrt{2}}
\end{aligned}$$

प्रश्न 23.

$$\int_0^{\pi/2} \frac{\sin x \cos x \, dx}{\cos^2 x + 3 \cos x + 2}$$

हल :

$$\int_0^{\pi/2} \frac{\sin x \cos x}{\cos^2 x + 3 \cos x + 2} \, dx$$

माना $\cos x = t$

तौ $-\sin x \, dx = dt$

या $\sin x \, dx = -dt$

जब $x = 0$, तौ $t = 1$

जब $x = \frac{\pi}{2}$, तौ $t = 0$

$$\begin{aligned}
\int_0^{\pi/2} \frac{\sin x \cos x}{\cos^2 x + 3 \cos x + 2} \, dx \\
&= -\int_1^0 \frac{t}{t^2 + 3t + 2} \, dt \\
&= -\int_1^0 \frac{t}{(t+1)(t+2)} \, dt
\end{aligned}$$

$$\text{अब } \frac{t}{(t+1)(t+2)} = \frac{A}{t+1} + \frac{B}{t+2}$$

$$t = A(t+2) + B(t+1)$$

$$= At + 2A + Bt + B$$

$$= (A+B)t + 2A + B$$

तुलना करने पर,

$$A + B = 1, 2A + B = 0$$

$$\Rightarrow A + (A+B) = 0$$

$$\Rightarrow A + 1 = 0$$

$$\Rightarrow A = -1$$

$$\Rightarrow -1 + B = 1$$

$$\Rightarrow B = 2$$

$$\therefore \frac{t}{(t+1)(t+2)} = \frac{1}{t+1} + \frac{2}{t+2}$$

$$= -\int_1^0 \left[-\frac{1}{(t+1)} \right] dt - \int_1^0 \left[\frac{2}{(t+2)} \right] dt$$

$$= \int_1^0 \frac{1}{t+1} dt - 2 \int_1^0 \frac{1}{t+2} dt$$

$$= [\log(t+1)]_1^0 - 2[\log(t+2)]_1^0$$

$$= [\log(0+1) - \log(1+1)] - 2[\log(0+2) - \log(1+2)]$$

$$= \log 1 - \log 2 - 2 \log 2 + 2 \log 3$$

$$= 0 - \log 2 - \log 2^2 + \log 3^2$$

$$= \log 9 - \log 2 \times 4$$

$$= \log \frac{9}{8}$$

प्रश्न 24.

$$\int_0^3 \sqrt{\frac{x}{3-x}} dx$$

हल :

$$\int_0^3 \sqrt{\frac{x}{3-x}} dx$$

$$\text{माना } x = 3 \sin^2 \theta$$

$$\therefore dx = 3 \sin^2 \theta d\theta$$

$$\text{जब } x = 0, \text{ तौ } \theta = \frac{\pi}{2}$$

जब $x = 0$, तो $\theta = 0$

$$\begin{aligned}\therefore \int_0^3 \sqrt{\frac{x}{3-x}} dx &= \int_0^{\pi/2} \sqrt{\frac{3\sin^2 \theta}{3(1-\sin^2 \theta)}} \cdot 3\sin 2\theta d\theta \\ &= 3 \int_0^{\pi/2} \frac{\sin \theta}{\cos \theta} \times 2\sin \theta \cos \theta d\theta \\ &= 3 \int_0^{\pi/2} 2\sin^2 \theta d\theta \\ &= 3 \int_0^{\pi/2} (1 - \cos 2\theta) d\theta \\ &= 3 \left(\theta - \frac{\sin 2\theta}{2} \right)_0^{\pi/2} \\ &= 3 \left(\frac{\pi}{2} - \frac{1}{2} \sin \pi \right) - (0 - 0) \\ &= 3 \left(\frac{\pi}{2} - 0 \right) = \frac{3\pi}{2}\end{aligned}$$

प्रश्न 25.

$$\int_0^1 \frac{x^2}{1+x^2} dx$$

हल :

$$\begin{aligned}\int_0^1 \frac{x^2}{1+x^2} dx &= \int_0^1 \frac{(1+x^2)-1}{(1+x^2)} dx \\ &= \int_0^1 dx - \int_0^1 \frac{1}{1+x^2} dx \\ &= [x]_0^1 - [\tan^{-1} x]_0^1 \\ &= (1 - 0) - (\tan^{-1} 1 - \tan^{-1} 0) \\ &= 1 - \left(\frac{\pi}{4} - 0 \right) \\ &= 1 - \frac{\pi}{4}\end{aligned}$$

प्रश्न 26.

$$\int_1^2 \frac{dx}{(x+1)(x+2)}$$

हल :

$$\begin{aligned} \int_1^2 \frac{dx}{(x+1)(x+2)} &= \int_1^2 \frac{(x+2-x-1)}{(x+1)(x+2)} dx \\ &= \int_1^2 \frac{(x+2)}{(x+1)(x+2)} dx - \int_1^2 \frac{(x+1)}{(x+1)(x+2)} dx \\ &= \int_1^2 \frac{1}{x+1} dx - \int_1^2 \frac{1}{x+2} dx \\ &= [\log(x+1)]_1^2 - [\log(x+2)]_1^2 \\ &= [\log(x+1)] - [\log(x+2)] \\ &= [\log(2+1) - \log(1+1)] - [\log(2+2) - \log(1+2)] \\ &= \log 3 - \log 2 - \log 4 + \log 3 \\ &= 2 \log 3 - (\log 2 + \log 4) \\ &= 2 \log 3 - \log 8 \\ &= \log \frac{3^2}{8} = \log \frac{9}{8} \end{aligned}$$

Ex 10.3

निम्नलिखित समाकलों के मान ज्ञात कीजिए

प्रश्न 1.

$$\int_{-2}^2 |2x+3| dx$$

हल :

माना

$$\begin{aligned} I &= \int_{-2}^2 |2x+3| dx \\ &= \int_{-2}^{-3/2} -(2x+3) dx + \int_{-3/2}^2 (2x+3) dx \\ [\because \text{अन्तराल } \left(-2, -\frac{3}{2}\right) \text{ में } x \text{ का मान ऋणात्मक है} \\ &\quad \therefore |2x+3| = -(2x+3)] \\ &= -\left[\frac{2x^2}{2} + 3x\right]_{-2}^{-3/2} + \left[\frac{2x^2}{2} + 3x\right]_{-3/2}^2 \\ &= (x^2 + 3x)_{-3/2}^2 - (x^2 + 3x)_{-2}^{-3/2} \\ &= \left[2^2 + 3 \times 2 - \left(\frac{-3}{2}\right)^2 - 3\left(\frac{-3}{2}\right)\right] \\ &\quad - \left[\left(\frac{-3}{2}\right)^2 + 3\left(\frac{-3}{2}\right) - (-2)^2 - 3(-2)\right] \\ &= \left[4 + 6 - \frac{9}{4} + \frac{9}{2}\right] - \left[\frac{9}{4} - \frac{9}{2} - 4 + 6\right] \\ &= 4 + 6 - \frac{9}{4} + \frac{9}{2} - \frac{9}{4} + \frac{9}{2} + 4 - 6 \\ &= 8 + 9 - \frac{9}{2} = 8 + \frac{9}{2} = \frac{16+9}{2} = \frac{25}{2} \end{aligned}$$

प्रश्न 2.

$$\int_{-2}^2 |1-x^2| dx$$

हल :

माना

$$\begin{aligned} I &= \int_{-2}^2 |1-x^2| dx \\ \Rightarrow I &= \int_{-2}^{-1} |1-x^2| dx + \int_{-1}^1 |1-x^2| dx + \int_1^2 |1-x^2| dx \\ &= -\int_{-2}^{-1} (1-x^2) dx + \int_{-1}^1 (1-x^2) dx - \int_1^2 (1-x^2) dx \\ &= -\left[x - \frac{x^3}{3}\right]_{-2}^{-1} + \left[x - \frac{x^3}{3}\right]_{-1}^1 - \left[x - \frac{x^3}{3}\right]_1^2 \\ &= -\left[1 + \frac{1}{3} - \left(-2 + \frac{8}{3}\right)\right] + \left[1 - \frac{1}{3} - \left(-1 + \frac{1}{3}\right)\right] \\ &\quad \left[-2 - \frac{8}{3} - \left(1 - \frac{1}{3}\right)\right] \\ &= -\left(\frac{-4}{3}\right) + \frac{4}{3} - \left(\frac{-4}{3}\right) \\ &= \frac{4}{3} + \frac{4}{3} + \frac{4}{3} = 4 \end{aligned}$$

प्रश्न 3.

$$\int_1^4 f(x) dx, \text{ जहाँ } f(x) = \begin{cases} 7x+3, & 1 \leq x \leq 3 \\ 8x, & 3 \leq x \leq 4 \end{cases}$$

हल :

माना

$$\begin{aligned} I &= \int_1^4 f(x) dx \\ I &= \int_1^3 (7x+3) dx + \int_3^4 8x dx \\ &= \left[7\left(\frac{x^2}{2}\right) + 3x\right]_1^3 + 8\left[\frac{x^2}{2}\right]_3^4 \\ &= \frac{7}{2}(3^2 - 1^2) + 3(3 - 1) + 4(4^2 - 3^2) \\ &= \frac{7}{2} \times 8 + 6 + 4 \times 7 \end{aligned}$$

$$= 28 + 6 + 28$$

$$= 62$$

प्रश्न 4.

$$\int_0^3 [x] dx,$$

हल : माना

$$I = \int_0^3 [x] dx$$

$$= \int_0^1 [x] dx + \int_1^2 [x] dx + \int_2^3 [x] dx$$

$$= \int_0^1 0 dx + \int_1^2 1 dx + \int_2^3 2 dx$$

$$= 0 + [x]_1^2 + (2x)_2^3$$

$$= 0 + (2 - 1) + (6 - 4)$$

$$= 0 + 1 + 2$$

$$= 3$$

प्रश्न 5.

$$\int_{-\pi/4}^{\pi/4} x^5 \cos^2 x dx$$

हल : माना

$$I = \int_{-\pi/4}^{\pi/4} x^5 \cos^2 x dx$$

$$\text{यहाँ } f(x) = x^5 \cos^2 x$$

$$\text{अब } f(x) = (-x)^5 \cos^2 (-x)$$

$$= -x^5 \cos^2 x$$

$$= -f(x)$$

अतः यह एक विषम फलन है।

$$\therefore \int_{-\pi/4}^{\pi/4} x^5 \cos^2 x dx = 0$$

प्रश्न 6.

$$\int_{-\pi}^{\pi} \frac{\sin x \cos x}{1 + \cos^2 x} dx$$

हल : माना

$$I = \int_{-\pi}^{\pi} \frac{\sin x \cos x}{1 + \cos^2 x} dx$$

यहाँ $f(x) = \frac{\sin x \cos x}{1 + \cos^2 x}$

$$\begin{aligned} \Rightarrow f(-x) &= \frac{\sin(-x) \cdot \cos(-x)}{1 + [\cos(-x)]^2} \\ &= \frac{-\sin x \cos x}{1 + \cos^2 x} \end{aligned}$$

[∵ sin x एक विषम फलन है तथा cos x एक सम फलन है]

$$\therefore \sin(-x) = -\sin x$$

$$\cos(-x) = \cos x$$

$$= -f(x)$$

अतः यह एक विषम फलन है।

$$\therefore \int_{-\pi}^{\pi} \frac{\sin x \cos x}{1 + \cos^2 x} dx = 0$$

प्रश्न 7.

$$\int_{\pi/4}^{3\pi/4} \frac{\sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx$$

हल : माना

$$I = \int_{\pi/4}^{3\pi/4} \frac{\sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx \quad \dots(i)$$

$$\Rightarrow I = \int_{\pi/4}^{3\pi/4} \frac{\sqrt{\sin\left(\frac{3\pi}{4} - \frac{\pi}{4} - x\right)}}{\sqrt{\cos\left(\frac{3\pi}{4} - \frac{\pi}{4} - x\right)} + \sqrt{\sin\left(\frac{3\pi}{4} - \frac{\pi}{4} - x\right)}} dx$$

$$= \int_{\pi/4}^{3\pi/4} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\cos\left(\frac{\pi}{2} - x\right)} + \sqrt{\sin\left(\frac{\pi}{2} - x\right)}} dx$$

$$= \int_{\pi/4}^{3\pi/4} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \quad \dots(ii)$$

समीकरण (i) व (ii) को जोड़ने पर

$$\begin{aligned}
 2I &= \int_{\pi/4}^{3\pi/4} \frac{\sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx \\
 &\quad + \int_{\pi/4}^{3\pi/4} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx \\
 &= \int_{\pi/4}^{3\pi/4} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx = \int_{\pi/4}^{3\pi/4} 1 dx \\
 &= [x]_{\pi/4}^{3\pi/4} = \frac{3\pi}{4} + \frac{\pi}{4} = \pi
 \end{aligned}$$

$$\Rightarrow I = \frac{\pi}{2}$$

$$\Rightarrow \int_{\pi/4}^{3\pi/4} \frac{\sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx = \frac{\pi}{2}$$

प्रश्न 8.

$$\int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}} dx$$

हल : माना

$$I = \int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}} dx \quad \dots(i)$$

$$I = \int_0^{\pi} \frac{e^{\cos(\pi-x)}}{e^{\cos(\pi-x)} + e^{-\cos(\pi-x)}} dx$$

$$I = \int_0^{\pi} \frac{e^{-\cos x}}{e^{\cos x} + e^{-\cos x}} dx \quad \dots(ii)$$

समीकरण (i) व (ii) को जोड़ने पर

$$\begin{aligned}
 2I &= \int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}} dx \\
 &\quad + \int_0^{\pi} \frac{e^{-\cos x}}{e^{\cos x} + e^{-\cos x}} dx
 \end{aligned}$$

$$\Rightarrow 2I = \int_0^{\pi} \frac{e^{\cos x} + e^{-\cos x}}{e^{\cos x} + e^{-\cos x}} dx$$

$$\begin{aligned} \Rightarrow 2I &= \int_0^{\pi} 1 \, dx \\ \Rightarrow 2I &= [x]_0^{\pi} = (\pi - 0) = \pi \\ \therefore I &= \frac{\pi}{2} \\ \Rightarrow \int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}} \, dx &= \frac{\pi}{2} \end{aligned}$$

प्रश्न 9.

$$\int_0^{\pi/2} \sin 2x \cdot \log \tan x \, dx$$

हल : माना

$$I = \int_0^{\pi/2} \sin 2x \log \tan x \, dx \quad \dots(i)$$

$$I = \int_0^{\pi/2} \sin 2\left(\frac{\pi}{2} - x\right) \log \tan\left(\frac{\pi}{2} - x\right) \, dx$$

$$I = \int_0^{\pi/2} \sin 2x \log \cot x \, dx \quad \dots(ii)$$

समीकरण (i) व (ii) को जोड़ने पर

$$\begin{aligned} 2I &= \int_0^{\pi/2} \sin 2x \log \tan x \, dx \\ &\quad + \int_0^{\pi/2} \sin 2x \log \cot x \, dx \end{aligned}$$

$$\Rightarrow 2I = \int_0^{\pi/2} \sin 2x (\log \tan x + \log \cot x) \, dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} \sin 2x \log(\tan x \times \cot x) \, dx$$

$$\Rightarrow I = \frac{1}{2} \int_0^{\pi/2} \sin 2x \times \log 1 \, dx$$

$$\Rightarrow I = \frac{1}{2} \int_0^{\pi/2} \sin 2x \times 0 \, dx$$

$$\Rightarrow I = 0$$

$$\therefore \int_0^{\pi/2} \sin 2x \log \tan x \, dx = 0$$

प्रश्न 10.

$$\int_{-1}^1 \log \left[\frac{2-x}{2+x} \right] dx$$

हल : माना

$$\text{माना } f(x) = \log \left(\frac{2-x}{2+x} \right)$$

$$\begin{aligned} \text{तब } f(-x) &= \log \left(\frac{2-(-x)}{2+(-x)} \right) \\ &= \log \left(\frac{2+x}{2-x} \right) = \log \left(\frac{2-x}{2+x} \right)^{-1} \\ &= -\log \left(\frac{2-x}{2+x} \right) \\ &= -f(x) \end{aligned}$$

$\therefore f(x)$ विषम है।

$$\therefore \int_a^a f(x) dx = 0$$

जब $f(x)$ विषम है।

$$\therefore \int_{-1}^1 \log \left(\frac{2-x}{2+x} \right) dx = 0$$

प्रश्न 11.

$$\int_0^1 \log \left(\frac{1}{x} - 1 \right) dx$$

हल : माना

$$\begin{aligned} I &= \int_0^1 \log \left(\frac{1}{x} - 1 \right) dx \\ &= \int_0^1 \log \left(\frac{1-x}{x} \right) dx \\ &= \int_0^1 \log \frac{x}{(1-x)} dx = -I \end{aligned}$$

$$2I = 0 \Rightarrow I = 0$$

$$\int_0^1 \log \left(\frac{1}{x} - 1 \right) dx = 0$$

प्रश्न 12.

$$\int_{\pi/6}^{\pi/3} \frac{dx}{1 + \sqrt{\tan x}}$$

हल :

माना

$$I = \int_{\pi/6}^{\pi/3} \frac{dx}{1 + \sqrt{\tan x}}$$

$$\Rightarrow I = \int_{\pi/6}^{\pi/3} \frac{1}{1 + \sqrt{\frac{\sin x}{\cos x}}} dx$$

$$\Rightarrow I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx \quad \dots(i)$$

$$\Rightarrow I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos\left(\frac{\pi}{3} + \frac{\pi}{6} - x\right)}}{\sqrt{\cos\left(\frac{\pi}{3} + \frac{\pi}{6} - x\right)} + \sqrt{\sin\left(\frac{\pi}{3} + \frac{\pi}{6} - x\right)}} dx$$

$$\Rightarrow I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos\left(\frac{\pi}{2} - x\right)}}{\sqrt{\cos\left(\frac{\pi}{2} - x\right)} + \sqrt{\sin\left(\frac{\pi}{2} - x\right)}} dx$$

$$\Rightarrow I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \quad \dots(ii)$$

समीकरण (i) व (ii) को जोड़ने पर

$$2I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

$$I = \frac{1}{2} \int_{\pi/6}^{\pi/3} (1) dx = \frac{1}{2} [x]_{\pi/6}^{\pi/3}$$

$$= \frac{1}{2} \left(\frac{\pi}{3} - \frac{\pi}{6} \right) = \frac{1}{2} \left(\frac{\pi}{6} \right) = \frac{\pi}{12}$$

प्रश्न 13.

$$\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx$$

हल : माना

$$I = \int_0^{\pi/2} \frac{\sin x}{(\sin x + \cos x)} dx \quad \dots(i)$$

$$I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\left[\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)\right]} dx$$

$$I = \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x} dx \quad \dots(ii)$$

समीकरण (i) व (ii) को जोड़ने पर

$$2I = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x} dx$$

$$2I = \int_0^{\pi/2} 1 dx$$

$$2I = [x]_0^{\pi/2} = \frac{\pi}{2} - 0$$

$$I = \frac{\pi}{4}$$

$$\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx = \frac{\pi}{4}$$

प्रश्न 14.

$$\int_0^{\pi/2} \log \sin 2x dx$$

हल : माना

$$\int_0^{\pi/2} \log \sin 2x dx$$

$$\int_0^{\pi/2} \log 2 dx + \int_0^{\pi/2} \log \sin x dx + \int_0^{\pi/2} \log \cos x dx$$

$$\int_0^{\pi/2} \log 2 dx + 2 \int_0^{\pi/2} \log \sin x dx$$

$$\frac{\pi}{2} \log 2 + 2 \int_0^{\pi/2} \log (\sin x) dx$$

$$\begin{aligned}
& 2 \int_0^{\pi/2} \log (\sin x) dx \\
&= \int_0^{\pi/2} \log (\sin 2x) dx - \frac{\pi}{2} \log 2 \\
&= \frac{1}{2} \int_0^{\pi} \log (\sin t) dt + \frac{\pi}{2} \log \left(\frac{1}{2} \right) \\
&= \frac{1}{2} \times 2 \int_0^{\pi/2} \log (\sin t) dt + \frac{\pi}{2} \log \frac{1}{2} \\
&= \int_0^{\pi/2} \log (\sin t) dt + \frac{\pi}{2} \log \frac{1}{2} \\
&= \int_0^{\pi/2} \log (\sin x) dx + \frac{\pi}{2} \log \frac{1}{2} \\
\Rightarrow \int_0^{\pi/2} \log (\sin x) dx &= \frac{\pi}{2} \log \frac{1}{2}
\end{aligned}$$

प्रश्न 15.

$$\int_{\pi/4}^{\pi/4} \frac{\left(x + \frac{\pi}{4} \right)}{2 - \cos 2x} dx$$

हल : माना

$$\begin{aligned}
I &= \int_{\pi/4}^{\pi/4} \frac{x + \frac{\pi}{4}}{2 - \cos 2x} dx \quad \dots(i) \\
&= \int_{\pi/4}^{\pi/4} \frac{x}{2 - \cos 2x} dx + \frac{\pi}{4} \int_{\pi/4}^{\pi/4} \frac{1}{2 - \cos 2x} dx \\
&= 0 + \frac{\pi}{4} \times 2 \int_0^{\pi/4} \frac{1}{2 - \cos 2x} dx \\
&\quad \left(\because \frac{x}{2 - \cos 2x}, x \text{ का विषम फलन तथा} \right. \\
&\quad \left. \frac{1}{2 - \cos 2x}, x \text{ का सम फलन है} \right) \\
&= \frac{\pi}{2} \int_0^{\pi/4} \frac{1}{1 + 2 \sin^2 x} dx \quad \left(\because \cos 2x = 1 - 2 \sin^2 x \right)
\end{aligned}$$

$$= \frac{\pi}{2} \int_0^{\pi/4} \frac{\sec^2 x}{1+3 \tan^2 x} dx$$

($\cos^2 x$ से भाग देने पर तथा $\sec^2 x = 1 + \tan^2 x$
 दर में रखने पर)

$$= \frac{\pi}{2} \int_0^1 \frac{dt}{1+(\sqrt{3}t)^2}$$

जहाँ $t = \tan x$ तथा $dt = \sec^2 x dx$

जब $x = 0$, तो $t = \tan 0 = 0$

$$\text{जब } x = \frac{\pi}{4}, \text{ तो } t = \tan \frac{\pi}{4} = 1$$

$$= \frac{\pi}{2} \left[\frac{1}{\sqrt{3}} \tan^{-1} (\sqrt{3}t) \right]_0^1$$

$$= \frac{\pi}{2} \left[\frac{1}{\sqrt{3}} \tan^{-1} \sqrt{3} - 0 \right]$$

$$= \frac{\pi}{2} \left[\frac{1}{\sqrt{3}} \times \frac{\pi}{3} \right] = \frac{\pi^2}{6\sqrt{3}}$$

प्रश्न 16.

$$\int_0^{\pi} \log (1 - \cos x) dx$$

हल : माना

$$I = \int_0^{\pi} \log (1 + \cos x) dx \quad \dots(i)$$

$$= \int_0^{\pi} \log [1 + \cos (\pi - x)] dx$$

$$I = \int_0^{\pi} \log (1 + \cos x) dx \quad \dots(ii)$$

(i) तथा (ii) को जोड़ने

$$\begin{aligned}
2I &= \int_0^{\pi} \log(1 + \cos x) dx + \int_0^{\pi} \log(1 + \cos x) dx \\
&= \int_0^{\pi} \log(1 + \cos x)(1 + \cos x) dx \\
&= \int_0^{\pi} \log(1 - \cos^2 x) dx \\
&= \int_0^{\pi} \log \sin^2 x dx \\
&= 2 \int_0^{\pi} \log \sin x dx
\end{aligned}$$

या $\therefore I = \int_0^{\pi} \log \sin x dx$

$$= 2 \int_0^{\pi/2} \log \sin x dx = 2I_1 \quad \dots(\text{iii})$$

$$\left[\because \int_0^{2a} f(x) dx = 2 \int_0^a f(x) dx \right]$$

जब $f(2a - x) = f(x)$

अब $I_1 = \int_0^{\pi/2} \log \sin x dx \quad \dots(\text{iv})$

या $I_1 = \int_0^{\pi/2} \log \sin \left(\frac{\pi}{2} - x \right) dx$

या $I_1 = \int_0^{\pi/2} \log \cos x dx \quad \dots(\text{v})$

$$\left[\because \int_0^a f(x) dx = \int_0^a f(a - x) dx \right]$$

(iv) तथा (v) को जोड़ने

$$\begin{aligned}
2I_1 &= \int_0^{\pi/2} \log \sin x dx + \int_0^{\pi/2} \log \cos x dx \\
&= \int_0^{\pi/2} [\log \sin x + \log \cos x] dx \\
&= \int_0^{\pi/2} \log(\sin x \cos x) dx \\
&= \int_0^{\pi/2} \log \left(\frac{2 \sin x \cos x}{2} \right) dx \\
&= \int_0^{\pi/2} [\log(2 \sin x \cos x) - \log 2] dx
\end{aligned}$$

$$\text{या } 2I_1 = \int_0^{\pi/2} \log \sin 2x \, dx - \log 2 \int_0^{\pi/2} dx$$

$$\begin{aligned} 2I_1 &= \int_0^{\pi/2} \log \sin 2x - \log 2 [x]_0^{\pi/2} \\ &= \int_0^{\pi/2} \log \sin 2x \, dx - \log 2 \left[\frac{\pi}{2} - 0 \right] \end{aligned}$$

$$2I_1 = \int_0^{\pi/2} \log \sin 2x - \frac{\pi}{2} \log 2 \quad \dots(\text{vi})$$

$$\text{या } 2I_1 = I_2 - \frac{\pi}{2} \log 2$$

$$\text{अब } I_2 = \int_0^{\pi/2} \log \sin 2x \, dx$$

$2x = t$ रखने पर,

$$2 \, dx = dt \text{ या } dx = \frac{1}{2} dt$$

जब $x = 0$ तब $t = 0$, जब $x = \frac{\pi}{2}$ तब $t = \pi$

$$\begin{aligned} \therefore I_2 &= \int_0^{\pi} \log \sin t \frac{dt}{2} \\ &= \frac{1}{2} \int_0^{\pi} \log \sin t \, dt \\ &= \frac{1}{2} \times 2 \int_0^{\pi/2} \log (\sin t) \, dt \\ &= \int_0^{\pi/2} \log \sin x \, dx = I_1 \end{aligned}$$

$$\therefore I_2 = I_1$$

समीकरण (vi) से,

$$2I_1 = I_1 - \frac{\pi}{2} \log 2 \quad (\because I_2 = I_1)$$

$$\therefore I_1 = -\frac{\pi}{2} \log 2$$

I1 का मान समीकरण (iii) में रखने पर

$$I = 2I_1 = 2 \left(-\frac{\pi}{2} \log 2 \right) = -\pi \log 2$$

$$\Rightarrow I = -\pi \log 2$$

$$\Rightarrow I = \pi \log (2 - 1)$$

$$\Rightarrow I = \pi \log \frac{1}{2}$$

$$\int_0^{\pi} \log (1 - \cos x) dx = \pi \log \frac{1}{2}$$

प्रश्न 17.

$$\int_{\pi/4}^{\pi/4} \sin^2 x dx$$

हल : माना

$$\int_{\pi/4}^{\pi/4} \sin^2 x dx = 2 \int_0^{\pi/4} \sin^2 x dx$$

($\because \sin^2 x$ एक सम फलन है)

$$= 2 \int_0^{\pi/4} \left[\frac{1 - \cos 2x}{2} \right] dx = \int_0^{\pi/4} (1 - \cos 2x) dx$$

समीकरण (i) व (ii) को जोड़ने पर

$$= \left[x - \frac{\sin 2x}{2} \right]_0^{\pi/4} = \left[\frac{\pi}{4} - \frac{\sin \pi/2}{2} \right] - (0 - 0)$$

$$= \frac{\pi}{4} - \frac{1}{2}$$

प्रश्न 18.

$$\int_0^{\pi} \frac{x}{1 + \sin x} dx$$

हल : माना

$$I = \int_0^{\pi} \frac{x dx}{1 + \sin x} \quad \dots(1)$$

$$I = \int_0^{\pi} \frac{(\pi - x) dx}{1 + \sin(\pi - x)}$$

$$I = \int_0^{\pi} \frac{(\pi - x) dx}{1 + \sin x} \quad \dots(2)$$

(1) तथा (2) को जोड़ने पर

$$\begin{aligned}2I &= \int_0^{\pi} \frac{x \, dx}{1 + \sin x} + \int_0^{\pi} \frac{(\pi - x) \, dx}{1 + \sin x} \\&= \int_0^{\pi} \frac{(x + \pi - x) \, dx}{1 + \sin x} = \int_0^{\pi} \frac{\pi}{1 + \sin x} \\&= \pi \int_0^{\pi} \frac{1}{1 + \sin x} \times \left(\frac{1 - \sin x}{1 - \sin x} \right) dx \\&= \pi \int_0^{\pi} \frac{1 - \sin x}{1 - \sin^2 x} dx \\&= \pi \int_0^{\pi} \frac{1 - \sin x}{\cos^2 x} dx \\&= \pi \int_0^{\pi} \frac{1}{\cos^2 x} dx - \pi \int_0^{\pi} \frac{\sin x}{\cos^2 x} dx \\&= \pi \int_0^{\pi} \sec^2 x \, dx - \pi \int_0^{\pi} \sec x \tan x \, dx \\&= \pi [\tan x]_0^{\pi} - \pi [\sec x]_0^{\pi} \\&= \pi [\tan \pi - \tan 0] - \pi [\sec \pi - \sec 0] \\&= \pi \times 0 - \pi(-1-1) = 2\pi \\2I &= 2\pi \\ \therefore I &= \pi\end{aligned}$$

प्रश्न 19.

$$\int_0^{\pi} x \sin^3 x \, dx$$

हल : माना

$$\begin{aligned}\int_0^{\pi} x \sin^3 x \, dx &= \int_0^{\pi} x \left(\frac{3 \sin x - \sin 3x}{4} \right) dx \\&= \frac{3}{4} \int_0^{\pi} x \sin x \, dx - \frac{1}{4} \int_0^{\pi} x \sin 3x \, dx \\&= \frac{3}{4} I_1 - \frac{1}{4} I_2 \quad \dots(i)\end{aligned}$$

$$I_1 = \int_0^{\pi} x \sin x \, dx$$

$$\begin{aligned}
\int_1^{\text{II}} x \sin x \, dx &= x \int \sin x \, dx - \int \left[\frac{d}{dx} \cdot x \int \sin x \, dx \right] dx \\
&= x \cdot (-\cos x) - \int 1 \cdot (-\cos x) \, dx \\
&= -x \cos x + \int \cos x \, dx \\
&= -x \cos x + \sin x
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \int_0^{\pi} x \sin x \, dx &= [-x \cos x + \sin x]_0^{\pi} \\
&= (-\pi \cos \pi + \sin \pi) - (0 + 0) \\
&= -\pi(-1) + 0 = \pi \quad \dots(\text{ii})
\end{aligned}$$

अब $I_2 = \int_0^{\pi} x \sin 3x \, dx$

$$\begin{aligned}
\int_1^{\text{II}} x \sin 3x \, dx &= x \int \sin 3x \, dx - \int \left[\frac{d(x)}{dx} \int \sin 3x \, dx \right] dx \\
&= x \left(\frac{-\cos 3x}{3} \right) - \int \left(\frac{-\cos 3x}{3} \right) dx \\
&= \frac{-1}{3} x \cos 3x + \frac{1}{3} \frac{\sin 3x}{3} \\
&= \frac{-1}{3} x \cos 3x + \frac{1}{9} \sin 3x
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \int_0^{\pi} x \sin 3x \, dx &= \left[\frac{1}{9} \sin 3x - \frac{1}{3} x \cos 3x \right]_0^{\pi} \\
&= \left(\frac{1}{9} \sin 3\pi - \frac{1}{3} \pi \cos 3\pi \right) \\
&\quad - \left(\frac{1}{9} \sin 0 - \frac{1}{3} \times 0 \times \cos 0 \right) \\
&= 0 - \frac{\pi}{3} \times (-1) - 0 \\
&= \frac{\pi}{3} \quad \dots(\text{iii})
\end{aligned}$$

समी (ii) व (iii) से मान समी (i) में रखने पर

$$\begin{aligned}
\int_0^{\pi} x \sin^3 x \, dx &= \frac{3}{4} \pi - \frac{1}{4} \frac{\pi}{3} \\
&= \frac{9\pi - \pi}{3 \times 4} = \frac{2\pi}{3}
\end{aligned}$$

प्रश्न 20.

$$\int_0^{\pi/2} \log (\tan x + \cot x) dx$$

हल : माना

$$\begin{aligned} I &= \int_0^{\pi/2} \log (\tan x + \cot x) dx \\ &= \int_0^{\pi/2} \log \left(\frac{\sin^2 x + \cos^2 x}{\sin x \cos x} \right) dx \\ &= \int_0^{\pi/2} \log 1 dx - \int_0^{\pi/2} \log (\sin x \cos x) dx \\ &= 0 - \int_0^{\pi/2} \log \frac{\sin 2x}{2} dx \\ &= \int_0^{\pi/2} \log 2 dx - \int_0^{\pi/2} \log (\sin 2x) dx \\ &= \frac{\pi}{2} \log 2 - \int_0^{\pi/2} \log (\sin 2x) dx \quad \dots(i) \end{aligned}$$

$$\begin{aligned} &\int_0^{\pi/2} \log \sin 2x dx \\ &= \int_0^{\pi/2} \log 2 dx + \int_0^{\pi/2} \log \sin x dx + \int_0^{\pi/2} \log \cos x dx \\ &= \int_0^{\pi/2} \log 2 dx + 2 \int_0^{\pi/2} \log \sin x dx \\ &= \frac{\pi}{2} \log 2 + 2 \int_0^{\pi/2} \log (\sin x) dx \\ 2 \int_0^{\pi/2} \log (\sin x) dx &= \int_0^{\pi/2} \log (\sin 2x) dx - \frac{\pi}{2} \log 2 \\ &= \frac{1}{2} \int_0^{\pi/2} \log (\sin t) dt + \frac{\pi}{2} \log \left(\frac{1}{2} \right) \\ &= \frac{1}{2} \times 2 \int_0^{\pi/2} \log (\sin t) dt + \frac{\pi}{2} \log \frac{1}{2} \\ &= \int_0^{\pi/2} \log (\sin t) dt + \frac{\pi}{2} \log \frac{1}{2} \\ &= \int_0^{\pi/2} \log (\sin x) dx + \frac{\pi}{2} \log \frac{1}{2} \end{aligned}$$

$$\Rightarrow \int_0^{\pi/2} \log(\sin x) dx = \frac{\pi}{2} \log \frac{1}{2} \quad \dots(ii)$$

समी. (i) व (ii) से

$$\begin{aligned} \int_0^{\pi/2} \log(\tan x + \cot x) dx &= \frac{\pi}{2} \log 2 - \frac{\pi}{2} \log \frac{1}{2} \\ &= \frac{\pi}{2} \log 2 + \frac{\pi}{2} \log 2 \\ &= 2 \times \frac{\pi}{2} \log 2 = \pi \log 2 \end{aligned}$$

प्रश्न 21.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{1+e^x} dx$$

हल : माना

$$\begin{aligned} I &= \int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^x} dx \\ &= \int_{-\pi/2}^0 \frac{\cos x}{1+e^x} dx + \int_0^{\pi/2} \frac{\cos x}{1+e^x} dx \end{aligned}$$

$\int_{-\pi/2}^0 \frac{\cos x}{1+e^x} dx$ में x के स्थान पर x रखने पर,

$$\int_{-\pi/2}^0 \frac{\cos(-x)}{1+e^x} dx + \int_0^{\pi/2} \frac{e^x \cos x}{1+e^x} dx$$

$$\begin{aligned} \therefore I &= \int_0^{\pi/2} \frac{e^x \cos x}{1+e^{-x}} dx + \int_0^{\pi/2} \frac{\cos x}{1+e^x} dx \\ &= \int_0^{\pi/2} \frac{(1+e^x) \cos x}{(1+e^x)} dx \end{aligned}$$

$$= \int_0^{\pi/2} \cos x dx = [\sin x]_0^{\pi/2}$$

$$= \sin \frac{\pi}{2} - \sin 0 = 1 - 0 = 1$$

$$\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^x} dx = 1$$

प्रश्न 22.

$$\int_a^b \frac{f(x)}{f(x) - f(a+b-x)} dx$$

हल : माना

$$I = \int_a^b \frac{f(x) dx}{f(x) + f(a+b-x)} \quad \dots(i)$$

$$\Rightarrow I = \int_a^b \frac{f(a+b-x)}{f(a+b-x) + f[a+b-(a+b-x)]} dx$$

$$\Rightarrow I = \int_a^b \frac{f(a+b-x)}{f(a+b-x) + f(x)} dx \quad \dots(ii)$$

समीकरण (i) व (ii) को जोड़ने पर

$$2I = \int_a^b \frac{f(x) + f(a+b-x)}{f(a+b-x) + f(x)} dx$$

$$\Rightarrow 2I = \int_a^b 1 dx = [x]_a^b$$

$$\Rightarrow 2I = b - a$$

$$\Rightarrow I = \frac{b-a}{2}$$

$$\therefore \int_a^b \frac{f(x)}{f(x) - f(a+b-x)} dx = \frac{b-a}{2}$$

Miscellaneous Exercise

निम्नलिखित का समाकल कीजिए

प्रश्न 1.

$$\int_0^{\pi/4} \sqrt{1 + \sin 2x} \, dx$$

का मान है

- (a) $2 \int_0^a \sin 3x \cdot x \, dx$ (b) 0
(c) a^2 (d) 1

हल : माना

$$\begin{aligned} & \int_0^{\pi/4} \sqrt{1 + \sin 2x} \, dx \\ &= \int_0^{\pi/4} \sqrt{\sin^2 x + \cos^2 x + 2 \sin x \cos x} \, dx \\ &= \int_0^{\pi/4} (\sqrt{\sin x + \cos x})^2 \, dx \\ &= \int_0^{\pi/4} (\sin x + \cos x) \, dx \\ &= [-\cos x + \sin x]_0^{\pi/4} \\ &= (\sin x - \cos x)_0^{\pi/4} \\ &= \left(\sin \frac{\pi}{4} - \cos \frac{\pi}{4} \right) - (\sin 0 - \cos 0) \\ &= \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right) - (0 - 1) = 1 \end{aligned}$$

अतः विक (d) सही है।

प्रश्न 2.

$$\int_2^7 \frac{\sqrt{x}}{\sqrt{x} + \sqrt{7-x}} \, dx$$

का मान है

(a) 3

(b) 2

(d) $\frac{3}{2}$

(c) $\frac{1}{2}$

हल : माना

$$I = \int_2^5 \frac{\sqrt{x}}{\sqrt{x} + \sqrt{7-x}} dx \quad \dots(1)$$

$$I = \int_2^5 \frac{\sqrt{2+5-x}}{\sqrt{2+5-x} + \sqrt{7-2-5+x}} dx$$

$$I = \int_2^5 \frac{\sqrt{7-x}}{\sqrt{7-x} + \sqrt{x}} dx \quad \dots(2)$$

समीकरण (1) व (2) को जोड़ने पर।

$$\begin{aligned} 2I &= \int_2^5 \frac{\sqrt{x} + \sqrt{7-x}}{\sqrt{7-x} + \sqrt{x}} dx \\ &= \int_2^5 1 dx = [x]_2^5 = 5 - 2 \\ &= 3 \end{aligned}$$

$$I = \frac{3}{2}$$

$$\Rightarrow \int_2^5 \frac{\sqrt{x}}{\sqrt{x} + \sqrt{7-x}} dx = \frac{3}{2}$$

अतः विक (c) सही है।

प्रश्न 3.

$$\int_a^b \frac{f(x+c)}{x+c} dx$$

का मान है

(a) $\int_a^b f(x+c) dx$

(b) $\int_a^b f(x) dx$

(c) $\int_a^{b-2c} f(x) dx$

(d) $\int_a^b f(x+2c) dx$

हल : अतः विक (b) सही है।

प्रश्न 4. यदि

$$A(x) = \int_0^x \theta^2 d\theta$$

हो, तो A(3) का मान होगा

- (a) 9
- (b) 27
- (c) 3
- (d) 81

हल :

$$A(x) = \int_0^x \theta^2 d\theta$$

$$= \left[\frac{\theta^3}{3} \right]_0^x$$

$$= \frac{1}{3}x^3$$

$$A(3) = \frac{1}{3} \times (3)^3 = \frac{1}{3} \times 3 \times 3 \times 3$$

$$= 3 \times 3 = 9$$

अतः विकल्प (a) सही है।

प्रश्न 5.

$$\int_1^2 \frac{(x+3)}{x(x+2)} dx$$

हल : माना

$$\frac{x+3}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2}$$

$$\frac{x+3}{x(x+2)} = \frac{A(x+2) + Bx}{x(x+2)}$$

$$x+3 = (A+B)x + 2A$$

तुलना से, $A+B=1$, $2A=3$

$$\Rightarrow A+B=1, A=3/2$$

$$B=1-3/2=-1/2$$

$$\therefore \int_1^2 \frac{x+3}{x(x+2)} dx = \int_1^2 \frac{3}{2x} dx - \int_1^2 \frac{1}{2(x+2)} dx$$

$$\Rightarrow \int_1^2 \frac{x+3}{x(x+2)} dx = \frac{3}{2}(\log x)_1^2 - \frac{1}{2}[\log(x+2)]_1^2$$

$$\Rightarrow \int_1^2 \frac{x+3}{x(x+2)} dx = \frac{3}{2} \log 2 - \frac{1}{2} \log \frac{4}{3}$$

$$\Rightarrow \int_1^2 \frac{x+3}{x(x+2)} dx = \frac{1}{2} \left[\log 2^3 - \log \frac{2^2}{3} \right]$$

$$\Rightarrow \int_1^2 \frac{x+3}{x(x+2)} dx = \frac{1}{2} \log \left(\frac{2^3 \times 3}{2^2} \right) = \frac{1}{2} \log 6$$

$$\text{अतः } \int_1^2 \frac{x+3}{x(x+2)} dx = \frac{1}{2} \log 6$$

प्रश्न 6.

$$\int_1^2 \frac{xe^x}{(1+x)^2} dx$$

हल : माना

$$\begin{aligned} I &= \int_1^2 \frac{xe^x}{(1+x)^2} dx \\ &= \int_1^2 \frac{(x+1)-1}{(1+x)^2} e^x dx \\ &= \int_1^2 \left(\frac{1}{1+x} - \frac{1}{(1+x)^2} \right) e^x dx \\ &= \left[\frac{e^x}{1+x} \right]_1^2 \\ &= \frac{e^2}{1+2} - \frac{e^1}{1+1} \\ &= \frac{e^2}{3} - \frac{e}{2} = \frac{e}{6} (2e - 3) \end{aligned}$$

प्रश्न 7.

$$\int_0^{\frac{\pi}{2}} e^x \left(\frac{1 + \sin x}{1 + \cos x} \right) dx$$

हल : माना

$$I = \int_0^{\pi/2} e^x \left(\frac{1 + \sin x}{1 + \cos x} \right) dx$$

$$\text{तब } \int e^x \left(\frac{1 + \sin x}{1 + \cos x} \right) dx$$

$$= \int e^x \left(\frac{1 + 2 \sin \frac{x}{2} \cos \frac{x}{2}}{2 \cos^2 \frac{x}{2}} \right) dx$$

$$= \frac{1}{2} \int e^x \left(\sec^2 \frac{x}{2} \right) dx + \int e^x \left(\tan \frac{x}{2} \right) dx$$

$$= \int \tan \frac{x}{2} \cdot e^x dx + \frac{1}{2} \int e^x \sec^2 \frac{x}{2} dx$$

$$= \tan \frac{x}{2} \cdot e^x - \int \sec^2 \frac{x}{2} \cdot \frac{1}{2} e^x dx + \frac{1}{2} \int e^x \sec^2 \frac{x}{2} dx$$

$$= e^x \tan \frac{x}{2} - \frac{1}{2} \int e^x \sec^2 x dx + \frac{1}{2} \int e^x \sec^2 \frac{x}{2} dx$$

$$= e^x \tan \frac{x}{2}$$

$$\begin{aligned} \therefore \int_0^{\pi/2} e^x \left(\frac{1 + \sin x}{1 + \cos x} \right) dx &= \left[e^x \tan \frac{x}{2} \right]_0^{\pi/2} \\ &= \left[e^{\pi/2} \tan \frac{\pi}{4} - e^0 \tan 0 \right] \end{aligned}$$

$$= e^{\pi/2} \times 1 - 0$$

$$= e^{\pi/2}$$

प्रश्न 8.

$$\int_{1/3}^1 \frac{(x - x^3)^{1/3}}{x^4} dx$$

हल :

$$\begin{aligned} \int_{1/3}^1 \frac{(x - x^3)^{1/3}}{x^4} dx &= \int_{1/3}^1 \frac{x \left(\frac{1}{x^2} - 1 \right)^{1/3}}{x^4} dx \\ &= \int_{1/3}^1 \frac{\left(\frac{1}{x^2} - 1 \right)^{1/3}}{x^3} dx \end{aligned}$$

अब माना $\frac{1}{x^2} - 1 = t$ जब $x = \frac{1}{3}$, तो $t = 8$

$$\frac{dx}{x^3} = \frac{-dt}{2} \text{ तथा } x = 1 \text{ तो } t = 0$$

$$\begin{aligned} \text{अतः } \frac{-1}{2} \int_8^0 (t)^{1/3} dt &= \frac{1}{2} \int_0^8 t^{1/3} dt \\ &= \frac{1}{2} \times \frac{3}{4} \times [t^{4/3}]_0^8 \\ &= \frac{3}{8} \times (8)^{4/3} = 6 \end{aligned}$$

प्रश्न 9.

$$\int_0^{\pi/2} x^2 \cos^2 x dx$$

हल :

$$\begin{aligned} &\int_0^{\pi/2} x^2 \cos^2 x dx \\ &= \int_0^{\pi/2} x^2 \left[\frac{1}{2}(1 + \cos 2x) \right] dx \\ &= \frac{1}{2} \int_0^{\pi/2} x^2 dx + \frac{1}{2} \int_0^{\pi/2} x^2 \cos 2x dx \quad \dots(i) \end{aligned}$$

$$\begin{aligned} \text{अब } \int x^2 \cos 2x dx \\ &= x^2 \int \cos 2x dx - \int \left[\frac{d}{dx}(x^2) \int \cos 2x dx \right] dx \\ &= x^2 \frac{\sin 2x}{2} - \int 2x \cdot \frac{\sin 2x}{2} dx \\ &= \frac{1}{2} x^2 \sin 2x - \int x \sin 2x dx \quad \dots(ii) \end{aligned}$$

$$\begin{aligned} &\int x \sin 2x dx \\ &= x \int \sin 2x dx - \int \left[\frac{d}{dx}(x) \int \sin 2x dx \right] dx \\ &= x \left(\frac{-\cos 2x}{2} \right) - \int 1 \times \left(\frac{-\cos 2x}{2} \right) dx \end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{2}x \cos 2x + \frac{1}{2} \int \cos 2x \, dx \\
&= \frac{1}{2} \cos 2x + \frac{1}{4} \sin 2x \quad \dots(\text{iii})
\end{aligned}$$

समीकरण (i) (ii) व (iii) को जोड़ने पर।

$$\begin{aligned}
\int_0^{\pi/2} x^2 \cos^2 x \, dx &= \frac{1}{6}(x^3)_0^{\pi/2} + \left[\frac{1}{4}x^2 \sin 2x + \frac{1}{8}x \cos 2x \right. \\
&\quad \left. - \frac{1}{8} \sin 2x \right]_0^{\pi/2} \\
&= \frac{1}{6} \times \frac{\pi^3}{8} + \frac{\pi^2}{16} \sin \pi + \frac{\pi}{8} \cos \pi - \frac{1}{8} \sin \pi \\
&= \frac{\pi^3}{6 \times 8} + 0 - \frac{\pi}{8} + 0 = \frac{\pi^2}{48} (\pi - 6)
\end{aligned}$$

प्रश्न 10.

$$\int_0^1 \tan^{-1} x \, dx$$

हल : माना

$$\begin{aligned}
I &= \int_0^1 \tan^{-1} x \, dx \\
&= \int_0^1 \tan^{-1} x \cdot \underset{\text{I}}{1} \underset{\text{II}}{dx} \quad \dots(\text{i})
\end{aligned}$$

$$\begin{aligned}
\int \tan^{-1} x \cdot 1 \, dx &= (\tan^{-1} x)x - \int \frac{1}{1+x^2} x \, dx \\
&= x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2} \, dx \\
&\quad (\text{माना } 1+x^2 = t) \\
&= x \tan^{-1} x - \frac{1}{2} \frac{dt}{t} \\
&= x \tan^{-1} x - \frac{1}{2} \log t
\end{aligned}$$

$$\int_0^1 \tan^{-1} x \, dx = \left[x \tan^{-1} x - \frac{1}{2} \log (1+x^2) \right]_0^1$$

$$\begin{aligned}
&= \left[1 \tan^{-1} x - \frac{1}{2} \log (1+x^2) \right] - 0 \\
&= \frac{\pi}{4} - \frac{1}{2} \log 2
\end{aligned}$$

प्रश्न 11.

$$\int_0^{\frac{\pi}{4}} \sin 3x \sin 2x \, dx$$

हल : माना

$$\begin{aligned}
&\int_0^{\pi/4} \sin 3x \sin 2x \, dx \\
&= \frac{1}{2} \int_0^{\pi/4} 2 \sin 3x \sin 2x \, dx \\
&= \frac{1}{2} \int_0^{\pi/4} [\cos (3x - 2x) - \cos (3x + 2x)] \, dx \\
&= \frac{1}{2} \int_0^{\pi/4} (\cos x - \cos 5x) \, dx \\
&= \frac{1}{2} \left[\sin x - \frac{\sin 5x}{5} \right]_0^{\pi/4} \\
&= \frac{1}{2} \left[\left(\sin \frac{\pi}{4} - \frac{1}{5} \sin \frac{5\pi}{4} \right) - \left(\sin 0 - \frac{1}{5} \sin 0 \right) \right] \\
&= \frac{1}{2} \left[\frac{1}{\sqrt{2}} - \frac{1}{5} \sin \frac{5\pi}{4} - 0 \right] \\
&= \frac{1}{2} \left[\frac{1}{\sqrt{2}} - \frac{1}{5} \sin \frac{5\pi}{4} \right] \\
&= \frac{1}{2} \left(\frac{1}{\sqrt{2}} + \frac{1}{5} \times \frac{1}{\sqrt{2}} \right) \\
&= \frac{1}{2} \left(\frac{1}{\sqrt{2}} + \frac{1}{5\sqrt{2}} \right) \\
&= \frac{6}{2 \times 5\sqrt{2}} = \frac{3}{5\sqrt{2}} = \frac{3\sqrt{2}}{10}
\end{aligned}$$

प्रश्न 12.

$$\int_{-2}^2 |1-x^2| dx$$

हल : माना

$$\begin{aligned} I &= \int_{-2}^2 |1-x^2| dx \\ &= \int_{-2}^{-1} |1-x^2| dx + \int_{-1}^1 |1-x^2| dx + \int_1^2 |1-x^2| dx \\ &= -\int_{-2}^{-1} (1-x^2) dx + \int_{-1}^1 (1-x^2) dx - \int_1^2 (1-x^2) dx \\ &= -\left[x + \frac{x^3}{3}\right]_{-2}^{-1} + \left[x - \frac{x^3}{3}\right]_{-1}^1 - \left[x - \frac{x^3}{3}\right]_1^2 \\ &= -\left[-1 + \frac{1}{3} - \left(-2 + \frac{8}{3}\right)\right] + \left[1 - \frac{1}{3} - \left(-1 + \frac{1}{3}\right)\right] \\ &\quad - \left[2 - \frac{8}{3} - \left(1 - \frac{1}{3}\right)\right] \\ &= -\left(-\frac{4}{3}\right) + \frac{4}{3} - \left(-\frac{4}{3}\right) = \frac{4}{3} + \frac{4}{3} + \frac{4}{3} \end{aligned}$$

प्रश्न 13.

$$\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x} dx$$

हल :

$$\begin{aligned} I &= \int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x} dx \\ I &= \int_{-\pi}^{\pi} \frac{2x + 2x \sin x}{1+\cos^2 x} dx \\ I &= \int_{-\pi}^{\pi} \frac{2x dx}{1+\cos^2 x} + \int_{-\pi}^{\pi} \frac{2x \sin x}{1+\cos^2 x} dx \\ I &= I_1 + I_2 \end{aligned}$$

$$\text{यहाँ } I_1 = \int_{-\pi}^{\pi} \frac{2x dx}{1+\cos^2 x} \text{ और } I_2 = \int_{-\pi}^{\pi} \frac{2x \sin x}{1+\cos^2 x} dx$$

$\therefore \frac{2x}{1+\cos^2 x}$ एक विषम फलन है $\frac{2x \sin x}{1+\cos^2 x}$ और एक सम फलन है।

$$\therefore I_1 = 0 \text{ और } I_2 = 2 \int_0^{\pi} \frac{2x \sin x}{1 + \cos^2 x}$$

$$\text{अतः } I_2 = 4 \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx \quad \dots(2)$$

$$\text{या } I_2 = 4 \int_0^{\pi} \frac{(\pi - x) \sin(\pi - x)}{1 + \cos^2(\pi - x)} dx$$

$$\left[\because \int_0^a f(x) dx = \int_0^a f(a-x) dx \right]$$

$$I_2 = 4 \int_0^{\pi} \frac{(\pi - x) \sin x}{1 + \cos^2 x} dx$$

$$I_2 = 4 \int_0^{\pi} \frac{\pi \sin x}{1 + \cos^2 x} dx - 4 \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx \quad \dots(3)$$

$$I_2 = 4\pi \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx - I_2$$

$$\therefore 2I_2 = 4\pi \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx$$

माना $\cos x \Rightarrow -\sin x dx = dt$

$\sin x dx = -dt$

जब $x = 0$, तो $t = 1$ तथा जब $x = \pi$ तो $t = -1$

$$\therefore 2I_2 = 4\pi \int_1^{-1} \frac{-dt}{1+t^2}$$

$$2I_2 = 4\pi \int_1^{-1} \frac{dt}{1+t^2}$$

$$\therefore I_2 = 2\pi [\tan^{-1} t]_1^{-1}$$

$$= 2\pi [\tan^{-1} 1 - \tan^{-1}(-1)]$$

$$= 2\pi \left[\frac{\pi}{4} - \left(-\frac{\pi}{4} \right) \right]$$

$$= 2\pi \left[\frac{\pi}{4} + \frac{\pi}{4} \right] = 2\pi \times \frac{2\pi}{4} = \frac{4\pi^2}{4}$$

$$I_2 = \pi^2$$

11 और 12 का मान समीकरण (1) में रखने पर

$$I = 0 + \pi^2 = \pi^2$$

अर्थात्
$$\int_{-\pi}^{\pi} \frac{2x(1 + \sin x)}{1 + \cos^2 x} = \pi^2$$

प्रश्न 14.

$$\int_0^1 \frac{\sin^{-1} x}{(1-x^2)^{3/2}} dx$$

हल : माना

$$I = \int \frac{\sin^{-1} x}{(1-x^2)^{3/2}} dx$$

$$I = \int \frac{\sin^{-1} x}{(1-x^2)\sqrt{1-x^2}} dx$$

माना

$$\sin^{-1} x = t \Rightarrow \frac{1}{\sqrt{1-x^2}} dx = dt$$

$$I = \int \frac{t dt}{(1-\sin^2 t)} = \int \frac{t dt}{\cos^2 t} = \int t \sec^2 t dt$$

$$I = \int t \sec^2 t dt$$

$$I = t \tan t - \int 1 \cdot \tan t dt$$

$$= t \tan t - \log \sec t + c$$

$$I = \sin x \tan(\sin) - \log \sec(\sin x) + c$$

$$\begin{aligned} I &= \sin^{-1} x \tan \left(\tan^{-1} \frac{x}{\sqrt{1-x^2}} \right) \\ &\quad - \log \sec \left(\sec^{-1} \frac{x}{\sqrt{1-x^2}} \right) + C \\ &= \frac{x \sin^{-1} x}{\sqrt{1-x^2}} - \log \frac{1}{\sqrt{1-x^2}} + C \\ &= \frac{x \sin^{-1} x}{\sqrt{1-x^2}} - \log (1-x^2)^{-\frac{1}{2}} + C \end{aligned}$$

$$= \frac{x \sin^{-1} x}{\sqrt{1-x^2}} + \frac{1}{2} \log(1-x^2) + C$$

अब $\int_0^{1/\sqrt{2}} \frac{\sin^{-1} x}{(1-x^2)^{3/2}} dx$

$$= \left[\frac{x \sin^{-1} x}{\sqrt{1-x^2}} + \frac{1}{2} \log(1-x^2) \right]_0^{1/\sqrt{2}}$$

$$= \left[\frac{\frac{1}{\sqrt{2}} \sin^{-1} \left(\frac{1}{\sqrt{2}} \right)}{\sqrt{1-\frac{1}{2}}} + \frac{1}{2} \log \left(1-\frac{1}{2} \right) \right] - (0+0)$$

$$= \frac{\frac{1}{2} \times \frac{\pi}{4}}{1-\frac{1}{2}} + \frac{1}{2} \log \left(1-\frac{1}{2} \right)$$

$$= \frac{\pi}{8} \times \frac{1}{\left(\frac{1}{2} \right)} + \frac{1}{2} \log \left(\frac{1}{2} \right)$$

$$= \frac{\pi}{4} + \frac{1}{2} \log \frac{1}{2} = \frac{\pi}{4} - \frac{1}{2} \log 2$$

प्रश्न 15.

$$\int_0^{\infty} (\cot^{-1} x) dx$$

हल : माना

$$\int \cot^{-1} x dx = \int \cot^{-1} x \cdot 1 dx$$

$$= \cot^{-1} x \cdot \int 1 dx - \int \left\{ \frac{d}{dx} (\cot^{-1} x) \cdot \int 1 dx \right\} dx$$

$$= x \cot^{-1} x + \int \frac{x}{1+x^2} dx \quad \left[\begin{array}{l} \text{माना } 1+x^2 = t \\ x dx = \frac{dt}{2} \end{array} \right]$$

$$= x \cot^{-1} x + \frac{1}{2} \log |t|$$

$$= x \cot^{-1} x + \frac{1}{2} \log(1+x^2)$$

प्रश्न में limit 0 से 1 रखते हैं, तब

$$\begin{aligned} \Rightarrow \int_0^1 \cot^{-1} x \, dx &= \left[x \cot^{-1} x + \frac{1}{2} \log (1+x^2) \right]_0^1 \\ &= \left[1 \times \frac{\pi}{4} + \frac{1}{2} \log 2 \right] - (0+0) \\ &= \frac{\pi}{4} + \frac{1}{2} \log 2 \end{aligned}$$

प्रश्न 16.

$$\int_0^\pi \frac{dx}{1-2a \cos x + a^2}, \quad a > 1$$

हल :

माना

$$\begin{aligned} I &= \int_0^\pi \frac{dx}{1-2a \cos x + a^2} \\ &= \int_0^\pi \frac{dx}{(1+a^2) \left(\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} \right) - 2a \left(\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} \right)} \\ &= \int_0^\pi \frac{dx}{(a-1)^2 \cos^2 \frac{x}{2} + (a+1)^2 \sin^2 \frac{x}{2}} \\ &= \frac{2}{(1+a)^2} \int_0^\infty \frac{dt}{\left(\frac{a-1}{a+1} \right)^2 + t^2} \quad \text{जहाँ } t = \tan \frac{x}{2} \\ &\quad \quad \quad 2dt = \sec^2 \frac{x}{2} dx \\ &= \frac{2}{(1+a)^2} \cdot \frac{a+1}{a-1} \left[\tan^{-1} \left(\frac{a+1}{a-1} t \right) \right]_0^\infty \\ &= \frac{2}{(a+1)(a-1)} [\tan^{-1} \infty - \tan^{-1} 0] \\ &= \frac{2}{a^2-1} \times \left(\frac{\pi}{2} - 0 \right) = \frac{\pi}{a^2-1} \end{aligned}$$

प्रश्न 17.

$$\int_0^{\pi} \frac{x dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{\pi^2}{2ab}$$

हल : माना

$$I = \int_0^{\pi} \frac{x dx}{a^2 \cos^2 x + b^2 \sin^2 x}$$

यहाँ $f(x) = \frac{1}{a^2 \cos^2 x + b^2 \sin^2 x}$

$$\begin{aligned} \therefore f(\pi - x) &= \frac{1}{a^2 \cos^2(\pi - x) + b^2 \sin^2(\pi - x)} \\ &= \frac{1}{a^2 \cos^2 x + b^2 \sin^2 x} = f(x) \end{aligned}$$

अतः के निष्कासन नियम से

$$\begin{aligned} I &= \frac{\pi}{2} \int_0^{\pi} \frac{1}{a^2 \cos^2 x + b^2 \sin^2 x} dx \\ &= \frac{\pi}{2} \times 2 \int_0^{\pi} \frac{1}{a^2 \cos^2 x + b^2 \sin^2 x} dx \\ &\quad \text{(गुणधर्म VII से)} \end{aligned}$$

$$= \pi \int_0^{\pi} \frac{\sec^2 x dx}{a^2 + b^2 \tan^2 x}$$

($\cos^2 x$ का अंश-हर में भाग देने पर)

जहाँ माना $\tan x = t \Rightarrow \sec^2 x dx = dt$

$x = 0$ तो $t = 0$, $x = \frac{\pi}{2}$ तो $t = \infty$

$$\begin{aligned} &= \pi \int_0^{\infty} \frac{dt}{a^2 + b^2 t^2} \\ &= \frac{\pi}{b^2} \int_0^{\infty} \frac{dt}{\left(\frac{a}{b}\right)^2 + t^2} = \frac{\pi}{b^2} \frac{1}{\left(\frac{a}{b}\right)} \left[\tan^{-1} \left(\frac{t}{\frac{a}{b}} \right) \right]_0^{\infty} \\ &= \frac{\pi}{b^2} \times \frac{b}{a} (\tan^{-1} \infty - \tan^{-1} 0) \\ &= \frac{\pi}{ab} \left(\frac{\pi}{2} - 0 \right) = \frac{\pi^2}{2ab} \end{aligned}$$

सिद्ध हुआ।