CHAPTER

1

Logarithm and Its Applications

- Exponential Function
- Logarithmic Function
- Fundamental Laws of Logarithms
- Logarithmic Equations
- Trigonometric Ratios for Compound Angles
- Logarithmic Inequalities
- Finding Logarithm
- Antilogarithm

EXPONENTIAL FUNCTION

Exponential functions are perhaps the most important class of functions in mathematics. We use this type of function in the calculation of interest on investments, growth and decline rates of populations, forensics investigations, and in many other applications.

Definition

$y = f(x) = a^x$, where $a > 0$; $a \neq 1$, and $x \in R$. Here $a^x > 0$ for $\forall\, x \in R$. Thus, the range of the function is $(0, \infty)$.

Exponential functions always have some positive number other than 1 as the base. If you think about it, having a negative number (such as –2) as the base would not be very useful, since the even powers would give you positive answers (such as "$(-2)^2 = 4$") and the odd powers would give you negative answers (such as "$(-2)^3 = -8$"), and what would you even do with the powers that are not whole numbers? Also, having 0 or 1 as the base would be a kind of dumb, since 0 and 1 to any power are just 0 and 1, respectively; what would be the point? This is why exponentials always have something positive and other than 1 as the base.

Graphs of Exponential Function

When $a > 1$

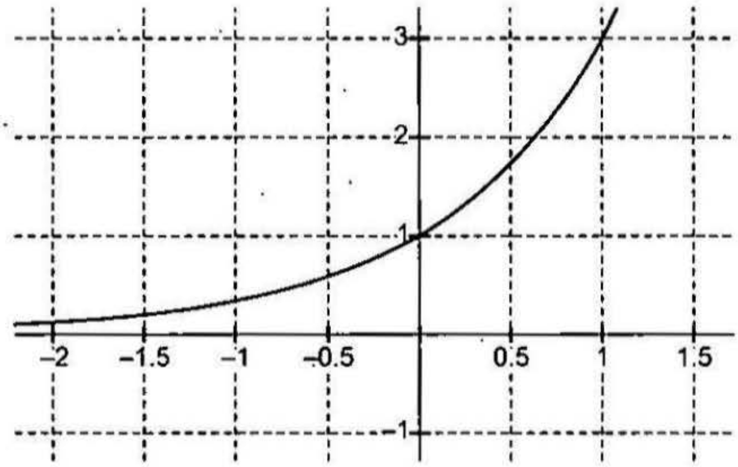


Fig. 1.1

From the graph, the function is increasing. For $x_2 > x_1 \Rightarrow a^{x_2} > a^{x_1}$.

When $0 < a < 1$,

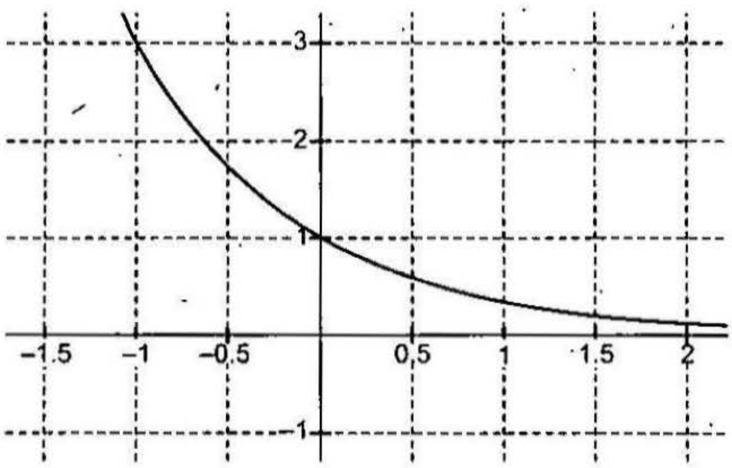


Fig. 1.2

From the graph, the function is decreasing. For $x_2 > x_1 \Rightarrow a^{x_2} < a^{x_1}$.

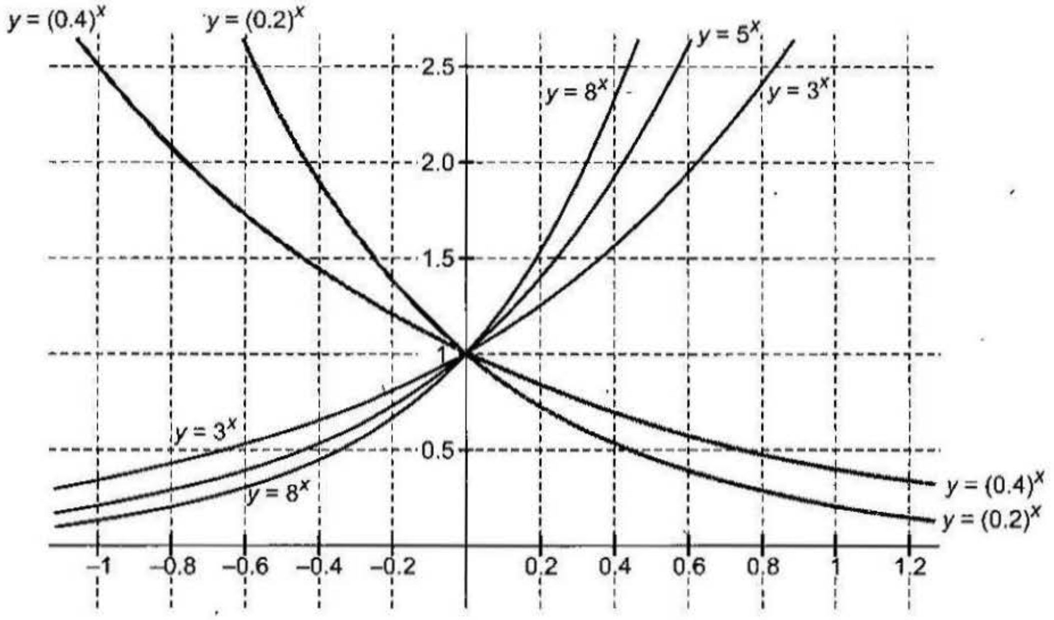


Fig. 1.3

LOGARITHMIC FUNCTION

The **logarithm** of a number to a given *base* is the *exponent* to which the base must be raised in order to produce that number. For example, the logarithm of 1000 to base 10 is 3, because 10 to the power of 3 is 1000, i.e., $10^3 = 1000$. Logarithm function is an inverse of exponential function. Hence, domain and range of the logarithmic functions are range and domain of exponential function, respectively.

Also graph of function can be obtained by taking the mirror image of the graph of the exponential function in the line $y = x$. If we consider point (x_1, y_1) on the graph of $y = a^x$, then we find point (y_1, x_1) on the graph of $y = \log_a x$.

Definition

Logarithmic function is defined as $y = \log_a x$, $a > 0$ and $a \neq 1$.
Domain : $(0, \infty)$,
Range : $(-\infty, \infty)$.

Graphs of Logarithm Function

When $a > 1$

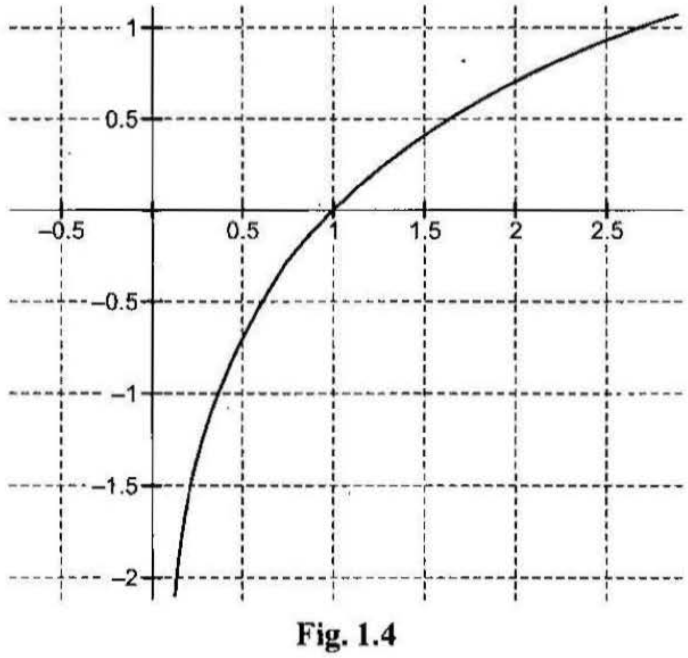


Fig. 1.4

When $a > 1$, $f(x) = \log_a x$ is an increasing function. Then for $x_2 > x_1 \Rightarrow \log_a x_2 > \log_a x_1$. Also $\log_a x_2 > x_1$ $\Rightarrow x_2 > a^{x_1}$

When $0 < a < 1$

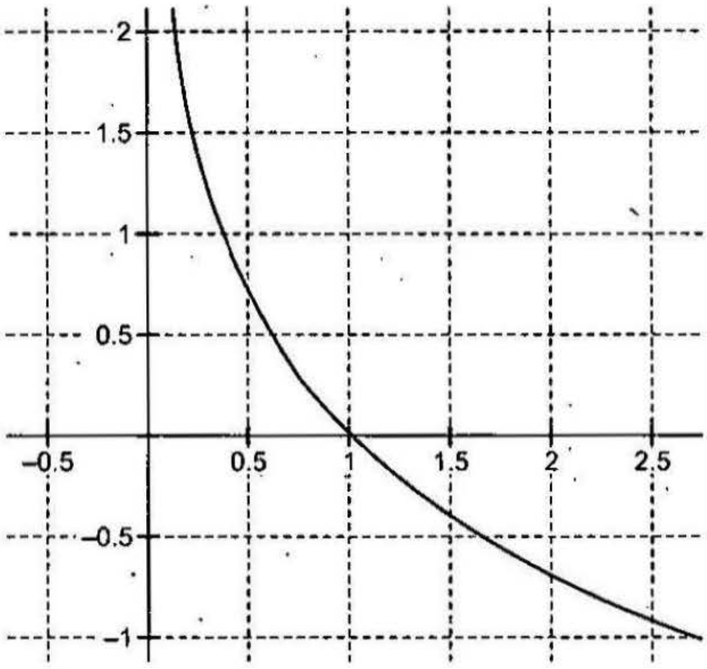


Fig. 1.5

When $0 < a < 1$, $f(x) = \log_a x$ is a decreasing function. Then for $x_2 > x_1 \Rightarrow \log_a x_2 < \log_a x_1$. Also $\log_a x_2 > x_1$ $\Rightarrow 0 < x_2 < a^{x_1}$

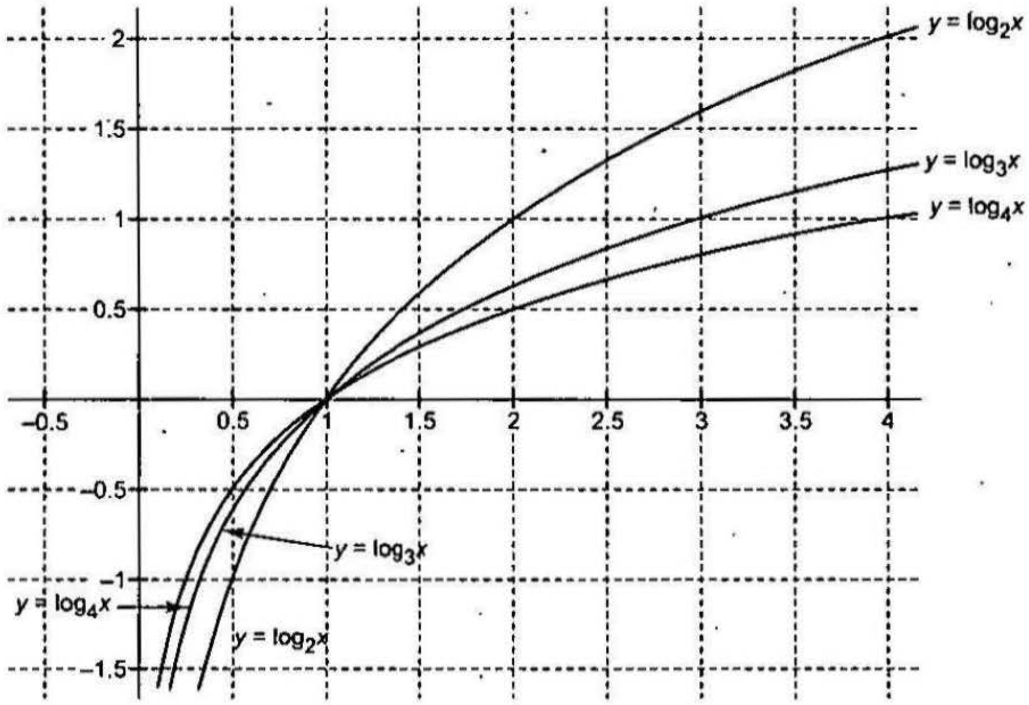


Fig. 1.6

The **natural logarithm** is the *logarithm* to the *base e*, where *e* is an *irrational* constant approximately equal to 2.718281828. Here, *e* is an irrational number. Also, *e* is defined exactly as $e = (1 + 1/m)^m$ as *m* increases to infinity. You can see how this definition produces *e* by inputting a large value of *m* like $m = 10{,}000{,}000$ to get $(1 + 1/10000000)^{10000000} = 2.7182817$ (rounded), which is very close to the actual value. The natural logarithm is generally written as $\ln(x)$, $\log_e(x)$.

The **common logarithm** is the *logarithm* with base 10. It is also known as the **decadic logarithm**, named after its base. It is indicated by $\log_{10}(x)$ On calculators, it is usually written as "log", but *mathematicians* usually mean *natural logarithm* rather than common logarithm when they write "log". To mitigate this ambiguity, the *ISO specification* is that $\log_{10}(x)$ should be **lg** (x) and $\log_e(x)$ should be **ln** (x).

FUNDAMENTAL LAWS OF LOGARITHMS

1. For $m, n, a > 0, a \neq 1$; $\log_a (mn) = \log_a m + \log_a n$
   Proof: Let $\log_a m = x$ and $\log_a n = y$.
   Then $\log_a m = x \Rightarrow a^x = m$ and, $\log_a n = y \Rightarrow a^y = n$
   $\therefore mn = a^x \cdot a^y$
   $\Rightarrow mn = a^{x+y} \Rightarrow \log_a(mn) = x + y \Rightarrow \log_a(mn) = \log_a m + \log_a n$
   In general for $x_1, x_2, \ldots, x_n$ are positive real numbers,
   $\log_a(x_1 x_2 \ldots x_n) = \log_a x_1 + \log_a x_2 + \ldots \log_a x_n$.

2. For $m, n, a > 0, a \neq 1$; $\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$

   Proof: Let $\log_a m = x \Rightarrow a^x = m$ and $\log_a n = y \Rightarrow a^y = n$

   $$\therefore \frac{m}{n} = \frac{a^x}{a^y} \Rightarrow \frac{m}{n} = a^{x-y} \Rightarrow \log_a \left(\frac{m}{n}\right) = x - y \Rightarrow \log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$$

3. For $m, n, a > 0, a \neq 1$; $\log_a(m^n) = n. \log_a m$
   Proof: Let $\log_a m = x \Rightarrow a^x = m \Rightarrow (a^x)^n = m^n \Rightarrow a^{xn} = m^n \Rightarrow \log_a(m^n) = nx \Rightarrow \log_a (m^n) = n. \log_a m$
4. $\log_a 1 = 0$.
   Proof: Since $a^0 = 1$. Therefore, by definition of log, we have $\log_a 1 = 0$.
5. $\log_a a = 1$
   Proof: Since $a^1 = a$. Therefore, by definition of log, we have $\log_a a = 1$. .

6. For $m, a, b > 0$ and $a \neq 1, b \neq 1$, then $\log_a m = \frac{\log_b m}{\log_b a}$

   Proof: Let $\log_a m = x$. Then, $a^x = m$
   Now, $a^x = m \Rightarrow \log_b(a^x) = \log_b m$ [Taking log to the base b]
   $\Rightarrow x \log_b a = \log_b m \Rightarrow \log_a m. \log_b a = \log_b m$ [$\because \log_m m = 1$]

   $$\Rightarrow \log_a m = \frac{\log_b m}{\log_b a}$$

   Replacing b by m in the above result, we get

   $$\log_a m = \frac{\log_m m}{\log_m a} \Rightarrow \log_a m = \frac{1}{\log_m a} \quad [\because \log_m m = 1]$$

7. For $a, n > 0$ and $a \neq 1$; $a^{\log_a n} = n$
   Proof: Let $\log_a n = x$. Then $a^x = n$. Therefore, $a^{\log_a n} = n$. [Putting the value of x in $a^x = n$]

   For example, $3^{\log_3 8} = 8$, $2^{3\log_2 5} = 2^{\log_2 5^3} = 5^3$, $5^{-2\log_5 3} = 5^{\log_5 3^{-2}} = 3^{-2} = 1/9$

8. $\log_{a^q} n^p = \frac{p}{q} \log_a n$, where $a, n > 0, a \neq 1$

   Proof: Let $\log_{a^q} n^p = x$ and $\log_a n = y$. Then, $(a^q)^x = n^p$ and $a^y = n$

Therefore, $a^{qx} = n^p$ and $a_y = n \Rightarrow a^{qx} = n^p$ and $(a^y)^p = n^p \Rightarrow a^{qx} = (a^y)^p \Rightarrow a^{qx} = a^{yp} \Rightarrow qx = yp \Rightarrow x = (p/2)y$

$\Rightarrow \log_{a^q} n^p = \dfrac{p}{q} \log_a n.$

9. $a^{\log_b c} = c^{\log_b a}$,

Proof: Let $a^{\log_b c} = p \Rightarrow \log_b c = \log_a p \Rightarrow \dfrac{\log c}{\log b} = \dfrac{\log p}{\log a} \Rightarrow \dfrac{\log a}{\log b} = \dfrac{\log p}{\log c} \Rightarrow \log_b a = \log_c p$

$\Rightarrow p = c^{\log_b a} \Rightarrow a^{\log_b c} = c^{\log_b a}$

Example 1.1 **Solve $(1/2)^{x^2-2x} < 1/4$.**

Sol. We have $(1/2)^{x^2-2x} < (1/2)^2$

It means $x^2 - 2x > 2$

$\Rightarrow (x-(1+\sqrt{3}))(x-(1-\sqrt{3})) > 0$

$\Rightarrow x > 1+\sqrt{3}$ or $x < 1-\sqrt{3}$

$\Rightarrow x \in (-\infty, 1-\sqrt{3}) \cup (1+\sqrt{3}, \infty)$

Example 1.2 **Solve $\dfrac{1-5^x}{7^{-x}-7} \geq 0$.**

Sol. $g(x) = \dfrac{5^x - 1}{7^{-x}-7} \leq 0$. Now $5^x - 1 = 0 \Rightarrow x = 0$ and $7^{-x} - 7 = 0 \Rightarrow x = -1$

Sign scheme of $g(x)$:

Fig. 1.7

Hence, from sign scheme of $g(x)$, $x \in (-\infty, -1) \cup [0, \infty)$.

Example 1.3 **Which of the following numbers are positive/negative?**

(i) $\log_2 7$ (ii) $\log_{0.2} 3$ (iii) log1/3(1/5)

(iv) $\log_4 3$ (v) $\log_2(\log_2 9)$

Sol. (i) Let $\log_2 7 = x \Rightarrow 7 = 2^x \Rightarrow x > 0$

(ii) Let $\log_{0.2} 3 = x \Rightarrow 3 = 0.2^x \Rightarrow x < 0$

(iii) Let $\log_{1/3}(1/5) = x \Rightarrow 1/5 = (1/3)^x \Rightarrow 5 = 3^x \Rightarrow x > 0$

(iv) Let $\log_4 3 = x \Rightarrow 3 = 4^x \Rightarrow x < 0$

(v) Let $\log_2(\log_2 9) = x \Rightarrow \log_2 9 = 2^x \Rightarrow 9 = 2^{2^x} \Rightarrow x > 0$

Example 1.4 **What is logarithm of $32\sqrt[5]{4}$ to the base $2\sqrt{2}$?**

Sol. $\log_{2\sqrt{2}} 32\sqrt[5]{4} = \log_{(2^{3/2})}(2^5 4^{\frac{1}{5}}) = \log_{(2^{3/2})}(2^{5+\frac{2}{5}}) = \dfrac{2}{3}\dfrac{27}{5}\log_2 2 = \dfrac{18}{5} = 3.6$

Example 1.5 **Find the value of $\log_5 \log_2 \log_3 \log_2 512$.**

Sol. $\log_5 \log_2 \log_3 \log_2 2^9 = \log_5 \log_2 \log_3 (9 \log_2 2)$

$= \log_5 \log_2 \log_3 3^2$ ($\log_a a = 1$ if $a > 1$ $a \neq 1$)

$= \log_5 \log_2 2 = \log_5 1 = 0.$

Example 1.6 **If $\log_{\sqrt{8}} b = 3\frac{1}{3}$, then find the value of b.**

Sol. $\log_{\sqrt{8}} b = 3\frac{1}{3} \Rightarrow \frac{2}{3}\log_2 b = \frac{10}{3} \Rightarrow \log_2 b = 5 \Rightarrow b = 2^5 = 32$

Example 1.7 **If $n > 1$, then prove that $= \frac{1}{\log_2 n} + \frac{1}{\log_3 n} + \cdots + \frac{1}{\log_{53} n} = \frac{1}{\log_{53\,!}\, n}$.**

Sol. The given expression is equal to $\log_n 2 + \log_n 3 + \cdots + \log_n 53 = \log_n (2.3 \ldots 53) = \log_n 53! = \frac{1}{\log_{53!} n}$

Example 1.8 **Which is greater $x = \log_3 5$ or $y = \log_{17} 25$?**

Sol. $\frac{1}{y} = \log_{25} 17 = \frac{1}{2}\log_5 17$ and $\frac{1}{x} = \log_5 3 = \frac{1}{2}\log_5 9$

$\therefore \frac{1}{y} > \frac{1}{x} \Rightarrow x > y$

Example 1.9 **$y = 2^{\frac{1}{\log_x 4}}$, then find x in terms of y.**

Sol. Since $y = 2^{\frac{1}{\log_x 4}}$, we get $\log_2 y = \frac{1}{\log_x 4}$ $\qquad (\because x > 0, x \neq 1)$

$\Rightarrow \log_2 y = \log_4 x = \frac{1}{2}\log_2 x$

$\Rightarrow 2\log_2 y = \log_2 x$

$\Rightarrow \log_2 y^2 = \log_2 x$

$\therefore x = y^2$

Example 1.10 **Find the value of $81^{(1/\log_5 3)} + 27^{\log_9 36} + 3^{4/\log_7 9}$.**

Sol. $81^{(1/\log_5 3)} + 27^{\log_9 36} + 3^{4/\log_7 9} = (3^4)^{\log_3 5} + (3^3)^{\log_{3^2}(6^2)} + 3^{4\log_9 7}$

$= 3^{\log_3 5^4} + (3^3)^{\log_3(6)} + 3^{4\log_{3^2} 7}$

$= 5^4 + 3^{\log_3 6^3} + 3^{2\log_3 7}$

$= 5^4 + 6^3 + 3^{\log_3 7^2}$

$= 625 + 216 + 7^2 = 890$

Example 1.11 **Prove that number $\log_2 7$ is an irrational number.**

Sol. Let $\log_2 7$ is a rational number

$\Rightarrow \log_2 7 = \frac{p}{q} \Rightarrow 7 = 2^{p/q} \Rightarrow 7^q = 2^p$ which is not possible for any integral values of p and q.

Hence, $\log_2 7$ is not rational.

Example 1.12 **Find the value of $\log_3 4 \log_4 5 \log_5 6 \log_6 7 \log_7 8 \log_8 9$.**

Sol. $\log_3 4 \log_4 5 \log_5 6 \log_6 7 \log_7 8 \log_8 9 = \frac{\log 4}{\log 3}\times\frac{\log 5}{\log 4}\times\frac{\log 6}{\log 5}\times\frac{\log 7}{\log 6}\times\frac{\log 8}{\log 7}\times\frac{\log 9}{\log 8}$

$$= \frac{\log 9}{\log 3} = \log_3 9 = 2$$

Example 1.13 **If $\log_e\left(\frac{a+b}{2}\right) = \frac{1}{2}(\log_e a + \log_e b)$, then find the relation between a and b.**

Sol. $\log_e\left(\frac{a+b}{2}\right) = \frac{1}{2}(\log_e a + \log_e b)$

$$\Rightarrow \log_e\left(\frac{a+b}{2}\right) = (\log_e \sqrt{ab}) \Rightarrow \frac{a+b}{2} = \sqrt{ab}$$

$$\Rightarrow a + b - 2\sqrt{ab} = 0 \Rightarrow (\sqrt{a}-\sqrt{b})^2 = 0 \Rightarrow a = b$$

Example 1.14 **If $a^x = b, b^y = c, c^z = a$, then find the value of xyz.**

Sol. $a^x = b, b^y = c, c^z = a \Rightarrow x = \log_a b, y = \log_b c, z = \log_c a$

$$\Rightarrow xyz = (\log_a b)(\log_b c)(\log_c a) = \frac{\log b}{\log a}\frac{\log c}{\log b}\frac{\log a}{\log c} = 1$$

Example 1.15 **If $\frac{x(y+z-x)}{\log x} = \frac{y(z+x-y)}{\log y} = \frac{z(x+y-z)}{\log z}$, prove that $x^y y^x = z^y y^z = x^z z^x$.**

Sol. Let $\frac{x(y+z-x)}{\log_a x} = \frac{y(z+x-y)}{\log_a y} = \frac{z(x+y-z)}{\log^a z} = k$

$$\Rightarrow \log_a x = \frac{x(y+z-x)}{k} \Rightarrow x = a^{\frac{x(y+y-z)}{k}}$$

Similarly, $y = a^{\frac{y(z+x-z)}{k}}$ and $z = a^{\frac{z(x+y-z)}{k}}$

Now $x^y y^x = a^{\frac{xy(y+z-x)}{k}} a^{\frac{yx(z+x-y)}{k}} = a^{\frac{xy^2+xyz-x^2y+xyz+x^2y-xy^2}{k}} = a^{\frac{2xyz}{k}}$

Similarly, $z^y y^z = x^z z^x = a^{\frac{2xyz}{k}}$.

Example 1.16 **Which of the following is greater: $m = (\log_2 5)^2$ or $n = \log_2 20$?**

Sol. $m - n = (\log_2 5)^2 - [\log_2 5 + 2]$
Let $\log_2 5 = x \Rightarrow m - n = x^2 - x - 2 = (x-2)(x+1) = (\log_2 5 - 2)(\log_2 5 + 1) > 0$
Hence, $m > n$.

Example 1.17 **If $\log_{12} 27 = a$, then find $\log_6 16$ in terms of a.**

a. $2\left(\frac{3-a}{3+a}\right)$ b. $3\left(\frac{3-a}{3+a}\right)$ c. $4\left(\frac{3-a}{3+a}\right)$ d. $5\left(\frac{3-a}{3+a}\right)$

Sol. Since $a = \log_{12} 27 = \log_{12} (3)^3 = 3 \log_{12} 3$, we get

$$\frac{3}{\log_3 12} = \frac{3}{1+\log_3 4} = \frac{3}{1+2\log_3 2}$$

$$\therefore \log_3 2 = \frac{3-a}{2a}$$

Then, $\log_6 16 = \log_6 2^4 = 4\log_6 2 = \frac{4}{\log_2 6} = \frac{4}{1+\log_2 3} = \frac{4}{1+\frac{2a}{3-a}} = 4\left(\frac{3-a}{3+a}\right)$

Example 1.18 Simplify $\frac{1}{1+\log_a bc} + \frac{1}{1+\log_b ca} + \frac{1}{1+\log_c ab}$.

Sol. $\frac{1}{1+\log_a bc} + \frac{1}{1+\log_b ca} + \frac{1}{1+\log_c ab}$

$$= \frac{\log a}{\log a + \log b + \log c} + \frac{\log b}{\log a + \log b + \log c} + \frac{\log c}{\log a + \log b + \log c} = 1$$

Example 1.19 If $y^2 = xz$ and $a^x = b^y = c^z$, then prove that $\log_b a = \log_c b$.

Sol. $a^x = b^y = c^z \Rightarrow x \log a = y \log b = z \log c$

$$\therefore \frac{y}{x} = \frac{z}{y} \Rightarrow \frac{\log a}{\log b} = \frac{\log b}{\log c}$$

$$\Rightarrow \log_b a = \log_c b$$

Example 1.20 Suppose $x, y, z > 0$ and not equal to 1 and $\log x + \log y + \log z = 0$. Find the value of $x^{\frac{1}{\log y}+\frac{1}{\log z}} \times y^{\frac{1}{\log z}+\frac{1}{\log x}} \times z^{\frac{1}{\log x}+\frac{1}{\log y}}$ (base 10)

Sol. Let $K = x^{\frac{1}{\log y}+\frac{1}{\log z}} \times y^{\frac{1}{\log z}+\frac{1}{\log x}} \times z^{\frac{1}{\log x}+\frac{1}{\log y}}$

$$\log K = \log x\left[\frac{1}{\log y} + \frac{1}{\log z}\right] + \log y\left[\frac{1}{\log z} + \frac{1}{\log x}\right] + \log z\left[\frac{1}{\log x} + \frac{1}{\log y}\right] \quad \text{(i)}$$

Putting $\log x + \log y + \log z = 0$ (given), we get

$$\frac{\log x}{\log y} + \frac{\log z}{\log y} = -1; \frac{\log y}{\log x} + \frac{\log z}{\log x} = -1 \text{ and } \frac{\log x}{\log z} + \frac{\log y}{\log z} = -1$$

Therefore, R.H.S. of Eq. (i) $= -3 \Rightarrow \log_{10} K = -3 \Rightarrow K = 10^{-3}$

Example 1.21 Prove that $\frac{2^{\log_{2^{1/4}} x} - 3^{\log_{27}(x^2+1)^3} - 2x}{7^{4\log_{49} x} - x - 1} > 0; \forall x \in R$.

Sol. $y = \frac{2^{\log_{2^{1/4}} x} - 3^{\log_{27}(x^2+1)^3} - 2x}{7^{4\log_{49} x} - x - 1}$

$$= \frac{x^4 - (x^2+2x+1)}{x^2 - x - 1}$$

$= x^2 + x + 1$

$= (x + 1/2)^2 + 3/4 > 0; \forall x \in R$

Concept Application Exercise 1.1

1. Prove that $\log_7 \log_7 \sqrt{7\sqrt{(7\sqrt{7})}} = 1 - 3 \log_7 2$.
2. Solve for x and y : $y^x = x^y$; $x = 2y$.
3. Find the value of $3^{2 \log_9 3}$.
4. If $\log_{10} x = y$, then find $\log_{1000} x^2$ in terms of y.
5. If $\log_7 2 = m$, then find $\log_{49} 28$ in terms of m.
6. Find the value of $\sqrt{(\log_{0.5}^2 4)}$.
7. Find the value of $7\log\left(\frac{16}{15}\right)+5\log\left(\frac{25}{24}\right)+3\log\left(\frac{81}{80}\right)$.
8. If $a^2 + b^2 = 7\,ab$, prove that $\log\left(\frac{a+b}{3}\right) = \frac{1}{2}(\log a + \log b)$.
9. Prove the following identities:

   (i) $\frac{\log_a n}{\log_{ab} n} = 1 + \log_a b$ (ii) $\log_{ab} x = \frac{\log_a x \log_b x}{\log_a x + \log_b x}$
10. If $\log_a (ab) = x$, then evaluate $\log_b (ab)$ in terms of x.
11. Compute $\log_{ab} (\sqrt[3]{a} / \sqrt{b})$ if $\log_{ab} a = 4$.
12. If $a^x = b^y = c^z = d^w$, show that $\log_a (bcd) = x\left(\frac{1}{y}+\frac{1}{z}+\frac{1}{w}\right)$.
13. Solve for x: $11^{4x-5} \cdot 3^{2x} = 5^{3-x} \cdot 7^{-x}$.
14. If $\log_{\sqrt{b}} n = 2$ and $\log_n 2b = 2$, then find the value of b.
15. Suppose that a and b are positive real numbers such that $\log_{27}a + \log_9 b = 7/2$ and $\log_{27}b + \log_9 a = 2/3$. Then find the value of ab.

LOGARITHMIC EQUATIONS

While solving logarithmic equations, we tend to simplify the equation. Solving equation after simplification may give some roots which are not defining all the terms in the initial equation. Thus, while solving equations involving logarithmic function, we must take care of domain of the equation.

Example 1.22 **Solve $\log_4 8 + \log_4(x+3) - \log_4(x-1) = 2$.**

Sol. $\log_4 8 + \log_4(x+3) - \log_4(x-1) = 2$

$$\Rightarrow \log_4 \frac{8(x+3)}{x-1} = 2 \Rightarrow \frac{8(x+3)}{x-1} = 4^2$$

$$\Rightarrow x + 3 = 2x - 2 \Rightarrow x = 5$$

Also for $x = 5$ all terms of the equation are defined.

Example 1.23 **Solve $\log(-x) = 2\log(x+1)$.**

Sol. By definition, $x < 0$ and $x + 1 > 0$. $\Rightarrow -1 < x < 0$
Now $\log(-x) = 2\log(x+1) \Rightarrow -x = (x+1)^2 \Rightarrow x^2 + 3x + 1 = 0$

$$\Rightarrow x = \frac{-3+\sqrt{5}}{2}, \frac{-3-\sqrt{5}}{2}$$

Hence, $x = \frac{-3+\sqrt{5}}{2}$ is the only solution.

Example 1.24 **Solve $\log_2(3x-2) = \log_{1/2} x$.**

Sol. $\log_2(3x-2) = \log_{1/2} x = \frac{\log_2 x}{\log_2 2^{-1}} = \log_2 x^{-1}$

$\Rightarrow 3x - 2 = x^{-1} \Rightarrow 3x^2 - 2x = 1 \Rightarrow x = 1$ or $x = -1/3$. But $\log_2(3x-2)$ and $\log_{1/2} x$ are meaningful if $x > 2/3$.
Hence, $x = 1$.

Example 1.25 **Solve $2^{x+2}27^{x/(x-1)} = 9$.**

Sol. Taking log of both sides, we have $(x+2)\log 2 + \frac{x}{x-1}\log 27 = \log 9$

$$\Rightarrow (x+2)\log 2 + \frac{x}{x-1} 3\log 3 = 2\log 3$$

$$\Rightarrow (x+2)\log 2 + \left(\frac{3x}{x-1} - 2\right)\log 3 = 0$$

$$\Rightarrow (x+2)\left[\log 2 + \frac{\log 3}{x-1}\right] = 0$$

$$\Rightarrow x = -2 \text{ or } x - 1 = -\frac{\log 3}{\log 2}$$

$$\Rightarrow x = -2, 1 - \frac{\log 3}{\log 2}.$$

Example 1.26 **Solve $\log_2(4 \times 3^x - 6) - \log_2(9^x - 6) = 1$.**

Sol. $\log_2(4 \times 3^x - 6) - \log_2(9^x - 6) = 1$

$$\Rightarrow \log_2 \frac{4 \times 3^x - 6}{9^x - 6} = 1$$

$$\Rightarrow \frac{4 \times 3^x - 6}{9^x - 6} = 2$$

$\Rightarrow 4y - 6 = 2y^2 - 12$ (putting $3^x = y$)
$\Rightarrow y^2 - 2y - 3 = 0$
$\Rightarrow y = -1, 3$
$\Rightarrow 3^x = 3$
$\Rightarrow x = 1$

Example 1.27 **Solve $6(\log_x 2 - \log_4 x) + 7 = 0$.**

Sol. $6(\log_x 2 - \log_4 x) + 7 = 0$

$$\Rightarrow 6\left(\log_x 2 - \frac{1}{2}\log_2 x\right) + 7 = 0$$

$$\Rightarrow 6\left(\frac{1}{y} - \frac{y}{2}\right) + 7 = 0 \text{ (where } y = \log_2 x)$$

$$\Rightarrow 6\left(\frac{2 - y^2}{2y}\right) + 7 = 0$$

$$\Rightarrow 3\left(\frac{2 - y^2}{y}\right) + 7 = 0$$

$\Rightarrow 6 - 3y^2 + 7y = 0$
$\Rightarrow 3y^2 - 7y - 6 = 0$
$\Rightarrow 3y^2 + 2y - 9y - 6 = 0$
$\Rightarrow (y - 3)(3y + 2) = 0$
$\Rightarrow y = 3$ or $y = -2/3$
$\Rightarrow \log_2 x = 3$ or $-2/3$
$\Rightarrow x = 8$ or $x = 2^{-2/3}$

Example 1.28 **Find the number of solution to equation $\log_2 (x + 5) = 6 - x$.**

Sol. Here, $x + 5 = 2^{6-x}$.

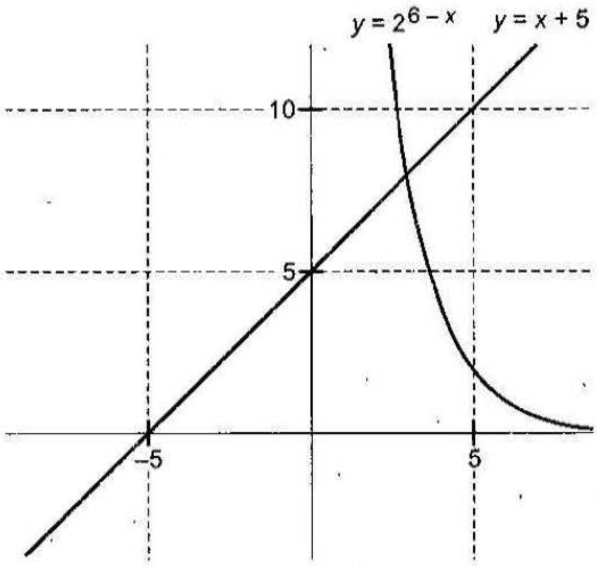


Fig. 1.8

Now graph of $y = x + 5$ and $y = 2^{6-x}$ intersect only once.
Hence, there is only one solution.

Example 1.29 **Solve $4^{\log_2 \log x} = \log x - (\log x)^2 + 1$ (base is e).**

Sol. $\log_2 \log x$ is meaningful if $x > 1$.

Since $4^{\log_2 \log x} = 2^{2\log_2 \log x} = (2^{\log_2 \log x})^2 = (\log x)^2$ $(a^{\log_a x} = x, a > 0, a \neq 1)$

So the given equation reduces to $2(\log x)^2 - \log x - 1 = 0$.

Therefore, $\log x = 1$, $\log x = 1/2$. But for $x > 1$,

$\log x > 0$ so $\log \log x = 1$, i.e., $x = e$.

Example 1.30 **Solve** $4\log_{x/2}\left(\sqrt{x}\right)+2\log_{4x}\left(x^2\right)=3\log_{2x}\left(x^3\right)$

Sol. $\dfrac{4\log_2\sqrt{x}}{\log_2(x/2)}+\dfrac{2\log_2(x^2)}{\log_2(4x)}=\dfrac{3\log_2(x^3)}{\log_2(2x)}$

$\Rightarrow \dfrac{4\times\frac{1}{2}\log_2(x)}{\log_2 x-1}+\dfrac{4\log_2(x)}{2+\log_2(x)}=\dfrac{9\log_2(x)}{1+\log_2(x)}$

Let $\log_2 x = t$, given equation reduces to

$\dfrac{2t}{t-1}+\dfrac{4t}{t+2}=\dfrac{9t}{t+1}$

$\Rightarrow t=0$ or $\dfrac{2}{t-1}+\dfrac{4}{t+2}=\dfrac{9}{t+1}$

$\Rightarrow \dfrac{2t+4+4t-4}{(t-1)(t+2)}=\dfrac{9}{t+1}$

$\Rightarrow t^2+t-6=0$

$\Rightarrow (t+3)(t-2)=0$

$\Rightarrow t=0, 2$ or -3

$\Rightarrow x=1, 4, 1/8$

Example 1.31 **Solve** $4^{\log_9 x}-6x^{\log_9 2}+2^{\log_3 27}=0.$

Sol. Let $2^{\log_9 x}=y$, we get $y^2-6y+8=0$

$\Rightarrow y=4$ or 2

If $2^{\log_9 x}=2^2 \Rightarrow \log_9 x=2 \Rightarrow x=81$

If $2^{\log_9 x}=2^1 \Rightarrow \log_9 x=1 \Rightarrow x=9$

Concept Application Exercise 1.2

1. Solve $\log_2(25^{x+3}-1)=2+\log_2(5^{x+3}+1)$.
2. Solve $\log_4(2\times 4^{x-2}-1)+4=2x$.
3. Solve $\log_5(5^{1/x}+125)=\log_5 6+1+\dfrac{1}{2x}$.
4. Solve $\log_4(x-1)=\log_2(x-3)$.
5. Solve $\log_6 9-\log_9 27+\log_8 x=\log_{64}x-\log_6 4$.
6. Solve $\log_2\left(2\sqrt{17-2x}\right)=1-\log_{1/2}(x-1)$.
7. Solve $3\log_x 4+2\log_{4x}4+3\log_{16x}4=0$.
8. Solve $(\log_3 x)(\log_5 9)-\log_x 25+\log_3 2=\log_3 54$.
9. Solve $(x^{\log_{10}3})^2-(3^{\log_{10}x})-2=0$.

LOGARITHMIC INEQUALITIES

Standard Logarithmic Inequalities

1. If $\log_a x > \log_a y \Rightarrow \begin{cases} x > y, \text{ if } a > 1 \\ 0 < x < y, \text{ if } 0 < a < 1 \end{cases}$

2. If $\log_a x > y \Rightarrow \begin{cases} x > a^y, \text{ if } a > 1 \\ 0 < x < a^y, \text{ if } 0 < a < 1. \end{cases}$

3. $\log_a x > 0 \Rightarrow x > 1$ and $a > 1$
   or $0 < x < 1$ and $0 < a < 1$

Frequently Used Inequalities

1. $(x-a)(x-b) < 0\ (a < b) \Rightarrow a < x < b$
2. $(x-a)(x-b) > 0\ (a < b) \Rightarrow x < a$ or $x > b$
3. $|x| < a \Rightarrow -a < x < a$
4. $|x| > a \Rightarrow x < -a$ or $x > a$

Example 1.32 **Solve $\log_2(x-1) > 4$.**

Sol. $\log_2(x-1) > 4 \Rightarrow x-1 > 2^4 \Rightarrow x > 17$

Example 1.33 **Solve $\log_3(x-2) \le 2$.**

Sol. $\log_3(x-2) \le 2 \Rightarrow 0 < x-2 \le 3^2 \Rightarrow 2 < x \le 11$

Example 1.34 **Solve $\log_{0.3}(x^2-x+1) > 0$.**

Sol. $\log_{0.3}(x^2-x+1) > 0 \Rightarrow 0 < x^2-x+1 < (0.3)^0$

$\Rightarrow 0 < x^2-x+1 < 1 \Rightarrow x^2-x+1 > 0$ and $x^2-x < 0 \Rightarrow x(x-1) < 0$

$\Rightarrow 0 < x < 1$ (as $x^2-x+1 = (x-1/2)^2 + 3/4 > 0$ for all real x)

Example 1.35 **Solve $1 < \log_2(x-2) \le 2$.**

Sol. $1 < \log_2(x-2) \le 2 \Rightarrow 2^1 < x-2 \le 2^2$

$\Rightarrow 4 < x < 6$

Example 1.36 **Solve $\log_2|x-1| < 1$.**

Sol. $\log_2|x-1| < 1 \Rightarrow 0 < |x-1| < 2^1$

$\Rightarrow -2 < x-1 < 2$ and $x-1 \ne 0$

$\Rightarrow -1 < x < 3$ and $x \ne 1$

$\Rightarrow x \in (-1, 3) - \{1\}$

Example 1.37 **Solve $\log_{0.2}|x-3| \ge 0$.**

Sol. $\log_{0.2}|x-3| \ge 0 \Rightarrow 0 < |x-3| \le (0.2)^0$

$\Rightarrow 0 < |x-3| \le 1 \Rightarrow -1 < x-3 \le 1$ and $x-3 \ne 0$

$\Rightarrow 2 < x \le 4$ and $x \ne 3 \Rightarrow x \in (2, 4] - \{3\}$

Example 1.38 Solve $\log_2 \frac{x-1}{x-2} > 0$.

Sol. $\log_2 \frac{x-1}{x-2} > 0 \Rightarrow \frac{x-1}{x-2} > 2^0$

$\Rightarrow \frac{x-1}{x-2} > 1 \Rightarrow \frac{x-1}{x-2} - 1 > 0$

$\Rightarrow \frac{x-1-x+2}{x-2} > 0 \Rightarrow \frac{1}{x-2} > 0 \Rightarrow x > 2$

Example 1.39 Solve $\log_{0.5} \frac{3-x}{x+2} < 0$.

Sol. $\log_{0.5} \frac{3-x}{x+2} < 0 \Rightarrow \frac{3-x}{x+2} > (0.5)^0$

$\Rightarrow \frac{3-x}{x+2} > 1 \Rightarrow \frac{3-x}{x+2} - 1 > 0$

$\Rightarrow \frac{3-x-x-2}{x+2} > 0 \Rightarrow \frac{2x-1}{x+2} < 0$

$\Rightarrow -2 < x < 1/2$

Example 1.40 Solve $\log_3(2x^2+6x-5) > 1$.

Sol. $\log_3(2x^2+6x-5) > 1 \Rightarrow 2x^2+6x-5 > 3^1$

$\Rightarrow 2x^2+6x-8 > 0 \Rightarrow x^2+3x-4 > 0$

$\Rightarrow (x-1)(x+4) > 0 \Rightarrow x < -4$ or $x > 1$

Example 1.41 Solve $\log_{0.04}(x-1) \ge \log_{0.2}(x-1)$.

Sol. $\log_{0.04}(x-1) \ge \log_{0.2}(x-1)$

$\Rightarrow \log_{(0.2^2)}(x-1) \ge \log_{0.2}(x-1)$

$\Rightarrow \frac{1}{2}\log_{0.2}(x-1) \ge \log_{0.2}(x-1)$

$\Rightarrow \log_{0.2}(x-1) \ge 2\log_{0.2}(x-1)$

$\Rightarrow \log_{0.2}(x-1) \ge \log_{0.2}(x-1)^2$

$\Rightarrow (x-1) \le (x-1)^2$

$\Rightarrow (x-1)^2-(x-1) \ge 0$

$\Rightarrow (x-1)(x-1-1) \ge 0$

$\Rightarrow (x-1)(x-2) \ge 0$

$\Rightarrow x \le 1$ or $x \ge 2$

Also, $x > 1$; hence, $x \ge 2$.

Example 1.42 Solve $\log_{(x+3)}(x^2-x) < 1$.

Sol. $\log_{x+3}(x^2-x) < 1$

$x(x-1) > 0 \Rightarrow x > 1$ or $x < 0$ (i)

If $x+3 > 1 \Rightarrow x > -2$

then $x^2 - x < x + 3$

$\Rightarrow x^2 - 2x - 3 < 0$

$\Rightarrow (x-3)(x+1)<0$

Hence, $x \in (-1, 0) \cup (1, 3)$

If $0 < x+3 < 1 -3 < x < -2$, then (i)

$x^2 - x > x + 3$

$\Rightarrow x^2 - 2x - 3 > 0$

$\Rightarrow (x-3)(x+1) > 0$ (ii)

$\Rightarrow x \in (-3, -2)$

Example 1.43 **Solve $2 \log_3 x - 4 \log_x 27 \le 5$ $(x > 1)$.**

Sol. Let $\log_3 x = y \Rightarrow x = 3y$ (i)

Therefore, the given inequality $2 \log_3 x - 12 \log_x 3 \le 5$

$$\Rightarrow 2y - \frac{12}{y} \le 5$$

$\Rightarrow 2y^2 - 5y - 12 \le 0$ (as $x > 1 \Rightarrow y > 0$)

$\Rightarrow (2y+3)(y-4) \le 0$

$$\Rightarrow y \in \left[-\frac{3}{2}, 4\right] \Rightarrow -\frac{3}{2} \le \log_3 x \le 4$$

$\Rightarrow 3^{-3/2} \le x \le 81$

Concept Application Exercise 1.3

1. Solve $\log_3|x| > 2$.
2. Solve $\log_2 \dfrac{x-4}{2x+5} < 1$.
3. Solve $\log_{10}(x^2 - 2x - 2) \le 0$.
4. Let $f(x) = \sqrt{\log_{10} x^2}$. Find the set of all values of x for which $f(x)$ is real.
5. Solve $2^{\log_2(x-1)} > x + 5$.
6. Solve $\log_2 |4 - 5x| > 2$.
7. Solve $\log_{0.2} \dfrac{x+2}{x} \le 1$.
8. Solve $\log_{1/2} (x^2 - 6x + 12) \ge -2$.
9. Solve $\log_{1-x} (x-2) \ge -1$.
10. Solve $\log_3(x+2)(x+4) + \log_{1/3}(x+2) < \dfrac{1}{2} \log_{\sqrt{3}} 7$.
11. Solve $\log_x (x^2 - 1) \le 0$.

FINDING LOGARITHM

To calculate the logarithm of any positive number in decimal form, we always express the given positive number in decimal form as the product of an integral power of 10 and a number between 1 and 10, i.e., any positive number k in decimal form is written in the form as

$$K = m \times 10^p,$$

where p is an integer and $1 \le m < 10$. This is called the standard from of k.

Characteristic and Mantissa of a Logarithm

Let n be a positive real number and let $m \times 10^p$ be the standard fom of n. Then $n = m \times 10^p$ where p is an integer and m is a real number between 1 and 10, i.e., $1 \leq m < 10$

$$\Rightarrow \quad \log_{10} n = \log_{10}(m \times 10^p)$$
$$= \log_{10} m + \log_{10} 10^p$$
$$= \log_{10} m + p \log_{10} 10$$
$$= p + \log_{10} m$$

Here p is an integer and $1 \leq m < 10$. Now, $1 \leq m < 10$

$\Rightarrow \quad \log_{10} 1 \leq \log_{10} m < \log_{10} 10$

$\Rightarrow \quad 0 \leq \log_{10} m < 1.$

Thus, the logarithm of positive real number n consists of two parts:

(i) The integral part p, which is positive, negative or zero, is called characteristic.

(ii) The decimal part $\log m$, which is a real number between 0 and 1, is called mantissa.

Thus, $\log n$ = Characteristic + Mantissa.

Note that it is only the characteristic that changes when the decimal point is moved. An advantage of using the base 10 is thus revealed: if the characteristic is known, the decimal point may easily be placed. If the number is known, the characteristic may be determined by inspection; that is, by observing the location of the decimal point.

Although an understanding of the relation of the characteristic to the powers of 10 is necessary for thorough comprehension of logarithms, the characteristic may be determined mechanically by the application of the following rules:

1. For a number greater than 1, the characteristic is positive and is one less than the number of digits to the left of the decimal point in the number.
2. For a positive number less than 1, the characteristic is negative and has an absolute value one more than the number of zeros between the decimal point and the first non-zero digit of the number.

Example 1.44 **Write the characteristic of each of the following numbers by using their standard forms:**

(i) 1235.5 (ii) 346.41 (iii) 62.723 (iv) 7.12345

(v) 0.35792 (vi) 0.034239 (viii) 0.002385 (viii) 0.0009468

Sol.

Number	Standard Form	Characteristic
1235.5	1.2355×10^3	3
346.41	3.4641×10^2	2
62.723	6.2723×10^1	1
7.12345	7.12345×10^0	0
0.35792	3.5792×10^{-1}	–1
0.034239	3.4239×10^{-2}	–2
0.002385	2.385×10^{-3}	–3
0.0009468	9.468×10^{-4}	–4

Mantissa of the Logarithm of a Given Number

The logarithm table is used to find the mantissa of logarithms of numbers. It contains 90 rows and 20 column.

Every row beings with a two-digit number 10, 11, 12, ..., 98, 99 and every column is headed by a one-digit number 0, 1, 2, 3, ..., 9. On the right of the table, we have a big column which is divided into 9 sub-columns headed by the digit 1, 2, 3, ..., 9. This column is called the column of mean differences.

Note that the position of the decimal point in a number is immaterial for finding the mantissa. To find the mantissa of a number, we consider first four digits from the left most side of the number. If the number in the decimal form is less than one and it has four or more consecutive zeros to the right of the decimal point, then its mantissa is calculated with the help of the number formed by digits beginning with the first non-zero digit. For example, to find the mantissa of 0.000032059, we consider the number 3205. If the given number has only one digit, we replace it by a two-digit number obtained by adjoining zero to the right of the number. Thus, 2 is to be replaced by 20 for finding the mantissa.

Significant Digits

The digits used to compute the mantissa of a given number are called its significant digits.

Example 1.45 **Write the significant digits in each of the following numbers to compute the mantissa of their logarithms :**

(i) 3.239 **(ii) 8** **(iii) 0.9** **(iv) 0.02**

(v) 0.0367 **(vi) 89** **(vii) 0.0003** **(viii) 0.00075**

Sol.

Number	Significant digits to find the mantissa of its logarithm
3.239	3239
8	80
0.9	90
0.02	20
0.0367	367
89	89
0.0003	30
0.00075	75

NEGATIVE CHARACTERISTICS

When a characteristic is negative, such as -2, we do not perform the subtraction, since this would involve a negative mantissa. There are several ways of indicating a negative characteristic. Mantissas as presented in the table in the appendix are always positive and the sign of the characteristic is indicated separately. For example, where $\log 0.023 = \bar{2}.36173$, the bar over the 2 indicates that only the characteristic is negative, that is, the logarithm is $-2 + 0.36173$.

Example 1.46 **Find the mantissa of the logarithm of the number 5395.**

Sol. To find the mantissa of log 5395, we first look into the row starting with 53. In this row, look at the number in the column headed by 9. The number is 7316.

	0	1	2	3	4	5	6	7	8	9	Mean Differences								
											1	2	3	4	5	6	7	8	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7

Fig. 1.9

Now, move to the column of mean differences and look under the column headed by 5 in the row corresponding to 53. We see the number 4 there. Add this number 4 to 7316 to get 7320. This is the required mantissa of log 5395.

If we wish to find the log 5395, then we compute its characteristic also.

Clearly, the characteristic is 3. So, log 5395 = 3.7320.

Example 1.47 **Find the mantissa of the logarithm of the number 0.002359.**

Sol. The first four digits beginning with the first non-zero digit on the right of the decimal point form the number 2359. To find the mantissa of log (0.002359), we first look in the row starting with 23. In this row, look at the number in the column headed by 5. The number is 3711.

	0	1	2	3	4	5	6	7	8	9	Mean Differences								
											1	2	3	4	5	6	7	8	9
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	15	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15

Fig. 1.10

Now, move to the column of mean difference and look under the column headed by 9 the row corresponding to 23. We see the number 17 there.

Add this number to 3711. We get the number 3728. This is the required mantissa of log (0.002359).

Mantissa of log 23.598, log 2.3598 and 0.023598 is the same (only characteristic are different).

Example 1.48 **Use logarithm tables to find the logarithm of the following numbers:**

(i) 25795 **(ii) 25.795**

Sol.

(i) The characteristic of the logarithm of 25795 is 4.
To find the mantissa of the logarithm of 25795, we take the first four digits.
The number formed by the first four digits is 2579. Now, we look in the row starting with 25. In this row, look at the number in the column headed by 7. The number is 4099. Now, move to the column of mean differences and look under the column headed by 9 in the row corresponding to 25. We see that the number there is 15.
Add this number to 4099. We get the number 4114. This is the required mantissa. Hence, log (25795) = 4.4114

(ii) The characteristic of the logarithm of 25.795 is 1, because there are two digits to the left of the decimal point. The mantissa is the same as in the above question. Hence, log 25.795 = 1.4114.

Similarly, $\log 2.5795 = 0.4114$. and $\log(0.25795) = -1 + 0.4114 = \bar{1}.4114$
Here $-1 + 04114$ cannot be written as -1.4114, as -1.4114 is a negative number of magnitude 1.4114, whereas $-1 + 0.4114$ is equal to -05886. In order to avoid this confusion, we write $\bar{1}$ for -1 and thus $\log(0.25795) = \bar{1}.4114$.

ANTILOGARITHM

The positive number n is called the antilogarithm of a number m if $\log n = m$. If n is antilogarithm of m, we write $n = \text{antilog } m$. For example,

(i) $\log 100 = 2 \quad \Leftrightarrow \quad \text{antilog } 2 = 100$

(ii) $\log 431.5 = 2.6350 \quad \Leftrightarrow \quad \text{antilog}(2.6350) = 431.5$

(iii) $\log 0.1257 = \bar{1}.993 \quad \Leftrightarrow \quad \text{antilog}(\bar{1}.993) = 0.1257$

To find the antilog of a given number, we use the antilogarithm tables given at the end of the book. To find n, when $\log n$ is given, we use only the mantissa part. The characteristic is used only in determining the number of digits in the integral part or the number of zeros on the right side of the decimal point in the required number.

To Find Antilog of a Number

Step I: Determine whether the decimal part of the given number is positive or negative. If it is negative, make it positive by adding 1 to the decimal part and by subtracting 1 from the integral part.
For example, in -2.5983, the decimal part is -0.5983 which is negative. So, write

$$\begin{aligned} -2.5983 &= -2 - 0.5983 \\ &= -2 - 1 + 1 - 0.5983 \\ &= -3 + 0.4017 \\ &= \bar{3}.4017 \end{aligned}$$

Step II: In the antilogarithm table, look into the row containing the first two digits in the decimal part of the given number.

Step III: In the row obtained in step II, look at the number in the column headed by the third digit in the decimal part.

Step IV: In the row chosen in step III, move in the column of mean differences and look at the number in the column headed by the fourth digit in the decimal part. Add this number to number obtained in step III.

Step V: Obtain the integral part (Characteristic) of the given number.
If the characteristic is positive and is equal to n, then insert decimal point after $(n + 1)$ digits in the number obtained in step IV.

If $n > 4$, then write zeros on the right side to get $(n+1)$ digits.

If the characteristic is negative and is equal to $-n$ or n, then on the right side of decimal point write $(n-1)$ consecutive zeros and then write the number obtained in step IV.

Example 1.49 **Find the antilogarithm of each of the following :**

(i) 2.7523 **(ii) 3.7523** **(iii) 5.7523** **(iv) 0.7523**

(v) $\bar{1}.7523$ **(vi) $\bar{2}.7523$** **(vii) $\bar{3}.7523$**

Sol.

(i) The mantissa of 2.7523 is positive and is equal to 0.7523.
Now, look into the row starting 0.75. In this row, look at the number in the column headed by 2. The number is 5649. Now in the same row move in the column of mean differences and look at the number in the column headed by 3. The number there is 4. Add this number to 5649 to get 5653. The characteristic is 2. So, the decimal point is put after 3 digits to get 565.3.
Hence, antilog (2.7523) = 565. 3.

(ii) The mantissa of 3.7523 is the same as the mantissa of the number in (i), but the characteristic is 3.
Hence, antilog (3.7523) = 5653.0.

(iii) The mantissa of 5.7523 is the same as the mantissa of the number in (i), but the characteristic is 5.
Hence, antilog (5.7523) = 565300.0.

(iv) Proceeding as above , we have antilog (0.7523) = 5.653.

(v) In this case, the characteristic is $\bar{1}$, i.e., -1.

Hence, antilog $(\bar{1}.7523) = 0.5653$.

(vi) In this case, the characteristic is $\bar{2}$, i.e., -2. So, we write one zero on the right side of the decimal point.

Hence, antilog $(\bar{2}.7523) = 0.05653$.

(vii) Proceeding as above, antilog $(\bar{3}.7523) = 0.005653$.

Example 1.50 **Evaluate $\sqrt[3]{72.3}$, if $\log 0.723 = \bar{1}.8591$.**

Sol. Let $x = \sqrt[3]{72.3}$.

Then, $\log x = (72.3)^{1/3} \Rightarrow \log x = \frac{1}{3}\log 72.3$

$\Rightarrow \log x = \frac{1}{3} \times 1.8591 \Rightarrow \log x = 0.6197$

$\Rightarrow x = \text{antilog}\,(0.6197)$

$\Rightarrow x = 4.166$ (using antilog table)

Example 1.51 **Using logarithms, find the value of 6.45 × 981.4.**

Sol. Let $x = 6.45 \times 981.4$,

Then, $\lg x = \log(6.45 \times 981.4)$

$= \log 6.45 + \log 981.4$

$= 0.8096 + 2.9919$ (using log table)

$= 3.8015$

$\therefore x = \text{antilog}\,(3.8015) = 6331$ (using antilog table)

Example 1.52 **Let $x=(0.15)^{20}$. Find the characteristic and mantissa of the logarithm of x to the base 10. Assume $\log_{10}2=0.301$ and $\log_3 10=0.477$.**

Sol. $\log x=\log(0.15)^{20}=20\log\left(\frac{15}{100}\right)$

$$=20[\log 15-2]$$
$$=20[\log 3+\log 5-2]$$
$$=20[\log 3+1-\log 2-2]\left(\because \log_{10}5=\log_{10}\frac{10}{2}\right)$$
$$=20[-1+\log 3-\log 2]$$
$$=20[-1+0.477-0.301]$$
$$=-20\times 0.824=-16.48=\overline{17}.52$$

Hence, characteristic $=-17$ and mantissa $=0.52$.

Example 1.53 **In the 2001 census, the population of India was found to be 8.7×10^7. If the population increases at the rate of 2.5% every year, what would be the population in 2011 ?**

Sol. Here, $P_0=8.7\times 10^7$, $r=2.5$ and $n=10$.

Let P be the population in 2011.

Then, $P=P_0\left(1+\frac{r}{100}\right)^n$

$$=8.7\times 10^7\left(1+\frac{2.5}{100}\right)^{10}$$
$$=8.7\times 10^7(1.025)^{10}$$

Taking log of both sides, we get

$$\log P=\log[8.7\times 10^7(1.025)^{10}]$$
$$=\log 8.7+\log 10^7+\log(1.025)^{10}$$
$$=\log 8.7+7\log 10+10\log(1.025)$$
$$=0.9395+7+0.1070$$
$$=8.0465$$

$\Rightarrow$ $P=\text{antilog}(8.0465)=1.113\times 10^8$ (using antilog table)

Example 1.54 **Find the compound interest on ₹ 12000 for 10 years at the rate of 12% per annum compounded annually.**

Sol. We know that the amount A at the end of n years at the rate of r% per annum when the interest is compounded annually is given by

$$A=P\left(1+\frac{r}{100}\right)^n.$$

Here, $P=₹\,12000$, $r=12$ and $n=10$.

$$\therefore A=₹\left[12000\left(1+\frac{12}{100}\right)^{10}\right]$$

$$= ₹\left[12000\left(1+\frac{3}{25}\right)^{10}\right]$$

$$= ₹\left[12000\left(\frac{25+3}{25}\right)^{10}\right]$$

$$= ₹\left[12000\left(\frac{28}{25}\right)^{10}\right]$$

Now, $A = ₹\ 12000\left(\frac{28}{25}\right)^{10}$

$\Rightarrow \log A = \log 12000 + 10(\log 28 - \log 25)$

$= 4.0792 + 10(1.4472 - 1.3979)$

$= 4.0792 + 0.493 \Rightarrow \log A = 4.5722$

$\Rightarrow A = \text{antilog}(4.5722) = 37350.$

So, the amount after 10 years is ₹ 37350.

Hence, compound interest = ₹ (37350 − 12000) = ₹ 25350.

Example 1.55 **If P is the number of natural numbers whose logarithms to the base 10 have the characteristic p and Q is the number of natural numbers logarithms of whose reciprocals to the base 10 have the characteristic $-q$, then find the value of $\log_{10}P - \log_{10}Q$.**

Sol. $10^p \le P < 10^{p+1} \Rightarrow P = 10^{p+1} - 10^p \Rightarrow P = 9 \times 10^p$

Similarly, $10^{q-1} < Q \le 10^q \Rightarrow Q = 10^q - 10^{q-1} = 10^{q-1}(10-1) = 9 \times 10^{q-1}$

$\therefore \log_{10}P - \log_{10}Q = \log_{10}(P/Q) = \log_{10}10^{p-q+1} = p - q + 1$

Example 1.56 **Let L denote antilog$_{32}$ 0.6 and M denote the number of positive integers which have the characteristic 4, when the base of log is 5 and N denote the value of $49^{(1-\log_7 2)} + 5^{-\log_5 4}$. Find the value of LM/N.**

Sol. $L = \text{antilog}_{32} 0.6 = (32)^{6/10} = 2^{(5\times6)/10} = 2^3 = 8$

M = Integer from 625 to 3125 = 2500

$$N = 49^{(1-\log_7 2)} + 5^{-\log_5 4}$$

$$= 49 \times 7^{-2\log_7 2} + 5^{-\log_5 4}$$

$$= 49 \times \frac{1}{4} + \frac{1}{4} = \frac{50}{4} = \frac{25}{2}$$

$$\therefore \frac{LM}{N} = \frac{8 \times 2500 \times 2}{25} = 1600$$

EXERCISES

Subjective Type

Solutions on page 1.30

1. If $x = \log_{2a} a$, $y = \log_{3a} 2a$, $z = \log_{4a} 3a$, prove that $1 + xyz = 2yz$.
2. Solve the equations for x and y: $(3x)^{\log 3} = (4y)^{\log 4}$, $4^{\log x} = 3^{\log y}$.
3. If $a = \log_{12} 18$, $b = \log_{24} 54$, then find the value of $ab + 5(a-b)$.

4. If $y = a^{\frac{1}{1-\log_a x}}$ and $z = a^{\frac{1}{1-\log_a y}}$, then prove that $x = a^{\frac{1}{1-\log_a z}}$.
5. Solve $\log_x 2 \log_{2x} 2 = \log_{4x} 2$.
6. Let a, b, c, d be positive integers such that $\log_a b = 3/2$ and $\log_c d = 5/4$. If $(a - c) = 9$, then find the value of $(b - d)$.
7. Solve $\sqrt{\log(-x)} = \log\sqrt{x^2}$ (base is 10).
8. If $a \geq b > 1$, then find the largest possible value of the expression $\log_a(a/b) + \log_b(b/a)$.
9. Solve $3^{(\log_9 x)^2 - \frac{9}{2}\log_9 x + 5} = 3\sqrt{3}$.
10. Solve the inequality $\sqrt{\log_2\left(\frac{2x-3}{x-1}\right)} < 1$.
11. Find the number of solutions of equation $2^x + 3^x + 4^x - 5^x = 0$.
12. Solve $x^{\log_y x} = 2$ and $y^{\log_x y} = 16$.
13. Solve $\log_{2x}2 + \log_4 2x = -3/2$.
14. Solve for x: $(2x)^{\log_b 2} = (3x)^{\log_b 3}$.
15. If $\log_b a \log_c a + \log_a b \log_c b + \log_a c \log_b c = 3$ (where a, b, c are different positive real numbers $\neq 1$), then find the value of abc.
16. If $\frac{\log_a N}{\log_c N} = \frac{\log_a N - \log_b N}{\log_b N - \log_c N}$, where $N > 0$ and $N \neq 1$, $a, b, c > 0$ and not equal to 1, then prove that $b^2 = ac$.
17. Given a and b are positive numbers satisfying $4(\log_{10} a)^2 + (\log_2 b)^2 = 1$, then find the range of values of a and b.

Objective Type

Solutions on page 1.35

1. $\log_4 18$ is

   a. a rational number **b.** an irrational number **c.** a prime number **d.** none of these

2. If $\log_4 5 = a$ and $\log_5 6 = b$, then $\log_3 2$ is equal to

   a. $\frac{1}{2a+1}$ **b.** $\frac{1}{2b+1}$ **c.** $2ab+1$ **d.** $\frac{1}{2ab-1}$

3. The value of x satisfying $\sqrt{3}^{-4+2\log_{\sqrt{5}} x} = 1/9$ is

   a. 2 **b.** 3 **c.** 4 **d.** none of these

4. $x^{\log_5 x} > 5$ implies

   a. $x \in (0, \infty)$ **b.** $x \in (0, 1/5) \cup (5, \infty)$ **c.** $x \in (1, \infty)$ **d.** $x \in (1, 2)$

5. The number $N = 6\log_{10} 2 + \log_{10} 31$ lies between two successive integers whose sum is equal to

   a. 5 **b.** 7 **c.** 9 **d.** 10

6. The value of $49^{(1-\log_7 2)} + 5^{-\log_5 4}$ is

   a. 27/2 **b.** 25/2 **c.** 625/16 **d.** none of these

7. If $\sqrt{\log_2 x} - 0.5 = \log_2 \sqrt{x}$, then x equals

a. odd integer   b. prime number   c. composite number   d. irrational

8. If $\log_y x + \log_x y = 1$, $x^2 + y = 12$, then the value of xy is

a. 9   b. 12   c. 15   d. 21

9. If $\log_a 3 = 2$ and $\log_b 8 = 3$, then $\log_a b$ is

a. $\log_3 2$ b. $\log_2 3$ c. $\log_3 4$ d. $\log_4 3$

10. If $(4)^{\log_9 3} + (9)^{\log_2 4} = (10)^{\log_x 83}$, then x is equal to

a. 2   b. 3   c. 10   d. 30

11. The value of $\log ab - \log|b| =$

a. $\log a$ b. $\log |a|$ c. $-\log a$ d. none of these

12. If $(x+1)^{\log_{10}(x+1)} = 100(x+1)$, then

a. all the roots are positive real numbers.   b. all the roots lie in the interval (0, 100)

c. all the roots lie in the interval [−1, 99]   d. none of these

13. If a, b, c are distinct positive numbers different from 1 such that $(\log_b a \log_c a - \log_a a) + (\log_a b \log_c b - \log_b b) + (\log_a c \log_b c - \log_c c) = 0$, then abc is

a. 0   b. e   c. 1   d. none of these

14. Given that log (2) = 0.3010..., the number of digits in the number 2000^{2000} is

a. 6601   b. 6602   c. 6603   d. 6604

15. The value of $\dfrac{\log_2 24}{\log_{96} 2} - \dfrac{\log_2 192}{\log_{12} 2}$ is

a. 3   b. 0   c. 2   d. 1

16. The set of all values of x satisfying $x^{\log_x (1-x)^2} = 9$ is

a. a subset of R containing N b. a subset of R containing Z (set of all integers)

c. is a finite set containing at least two elements   d. a finite set

17. If $\ln\left(\dfrac{a+b}{3}\right) = \left(\dfrac{\ln a + \ln b}{2}\right)$, then $\dfrac{a}{b} + \dfrac{b}{a}$ is equal to

a. 1   b. 3   c. 5   d. 7

18. The value of b for which the equation $2 \log_{1/25}(bx + 28) = -\log_5 (12 - 4x - x^2)$ has coincident roots if

a. $b = -12$ b. $b = 4$ c. $b = 4$ or $b = -12$ d. $b = -4$ or $b = 12$

19. If $S = \{x \in N : 2 + \log_2 \sqrt{x+1} > 1 - \log_{1/2} \sqrt{4 - x^2}\}$, then

a. $S = \{1\}$ b. $S = Z$ c. $S = N$ d. none of these

20. If $a^4 \cdot b^5 = 1$, then the value of $\log_a(a^5 b^4)$ equals

a. 9/5   b. 4   c. 5   d. 8/5

21. If the equation $2^x + 4^y = 2^y + 4^x$ is solved for y in terms of x, where $x < 0$, then the sum of the solutions is

a. $x \log_2(1 - 2^x)$ b. $x + \log_2(1 - 2^x)$ c. $\log_2(1 - 2^x)$ d. $x \log_2(2^x + 1)$

22. The minimum value of the expression $2\log_{10}x - \log_x 0.01$, where $x > 1$, is

a. 2 **b.** 0.1 **c.** 4 **d.** 1

23. The value of $3^{\log_4 5} - 5^{\log_4 3}$ is

a. 0 **b.** 1 **c.** 2 **d.** none of these

24. If a, b, c are consecutive positive integers and $\log(1 + ac) = 2K$, then the value of K is

a. $\log b$ **b.** $\log a$ **c.** 2 **d.** 1

25. If $\log_2 x + \log_x 2 = \frac{10}{3} = \log_2 y + \log_y 2$ and $x \neq y$ then $x + y =$

a. 2 **b.** 65/8 **c.** 37/6 **d.** none of these

26. If $\log_{10}\left[\frac{1}{2^x + x - 1}\right] = x[\log_{10} 5 - 1]$, then $x =$

a. 4 **b.** 3 **c.** 2 **d.** 1

27. If $S = \{x \in R : (\log_{0.6} 0.216)\log_5(5 - 2x) \le 0\}$, then S is equal to

a. $[2.5, \infty)$ **b.** $[2, 2.5)$ **c.** $(2, 2.5)$ **d.** $(0, 2.5)$

28. Solution set of the inequality $\log_3(x+2)(x+4) + \log_{1/3}(x+2) < (1/2)\log_{\sqrt{3}} 7$ is

a. $(-2, -1)$ **b.** $(-2, 3)$ **c.** $(-1, 3)$ **d.** $(3, \infty)$

29. If $\log_3\{5 + 4\log_3(x-1)\} = 2$, then x is equal to

a. 2 **b.** 4 **c.** 8 **d.** $\log_2 16$

30. If $2x^{\log_4 3} + 3^{\log_4 x} = 27$, then x is equal to

a. 2 **b.** 4 **c.** 8 **d.** 16

31. Equation $\log_4(3 - x) + \log_{0.25}(3 + x) = \log_4(1 - x) + \log_{0.25}(2x + 1)$ has

a. only one prime solution **b.** two real solutions

c. no real solution **d.** none of these

32. The value of $\frac{1 + 2\log_3 2}{(1 + \log_3 2)^2} + (\log_6 2)^2$ is

a. 2 **b.** 3 **c.** 4 **d.** 1

33. Product of roots of the equation $\frac{\log_8(8/x^2)}{(\log_8 x)^2} = 3$ is

a. 1 **b.** 1/2 **c.** 1/3 **d.** 1/4

34. Let $a > 1$ be a real number. Then the number of roots equation $a^{2\log_2 x} = 5 + 4x^{\log_2 a}$ has

a. 2 **b.** infinite **c.** 0 **d.** 1

35. If $(21.4)^a = (0.00214)^b = 100$, then the value of $\frac{1}{a} - \frac{1}{b}$ is

a. 0 **b.** 1 **c.** 2 **d.** 4

36. The solution set of the inequality $\log_{10}(x^2-16) \le \log_{10}(4x-11)$ is
a. $(4, \infty)$ b. $(4, 5]$ c. $(11/4, \infty)$ d. $(11/4, 5)$

37. The number of roots of the equation $\log_{3\sqrt{x}} x + \log_{3x} \sqrt{x} = 0$ is
a. 1 b. 2 c. 3 d. 0

38. The set of all x satisfying the equation $x^{\log_3 x^2 + (\log_3 x)^2 - 10} = 1/x^2$ is
a. $\{1, 9\}$ b. $\{1, 9, 1/81\}$ c. $\{1, 4, 1/81\}$ d. $\{9, 1/81\}$

39. If $xy^2 = 4$ and $\log_3(\log_2 x) + \log_{1/3}(\log_{1/2} y) = 1$, then x equals
a. 4 b. 8 c. 16 d. 64

40. If $2^{x+y} = 6^y$ and $3^{x-1} = 2^{y+1}$, then the value of $(\log 3 - \log 2)/(x-y)$ is
a. 1 b. $\log_2 3 - \log_3 2$ c. $\log(3/2)$ d. none of these

41. If $\log_2 x + \log_2 y \ge 6$, then the least value of $x + y$ is
a. 4 b. 8 c. 16 d. 32

42. If $f(x) = \log\left(\frac{1+x}{1-x}\right)$, then
a. $f(x_1) . f(x_2) = f(x_1 + x_2)$ b. $f(x+2) - 2f(x+1) + f(x) = 0$
c. $f(x) + f(x+1) = f(x^2 + x)$ d. $f(x_1) + f(x_2) = f\left(\frac{x_1 + x_2}{1 + x_1 x_2}\right)$

43. Which of the following is not the solution of $\log_x\left(\frac{5}{2} - \frac{1}{x}\right) > \left(\frac{5}{2} - \frac{1}{x}\right)$?
a. $\left(\frac{2}{5}, \frac{1}{2}\right)$ b. $(1, 2)$ c. $\left(\frac{2}{5}, \frac{3}{4}\right)$ d. none of these

44. If x_1 and x_2 are the roots of the equation $e^2 \cdot x^{\ln x} = x^3$ with $x_1 > x_2$, then
a. $x_1 = 2x_2$ b. $x_1 = x_2^2$ c. $2x_1 = x_2^2$ d. $x_1^2 = x_2^3$

45. The number of real values of the parameter k for which $(\log_{16} x)^2 - \log_{16} x + \log_{16} k = 0$ with real coefficients will have exactly one solution is
a. 2 b. 1 c. 4 d. none of these

Multiple Correct Answers Type

Solutions on page 1.41

Each question has four choices a, b, c, and d, out of which one or more answers are correct.

1. For $a > 0, \ne 1$, the roots of the equation $\log_{ax} a + \log_x a^2 + \log_{a^2 x} a^3 = 0$ are given by
a. $a^{-4/3}$ b. $a^{-3/4}$ c. a d. $a^{-1/2}$

2. The real solutions of the equation $2^{x+2} . 5^{6-x} = 10^{x^2}$ is/are
a. 1 b. 2 c. $-\log_{10}(250)$ d. $\log_{10} 4 - 3$

3. If $\frac{\log x}{b-c} = \frac{\log y}{c-a} = \frac{\log z}{a-b}$, then which of the following is/are true?
a. $xyz = 1$ b. $x^a y^b z^c = 1$ c. $x^{b+c} y^{c+a} z^{a+b} = 1$ d. $xyz = x^a y^b z^c$

4. If $\log_k x \cdot \log_5 k = \log_x 5$, $k \neq 1$, $k > 0$, then x is equal to

a. k **b.** $1/5$ **c.** 5 **d.** none of these

5. If $p, q \in N$ satisfy the equation $x^{\sqrt{x}} = \left(\sqrt{x}\right)^x$; then p and q are

a. relatively prime **b.** twin prime
c. coprime **d.** if $\log_q p$ is defined, then $\log_p q$ is not and vice versa

6. Which of the following, when simplified, reduces to unity?

a. $\log_{10}5 \cdot \log_{10}20 + (\log_{10}2)^2$ **b.** $\dfrac{2\log 2 + \log 3}{\log 48 - \log 4}$

c. $-\log_5 \log_3 \sqrt{\sqrt[5]{9}}$ **d.** $\dfrac{1}{6} \log_{\frac{\sqrt{3}}{2}} \left(\dfrac{64}{27}\right)$

7. If $\log a\, x = b$ for permissible values of a and x, then identify the statement (s) which can be correct.

a. If a and b are two irrational numbers, then x can be rational.
b. If a is rational and b is irrational, then x can be rational.
c. If a is irrational and b is rational, then x can be rational.
d. If a is rational and b is rational, then x can be rational.

8. The equation $\log_{x+1}(x - 0.5) = \log_{x-0.5}(x + 1)$ has

a. two real solutions **b.** no prime solution **c.** one integral solution **d.** no irrational solution

9. The equation $\sqrt{1 + \log_x \sqrt{27}}\, \log_3 x + 1 = 0$ has

a. no integral solution **b.** one irrational solution **c.** two real solutions **d.** no prime solution

10. If $\log_{1/2}(4 - x) \geq \log_{1/2}2 - \log_{1/2}(x - 1)$, then x belongs to

a. $(1, 2]$ **b.** $[3, 4)$ **c.** $(1, 3]$ **d.** $[1, 4)$

11. If the equation $x^{\log_a x^2} = \dfrac{x^{k-2}}{a^k}$, $a \neq 0$, has exactly one solution for x, then the value of k is/are

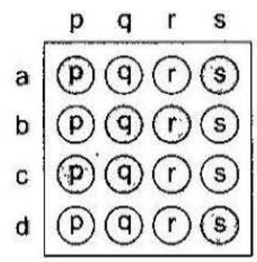
a. $6 + 4\sqrt{2}$ **b.** $2 + 6\sqrt{3}$ **c.** $6 - 4\sqrt{2}$ **d.** $2 - 6\sqrt{3}$

Matrix-Match Type

Solutions on page 1.44

Each question contains statements given in two columns which have to be matched.

Statements (a, b, c, d) in column I have to be matched with statements (p, q, r, s) in column II. If the correct matches are a-p, a-s, b-q, b-r, c-p, c-q and d-s, then the correctly bubbled 4 × 4 matrix should be as follows:



1.

Column I	Column II
a. The smallest integer greater than $\frac{1}{\log_3 \pi} + \frac{1}{\log_4 \pi}$ is	p. 10
b. Let $3^a = 4, 4^b = 5, 5^c = 6, 6^d = 7, 7^e = 8$ and $8^f = 9$. Then the value of the product ($abcdef$) is	q. 3
c. Characteristic of the logarithm of 2008 to the base 2 is	r. 1
d. If $\log_2(\log_2(\log_3 x)) = \log_2(\log_3(\log_2 y)) = 0$, then the value of $(x - y)$ is	s. 2

2.

Column I	Column II
a. The value of $\log_2 \log_2 \log_4 256 + 2\log \sqrt{2}^2$ is	p. 1
b. If $\log_3(5x-2) - 2\log_3\sqrt{3x+1} = 1 - \log_3 4$, then $x =$	q. 6
c. Product of roots of the equation $7^{\log_7(x^2-4x+5)} = (x-1)$ is	r. 3
d. Number of integers satisfying $\log_2 \sqrt{x} - 2(\log_{1/4} x)^2 x + 1 > 0$ are	s. 5

3.

Column I	Column II
a. $2^{\log_{(2\sqrt{2})} 15}$ is	p. rational
b. $\sqrt[3]{\left(5^{1/\log_7 5} + \frac{1}{\sqrt{(-\log_{10} 0.1)}}\right)}$ is	q. irrational
c. $\log_3 5 \cdot \log_{25} 27$ is	r. composite
d. Product of roots of equation $x^{\log_{10} x} = 100x$ is	s. prime

Integer Type

Solutions on page 1.45

1. If $\log_a b = 2$; $\log_b c = 2$ and $\log_3 c = 3 + \log_3 a$, then the value of $c/(ab)$ is ________.
2. The value of $(\log_{10}2)^3 + \log_{10}8 \cdot \log_{10}5 + (\log_{10}5)^3$ is ________.
3. If $\log_4 A = \log_6 B = \log_9(A + B)$, then $\left[4\frac{B}{A}\right]$ (where $[\cdot]$ represents the greatest integer function) equals ________.
4. Integral value of x which satisfies the equation $\log_6 54 + \log_x 16 = \log_{\sqrt{2}} x - \log_{36}\frac{4}{9}$ is ________.
5. If $a = \log_{245}175$ and $b = \log_{1715}875$, then the value of $\frac{1-ab}{a-b}$ is ________.
6. The difference of roots of the equation $(\log_{27}x^3)^2 = \log_{27}x^6$ is ________.
7. Sum of all integral values of x satisfying the inequality $(3^{5/2\log_3(12-3x)}) - (3^{\log_2 x}) > 32$ is ________.
8. The least integer greater than $\log_2 15 \cdot \log_{1/6}2 \cdot \log_3 1/6$ is ________.
9. The reciprocal of $\frac{2}{\log_4(2000)^6} + \frac{3}{\log_5(2000)^6}$ is ________.
10. Sum of integers satisfying $\sqrt{\log_2 x - 1} - 1/2\log_2(x^3) + 2 > 0$ is ________.
11. Number of integers satisfying the inequality $\log_{1/2}|x-3| > -1$ is ________.
12. Number of integers ≤ 10 satisfying the inequality $2\log_{1/2}(x-1) \le \frac{1}{3} - \frac{1}{\log_{x^2-x}8}$ is ________.
13. The value of $\log_{\left(\sqrt{3+2\sqrt{2}}+\sqrt{3-2\sqrt{2}}\right)} 2^9$ is ________.

ANSWERS AND SOLUTIONS

Subjective Type

1. $1 + xyz = 1 + (\log_{2a} a)(\log_{3a} 2a)(\log_{4a} 3a)$

$$= 1 + \frac{\log a}{\log 2a}\frac{\log 2a}{\log 3a}\frac{\log 3a}{\log 4a}$$

$$= 1 + \frac{\log a}{\log 4a}$$

$= \log_{4a}4a + \log_{4a}a$

$= \log_{4a}4a^2$

$= 2\log_{4a}2a$

$= 2(\log_{3a}2a)(\log_{4a}3a) = 2yz$

2. $(3x)^{\log 3} = (4y)^{\log 4}$, $4^{\log x} = 3^{\log y}$

$\Rightarrow (\log 3)(\log 3x) = (\log 4)(\log 4y)$ and $(\log x)(\log 4) = (\log y)(\log 3)$

$\Rightarrow (\log 3)(\log 3 + \log x) = (\log 4)(\log 4 + \log y)$ and $(\log x)(\log 4) = (\log y)(\log 3)$

$\Rightarrow (\log 3)(\log 3 + p) = (\log 4)(\log 4 + q)$ and $p(\log 4) = q(\log 3)$ (where $p = \log x$ and $q = \log y$)

$\Rightarrow (\log 3)\left(\log 3 + \frac{q \log 3}{\log 4}\right) = \log 4(\log 4 + q)$ (eliminating p)

$\Rightarrow (\log 3)^2 - (\log 4)^2 = \frac{(\log 4)^2 - (\log 3)^2}{\log 4} q$

$\Rightarrow q = -\log 4 \Rightarrow \log y = \log 4^{-1} \Rightarrow y = 1/4$

Now $p(\log 4) = q(\log 3)$

$\Rightarrow p(\log 4) = -(\log 4)(\log 3)$

$\Rightarrow p = -\log 3$

$\Rightarrow \log x = \log 3^{-1}$

$\Rightarrow x = 1/3$

3. We have $a = \log_{12} 18 = \frac{\log_2 18}{\log_2 12} = \frac{1 + 2\log_2 3}{2 + \log_2 3}$ and $b = \log_{24} 54 = \frac{\log_2 54}{\log_2 24} = \frac{1 + 3\log_2 3}{3 + \log_2 3}$

Putting $x = \log_2 3$, we have

$$ab + 5(a - b) = \frac{1 + 2x}{2 + x} \cdot \frac{1 + 3x}{3 + x} + 5\left(\frac{1 + 2x}{2 + x} - \frac{1 + 3x}{3 + x}\right)$$

$$= \frac{6x^2 + 5x + 1 + 5(-x^2 + 1)}{(x + 2)(x + 3)}$$

$$= \frac{x^2 + 5x + 6}{(x + 2)(x + 3)} = 1$$

4. $\log_a y = \frac{1}{1 - \log_a x}$, therefore, $1 - \log_a y = 1 - \frac{1}{1 - \log_a x} = \frac{-\log_a x}{1 - \log_a x}$

or $\frac{1}{1 - \log_a y} = \frac{1 - \log_a x}{-\log_a x}$ (i)

But $z = a^{\frac{1}{1 - \log_a y}} \Rightarrow \log_a z = \frac{1}{1 - \log_a y} = -\frac{1}{\log_a x} + 1$

$\Rightarrow \frac{1}{\log_a x} = 1 - \log_a z \Rightarrow \log_a x = \frac{1}{1 - \log_a z}$

$\Rightarrow x = a^{\frac{1}{1 - \log_a z}}$

5. Since $\log_x 2 \log_{2x} 2 = \log_{4x} 2$, we have $x > 0$, $2x > 0$ and $4x > 0$ and $x \neq 1$, $2x \neq 1$, $4x \neq 1$

$\Rightarrow x > 0$ and $x \neq 1, \frac{1}{2}, \frac{1}{4}$

Then, $\frac{1}{\log_2 x} \cdot \frac{1}{\log_2 2x} = \frac{1}{\log_2 4x}$

$\Rightarrow \log_2 x \cdot \log_2 2x = \log_2 4x$

$\Rightarrow \log_2 x(1 + \log_2 x) = (2 + \log_2 x)$

$\Rightarrow (\log_2 x)^2 = 2 \Rightarrow \log_2 x = \pm\sqrt{2}$

$\therefore x = 2^{\pm\sqrt{2}}$, i.e., $\{2^{-\sqrt{2}}, 2^{\sqrt{2}}\}$

6. $b = a^{3/2}$ and $d = c^{5/4}$

Let $a = x^2$ and $c = y^4$, $x, y \in N$

$\Rightarrow b = x^3$; $d = y^5$

Given $a - c = 9$, then $x^2 - y^4 = 9$

$\Rightarrow (x - y^2)(x + y^2) = 9$. Hence, $x - y^2 = 1$ and $x + y^2 = 9$.

(No other combination in the set of + ve integers will be possible.)

$x = 5$ and $y = 2$. Therefore, $b - d = x^3 - y^5 = 125 - 32 = 93$.

7. Since the equation can be satisfied only for $x < 0$, hence $\sqrt{x^2} = |x| = -x$. That is,

$\sqrt{\log(-x)} = \log(-x) \Rightarrow \log(-x) = [\log(-x)]^2$

$\Rightarrow \log(-x)[1 - \log(-x)] = 0$

if $\log(-x) = 0 \Rightarrow -x = 1 \Rightarrow x = -1$

if $\log_{10}(-x) = 1 \Rightarrow -x = 10 \Rightarrow x = -10$

8. Let $x = \log_a(a/b) + \log_b(b/a) = \log_a a - \log_a b + \log_b b - \log_b a = 2 - (\log_b a + \log_a b)$

$$= -\left(\sqrt{\log_b a} - \sqrt{\log_a b}\right)^2 \le 0$$

Hence, the maximum value is 0.

9. We have $(\log_9 x)^2 - \dfrac{9}{2}\log_9 x + 5 = \dfrac{3}{2}$ (taking log on both sides to the base 3)

Putting $\log_9 x = y$, we have $y^2 - \dfrac{9}{2}y + 5 = \dfrac{3}{2}$

$\Rightarrow 2y^2 - 9y + 7 = 0$, i.e., $(2y - 7)(y - 1) = 0$

$\Rightarrow y = 7/2, 1$

Therefore, either $\log_9 x = 1$ or $\log_9 x = 7/2$

i.e., either $x = 9$ or $x = 9^{7/2} = 3^7$

10. Inequality is true if

$$0 \le \log_2\left(\frac{2x-3}{x-1}\right) < 1 \Rightarrow 1 \le \frac{2x-3}{x-1} < 2$$

Now $\dfrac{2x-3}{x-1} - 2 < 0 \Rightarrow \dfrac{2x-3-2x+2}{x-1} < 0$

$$\Rightarrow \frac{-1}{x-1} < 0 \Rightarrow \frac{1}{x-1} > 0 \Rightarrow x > 1 \quad \text{(i)}$$

and $\frac{2x-3}{x-1} \geq 1 \Rightarrow \frac{2x-3}{x-1} - 1 \geq 0$

$\Rightarrow \frac{2x-3-x+1}{x-1} \geq 0 \Rightarrow \frac{x-2}{x-1} \geq 0 \Rightarrow x \geq 2$ or $x < 1$ (ii)

Taking intersection of Eqs. (i) and (ii), we have $x \geq 2$.

11. $2^x + 3^x + 4^x - 5^x = 0 \Rightarrow 2^x + 3^x + 4^x = 5^x$

$$\Rightarrow \left(\frac{2}{5}\right)^x + \left(\frac{3}{5}\right)^x + \left(\frac{4}{5}\right)^x = 1$$

Now the number of solutions of the equation is equal to number of times.

$y = \left(\frac{2}{5}\right)^x + \left(\frac{3}{5}\right)^x + \left(\frac{4}{5}\right)^x$ and $y = 1$ intersect.

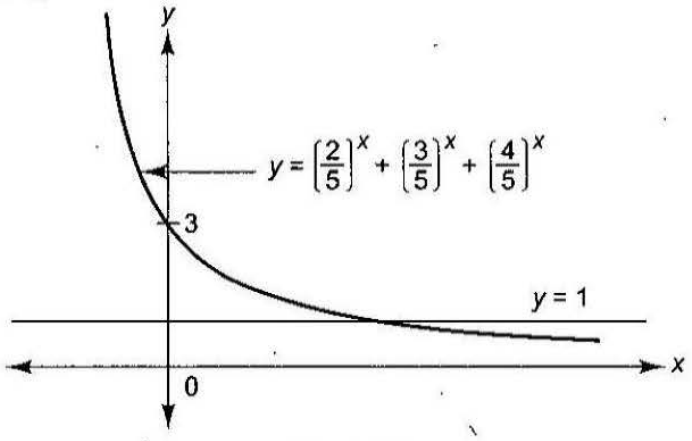


Fig. 1.11

From the graph, equation has only one solution.

12. Let $\log_y x = t \Rightarrow x = y^t$ (i)

Also $x^t = 2$ and $y^{1/t} = 2^4$

$\Rightarrow x = 2^{1/t}$ (ii)

$\Rightarrow y = 2^{4t}$ (iii)

Putting the values of x and y in Eq. (i), we have

$2^{1/t} = 2^{4t^2} \Rightarrow 4t^3 = 1$

$\therefore t = \left(\frac{1}{4}\right)^{1/3}$ (iv)

Using Eq. (iv) in Eq. (ii), we get $x = (2)^{(4)^{1/3}} = 2^{\sqrt[3]{4}}$.

Using Eq. (iv) in Eq. (iii), we get $y = (2)^{(4)^{4/3}}$.

13. Given equation is

$$\frac{1}{1+\log_2 x} + \frac{\log_2 2x}{2} = -\frac{3}{2}$$

$$\Rightarrow \frac{1}{1+\log_2 x} + \frac{1+\log_2 x}{2} = -\frac{3}{2}$$

Let $1 + \log_2 x = y \Rightarrow \frac{1}{y} + \frac{y}{2} = -\frac{3}{2}$

$\Rightarrow 2 + y^2 + 3y = 0 \Rightarrow y = -1$ or -2
$\Rightarrow 1 + \log_2 x = -1$ or -2
$\Rightarrow \log_2 x = -2$ or -3
$\Rightarrow x = 2^{-2}$ or 2^{-3}

14. $(2x)^{\log_b 2} = (3x)^{\log_b 3}$

$\Rightarrow \log_b 2[\log 2 + \log x] = \log_b 3[\log 3 + \log x]$

$\Rightarrow (\log_b 2)(\log 2) - \log_b 3 \cdot \log 3 = (\log_b 3 - \log_b 2) \log x$

$$\Rightarrow \frac{\log 2}{\log b}\cdot \log 2 - \frac{\log 3}{\log b}\cdot \log 3 = \left(\frac{\log 3}{\log b} - \frac{\log 2}{\log b}\right)\log x$$

$$\Rightarrow \frac{(\log 2)^2 - (\log 3)^2}{\log b} = \left(\frac{\log 3 - \log 2}{\log b}\right)\log x$$

$\Rightarrow \log x = -(\log 3 + \log 2) = \log (6)^{-1}$
$\Rightarrow x = 1/6$

15. $\log_b a \,.\, \log_c a + \log_a b \,.\, \log_c b + \log_a c \,.\, \log_b c = 3$

$$\Rightarrow \frac{\log a}{\log b}\frac{\log a}{\log c} + \frac{\log b}{\log a}\frac{\log b}{\log c} + \frac{\log c}{\log a}\frac{\log c}{\log b} = 3$$

$\Rightarrow (\log a)^3 + (\log b)^3 + (\log c)^3 = 3(\log a)(\log b)(\log c)$

$\Rightarrow \log a + \log b + \log c = 0$ (as a, b, c are different)

$\Rightarrow \log abc = 0 \Rightarrow abc = 1$

16. $$\frac{\log_a N}{\log_c N} = \frac{\log_a N - \log_b N}{\log_b N - \log_c N}$$

$$\Rightarrow \frac{\log_N c}{\log_N a} = \frac{\frac{1}{\log_N a} - \frac{1}{\log_N b}}{\frac{1}{\log_N b} - \frac{1}{\log_N c}}$$

$$\Rightarrow \frac{\log_N c}{\log_N a} = \frac{\log_N c}{\log_N a} \times \frac{\log_N b - \log_N a}{\log_N c - \log_N b}$$

$$\Rightarrow \frac{\log_N b - \log_N a}{\log_N c - \log_N b} = 1$$

$\Rightarrow \log_N b - \log_N a = \log_N c - \log_N b$
$\Rightarrow b/a = c/b$
$\Rightarrow b^2 = ac$

17. $(\log_2 b)^2 = 1 - (2 \log_{10} a)^2 \geq 0$
$\Rightarrow (2 \log_{10} a)^2 - 1 \leq 0$
$\Rightarrow (2 \log_{10} a - 1)(2 \log_{10} a + 1) \leq 0$

$\Rightarrow \log_{10}a \in \left[-\frac{1}{2}, \frac{1}{2}\right]; a \in \left[\frac{1}{\sqrt{10}}, \sqrt{10}\right]$

Similarly, $(\log_{10}a)^2 = \frac{1-(\log_{10}b)^2}{4} \geq 0 \Rightarrow (\log_{10}b)^2 - 1 \leq 0$

$\Rightarrow (\log_{10}b - 1)(\log_{10}b + 1) \leq 0$

$\Rightarrow \log_{10}b \in [-1, 1] \Rightarrow b \in \left[\frac{1}{10}, 10\right]$

Objective Type

1. b. Let $\log_4 18 = p/q$, where $p, q \in I$

$\Rightarrow \log_4 9 + \log_4 2 = \frac{p}{q} \Rightarrow \frac{1}{2} \times 2\log_2 3 + \frac{1}{2} = \frac{p}{q} \Rightarrow \log_2 3 = \frac{p}{q} - \frac{1}{2} = \frac{m}{n}$ (say)

where $m, n \in I$ and $n \neq 0 \Rightarrow 3 = (2)^{m/n} \Rightarrow 3^n = 2^m$ (possible only when $m = n = 0$ which is not true)
Hence, $\log_4 18$ is an irrational number.

2. d. Here, $5 = 4^a$ and $6 = 5^b$.
Let $\log_3 2 = x$, then $2 = 3^x$.
Now, $6 = 5^b = (4^a)^b = 4^{ab}$ or $3 = 2^{2ab-1}$
Therefore, $2 = (2^{2ab-1})^x = 2^{x(2ab-1)} \Rightarrow x(2ab-1) = 1$.

3. d. $3^{-2}3^{\log_{\sqrt{3}} x} = 3^{-2} \Rightarrow 3^{\log_{\sqrt{3}} x} = 1 \Rightarrow \log_{\sqrt{5}} x = 0 \Rightarrow x = 1$

4. b. Taking logarithm with base 5, we have

$x^{\log_5 x} > 5 \Rightarrow (\log_5 x)(\log_5 x) > 1 \Rightarrow (\log_5 x - 1)(\log_5 x + 1) > 0 \Rightarrow \log_5 x > 1$ or $\log_5 x < -1 \Rightarrow x > 5$ or $x < 1/5$
Also we must have $x > 0$. Thus, $x \in (0, 1/5) \cup (5, \infty)$.

5. b. $N = \log_{10} 64 + \log_{10} 31 = \log_{10} 1984$. Therefore, $3 < N < 4 \Rightarrow 7$.

6. b. $49^{(1-\log_7 2)} + 5^{-\log_5 4} = 49 \times 7^{-2\log_7 2} + 5^{-\log_5 4} = 49 \times \frac{1}{4} + \frac{1}{4} = \frac{50}{4} = \frac{25}{2}$

7. b. $\sqrt{\log_2 x} - 0.5 = \log_2 \sqrt{x}$

$\Rightarrow \sqrt{\log_2 x} - 0.5 = 0.5 \log_2 x \Rightarrow y - 0.5 = 0.5y^2 \Rightarrow y^2 - 2y + 1 = 0 \Rightarrow y = 1 \Rightarrow \log_2 x = 1 \Rightarrow x = 2$

8. a. Let $t = \log_y x (x, y > 0$, and $\neq 1)$, then $t + \frac{1}{t} = 2$ or $(t-1)^2 = 0$

$\therefore t = \log_y x = 1$, i.e., $x = y$. We get $x^2 + x - 12 = 0$ $x = -4, 3$.
$x = 3$ only (-4 rejected)

9. c. $\log_b 8 = 3 \Rightarrow 3 \log_b 2 = 3 \Rightarrow \log_b 2 = 1$
$\log_a b = \log_2 b . \log_a 2 = \log_2 b. \log_3 2. \log_a 3 = 1.\log_3 2.2 = 2 \log_3 2 = \log_3 4$

10. c. $(4)^{\log_9 3} + (9)^{\log_2 4} = (10)^{\log_x 83}$

$\Rightarrow (4)^{\frac{1}{2}\log_3 3} + (9)^{2\log_2 2} = (10)^{\log_x 83} \Rightarrow 2 + 81 = (10)^{\log_x 83} \Rightarrow 83 = (10)^{\log_x 83} \Rightarrow x = 10$

11. b. log ab is defined if $ab > 0$ or a and b have the same sign.

Case (i): $a, b > 0$

$\Rightarrow \log ab - \log|b| = \log a + \log b - \log b = \log a$ (i)

Case (ii): $a. b < 0$

$\Rightarrow \log ab - \log|b| = \log(-a) + \log(-b) - \log(-b) = \log(-a)$ (ii)

From Eqs. (i) and (ii), we have $\log ab - \log|b| = \log|a|$.

12. c. $(x+1)^{\log_{10}(x+1)} = 100(x+1) \Rightarrow \log_{10}(x+1)^{\log_{10}(x+1)} = \log_{10}(100(x+1))$

$\log_{10}(x+1)\log_{10}(x+1) = 2 + \log_{10}(x+1)$

Let $\log_{10}(x+1) = y$

$\Rightarrow y^2 - y - 2 = 0 \Rightarrow y = 2 \text{ or } -1 \Rightarrow \log_{10}(x+1) = 2, -1 \Rightarrow x + 1 = 100, 1/10 \Rightarrow x = 99 \text{ or } -9/10$

13. c. $(\log_b a \log_c a - 1) + (\log_a b . \log_c b - 1) + (\log_a c \log_b c - 1) = 0$

$$\Rightarrow \frac{\log a}{\log b}\cdot\frac{\log a}{\log c} + \frac{\log b}{\log a}\cdot\frac{\log b}{\log c} + \frac{\log c}{\log a}\cdot\frac{\log c}{\log b} = 3$$

$\Rightarrow (\log a)^3 + (\log b)^3 + (\log c)^3 = 3 \log a \log b \log c$

$\Rightarrow (\log a + \log b + \log c) = 0$ [$\because$ If $a^3 + b^3 + c^3 - 3abc = 0$, then $a + b + c = 0$ if $a \neq b \neq c$]

$\Rightarrow \log abc = \log 0 \Rightarrow abc = 1$

14. c. Let $x = 2000^{2000}$

$\log x = 2000 \log_{10}(2000) = 2000(\log_{10}2 + 3) = 2000(3.3010) = 6602$

Therefore, the number of digits is 6603.

15. a. Let $\log_2 12 = a$, then $\dfrac{1}{\log_{96} 2} = \log_2 96 = \log_2 2^3 \times 12 = 3 + a$

$\log_2 24 = 1 + a \Rightarrow \log_2 192 = \log_2(16 \times 12) = 4 + a$ and $\dfrac{1}{\log_{12} 2} = \log_2 12 = a.$

Therefore, the given expression $= (1 + a)(3 + a) - (4 + a)a = 3.$

16. d. Taking logarithm of both the sides with base 3, we have

$$\log_x (1-x)^2 \log_3 x = 2 \Rightarrow \frac{\log_3(1-x)^2}{\log_3 x}\log_3 x = 2$$

$\Rightarrow \log_3(1-x)^2 = 2 \Rightarrow (1-x)^2 = 9$ (clearly $x \neq 3$)

$\Rightarrow x = 4, -2$. But $x > 0$, hence the solution set is $\{4\}$.

17. d. $$\ln\left(\frac{a+b}{3}\right) = \frac{\ln ab}{2} = \ln\sqrt{ab} \Rightarrow \frac{a+b}{3} = \sqrt{ab} \Rightarrow a^2 + 2ab + b^2 = 9ab \Rightarrow \frac{a}{b} + 2 + \frac{b}{a} = 9$$

$$\therefore \frac{a}{b} + \frac{b}{a} = 7$$

18. c. $$\frac{2\log_5(bx+28)}{\log_5(1/5)^2} = -\log_5(12 - 4x - x^2) \Rightarrow bx + 28 = 12 - 4x - x^2 \Rightarrow x^2 + (b+4)x + 16 = 0$$

For coincident roots, $D = 0 \Rightarrow (b+4)^2 = 4(16) \Rightarrow b + 4 = \pm 8$

19. a. $2 + \log_2 \sqrt{x+1} > 1 - \log_{1/2} \sqrt{4-x^2}$

$\Rightarrow 1 + \log_2 \sqrt{x+1} - \log_2 \sqrt{4-x^2} > 0$

$\Rightarrow \log_2 2 + \log_2 \sqrt{x+1} - \log_2 \sqrt{4-x^2} > 0$

$\Rightarrow \log_2 \dfrac{2\sqrt{x+1}}{\sqrt{4-x^2}} > 0 \Rightarrow \dfrac{2\sqrt{x+1}}{\sqrt{4-x^2}} > 1$

$\Rightarrow 4(x+1)^2 > 4 - x^2 \Rightarrow 4x^2 + 8x + 4 > 4 - x^2 \Rightarrow 5x^2 + 8x > 0 \Rightarrow x > 0$ (i)

Also $x + 1 > 0$ and $4 - x^2 > 0$

$\Rightarrow x > -1$ and $-2 < x < 2$ (ii)

From Eqs. (i) and (ii), $0 < x < 2$

$\Rightarrow x = 1$ as $x \in N$

20. a. Given $4 \log_a a + 5\log_a b = 0 \Rightarrow \log_a b = -4/5$ (i)

Now $\log_a(a^5 b^4) = 5 + 4 \log_a b = 5 + 4\left(-\dfrac{4}{5}\right) = 5 - \dfrac{16}{5} = \dfrac{9}{5}$

21. b. $2^{2y} - 2^y + 2^x(1-2^x) = 0$

Putting $2^y = t$, we get

$t^2 - t + 2^x(1-2^x) = 0$ where $t_1 = 2^{y_1}$ and $t_2 = 2^{y_2}$

$t_1 t_2 = 2^x(1-2^x)$

$2^{y_1+y_2} = 2^x(1-2^x)$

$y_1 + y_2 = x + \log_2(1-2^x)$

22. c. $2 \log_{10} x - \dfrac{\log_{10} 0.01}{\log_{10} x}, x > 1$

$\Rightarrow 2 \log_{10} x + \dfrac{2}{\log_{10} x} = 2\left[\dfrac{1}{y} + y\right] \geq 4$

(where $\log_{10} x = y$)

23. a. Let $3^{\log_4 5} = a \Rightarrow \log_4 5 = \log_3 a \Rightarrow \dfrac{\log 5}{\log 4} = \dfrac{\log a}{\log 3}$

$\Rightarrow \dfrac{\log a}{\log 5} = \dfrac{\log 3}{\log 4} \Rightarrow \log_5 a = \log_4 3$

$\Rightarrow a = 5^{\log_4 3} \Rightarrow 3^{\log_4 5} - 5^{\log_4 3} = 0$

24. a. Let $a = x - 1, b = x, c = x + 1$

Now $\log(1 + ac) = \log[1 + (x-1)(x+1)] = \log x^2 = 2\log x = 2\log b \Rightarrow K = \log b$

25. d. $\dfrac{10}{3} = 3 + \dfrac{1}{3}$. The given equation is of the form $p + \dfrac{1}{p} = 3 + \dfrac{1}{3} = q + \dfrac{1}{q}$, where $p \neq q$ as $x \neq y$

$\Rightarrow \log_2 x = 3.\ \log_2 y = 1/3 \Rightarrow x = 2^3, y = 2^{1/3} \Rightarrow x + y = 8 + 2^{1/3}$.

26. d. R.H.S. $= x = [\log_{10} 5 - \log_{10} 10] = x \log_{10} \dfrac{5}{10} = \log_{10} \dfrac{1}{2^x} \Rightarrow \dfrac{1}{2^x + x - 1} = \dfrac{1}{2^x}$, therefore $x - 1 = 0$ or $x = 1$.

27. b. $(\log_{(0.6)} (0.6)^3) \log_5(5-2x) \le 0 \Rightarrow 5 - 2x \le 1 \Rightarrow x \ge 2$ (i)
Also, $5 - 2x > 0$ (ii)
From Eqs. (i) and (ii), we have x [2, 2.5)

28. b. $(x+2)(x+4) > 0$ and $x + 2 > 0$
$\Rightarrow x > -2$
Now the inequality can be written as $\log_3 (x+2)(x+4) - \log_3(x+2) < \log_3 7$
$\Rightarrow \log_3 (x+4) < \log_3 7 \Rightarrow x + 4 < 7$ or $x < 3$

29. b. We must have $x - 1 > 0 \Rightarrow x > 1$ (i)
and $5 + 4 \log_3 (x-1) > 0 \Rightarrow 4 \log_3 (x-1) > -5$

$\Rightarrow \log_3 (x-1) > -\dfrac{5}{4}$

$\Rightarrow x - 1 > 3^{-5/4} \Rightarrow x > 1 + 3^{-5/4}$ (ii)
From Eqs. (i) and (ii), we get $x > 1 + 3^{-5/4}$. Therefore, $5 + 4 \log_3 (x-1) = 9 \Rightarrow 4 \log_3 (x-1) = 4$
$\Rightarrow \log_3 (x-1) = 1 \Rightarrow x - 1 = 3 \Rightarrow x = 4$

30. d. $2\, x^{\log_4 3} + 3^{\log_4 x} = 27 \Rightarrow 2.\, 3^{\log_4 x} + 3^{\log_4 x} = 27 \Rightarrow 3^{\log_4 x} = 9 = 3^2 \Rightarrow \log_4 x = 2$, therefore $x = 4^2 = 16$.

31. d. $\log_4 (3-x) + \log_{0.25} (3+x) = \log_4 (1-x) + \log_{0.25} (2x+1)$

$\Rightarrow \log_4 (3-x) - \log_4 (3+x) = \log_4 (1-x) - \log_4 (2x+1)$

$\Rightarrow \log_4 (3-x) + \log_4 (2x+1) = \log_4 (1-x) + \log_4 (3+x)$

$\Rightarrow (3-x)(2x+1) = (1-x)(3+x)$

$\Rightarrow 3 + 5x - 2x^2 = 3 - 2x - x^2$

$\Rightarrow x^2 - 7x = 0$

$\Rightarrow x = 0, 7$

Only $x = 0$ is the solution and $x = 7$ is to be rejected.

32. d. $\dfrac{1 + 2\log_3 2}{(1+\log_3 2)^2} + \dfrac{(\log_3 2)^2}{(1+\log_3 2)^2} = \dfrac{(1+\log_3 2)^2}{(1+\log_3 2)^2} = 1$

33. d. Let $\log_8 x = y$, then the given equation reduces to $(1-2y)/y^2 = 3$.

$\Rightarrow 3y^2 + 2y - 1 = 0 \Rightarrow 3y^2 + 3y - y - 1 = 0$

$\Rightarrow 3y(y+1) - 1(y+1) = 0 \Rightarrow \log_8 x = y = 1/3, -1$

$\Rightarrow x = 2, 1/8$

34. d. Given equation can be written as $(a^{\log_2 x})^2 = 5 + 4\, a^{\log_2 x}$

Let $a^{\log_2 x} = t$, then the given equation is $t^2 - 4t - 5 = 0$. We get $(t-5)(t+1) = 0$
$\Rightarrow t = 5$ or $t = -1$ (rejected)

$\therefore\ a^{\log_2 x} = 5 \Rightarrow x^{\log_2 a} = 5 \Rightarrow x = 5^{\log_a 2}$

35. c. $(21.4)^a = 100 \Rightarrow a \log(21.4) = 2$

$\therefore \log(21.4) = 2/a$ (i)

Again $(0.00214)^b = 100$, we get $b(\log 0.00214) = 2$
$\Rightarrow b \log(21.4 \times 10^{-4}) = 2$

$$\Rightarrow b = \frac{2}{\log 21.4 - 4} = \frac{2}{\frac{2}{a} - 4} = \frac{1}{\frac{1}{a} - 2}$$

$$\Rightarrow b = \frac{a}{1-2a}; \frac{1}{b} = \frac{1-2a}{a}$$

$$\Rightarrow \frac{1}{a} - \frac{1}{b} = 2$$

36. b. $x^2 - 16 \le 4x - 11 \Rightarrow x^2 - 4x - 5 \le 0 \Rightarrow (x-5)(x+1) \le 0 \Rightarrow -1 \le x \le 5$ (i)

Also $x^2 - 16 > 0 \Rightarrow x < -4$ or $x > 4$ (ii)

And $4x - 11 > 0 \Rightarrow x > 11/4$ (iii)

From Eqs. (i), (ii) and (iii), we have $x \in (4, 5]$.

37. b. $$\frac{\log x}{\log 3 + (1/2)\log x} + \frac{(1/2)\log x}{\log 3 + \log x}$$

$$\Rightarrow \frac{\log_3 x}{1 + (1/2)\log_3 x} + \frac{1}{2}\frac{\log_3 x}{(1 + \log_3 x)} = 0$$

Let $\log_3 x = y$, we get $\frac{y}{1 + (y/2)} + \frac{y}{2(1 + y)} = 0$

$$\Rightarrow y\left(\frac{2}{2+y} + \frac{1}{2(1+y)}\right) = 0$$

$\Rightarrow y[4 + 4y + 2 + y] = 0$

$\Rightarrow y = 0$ or $y = -6/5$

$\Rightarrow \log_3 x = 0$ or $\log_3 x = -6/5$

$\Rightarrow x = 1$ or $x = 3^{-6/5}$

38. b. Taking log of both the side with base 3, we have $(\log_3 x^2 + (\log_3 x)^2 - 10)(\log_3 x) = -2\log_3 x$.

$\Rightarrow \log_3 x = 0$ or $2\log_3 x + (\log_3 x)^2 - 8 = 0$

$\Rightarrow x = 1$, $\log_3 x = -1 \pm 3$ or $\log_3 x = 2$, $\log_3 x = -4$.

Hence, $x = 1, 3^2, 3^{-4} = 1, 9, 1/81$.

39. d. $\log_3(\log_2 x) + \log_{1/3}(\log_{1/2} y) = 1$

$\Rightarrow \log_3(\log_2 x) - \log_3(\log_{1/2} y) = 1$

$\Rightarrow \log_3(\log_2(4/y^2)) - \log_3(\log_{1/2} y) = 1$

$\Rightarrow \log_2(4/y^2) = 3(\log_{1/2} y)$

$\Rightarrow \log_2(4/y^2) = -3(\log_2 y) \Rightarrow \log_2(4/y^2) + (\log_2 y^3) = 0 \Rightarrow 4y = 1 \Rightarrow y = 1/4 \Rightarrow x = 64$

40. c. Taking log, we have $(x + y)\log 2 = y(\log 2 + \log 3)$, therefore $x \log 2 = y \log 3$.

or $\frac{x}{\log 3} = \frac{y}{\log 2} = \frac{x - y}{\log 3 - \log 2} = \lambda$ say (i)

Also $(x - 1)\log 3 = (y + 1)\log 2$.

or $x\log 3 - y\log 2 = \log 3 + \log 2$

Using Eq. (i), we get $\lambda[(\log 3)^2 - (\log 2)^2] = \log 3 + \log 2$

$\lambda = \frac{1}{\log 3 - \log 2}$, therefore $\frac{1}{\lambda} = \log 3 - \log 2 = \log\frac{3}{2}$

41. c. Given $\log_2 x + \log_2 y \geq 6 \Rightarrow \log_2 (xy) \geq 6$

$\Rightarrow xy \geq 64$

Also to define $\log_2 x$ and $\log_2 y$, we have $x > 0, y > 0$.

Since A.M. ≥ G.M.

$$\therefore \frac{x+y}{2} \geq \sqrt{xy} \Rightarrow x+y \geq 2\sqrt{xy} \geq 16$$

42. d.
$$f(x_1)+f(x_2) = \log\left(\frac{1+x_1}{1-x_1}\cdot\frac{1+x_2}{1-x_2}\right)$$
$$= \log\left(\frac{1+x_1x_2+x_1+x_2}{1+x_1x_2-x_1-x_2}\right)$$
$$= \log\left(\frac{1+\frac{x_1+x_2}{1+x_1x_2}}{1-\frac{x_1+x_2}{1+x_1x_2}}\right) = f\left(\frac{x_1+x_2}{1+x_1x_2}\right)$$

43. c. Given inequality is defined if $x > 2/5;\ x \neq 1$

Case I: If $x > 1 \Rightarrow \frac{5}{2} - \frac{1}{x} > x \Rightarrow x + \frac{1}{x} < \frac{5}{2}$

$\Rightarrow 2(x^2+1) < 5x$

$\Rightarrow 2x^2 - 5x + 2 < 0$

$\Rightarrow 2x^2 - 4x - x + 2 < 0$

$\Rightarrow (x-2)(2x-1) < 0$

$\Rightarrow x \in (1, 2)$ (i)

Case II: $\frac{5}{2} < x < 1$, then $(x-2)(2x-1) > 0$.

$\Rightarrow x \in \left(\frac{2}{5}, \frac{1}{2}\right)$ (ii)

From Eqs. (i) and (ii), we get $x \in \left(\frac{2}{5}, \frac{1}{2}\right) \cup (1, 2)$.

44. $e^2 \cdot x^{\ln x} = x^3$

Taking log on both sides, we get $\ln(e^2 \cdot x^{\ln x}) = \ln(x^3)$

$\Rightarrow (\ln x)^2 - 3\ln x + 2 = 0$

$\Rightarrow (\ln x - 2)(\ln x - 1) = 0$

If $\ln x = 2 \Rightarrow x = e^2$

If $\ln x = 1 \Rightarrow x = e$

Since $x_1 > x_2$, we get $x_1 = e^2$ and $x_2 = e$

$\Rightarrow x_2^2 = x_1$

45. a. $\log_{16} x = \frac{1 \pm \sqrt{1 - 4\log_{16} k}}{2}$. For exactly one solution, $4\log_{16} k = 1$.

$\therefore k^4 = 16$, i.e., $k = 2, -2, 2i, -2i$.

Multiple Correct Answers Type

1. a, d.

$\log_{ax} a + \log_x a^2 + \log_{a^2x} a^3 = 0$

$\Rightarrow \dfrac{1}{\log_a ax} + \dfrac{2}{\log_a x} + \dfrac{3}{(\log_a a^2x)} = 0$

$\Rightarrow \dfrac{1}{\log_a a + \log_a x} + \dfrac{2}{\log_a x} + \dfrac{3}{(2 + \log_a x)} = 0$

Let $\log_a x = y$, we have $\dfrac{1}{y+1} + \dfrac{2}{y} + \dfrac{3}{2+y} = 0$

$\Rightarrow 6y^2 + 11y + 4 = 0$

$\Rightarrow y = \log_a x = -\dfrac{1}{2}, -\dfrac{4}{3}$

$\Rightarrow x = a^{-4/3}, a^{-1/2}$

2. b, c, d.

$2^{x+2}.5^{6-x} = 2^{x^2} \cdot 5^{x^2}$

$\Rightarrow 5^{6-x-x^2} = 2^{x^2-x-2}$

$\Rightarrow (6-x-x^2)\log_{10} 5 = (x^2 - x - 2)\log_{10} 2$ (base 10)

$\Rightarrow (6-x-x^2)[1 - \log_2 10] = (x^2 - x - 2)\log_{10} 2$

$\Rightarrow 6 - x - x^2 = (\log_{10} 2)[(x^2 - x - 2) - x^2 - x + 6]$

$\Rightarrow 6 - x - x^2 = (\log_{10} 2)[4 - 2x]$

$\Rightarrow x^2 + x - 6 = 2(\log_{10} 2)(x - 2)$

$\Rightarrow (x+3)(x-2) = (\log_{10} 4)(x-2)$

Therefore, either $x = 2$ or $x + 3 = \log_{10} 4$

$\Rightarrow x = \log_{10} 4 - 3 = \log_{10}\left(\dfrac{4}{1000}\right)$; $x = -\log_{10}(250)$

3. a, b, c, d.

Let $\dfrac{\log_k x}{b-c} = \dfrac{\log_k y}{c-a} = \dfrac{\log_k z}{a-b} = p$

$\Rightarrow x = k^{p(b-c)}, y = k^{p(c-a)}, z = k^{p(a-b)}$

$\Rightarrow xyz = k^{p(b-c)} k^{p(c-a)} k^{p(a-b)} = k^{p(b-c)+p(c-a)+p(a-b)} = k^0 = 1$

$x^a y^b z^c = k^{pa(b-c)} k^{pb(c-a)} k^{pc(a-b)} = k^0 = 1$

$x^{b+c} y^{c+a} z^{a+b} = k^{p(b+c)(b-c)} k^{p(c+a)(c-a)} k^{p(a+b)(a-b)} = k^0 = 1$

4. b, c.

$\log_k x \,.\, \log_5 k = \log_x 5$

$\Rightarrow \dfrac{\log x \log k}{\log k \log 5} = \log_x 5$

$\Rightarrow \dfrac{\log x}{\log 5} = \log_x 5$

$\Rightarrow \log_5 x = \dfrac{1}{\log_5 x}$

$\Rightarrow (\log_5 x)^2 = 1 \Rightarrow \log_5 x = \pm 1$

$\Rightarrow x = 5^{\pm 1} \Rightarrow x = \dfrac{1}{5}, 5$

5. a, c, d.

$x^{\sqrt{x}} = \left(\sqrt{x}\right)^x, p, q \in N$

$\Rightarrow \sqrt{x} \log x = x \log \sqrt{x}$

$\Rightarrow \log x\left[\sqrt{x} - \dfrac{x}{2}\right] = 0$

$\Rightarrow \log x = 0 \text{ or } \left[\sqrt{x} - \dfrac{x}{2}\right] = 0$

$\Rightarrow x = 1 \text{ or } 4$

6. a, b, c.

a. $\log_{10}\left(\dfrac{10}{2}\right) . \log_{10}(10 \times 2) + (\log_{10}2)^2 = (1 - \log_{10}2)(1 + \log_{10}2) + (\log_{10}2)^2 = 1$

b. $\dfrac{\log 2^2 \times 3}{\log(48/4)} = 1$

c. $-\log_5 \log_3 9^{1/10} = -\log_5 \log_3 3^{1/5} = -\log_5(1/5) = 1$

d. $\dfrac{1}{6} \log_{\frac{\sqrt{3}}{2}}\left(\dfrac{64}{27}\right) = \dfrac{1}{6} \log_{\frac{\sqrt{3}}{2}}\left(\frac{\sqrt{3}}{2}\right)^{-6} = -1$

7. a, b, c, d.

$\log_a x = b \Rightarrow x = a^b$

a. For $a = \sqrt{2}^{\sqrt{2}} \notin Q$ and $b = \sqrt{2} \notin Q; x = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$ which is rational.

b. For $a = 2 \in Q$ and $b = \log_2 3 \notin Q; x = 3$ which is rational.

c. For $a = \sqrt{2}$ and $b = 2; x = 2$

d. The option is obviously correct.

8. b, c, d.

$\dfrac{\log_2(x-0.5)}{\log_2(x+1)} = \dfrac{\log_2(x+1)}{\log_2(x-0.5)}$

$\Rightarrow [\log_2(x+1)]^2 = [\log_2(x-0.5)^2]$

$\Rightarrow \log_2(x+1) = \log_2(x-0.5) \text{ or } -\log_2(x-0.5)$

If $\log_2(x+1) = \log_2(x-0.5) \Rightarrow x + 1 = x - 0.5 \Rightarrow$ no solution

If $\log_2(x+1) = \log(x-0.5)^{-1}$

$$\Rightarrow x+1=\frac{1}{x-(1/2)}=\frac{2}{2x-1}$$

$\Rightarrow (x+1)(2x-1)=2$

$\Rightarrow 2x^2+x-3=0$

$\Rightarrow 2x^2+3x-2x-3=0$

$\Rightarrow (x-1)(2x+3)=0$

$\Rightarrow x=1 (x=-3/2$ rejected)

9. a, d.

$$\left(\sqrt{1+\frac{3}{2\log_3 x}}\right)\log_3 x+1=0$$

Let $\log_3 x=y$, we get

$$\left(\sqrt{1+\frac{3}{2y}}\right)y=-1 \Rightarrow \left(1+\frac{3}{2y}\right)=\frac{1}{y^2} \Rightarrow \frac{2y+3}{2y}=\frac{1}{y^2}$$

$\Rightarrow 2y^2+3y-2=0 \Rightarrow 2y^2+4y-y-2=0 \Rightarrow (y+2)(2y-1)=0$

$y=1/2$ or $y=-2 \Rightarrow x=3^{1/2}$ (rejected) or $x=1/9$

10. a, b.

$\log_{1/2}(4-x)\ge \log_{1/2}2-\log_{1/2}(x-1)$

$\Rightarrow \log_{1/2}(4-x)(x-1)\ge \log_{1/2}2$

$\Rightarrow (4-x)(x-1)\le 2$

$\Rightarrow x^2-5x+6\ge 0$

$\Rightarrow (x-3)(x-2)\ge 0$

$\Rightarrow x\ge 3$ or $x\le 2$

But $x\in(1,4)$

$\Rightarrow x\in(1,2]\cup[3,4)$

11. a, c.

$(\log_a x^2)\log_a x=(k-2)\log_a x-k$ (taking log on base a)

Let $\log_a x=t$, we get

$2t^2-(k-2)t+k=0$

Putting $D=0$ (has only one solution), we have

$(k-2)^2-8k=0$

$\Rightarrow k^2-12k+4=0$

$$\Rightarrow k=\frac{12\pm\sqrt{128}}{2}$$

$\Rightarrow k=6\pm 4\sqrt{2}$

Matrix-Match Type

1. a → q; b → s; c → p; d → r

a. We have $\frac{1}{\log_3 \pi} + \frac{1}{\log_4 \pi} = \log_\pi 3 + \log_\pi 4 = \log_\pi 12$

But $\pi^2 < 12 < \pi^3$, we have $2 < \log_\pi 12 < 3$.

b. $3^a = 4$; $a = \log_3 4$

Similarly, $b = \log_4 5$ etc.

Hence, $abcdef = \log_3 4 \cdot \log_4 5 \cdot \log_5 6 \cdot \log_6 7 \cdot \log_7 8 \cdot \log_8 9 = \log_3 9 = 2$

c. We have to find characteristic of $\log_2 2008$.

We know that $\log_2 1024 = 10$ and $\log_2 2048 = 11$, therefore

$10 < \log_2 2008 < 11$

Hence, it has characteristic = 10.

d. $\log_2(\log_2(\log_3 x)) = 0 \Rightarrow \log_2(\log_3 x) = 1 \Rightarrow \log_3 x = 2 \Rightarrow x = 9$

Similarly, we have $\log_3(\log_2 y) = 1$

$\Rightarrow \log_2 y = 3 \Rightarrow y = 8$

Therefore, $x - y = 1$.

2. a → s; b → p; c → q; d → r

a. $A = \log_2 \log_2 \log_4 256 + 2\log_{\sqrt{2}} 2$

$= \log_2 \log_2 \log_4 4^4 + 2\log_{2^{1/2}} 2$

$= \log_2 \log_2 4 + 4\log_2 2 + 4 = 1 + 4 = 5$

b. $\log_3(5x-2) - 2\log_3\sqrt{3x+1} = 1 - \log_3 4$

$\Rightarrow \log_3(5x-2) - \log_3(3x+1) + \log_3 4 = 1$

$\Rightarrow \log_3\left(\frac{(5x-2)(4)}{3x+1}\right) = 1 \Rightarrow \frac{(5x-2)(4)}{3x+1} = 3 \Rightarrow x = 1$

c. $7^{\log_7(x^2-4x+5)} = (x-1)$

$\Rightarrow x^2 - 4x + 5 = x - 1$

$\Rightarrow x^2 - 5x + 6 = 0$

$\Rightarrow (x-2)(x-3) = 0 \Rightarrow x = 2$ or $x = 3$

Also we must have $x^2 - 4x + 5 > 0$ and $x - 1 > 0$

$\Rightarrow x > 1$ (as $x^2 - 4x + 5 > 0$ is true for all real numbers)

d. $x > 0$, $\frac{1}{2}\log_2 x - 2\left(\frac{\log_2 x}{2}\right)^2 + 1 > 0$

$\Rightarrow \log_2 x - (\log_2 x)^2 + 2 > 0$

$\Rightarrow (\log_2 x)^2 - \log_2 x - 2 < 0$

Let $\log_2 x = t$, we have $t^2 - t - 2 < 0$

$\Rightarrow (t-2)(t+1) < 0 \Rightarrow -1 < t < 2$

$\Rightarrow -1 < \log_2 x < 2 \Rightarrow \frac{1}{2} < x < 4$

Hence, the number of integers is 3, i.e., $\{1, 2, 3\}$.

3. a → q; b → p, s; c → p; d → p, r

a. $2^{\log_{(2\sqrt{2})}15} = 2^{\log_{2^{3/2}}15} = 2^{2/3\log_2 15} = 2^{\log_2 15^{2/3}} = 15^{2/3}$

b. $\sqrt[3]{\left(5^{1/\log_7 5} + \frac{1}{\sqrt{(-\log_{10} 0.1)}}\right)} = \sqrt[3]{\left(5^{\log_5 7} + \frac{1}{\sqrt{(\log_{10} 0.1^{-1})}}\right)} = \sqrt[3]{\left(7 + \frac{1}{\sqrt{\log_{10} 10}}\right)} = \sqrt[3]{(7+1)} = 2$

c. $\log_3 5 \cdot \log_{25} 27 = \frac{\log 5}{\log 3}\frac{\log 27}{\log 25} = \frac{\log 5}{\log 3} \times \frac{3}{2}\frac{\log 3}{\log 5} = \frac{3}{2}$

d. $(\log_{10} x)^2 = \log 100x = 2 + \log 10x$

Putting $\log x = t$, we get $t^2 = 2 + t$

$\Rightarrow t^2 - t - 2 = 0$

$\Rightarrow t = 2$ or $t = -1$

$\Rightarrow \log 10x = 2$ or $\log 10x = -1$

$\Rightarrow x = 100$ or $x = 1/10$

Hence, the product of roots is 10.

Integer Type

1.(3) $\log_3 c = 3 + \log_3 a \Rightarrow \log_3 \frac{c}{a} = 3 \Rightarrow c = 27a$ (i)

$\log_a b = 2;\ \log_b c = 2$

$\Rightarrow \log_a b \cdot \log_b c = 4 \Rightarrow \log_a c = 4 \Rightarrow c = a^4$ (ii)

From Eqs. (i) and (ii), we get $a = 3, c = 81$.

From relation (i), we have $b = a^2 = 9$.

Hence, $c/(ab) = 3$.

2.(1) Let $\log_2 10 = p$ and $\log_5 10 = q$

Hence, $p + q = 1$

$x = p^3 + 3pq + q^3$

$= (p+q)^3 - 3pq(p+q) + 3pq$

$= 1 - 3pq + 3pq$

$= 1$

3.(6) Let $\log_4 A = \log_6 B = \log_9(A+B) = x$

$\Rightarrow A = 4^x; B = 6^x$ and $A + B = 9^x$

$A + B = 9^x \Rightarrow 4^x + 6^x = 9^x$

$\Rightarrow 2^{2x} + 2^x \cdot 3^x = 3^{2x}$

$\Rightarrow \left(\frac{3}{2}\right)^{2x} - \left(\frac{3}{2}\right)^x - 1 = 0$

$\Rightarrow \left(\frac{3}{2}\right)^x = \frac{1+\sqrt{5}}{2}$

$$\Rightarrow \frac{B}{A} = \left(\frac{3}{2}\right)^x = \frac{1+\sqrt{5}}{2}$$

$$\Rightarrow 4\frac{B}{A} = 4\left(\frac{1+\sqrt{5}}{2}\right)$$

$$\Rightarrow \left[4\frac{B}{A}\right] = 6$$

4.(4) $\log_6 54 + \log_x 16 = \log_{\sqrt{2}} x - \log_{36} \frac{4}{9}$

$$\Rightarrow 1 + 2\log_6 3 + \log_x 16 = 2\log_2 x - \log_6 \frac{2}{3}$$

$$\Rightarrow 1 + 2\log_6 3 + \log_x 16 = 2\log_2 x - \log_6 2 + \log_6 3$$

$$\Rightarrow 1 + \log_x 16 = 2\log_2 x - (\log_6 2 + \log_6 3)$$

$$\Rightarrow 1 + \log_x 16 = 2\log_2 x - 1$$

$$\Rightarrow \frac{4}{\log_2 x} = 2\log_2 x - 2$$

Let $\log_2 x = t$, we have $(t-2)(t+1) = 0$

$\Rightarrow t = 2$ or $t = -1$

$\Rightarrow x = 4$ or $1/2$

5.(5) $a = \frac{\log_5 175}{\log_5 245} = \frac{2+\log_5 7}{1+2\log_5 7}$

$$\Rightarrow a + 2a\log_5 7 = 2 + \log_5 7$$

$$\Rightarrow \log_5 7 = \frac{a-2}{1-2a} \quad \text{(i)}$$

$$b = \frac{\log_5 875}{\log_5 1715} = \frac{3+\log_5 7}{1+3\log_5 7}$$

$$\Rightarrow b + 3b\log_5 7 = 3 + \log_5 7$$

$$\Rightarrow \log_5 7 = \frac{b-3}{1-3b} \quad \text{(ii)}$$

From Eqs. (i) and (ii); we get $\frac{a-2}{1-2a} = \frac{b-3}{1-3b} \Rightarrow \frac{1-ab}{a-b} = 5$.

6.(8) $(\log_{27} x^3)^2 = \log_{27} x^6$

$$\Rightarrow (3\log_{27} x)^2 = 6\log_{27} x$$

$$\Rightarrow 3\log_{27} x\,(3\log_{27} x - 2) = 0$$

$$\Rightarrow x = 1 \text{ or } \log_{27} x = \frac{2}{3}$$

$$\Rightarrow x = (27)^{2/3} = 9$$

Difference $= 9 - 1 = 8$

7.(3) We must have $12-3x>0$ and $x>0 \Rightarrow x \in (0,4)$
Therefore, the integral values are 1, 2, 3.

For $x=1$; $\left(3^{\frac{5}{2}\log_3 9}\right)-\left(3^{\log_2 1}\right)=3^5-3^0>32$

Similarly, $x=2$ satisfies but not $x=3$
Hence, the required sum = 3.

8.(3) $\log_2 15 . \log_{1/6} 2 . \log_3 1/6$.

$=\dfrac{\log 15}{\log 2}\cdot\dfrac{\log 2}{-\log 6}\cdot\dfrac{-\log 6}{\log 3}$

$=\log_3 15$

9.(6) Let $N=2\log_x 4+3\log_x 5$; where $x=(2000)^6$
$=\log_x 4^2+\log_x 5^3$

$=\log_x 4^2\cdot 5^3=\log_{(2000)^6}(2000)=\dfrac{1}{6}$

Hence reciprocal of given value is 6

10.(5) $\sqrt{\log_2 x-1}-\dfrac{3}{2}\log_2 x+2>0\,(x>0)$

$\Rightarrow \sqrt{\log_2 x-1}-\dfrac{3}{2}\left(\log_2 x-1\right)+\dfrac{1}{2}>0$ (i)

Let $\sqrt{\log_2 x-1}=t\geq 0$, we have (ii)
$\log_2 x \,^3\, 1 \Rightarrow x\geq 2$

Then from Eq. (i), we have $t-\dfrac{3}{2}t^2+\dfrac{1}{2}>0$

$\Rightarrow 3t^2-2t-1<0$
$\Rightarrow 1/3<t<1$ (iii)
From Eqs. (i) and (ii), we have $0\leq t<1$.

$0\leq\sqrt{\log_2 x-1}<1$
$0\leq\log_2 x-1<1$
$1\leq\log_2 x<2$
$2\leq x<4$
Hence, the integral values are 2 and 3, and their sum is 5.

11.(2) $\log_{1/2}|x-3|>-1$
$\Rightarrow |x-3|<2$
$\Rightarrow -2<x-3<2$
$\Rightarrow 1<x<5, x\neq 3$
$\therefore x\in\{2,4\}$

12.(9) We must be $x>1$

$2\log_{1/2}(x-1)\leq\dfrac{1}{3}-\dfrac{1}{\log_{x^2-x} 8}$

$$\Rightarrow \frac{1}{3} - \frac{\log_2(x^2 - x)}{3} + 2\log_2(x-1) \geq 0$$

$$\Rightarrow \log_2 2 - \log_2(x^2 - x) + 6\log_2(x-1) \geq 0$$

$$\Rightarrow \log_2 \frac{2(x-1)^6}{x(x-1)} \geq 0$$

$$\Rightarrow \frac{2(x-1)^5}{x} \geq 1$$

Putting $x - 1 = y$, we have $y > 0$.

$$\Rightarrow \frac{2y^5}{y+1} - 1 \geq 0$$

$$\Rightarrow \frac{2y^5 - y - 1}{y+1} \geq 0$$

$$\Rightarrow \frac{2y^5 - 2y + y - 1}{y+1} \geq 0$$

$$\Rightarrow \frac{2y(y^4 - 1) + y - 1}{y+1} \geq 0$$

$$\Rightarrow \frac{(y-1)\left[2y(y+1)(y^2+1)+1\right]}{y+1} \geq 0$$

$$\Rightarrow \frac{y-1}{y+1} \geq 0 \Rightarrow y \geq 1$$

$$\Rightarrow x \geq 2$$

13.(6) $3 + 2\sqrt{2} = \left(\sqrt{2}+1\right)^2$ and $3 - 2\sqrt{2} = \left(\sqrt{2}-1\right)^2$

$$\Rightarrow \log_{\left(\sqrt{3+2\sqrt{2}} + \sqrt{3-2\sqrt{2}}\right)} 2^9 = \frac{1}{\log_{2^9}\left(\left(\sqrt{2}+1\right) + \left(\sqrt{2}-1\right)\right)}$$

$$= \frac{1}{\log_{2^9} 2^{3/2}}$$

$$= \frac{9}{3/2} = 6$$