CHAPTER

4

Inverse Trigonometric Functions

- Introduction
- Properties and Important Formulas of Inverse Functions

INTRODUCTION

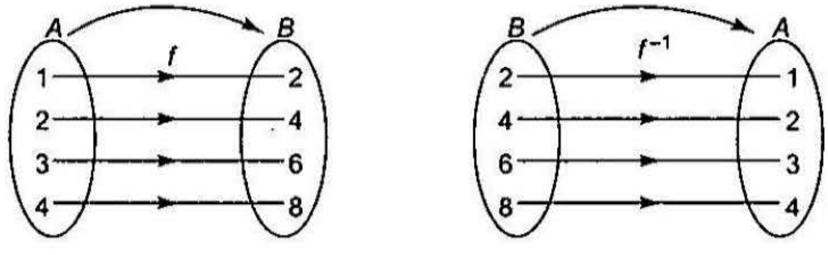


Fig. 4.1

If $f: X \to Y$ is a function defined by $y = f(x)$ such that f is both one-one and onto, then there exists a unique function $g: Y \to X$ such that for each $y \in Y$, $g(y) = x$ if and only if $y = f(x)$. The function g so defined is called the inverse of f and denoted by f^{-1}. Also if g is the inverse of f, then f is the inverse of g; and the two functions f and g are said to be inverses of each other.

The condition for existence of inverse of a function is that the function must be one-one and onto. Whenever an inverse function is defined, the range of the original function becomes the domain of the inverse function and domain of the original function becomes the range of the inverse function.

We know that trigonometric functions are many-one in their actual domain. Hence, for inverse functions to get defined, the actual domain of trigonometric functions must be restricted to make the function one-one.

Inverse Circular Functions

Since the domain of sine function is the set of all real numbers and range is $[-1, 1]$, if we restrict its domain to $[-\pi/2, \pi/2]$, then it becomes one-one and onto within the range $[-1, 1]$. Actually, sine function can be restricted to any of the intervals $[-3\pi/2, -\pi/2]$, $[-\pi/2, \pi/2]$, $[\pi/2, 3\pi/2]$, etc. It becomes one-one and its range is $[-1, 1]$. We can, therefore, define the inverse of sine function in each of these intervals. We denote the inverse of sine function by $\sin^{-1}$ (arc sine function). Thus, $\sin^{-1}$ is a function whose domain is $[-1, 1]$ and the range could be any of the intervals $[-3\pi/2, -\pi/2]$, $[-\pi/2, \pi/2]$ or $[\pi/2, 3\pi/2]$ and so on. Corresponding to each such interval, we get a branch of the function $\sin^{-1}$. The branch with range $[-\pi/2, \pi/2]$ is called the principal value branch, whereas other intervals as range give different branches of $\sin^{-1}$. When we refer to the function $\sin^{-1}$, we take it as the function whose domain is $[-1, 1]$ and range is $[-\pi/2, \pi/2]$. We write $\sin^{-1}$: $[-1, 1] \to [-\pi/2, \pi/2]$.

From the definition of the inverse functions, it follows that $\sin(\sin^{-1}x) = x$ if $-1 \le x \le 1$ and $\sin^{-1}(\sin x) = x$ if $-\pi/2 \le x \le \pi/2$.

If any point (x_1, y_1) lies on the curve $y = f(x)$, then corresponding to it (y_1, x_1) lies on $y = f^{-1}(x)$. Since points (x_1, y_1) and the (y_1, x_1) are symmetrical about the line $y = x$, the graphs of $y = f(x)$ and $y = f^{-1}(x)$ are symmetrical about the line $y = x$.

Note:

- $\sin^{-1}x$ is *entirely different from* $(\sin x)^{-1}$. *The former is the measure of an angle in radians whose* sine *is x while the latter is* $\dfrac{1}{\sin x}$.

Domain, Range and Graphs of Inverse Trigonometric Functions

With reference to the preceding discussion, the domain, range and graphs of inverse trigonometric functions can be summarized as follows.

In the following figures, dotted line graphs are of trigonometric functions and solid line graphs are of corresponding inverse trigonometric functions.

$f(x) = \sin^{-1}x$
Domain: $[-1, 1]$
Range (principal values): $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

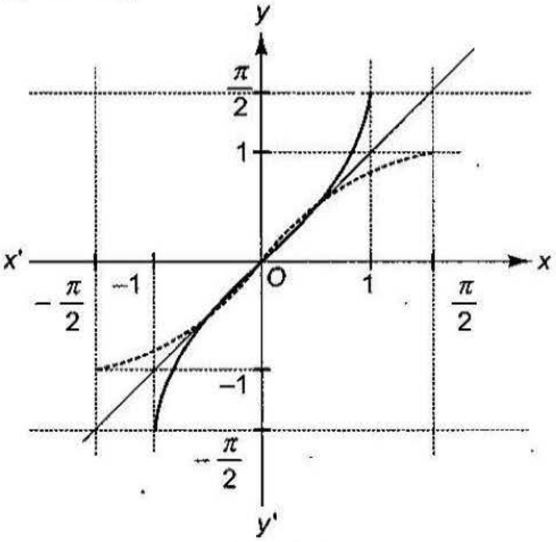


Fig. 4.2

$f(x) = \cos^{-1}x$
Domain: $[-1, 1]$
Range (principal values): $[0, \pi]$

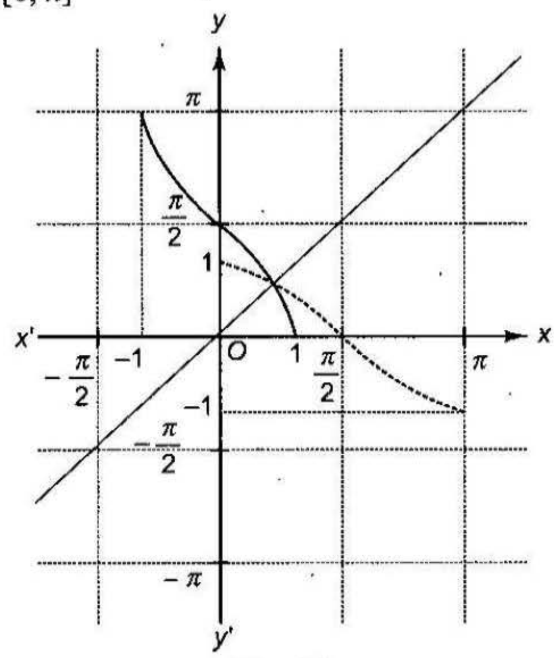


Fig. 4.3

$f(x) = \tan^{-1}x$
Domain: R
Range (principal values): $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

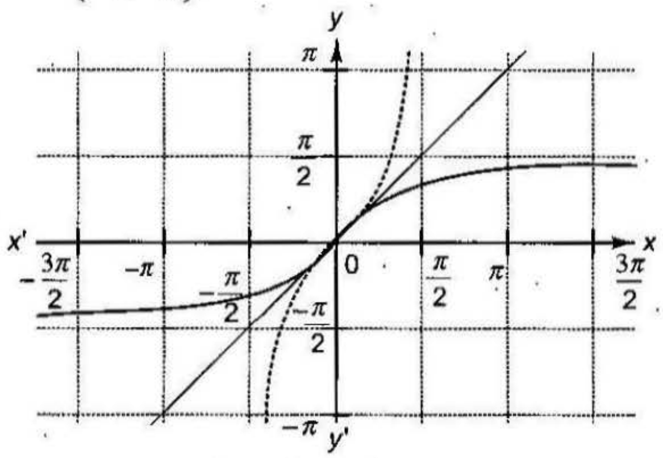


Fig. 4.4

$f(x) = \cot^{-1}x$
Domain: R
Range (principal values): $(0, \pi)$

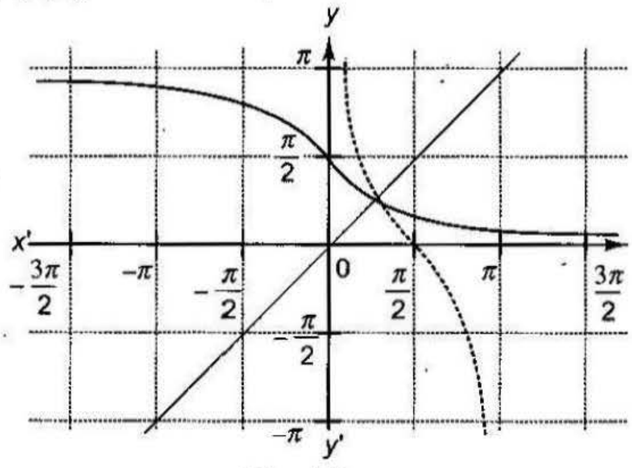


Fig. 4.5

$f(x) = \sec^{-1}x$
Domain: $(-\infty, -1] \cup [1, \infty)$
Range (principal values): $[0, \pi] - \{\pi/2\}$

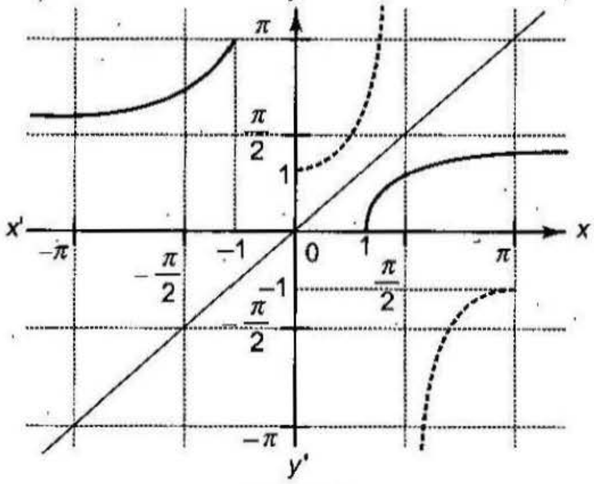


Fig. 4.6

$f(x) = \text{cosec}^{-1}x$

Domain: $(-\infty, -1] \cup [1, \infty)$

Range (principal values): $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

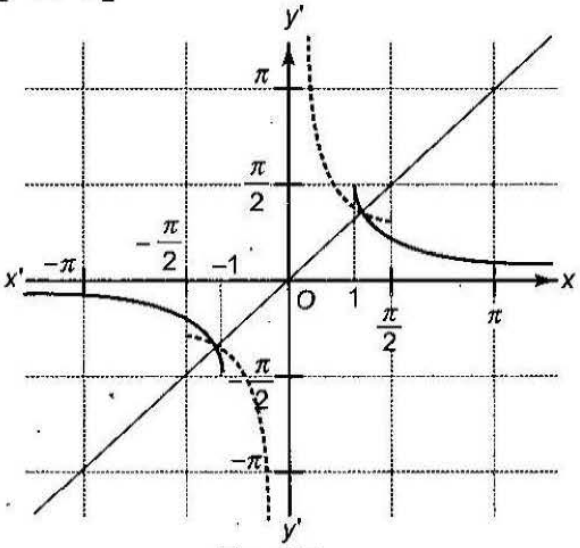
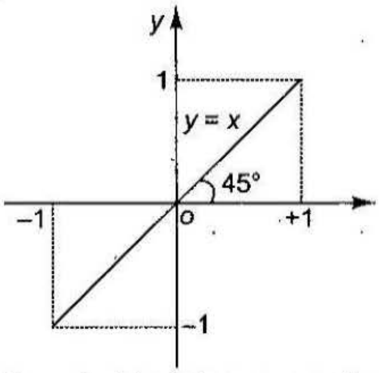


Fig. 4.7

PROPERTIES AND IMPORTANT FORMULAS OF INVERSE FUNCTIONS

Property 1

i. $\sin(\sin^{-1} x) = x$, for all $x \in [-1, 1]$

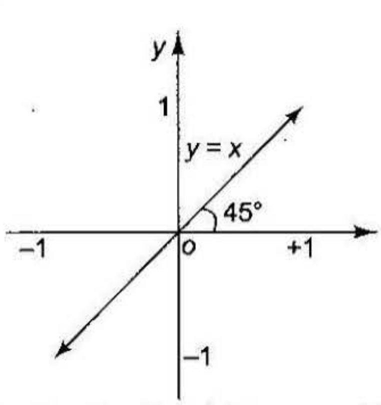


Graph of $y = \sin(\sin^{-1}x)$ or $y = \cos(\cos^{-1}x)$

Fig. 4.8

ii. $\cos(\cos^{-1} x) = x$, for all $x \in [-1, 1]$

iii. $\tan(\tan^{-1} x) = x$, for all $x \in R$

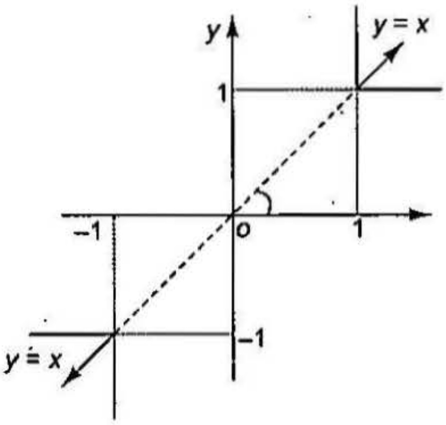


Graph of $y = \tan(\tan^{-1}x)$ or $y = \cot(\cot^{-1}x)$

Fig. 4.9

iv. $\cot(\cot^{-1} x) = x$, for all $x \in R$

v. $\text{cosec}(\text{cosec}^{-1} x) = x$, for all $x \in (-\infty, -1] \cup [1, \infty)$



Graph of $y = \sec(\sec^{-1}x)$ or $y = \text{cosec}(\text{cosec}^{-1}x)$

Fig. 4.10

vi. $\sec(\sec^{-1} x) = x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

Property 2

i. $\sin^{-1}(\sin x) = x$, for all $x \in [-\pi/2, \pi/2]$

$\sin^{-1}(\sin x)$ is defined when $\sin x \in [-1, 1]$ which is true $\forall\, x \in R$.

But range of $\sin^{-1}x$ is $[-\pi/2, \pi/2]$, hence $\sin^{-1}(\sin x) = x$ is true only for $x \in [-\pi/2, \pi/2]$.

With the same reasoning, we have the following results.

ii. $\cos^{-1}(\cos x) = x$, for all $x \in [0, \pi]$

iii. $\tan^{-1}(\tan x) = x$, for all $x \in (-\pi/2, \pi/2)$

iv. $\cot^{-1}(\cot x) = x$, for all $x \in (0, \pi)$

v. $\text{cosec}^{-1}(\text{cosec}\, x) = x$, for all $x \in [-\pi/2, \pi/2] - \{0\}$

vi. $\sec^{-1}(\sec x) = x$, for all $x \in [0, \pi] - \{\pi/2\}$

Graph of $y = \sin^{-1}(\sin x)$

For x not lying in the principal domain, we have the following method to draw the graph.

Consider $y = \sin^{-1}(\sin x) \Rightarrow \sin y = \sin x \Rightarrow y = n\pi + (-1)^n x, n \in Z$

Now, keeping in mind that $y \in [-\pi/2, \pi/2]$, we have the following table:

Value of n	Relation	Range of x
...	...	...
...	...	...
$n = -2$	$y = -2\pi + x$	$x \in [3\pi/2, 5\pi/2]$
$n = -1$	$y = -\pi - x$	$x \in [-3\pi/2, -\pi/2]$
$n = 0$	$y = x$	$x \in [-\pi/2, \pi/2]$
$n = 1$	$y = \pi - x$	$x \in [\pi/2, 3\pi/2]$
$n = 2$	$y = 2\pi + x$	$x \in [-5\pi/2, -3\pi/2]$
...	...	...
...	...	...

From the preceding information, we can plot the graph of $y = \sin^{-1}(\sin x)$ as follows (Fig. 4.11):

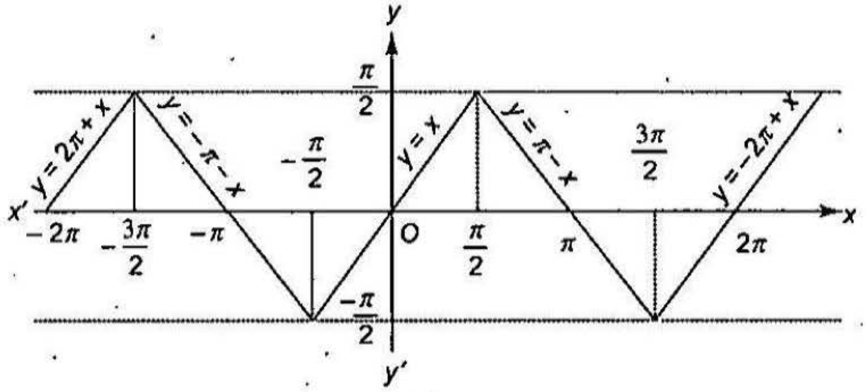


Fig. 4.11

Graph of $y = \cos^{-1}(\cos x)$

$y = \cos^{-1}(\cos x)$

$\Rightarrow \quad \cos y = \cos x \quad \Rightarrow \quad y = 2n\pi \pm x,\ n \in Z$

Now, keeping in mind that $y \in [0, \pi]$, we can plot the graph of $y = \sin^{-1}(\sin x)$ as follows (Fig. 4.12):

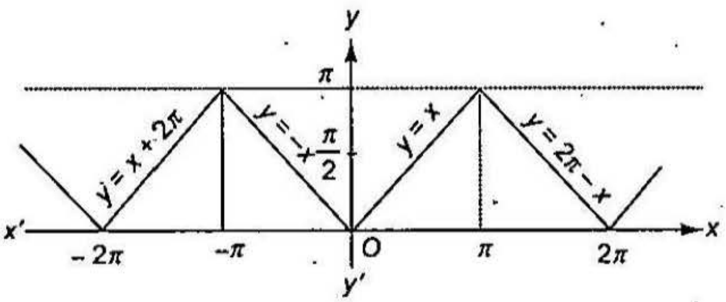


Fig. 4.12

Graph of $y = \tan^{-1}(\tan x)$

$y = \tan^{-1}(\tan x)$

$\Rightarrow \quad \tan y = \tan x \quad \Rightarrow \quad y = n\pi + x,\ n \in Z$

Now, keeping in mind that $y \in (-\pi/2, \pi/2)$, we can plot the graph of $y = \tan^{-1}(\tan x)$ as follows (Fig. 4.13):

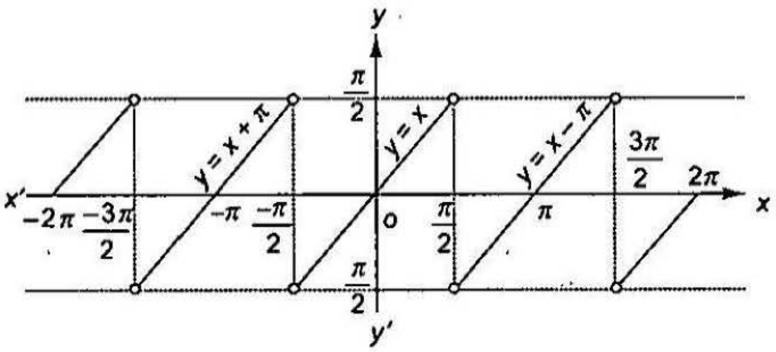


Fig. 4.13

Graph of $y = \cot^{-1}(\cot x)$

$y = \cot^{-1}(\cot x) \quad \Rightarrow \quad \cot y = \cot x$

$\Rightarrow \quad \tan y = \tan x \quad \Rightarrow \quad y = n\pi + x,\ n \in Z$

Now, keeping in mind that $y \in (0, \pi)$, we can plot the graph of $y = \cot^{-1}(\cot x)$ as follows (Fig. 4.14):

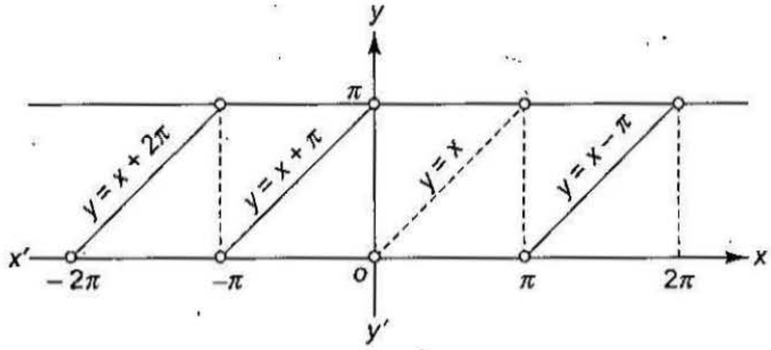


Fig. 4.14

Graph of $y = \text{cosec}^{-1}(\text{cosec}\, x)$

$y = \text{cosec}^{-1}(\text{cosec}\, x) \Rightarrow \quad \text{cosec}\, y = \text{cosec}\, x \quad \Rightarrow \quad \sin y = \sin x$

Hence, graph of $y = \text{cosec}^{-1}(\text{cosec}\, x)$ is the same as that of $y = \sin^{-1}(\sin x)$, but excluding points $x = n\pi, n \in Z$.

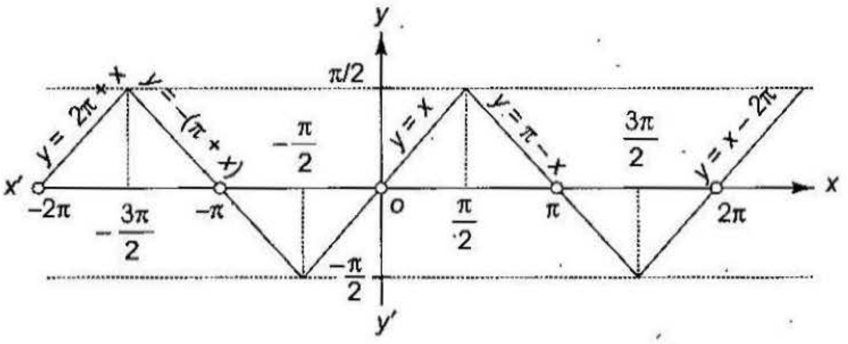


Fig. 4.15

Graph of $f(x) = \sec^{-1}(\sec x)$

$y = \sec^{-1}(\sec x) \Rightarrow \quad \sec y = \sec x \quad \Rightarrow \quad \cos y = \cos x$

Hence, graph of $y = \sec^{-1}(\sec x)$ is the same as that of $y = \cos^{-1}(\cos x)$, but excluding points $x = (2n + 1)\pi/2, n \in Z$.

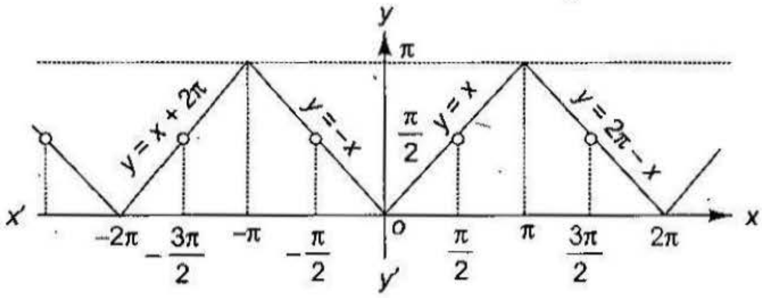


Fig. 4.16

Example 4.1 **Evaluate the following:**

i. $\sin^{-1}(\sin \pi/4)$ ii. $\cos^{-1}(2\pi/3)$ iii. $\tan^{-1}(\tan \pi/3)$

Sol. We know that

$\sin^{-1}(\sin \theta) = \theta$, if $-\pi/2 \le \theta \le \pi/2$,

$\cos^{-1}(\cos \theta) = \theta$, if $0 \le \theta \le \pi$

and $\tan^{-1}(\tan\theta) = \theta$, if $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

Then

i. $\sin^{-1}\left(\sin\frac{\pi}{4}\right) = \frac{\pi}{4}$

ii. $\cos^{-1}\left(\cos\frac{2\pi}{3}\right) = \frac{2\pi}{3}$

iii. $\tan^{-1}\left(\tan\frac{\pi}{3}\right) = \frac{\pi}{3}$

Example 4.2 **Evaluate the following:**

i. $\sin^{-1}\left(\frac{2\pi}{3}\right)$ **ii.** $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ **iii.** $\tan^{-1}\left(\tan\frac{2\pi}{3}\right)$ **iv.** $\cos\left(\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) + \frac{\pi}{6}\right)$

Sol.

i. $\sin^{-1}\left(\sin\frac{2\pi}{3}\right) \neq \frac{2\pi}{3}$, as $\frac{2\pi}{3}$ does not lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$

Now, $\sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \sin^{-1}\left(\sin\left(\pi - \frac{\pi}{3}\right)\right) = \sin^{-1}\left(\sin\frac{\pi}{3}\right) = \frac{\pi}{3}$

ii. $\cos^{-1}\left(\cos\frac{7\pi}{6}\right) \neq \frac{7\pi}{6}$, as $\frac{7\pi}{6}$ does not lie between 0 and π

Now, $\cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left(\cos\left(2\pi - \frac{5\pi}{6}\right)\right) = \cos^{-1}\left(\cos\frac{5\pi}{6}\right) = \frac{5\pi}{6}$

iii. $\tan^{-1}\left(\tan\frac{2\pi}{3}\right) \neq \frac{2\pi}{3}$, because $\frac{2\pi}{3}$ does not lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$

Now, $\tan^{-1}\left(\tan\frac{2\pi}{3}\right) = \tan^{-1}\left(\tan\left(\pi - \frac{\pi}{3}\right)\right) = \tan^{-1}\left(-\tan\frac{\pi}{3}\right) = \tan^{-1}\left(\tan\left(-\frac{\pi}{3}\right)\right) = -\frac{\pi}{3}$

iv. $\cos\left(\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) + \frac{\pi}{6}\right) = \cos\left(\frac{5\pi}{6} + \frac{\pi}{6}\right) = \cos(\pi) = -1$

Example 4.3 **Evaluate the following:**

i. $\sin^{-1}(\sin 10)$ **ii.** $\sin^{-1}(\sin 5)$ **iii.** $\cos^{-1}(\cos 10)$ **iv.** $\tan^{-1}(\tan(-6))$

Sol i. Here, $\theta = 10$ rad does not lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

But, $3\pi - \theta$, i.e., $3\pi - 10$ lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

Also, $\sin(3\pi - 10) = \sin 10$

$\therefore \quad \sin^{-1}(\sin 10) = \sin^{-1}(\sin(3\pi - 10)) = (3\pi - 10)$

ii. Here, $\theta = 5$ rad. Clearly, it does not lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. But both $2\pi - 5$ and $5 - 2\pi$ lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Therefore,

$\sin(5 - 2\pi) = \sin(-(2\pi - 5)) = -\sin(2\pi - 5) = -(-\sin 5) = \sin 5$

$\Rightarrow \quad \sin^{-1}(\sin 5) = \sin^{-1}(\sin(5 - 2\pi)) = 5 - 2\pi$

iii. We know that $\cos^{-1}(\cos\theta) = \theta$, if $0 \le \theta \le \pi$
Here, $\theta = 10$ rad.
Clearly, it does not lie between 0 and π.
However, $(4\pi - 10)$ lies between 0 and π such that $\cos(4\pi - 10) = \cos 10$
$\Rightarrow \quad \cos^{-1}(\cos 10) = \cos^{-1}(\cos(4\pi - 10)) = 4\pi - 10$

iv. we know that $\tan^{-1}(\tan\theta) = \theta$, if $-\pi/2 < \theta < \pi/2$.
Here, $\theta = -6$ rad does not lie between $-\pi/2$ and $\pi/2$. We find that $2\pi - 6$ lies between $-\pi/2$ and $\pi/2$ such that
$\tan(2\pi - 6) = -\tan 6 = \tan(-6)$
$\therefore \quad \tan^{-1}(\tan(-6)) = \tan^{-1}(\tan(2\pi - 6)) = 2\pi - 6$

Example 4.4 **Evaluate the following:**

i. $\sin\left(\cos^{-1}\frac{3}{5}\right)$ **ii.** $\cos\left(\tan^{-1}\frac{3}{4}\right)$ **iii.** $\sin\left(\frac{\pi}{2} - \sin^{-1}\left(\frac{1}{2}\right)\right)$

Sol.

i. Let $\cos^{-1} 3/5 = \theta$.
Then, $\cos\theta = 3/5 \quad \Rightarrow \quad \sin\theta = 4/5$
$\therefore \sin(\cos^{-1} 3/5) = \sin\theta = 4/5$

ii. Let $\tan^{-1} 3/4 = \theta$.

Then, $\tan\theta = 3/4 \quad \Rightarrow \quad \cos\theta = 4/5 \qquad \left(\because \text{as } \cos^2\theta = \frac{1}{1 + \tan^2\theta}\right)$

$\therefore \cos(\tan^{-1}(3/4)) = \cos\theta = 4/5$

iii. $\sin\left(\frac{\pi}{2} - \sin^{-1}\left(-\frac{1}{2}\right)\right) = \sin\left(\frac{\pi}{2} - \left(-\frac{\pi}{6}\right)\right) = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

Example 4.5 **If $\cos^{-1}\lambda + \cos^{-1}\mu + \cos^{-1}\gamma = 3\pi$, then find the value of $\lambda\mu + \mu\gamma + \gamma\lambda$.**

Sol.

We know that $0 \le \cos^{-1} x \le \pi$.
Hence, from the question
$\cos^{-1}\lambda = \pi$, $\cos^{-1}\mu = \pi$, $\cos^{-1}\gamma = \pi$
[$\because \cos^{-1}\lambda + \cos^{-1}\mu + \cos^{-1}\gamma = 3\pi$ is possible only when each term attains its maximum.]
$\Rightarrow \quad \lambda = \mu = \gamma = -1 \quad \Rightarrow \quad \lambda\mu + \mu\gamma + \gamma\lambda = 3$

Example 4.6 **If $\cos(2\sin^{-1} x) = \frac{1}{9}$, then find the values of x.**

Sol. Let $\sin^{-1} x = \theta$

$\therefore \cos 2\theta = \frac{1}{9}$

$\Rightarrow \quad 1 - 2\sin^2\theta = 9 \quad \Rightarrow \quad 1 - 2x^2 = \frac{1}{9} \quad \Rightarrow \quad x^2 = \frac{4}{9} \quad \Rightarrow \quad x = \pm\frac{2}{3}$

Example 4.7 **Find the value of $\sin\left(\frac{1}{2}\cot^{-1}\left(-\frac{3}{4}\right)\right)$.**

Sol.

Let $\cot^{-1}(-3/4) = \theta \quad \Rightarrow \quad \cot\theta = -3/4 \quad \Rightarrow \quad \theta \in (\pi/2, \pi)$
$\Rightarrow \quad \cos\theta = -3/5\ (\theta \in (\pi/2, \pi)) \quad \Rightarrow \quad 1 - 2\sin^2(\theta/2) = -3/5$

$\therefore\quad \sin^2 \theta/2 = 4/5$ or $\sin\theta/2 = 2/\sqrt{5}$

Example 4.8 **Find the number of solutions of the equation $\cos(\cos^{-1}x) = \text{cosec}(\text{cosec}^{-1}x)$.**

Sol.

$\cos(\cos^{-1}x) = x$ for $x \in [-1, 1]$

$\text{cosec}(\text{cosec}^{-1}x) = x$ for $x \in (-\infty, -1] \cup [1, \infty)$

$\Rightarrow \cos(\cos^{-1}x) = \text{coesec}(\text{cosec}^{-1}x)$ for $x = \pm 1$ only.

Hence, there are two roots only.

Example 4.9 **Find the range of $f(x) = |3\tan^{-1}x - \cos^{-1}(0)| - \cos^{-1}(-1)$.**

Sol.

$f(x) = |3\tan^{-1}x - \cos^{-1}(0)| - \cos^{-1}(-1) = |3\tan^{-1}x - (\pi/2)| - \pi$

Now $-\dfrac{\pi}{2} < \tan^{-1} x < \dfrac{\pi}{2}$

$\Rightarrow -\dfrac{3\pi}{2} < 3\tan^{-1} x < \dfrac{3\pi}{2}$

$\Rightarrow -2\pi < 3\tan^{-1} x - \dfrac{\pi}{2} < \pi$

$\Rightarrow 0 \le \left|3\tan^{-1} x - \dfrac{\pi}{2}\right| < 2\pi$

$\Rightarrow -\pi \le \left|3\tan^{-1} x - \dfrac{\pi}{2}\right| - \pi < \pi$

Concept Application Exercise 4.1

1. Find the principal of

   i. $\text{cosec}^{-1}(-1)$ ii. $\cot^{-1}\left(-\dfrac{1}{\sqrt{3}}\right)$

2. Find the principal value of

   i. $\sin^{-1}(\sin 3)$ ii. $\sin^{-1}(\sin 100)$ iii. $\cos^{-1}(\cos 20)$ iv. $\cot^{-1}(\cot 4)$

3. Evaluate the following:

   i. $\sin(\cot^{-1} x)$ ii. $\sin\left(\dfrac{\pi}{2} - \sin^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)\right)$

4. If $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \dfrac{3\pi}{2}$, then find the value $x^2 + y^2 + z^2$.

5. If $(\sin^{-1} x)^2 + (\sin^{-1} y)^2 + (\sin^{-1} z)^2 = \dfrac{3\pi^2}{4}$, then find the minimum value of $x + y + z$.

6. Find the value of $\tan\left[\dfrac{1}{2}\cos^{-1}\dfrac{\sqrt{5}}{3}\right]$.

Property 3

i. $\sin^{-1}(-x) = -\sin^{-1}(x)$, for all $x \in [-1, 1]$
ii. $\cos^{-1}(-x) = \pi - \cos^{-1} x$, for all $x \in [-1, 1]$
iii. $\tan^{-1}(-x) = -\tan^{-1} x$, for all $x \in R$
iv. $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1} x$, for all $x \in (-\infty, -1] \cup [1, \infty)$
v. $\sec^{-1}(-x) = \pi - \sec^{-1} x$, for all $x \in (-\infty, -1] \cup [1, \infty)$
vi. $\cot^{-1}(-x) = \pi - \cot^{-1} x$, for all $x \in R$

Proof:

i. Clearly, $-x \in [-1, 1]$ for all $x \in [-1, 1]$
Let $\sin^{-1}(-x) = \theta$ (i)
$\Rightarrow \quad -x = \sin\theta$
$\Rightarrow \quad x = -\sin\theta$
$\Rightarrow \quad x = \sin(-\theta)$
$\Rightarrow \quad -\theta = \sin^{-1} x$ $[\because x \in [-1, 1] \text{ and } -\theta \in [-\pi/2, \pi/2] \text{ for all } \theta \in [-\pi/2, \pi/2]]$
$\Rightarrow \quad \theta = -\sin^{-1} x$ (ii)
From Eqs. (i) and (ii), we get $\sin^{-1}(-x) = -\sin^{-1} x$

Proof:

ii. Clearly, $-x \in [-1, 1]$ for all $x \in [-1, 1]$
Let $\cos^{-1}(-x) = \theta$ (i)
$\Rightarrow \quad -x = \cos\theta$
$\Rightarrow \quad x = -\cos\theta$
$\Rightarrow \quad x = \cos(\pi - \theta)$
$\Rightarrow \quad \cos^{-1} x = \pi - \theta$ $[\because x \in [-1, 1] \text{ and } \pi - \theta \in [0, \pi] \text{ for all } \theta \in [0, \pi]]$
$\Rightarrow \quad \theta = \pi - \cos^{-1} x$ (ii)
From Eqs. (i) and (ii), we get
$\cos^{-1}(-x) = \pi - \cos^{-1} x$
Similarly, we can prove other results.

Property 4

i. $\sin^{-1}\left(\frac{1}{x}\right) = \text{cosec}^{-1} x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

ii. $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

iii. $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1} x, & \text{for } x > 0 \\ -\pi + \cot^{-1} x, & \text{for } x < 0 \end{cases}$

Proof:

i. Let $\text{cosec}^{-1} x = \theta$ (i)
where $\theta \in [-\pi/2, \pi/2] - \{0\}$ and $x \in (-\infty, -1] \cup [1, \infty)$
$\Rightarrow \quad x = \text{cosec}\,\theta \quad \Rightarrow \quad \frac{1}{x} = \sin\theta \quad \Rightarrow \quad \theta = \sin^{-1}\frac{1}{x}$ (ii)
From Eqs. (i) and (ii), we get $\sin^{-1}(1/x) = \text{cosec}^{-1} x$.

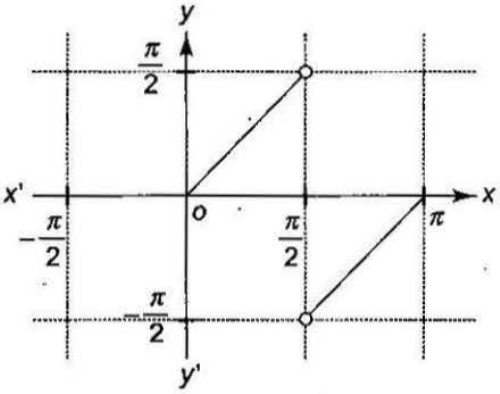
ii. Let $\sec^{-1} x = \theta$ (i)
where $\theta \in [0, \pi] - \{\pi/2\}$ and $x \in (-\infty, -1] \cup [1, \infty)$
Now, $\sec^{-1} x = \theta$

$$\Rightarrow \quad x = \sec\theta \quad \Rightarrow \quad \frac{1}{x} = \cos\theta \quad \Rightarrow \quad \theta = \cos^{-1}\frac{1}{x} \tag{ii}$$

From Eqs.(i) and (ii), we get $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x$.

iii. Let $\cot^{-1} x = \theta$, where $\theta \in (0, \pi)$ and $x \in R$

$$\Rightarrow \quad x = \cot\theta \quad \Rightarrow \quad \frac{1}{x} = \tan\theta \quad \Rightarrow \quad \tan^{-1}\left(\frac{1}{x}\right) = \tan^{-1}(\tan\theta)$$



Graph of $y = \tan^{-1}(\tan x)$

Fig.4.17

From the graph,

$$\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases}\theta, & 0<\theta<\pi/2 \\ -\pi+\theta, & \pi/2<\theta<\pi\end{cases} = \begin{cases}\cot^{-1}x, & 0<\cot^{-1}x<\pi/2 \\ -\pi+\cot^{-1}x, & \pi/2<\cot^{-1}x<\pi\end{cases} = \begin{cases}\cot^{-1}x, & x>0 \\ -\pi+\cot^{-1}x, & x<0\end{cases}$$

Example 4.10 **Prove that $\tan^{-1} x + \tan^{-1}\frac{1}{x} = \begin{cases}\pi/2, & \text{if } x>0 \\ -\pi/2, & \text{if } x<0\end{cases}$**

Sol.

We know that $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases}\cot^{-1}x, & x>0 \\ -\pi+\cot^{-1}x, & x<0\end{cases}$

$$\Rightarrow \quad \tan^{-1}x + \tan^{-1}\frac{1}{x} = \begin{cases}\tan^{-1}x+\cot^{-1}x, & x>0 \\ -\pi+\cot^{-1}x+\tan^{-1}x, & x<0\end{cases} = \begin{cases}\frac{\pi}{2}, & x>0 \\ -\pi+\frac{\pi}{2}, & x<0\end{cases} = \begin{cases}\frac{\pi}{2}, & x>0 \\ -\frac{\pi}{2}, & x<0\end{cases}$$

Example 4.11 **If $x > y > z > 0$, then find the value of $\cot^{-1}\frac{xy+1}{x-y} + \cot^{-1}\frac{yz+1}{y-z} + \cot^{-1}\frac{zx+1}{z-x}$.**

Sol.

$$\cot^{-1}\frac{xy+1}{x-y} + \cot^{-1}\frac{yz+1}{y-z} + \cot^{-1}\frac{zx+1}{z-x}$$

$= \tan^{-1}\dfrac{x-y}{1+xy} + \tan^{-1}\dfrac{y-z}{1+yz} + \pi + \tan^{-1}\dfrac{z-x}{1+zx} \quad \left[\because \tan^{-1}\left(\dfrac{1}{x}\right) = \begin{cases} \cot^{-1}x, & \text{for } x>0 \\ -\pi+\cot^{-1}x, & \text{for } x<0 \end{cases}\right]$

$= \tan^{-1}x - \tan^{-1}y + \tan^{-1}y - \tan^{-1}z + \pi + \tan^{-1}z - \tan^{-1}x$

$= \pi$

Example 4.12 **Find the value of x for which $\sec^{-1}x + \sin^{-1}x = \dfrac{\pi}{2}$.**

Sol.

We know that $\sec^{-1}x$ is defined for $x \in (-\infty, -1] \cup [1, \infty)$.
But $\sin^{-1}x$ is defined for $x \in [-1, 1]$

Hence, $\sec^{-1}x + \sin^{-1}x = \dfrac{\pi}{2}$ for $x = \pm 1$.

Concept Application Exercise 4.2

1. If $\tan^{-1}\left(\dfrac{1}{y}\right) = -\pi + \cot^{-1}y$, where $y = x^2 - 3x + 2$, then find the value of x.
2. If $\alpha \in \left(-\dfrac{\pi}{2}, 0\right)$, then find the value of $\tan^{-1}(\cot\alpha) - \cot^{-1}(\tan\alpha)$.

Property 5

i. $\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}$, for all $x \in [-1, 1]$

ii. $\tan^{-1}x + \cot^{-1}x = \dfrac{\pi}{2}$, for all $x \in R$

iii. $\sec^{-1}x + \operatorname{cosec}^{-1}x = \dfrac{\pi}{2}$ for all $x \in (-\infty, -1] \cup [1, \infty)$

Proof:

i. Let $\sin^{-1}x = \theta$ (i)

where $\theta \in [-\pi/2, \pi/2]$

$\Rightarrow -\dfrac{\pi}{2} \le -\theta \le \dfrac{\pi}{2}$

$\Rightarrow 0 \le \dfrac{\pi}{2} - \theta \le \pi$

$\Rightarrow \dfrac{\pi}{2} - \theta \in [0, \pi]$

Now, $\sin^{-1}x = \theta$

$\Rightarrow x = \sin\theta$

$\Rightarrow x = \cos\left(\dfrac{\pi}{2} - \theta\right)$

$\Rightarrow \quad \cos^{-1}x = \frac{\pi}{2} - \theta$ $\quad [\because x \in [-1, 1] \text{ and } (\pi/2 - \theta) \in [0, \pi]]$

$\Rightarrow \quad \theta + \cos^{-1}x = \frac{\pi}{2}$ (ii)

From Eqs. (i) and (ii), we get $\sin^{-1}x + \cos^{-1}x = \pi/2$. Similarly, we get the other results.

Example 4.13 **If $\sin^{-1}x = \pi/5$, for some $x \in (-1, 1)$, then find the value of $\cos^{-1}x$.**

Sol.

$$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2} \quad \Rightarrow \quad \cos^{-1}x = \frac{\pi}{2} - \sin^{-1}x = \frac{\pi}{2} - \frac{\pi}{5} = \frac{3\pi}{10}$$

Example 4.14 **If $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1}x\right) = 1$, then find the value of x.**

Sol.

$$\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1}x\right) = 1 = 1$$

$$\Rightarrow \quad \sin^{-1}\frac{1}{5} + \cos^{-1}x = \frac{\pi}{2}$$

$$\Rightarrow \quad \sin^{-1}\frac{1}{5} = \frac{\pi}{2} - \cos^{-1}x$$

$$\Rightarrow \quad \sin^{-1}\frac{1}{5} = \sin^{-1}x$$

$$\Rightarrow \quad x = \frac{1}{5}$$

Example 4.15 **Solve $\sin^{-1}x \le \cos^{-1}x$.**

Sol.

$$\cos^{-1}x \ge \sin^{-1}x \Rightarrow \frac{\pi}{2} \ge 2\sin^{-1}x \quad \Rightarrow \quad \sin^{-1}x \le \frac{\pi}{4}$$

$$\Rightarrow \quad -1 \le x \le \sin\left(\frac{\pi}{4}\right) \quad \Rightarrow \quad x \in \left[-1, \frac{1}{\sqrt{2}}\right]$$

Example 4.16 **Find the range of $f(x) = \sin^{-1}x + \tan^{-1}x + \cos^{-1}x$.**

Sol.

Clearly, the domain of the function is $[-1, 1]$.

Also, $\tan^{-1}x \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ for $x \in [-1, 1]$.

Now, $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ for $x \in [-1, 1]$.

Thus, $f(x) = \tan^{-1}x + \frac{\pi}{2}$, where $x \in [-1, 1]$.

Hence, the range is $\left[-\frac{\pi}{4} + \frac{\pi}{2}, \frac{\pi}{4} + \frac{\pi}{2}\right] = \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$.

Example 4.17 **Find the minimum value of $(\sec^{-1} x)^2 + (\text{cosec}^{-1} x)^2$.**

Sol.

Let $I = (\sec^{-1} x)^2 + (\text{cosec}^{-1} x)^2$

$$= (\sec^{-1} x + \text{cosec}^{-1} x)^2 - 2\sec^{-1} x \,\text{cosec}^{-1} x$$

$$= \frac{\pi^2}{4} - 2\sec^{-1} x\left(\frac{\pi}{2} - \sec^{-1} x\right)$$

$$= \frac{\pi^2}{4} + 2\left(\sec^{-1} x\right)^2 - \pi \sec^{-1} x$$

$$= \frac{\pi^2}{4} + 2\left[\left(\sec^{-1} x\right)^2 - 2\frac{\pi}{4}\sec^{-1} x + \left(\frac{\pi}{4}\right)^2\right] - \frac{\pi^2}{8}$$

$$= 2\left(\sec^{-1} x - \frac{\pi}{4}\right)^2 + \frac{\pi^2}{8} \Rightarrow I \geq \frac{\pi^2}{8}$$

Example 4.18 **Solve $\sin^{-1}\frac{14}{|x|} + \sin^{-1}\frac{2\sqrt{15}}{|x|} = \frac{\pi}{2}$.**

Sol.

$$\sin^{-1}\frac{14}{|x|} + \sin^{-1}\frac{2\sqrt{15}}{|x|} = \frac{\pi}{2}$$

$$\Rightarrow \quad \sin^{-1}\frac{14}{|x|} = \frac{\pi}{2} - \sin^{-1}\frac{2\sqrt{15}}{|x|} = \cos^{-1}\frac{2\sqrt{15}}{|x|} = \sin^{-1}\sqrt{1-\left(\frac{2\sqrt{15}}{|x|}\right)^2}$$

For $0 \leq \frac{2\sqrt{15}}{|x|} \leq 1$ or $|x| \geq 2\sqrt{15}$, we have

$$\left(\frac{14}{|x|}\right)^2 = 1-\left(\frac{2\sqrt{15}}{|x|}\right)^2 \quad \Rightarrow \quad |x| = 16 \quad \Rightarrow \quad x = \pm 16 \quad \text{which satisfy } |x| \geq 2\sqrt{15}.$$

Example 4.19. **If $\alpha = \sin^{-1}(\cos(\sin^{-1} x))$ and $\beta = \cos^{-1}(\sin(\cos^{-1} x))$, then find $\tan\alpha \cdot \tan\beta$.**

Sol.

$$\beta = \cos^{-1}\left(\sin\left(\frac{\pi}{2} - \sin^{-1} x\right)\right) = \cos^{-1}[\cos(\sin^{-1} x)] \text{ also } \alpha = \sin^{-1}[\cos(\sin^{-1} x)]$$

$$\alpha + \beta = \pi/2 \Rightarrow \tan\alpha = \cot\beta \Rightarrow \tan\alpha \cdot \tan\beta = 1$$

Example 4.20 **Find the value of $\sin^{-1} x + \sin^{-1}\frac{1}{x} + \cos^{-1} x + \cos^{-1}\frac{1}{x}$.**

Sol.

$\sin^{-1} x + \sin^{-1}\frac{1}{x} + \cos^{-1} x + \cos^{-1}\frac{1}{x}$ is defined only when $x, \frac{1}{x} \in [-1, 1]$

which is possible only when $x = \pm 1$

for which $\sin^{-1} x + \sin^{-1}\frac{1}{x} + \cos^{-1} x + \cos^{-1}\frac{1}{x} = \pi$

Concept Application Exercise 4.3

1. If $\sin^{-1} x + \sin^{-1} y = \frac{2\pi}{3}$, then find the value of $\cos^{-1} x + \cos^{-1} y$.
2. Solve $\cot^{-1} x + \tan^{-1} 3 = \frac{\pi}{2}$.
3. Solve $\sec^{-1}x > \operatorname{cosec}^{-1}x$.
4. Solve $\tan^{-1}x > \cot^{-1}x$.
5. Solve $2\cos^{-1} x + \sin^{-1} x = \frac{11\pi}{6}$.
6. Solve $(\tan^{-1} x)^2 + (\cot^{-1} x)^2 = \frac{5\pi^2}{8}$.

Property 6

i. For $x > 0$,

$$\sin^{-1} x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{x}{\sqrt{1-x^2}} = \cot^{-1}\frac{\sqrt{1-x^2}}{x} = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$$

Refer the following diagram for the proof.

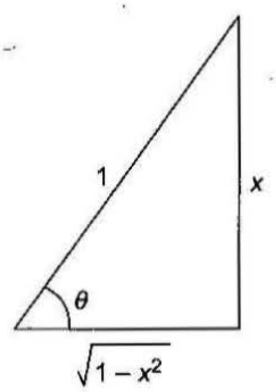


Fig. 4.18

ii. For $x > 0$,

$$\cos^{-1} x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \sec^{-1}\frac{1}{x} = \operatorname{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$$

Refer the following diagram for the proof.

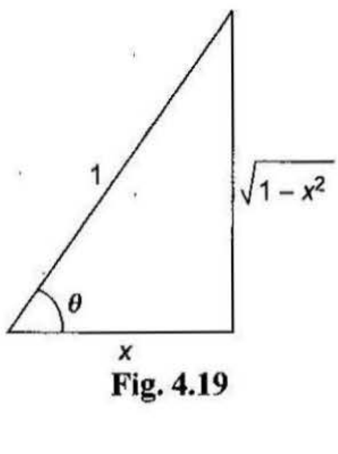


Fig. 4.19

iii. For $x > 0$,

$$\tan^{-1} x = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = \cot^{-1}\left(\frac{1}{x}\right)$$

$$= \sec^{-1}\sqrt{1+x^2} = \text{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right)$$

Refer the following diagram for the proof.

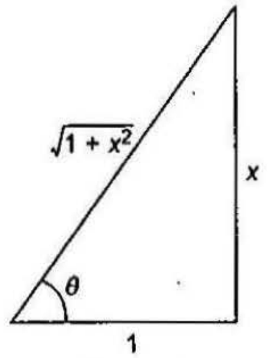


Fig. 4.20

Example 4.21 **Find $\tan^{-1}\frac{x}{\sqrt{a^2-x^2}}$, in terms of $\sin^{-1}$ where $x \in (0, a)$.**

Sol.

$$\tan^{-1}\frac{x}{\sqrt{a^2-x^2}} = \tan^{-1}\left(\frac{a\sin\theta}{a\cos\theta}\right) \qquad \text{[putting } x = a\sin\theta]$$

$$= \tan^{-1}(\tan\theta) = \theta = \sin^{-1}\left(\frac{x}{a}\right)$$

Example 4.22 **Prove that $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \frac{1}{2}\tan^{-1}x$.**

Sol.

$$\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \tan^{-1}\left[\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right] \qquad \text{[putting } x = \tan\theta]$$

$$= \tan^{-1}\left[\frac{\sec\theta-1}{\tan\theta}\right]$$

$$= \tan^{-1}\left[\frac{1-\cos\theta}{\sin\theta}\right]$$

$$= \tan^{-1}\left[\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right]$$

$$= \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x$$

Example 4.23 **Simplify $\sin \cot^{-1} \tan \cos^{-1} x$.**

Sol.

Let $\cos^{-1} x = \theta$

$\Rightarrow \quad x = \cos \theta$

$\Rightarrow \quad \sec \theta = \dfrac{1}{x}$

$\Rightarrow \quad \tan \theta = \sqrt{\sec^2 \theta - 1} = \sqrt{\dfrac{1}{x^2} - 1} = \dfrac{1}{|x|}\sqrt{1 - x^2}$

Now, $\sin \cot^{-1} \tan \theta = \sin \cot^{-1} \left(\dfrac{1}{|x|}\sqrt{1 - x^2}\right)$.

Again, putting $x = \sin \theta$, we get

$$\sin \cot^{-1} \left(\frac{1}{|x|}\sqrt{1 - x^2}\right) = \sin \cot^{-1} \left(\frac{\sqrt{1 - \sin^2 \theta}}{|\sin \theta|}\right) = \sin \cot^{-1} |\cot \theta| = \sin \theta = x$$

Example 4.24 **Prove that $\operatorname{cosec} (\tan^{-1} (\cos (\cot^{-1} (\sec(\sin^{-1} a))))) = \sqrt{3 - a^2}$, where $a \in [0, 1]$.**

Sol.

Here $x = \operatorname{cosec}(\tan^{-1} (\cos (\cot^{-1} (\sec (\sin^{-1} a)))))$

$$= \operatorname{cosec}\left(\tan^{-1}\left(\cos\left(\cot^{-1}\left(\frac{1}{\sqrt{1-a^2}}\right)\right)\right)\right)$$

$$= \operatorname{cosec}\left(\tan^{-1}\left(\frac{1}{\sqrt{2-a^2}}\right)\right)$$

$$= \sqrt{3-a^2} \qquad \text{(i)}$$

Example 4.25 **If $x < 0$, then prove that $\cos^{-1}x = \pi - \sin^{-1} \sqrt{1 - x^2}$.**

Sol.

Let $x = \cos \theta \quad \Rightarrow \quad \cos \theta = x$

Since $x < 0$, $\theta \in [\pi/2, \pi]$

Now, $\sin^{-1} \sqrt{1-x^2} = \sin^{-1} \sqrt{1-\cos^2 \theta}$

$= \sin^{-1} (\sin \theta) \neq \theta \qquad (\because \theta \notin [-\pi/2, \pi/2])$

$= \sin^{-1} (\sin(\pi - \theta)) = \pi - \theta$

$\Rightarrow \quad \cos^{-1}x = \pi - \sin^{-1} \sqrt{1-x^2}$

Example 4.26 **Prove that $\cos^{-1}\left\{\sqrt{\dfrac{1+x}{2}}\right\} = \dfrac{\cos^{-1} x}{2}$, $-1 < x < 1$.**

Sol.

Let $x = \cos \theta$, where $\theta \in [0, \pi]$

$$\Rightarrow \cos^{-1}\left\{\sqrt{\frac{1+x}{2}}\right\} = \cos^{-1}\left\{\sqrt{\frac{1+\cos\theta}{2}}\right\}$$

$$= \cos^{-1}\left\{\sqrt{\frac{2\cos^2\frac{\theta}{2}}{2}}\right\}$$

$$= \cos^{-1}\left(\cos\frac{\theta}{2}\right)$$

$$= \frac{\theta}{2}$$

$$= \frac{\cos^{-1} x}{2}$$

Example 4.27 **Prove that** $\tan^{-1}\left\{\frac{x}{a+\sqrt{a^2-x^2}}\right\} = \frac{1}{2}\sin^{-1}\frac{x}{a}, -a < x < a.$

Sol.

Let $x = a\sin\theta$, since $-a < x < a$

$\Rightarrow -a < a\sin\theta < a \Rightarrow -1 < \sin\theta < 1 \Rightarrow \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\Rightarrow \tan^{-1}\left\{\frac{x}{a+\sqrt{a^2-x^2}}\right\} = \tan^{-1}\left\{\frac{a\sin\theta}{a+\sqrt{a^2-\sin^2\theta}}\right\}$$

$$= \tan^{-1}\left\{\frac{\sin\theta}{1+\cos\theta}\right\}$$

$$= \tan^{-1}\left\{\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}\right\}$$

$$= \tan^{-1}\left\{\tan\frac{\theta}{2}\right\}$$

$$= \frac{\theta}{2}$$

$$= \frac{1}{2}\sin^{-1}\frac{x}{a}$$

Example 4.28 **Prove that** $\sin^{-1}\left\{\frac{\sqrt{1+x}+\sqrt{1-x}}{2}\right\} = \frac{\pi}{4} + \frac{\sin^{-1} x}{2}, 0 < x < 1.$

Sol.

Let $x = \sin\theta$. Since $0 < x < 1 \Rightarrow 0 < \sin\theta < 1 \Rightarrow \theta \in \left(0, \frac{\pi}{2}\right)$

$$\Rightarrow \sin^{-1}\left\{\frac{\sqrt{1+x}+\sqrt{1-x}}{2}\right\} = \sin^{-1}\left\{\frac{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}{2}\right\}$$

$$= \sin^{-1}\left\{\frac{\sqrt{2\cos^2\frac{\theta}{2}}+\sqrt{2\sin^2\frac{\theta}{2}}}{2}\right\}$$

$$= \sin^{-1}\left\{\frac{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{\sqrt{2}}\right\}$$

$$= \sin^{-1}\left\{\sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right\}$$

$$= \frac{\pi}{4}+\frac{\theta}{2} \qquad \left[\because \theta\in\left(0,\frac{\pi}{2}\right)\Rightarrow\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\in\left(\frac{\pi}{4},\frac{\pi}{2}\right)\right]$$

$$= \frac{\pi}{4}+\frac{\sin^{-1}x}{2}$$

Example 4.29 **Prove that $\cos^{-1}\left(\frac{1-x^{2n}}{1+x^{2n}}\right) = 2\tan^{-1}x^n, 0 < x < \infty$.**

Sol.

Since $0 < x < \infty$; $0 < x^n < \infty$

Let $x^n = \tan\theta \Rightarrow \theta \in (0, \pi/2)$

$$\cos^{-1}\left(\frac{1-x^{2n}}{1+x^{2n}}\right) = \cos^{-1}\left(\frac{1-\tan^2\theta}{1-\tan^2\theta}\right)$$

$$= \cos^{-1}(\cos 2\theta)$$

$$= 2\theta$$

$$= 2\tan^{-1}x^n$$

Concept Application Exercise 4.4

1. Evaluate $\tan^{-1}\left(\frac{\sqrt{1+a^2x^2}-1}{ax}\right)$, where $x \neq 0$.

2. Express $\sin^{-1}\frac{\sqrt{x}}{\sqrt{x+a}}$ as a function of $\tan^{-1}$.

3. If $x < 0$, then prove that $\cos^{-1}x = \pi + \tan^{-1}\frac{\sqrt{1-x^2}}{x}$.

4. If $\tan(\cos^{-1}x) = \sin\left(\cot^{-1}\frac{1}{2}\right)$, then find the value of x.

5. Evaluate $\sin^{-1}\left(\frac{x+\sqrt{1-x^2}}{\sqrt{2}}\right)$, where $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$.

6. Prove that $\sin\left[2\tan^{-1}\left\{\sqrt{\frac{1-x}{1+x}}\right\}\right] = \sqrt{1-x^2}$.

Property 7

i. $\tan^{-1}x + \tan^{-1}y = \begin{cases} \tan^{-1}\left(\frac{x+y}{1-xy}\right), \text{if } xy<1 & \text{if } xy<1 \\ \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right), & \text{if } x>0,\ y>0 \text{ and } xy>1 \\ -\pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right), & \text{if } x<0, y<0 \text{ and } xy>1 \end{cases}$

ii. $\tan^{-1}x - \tan^{-1}y = \begin{cases} \tan^{-1}\left(\frac{x-y}{1+xy}\right), & \text{if } xy>-1 \\ \pi + \tan^{-1}\left(\frac{x-y}{1+xy}\right), & \text{if } x>0,\ y<0 \text{ and } xy<-1 \\ -\pi + \tan^{-1}\left(\frac{x-y}{1+xy}\right), & \text{if } x<0,\ y>0 \text{ and } xy<-1 \end{cases}$

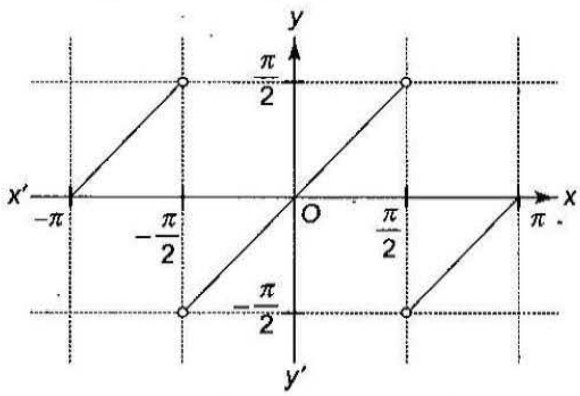
Proof:

i. Let $\tan^{-1}x = A$ and $\tan^{-1}y = B$, where $A, B \in (-\pi/2, \pi/2)$.

Now, $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{x+y}{1-xy}$

$\Rightarrow \quad \tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1}\tan(A+B)$

$= \tan^{-1}\tan\alpha$, where $\alpha \in (-\pi, \pi)$



Graph of $y = \tan^{-1}(\tan x)$

Fig. 4.21

From the graph,

$$\tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1}(\tan\alpha) = \begin{cases} \alpha+\pi, & -\pi<\alpha<(-\pi/2) \\ \alpha, & (-\pi/2)\le\alpha\le(\pi/2) \\ \alpha-\pi, & (\pi/2)<\alpha<\pi \end{cases}$$

$$= \begin{cases} \tan^{-1}x+\tan^{-1}y+\pi, & -\pi<\tan^{-1}x+\tan^{-1}y<(-\pi/2) \\ \tan^{-1}x+\tan^{-1}y, & (-\pi/2)\le\tan^{-1}x+\tan^{-1}y\le(\pi/2) \\ \tan^{-1}x+\tan^{-1}y-\pi, & (\pi/2)<\tan^{-1}x+\tan^{-1}y<\pi \end{cases}$$

Case I

$-\pi<\tan^{-1}x+\tan^{-1}y<(-\pi/2) \quad \Rightarrow \quad x<0, y<0$

Also, $\tan^{-1}x<(-\pi/2)-\tan^{-1}y$

$\Rightarrow \quad \tan^{-1}x<-\left((\pi/2)-\tan^{-1}(-y)\right) \quad \Rightarrow \quad x<-\tan(-1/y) \quad \Rightarrow \quad x<(1/y) \quad \Rightarrow \quad xy>1$

Case II

$(\pi/2)<\tan^{-1}x+\tan^{-1}y<\pi \quad \Rightarrow \quad x,y>0$

Also, $\tan^{-1}x>(\pi/2)-\tan^{-1}y \quad \Rightarrow \quad \tan^{-1}x>\tan^{-1}(1/y) \quad \Rightarrow \quad x>(1/y) \quad \Rightarrow \quad xy>1.$

Case III

$(-\pi/2)\le\tan^{-1}x+\tan^{-1}x\le(\pi/2) \quad \Rightarrow \quad xy<1$

This property can be proved by replacing y by $-y$.

Example 4.30 **Find the value of $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}$.**

Sol.

Here, $\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}<1 \Rightarrow \quad \tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\tan^{-1}\left(\frac{(1/2)+(1/3)}{1-(1/2)\times(1/3)}\right)=\tan^{-1}1=\frac{\pi}{4}$

Example 4.31 **If two angles of a triangle are $\tan^{-1}(2)$ and $\tan^{-1}(3)$, then find the third angle.**

Sol.

Given two angles are $\tan^{-1}(2)$ and $\tan^{-1}(3)$. Now $(2)(3)>1$

$\Rightarrow \quad \tan^{-1}(2)$ and $\tan^{-1}(3) = \pi+\tan^{-1}\left(\frac{2+3}{1-2\times3}\right) = \pi+\tan^{-1}(-1) = \pi-\frac{\pi}{4}=\frac{3\pi}{4}$. Hence, the third angle is $\pi-\frac{3\pi}{4}=\frac{\pi}{4}$.

Example 4.32 **Solve $\tan^{-1}\frac{x-1}{x+2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$.**

Sol.

$$\tan^{-1}\frac{x-1}{x+2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$$

$$\Rightarrow \quad \tan^{-1}\left[\frac{\dfrac{x-1}{x+2}+\dfrac{x+1}{x+2}}{1-\left(\dfrac{x-1}{x+2}\right)\left(\dfrac{x+1}{x+2}\right)}\right]=\frac{\pi}{4}$$

$$\Rightarrow \quad \left[\frac{2x\,(x+2)}{x^2+4+4x-x^2+1}\right]=\tan\frac{\pi}{4}$$

$$\Rightarrow \quad \frac{2x\,(x+2)}{4x+5}=\tan\frac{\pi}{4}=1 \qquad \Rightarrow \quad 2x^2+4x=4x+5$$

$$\Rightarrow \quad x=\pm\sqrt{\frac{5}{2}}$$

But for $x=-\sqrt{\dfrac{5}{2}}$, L.H.S. is negative. Hence $x=\sqrt{\dfrac{5}{2}}$.

Example 4.33 **Find the value of $\tan^{-1}\left(\dfrac{1}{2}\tan 2A\right)+\tan^{-1}(\cot A)+\tan^{-1}(\cot^3 A)$, for $0<A<\dfrac{\pi}{4}$.**

Sol.

For $0<A<(\pi/4)$, $\cot A>1 \qquad \Rightarrow \quad (\cot A)(\cot^3 A)>1$

Then $\tan^{-1}\left(\dfrac{1}{2}\tan 2A\right)+\tan^{-1}(\cot A)+\tan^{-1}(\cot^3 A)$

$$=\tan^{-1}\left(\frac{\tan A}{1-\tan^2 A}\right)+\pi+\tan^{-1}\left(\frac{\cot A+\cot^3 A}{1-\cot^4 A}\right)$$

$$=\tan^{-1}\left(\frac{\tan A}{1-\tan^2 A}\right)+\pi+\tan^{-1}\left(\frac{\cot A}{1-\cot^2 A}\right)$$

$$=\tan^{-1}\left(\frac{\tan A}{1-\tan^2 A}\right)+\pi+\tan^{-1}\left(\frac{\tan A}{\tan^2 A-1}\right)=\pi$$

Example 4.34 **Simplify $\tan^{-1}\left[\dfrac{3\sin 2\alpha}{5+3\cos 2\alpha}\right]+\tan^{-1}\left[\dfrac{\tan\alpha}{4}\right]$, where $-\dfrac{\pi}{2}<\alpha<\dfrac{\pi}{2}$.**

Sol.

$$\tan^{-1}\left[\frac{3\sin 2\alpha}{5+3\cos 2\alpha}\right]+\tan^{-1}\left[\frac{\tan\alpha}{4}\right]=\tan^{-1}\left(\frac{6\tan\alpha}{8+2\tan^2\alpha}\right)+\tan^{-1}\left(\frac{\tan\alpha}{4}\right)$$

$$=\tan^{-1}\left(\frac{\dfrac{3\tan\alpha}{4+\tan^2\alpha}+\dfrac{\tan\alpha}{4}}{1-\dfrac{3\tan^2\alpha}{16+4\tan^2\alpha}}\right) \qquad \left[\because \frac{3\tan^2\alpha}{16+4\tan^2\alpha}<1\right]$$

$$=\tan^{-1}\left(\frac{12\tan\alpha+4\tan\alpha+\tan^3\alpha}{16+\tan^2\alpha}\right)$$

$$=\tan^{-1}(\tan\alpha)=\alpha$$

Example 4.35 If $a_1, a_2, a_3, \ldots, a_n$ is an A.P. with common difference d, then prove that

$$\tan\left[\tan^{-1}\left(\frac{d}{1+a_1a_2}\right)+\tan^{-1}\left(\frac{d}{1+a_2a_3}\right)+\cdots+\tan^{-1}\left(\frac{d}{1+a_{n-1}a_n}\right)\right]=\frac{(n-1)d}{1+a_1a_n}.$$

Sol.

We have

$$\tan^{-1}\left(\frac{d}{1+a_1a_2}\right)+\tan^{-1}\left(\frac{d}{1+a_2a_3}\right)+\cdots+\tan^{-1}\left(\frac{d}{1+a_{n-1}a_n}\right)$$

$$=\tan^{-1}\left(\frac{a_2-a_1}{1+a_1a_2}\right)+\tan^{-1}\left(\frac{a_3-a_2}{1+a_2a_3}\right)+\cdots+\tan^{-1}\left(\frac{a_n-a_{n-1}}{1+a_{n-1}a_n}\right)$$

$$=(\tan^{-1}a_2-\tan^{-1}a_1)+(\tan^{-1}a_3-\tan^{-1}a_2)+\cdots+(\tan^{-1}a_n-\tan^{-1}a_{n-1})$$

$$=\tan^{-1}a_n-\tan^{-1}a_1=\tan^{-1}\left(\frac{a_n-a_1}{1+a_na_1}\right)=\tan^{-1}\left(\frac{(n-1)d}{1+a_1a_n}\right).$$

$$\Rightarrow\quad \tan\left[\tan^{-1}\left(\frac{d}{1+a_1a_2}\right)+\tan^{-1}\left(\frac{d}{1+a_2a_3}\right)+\cdots+\tan^{-1}\left(\frac{d}{1+a_{n-1}a_n}\right)\right]=\frac{(n-1)d}{1+a_1a_n}$$

Example 4.36 Solve the equation $\tan^{-1}2x+\tan^{-1}3x=\pi/4$.

Sol. $\tan^{-1}2x+\tan^{-1}3x=\dfrac{\pi}{4}$

$$\tan^{-1}\left(\frac{2x+3x}{1-6x^2}\right)=\frac{\pi}{4}$$

$$\Rightarrow\quad \frac{5x}{1-6x^2}=1$$

$$\Rightarrow\quad 6x^2+5x-1=0 \qquad \text{(i)}$$

$\Rightarrow\quad (6x-1)(x+1)=0 \;\Rightarrow\; x=1/6$ or -1, but $x=-1$ does not satisfy Eq. (i). Hence, $x=1/6$.

Concept Application Exercise 4.5

1. Find the value of $\sin^{-1}\left(\dfrac{3}{5}\right)+\tan^{-1}\left(\dfrac{1}{7}\right)$.
2. Find the value of $\tan\left[\cos^{-1}\left(\dfrac{4}{5}\right)+\tan^{-1}\left(\dfrac{2}{3}\right)\right]$.
3. If $x>y>0$, then find the value of $\tan^{-1}\dfrac{x}{y}+\tan^{-1}\left[\dfrac{x+y}{x-y}\right]$.
4. Find the sum: $\tan^{-1}\dfrac{c_1x-y}{c_1y+x}+\tan^{-1}\dfrac{c_2-c_1}{1+c_2c_1}+\tan^{-1}\dfrac{c_3-c_2}{1+c_3c_2}+\cdots+\tan^{-1}\dfrac{1}{c_n}$.
5. If $x+y+z=xyz$, and $x, y, z>0$, then find the value of $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z$.
6. Find the value of $\tan^{-1}\left(\sqrt{\dfrac{a\lambda}{bc}}\right)+\tan^{-1}\left(\sqrt{\dfrac{b\lambda}{ca}}\right)+\tan^{-1}\left(\sqrt{\dfrac{c\lambda}{ab}}\right)$, where $a, b, c\in R^+$ and $\lambda=a+b+c$.

Property 8

i. $\sin^{-1}x+\sin^{-1}y=\begin{cases}\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right), & x\ge 0,\ y\ge 0 \text{ and } x^2+y^2\le 1\\ \pi-\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right), & x\ge 0,\ y\ge 0 \text{ and } x^2+y^2>1\end{cases}$

Proof:

Let $\sin^{-1}x = A$ and $\sin^{-1}y = B$, where $x \ge 0$ and $y \ge 0$

$\Rightarrow \quad A, B \in [0, \pi/2] \quad \Rightarrow \quad A + B \in [0, \pi]$

Now, $\sin(A + B) = \sin A \cos B + \sin B \cos A = x\sqrt{1-y^2}+y\sqrt{1-x^2}$

$\Rightarrow \quad \sin^{-1}(\sin(A+B)) = \sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$

$$\Rightarrow \quad \sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)=\begin{cases}A+B, & 0\le A+B\le(\pi/2)\\ \pi-(A+B), & (\pi/2)<A+B\le\pi\end{cases} \qquad \text{(i)}$$

Now, $A + B \le (\pi/2) \Rightarrow A \le (\pi/2) - B$

$\Rightarrow \quad \sin A \le \cos B \quad \Rightarrow \quad x \le \sqrt{1-y^2} \quad \Rightarrow \quad x^2+y^2\le 1$

And $A + B > \dfrac{\pi}{2} \Rightarrow x^2+y^2>1$

Hence from Eq. (i), we get

$$\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)=\begin{cases}\sin^{-1}x+\sin^{-1}y, & x^2+y^2\le 1\\ \pi-(\sin^{-1}x+\sin^{-1}y), & x^2+y^2>1\end{cases}$$

$$\Rightarrow \quad \sin^{-1}x+\sin^{-1}y=\begin{cases}\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right), & x\ge 0,\ y\ge 0 \text{ and } x^2+y^2\le 1\\ \pi-\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right), & x\ge 0,\ y\ge 0 \text{ and } x^2+y^2>1\end{cases}$$

Note:

For $x < 0$ and $y < 0$, these identities can be used with the help of property 3, i.e., change x and y to $-x$ and $-y$ which are positive.

ii. $\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left(xy-\sqrt{1-x^2}\sqrt{1-y^2}\right), x\ge 0, y\ge 0$

Proof:

Let $\cos^{-1}x = A$ and $\cos^{-1}y = B$, where $x \ge 0$ and $y \ge 0$

$\Rightarrow \quad A, B \in [0, \pi/2] \quad \Rightarrow \quad A + B \in [0, \pi]$

Now, $\cos(A + B) = \cos A \cos B - \sin B \sin A = xy-\sqrt{1-y^2}\sqrt{1-x^2}$

$$\Rightarrow \quad \cos^{-1}(\cos(A+B)) = \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right)$$

$$\Rightarrow \quad A + B = \cos^{-1}x + \cos^{-1}y = \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right)$$

iii. $\cos^{-1}x - \cos^{-1}y = \begin{cases} \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right), & x \ge 0,\ y \ge 0 \text{ and } x \le y \\ -\cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right), & x \ge 0,\ y \ge 0 \text{ and } x > y \end{cases}$

Proof:

Let $\cos^{-1}x = A$ and $\cos^{-1}y = B$, where $x \ge 0$ and $y \ge 0$

$\Rightarrow \quad A, B \in [0, \pi/2]$

If $x \le y$, then $\cos^{-1}x \ge \cos^{-1}y$ ($\because \cos^{-1}$ is a decreasing function)

$\Rightarrow \quad A \ge B \qquad \Rightarrow \quad A - B \in [0, \pi/2]$

Now, $\cos(A-B) = \cos A \cos B + \sin B \sin A = xy - \sqrt{1-y^2}\sqrt{1-x^2}$

$$\Rightarrow \quad \cos^{-1}(\cos(A-B)) = \cos^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$$

$$\Rightarrow \quad A - B = \cos^{-1}x - \cos^{-1}y = \cos^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$$

If $x > y$, then $\cos^{-1}x < \cos^{-1}y$

$\Rightarrow \quad A < B \qquad \Rightarrow \quad A - B \in [-\pi/2, 0)$

$$\Rightarrow \quad \cos^{-1}x - \cos^{-1}y = -\cos^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$$

Note:

For $x < 0$ and $y < 0$, these identities can be used with the help of property 3. i.e., change x and y to $-x$ and $-y$ which are positive.

Example 4.37 **Find the value of $\cot^{-1}\frac{3}{4} + \sin^{-1}\frac{5}{13}$.**

Sol. $\cot^{-1}\frac{3}{4} + \sin^{-1}\frac{5}{13} = \sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13}$

$$= \sin^{-1}\left(\frac{4}{5}\sqrt{1-\left(\frac{5}{13}\right)^2} + \frac{5}{13}\sqrt{1-\left(\frac{4}{5}\right)^2}\right) = \sin^{-1}\left(\frac{4}{5}\frac{12}{13} + \frac{5}{13}\frac{3}{5}\right) = \sin^{-1}\frac{63}{65}$$

Example 4.38 **Solve $\sin^{-1}x + \sin^{-1}2x = \frac{\pi}{3}$.**

Sol. $\sin^{-1}x + \sin^{-1}2x = \frac{\pi}{3}$ (i)

$$\sin^{-1}2x = \sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1}x = \sin^{-1}\left[\frac{\sqrt{3}}{2}\sqrt{1-x^2} - x\sqrt{1-\frac{3}{4}}\right]$$

$\Rightarrow \quad 2x = \frac{\sqrt{3}}{2}\sqrt{1-x^2} - \frac{x}{2}$

$\Rightarrow \quad \left(\frac{5x}{2}\right)^2 = \frac{3}{4}\left(1-x^2\right) \quad \Rightarrow \quad 28x^2 = 3$

$\Rightarrow \quad x = \sqrt{\frac{3}{28}} = \frac{1}{2}\sqrt{\frac{3}{7}} \qquad \left(\because x = -\frac{1}{2}\sqrt{\frac{3}{7}} \text{ makes L.H.S. of Eq. (i) negative}\right)$

Property 9

i. $$2\sin^{-1}x = \begin{cases} \sin^{-1}\left(2x\sqrt{1-x^2}\right), & -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}} \\ \pi - \sin^{-1}\left(2x\sqrt{1-x^2}\right), & x > \frac{1}{\sqrt{2}} \\ -\pi - \sin^{-1}\left(2x\sqrt{1-x^2}\right), & x < -\frac{1}{\sqrt{2}} \end{cases}$$

ii. $$3\sin^{-1}x = \begin{cases} \sin^{-1}\left(3x-4x^3\right), & -\frac{1}{2} \le x \le \frac{1}{2} \\ \pi - \sin^{-1}\left(3x-4x^3\right), & x > \frac{1}{2} \\ -\pi - \sin^{-1}\left(3x-4x^3\right), & x < -\frac{1}{2} \end{cases}$$

Proof:

Let $x = \sin\theta$, $\theta \in [-\pi/2, \pi/2] \quad \Rightarrow \quad \theta = \sin^{-1}x$

Now, $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = \sin^{-1}(2\sin\theta\cos\theta)$

$= \sin^{-1}(\sin 2\theta)$

$= \sin^{-1}(\sin\alpha)$, where $\alpha \in [-\pi, \pi]$

Now, consider the graph of $y = \sin^{-1}(\sin\alpha)$, where $\alpha \in [-\pi, \pi]$

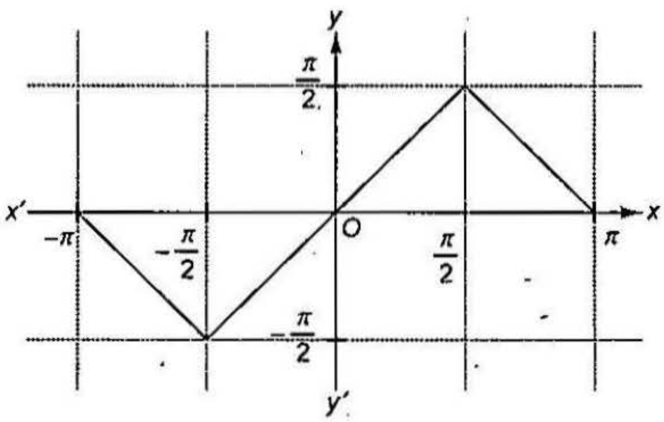


Fig. 4.22

From the graph,

$$\sin^{-1}\left(2x\sqrt{1-x^2}\right)=\sin^{-1}(\sin\alpha)$$

$$=\begin{cases}-\alpha-\pi, & -\pi<\alpha<(-\pi/2)\\ \alpha, & (-\pi/2)\le\alpha\le(\pi/2)\\ -\alpha+\pi, & (\pi/2)<\alpha<\pi\end{cases}$$

$$=\begin{cases}-2\sin^{-1}(x-\pi), & -\pi\le 2\sin^{-1}x<(-\pi/2)\\ 2\sin^{-1}x, & -(\pi/2)\le 2\sin^{-1}x\le(\pi/2)\\ -2\sin^{-1}(x+\pi), & (\pi/2)<2\sin^{-1}x<\pi\end{cases}$$

$$=\begin{cases}-2\sin^{-1}(x-\pi), & (-\pi/2)\le\sin^{-1}x<(-\pi/4)\\ 2\sin^{-1}x, & (-\pi/4)\le\sin^{-1}x\le(\pi/4)\\ -2\sin^{-1}(x+\pi), & (\pi/4)<\sin^{-1}x<(\pi/2)\end{cases}$$

$$=\begin{cases}-2\sin^{-1}x-\pi, & x<-\dfrac{1}{\sqrt{2}}\\ 2\sin^{-1}x, & -\dfrac{1}{\sqrt{2}}\le x\le\dfrac{1}{\sqrt{2}}\\ -2\sin^{-1}(x+\pi), & x>\dfrac{1}{\sqrt{2}}\end{cases}$$

$$\Rightarrow\ 2\sin^{-1}x=\begin{cases}\sin^{-1}\left(2x\sqrt{1-x^2}\right), & -\dfrac{1}{\sqrt{2}}\le x\le\dfrac{1}{\sqrt{2}}\\ \pi-\sin^{-1}\left(2x\sqrt{1-x^2}\right), & x>\dfrac{1}{\sqrt{2}}\\ -\pi-\sin^{-1}\left(2x\sqrt{1-x^2}\right), & x<-\dfrac{1}{\sqrt{2}}\end{cases}$$

Property 10

i. $$2\tan^{-1}x=\begin{cases}\tan^{-1}\left(\dfrac{2x}{1-x^2}\right), & \text{if } -1<x<1\\ \pi+\tan^{-1}\left(\dfrac{2x}{1-x^2}\right), & \text{if } x>1\\ -\pi+\tan^{-1}\left(\dfrac{2x}{1-x^2}\right), & \text{if } x<-1\end{cases}$$

ii. $$3\tan^{-1}x=\begin{cases}\tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right), & \text{if } -\dfrac{1}{\sqrt{3}}<x<\dfrac{1}{\sqrt{3}}\\ \pi+\tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right), & \text{if } x>\dfrac{1}{\sqrt{3}}\\ -\pi+\tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right), & \text{if } x<-\dfrac{1}{\sqrt{3}}\end{cases}$$

Proof:

i. In Property 7 (i) replace y by x.

From this information, we can also draw the graph of $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ as follows (Fig. 4.23).

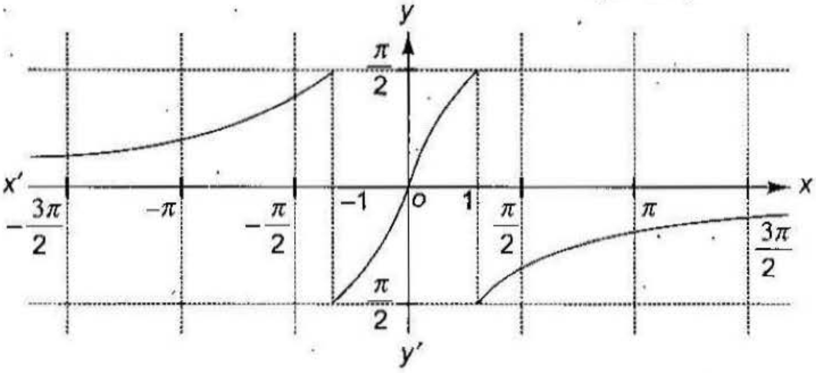


Fig. 4.23

Example 4.39 **Find the value of $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99}$.**

Sol.

$$4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99} = 2\tan^{-1}\left[\frac{\frac{2}{5}}{1-\frac{1}{25}}\right] - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99}$$

$$= 2\tan^{-1}\left(\frac{5}{12}\right) + \tan^{-1}\left[\frac{\frac{1}{99}-\frac{1}{70}}{1+\frac{1}{99}\times\frac{1}{70}}\right]$$

$$= \tan^{-1}\left[\frac{\frac{5}{6}}{1-\frac{25}{144}}\right] + \tan^{-1}\left(\frac{-29}{6931}\right)$$

$$= \tan^{-1}\left(\frac{120}{119}\right) - \tan^{-1}\left(\frac{1}{239}\right)$$

$$= \tan^{-1}\left[\frac{\frac{120}{119}-\frac{1}{239}}{1+\frac{120}{119}\times\frac{1}{239}}\right] = \tan^{-1}(1) = \frac{\pi}{4}$$

Property 11

$$\text{i. } 2\tan^{-1}x = \begin{cases} \sin^{-1}\left(\frac{2x}{1+x^2}\right), & \text{if } -1 \le x \le 1 \\ \pi - \sin^{-1}\left(\frac{2x}{1+x^2}\right), & \text{if } x > 1 \\ -\pi - \sin^{-1}\left(\frac{2x}{1+x^2}\right), & \text{if } x < -1 \end{cases}$$

ii. $2\tan^{-1}x = \begin{cases} \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right), & \text{if } 0 \le x < \infty \\ -\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right), & \text{if } -\infty < x \le 0 \end{cases}$

Proof:

i. Let $x = \tan\theta$, $\theta \in (-\pi/2, \pi/2)$ $\Rightarrow$ $\theta = \tan^{-1}x$

Now, $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right) = \sin^{-1}\left(\dfrac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = \sin^{-1}(\sin\alpha)$, where $\alpha \in (-\pi, \pi)$.

Now, consider the graph of $y = \sin^{-1}(\sin\alpha)$, where $\alpha \in (-\pi, \pi)$.

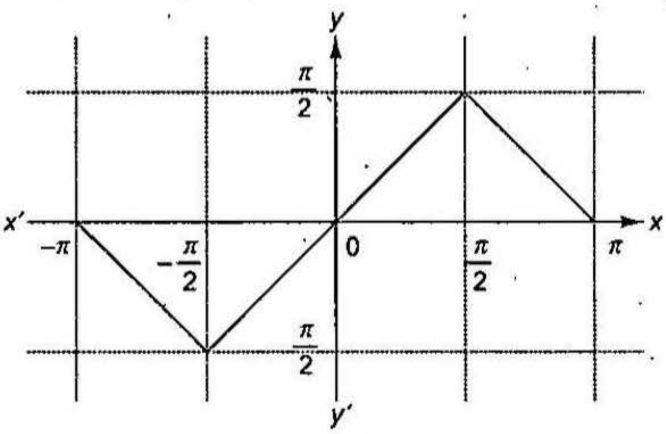


Fig. 4.24

From the graph,

$$\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \sin^{-1}(\sin\alpha)$$

$$= \begin{cases} -\alpha-\pi, & -\pi < \alpha < (-\pi/2) \\ \alpha, & (-\pi/2) \le \alpha \le (\pi/2) \\ -\alpha+\pi, & (\pi/2) < \alpha < \pi \end{cases}$$

$$= \begin{cases} -2\tan^{-1}(x-\pi) & -\pi < 2\tan^{-1}x < -(\pi/2) \\ 2\tan^{-1}x, & (-\pi/2) \le 2\tan^{-1}x \le (\pi/2) \\ -2\tan^{-1}(x+\pi), & (\pi/2) < 2\tan^{-1}x < \pi \end{cases}$$

$$= \begin{cases} -2\tan^{-1}(x-\pi), & (-\pi/2) < \tan^{-1}x < (-\pi/4) \\ 2\tan^{-1}x, & (-\pi/4) \le \tan^{-1}x \le (\pi/4) \\ -2\tan^{-1}(x+\pi), & (\pi/4) < \tan^{-1}x < (\pi/2) \end{cases}$$

$$= \begin{cases} -2\tan^{-1}(x-\pi), & x < -1 \\ 2\tan^{-1}x, & -1 \le x \le 1 \\ -2\tan^{-1}(x+\pi), & x > 1 \end{cases}$$

From this information, we can also draw the graph of $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ as follows (Fig. 4.25).

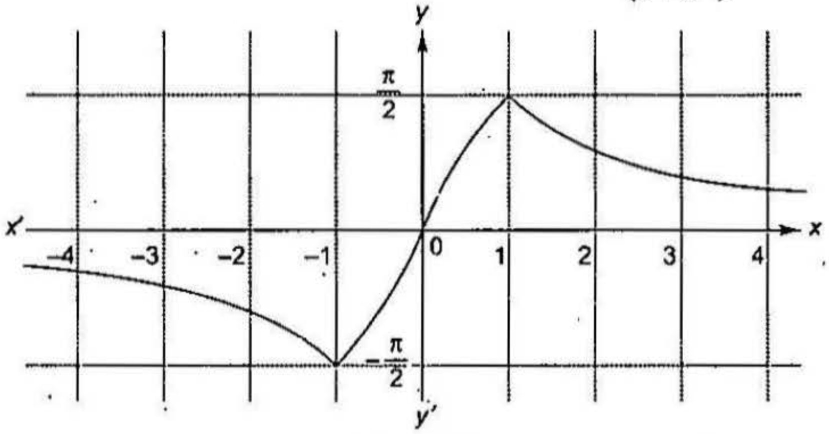


Fig. 4.25

ii. Let $x = \tan\theta$, $\theta \in (-\pi/2, \pi/2)$ $\Rightarrow$ $\theta = \tan^{-1}x$

Now, $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \cos^{-1}(\cos 2\theta) = \cos^{-1}(\cos\alpha)$, where $\alpha \in (-\pi, \pi)$.

Now, consider the graph of $y = \cos^{-1}(\cos\alpha)$, where $\alpha \in (-\pi, \pi)$.

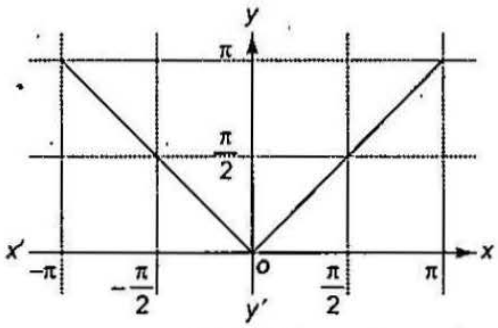


Fig. 4.26

From the graph,

$$\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \cos^{-1}(\cos\alpha)$$

$$= \begin{cases} -\alpha, & -\pi < \alpha < 0 \\ \alpha, & 0 \le \alpha < \pi \end{cases}$$

$$= \begin{cases} -2\tan^{-1}x, & -\pi < 2\tan^{-1}x < 0 \\ 2\tan^{-1}x, & 0 \le 2\tan^{-1}x < \pi \end{cases}$$

$$= \begin{cases} -2\tan^{-1}x, & (-\pi/2) < \tan^{-1}x < 0 \\ 2\tan^{-1}x, & 0 \le \tan^{-1}x < (\pi/2) \end{cases}$$

$$= \begin{cases} -2\tan^{-1} x, & x < 0 \\ 2\tan^{-1} x, & x \geq 0 \end{cases}$$

From this information, we can also draw the graph of $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ as follows (Fig. 4.27).

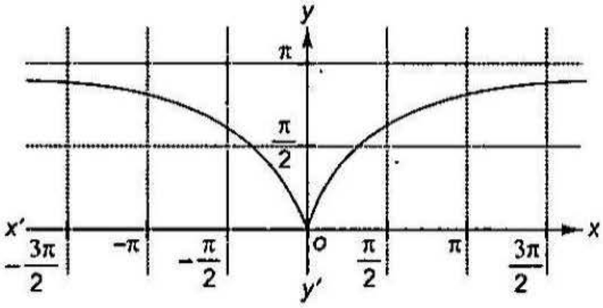


Fig. 4.27

Example 4.40 If $\sin^{-1}\frac{2x}{1+x^2} = \tan^{-1}\frac{2x}{1-x^2}$, then find the values of x.

Sol.

By referring to the graphs of $y = \sin^{-1}\frac{2x}{1+x^2}$ and $y = \sin^{-1}\frac{2x}{1+x^2}$, we get $-1 < x < 1$.

Example 4.41 If $\sin^{-1}\left(\frac{4x}{x^2+4}\right) + 2\tan^{-1}\left(-\frac{x}{2}\right)$ is independent of x, find the values of x.

Sol.

$$\sin^{-1}\left(\frac{4x}{x^2+4}\right) + 2\tan^{-1}\left(-\frac{x}{2}\right) = \sin^{-1}\left(\frac{2\times\frac{x}{2}}{\left(\frac{x}{2}\right)^2+1}\right) - 2\tan^{-1}\frac{x}{2}$$

$$= 2\tan^{-1}\frac{x}{2} - 2\tan^{-1}\frac{x}{2} = 0$$

$$\Rightarrow \left|\frac{x}{2}\right| \leq 1 \Rightarrow |x| \leq 2 \Rightarrow -2 \leq x \leq 2$$

Example 4.42 If $\cos^{-1}\frac{6x}{1+9x^2} = -\frac{\pi}{2} + 2\tan^{-1} 3x$, then find the values of x.

Sol.

$$\cos^{-1}\frac{6x}{1+9x^2} = -\frac{\pi}{2} + 2\tan^{-1} 3x$$

$$\Rightarrow \frac{\pi}{2} - \sin^{-1}\frac{6x}{1+9x^2} = -\frac{\pi}{2} + 2\tan^{-1} 3x$$

$$\sin^{-1}\frac{6x}{1+9x^2}=\pi-2\tan^{-1}3x$$

$$\sin^{-1}\frac{2\times 3x}{1+(3x)^2}=\pi-2\tan^{-1}3x$$

It is true when $3x>1 \quad\Rightarrow\quad x>\frac{1}{3}$

i.e., $x\in\left(\frac{1}{3},\infty\right)$

Example 4.43 **If $(x-1)(x^2+1)>0$, then find the value of $\sin\left(\frac{1}{2}\tan^{-1}\frac{2x}{1-x^2}-\tan^{-1}x\right)$.**

Sol.

$(x-1)(x^2+1)>0 \quad\Rightarrow\quad x>1$

$$\therefore\quad \sin\left[\frac{1}{2}\tan^{-1}\left(\frac{2x}{1-x^2}\right)-\tan^{-1}x\right]=\sin\left[\frac{1}{2}\left(-\pi+2\tan^{-1}x\right)-\tan^{-1}x\right]=\sin\left(-\frac{\pi}{2}\right)=-1$$

Example 4.44 **Solve $\cos^{-1}\left(\frac{1}{2}x^2+\sqrt{1-x^2}\sqrt{1-\frac{x^2}{4}}\right)=\cos^{-1}\frac{x}{2}-\cos^{-1}x$.**

Sol.

$$\cos^{-1}\left(\frac{1}{2}x^2+\sqrt{1-x^2}\sqrt{1-\frac{x^2}{4}}\right)=\cos^{-1}\left(\frac{x}{2}x+\sqrt{1-x^2}\sqrt{1-\left(\frac{x}{2}\right)^2}\right)$$

For $\cos^{-1}\left(\frac{1}{2}x^2+\sqrt{1-x^2}\sqrt{1-\frac{x^2}{4}}\right)=\cos^{-1}\frac{x}{2}-\cos^{-1}x$

L.H.S. > 0, hence R.H.S. > 0 $\quad\Rightarrow\quad \cos^{-1}\frac{x}{2}-\cos^{-1}x>0$

Since $\cos^{-1}x$ is a decreasing function, we get

$\frac{x}{2}\le x \quad\Rightarrow\quad \frac{x}{2}\ge 0\Rightarrow x\ge 0 \quad\Rightarrow\quad x\in[0,1]$

Example 4.45 **If $x\in\left(0,\frac{\pi}{2}\right)$, then show that**

$$\cos^{-1}\left(\frac{7}{2}(1+\cos 2x)+\sqrt{(\sin^2x-48\cos^2x)}\sin x\right)=x-\cos^{-1}(7\cos x).$$

Sol.

$$y = \cos^{-1}\left(\frac{7}{2}(1+\cos 2x) + \sqrt{(\sin^2 x - 48\cos^2 x)}\sin x\right)$$

$$= \cos^{-1}\left((7\cos x)(\cos x) + \sqrt{1-49\cos^2 x}\sqrt{1-\cos^2 x}\right)$$

$$= \cos^{-1}(\cos x) - \cos^{-1}(7\cos x) \qquad [\because \cos x < 7\cos x]$$

$$= x - \cos^{-1}(7\cos x)$$

Example 4.46 **Prove that** $2\cos^{-1}x = \begin{cases} 2\pi - \cos^{-1}\left(2x^2-1\right), & \text{if } -1 \le x < 0 \\ \cos^{-1}\left(2x^2-1\right), & \text{if } 0 \le x \le 1 \end{cases}$.

Sol.

Let $\cos^{-1}x = \theta$, where $\theta \in [0, \pi] \quad \Rightarrow \quad \cos\theta = x$

Now, $\cos^{-1}(2x^2 - 1) = \cos^{-1}(2\cos^2\theta - 1) = \cos^{-1}(\cos 2\theta) = \cos^{-1}(\cos\alpha)$, where $\alpha \in [0, 2\pi]$

Refer the graph of $y = \cos^{-1}(\cos\alpha)$, $\alpha \in [0, 2\pi]$:

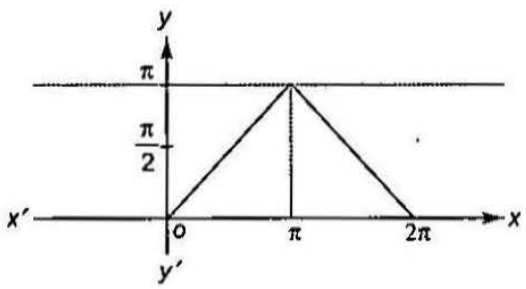


Fig. 4.28

From the graph,

$$\cos^{-1}(2x^2-1) = \begin{cases} \alpha, & \text{if } 0 \le \alpha < \pi \\ 2\pi - \alpha, & \text{if } \pi \le \alpha \le 2\pi \end{cases}$$

$$= \begin{cases} 2\cos^{-1}x, & \text{if } 0 \le 2\cos^{-1}x \le \pi \\ 2\pi - \cos^{-1}x, & \text{if } \pi < 2\cos^{-1}x \le 2\pi \end{cases}$$

$$= \begin{cases} 2\cos^{-1}x, & \text{if } 0 \le \cos^{-1}x \le (\pi/2) \\ 2\pi - \cos^{-1}x, & \text{if } (\pi/2) < \cos^{-1}x \le \pi \end{cases}$$

$$= \begin{cases} 2\cos^{-1}x, & \text{if } 0 \le x \le 1 \\ 2\pi - \cos^{-1}x, & \text{if } -1 \le x < 0 \end{cases}$$

$$\Rightarrow \quad 2\cos^{-1}x = \begin{cases} 2\pi - \cos^{-1}\left(2x^2-1\right), & \text{if } -1 \le x < 0 \\ \cos^{-1}\left(2x^2-1\right), & \text{if } 0 \le x \le 1 \end{cases}$$

EXERCISES

Subjective Type

Solutions on page 4.54

1. Solve $2\cos^{-1}x = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$.

2. Find the domain for $f(x) = \sin^{-1}\left(\dfrac{1+x^2}{2x}\right)$.

3. Find the range of $f(x) = \cot^{-1}(2x - x^2)$.

4. Find the sum $\cot^{-1}2 + \cot^{-1}8 + \cot^{-1}18 + \cdots \infty$.

5. Find the sum $\operatorname{cosec}^{-1}\sqrt{10} + \operatorname{cosec}^{-1}\sqrt{50} + \operatorname{cosec}^{-1}\sqrt{170} + \cdots + \operatorname{cosec}^{-1}\sqrt{(n^2+1)(n^2+2n+2)}$.

6. Find the number of positive integral solutions of the equation

   $\tan^{-1}x + \cos^{-1}\dfrac{y}{\sqrt{1-y^2}} = \sin^{-1}\dfrac{3}{\sqrt{10}}$.

7. If $\tan^{-1}y = 4\tan^{-1}x \left(|x| < \tan\dfrac{\pi}{8}\right)$, find y as an algebraic function of x, and hence, prove that $\tan \pi/8$ is a root of the equation $x^4 - 6x^2 + 1 = 0$.

8. If x_1, x_2, x_3 and x_4 are the roots of the equation $x^4 - x^3\sin 2\beta + x^2\cos 2\beta - x\cos\beta - \sin\beta = 0$, prove that $\tan^{-1}x_1 + \tan^{-1}x_2 + \tan^{-1}x_3 + \tan^{-1}x_4 = n\pi + (\pi/2) - \beta$, where n is an integer.

9. Solve for real values of x: $\dfrac{(\sin^{-1}x)^3 + (\cos^{-1}x)^3}{(\tan^{-1}x + \cot^{-1}x)^3} = 7$.

10. Find the set of values of parameter a so that the equation $(\sin^{-1}x)^3 + (\cos^{-1}x)^3 = a\pi^3$ has a solution.

11. If $p > q > 0$ and $pr < -1 < qr$, then find the value of $\tan^{-1}\dfrac{p-q}{1+pq} + \tan^{-1}\dfrac{q-r}{1+qr} + \tan^{-1}\dfrac{r-p}{1+rp}$.

12. Solve the equation $\sqrt{|\sin^{-1}|\cos x|| + |\cos^{-1}|\sin x||} = \sin^{-1}|\cos x| - \cos^{-1}|\sin x|$.

13. Solve the equation $\tan^{-1}\dfrac{x+1}{x-1} + \tan^{-1}\dfrac{x-1}{x} = \tan^{-1}(-7)$.

14. Solve the equation $\sin^{-1}6x + \sin^{-1}6\sqrt{3}x = -\pi/2$.

15. If $0 < a_1 < a_2 < \cdots < a_n$, then prove that

$$\tan^{-1}\left(\frac{a_1x-y}{x+a_1y}\right) + \tan^{-1}\left(\frac{a_2-a_1}{1+a_2a_1}\right) + \tan^{-1}\left(\frac{a_3-a_2}{1+a_3a_2}\right) + \cdots + \tan^{-1}\left(\frac{a_n-a_{n-1}}{1+a_na_{n-1}}\right) + \tan^{-1}\left(\frac{1}{a_n}\right) = \tan^{-1}\frac{x}{y}.$$

Objective Type

Solutions on page 4.59

Each question has four choices a, b, c and d, out of which *only one* is correct.

1. The principal value of $\sin^{-1}(\sin 10)$ is

   a. 10 b. $10-3\pi$ c. $3\pi-10$ d. none of these

2. $\cos^{-1}\left(\cos\left(\frac{5\pi}{4}\right)\right)$ is given by

   a. $\frac{5\pi}{4}$ b. $\frac{3\pi}{4}$ c. $\frac{-\pi}{4}$ d. none of these.

3. The value of $\sin^{-1}(\sin 12) + \cos^{-1}(\cos 12)$ is equal to

   a. zero b. $24-2\pi$ c. $4\pi-24$ d. none of these

4. The value of the expression $\sin^{-1}\left(\sin\frac{22\pi}{7}\right)+\cos^{-1}\left(\cos\frac{5\pi}{3}\right)+\tan^{-1}\left(\tan\frac{5\pi}{7}\right)+\sin^{-1}(\cos 2)$ is

   a. $\frac{17\pi}{42}-2$ b. -2 c. $\frac{-\pi}{21}-2$ d. none of these

5. The value of $\sin^{-1}(\cos(\cos^{-1}(\cos x)+\sin^{-1}(\sin x)))$, where $x \in \left(\frac{\pi}{2}, \pi\right)$, is equal to

   a. $\frac{\pi}{2}$ b. $-\pi$ c. π d. $-\frac{\pi}{2}$

6. $\cos^{-1}(\cos(2\cot^{-1}(\sqrt{2}-1)))$ is equal to

   a. $\sqrt{2}-1$ b. $\frac{\pi}{4}$ c. $\frac{3\pi}{4}$ d. none of these

7. The value of $\sin^{-1}\left(\cot\left(\sin^{-1}\sqrt{\frac{2-\sqrt{3}}{4}}+\cos^{-1}\frac{\sqrt{12}}{4}+\sec^{-1}\sqrt{2}\right)\right)$ is

   a. 0 b. $\frac{\pi}{2}$ c. $\frac{\pi}{3}$ d. none of these

8. The value of $\cos^{-1}\sqrt{\frac{2}{3}} - \cos^{-1}\frac{\sqrt{6}+1}{2\sqrt{3}}$ is equal to

   a. $\frac{\pi}{3}$ b. $\frac{\pi}{4}$ c. $\frac{\pi}{2}$ d. $\frac{\pi}{6}$

9. The value of $\cos\left(\frac{1}{2}\cos^{-1}\frac{1}{8}\right)$ is

   a. $\frac{3}{4}$ b. $-\frac{3}{4}$ c. $\frac{1}{16}$ d. $\frac{1}{4}$

10. If $\tan(x+y) = 33$ and $x = \tan^{-1} 3$, then y will be

   a. 0.3 b. $\tan^{-1}(1.3)$ c. $\tan^{-1}(0.3)$ d. $\tan^{-1}\left(\frac{1}{18}\right)$

11. The value of $\tan\left(\frac{1}{2}\cos^{-1}\left(\frac{\sqrt{5}}{3}\right)\right)$ is

a. $\frac{3+\sqrt{5}}{2}$ b. $3+\sqrt{5}$ c. $\frac{1}{2}(3-\sqrt{5})$ d. none of these

12. $\tan^{-1}\left[\frac{\cos x}{1+\sin x}\right]$ is equal to

a. $\frac{\pi}{4}-\frac{x}{2}$, for $x \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$ b. $\frac{\pi}{4}-\frac{x}{2}$, for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

c. $\frac{\pi}{4}-\frac{x}{2}$, for $x \in \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$ d. $\frac{\pi}{4}-\frac{x}{2}$, for $x \in \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right)$

13. If $f(x) = x^{11} + x^9 - x^7 + x^3 + 1$ and $f(\sin^{-1}(\sin 8)) = \alpha$, α is a constant, then $f(\tan^{-1}(\tan 8))$ is equal to

a. α b. $\alpha-2$ c. $\alpha+2$ d. $2-\alpha$

14. The maximum value of $f(x) = \tan^{-1}\left(\frac{(\sqrt{12}-2)x^2}{x^4+2x^2+3}\right)$ is

a. 18° b. 36° c. 22.5° d. 15°

15. The value of $\sin^{-1}\left[x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}\right]$ is equal to

a. $\sin^{-1} x + \sin^{-1} \sqrt{x}$ b. $\sin^{-1} x - \sin^{-1} \sqrt{x}$ c. $\sin^{-1} \sqrt{x} - \sin^{-1} x$ d. none of these

16. $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right)$ is

a. $\frac{\pi}{2}$ b. $\frac{\pi}{3}$ c. $\frac{\pi}{4}$ d. $\frac{\pi}{4}$ or $-\frac{3\pi}{4}$

17. If $\tan^{-1}\frac{a+x}{a} + \tan^{-1}\frac{a-x}{a} = \frac{\pi}{6}$, then $x^2 =$

a. $2\sqrt{3}a$ b. $\sqrt{3}a$ c. $2\sqrt{3}a^2$ d. none of these

18. If $\cot^{-1}\frac{n}{\pi} > \frac{\pi}{6}$, $n \in N$, then the maximum value of n is

a. 6 b. 7 c. 5 d. none of these

19. $\sec^2(\tan^{-1} 2) + \text{cosec}^2(\cot^{-1} 3)$ is equal to

a. 5 b. 13 c. 15 d. 6

20. If $A = \tan^{-1}\left(\frac{x\sqrt{3}}{2K-x}\right)$ and $B = \tan^{-1}\left(\frac{2x-K}{K\sqrt{3}}\right)$, then the value of $A - B$ is

a. 0° b. 45° c. 60° d. 30°

21. The value of $\sec\left[\tan^{-1}\frac{b+a}{b-a} - \tan^{-1}\frac{a}{b}\right]$ is

a. 2 b. $\sqrt{2}$ c. 4 d. 1

22. If $a\sin^{-1}x - b\cos^{-1}x = c$, then $a\sin^{-1}x + b\cos^{-1}x$ is equal to

a. 0 b. $\frac{\pi ab + c(b-a)}{a+b}$ c. $\frac{\pi}{2}$ d. $\frac{\pi ab + c(a-b)}{a+b}$

23. The number of solution of the equation $\cos^{-1}\left(\frac{1+x^2}{2x}\right) - \cos^{-1}x = \frac{\pi}{2} + \sin^{-1}x$ is given by

a. 0 b. 1 c. 2 d. 3

24. The sum of the solutions of the equation $2\sin^{-1}\sqrt{x^2+x+1} + \cos^{-1}\sqrt{x^2+x} = \frac{3\pi}{2}$ is

a. 0 b. -1 c. 1 d. 2

25. The number of solutions of the equation $\tan^{-1}(1+x) + \tan^{-1}(1-x) = \frac{\pi}{2}$ is

a. 2 b. 3 c. 1 d. 0

26. The number of solution of the equation $\sin^{-1}x + \sin^{-1}(1-x) = \cos^{-1}x$ is

a. 1 b. 0 c. 2 d. none of these

27. If $\tan^{-1}\frac{1-x}{1+x} = \frac{1}{2}\tan^{-1}x$, then x is equal to

a. 1 b. $\sqrt{3}$ c. $\frac{1}{\sqrt{3}}$ d. none of these

28. For the equation $\cos^{-1}x + \cos^{-1}2x + \pi = 0$, the number of real solution is

a. 1 b. 2 c. 0 d. ∞

29. The value of 'a', for which $ax^2 + \sin^{-1}(x^2-2x+2) + \cos^{-1}(x^2-2x+2) = 0$ has a real solution is

a. $\frac{\pi}{2}$ b. $-\frac{\pi}{2}$ c. $\frac{2}{\pi}$ d. $-\frac{2}{\pi}$

30. The number of real solutions of the equation $\tan^{-1}\sqrt{x^2-3x+2} + \cos^{-1}\sqrt{4x-x^2-3} = \pi$ is

a. one b. two c. zero d. infinite

31. If $3\tan^{-1}\left(\frac{1}{2+\sqrt{3}}\right) - \tan^{-1}\frac{1}{x} = \tan^{-1}\frac{1}{3}$, then x is equal to

a. 1 b. 2 c. 3 d. $\sqrt{2}$

32. If $\tan^{-1}x + 2\cot^{-1}x = \frac{2\pi}{3}$, then x is equal to

a. $\frac{\sqrt{3}-1}{\sqrt{3}+1}$ b. 3 c. $\sqrt{3}$ d. $\sqrt{2}$

33. If $3\sin^{-1}\left(\frac{2x}{1+x^2}\right) - 4\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) + 2\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{\pi}{3}$, where $|x| < 1$, then x is equal to

a. $\frac{1}{\sqrt{3}}$ b. $-\frac{1}{\sqrt{3}}$ c. $\sqrt{3}$ d. $-\frac{\sqrt{3}}{4}$ e. $\frac{\sqrt{3}}{2}$

34. If $\sin^{-1}\left(\frac{5}{x}\right) + \sin^{-1}\left(\frac{12}{x}\right) = \frac{\pi}{2}$, then x is equal to

a. $\frac{7}{13}$ b. $\frac{4}{3}$ c. 13 d. $\frac{13}{7}$

35. If $\sin^{-1}(x-1)+\cos^{-1}(x-3)+\tan^{-1}\left(\frac{x}{2-x^2}\right)=\cos^{-1}k+\pi$, then the value of k is

a. 1　b. $-\frac{1}{\sqrt{2}}$　c. $\frac{1}{\sqrt{2}}$　d. none of these

36. If $\cot^{-1}x+\cot^{-1}y+\cot^{-1}z=\frac{\pi}{2}$, $x, y, z>0$ and $xy<1$, then $x+y+z$ is also equal to

a. $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$　b. xyz　c. $xy+yz+zx$　d. none of these

37. If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$, then

a. $x^2+y^2+z^2+xyz=0$　b. $x^2+y^2+z^2+2xyz=0$
c. $x^2+y^2+z^2+xyz=1$　d. $x^2+y^2+z^2+2xyz=1$

38. If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\frac{\pi}{2}$, then

a. $x+y+z-xyz=0$　b. $x+y+z+xyz=0$　c. $xy+yz+zx+1=0$　d. $xy+yz+zx-1=0$

39. If $\cos^{-1}\sqrt{p}+\cos^{-1}\sqrt{1-p}+\cos^{-1}\sqrt{1-q}=\frac{3\pi}{4}$, then the value of q is

a. 1　b. $\frac{1}{\sqrt{2}}$　c. $\frac{1}{3}$　d. $\frac{1}{2}$

40. If $\sin^{-1}a+\sin^{-1}b+\sin^{-1}c=\pi$, then the value of $a\sqrt{(1-a^2)}+b\sqrt{(1-b^2)}+c\sqrt{(1-c^2)}$ will be

a. $2abc$　b. abc　c. $\frac{1}{2}abc$　d. $\frac{1}{3}abc$

41. If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\pi$, then $x^4+y^4+z^4+4x^2y^2z^2=K(x^2y^2+y^2z^2+z^2x^2)$, where K is equal to

a. 1　b. 2　c. 4　d. none of these

42. If $\cos^{-1}x-\cos^{-1}\frac{y}{2}=\alpha$, then $4x^2-4xy\cos\alpha+y^2$ is equal to

a. 4　b. $2\sin^2\alpha$　c. $-4\sin^2\alpha$　d. $4\sin^2\alpha$

43. The value of x which satisfies equation $2\tan^{-1}2x=\sin^{-1}\frac{4x}{1+4x^2}$ is valid in the interval

a. $\left[\frac{1}{2},\infty\right)$　b. $\left(-\infty,-\frac{1}{2}\right]$　c. $[-1, 1]$　d. $\left[-\frac{1}{2},\frac{1}{2}\right]$

44. If $x\in[-1, 0)$, then $\cos^{-1}(2x^2-1)-2\sin^{-1}x$ is equal to

a. $-\frac{\pi}{2}$　b. π　c. $\frac{3\pi}{2}$　d. -2π

45. If $2\sin^{-1}x=\sin^{-1}(2x\sqrt{1-x^2})$, then

a. $[-1, 1]$　b. $\left[-\frac{1}{\sqrt{2}}, 1\right]$　c. $\left[-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right]$　d. none of these

46. If $x_1=2\tan^{-1}\left(\frac{1+x}{1-x}\right)$, $x_2=\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, where $x\in(0, 1)$, then x_1+x_2 is equal to

a. 0　b. 2π　c. π　d. none of these

47. The value of $\sin(2\sin^{-1}(0.8))$ is equal to

a. $\sin 1.2°$ b. $\sin 1.6°$ c. 0.48 d. 0.96

48. The value of $\tan(\sin^{-1}(\cos(\sin^{-1}x)))\tan(\cos^{-1}(\sin(\cos^{-1}x)))$, where $x \in (0, 1)$, is equal to

a. 0 b. 1 c. –1 d. none of these

49. If $\sin^{-1}\left(\frac{2a}{1+a^2}\right)+\sin^{-1}\left(\frac{2b}{1+b^2}\right)=2\tan^{-1}x$, then x is equal to $[a, b \in (0, 1)]$

a. $\frac{a-b}{1+ab}$ b. $\frac{b}{1+ab}$ c. $\frac{b}{1-ab}$ d. $\frac{a+b}{1-ab}$

50. If x takes negative permissible value, then $\sin^{-1}x$ is equal to

a. $\cos^{-1}\sqrt{1-x^2}$ b. $-\cos^{-1}\sqrt{1-x^2}$ c. $\cos^{-1}\sqrt{x^2-1}$ d. $\pi-\cos^{-1}\sqrt{1-x^2}$

51. If $x^2+y^2+z^2=r^2$, then $\tan^{-1}\left(\frac{xy}{zr}\right)+\tan^{-1}\left(\frac{yz}{xr}\right)+\tan^{-1}\left(\frac{xz}{yr}\right)$ is equal to

a. π b. $\frac{\pi}{2}$ c. 0 d. none of these

52. If $f(x)=\sin^{-1}\left(\frac{\sqrt{3}}{2}x-\frac{1}{2}\sqrt{1-x^2}\right)$, $-\frac{1}{2}\le x\le 1$, then $f(x)$ is equal to

a. $\sin^{-1}\left(\frac{1}{2}\right)-\sin^{-1}(x)$ b. $\sin^{-1}x-\frac{\pi}{6}$ c. $\sin^{-1}x+\frac{\pi}{6}$ d. none of these

53. If $x \in (0, 1)$, then the value of $\tan^{-1}\left(\frac{1-x^2}{2x}\right)+\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ is equal to

a. $-\frac{\pi}{2}$ b. zero c. $\frac{\pi}{2}$ d. π

54. The trigonometric equation $\sin^{-1}x = 2\sin^{-1}a$ has a solution for

a. all real values b. $|a|<\frac{1}{2}$ c. $|a|\le\frac{1}{\sqrt{2}}$ d. $\frac{1}{2}<|a|<\frac{1}{\sqrt{2}}$

55. If $2\tan^{-1}x=\pi+\tan^{-1}\left(\frac{2x}{1-x^2}\right)$, then

a. $x>1$ b. $x<1$ c. $x>-1$ d. $-1<x<1$

56. $\sum_{r=1}^{n}\sin^{-1}\left(\frac{\sqrt{r}-\sqrt{r-1}}{\sqrt{r(r+1)}}\right)$ is equal to

a. $\tan^{-1}(\sqrt{n})-\frac{\pi}{4}$ b. $\tan^{-1}(\sqrt{n+1})-\frac{\pi}{4}$ c. $\tan^{-1}(\sqrt{n})$ d. $\tan^{-1}(\sqrt{n+1})$

57. $\sum_{r=1}^{n}\tan^{-1}\left(\frac{2^{r-1}}{1+2^{2r-1}}\right)$ is equal to

a. $\tan^{-1}(2^n)$ b. $\tan^{-1}(2^n)-\frac{\pi}{4}$ c. $\tan^{-1}(2^{n+1})$ d. $\tan^{-1}(2^{n+1})-\frac{\pi}{4}$

58. $\sum_{m=1}^{n} \tan^{-1}\left(\frac{2m}{m^4+m^2+2}\right)$ is equal to

a. $\tan^{-1}\left(\frac{n^2+n}{n^2+n+2}\right)$ b. $\tan^{-1}\left(\frac{n^2-n}{n^2-n+2}\right)$ c. $\tan^{-1}\left(\frac{n^2+n+2}{n^2+n}\right)$ d. none of these

59. The value of $\sum_{r=0}^{\infty} \tan^{-1}\left(\frac{1}{1+r+r^2}\right)$ is equal to

a. $\frac{\pi}{2}$ b. $\frac{3\pi}{4}$ c. $\frac{\pi}{4}$ d. none of these

60. If $\sin^{-1} x = \theta + \beta$ and $\sin^{-1} y = \theta - \beta$, then $1 + xy$ is equal to

a. $\sin^2\theta + \sin^2\beta$ b. $\sin^2\theta + \cos^2\beta$ c. $\cos^2\theta + \cos^2\theta$ d. $\cos^2\theta + \sin^2\beta$

61. If $u = \cot^{-1}\sqrt{\tan\alpha} - \tan^{-1}\sqrt{\tan\alpha}$, then $\tan\left(\frac{\pi}{4} - \frac{u}{2}\right)$ is equal to

a. $\sqrt{\tan\alpha}$ b. $\sqrt{\cot\alpha}$ c. $\tan\alpha$ d. $\cot\alpha$

62. of $\tan\left[\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\frac{a}{b}\right] + \tan\left[\frac{\pi}{4} - \frac{1}{2}\cos^{-1}\frac{a}{b}\right]$ is equal to

a. $\frac{2a}{b}$ b. $\frac{2b}{a}$ c. $\frac{a}{b}$ d. $\frac{b}{a}$

63. The value $2\tan^{-1}\left[\sqrt{\frac{a-b}{a+b}}\tan\frac{\theta}{2}\right]$ is equal to

a. $\cos^{-1}\left(\frac{a\cos\theta+b}{a+b\cos\theta}\right)$ b. $\cos^{-1}\left(\frac{a+b\cos\theta}{a\cos\theta+b}\right)$ c. $\cos^{-1}\left(\frac{a\cos\theta}{a+b\cos\theta}\right)$ d. $\cos^{-1}\left(\frac{b\cos\theta}{a\cos\theta+b}\right)$

64. $\cot^{-1}\left[\frac{\sqrt{1-\sin x}+\sqrt{1+\sin x}}{\sqrt{1-\sin x}-\sqrt{1+\sin x}}\right]$ $\left(\text{where } x \in \left[0, \frac{\pi}{2}\right]\right)$ is equal to

a. $\pi - x$ b. $2\pi - x$ c. $\frac{x}{2}$ d. $\pi - \frac{x}{2}$

65. The value of $\tan^{-1}\left(\frac{x\cos\theta}{1-x\sin\theta}\right) - \cot^{-1}\left(\frac{\cos\theta}{x-\sin\theta}\right)$ is

a. 2θ b. θ c. $\theta/2$ d. independent of θ

66. If $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then the value of $\tan^{-1}\left(\frac{\tan x}{4}\right) + \tan^{-1}\left(\frac{3\sin 2x}{5+3\cos 2x}\right)$ is

a. $x/2$ b. $2x$ c. $3x$ d. x

67. If $\cot^{-1}\left(\sqrt{\cos\alpha}\right) - \tan^{-1}\left(\sqrt{\cos\alpha}\right) = x$, then $\sin x$ is

a. $\tan^2\frac{\alpha}{2}$ b. $\cot^2\frac{\alpha}{2}$ c. $\tan a$ d. $\cot\frac{\alpha}{2}$

68. $\tan\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x\right)+\tan\left(\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x\right)$, $x \neq 0$, is equal to

a. x b. $2x$ c. $\frac{2}{x}$ d. none of these

69. The least and the greatest values of $(\sin^{-1}x)^3+(\cos^{-1}x)^3$ are

a. $\frac{-\pi}{2}, \frac{\pi}{2}$ b. $\frac{-\pi^3}{8}, \frac{\pi^3}{8}$ c. $\frac{\pi^3}{32}, \frac{7\pi^3}{8}$ d. none of these

70. Range of $f(x)=\sin^{-1}x+\tan^{-1}x+\sec^{-1}x$ is

a. $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$ b. $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$ c. $\left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\}$ d. none of these

71. Range of $\tan^{-1}\left(\frac{2x}{1+x^2}\right)$ is

a. $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ b. $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ c. $\left(-\frac{\pi}{2}, \frac{\pi}{4}\right]$ d. $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

72. If $[\cot^{-1}x]+[\cos^{-1}x]=0$, where $[\cdot]$ denotes the greatest integer function, then the complete set of values of x is

a. $(\cos 1, 1]$ b. $(\cos 1, \cos 1)$ c. $(\cot 1, 1]$ d. none of these

73. $\sin^{-1}(\sin 5) > x^2-4x$ holds if

a. $x=2-\sqrt{9-2\pi}$ b. $x=2+\sqrt{9-2\pi}$

c. $x>2+\sqrt{9-2\pi}$ d. $x \in (2-\sqrt{9-2\pi}, 2+\sqrt{9-2\pi})$

74. The value of $\frac{\alpha^3}{2}\operatorname{cosec}^2\left(\frac{1}{2}\tan^{-1}\frac{\alpha}{\beta}\right)+\frac{\beta^3}{2}\sec^2\left(\frac{1}{2}\tan^{-1}\left(\frac{\beta}{\alpha}\right)\right)$ is equal to

a. $(\alpha-\beta)(\alpha^2+\beta^2)$ b. $(\alpha+\beta)(\alpha^2-\beta^2)$ c. $(\alpha+\beta)(\alpha^2+\beta^2)$ d. none of these

75. The value of $\lim_{|x|\to\infty}\cos(\tan^{-1}(\sin(\tan^{-1}x)))$ is equal to

a. -1 b. $\sqrt{2}$ c. $-\frac{1}{\sqrt{2}}$ d. $\frac{1}{\sqrt{2}}$

76. $\sin^{-1}(3x-2-x^2)+\cos^{-1}(x^2-4x+3)=\frac{\pi}{4}$ can have a solution for $x \in$

a. $[1,2]$ b. $\left(\frac{3+\sqrt{5}}{2}, 2+\sqrt{2}\right)$ c. $\left(\frac{3-\sqrt{5}}{2}, 2-\sqrt{2}\right)$ d. none of these

77. If $2^{2\pi/\sin^{-1}x}-2(a+2)2^{\pi/\sin^{-1}x}+8a<0$ for at least one real x, then

a. $\frac{1}{8}\le a<2$ b. $a<2$ c. $a \in R-\{2\}$ d. $a \in \left[0, \frac{1}{8}\right) \cup (2, \infty)$

78. The number of integral values of k for which the equation $\sin^{-1}x+\tan^{-1}x=2k+1$ has a solution is

a. 1 b. 2 c. 3 d. 4

79. If $\tan^{-1}(\sin^2\theta - 2\sin\theta + 3) + \cot^{-1}\left(5^{\sec^2 y} + 1\right) = \frac{\pi}{2}$, then the value of $\cos^2\theta - \sin\theta$ is equal to

a. 0 b. –1 c. 1 d. none of these

80. Complete solution set of $[\cot^{-1}x] + 2[\tan^{-1}x] = 0$, where $[\cdot]$ denotes the greatest integer function, is equal to

a. $(0, \cot 1)$ b. $(0, \tan 1)$ c. $(\tan 1, \infty)$ d. $(\cot 1, \tan 1)$

81. Let $\begin{vmatrix} \tan^{-1}x & \tan^{-1}2x & \tan^{-1}3x \\ \tan^{-1}3x & \tan^{-1}x & \tan^{-1}2x \\ \tan^{-1}2x & \tan^{-1}3x & \tan^{-1}x \end{vmatrix} = 0$, then the number of values of x satisfying the equation is

a. 1 b. 2 c. 3 d. 4

82. Which of the following is the solution set of the equation $2\cos^{-1}x = \cot^{-1}\left(\frac{2x^2-1}{2x\sqrt{1-x^2}}\right)$?

a. $(0,1)$ b. $(-1,1)-\{0\}$ c. $(-1,0)$ d. $[-1,1]$

83. The values of x satisfying the equation $\sin(\tan^{-1}x) = \cos(\cot^{-1}(x+1))$ is

a. $\frac{1}{2}$ b. $-\frac{1}{2}$ c. $\sqrt{2}-1$ d. no finite value

84. There exists a positive real number x satisfying $\cos(\tan^{-1}x) = x$. Then the value of $\cos^{-1}\left(\frac{x^2}{2}\right)$ is

a. $\frac{\pi}{10}$ b. $\frac{\pi}{5}$ c. $\frac{2\pi}{5}$ d. $\frac{4\pi}{5}$

85. The range of values of p for which the equation $\sin\cos^{-1}(\cos(\tan^{-1}x)) = p$ has a solution is

a. $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ b. $[0,1)$ c. $\left(\frac{1}{\sqrt{2}}, 1\right)$ d. $(-1, 1)$

86. Sum of roots of the equation $\sin^{-1}x - \cos^{-1}x = \sin^{-1}(3x-2)$ is

a. 3/2 b. 1 c. 1/2 d. 2

87. The solution set of the equation $\sin^{-1}\sqrt{1-x^2} + \cos^{-1}x = \cot^{-1}\frac{\sqrt{1-x^2}}{x} - \sin^{-1}x$ is

a. $[-1,1]-\{0\}$ b. $(0,1] \cup \{-1\}$ c. $[-1,0) \cup \{1\}$ d. $[-1,1]$

88. The number of real solutions of the equation $\sqrt{1+\cos 2x} = \sqrt{2}\sin^{-1}(\sin x)$, $-\pi \le x \le \pi$, is

a. 0 b. 1 c. 2 d. infinite

89. The equation $3\cos^{-1}x - \pi x - \frac{\pi}{2} = 0$ has

a. one negative solution b. one positive solution
c. no solution d. more than one solution

90. If $\left|\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right| < \frac{\pi}{3}$, then

a. $x \in \left[-\frac{1}{3}, \frac{1}{\sqrt{3}}\right]$ b. $x \in \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ c. $x \in \left(0, \frac{1}{\sqrt{3}}\right)$ d. none of these

91. If $\sin^{-1} x + \sin^{-1} y = \frac{\pi}{2}$, then $\frac{1+x^4+y^4}{x^2 - x^2y^2 + y^2}$ is equal to

a. 1 b. 2 c. $\frac{1}{2}$ d. none of these

92. The value of $\sin^{-1}(x^2 - 4x + 6) + \cos^{-1}(x^2 - 4x + 6)$ for all $x \in R$ is

a. $\frac{\pi}{2}$ b. π c. 0 d. none of these

93. The product of all values of x satisfying the equation

$\sin^{-1} \cos\left(\frac{2x^2 + 10|x| + 4}{x^2 + 5|x| + 3}\right) = \cot\left(\cot^{-1}\left(\frac{2-18|x|}{9|x|}\right)\right) + \frac{\pi}{2}$ is

a. 9 b. –9 c. –3 d. –1

94. The value of $2 \tan^{-1}(\text{cosec} \tan^{-1} x - \tan \cot^{-1} x)$ is equal to

a. $\cot^{-1} x$ b. $\cot^{-1} \frac{1}{x}$ c. $\tan^{-1} x$ d. none of these

Multiple Correct Answers Type

Solutions on page 4.80

Each question has four choices a, b, c and d, out of which *one or more* answers are correct.

1. If $\alpha, \beta (\alpha < \beta)$ are the roots of the equation $6x^2 + 11x + 3 = 0$, then which of the following are real?

a. $\cos^{-1} \alpha$ b. $\sin^{-1} \beta$ c. $\text{cosec}^{-1} \alpha$ d. Both $\cot^{-1} \alpha$ and $\cot^{-1} \beta$

2. $2 \tan^{-1}(-2)$ is equal to

a. $-\cos^{-1}\left(\frac{-3}{5}\right)$ b. $-\pi + \cos^{-1}\frac{3}{5}$ c. $-\frac{\pi}{2} + \tan^{-1}\left(-\frac{3}{4}\right)$ d. $-\pi + \cot^{-1}\left(-\frac{3}{4}\right)$

3. If α, β and γ are the roots of $\tan^{-1}(x-1) + \tan^{-1} x + \tan^{-1}(x+1) = \tan^{-1} 3x$, then

a. $\alpha + \beta + \gamma = 0$
b. $\alpha\beta + \beta\gamma + \gamma\alpha = -1/4$
c. $\alpha\beta\gamma = 1$
d. $|\alpha - \beta|_{max} = 1$

4. If $f(x) = \sin^{-1} x + \sec^{-1} x$ is defined, then which of the following value/values is/are in its range?

a. $-\pi/2$ b. $\pi/2$ c. π d. $3\pi/2$

5. If $(\sin^{-1} x + \sin^{-1} w)(\sin^{-1} y + \sin^{-1} z) = \pi^2$, then $D = \begin{vmatrix} x^{N_1} & y^{N_2} \\ z^{N_3} & w^{N_4} \end{vmatrix}$ $(N_1, N_2, N_3, N_4 \in N)$

a. has a maximum value of 2
b. has a minimum value of 0
c. 16 different D are possible
d. has a minimum value of –2

6. Indicate the relation which can hold in their respective domain for infinite values of x.

a. $\tan|\tan^{-1} x| = |x|$ b. $\cot|\cot^{-1} x| = |x|$ c. $\tan^{-1}|\tan x| = |x|$ d. $\sin|\sin^{-1} x| = |x|$

7. If α is a real number for which $f(x) = \log_e \cos^{-1} x$ is defined, then a possible value of $[\alpha]$ (where $[\cdot]$ denotes the greatest integer function) is

a. 0 b. 1 c. – 1 d. –2

8. Which of the following is a rational number?

a. $\sin\left(\tan^{-1} 3 + \tan^{-1}\frac{1}{3}\right)$

b. $\cos\left(\frac{\pi}{2} - \sin^{-1}\frac{3}{4}\right)$

c. $\log_2\left(\sin\left(\frac{1}{4}\sin^{-1}\frac{\sqrt{63}}{8}\right)\right)$

d. $\tan\left(\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right)$

9. If $z = \sec^{-1}\left(x + \frac{1}{x}\right) + \sec^{-1}\left(y + \frac{1}{y}\right)$, where $xy < 0$, then the possible values of z is (are)

a. $\frac{8\pi}{10}$ b. $\frac{7\pi}{10}$ c. $\frac{9\pi}{10}$ d. $\frac{21\pi}{20}$

10. If $f(x) = (\sin^{-1} x)^2 + (\cos^{-1} x)^2$, then

a. $f(x)$ has the least value of $\frac{\pi^2}{8}$

b. $f(x)$ has the greatest value of $\frac{5\pi^2}{8}$

c. $f(x)$ has the least value of $\frac{\pi^2}{16}$

d. $f(x)$ has the greatest value of $\frac{5\pi^2}{4}$

11. If $\sin^{-1} x + \sin^{-1} y = \frac{\pi}{2}$ and $\sin 2x = \cos 2y$, then

a. $x = \frac{\pi}{8} + \sqrt{\frac{1}{2} - \frac{\pi^2}{64}}$ b. $y = \sqrt{\frac{1}{2} - \frac{\pi^2}{64}} - \frac{\pi}{12}$ c. $x = \frac{\pi}{12} + \sqrt{\frac{1}{2} - \frac{\pi^2}{64}}$ d. $y = \sqrt{\frac{1}{2} - \frac{\pi^2}{64}} - \frac{\pi}{8}$

12. If $\cot^{-1}\left(\frac{n^2 - 10n + 21.6}{\pi}\right) > \frac{\pi}{6}$, $n \in N$, then n can be

a. 3 b. 2 c. 4 d. 8

13. If $S_n = \cot^{-1}(3) + \cot^{-1}(7) + \cot^{-1}(13) + \cot^{-1}(21) + \cdots n$ terms, then

a. $S_{10} = \tan^{-1}\frac{5}{6}$ b. $S_\infty = \frac{\pi}{4}$ c. $S_6 = \sin^{-1}\frac{4}{5}$ d. $S_{20} = \cot^{-1} 1.1$

14. The value of k $(k > 0)$ such that the length of the longest interval in which the function $f(x) = \sin^{-1}|\sin kx| + \cos^{-1}(\cos kx)$ is constant is $\pi/4$ is/are

a. 8 b. 4 c. 12 d. 16

15. Equation $1 + x^2 + 2x \sin(\cos^{-1} y) = 0$ is satisfied by

a. exactly one value of x

b. exactly two values of x

c. exactly one value of y

d. exactly two values of y

16. To the equation $2^{2\pi/\cos^{-1}x} - \left(a + \frac{1}{2}\right)2^{\pi/\cos^{-1}x} - a^2 = 0$ has only one real root, then

a. $1 \le a \le 3$ b. $a \ge 1$ c. $a \le -3$ d. $a \ge 3$

17. If $\sin^{-1}\left(a - \frac{a^2}{3} + \frac{a^3}{9} + \cdots\right) + \cos^{-1}(1 + b + b^2 + \cdots) = \frac{\pi}{2}$, then

a. $b = \frac{2a-3}{3a}$ b. $b = \frac{3a-2}{2a}$ c. $a = \frac{3}{2-3b}$ d. $a = \frac{2}{3-2b}$

18. If $2\tan^{-1}x + \sin^{-1}\frac{2x}{1+x^2}$ is independent of x, then

a. $x > 1$ b. $x < -1$ c. $0 < x < 1$ d. $-1 < x < 0$

19. $\cos^{-1}x + \cos^{-1}\left(\frac{x}{2} + \frac{1}{2}\sqrt{3-3x^2}\right)$ is equal to

a. $\frac{\pi}{3}$ for $x \in \left[\frac{1}{2}, 1\right]$ b. $\frac{\pi}{3}$ for $x \in \left[0, \frac{1}{2}\right]$

c. $2\cos^{-1}x - \cos^{-1}\frac{1}{2}$ for $x \in \left[\frac{1}{2}, 1\right]$ d. $2\cos^{-1}x - \cos^{-1}\frac{1}{2}$ for $x \in \left[0, \frac{1}{2}\right]$

20. Which of the following quantities is/are positive?

a. $\cos(\tan^{-1}(\tan 4))$ b. $\sin(\cot^{-1}(\cot 4))$ c. $\tan(\cos^{-1}(\cos 5))$ d. $\cot(\sin^{-1}(\sin 4))$

21. If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$, then

a. $x^2 + y^2 + z^2 + 2xyz = 1$ b. $2(\sin^{-1}x + \sin^{-1}y + \sin^{-1}z) = \cos^{-1}x + \cos^{-1}y + \cos^{-1}z$

c. $xy + yz + zx = x + y + z - 1$ d. $\left(x + \frac{1}{x}\right) + \left(y + \frac{1}{y}\right) + \left(z + \frac{1}{z}\right) \geq 6$

22. Which one of the following quantities is/are positive?

a. $\cos(\tan^{-1}(\tan 4))$ b. $\sin(\cot^{-1}(\cot 4))$ c. $\tan(\cos^{-1}(\cos 5))$ d. $\cot(\sin^{-1}(\sin 4))$

23. Which of the following is/are the value of $\cos\left[\frac{1}{2}\cos^{-1}\left(\cos\left(-\frac{14\pi}{5}\right)\right)\right]$?

a. $\cos\left(-\frac{7\pi}{5}\right)$ b. $\sin\left(\frac{\pi}{10}\right)$ c. $\cos\left(\frac{2\pi}{5}\right)$ d. $-\cos\left(\frac{3\pi}{5}\right)$

Reasoning Type

Solutions on page 4.87

Each question has four choices a, b, c and d, out of which *only one* is correct. Each question contains STATEMENT 1 and STATEMENT 2.

a. Both the statements are TRUE, and STATEMENT 2 is the correct explanation of STATEMENT 1
b. Both the statements are TRUE but STATEMENT 2 is NOT the correct explanation of STATEMENT 1
c. STATEMENT 1 is TRUE and STATEMENT 2 is FALSE
d. STATEMENT 1 is FALSE and STATEMENT 2 is TRUE

1. **Statement 1:** Number of roots of the equation $\cot^{-1}x + \cos^{-1}2x + \pi = 0$ is zero.
Statement 2: Range of $\cot^{-1}x$ and $\cos^{-1}x$ is $(0, \pi)$ and $[0, \pi]$, respectively.

2. **Statement 1:** Range of $f(x) = \tan^{-1}x + \sin^{-1}x + \cos^{-1}x$ is $(0, \pi)$.
Statement 2: $f(x) = \tan^{-1}x + \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2} + \tan^{-1}x$, for $x \in [-1, 1]$.

3. **Statement 1:** $\operatorname{cosec}^{-1}\left(\frac{1}{2} + \frac{1}{\sqrt{2}}\right) > \sec^{-1}\left(\frac{1}{2} + \frac{1}{\sqrt{2}}\right)$.
Statement 2: $\operatorname{cosec}^{-1}x < \sec^{-1}x$ if $1 \leq x < \sqrt{2}$.

4. **Statement 1:** $\sin^{-1}\left(\frac{1}{\sqrt{e}}\right) > \tan^{-1}\left(\frac{1}{\sqrt{\pi}}\right)$.
Statement 2: $\sin^{-1}x > \tan^{-1}y$ for $x > y$, $\forall\ x, y \in (0, 1)$.

5. **Statement 1:** Principal value of $\cos^{-1}(\cos 30)$ is $30 - 9\pi$.
   Statement 2: $30 - 9\pi \in [0, \pi]$.

6. Let $f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

   Statement 1: $f'(2) = -\frac{2}{5}$

   Statement 2: $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \pi - 2\tan^{-1}x, \forall x > 1$

7. **Statement 1:** Domain of $\tan^{-1}x$ and $\cot^{-1}x$ is R.
   Statement 2: $f(x) = \tan x$ and $g(x) = \cot x$ are unbounded functions.

8. **Statement 1:** $\tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \frac{\pi}{4}$

   Statement 2: For $x > 0, y > 0$, $\tan^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y-x}{y+x}\right) = \frac{\pi}{4}$

9. **Statement 1:** Principal value of $\sin^{-1}(\sin 3)$ can be 3 if we restrict the domain of $f(x) = \sin x$ to $[\pi/2, 3\pi/2]$.
   Statement 2: The restriction that the principal values of $\sin^{-1}(\sin x)$ is $[-\pi/2, -\pi/2]$ is a matter of convention. We could have allowed principal values $[\pi/2, 3\pi/2]$ without affecting the condition required for definition of inverse function.

Linked Comprehension Type

Solutions on page 4.88

Based upon each paragraph, three multiple choice questions have to be answered. Each question has four choices a, b, c and d, out of which only one is correct.

For Problems 1–3

For $x, y, z, t \in R$, $\sin^{-1}x + \cos^{-1}y + \sec^{-1}z \geq t^2 - \sqrt{2\pi}\, t + 3\pi$

1. The value of $x + y + z$ is equal to
   a. 1 b. 0 c. 2 d. –1
2. The principal value of $\cos^{-1}(\cos 5t^2)$ is
   a. $\frac{3\pi}{2}$ b. $\frac{\pi}{2}$ c. $\frac{\pi}{3}$ d. $\frac{2\pi}{3}$
3. The value of $\cos^{-1}(\min\{x, y, z\})$ is
   a. 0 b. $\frac{\pi}{2}$ c. π d. $\frac{\pi}{3}$

For Problems 4 – 6

$ax + b(\sec(\tan^{-1}x)) = c$ and $ay + b(\sec(\tan^{-1}y)) = c$

4. The value of xy is
   a. $\frac{2ab}{a^2 - b^2}$ b. $\frac{c^2 - b^2}{a^2 - b^2}$ c. $\frac{c^2 - b^2}{a^2 + b^2}$ d. none of these
5. The value of $x + y$ is
   a. $\frac{2ac}{a^2 - b^2}$ b. $\frac{c^2 - b^2}{a^2 - b^2}$ c. $\frac{c^2 - b^2}{a^2 + b^2}$ d. none of these

6. The value of $\dfrac{x+y}{1-xy}$ is

a. $\dfrac{2ab}{a^2-c^2}$ b. $\dfrac{2ac}{a^2-c^2}$ c. $\dfrac{c^2-b^2}{a^2+b^2}$ d. none of these

For Problems 7 – 9

Consider the system of equations $\cos^{-1}x + (\sin^{-1}y)^2 = \dfrac{p\pi^2}{4}$ and $\left(\cos^{-1}x\right)\left(\sin^{-1}y\right)^2 = \dfrac{\pi^4}{16}, p \in Z$.

7. The value of p for which system has a solution is

a. 1 b. 2 c. 0 d. -1

8. The value of x which satisfies the system of equations is

a. $\cos\dfrac{\pi^2}{8}$ b. $\sin\dfrac{\pi^2}{4}$ c. $\cos\dfrac{\pi^2}{2}$ d. none of these

9. Which of the following is not the value of y that satisfies the system of equations?

a. 1 b. -1 c. $\dfrac{1}{2}$ d. none of these

For Problems 10 – 12

Let $\cos^{-1}(4x^3-3x) = a + b\cos^{-1}x$.

10. If $x \in \left[-\dfrac{1}{2}, -1\right)$, then the value of $a + b\pi$ is

a. 2π b. 3π c. π d. -2π

11. If $x \in \left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$, then the principal value of $\sin^{-1}\left(\sin\dfrac{a}{b}\right)$ is

a. $-\dfrac{\pi}{3}$ b. $\dfrac{\pi}{3}$ c. $-\dfrac{\pi}{6}$ d. $\dfrac{\pi}{6}$

12. If $x \in \left(\dfrac{1}{2}, 1\right]$, then the value of $\lim\limits_{y\to a} b\cos(y)$ is

a. $-1/3$ b. -3 c. $\dfrac{1}{3}$ d. 3

Matrix-Match Type

Solutions on page 4.91

Each question contains statements given in two columns which have to be matched. Statements a, b, c, d in column I have to be matched with statements p, q, r, s in column II. If the correct matches are a-p, a-s, b-q, b-r, c-p, c-q and d-s, then the correctly bubbled 4 × 4 matrix should be as follows:

	p	q	r	s
a	(p)	(q)	(r)	(s)
b	(p)	(q)	(r)	(s)
c	(p)	(q)	(r)	(s)
d	(p)	(q)	(r)	(s)

1.

Column I	**Column II**
a. $\cos^{-1}(4x^3-3x)=3\cos^{-1}x$, then x can take values	**p.** $[1/2, 1]$
b. $\sin^{-1}(3x-4x^3)=3\sin^{-1}x$, then x can take values	**q.** $[-1/2, 0]$
c. $\cos^{-1}(4x^3-3x)=3\sin^{-1}x$, then x can take values	**r.** $[0, \sqrt{3}/2]$
d. $\sin^{-1}(3x-4x^3)=3\cos^{-1}x$, then x can take values	**s.** $[0, 1/2]$

2.

Column I	**Column II**
a. $(\sin^{-1}x)^2+(\sin^{-1}y)^2=\frac{\pi^2}{2}$ $\Rightarrow x^3+y^3=$	**p.** 1
b. $(\cos^{-1}x)^2+(\cos^{-1}y)^2=2\pi^2$ $\Rightarrow x^5+y^5$	**q.** -2
c. $(\sin^{-1}x)^2(\cos^{-1}y)^2=\frac{\pi^4}{4}\Rightarrow \lvert x-y\rvert$	**r.** 0
d. $\lvert\sin^{-1}x-\sin^{-1}y\rvert=\pi\Rightarrow x^y$	**s.** 2

3.

Column I	**Column II**
a. $x\in[\pi,2\pi]\Rightarrow\lvert\tan^{-1}(\tan x)\rvert$ can be	**p.** $\lvert x-2\pi\rvert$
b. $x\in[\pi,2\pi]\Rightarrow\lvert\cot^{-1}(\cot x)\rvert$ can be	**q.** $\lvert x-\pi\rvert$
c. $x\in[-\pi,\pi]\Rightarrow\lvert\sin^{-1}(\sin x)\rvert$ can be	**r.** $\lvert x\rvert$
d. $x\in[-\pi,\pi]\Rightarrow\lvert\cos^{-1}(\cos x)\rvert$ can be	**s.** $\lvert x+\pi\rvert$

4.

Column I	**Column II**
a. $\sin^{-1}\frac{4}{5}+2\tan^{-1}\frac{1}{3}=$	**p.** $\pi/6$
b. $\sin^{-1}\frac{12}{13}+\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{63}{16}=$	**q.** $\pi/2$
c. If $A=\tan^{-1}\frac{x\sqrt{3}}{2\lambda-x}$ and $B=\tan^{-1}\left(\frac{2x-\lambda}{\lambda\sqrt{3}}\right)$, then the value of $A-B$ is	**r.** $\pi/4$
d. $\tan^{-1}\frac{1}{7}+2\tan^{-1}\frac{1}{3}=$	**s.** π

5.

Column I	Column II
a. Range of $f(x) = \sin^{-1} x + \cos^{-1} x + \cot^{-1} x$ is	p. $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
b. Range of $f(x) = \cot^{-1} x + \tan^{-1} x + \operatorname{cosec}^{-1} x$ is	q. $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$
c. Range of $f(x) = \cot^{-1} x + \tan^{-1} x + \cos^{-1} x$ is	r. $\{0, \pi\}$
d. Range of $f(x) = \sec^{-1} x + \operatorname{cosec}^{-1} x + \sin^{-1} x$ is	s. $\left[\frac{3\pi}{4}, \frac{5\pi}{4}\right]$

6.

Column I	Column II
a. $\sin^{-1} x + x > 0$, for	p. $x < 0$
b. $\cos^{-1} x - x \geq 0$, for	q. $x \in (0, 1]$
c. $\tan^{-1} x + x < 0$, for	r. $x \in [-1, 0)$
d. $\cot^{-1} x + x > 0$, for	s. $x > 0$

Integer Type

Solutions on page 4.94

1. The solution set of inequality $(\cot^{-1}x)(\tan^{-1}x) + \left(2 - \frac{\pi}{2}\right)\cot^{-1}x - 3\tan^{-1}x - 3\left(2 - \frac{\pi}{2}\right) > 0$ is (a, b), then the value of $\cot^{-1}a + \cot^{-1}b$ is ________.
2. If $x = \sin^{-1}(a^6 + 1) + \cos^{-1}(a^4 + 1) - \tan^{-1}(a^2 + 1)$, $a \in R$, then the value of $\sec^2 x$ is ________.
3. If the roots of the equation $x^3 - 10x + 11 = 0$ are u, v and w. Then the value of $3\operatorname{cosec}^2(\tan^{-1}u + \tan^{-1}v + \tan^{-1}w)$ is ________.
4. Number of values of x for which $\sin^{-1}\left(x^2 - \frac{x^4}{3} + \frac{x^6}{9} \cdots\right) + \cos^{-1}\left(x^4 - \frac{x^8}{3} + \frac{x^{12}}{9} \cdots\right) = \frac{\pi}{2}$, where $0 \leq |x| < \sqrt{3}$, is ________.
5. If the domain of the function $f(x) = \sqrt{3\cos^{-1}(4x) - \pi}$ is $[a, b]$, then the value of $4a + 64b$ is ________.
6. If $0 < \cos^{-1} x < 1$ and $1 + \sin(\cos^{-1} x) + \sin^2(\cos^{-1} x) + \sin^3(\cos^{-1} x) + \cdots \infty = 2$, then the value of $12x^2$ is ________.
7. If $\tan^{-1}\left(x + \frac{3}{x}\right) - \tan^{-1}\left(x - \frac{3}{x}\right) = \tan^{-1}\frac{6}{x}$, then the value of x^4 is ________.
8. If range of the function $f(x) = \sin^{-1}x + 2\tan^{-1}x + x^2 + 4x + 1$ is $[p, q]$, then the value of $(p + q)$ is ________.
9. If n is the number of terms of the series $\cot^{-1} 3$, $\cot^{-1} 7$, $\cot^{-1} 13$, $\cot^{-1} 21$, ..., whose sum is $\frac{1}{2}\cos^{-1}\left(\frac{24}{145}\right)$, then the value of $n - 5$ is ________.

10. If the area enclosed by the curves $f(x) = \cos^{-1}(\cos x)$ and $g(x) = \sin^{-1}(\cos x)$ in $x \in [9\pi/4, 15\pi/4]$ is $a\pi^2/b$ (where a and b are coprime), then the value of $(a-b)$ is ________.

11. Absolute value of sum of all integers in the domain of $f(x) = \cot^{-1}\sqrt{(x+3)\,x} + \cos^{-1}\sqrt{x^2+3x+1}$ is ________.

12. The least value of $(1 + \sec^{-1}x)(1 + \cos^{-1}x)$ is ________.

13. Let $\cos^{-1}(x) + \cos^{-1}(2x) + \cos^{-1}(3x)$ be π. If x satisfies the equation $ax^3 + bx^2 + cx - 1 = 0$, then the value of $(b-a-c)$ is ________.

14. Number of integral values of x satisfying the equation $\tan^{-1}(3x) + \tan^{-1}(5x) = \tan^{-1}(7x) + \tan^{-1}(2x)$ is ________.

Archives

Solutions on page 4.98

Subjective

1. Find the value of $\cos(2\cos^{-1}x + \sin^{-1}x)$ at $x = 1/5$, where $0 \le \cos^{-1}x \le \pi$ and $-\pi/2 \le \sin^{-1}x \le \pi/2$. **(IIT-JEE, 1981)**

2. Prove that $\cos\tan^{-1}\sin\cot^{-1}x = \sqrt{\dfrac{x^2+1}{x^2+2}}$. **(IIT-JEE, 2002)**

Objective

Fill in the blanks

1. Let a, b and c be positive real numbers. Let $\theta = \tan^{-1}\sqrt{\dfrac{a(a+b+c)}{bc}} + \tan^{-1}\sqrt{\dfrac{b(a+b+c)}{ca}}$ $+ \tan^{-1}\sqrt{\dfrac{c(a+b+c)}{ab}}$. Then $\tan\theta =$ ____. **(IIT-JEE, 1981)**

2. The numerical value of $\tan\left(2\tan^{-1}\left(\dfrac{1}{5}\right) - \dfrac{\pi}{4}\right)$ is equal to ____. **(IIT-JEE, 1984)**

3. The greater of the two angles $A = 2\tan^{-1}(2\sqrt{2}-1)$ and $B = 3\sin^{-1}(1/3) + \sin^{-1}(3/5)$ is ____. **(IIT-JEE, 1989)**

Multiple choice questions with one correct answer

1. The value of $\tan\left[\cos^{-1}\left(\dfrac{4}{5}\right) + \tan^{-1}\left(\dfrac{2}{3}\right)\right]$ is **(IIT-JEE, 1983)**

a. $\dfrac{6}{17}$ b. $\dfrac{7}{16}$ c. $\dfrac{16}{7}$ d. none of these

2. The principal value of $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ is **(IIT-JEE, 1986)**

a. $-\frac{2\pi}{3}$ b. $\frac{2\pi}{3}$ c. $\frac{4\pi}{3}$ d. $\frac{5\pi}{3}$
e. none of these

3. If we consider only the principal values of the inverse trigonometric functions, then the value of $\tan\left(\cos^{-1}\frac{1}{5\sqrt{2}}-\sin^{-1}\frac{4}{\sqrt{17}}\right)$ is **(IIT-JEE, 1994)**

a. $\frac{\sqrt{29}}{3}$ b. $\frac{29}{3}$ c. $\frac{\sqrt{3}}{29}$ d. $\frac{3}{29}$ **(IIT-JEE, 1999)**

4. The number of real solutions of $\tan^{-1}\sqrt{x(x+1)}+\sin^{-1}\sqrt{x^2+x+1}=\pi/2$ is
a. zero b. one c. two d. infinite
(IIT-JEE, 2001)

5. If $\sin^{-1}\left(x-\frac{x^2}{2}+\frac{x^3}{4}-\cdots\right)+\cos^{-1}\left(x^2-\frac{x^4}{2}+\frac{x^6}{4}-\cdots\right)=\frac{\pi}{2}$ for $0<|x|<\sqrt{2}$, then x equals

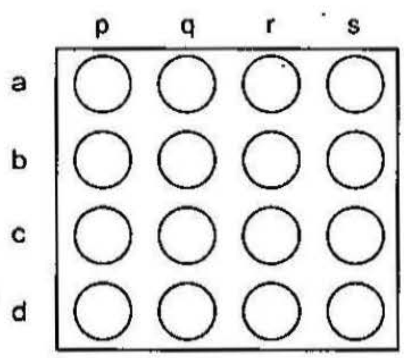
a. 1/2 b. 1 c. –1/2 d. –1

6. Domain of the definition of the function $f(x)=\sqrt{\sin^{-1}(2x)+\pi/6}$ is **(IIT-JEE, 2003)**
a. [–1/4, 1/2] b. [–1/2, 1/9] c. [–1/2, 1/2] d. [–1/4, 1/4]

7. The value of x for which $\sin(\cot^{-1}(1+x))=\cos(\tan^{-1}x)$ is **(IIT-JEE, 2004)**
a. 1/2 b. 1 c. 0 d. –1/2

Match the following type

1. The question contains statements given in two columns which have to be matched. Statements a, b, c, d in column I have to be matched with statements p, q, r, s in column II. The answers to these questions have to be appropriately bubbled as illustrated in the following example. If the correct matches are a-p, a-s, b-q, b-r, c-p, c-q, and d-s, then the correctly bubbled 4×4 matrix should be as follows:
(IIT-JEE, 2007)



Let (x, y) be such that $\sin^{-1}(ax)+\cos^{-1}(y)+\cos^{-1}(bxy)=\pi/2$. Match the statements in column I with statements in column II and indicate your answer by darkening the appropriate bubbles in the 4×4 matrix given in the ORS.

Column I	Column II
a. If $a = 1$ and $b = 0$, then (x, y)	**p.** lies on the circle $x^2 + y^2 = 1$
b. If $a = 1$ and $b = 1$, then (x, y)	**q.** lies on $(x^2 - 1)(y^2 - 1) = 0$
c. If $a = 1$ and $b = 2$, then (x, y)	**r.** lies on $y = x$
d. If $a = 2$ and $b = 2$, then (x, y)	**s.** lies on $(4x^2 - 1)(y^2 - 1) = 0$

ANSWERS AND SOLUTIONS

Subjective Type

1. Let $x = \cos y$, where $0 \le y \le \pi$, $|x| \le 1$

$$2\cos^{-1} x = \sin^{-1}\left(2x\sqrt{1-x^2}\right) \quad \text{(i)}$$

$\Rightarrow \quad 2\cos^{-1}(\cos y) = \sin^{-1}(2\cos y\sqrt{1-\cos^2 y})$

$= \sin^{-1}(2\cos y \sin y)$

$= \sin^{-1}(\sin 2y)$

$\Rightarrow \quad \sin^{-1}(\sin 2y) = 2y$ for $-\pi/4 \le y \le \pi/4$

and $2\cos^{-1}(\cos y) = 2y$ for $0 \le y \le \pi$

Thus, Eq. (i) holds only when

$y \in [0, \pi/4] \quad \Rightarrow \quad x \in [1/\sqrt{2}, 1]$

2. $f(x) = \sin^{-1}\left(\dfrac{1+x^2}{2x}\right)$ is defined for $-1 \le \dfrac{1+x^2}{2x} \le 1$ or $\left|\dfrac{1+x^2}{2x}\right| \le 1$

$\Rightarrow \quad |1 + x^2| \le |2x|$, for all x

$\Rightarrow \quad 1 + x^2 \le |2x|$, for all x (as $1 + x^2 > 0$)

$\Rightarrow \quad x^2 - 2|x| + 1 \le 0$

$\Rightarrow \quad |x|^2 - 2|x| + 1 \le 0$ (as $x^2 = |x^2|$)

$\Rightarrow \quad (|x| - 1)^2 \le 0$

But $(|x| - 1)^2$ is always either positive or zero.

Thus, $(|x| - 1)^2 = 0$

$\Rightarrow \quad |x| = 1 \Rightarrow x = \pm 1$

Hence, domain for $f(x)$ is $\{-1, 1\}$.

3. Let $\theta = \cot^{-1}(2x - x^2)$, where $\theta \in (0, \pi)$

$\Rightarrow \quad \cot\theta = 2x - x^2$, where $\theta \in (0, \pi)$

$= 1 - (1 - 2x + x^2)$, where $\theta \in (0, \pi)$

$= 1 - (1 - x)^2$, where $\theta \in (0, \pi)$

$\Rightarrow \quad \cot\theta \le 1$, where $\theta \in (0, \pi) \quad \Rightarrow \quad \dfrac{\pi}{4} \le \theta < \pi \quad \Rightarrow$ Range of $f(x)$ is $\left[\dfrac{\pi}{4}, \pi\right)$

4. Let t_n denotes the nth term of the series.

Then $t_n = \cot^{-1} 2n^2 = \tan^{-1} \dfrac{1}{2n^2} = \tan^{-1} \dfrac{(2n+1)-(2n-1)}{1+(4n^2-1)} = \tan^{-1}(2n+1) - \tan^{-1}(2n-1)$

Putting $n = 1, 2, 3, \ldots$, etc. in (1), we get

$t_1 = \tan^{-1} 3 - \tan^{-1} 1$

$t_2 = \tan^{-1} 5 - \tan^{-1} 3$

$t_3 = \tan^{-1} 7 - \tan^{-1} 5$

$\vdots$

$t_n = \tan^{-1}(2n+1) - \tan^{-1}(2n-1)$

Adding, we get

$S_n = \tan^{-1}(2n+1) - \tan^{-1} 1$

as $n \to \infty$, $\tan^{-1}(2n+1) \to \pi/2$

Hence, the required sum $= \dfrac{\pi}{4}$.

5. Let $\theta = \operatorname{cosec}^{-1} \sqrt{(n^2+1)(n^2+2n+2)}$

$\Rightarrow \operatorname{cosec}^2\theta = (n^2+1)(n^2+2n+2) = (n^2+1)^2 + 2n(n^2+1) + n^2 + 1 = (n^2+n+1)^2 + 1$

$\Rightarrow \cot^2\theta = (n^2+n+1)^2$

$$\Rightarrow \tan\theta = \frac{1}{n^2+n+1} = \frac{(n+1)-n}{1+(n+1)n}$$

$$\Rightarrow \theta = \tan^{-1}\left[\frac{(n+1)-n}{1+(n+1)n}\right] = \tan^{-1}(n+1) - \tan^{-1} n$$

Thus, sum of n terms of the given series

$= (\tan^{-1} 2 - \tan^{-1} 1) + (\tan^{-1} 3 - \tan^{-1} 2) + (\tan^{-1} 4 - \tan^{-1} 3) + \cdots + (\tan^{-1}(n+1) - \tan^{-1} n)$

$= \tan^{-1}(n+1) - \pi/4$

6. Here, $\tan^{-1} x + \cos^{-1} \dfrac{y}{\sqrt{1-y^2}} = \sin^{-1} \dfrac{3}{\sqrt{10}}$

$$\Rightarrow \tan^{-1} x + \tan^{-1}\left(\frac{1}{y}\right) = \tan^{-1}(3)$$

$$\Rightarrow \tan^{-1}\left(\frac{1}{y}\right) = \tan^{-1} 3 - \tan^{-1}(x)$$

$$\Rightarrow \tan^{-1}\left(\frac{1}{y}\right) = \tan^{-1}\left(\frac{3-x}{1+3x}\right)$$

$$\Rightarrow y = \frac{1+3x}{3-x}$$

As x, y are positive integers, $x = 1, 2$ and corresponding $y = 2, 7$.

Therefore, the solutions are $(x, y) = (1, 2)$ and $(2, 7)$, i.e., there are two solutions.

7. We have $\tan^{-1} y = 4 \tan^{-1} x$

$$\Rightarrow \tan^{-1} y = 2\tan^{-1} \frac{2x}{1-x^2} \text{ (as } |x| < 1)$$

$$= \tan^{-1} \frac{\frac{4x}{1-x^2}}{1 - \frac{4x^2}{(1-x^2)^2}}$$

$$= \tan^{-1} \frac{4x(1-x^2)}{x^4 - 6x^2 + 1} \quad \left(\text{as} \left|\frac{2x}{1-x^2}\right| < 1\right)$$

$$\Rightarrow \quad y = \frac{4x(1-x^2)}{x^4 - 6x^2 + 1}$$

If $x = \tan \frac{\pi}{8} \Rightarrow \tan^{-1} y = 4 \tan^{-1} x = \frac{\pi}{2} \Rightarrow y \to \infty \Rightarrow x^4 - 6x^2 + 1 = 0$

Hence, $\tan \frac{\pi}{8}$ is a root of $x^4 - 6x^2 + 1 = 0$.

8. x_1, x_2, x_3 and x_4 are the roots of the equation

$x^4 - x^3 \sin 2\beta + x^2 \cos 2\beta - x \cos \beta - \sin \beta = 0$.

$\Sigma x_1 = x_1 + x_2 + x_3 + x_4 = -\frac{(-\sin 2\beta)}{1} = \sin 2\beta$

$\Sigma x_1 x_2 = \cos 2\beta$

$\Sigma x_1 x_2 x_3 = \cos \beta$ and $x_1 x_2 x_3 x_4 = -\sin \beta$

Now, $\tan [\tan^{-1} x_1 + \tan^{-1} x_2 + \tan^{-1} x_3 + \tan^{-1} x_4]$

$$= \frac{\Sigma x_1 - \Sigma x_1 x_2 x_3}{1 - \Sigma x_1 x_2 + x_1 x_2 x_3 x_4} = \frac{\sin 2\beta - \cos \beta}{1 - \cos 2\beta - \sin \beta} = \frac{2 \sin \beta \cos \beta - \cos \beta}{2 \sin^2 \beta - \sin \beta} = \frac{\cos \beta (2 \sin \beta - 1)}{\sin \beta (2 \sin \beta - 1)} = \cot \beta$$

or $\tan [\tan^{-1} x_1 + \tan^{-1} x_2 + \tan^{-1} x_3 + \tan^{-1} x_4] = \tan\left(\frac{\pi}{2} - \beta\right)$

$$\Rightarrow \quad \tan^{-1} x_1 + \tan^{-1} x_2 + \tan^{-1} x_3 + \tan^{-1} x_4 = n\pi + \frac{\pi}{2} - \beta, n \in Z$$

9. As, $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2} \ \forall x \in R$

So, the given equation can be written as

$$(\sin^{-1} x)^3 + (\cos^{-1} x)^3 = 7\left(\frac{\pi^3}{8}\right)$$

$$\Rightarrow \quad (\sin^{-1} x + \cos^{-1} x)^3 - 3(\sin^{-1} x)(\cos^{-1} x)(\sin^{-1} x + \cos^{-1} x) = 7\left(\frac{\pi}{2}\right)^3$$

$$\Rightarrow \quad \left(\frac{\pi}{2}\right)^3 - 3(\sin^{-1} x)(\cos^{-1} x)\left(\frac{\pi}{2}\right) = 7\left(\frac{\pi}{2}\right)^3 \quad \Rightarrow \quad (\sin^{-1} x)(\cos^{-1} x) = -\frac{\pi^2}{2}$$

Now, as the maximum value of $\cos^{-1} x$ is π ($\because \cos^{-1} x$ is always ≥ 0) and the minimum value of $\sin^{-1} x$ is $-\pi/2$ and this happens at the same value of x, i.e., $x = -1$.

So, the minimum value of $(\sin^{-1} x)(\cos^{-1} x) = \left(-\frac{\pi}{2}\right)(\pi) = -\frac{\pi^2}{2}$.

So, if $(\sin^{-1} x)(\cos^{-1} x) = -\frac{\pi^2}{2}$, then $x = -1$ only.

10. $(\sin^{-1}x)^3 + (\cos^{-1} x)^3 = a\pi^3$

$\Rightarrow \quad (\sin^{-1} x + \cos^{-1} x)((\sin^{-1} x + \cos^{-1}x)^2 - 3 \sin^{-1} x \cos^{-1} x) = a\pi^3$

$\Rightarrow \quad \frac{\pi^2}{4} - 3 \sin^{-1} x \cos^{-1} x = 2a\pi^2$

$\Rightarrow \quad \sin^{-1} x \left(\frac{\pi}{2} - \sin^{-1} x\right) = \frac{\pi^2}{12}(1-8a)$

$\Rightarrow \quad (\sin^{-1} x)^2 - \frac{\pi}{2} \sin^{-1} x = -\frac{\pi^2}{12}(1-8a)$

$\Rightarrow \quad \left(\sin^{-1} x - \frac{\pi}{4}\right)^2 = \frac{\pi^2}{12}(8a-1) + \frac{\pi^2}{16}$

$= \frac{\pi^2}{48}(32a-1)$

Now, $\sin^{-1} x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\Rightarrow \quad -\frac{3\pi}{4} \le \sin^{-1} x - \frac{\pi}{4} \le \frac{\pi}{4}$

$\Rightarrow \quad 0 \le \left(\sin^{-1} x - \frac{\pi}{4}\right)^2 \le \frac{9\pi^2}{16}$

$\Rightarrow \quad 0 \le \frac{\pi^2}{48}(32a-1) \le \frac{9\pi^2}{16}$

$\Rightarrow \quad 0 \le 32a - 1 \le 27$

$\Rightarrow \quad \frac{1}{32} \le a \le \frac{7}{8}$

Thus, the required set of values of a is $\left[\frac{1}{32}, \frac{7}{8}\right]$.

11. Since $p, q > 0$, therefore $pq > 0$,

$\tan^{-1} \frac{p-q}{1+pq} = \tan^{-1} p - \tan^{-1} q$ (i)

Since $qr > -1$,

$\tan^{-1} \frac{q-r}{1+qr} = \tan^{-1} q - \tan^{-1} r$ (ii)

Since $pr < -1$ and $r < 0$,

$\tan^{-1} \frac{r-p}{1+rp} = \pi + \tan^{-1} r - \tan^{-1} p$ (iii)

On adding Eqs. (i), (ii) and (iii), we get

$\tan^{-1} \frac{p-q}{1+pq} + \tan^{-1} \frac{q-r}{1+qr} + \tan^{-1} \frac{r-p}{1+rp} = \pi$

12. Given $\sqrt{|\sin^{-1}|\cos x|| + |\cos^{-1}|\sin x||} = \sin^{-1}(|\cos x|) - \cos^{-1}|\sin x|$

When $x \in \left[0, \frac{\pi}{2}\right], 0 \le \frac{\pi}{2} - x \le \frac{\pi}{2}$, we have

$$\sqrt{\sin^{-1}\sin\left(\frac{\pi}{2}-x\right)+\cos^{-1}\cos\left(\frac{\pi}{2}-x\right)} = \sin^{-1}\left(\sin\left(\frac{\pi}{2}-x\right)\right)-\cos^{-1}\left(\cos\left(\frac{\pi}{2}-x\right)\right)$$

$$\Rightarrow \quad \sqrt{2\left(\frac{\pi}{2}-x\right)} = \frac{\pi}{2}-x-\frac{\pi}{2}+x$$

$$\Rightarrow \quad x = \frac{\pi}{2}$$

When $x \in \left[-\frac{\pi}{2}, 0\right], \frac{\pi}{2} \le \frac{\pi}{2} - x \le \pi$, we have

$$\sqrt{\sin^{-1}\left(\sin\left(\frac{\pi}{2}-x\right)\right)+\cos^{-1}\left(-\cos\left(\frac{\pi}{2}-x\right)\right)} = \sin^{-1}\left(\sin\left(\frac{\pi}{2}-x\right)\right)-\cos^{-1}\left(-\cos\left(\frac{\pi}{2}-x\right)\right)$$

$$\Rightarrow \quad \sqrt{\pi+2x} = 0 \qquad \Rightarrow \quad x = -\frac{\pi}{2}$$

13. Taking the tangents of both sides of the equation, we have

$$\frac{\tan\left[\tan^{-1}\frac{x+1}{x-1}\right]+\tan\left[\tan^{-1}\frac{x-1}{x}\right]}{1-\tan\left[\tan^{-1}\frac{x+1}{x-1}\right]\tan\left[\tan^{-1}\frac{x-1}{x}\right]} = \tan((\tan^{-1}(-7)) = -7$$

$$\Rightarrow \quad \frac{\frac{x+1}{x-1}+\frac{x-1}{x}}{1-\frac{x+1}{x-1}\frac{x-1}{x}} = -7$$

$$\Rightarrow \quad \frac{2x^2-x+1}{1-x} = -7$$

so that $x = 2$.

This value makes the left-hand side of the given equation positive, so there is no value of x strictly satisfying the given equation.

The value $x = 2$ is a solution of the equation $\tan^{-1}\frac{x+1}{x-1} + \tan^{-1}\frac{x-1}{x} = \pi + \tan^{-1}(-7)$.

14. Let us transfer $\sin^{-1} 6\sqrt{3}x$ to the right-hand side of the equation and calculate the sine of the both sides of the resulting equation:

$\sin(\sin^{-1} 6x) = \sin(-\sin^{-1} 6\sqrt{3}x - \pi/2)$

$\Rightarrow \quad 6x = -\sin(\sin^{-1} 6\sqrt{3}x + \sin^{-1} 1)$ [using $\sin^{-1}(-x) = -\sin^{-1}(x)$]

$= -\sin(\sin^{-1}\sqrt{1-108x^2})$ (i)

Squaring both sides, we get

$36x^2 = 1 - 108x^2 \quad \Rightarrow 144x^2 = 1$

whose roots are $x = 1/12$ and $x = -1/12$.

Let us verify:
Substituting $x = 1/12$ in the given equation, we get

$$\sin^{-1}\left(-\frac{1}{2}\right)+\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{6}-\frac{\pi}{3} = -\frac{\pi}{2}$$

Thus $x = 1/12$ is the roots of given equation.

Using Eq. (i), we get $x = \frac{-1}{12}$

$$\text{L.H.S.} = \frac{1}{2}$$

$$\text{R.H.S.} = -\sqrt{1-108x^2} = -1/2$$

Thus L.H.S. ≠ R.H.S. of Eq. (i)
Thus $x = -1/12$ is a root of the given equation.

15. Here,

$$\tan^{-1}\left(\frac{a_1x-y}{x+a_1y}\right) = \tan^{-1}\left(\frac{a_1-\frac{y}{x}}{1+a_1\frac{y}{x}}\right) = \tan^{-1}a_1 - \tan^{-1}\frac{y}{x}$$

$$\tan^{-1}\left(\frac{a_2-a_1}{1+a_2a_1}\right) = \tan^{-1}a_2 - \tan^{-1}a_1$$

$$\tan^{-1}\left(\frac{a_3-a_2}{1+a_3a_2}\right) = \tan^{-1}a_3 - \tan^{-1}a_2$$

$$\vdots$$

$$\tan^{-1}\left(\frac{a_n-a_{n-1}}{1+a_na_{n-1}}\right) = \tan^{-1}a_n - \tan^{-1}a_{n-1}$$

$$\tan^{-1}\left(\frac{1}{a_n}\right) = \cot^{-1}a_n$$

Adding, we get L.H.S. $= \tan^{-1}a_n + \cot^{-1}a_n - \tan^{-1}\frac{y}{x} = \frac{\pi}{2} - \tan^{-1}\frac{y}{x} =$ R.H.S.

$$= \cot^{-1}\frac{y}{x} = \tan^{-1}\frac{x}{y}$$

Objective Type

1. c. $\sin^{-1}(\sin 10) = \sin^{-1}[\sin(3\pi-10)] = 3\pi-10$

2. b. $\cos^{-1}\left(\cos\frac{5\pi}{4}\right) = \cos^{-1}\left(\cos\left(2\pi-\frac{5\pi}{4}\right)\right) = \cos^{-1}\left(\cos\frac{3\pi}{4}\right) = \frac{3\pi}{4}$

3. a. $\sin^{-1}(\sin 12) + \cos^{-1}(\cos 12) = \sin^{-1}(\sin(12-4\pi)) + \cos^{-1}(\cos(4\pi-12)) = 12-4\pi+4\pi-12 = 0$

4. a. $\sin^{-1}\sin\left(\frac{22\pi}{7}\right) = \sin^{-1}\sin\left(3\pi+\frac{\pi}{7}\right) = -\frac{\pi}{7}$

$$\cos^{-1}\cos\left(\frac{5\pi}{3}\right) = \cos^{-1}\cos\left(2\pi-\frac{\pi}{3}\right) = \frac{\pi}{3}$$

$\tan^{-1}\tan\left(\frac{5\pi}{7}\right) = \tan^{-1}\tan\left(\pi - \frac{2\pi}{7}\right) = -\frac{2\pi}{7}$

$\sin^{-1}\cos(2) = \frac{\pi}{2} - \cos^{-1}\cos 2 = \frac{\pi}{2} - 2$

Therefore, the required value $= -\frac{\pi}{7} + \frac{\pi}{3} - \frac{2\pi}{7} + \frac{\pi}{2} - 2$

$= \frac{(-18+35)\pi}{42} - 2 = \frac{17\pi}{42} - 2$

5. d. $\sin^{-1}(\cos(\cos^{-1}(\cos x) + \sin^{-1}(\sin x))) = \sin^{-1}(\cos(x + \pi - x))$ [as $x \in (\pi/2, \pi)$]

$= \sin^{-1}(\cos\pi) = \sin^{-1}(-1) = -\frac{\pi}{2}$

6. c. $\cos^{-1}(\cos(2\cot^{-1}(\sqrt{2}-1))) = \cos^{-1}\left(\cos\left(2\left(67.5^\circ\right)\right)\right)$

$= \cos^{-1}\left(\cos\left(135^\circ\right)\right) = 135^\circ = \frac{3\pi}{4}$

7. a. We have $\sin^{-1}\left(\cot\left(\sin^{-1}\sqrt{\frac{2-\sqrt{3}}{4}} + \cos^{-1}\frac{\sqrt{12}}{4} + \sec^{-1}\sqrt{2}\right)\right)$

$= \sin^{-1}\left(\cot\left(\sin^{-1}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right) + \cos^{-1}\frac{\sqrt{3}}{2} + \cos^{-1}\frac{1}{\sqrt{2}}\right)\right)$

$= \sin^{-1}[\cot(15^\circ + 30^\circ + 45^\circ)]$

$= \sin^{-1}(\cot(90^\circ)) = \sin^{-1}(0) = 0$

8. d. $\cos^{-1}\left(\sqrt{\frac{2}{3}}\right) - \cos^{-1}\left(\frac{\sqrt{6}+1}{2\sqrt{3}}\right)$

$= \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) - \tan^{-1}\left(\frac{\sqrt{3}-\sqrt{2}}{1+\sqrt{6}}\right) = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) - \left[\tan^{-1}\sqrt{3} - \tan^{-1}\sqrt{2}\right]$

$= \left(\tan^{-1}\frac{1}{\sqrt{2}} + \tan^{-1}\sqrt{2}\right) - \tan^{-1}\sqrt{3}$

$= \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$

9. a. Let $\cos^{-1}\left(\frac{1}{8}\right) = \theta$, where $0 < \theta < \pi$, then $\frac{1}{2}\cos^{-1}\frac{1}{8} = \frac{1}{2}\theta$

$\Rightarrow \cos\left(\frac{1}{2}\cos^{-1}\frac{1}{8}\right) = \cos\frac{\theta}{2}$

Now, $\cos^{-1}\frac{1}{8} = \theta \Rightarrow \cos\theta = \frac{1}{8} \Rightarrow \cos^2\frac{\theta}{2} = \frac{9}{16} \Rightarrow \cos\frac{\theta}{2} = \frac{3}{4}$

10. c. $x = \tan^{-1} 3 \Rightarrow \tan x = 3$

$\tan(x+y) = 33$

$\Rightarrow \quad \dfrac{\tan x+\tan y}{1-\tan x\tan y}=33$

$\Rightarrow \quad \dfrac{3+\tan y}{1-3\tan y}=33$

$\Rightarrow \quad 3+\tan y=33-99\tan y$

$\Rightarrow \quad 100\tan y=30$

$\Rightarrow \quad \tan y=0.3 \quad \Rightarrow \quad y=\tan^{-1}(0.3)$

11. c. Let $\cos^{-1}\left(\dfrac{\sqrt{5}}{3}\right)=\alpha$. Then $\cos\alpha=\dfrac{\sqrt{5}}{3}$, where $0<\alpha<\dfrac{\pi}{2}$

Now, $\tan\dfrac{\alpha}{2}=\sqrt{\dfrac{1-\cos\alpha}{1+\cos\alpha}}=\sqrt{\dfrac{1-\sqrt{5}/3}{1+\sqrt{5}/3}}=\sqrt{\dfrac{3-\sqrt{5}}{3+\sqrt{5}}}=\sqrt{\dfrac{(3-\sqrt{5})^2}{9-5}}=\dfrac{1}{2}(3-\sqrt{5})$

12. a. $\tan^{-1}\left[\dfrac{\cos x}{1+\sin x}\right]=\tan^{-1}\left[\dfrac{\sin[(\pi/2)-x]}{1+\cos[(\pi/2)-x]}\right]$

$=\tan^{-1}\left[\dfrac{2\sin[(\pi/4)-(x/2)]\cos[(\pi/4)-(x/2)]}{2\cos^2[(\pi/4)-(x/2)]}\right]$

$=\tan^{-1}\tan\left(\dfrac{\pi}{4}-\dfrac{x}{2}\right)=\dfrac{\pi}{4}-\dfrac{x}{2}$

$\Rightarrow \quad -\dfrac{\pi}{2}<\dfrac{\pi}{4}-\dfrac{x}{2}<\dfrac{\pi}{2}$

$\Rightarrow \quad -\dfrac{3\pi}{4}<-\dfrac{x}{2}<\dfrac{\pi}{4}$

$\Rightarrow -\dfrac{\pi}{4}<\dfrac{x}{2}<\dfrac{3\pi}{4}$

$\Rightarrow \quad -\dfrac{\pi}{2}<\dfrac{x}{2}<\dfrac{3\pi}{2}$

13. d. $f(x)+f(-x)=2$

Now $(\sin^{-1}(\sin 8))=3\pi-8=y$

and $(\tan^{-1}(\tan 8))=(8-3\pi)$

Hence, $f(y)+f(-y)=2$

Given $f(y)=\alpha$, we have $f(-y)=2-\alpha$.

14. d. $f(x)=\tan^{-1}\left(\dfrac{(\sqrt{12}-2)x^2}{x^4+2x^2+3}\right)$

$=\tan^{-1}\left(\dfrac{2(\sqrt{3}-1)}{x^2+\dfrac{3}{x^2}+2}\right)$

As $x^2+\frac{3}{x^2}\ge 2\sqrt{3}$ [using A.M. $\ge$ G.M.]

$\Rightarrow\ x^2+\frac{3}{x^2}+2\ge 2+2\sqrt{3}$

$\therefore (f(x))_{\max.}=\tan^{-1}\left(\frac{2(\sqrt{3}-1)}{2(\sqrt{3}+1)}\right)=\frac{\pi}{12}$

15. b. Let $x=\sin\theta$ and $\sqrt{x}=\sin\phi$, where $x\in[0,1]\Rightarrow\theta,\phi\in[0,\pi/2]$

$\Rightarrow\quad \theta-\phi\in\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$

Now, $\sin^{-1}\left(x\sqrt{1-x}-\sqrt{x}\sqrt{1-x^2}\right)=\sin^{-1}\left(\sin\theta\sqrt{1-\sin^2\phi}-\sin\phi\sqrt{1-\sin^2\theta}\right)$

$=\sin^{-1}(\sin\theta\cos\phi-\sin\phi\cos\theta)$

$=\sin^{-1}\sin(\theta-\phi)=\theta-\phi=\sin^{-1}(x)-\sin^{-1}\left(\sqrt{x}\right)$

16. c. $\tan^{-1}\frac{x}{y}-\tan^{-1}\left(\frac{x-y}{x+y}\right)=\tan^{-1}\frac{x}{y}-\tan^{-1}\left(\frac{1-(y/x)}{1+(y/x)}\right)$

$=\tan^{-1}\frac{x}{y}-\left(\tan^{-1}1-\tan^{-1}\frac{y}{x}\right)$

$=\tan^{-1}\frac{x}{y}+\tan^{-1}\frac{y}{x}-\frac{\pi}{4}$

$=\tan^{-1}\frac{x}{y}+\cot^{-1}\frac{x}{y}-\frac{\pi}{4}=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$

17. c. Given equation is $\tan^{-1}\frac{a+x}{a}+\tan^{-1}\frac{a-x}{a}=\frac{\pi}{6}$

$\Rightarrow\quad \tan^{-1}\left(\frac{\frac{a+x}{a}+\frac{a-x}{a}}{1-\frac{a+x}{a}\frac{a-x}{a}}\right)=\frac{\pi}{6}$

$\Rightarrow\quad \frac{2a^2}{x^2}=\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}\Rightarrow x^2=2\sqrt{3}a^2$

18. c. $\cot^{-1}\frac{n}{\pi}>\frac{\pi}{6}$

$\Rightarrow\quad \frac{n}{\pi}<\cot\frac{\pi}{6}$ [as $\cot^{-1}x$ is a decreasing function]

$\Rightarrow\quad \frac{n}{\pi}<\sqrt{3}\quad\Rightarrow\quad n<\sqrt{3}\,\pi\quad\Rightarrow\quad n<5.46\quad\Rightarrow$ maximum value of n is 5

19. c. Let $\tan^{-1}2=\alpha\quad\Rightarrow\quad\tan\alpha=2$

and $\cot^{-1}3=\beta\quad\Rightarrow\quad\cot\beta=3$

$\sec^2(\tan^{-1} 2) + \text{cosec}^2(\cot^{-1} 3) = \sec^2\alpha + \text{cosec}^2\beta = 1 + \tan^2\alpha + 1 + \cot^2\beta$

$$= 2 + (2)^2 + (3)^2 = 15$$

20. d. $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

$$= \frac{\dfrac{\sqrt{3}\,x}{2K - x} - \dfrac{2x - K}{\sqrt{3}\,K}}{1 + \dfrac{\sqrt{3}\,x}{2K - x}\cdot\dfrac{2x - K}{\sqrt{3}\,K}}$$

$$= \frac{3Kx - (2x - K)(2K - x)}{(2K - x)\sqrt{3}\,K + \sqrt{3}\,x(2x - K)}$$

$$= \frac{3Kx - (4Kx - 2x^2 - 2K^2 + Kx)}{2\sqrt{3}\,K^2 - \sqrt{3}Kx + 2\sqrt{3}\,x^2 - \sqrt{3}\,Kx}$$

$$= \frac{2x^2 - 2Kx + 2K^2}{2\sqrt{3}\,x^2 - 2\sqrt{3}\,Kx + 2\sqrt{3}\,K^2} = \frac{1}{\sqrt{3}} = \tan 30°$$

$\therefore A - B = 30°$

21. b. $\tan^{-1}\dfrac{b+a}{b-a} - \tan^{-1}\dfrac{a}{b} = \tan^{-1}\dfrac{\dfrac{b+a}{b-a} - \dfrac{a}{b}}{1 + \dfrac{b+a}{b-a}\dfrac{a}{b}}$

$$= \tan^{-1}\frac{b^2 + ab - ab + a^2}{b^2 - ab + ab + a^2} = \tan^{-1}\frac{a^2 + b^2}{a^2 + b^2} = \tan^{-1} 1 = \frac{\pi}{4}$$

Therefore, the required value $= \sec\left(\dfrac{\pi}{4}\right) = \sqrt{2}$

22. d. $a\sin^{-1} x - b\cos^{-1} x = c$

We have $b\sin^{-1}x + b\cos^{-1} x = \dfrac{b\pi}{2}$ $\Rightarrow$ $(a + b)\sin^{-1} x = \dfrac{b\pi}{2} + c$

$$\Rightarrow \quad \sin^{-1} x = \frac{\dfrac{(b\pi)}{2} + c}{a + b}$$

$$\Rightarrow \quad \cos^{-1} x = \frac{\pi ab + c(a - b)}{a + b}$$

23. b. $\cos^{-1}\left(\dfrac{1 + x^2}{2x}\right) = \dfrac{\pi}{2} + (\sin^{-1} x + \cos^{-1} x)$

$\Rightarrow \cos^{-1}\left(\dfrac{1 + x^2}{2x}\right) = \pi \quad \Rightarrow \left(\dfrac{1 + x^2}{2x}\right) = \cos\pi = -1 \quad \Rightarrow x^2 + 1 + 2x = 0 \quad \Rightarrow x = -1$

24. b. $0 \le x^2 + x + 1 \le 1$ and $0 \le x^2 + x \le 1$

$\therefore x = -1, 0$

For $x = -1$

L.H.S. $= 2\sin^{-1} 1 + \cos^{-1} 0 = \dfrac{3\pi}{2}$

$\therefore x = -1$ is a solution.

For $x = 0$, L.H.S. $= 2\sin^{-1} 1 + \cos^{-1} 0 = \dfrac{3\pi}{2}$

Therefore, $x = 0$ is a solution and sum of the solutions $= -1$.

25. c. $\tan^{-1}(1+x) + \tan^{-1}(1-x) = \dfrac{\pi}{2}$

$$\Rightarrow \quad \tan^{-1}(1+x) = \frac{\pi}{2} - \tan^{-1}(1-x)$$
$$= \cot^{-1}(1-x)$$
$$= \tan^{-1}\left(\frac{1}{1-x}\right)$$

$$\Rightarrow \quad 1 + x = \frac{1}{1-x} \Rightarrow 1 - x^2 = 1 \Rightarrow x = 0$$

26. c. We have $\sin^{-1} x + \sin^{-1}(1-x) = \cos^{-1} x$

$$\Rightarrow \quad \sin(\sin^{-1} x + \sin^{-1}(1-x)) = \sin(\cos^{-1} x)$$

$$\Rightarrow \quad x\sqrt{1-(1-x)^2} + \sqrt{1-x^2}\,(1-x) = \sqrt{1-x^2}$$

$$\Rightarrow \quad x\sqrt{1-(1-x)^2} = x\sqrt{1-x^2}$$

$$\Rightarrow \quad x = 0 \text{ or } 2x - x^2 = 1 - x^2 \quad \Rightarrow \quad x = 0 \text{ or } x = \frac{1}{2}$$

27. c. We have $\tan^{-1}\dfrac{1-x}{1+x} = \dfrac{1}{2}\tan^{-1} x$

$$\Rightarrow \quad \tan^{-1}\left[\frac{1-\tan\theta}{1+\tan\theta}\right] = \frac{1}{2}\theta \qquad \text{(putting } x = \tan\theta\text{)}$$

$$\Rightarrow \quad \tan^{-1}\left[\frac{\tan\dfrac{\pi}{4} - \tan\theta}{1 + \tan\dfrac{\pi}{4}\tan\theta}\right] = \frac{\theta}{2}$$

$$\Rightarrow \quad \tan^{-1}\tan\left(\frac{\pi}{4} - \theta\right) = \frac{\theta}{2}$$

$$\Rightarrow \quad \frac{\pi}{4} - \theta = \frac{\theta}{2}$$

$$\Rightarrow \quad \theta = \frac{\pi}{6} = \tan^{-1} x$$

$$\Rightarrow \quad x = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

28. c. We have $\cos^{-1} x + \cos^{-1}(2x) = -\pi$, which is not possible as $\cos^{-1}x$ and $\cos^{-1}2x$ never take negative values.

29. b. The given equation is $ax^2 + \sin^{-1}((x-1)^2 + 1) + \cos^{-1}((x-1)^2 + 1) = 0$.
Now, $-1 \le (x-1)^2 + 1 \le 1 \Rightarrow x = 1$

So, we have $a + \dfrac{\pi}{2} = 0 \Rightarrow a = -\dfrac{\pi}{2}$

30. c. Since $\sqrt{x^2-3x+2} \geq 0 \Rightarrow 0 \leq \tan^{-1}\sqrt{x^2-3x+2} < \frac{\pi}{2}$

and $\sqrt{4x-x^2-3} \geq 0 \Rightarrow 0 < \cos^{-1}\sqrt{4x-x^2-3} \leq \frac{\pi}{2}$

Adding, we have $0 < \text{L.H.S.} < \pi$

Therefore, the given equation has no solution.

31. b. The given equation can be written as

$3\tan^{-1}\left(2-\sqrt{3}\right) = \tan^{-1}\left(\frac{1}{x}\right) + \tan^{-1}\left(\frac{1}{3}\right)$

$\Rightarrow \quad 3(15°) = \tan^{-1}\frac{\frac{1}{x}+\frac{1}{3}}{1-\frac{1}{x}\frac{1}{3}} \quad \Rightarrow \quad 1 = \frac{3+x}{3x-1} \Rightarrow x = 2$

32. c. $\tan^{-1}x + 2\cot^{-1}x = \frac{2\pi}{3}$

$\Rightarrow \quad \tan^{-1}x = 2\left(\frac{\pi}{3} - \cot^{-1}x\right) = 2\left(\frac{\pi}{3} - \left(\frac{\pi}{2} - \tan^{-1}x\right)\right) = 2\left(-\frac{\pi}{6} + \tan^{-1}x\right)$

$\Rightarrow \quad \tan^{-1}x = \frac{\pi}{3} \quad \Rightarrow \quad x = \tan\frac{\pi}{3} = \sqrt{3}$

33. a. Put $x = \tan\theta$

$\therefore 3\sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta} - 4\cos^{-1}\frac{1-\tan^2\theta}{1+\tan^2\theta} + 2\tan^{-1}\frac{2\tan\theta}{1-\tan^2\theta} = \frac{\pi}{3}$

$\Rightarrow \quad 3\sin^{-1}(\sin 2\theta) - 4\cos^{-1}(\cos 2\theta) + 2\tan^{-1}(\tan 2\theta) = \frac{\pi}{3}$

$\Rightarrow \quad 3(2\theta) - 4(2\theta) + 2(2\theta) = \frac{\pi}{3} \quad \Rightarrow \quad 2\theta = \frac{\pi}{3} \quad \Rightarrow \quad \theta = \frac{\pi}{6} \quad \Rightarrow \quad x = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

34. c. Put $\sin^{-1}\frac{5}{x} = A \Rightarrow \frac{5}{x} = \sin A$

$\sin^{-1}\frac{12}{x} = B \Rightarrow \frac{12}{x} = \sin B \quad \Rightarrow \quad A + B = \frac{\pi}{2}$

$\Rightarrow \quad \sin A = \sin\left(\frac{\pi}{2} - B\right) = \cos B = \sqrt{1-\sin^2 B}$

$\Rightarrow \quad \frac{5}{x} = \sqrt{1-\frac{144}{x^2}} \Rightarrow \frac{169}{x^2} = 1$

$\Rightarrow \quad x^2 = 169 \quad \Rightarrow \quad x = 13$ [$\because x = -13$ does not satisfy the given equation]

35. c. $\sin^{-1}(x-1) \quad \Rightarrow \quad -1 \leq x-1 \leq 1 \quad \Rightarrow \quad 0 \leq x \leq 2$

$\cos^{-1}(x-3) \quad \Rightarrow \quad -1 \leq x-3 \leq 1 \quad \Rightarrow \quad 2 \leq x \leq 4$

$\therefore x = 2$

So, $\sin^{-1}(2-1) + \cos^{-1}(2-3) + \tan^{-1}\frac{2}{2-4} = \cos^{-1}k + \pi$

$\Rightarrow \quad \sin^{-1}1 + \cos^{-1}(-1) + \tan^{-1}(-1) = \cos^{-1}k + \pi$.

$\Rightarrow \quad \frac{\pi}{2}+\pi-\frac{\pi}{4}=\cos^{-1}k+\pi$

$\Rightarrow \quad \cos^{-1}k=\frac{\pi}{4} \Rightarrow k=\frac{1}{\sqrt{2}}$

36. b. $\cot^{-1}x+\cot^{-1}y+\cot^{-1}z=\frac{\pi}{2}$

$\Rightarrow \quad \frac{\pi}{2}-\tan^{-1}x+\frac{\pi}{2}-\tan^{-1}y+\frac{\pi}{2}-\tan^{-1}z=\frac{\pi}{2}$

$\Rightarrow \quad \tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi$

$\Rightarrow \quad \tan^{-1}x+\tan^{-1}y=\pi-\tan^{-1}z$

$\Rightarrow \quad \tan(\tan^{-1}x+\tan^{-1}y)=\tan(\pi-\tan^{-1}z)$

$\Rightarrow \quad \frac{x+y}{1-xy}=-z$

$\Rightarrow \quad x+y+z=xyz$

37. d. Given that $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$

$\Rightarrow \quad \cos^{-1}(x)+\cos^{-1}(y)=\pi-\cos^{-1}(z)$

$\Rightarrow \quad \cos^{-1}\left(xy-\sqrt{1-x^2}\sqrt{1-y^2}\right)=\cos^{-1}(-z)$

$\Rightarrow \quad xy-\sqrt{(1-x^2)(1-y^2)}=-z$

$\Rightarrow \quad (xy+z)=\sqrt{(1-x^2)(1-y^2)}$

Squaring both sides, we get $x^2+y^2+z^2+2xyz=1$

Trick: Put $x=y=z=\frac{1}{2}$ so that $\cos^{-1}\frac{1}{2}+\cos^{-1}\frac{1}{2}+\cos^{-1}\frac{1}{2}=\pi$.

38. d. Given that $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\frac{\pi}{2}$

$\Rightarrow \quad \tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-yz-xz}\right]=\frac{\pi}{2}$

Hence, $xy+yz+zx-1=0$.

39. d. Let $\alpha=\cos^{-1}\sqrt{p}$, $\beta=\cos^{-1}\sqrt{1-p}$ and $\gamma=\cos^{-1}\sqrt{1-q}$

$\Rightarrow \quad \cos\alpha=\sqrt{p}$, $\cos\beta=\sqrt{1-p}$ and $\cos\gamma=\sqrt{1-q}$

Therefore, $\sin\alpha=\sqrt{1-p}$, $\sin\beta=\sqrt{p}$ and $\sin\gamma=\sqrt{q}$

The given equation may be written as

$\alpha+\beta+\gamma=\frac{3\pi}{4}$

$\Rightarrow \quad \alpha+\beta=\frac{3\pi}{4}-\gamma$

$\Rightarrow \quad \cos(\alpha+\beta)=\cos\left(\frac{3\pi}{4}-\gamma\right)$

$\Rightarrow \quad \cos\alpha\cos\beta-\sin\alpha\sin\beta=\cos\left(\pi-\left(\frac{\pi}{4}+\gamma\right)\right)=-\cos\left(\frac{\pi}{4}+\gamma\right)$

$\Rightarrow \quad \sqrt{p}\sqrt{1-p}-\sqrt{1-p}\sqrt{p}=-\left(\frac{1}{\sqrt{2}}\sqrt{1-q}-\frac{1}{\sqrt{2}}\sqrt{q}\right)$

$\Rightarrow\ 0 = \sqrt{1-q} - \sqrt{q} \Rightarrow 1 - q = q \Rightarrow q = \frac{1}{2}$

40. a. Let $\sin^{-1} a = A$, $\sin^{-1} b = B$ and $\sin^{-1} c = C$

$\Rightarrow\ \sin A = a, \sin B = b, \sin C = c$

and $A + B + C = \pi \Rightarrow \sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$ (i)

$\Rightarrow\ \sin A \cos A + \sin B \cos B + \sin C \cos C = 2 \sin A \sin B \sin C$

$\Rightarrow\ \sin A \sqrt{(1-\sin^2 A)} + \sin B \sqrt{(1-\sin^2 B)} + \sin C \sqrt{1-\sin^2 C} = 2 \sin A \sin B \sin C$ (ii)

$\Rightarrow\ a\sqrt{(1-a^2)} + b\sqrt{(1-b^2)} + c\sqrt{(1-c)^2} = 2abc$

Trick: Let $a = \frac{1}{\sqrt{2}}, b = \frac{1}{\sqrt{2}}, c = 1.$

Then $a\sqrt{1-a^2} + b\sqrt{1-b^2} + c\sqrt{1-c^2} = \frac{1}{\sqrt{2}}\sqrt{1-\frac{1}{2}} + \frac{1}{\sqrt{2}}\sqrt{1-\frac{1}{2}} + 1\sqrt{1-1} = 1.$

41. b. Since $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \pi$

$\therefore \sin^{-1} x + \sin^{-1} y = \pi - \sin^{-1} z$

$\Rightarrow\ \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right) = \pi - \sin^{-1}(z)$

$\Rightarrow\ x\sqrt{1-y^2} + y\sqrt{1-x^2} = \sin(\pi - \sin^{-1}(z)) = \sin(\sin^{-1} z) = z$

$\Rightarrow\ x^2(1-y^2) = z^2 + y^2(1-x^2) - 2zy\sqrt{1-x^2} \Rightarrow (x^2-y^2-z^2)^2 = 4y^2z^2(1-x^2)$

$\Rightarrow\ x^4 + y^4 + z^4 - 2x^2y^2 - 2x^2z^2 + 2y^2z^2 = 4y^2z^2 - 4x^2y^2z^2$

$\Rightarrow\ x^4 + y^4 + z^4 + 4x^2y^2z^2 = 2(x^2y^2 + y^2z^2 + z^2x^2) \qquad \Rightarrow\ K = 2$

42. d. We have $\cos^{-1} x - \cos^{-1}\frac{y}{2} = \alpha$

$\Rightarrow\ x = \cos\left(\cos^{-1}\frac{y}{2} + \alpha\right) = \cos\left(\cos^{-1}\frac{y}{2}\right)\cos\alpha - \sin\left(\cos^{-1}\frac{y}{2}\right)\sin\alpha$

$= \frac{y}{2}\cos\alpha - \sqrt{1-\frac{y^2}{4}}\sin\alpha$

$\Rightarrow\ 2x = y\cos\alpha - \sin\alpha\sqrt{4-y^2}$

$\Rightarrow\ 2x - y\cos\alpha = -\sin\alpha\sqrt{4-y^2}$

Squaring, we get

$4x^2 + y^2\cos^2\alpha - 4xy\cos\alpha = 4\sin^2\alpha - y^2\sin^2\alpha$

$\Rightarrow\ 4x^2 - 4xy\cos\alpha + y^2 = 4\sin^2\alpha$

43. d. $2\tan^{-1} 2x = \sin^{-1}\frac{4x}{1+4x^2}$

$\Rightarrow\ -\frac{\pi}{2} \le 2\tan^{-1} 2x \le \frac{\pi}{2}$

$\Rightarrow\ -\frac{\pi}{4} \le \tan^{-1} 2x \le \frac{\pi}{4}$

$\Rightarrow \quad -1 \le 2x \le 1$

$\Rightarrow \quad -\dfrac{1}{2} \le x \le \dfrac{1}{2}$

44. b. $\cos^{-1}(2x^2-1) = 2\pi - 2\cos^{-1}x$ (as $x < 0$)

$\Rightarrow \quad \cos^{-1}(2x^2-1) - 2\sin^{-1}x = 2\pi - 2\cos^{-1}x - 2\sin^{-1}x$

$$= 2\pi - 2(\cos^{-1}x + \sin^{-1}x)$$

$$= 2\pi - 2\,\frac{\pi}{2} = \pi$$

45. c. $2\sin^{-1}x = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

Range of the right-hand angle is $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

$\Rightarrow \quad -\dfrac{\pi}{2} \le 2\sin^{-1}x \le \dfrac{\pi}{2}$

$\Rightarrow \quad \dfrac{-\pi}{4} \le \sin^{-1}x \le \dfrac{\pi}{4}$

$\Rightarrow \quad x \in \left[-\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right]$

46. c. $x_1 = 2\tan^{-1}\left(\dfrac{1+x}{1-x}\right)$ and $x_2 = \sin^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) = \tan^{-1}\left(\dfrac{1-x^2}{2x}\right)$

Now $\dfrac{1+x}{1-x} > 1 \quad \Rightarrow \quad x_1 = \pi + \tan^{-1}\left(\dfrac{2\left(\dfrac{1+x}{1-x}\right)}{1-\left(\dfrac{1+x}{1-x}\right)^2}\right) = \pi + \tan^{-1}\left(\dfrac{1-x^2}{-2x}\right) = \pi - \tan^{-1}\left(\dfrac{1-x^2}{2x}\right)$

$\Rightarrow \quad x_1 + x_2 = \pi$

47. d. $\sin(2\sin^{-1}(0.8)) = \sin(\sin^{-1}(2\times 0.8\sqrt{1-(0.8)^2})) = \sin(\sin^{-1}0.96) = 0.96$

48. b. $\tan(\sin^{-1}(\cos(\sin^{-1}x)))\tan(\cos^{-1}(\sin(\cos^{-1}x)))$

$= \tan(\sin^{-1}(\cos(\cos^{-1}\sqrt{1-x^2})))\tan\{\cos^{-1}(\sin(\sin^{-1}\sqrt{1-x^2}))\}$

$= \tan(\sin^{-1}\sqrt{1-x^2})\tan(\cos^{-1}\sqrt{1-x^2})$

$= \tan(\cos^{-1}x)\tan(\sin^{-1}x)$

$= \tan(\cos^{-1}x)\tan(\pi/2 - \cos^{-1}x) = \tan(\cos^{-1}x)\cot(\cos^{-1}x) = 1$

49. d. Since $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right) = 2\tan^{-1}x$ for $x \in (-1, 1)$

$\sin^{-1}\left(\dfrac{2a}{1+a^2}\right) + \sin^{-1}\left(\dfrac{2b}{1+b^2}\right) = 2\tan^{-1}x$

$\Rightarrow \quad 2\tan^{-1}a + 2\tan^{-1}b = 2\tan^{-1}x$

$\Rightarrow \quad \tan^{-1} a + \tan^{-1} b = \tan^{-1} x$

$\Rightarrow \quad \tan^{-1}\left(\dfrac{a+b}{1-ab}\right) = \tan^{-1} x$

$\Rightarrow \quad x = \dfrac{a+b}{1-ab}$

50. b. If $x < 0$, then $\sin^{-1}x < 0$ but $\cos^{-1}\sqrt{1-x^2}$ is always positive.

So, $\sin^{-1}x = -\cos^{-1}\sqrt{1-x^2}$.

51. b.

We have $\dfrac{xy}{zr}\dfrac{yz}{xr} = \dfrac{y^2}{r^2} = \dfrac{y^2}{x^2+y^2+z^2} < 1$

$$\Rightarrow \quad \tan^{-1}\left(\frac{xy}{zr}\right) + \tan^{-1}\left(\frac{yz}{xr}\right) + \tan^{-1}\left(\frac{xz}{yr}\right) = \tan^{-1}\left(\frac{\frac{xy}{zr}+\frac{yz}{xr}}{1-\frac{xy}{zr}\frac{yz}{xr}}\right) + \tan^{-1}\left(\frac{xz}{yr}\right)$$

$$= \tan^{-1}\left(\frac{\frac{y(x^2+z^2)}{xzr}}{\frac{r^2-y^2}{r^2}}\right) + \tan^{-1}\left(\frac{xz}{yr}\right)$$

$$= \tan^{-1}\left(\frac{\frac{yr(x^2+z^2)}{xz}}{(x^2+z^2)}\right) + \tan^{-1}\left(\frac{xz}{yr}\right)$$

$$= \tan^{-1}\left(\frac{yr}{xz}\right) + \tan^{-1}\left(\frac{xz}{yr}\right) = \frac{\pi}{2}$$

52. b. Let $x = \sin\theta$ where $-\dfrac{1}{2} \le x \le 1 \Rightarrow -\dfrac{\pi}{6} \le \theta \le \dfrac{\pi}{2}$

Then $f(x) = \sin^{-1}\left(\dfrac{\sqrt{3}}{2}x - \dfrac{1}{2}\sqrt{1-x^2}\right)$

$= \sin^{-1}\left(\dfrac{\sqrt{3}}{2}\sin\theta - \dfrac{1}{2}\cos\theta\right)$

$= \sin^{-1}\left(\sin\left(\theta - \dfrac{\pi}{6}\right)\right)$

$= \theta - \dfrac{\pi}{6} = \sin^{-1} x - \dfrac{\pi}{6}$ $\quad \left[\because \theta - \dfrac{\pi}{6} \in \left(\dfrac{-\pi}{3}, \dfrac{\pi}{3}\right)\right]$

53. c. Let $y = \tan^{-1}\left(\dfrac{1-x^2}{2x}\right) + \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

Put $x = \tan\theta$. As $x \in (0, 1)$, $\theta \in \left(0, \frac{\pi}{4}\right)$ and $\frac{\pi}{2} - 2\theta \in (0, \pi/2)$

$\therefore\quad y = \tan^{-1}(\cot 2\theta) + \cos^{-1}(\cos 2\theta) = \tan^{-1}\left(\tan\left(\frac{\pi}{2} - 2\theta\right)\right) + \cos^{-1}(\cos 2\theta) = \frac{\pi}{2} - 2\theta + 2\theta = \frac{\pi}{2}$

54. c. $\sin^{-1} x = 2\sin^{-1} a$

Now $-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}$

$\Rightarrow\quad -\frac{\pi}{2} \le 2\sin^{-1} a \le \frac{\pi}{2}$

$\Rightarrow\quad -\frac{\pi}{4} \le \sin^{-1} a \le \frac{\pi}{4}$

$\Rightarrow\quad \frac{1}{\sqrt{2}} \le a \le \frac{1}{\sqrt{2}} \qquad \Rightarrow\quad |a| \le \frac{1}{\sqrt{2}}$

55. a. Let $\tan^{-1} x = \theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2} \qquad \Rightarrow\quad -\pi < 2\theta < \pi$

Let $\frac{\pi}{2} < 2\theta < \pi \qquad \Rightarrow\quad \frac{\pi}{4} < \theta < \frac{\pi}{2} \qquad \Rightarrow\quad \frac{\pi}{4} < \tan^{-1} x < \frac{\pi}{2} \qquad \Rightarrow x > 1$

$\Rightarrow\quad \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = \tan^{-1}(\tan 2\theta) = \tan^{-1}(\tan(2\theta - \pi)) = 2\theta - \pi = 2\tan^{-1} x - \pi$

56. c. $\sin^{-1}\left(\frac{\sqrt{r}-\sqrt{r-1}}{\sqrt{r(r+1)}}\right) = \tan^{-1}\left(\frac{\sqrt{r}-\sqrt{r-1}}{1+\sqrt{r(r-1)}}\right)$

$\Rightarrow\quad \sum_{r=1}^{n} \sin^{-1}\left(\frac{\sqrt{r}-\sqrt{r-1}}{\sqrt{r(r+1)}}\right) = \sum_{r=1}^{n} (\tan^{-1}\sqrt{r} - \tan^{-1}\sqrt{r-1}) = \tan^{-1}\sqrt{n}$

57. b. $\sum_{r=1}^{n} \tan^{-1}\left(\frac{2^{r-1}}{1+2^{2r-1}}\right) = \sum_{r=1}^{n} \tan^{-1}\left(\frac{2^{r-1}}{1+2^r\, 2^{r-1}}\right)$

$= \sum_{r=1}^{n} \tan^{-1}\left(\frac{2^r - 2^{r-1}}{1+2^r\, 2^{r-1}}\right)$

$= \sum_{r=1}^{n} [\tan^{-1}(2^r) - \tan^{-1}(2^{r-1})] = \tan^{-1}(2^n) - \tan^{-1}(1) = \tan^{-1}(2^n) - \frac{\pi}{4}$

58. a. We have $\sum_{m=1}^{n} \tan^{-1}\left(\frac{2m}{m^4 + m^2 + 2}\right) = \sum_{m=1}^{n} \tan^{-1}\left(\frac{2m}{1+(m^2+m+1)(m^2-m+1)}\right)$

$= \sum_{m=1}^{n} \tan^{-1}\left(\frac{(m^2+m+1)-(m^2-m+1)}{1+(m^2+m+1)(m^2-m+1)}\right)$

$$= \sum_{m=1}^{n} [\tan^{-1}(m^2+m+1) - \tan^{-1}(m^2-m+1)]$$

$$= (\tan^{-1} 3 - \tan^{-1} 1) + (\tan^{-1} 7 - \tan^{-1} 3) + (\tan^{-1} 13 - \tan^{-1} 7) \quad + \cdots + [\tan^{-1}(n^2+n+1) - \tan^{-1}(n^2-n+1)]$$

$$= \tan^{-1}(n^2+n+1) - \tan^{-1} 1 = \tan^{-1}\left(\frac{n^2+n}{2+n^2+n}\right)$$

For $n \to \infty$, sum $= \tan^{-1}(1) = \frac{\pi}{4}$

59. a. $\tan^{-1}\left(\frac{1}{1+r+r^2}\right) = \tan^{-1}\left(\frac{r+1-r}{1+r(r+1)}\right)$

$$= \tan^{-1}(r+1) - \tan^{-1}(r)$$

$$\Rightarrow \sum_{r=0}^{n} [\tan^{-1}(r+1) - \tan^{-1}(r)] = \tan^{-1}(n+1) - \tan^{-1}(0)$$

$$= \tan^{-1}(n+1)$$

$$\Rightarrow \sum_{r=0}^{\infty} \tan^{-1}\left(\frac{1}{1+r+r^2}\right) = \tan^{-1}(\infty) = \frac{\pi}{2}$$

60. b. Obviously, $x = \sin(\theta+\beta)$ and $y = \sin(\theta-\beta)$

$$\Rightarrow 1 + xy = 1 + \sin(\theta+\beta)\sin(\theta-\beta) = 1 + \sin^2\theta - \sin^2\beta = \sin^2\theta + \cos^2\beta$$

61. a. Let $\sqrt{\tan\alpha} = \tan x$, then $u = \cot^{-1}(\tan x) - \tan^{-1}(\tan x) = \frac{\pi}{2} - x - x = \frac{\pi}{2} - 2x$

$$\Rightarrow 2x = \frac{\pi}{2} - u \qquad \Rightarrow \frac{\pi}{4} - \frac{u}{2}$$

$$\Rightarrow \tan x = \tan\left(\frac{\pi}{4} - \frac{u}{2}\right)$$

$$\Rightarrow \sqrt{\tan\alpha} = \tan\left(\frac{\pi}{4} - \frac{u}{2}\right)$$

62. b. $\tan\left[\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\frac{a}{b}\right] + \tan\left[\frac{\pi}{4} - \frac{1}{2}\cos^{-1}\frac{a}{b}\right]$

Let $\frac{1}{2}\cos^{-1}\frac{a}{b} = \theta \Rightarrow \cos 2\theta = \frac{a}{b}$

Thus, $\tan\left[\frac{\pi}{4} + \theta\right] + \tan\left[\frac{\pi}{4} - \theta\right] = \frac{1+\tan\theta}{1-\tan\theta} + \frac{1-\tan\theta}{1+\tan\theta}$

$$= \frac{1+\tan^2\theta + 2\tan\theta + 1 + \tan^2\theta - 2\tan\theta}{(1-\tan^2\theta)}$$

$$= \frac{2(1+\tan^2\theta)}{1-\tan^2\theta} = \frac{2}{\cos 2\theta} = \frac{2}{(a/b)} = \frac{2b}{a}$$

63. a. $2\tan^{-1}\left[\sqrt{\frac{a-b}{a+b}}\tan\frac{\theta}{2}\right] = \cos^{-1}\left[\frac{1-\left(\frac{a-b}{a+b}\right)\tan^2\frac{\theta}{2}}{1+\left(\frac{a-b}{a+b}\right)\tan^2\frac{\theta}{2}}\right]$ $\left[\because 2\tan^{-1}x = \cos^{-1}\frac{1-x^2}{1+x^2}\right]$

$$= \cos^{-1}\left[\frac{(a+b)-(a-b)\tan^2\frac{\theta}{2}}{(a+b)+(a-b)\tan^2\frac{\theta}{2}}\right]$$

$$= \cos^{-1}\left[\frac{a\left(1-\tan^2\frac{\theta}{2}\right)+b\left(1+\tan^2\frac{\theta}{2}\right)}{a\left(1+\tan^2\frac{\theta}{2}\right)+b\left(1-\tan^2\frac{\theta}{2}\right)}\right]$$

$$= \cos^{-1}\left[\frac{\frac{a\left(1-\tan^2\frac{\theta}{2}\right)}{1+\tan^2\frac{\theta}{2}}+b}{a+b\left(\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right)}\right]$$

$$= \cos^{-1}\left[\frac{a\cos\theta+b}{a+b\cos\theta}\right]$$

64. d. $\cot^{-1}\left[\frac{\sqrt{1-\sin x}+\sqrt{1+\sin x}}{\sqrt{1-\sin x}-\sqrt{1+\sin x}}\right]$

$$= \cot^{-1}\left[\frac{\left(\sqrt{1-\sin x}+\sqrt{1+\sin x}\right)}{\left(\sqrt{1-\sin x}-\sqrt{1+\sin x}\right)}\ \frac{\left(\sqrt{1-\sin x}+\sqrt{1+\sin x}\right)}{\left(\sqrt{1-\sin x}+\sqrt{1+\sin x}\right)}\right]$$

$$= \cot^{-1}\left[\frac{(1-\sin x)+(1+\sin x)+2\sqrt{1-\sin^2 x}}{(1-\sin x)-(1+\sin x)}\right] = \cot^{-1}\left[\frac{2(1+\cos x)}{-2\sin x}\right]$$

$$= \cot^{-1}\left[-\frac{2\cos^2(x/2)}{2\sin(x/2)\cos(x/2)}\right] = \cot^{-1}\left(-\cot\frac{x}{2}\right) = \cot^{-1}\left[\cot\left(\pi-\frac{x}{2}\right)\right] = \pi-\frac{x}{2}$$

65. b. $\tan^{-1}\left(\frac{x\cos\theta}{1-x\sin\theta}\right)-\cot^{-1}\left(\frac{\cos\theta}{x-\sin\theta}\right)=\tan^{-1}\left(\frac{x\cos\theta}{1-x\sin\theta}\right)-\tan^{-1}\left(\frac{x-\sin\theta}{\cos\theta}\right)$

$$=\tan^{-1}\left(\frac{\frac{x\cos\theta}{1-x\sin\theta}-\frac{x-\sin\theta}{\cos\theta}}{1+\left(\frac{x\cos\theta}{1-x\sin\theta}\right)\left(\frac{x-\sin\theta}{\cos\theta}\right)}\right)$$

$$=\tan^{-1}\left(\frac{x\cos^2\theta-x+\sin\theta+x^2\sin\theta-x\sin^2\theta}{\cos\theta-x\cos\theta\sin\theta+x^2\cos\theta-x\cos\theta\sin\theta}\right)$$

$$=\tan^{-1}\left(\frac{-x\sin^2\theta+\sin\theta+x^2\sin\theta-x\sin^2\theta}{\cos\theta-2x\cos\theta\sin\theta+x^2\cos\theta}\right)$$

$$=\tan^{-1}\left(\frac{-2x\sin^2\theta+\sin\theta+x^2\sin\theta}{\cos\theta-2x\cos\theta\sin\theta+x^2\cos\theta}\right)$$

$$=\tan^{-1}\left(\frac{\sin\theta(-2x\sin\theta+1+x^2)}{\cos\theta(1-2x\sin\theta+x^2)}\right)=\tan^{-1}(\tan\theta)=\theta$$

66. d. $\tan^{-1}\left(\frac{\tan x}{4}\right)+\tan^{-1}\left(\frac{3\sin 2x}{5+3\cos 2x}\right)=\tan^{-1}\left(\frac{\tan x}{4}\right)+\tan^{-1}\left(\frac{\frac{6\tan x}{1+\tan^2 x}}{5+\frac{3(1-\tan^2 x)}{1+\tan^2 x}}\right)$

$$=\tan^{-1}\left(\frac{\tan x}{4}\right)+\tan^{-1}\left(\frac{6\tan x}{8+2\tan^2 x}\right)$$

$$=\tan^{-1}\left(\frac{\tan x}{4}\right)+\tan^{-1}\left(\frac{3\tan x}{4+\tan^2 x}\right)$$

$$=\tan^{-1}\left(\frac{\frac{\tan x}{4}+\frac{3\tan x}{4+\tan^2 x}}{1-\frac{3\tan^2 x}{4(4+\tan^2 x)}}\right)\quad\left[\text{as }\left|\frac{\tan x}{4}\ \frac{3\tan x}{4+\tan^2 x}\right|<1\right]$$

$$=\tan^{-1}\left(\frac{16\tan x+\tan^3 x}{16+\tan^2 x}\right)$$

$$=\tan^{-1}(\tan x)=x$$

67. a. $\cot^{-1}\left(\sqrt{\cos\alpha}\right)-\tan^{-1}\left(\sqrt{\cos\alpha}\right)=x$

$$\Rightarrow \tan^{-1}\left(\frac{1}{\sqrt{\cos\alpha}}\right) - \tan^{-1}\left(\sqrt{\cos\alpha}\right) = x$$

$$\Rightarrow \tan^{-1}\frac{\frac{1}{\sqrt{\cos\alpha}} - \sqrt{\cos\alpha}}{1 + \frac{1}{\sqrt{\cos\alpha}}\sqrt{\cos\alpha}} = x$$

$$\Rightarrow \tan^{-1}\frac{1-\cos\alpha}{2\sqrt{\cos\alpha}} = x$$

$$\Rightarrow \tan x = \frac{1-\cos\alpha}{2\sqrt{\cos\alpha}}$$

$$\Rightarrow \cot x = \frac{2\sqrt{\cos\alpha}}{1-\cos\alpha}$$

$$\Rightarrow \operatorname{cosec} x = \sqrt{1 + \frac{4\cos\alpha}{(1-\cos\alpha)^2}} = \frac{1+\cos\alpha}{1-\cos\alpha}$$

$$\Rightarrow \sin x = \frac{1-\cos\alpha}{1+\cos\alpha} = \frac{2\sin^2(\alpha/2)}{2\cos^2(\alpha/2)} = \tan^2 \alpha/2$$

68. c. $$\tan\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x\right) + \tan\left(\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x\right) = \frac{1+\tan\left(\frac{1}{2}\cos^{-1}x\right)}{1-\tan\left(\frac{1}{2}\cos^{-1}x\right)} + \frac{1-\tan\left(\frac{1}{2}\cos^{-1}x\right)}{1+\tan\left(\frac{1}{2}\cos^{-1}x\right)}$$

$$= \frac{\left(1+\tan\left(\frac{1}{2}\cos^{-1}x\right)\right)^2 + \left(1-\tan\left(\frac{1}{2}\cos^{-1}x\right)\right)^2}{1-\tan^2\left(\frac{1}{2}\cos^{-1}x\right)}$$

$$= 2\frac{1+\tan^2\left(\frac{1}{2}\cos^{-1}x\right)}{1-\tan^2\left(\frac{1}{2}\cos^{-1}x\right)}$$

$$= \frac{2}{\cos(\cos^{-1}x)} = \frac{2}{x}$$

69. c. We have $(\sin^{-1}x)^3 + (\cos^{-1}x)^3 = (\sin^{-1}x + \cos^{-1}x)^3 - 3\sin^{-1}\cos^{-1}x\,(\sin^{-1}x + \cos^{-1}x)$

$$= \frac{\pi^3}{8} - 3(\sin^{-1}x\cos^{-1}x)\frac{\pi}{2}$$

$$= \frac{\pi^3}{8} - \frac{3\pi}{2}\sin^{-1}x\left(\frac{\pi}{2} - \sin^{-1}x\right)$$

$$= \frac{\pi^3}{8} - \frac{3\pi^2}{4}\sin^{-1}x + \frac{3\pi}{2}(\sin^{-1}x)^2$$

$$= \frac{\pi^3}{8} + \frac{3\pi}{2}\left[(\sin^{-1}x)^2 - \frac{\pi}{2}\sin^{-1}x\right]$$

$$= \frac{\pi^3}{8} + \frac{3\pi}{2}\left[\left(\sin^{-1}x - \frac{\pi}{4}\right)^2\right] - \frac{3\pi^3}{32}$$

$$= \frac{\pi^3}{32} + \frac{3\pi}{2}\left(\sin^{-1}x - \frac{\pi}{4}\right)^2$$

So, the least value is $\frac{\pi^3}{32}$ when $\left(\sin^{-1}x - \frac{\pi}{4}\right) = 0$.

And the greatest value occurs when $\left(\sin^{-1}x - \frac{\pi}{4}\right)^2 = \left(-\frac{\pi}{2} - \frac{\pi}{4}\right)^2 = \frac{9\pi^2}{16}$.

Therefore, the greatest value is $\frac{\pi^3}{32} + \frac{9\pi^2}{16} \times \frac{3\pi}{2} = \frac{7\pi^3}{8}$.

70. c. $f(x) = \sin^{-1}x + \tan^{-1}x + \sec^{-1}x$, clearly domain of $f(x)$ is $x = \pm 1$.

Thus, the range is $\{f(1), f(-1)\}$, i.e., $\left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\}$.

71. a. $1 + x^2 \geq 2|x| \Rightarrow \frac{2|x|}{1+x^2} \leq 1$

$\Rightarrow -1 \leq \frac{2x}{1+x^2} \leq 1 \Rightarrow \tan^{-1}\left(\frac{2x}{1+x^2}\right) \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

72. c. $[\cot^{-1}x] + [\cos^{-1}x] = 0$

As $\cos^{-1}x, \cot^{-1}x \geq 0$, $[\cot^{-1}x] = [\cos^{-1}x] = 0$

$[\cot^{-1}x] = 0 \Rightarrow x \in (\cot 1, \infty)$ (i)

$[\cos^{-1}x] = 0 \Rightarrow x \in (\cos 1, 1]$ (ii)

Hence, from Eqs. (i) and (ii), $x \in (\cot 1, 1]$.

73. d. $\frac{3\pi}{2} < 5 < \frac{5\pi}{2}$

$\Rightarrow \sin^{-1}(\sin 5) = 5 - 2\pi$

Given $\sin^{-1}(\sin 5) > x^2 - 4x$

$\Rightarrow x^2 - 4x + 4 < 9 - 2\pi$

$\Rightarrow (x-2)^2 < 9 - 2\pi$

$\Rightarrow -\sqrt{9-2\pi} < x - 2 < \sqrt{9-2\pi}$

$\Rightarrow 2 - \sqrt{9-2\pi} < x < 2 + \sqrt{9-2\pi}$

74. c. $\frac{\alpha^3}{2}\operatorname{cosec}^2\left(\frac{1}{2}\tan^{-1}\frac{\alpha}{\beta}\right) + \frac{\beta^3}{2}\sec^2\left(\frac{1}{2}\tan^{-1}\frac{\beta}{\alpha}\right)$

$$= \alpha^3 \frac{1}{1-\cos\left(\tan^{-1}\left(\frac{\alpha}{\beta}\right)\right)} + \beta^3 \frac{1}{1+\cos\left(\tan^{-1}\frac{\beta}{\alpha}\right)}$$

$$= \alpha^3 \frac{1}{1-\cos\left(\cos^{-1}\left(\frac{\beta}{\sqrt{\alpha^2+\beta^2}}\right)\right)} + \beta^3 \frac{1}{1+\cos\left(\cos^{-1}\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}\right)}$$

$$= \alpha^3 \frac{1}{1-\frac{\beta}{\sqrt{\alpha^2+\beta^2}}} + \beta^3 \frac{1}{1+\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}}$$

$$= \sqrt{\alpha^2+\beta^2}\left(\frac{\alpha^3}{\sqrt{\alpha^2+\beta^2}-\beta} + \frac{\beta^3}{\sqrt{\alpha^2+\beta^2}+\alpha}\right)$$

$$= \sqrt{\alpha^2+\beta^2}\left(\alpha^3 \frac{(\sqrt{\alpha^2+\beta^2}+\beta)}{\alpha^2} + \beta^3 \frac{(\sqrt{\alpha^2+\beta^2}-\alpha)}{\beta^2}\right)$$

$$= \sqrt{\alpha^2+\beta^2}\left[\alpha\left(\sqrt{\alpha^2+\beta^2}+\beta\right) + \beta\left(\sqrt{\alpha^2+\beta^2}-\alpha\right)\right]$$

$$= \sqrt{\alpha^2+\beta^2}(\alpha+\beta)\sqrt{\alpha^2+\beta^2}$$

$$= (\alpha+\beta)(\alpha^2+\beta^2)$$

75. d. $\lim_{|x|\to\infty} \cos(\tan^{-1}(\sin(\tan^{-1}x))) = \cos(\tan^{-1}(\sin(\tan^{-1}\infty)))$

$$= \cos(\tan^{-1}(\sin(\pi/2)))$$

$$= \cos(\tan^{-1}(1)) = \cos(\pi/4) = \frac{1}{\sqrt{2}}$$

76. d. $\sin^{-1}(-(x-1)(x-2)) + \cos^{-1}((x-3)(x-1)) = \frac{\pi}{4}$

For $x \in [1, 2] \Rightarrow \sin^{-1}(-(x-1)(x-2)) \in [0, \pi/2)$

and $\cos^{-1}((x-3)(x-1)) \in [\pi/2, \pi] \Rightarrow$ no solution in the given range

Also, $-1 \le 3x-2-x^2 \le 1$ and $-1 \le x^2-4x+3 \le 1 \Rightarrow 2-\sqrt{2} \le x \le \frac{3+\sqrt{5}}{2}$

77. d. $2^{2\pi/\sin^{-1}x} - 2(a+2)2^{\pi/\sin^{-1}x} + 8a < 0$

$(2^{\pi/\sin^{-1}x} - 4)(2^{\pi/\sin^{-1}x} - 2a) < 0$

Now $2^{\pi/\sin^{-1}x} \in \left(0, \frac{1}{4}\right] \cup [4, \infty)$

Now for $2^{\pi/\sin^{-1}x} \in \left(0, \frac{1}{4}\right]$, we have $(2^{\pi/\sin^{-1}x} - 4) < 0$

$\Rightarrow 2^{\pi/\sin^{-1}x} - 2a > 0$

$\Rightarrow 2a < 2^{\pi/\sin^{-1}x} \Rightarrow 2a < \frac{1}{4}$

$\Rightarrow \quad 0 \le a < \frac{1}{8}$

Similarly, for $2^{\pi/\sin^{-1}x} \in [4, \infty), a > 2$, we get

$a \in \left[0, \frac{1}{8}\right) \cup (2, \infty)$

78. b. Given that $\sin^{-1} x + \tan^{1} x = 2k + 1$

The range of the function $\sin^{-1} x + \tan^{-1} x$ is $\left[\frac{-3\pi}{4}, \frac{3\pi}{4}\right]$ [as both functions are increasing]

Therefore, the integral values of k are -1 and 0.

79. c. From the given equation $\sin^2 \theta - 2 \sin \theta + 3 = 5^{\sec^2 y} + 1$, we get

$(\sin \theta - 1)^2 + 2 = 5^{\sec^2 y} + 1$

L.H.S. ≤ 6, R.H.S. ≥ 6

Possible solution is $\sin \theta = -1$ when L.H.S. = R.H.S. $\Rightarrow \cos^2 \theta = 0 \Rightarrow \cos^2 \theta - \sin \theta = 1$

80. d. $[\cot^{-1} x] + 2[\tan^{-1} x] = 0 \Rightarrow [\cot^{-1} x] = 0, [\tan^{-1} x] = 0$

or $[\cot^{-1} x] = 2, [\tan^{-1} x] = -1$

Now $[\cot^{-1} x] = 0 \Rightarrow x \in (\cot 1, \infty)$

$[\tan^{-1} x] = 0 \Rightarrow x \in (0, \tan 1)$

Therefore, for $[\cot^{-1} x] = [\tan^{-1} x] = 0, x \in (\cot 1, \tan 1)$

$[\cot^{-1} x] = 2 \Rightarrow x \in (\cot 3, \cot 2]$

$[\tan^{-1} x] = -1 \Rightarrow x \in [-\tan 1, 0) \Rightarrow$ No such x exists.

Thus, the solution set is $(\cot 1, \tan 1)$.

81. a. Expanding, we have

$(\tan^{-1} x)^3 + (\tan^{-1} 2x)^3 + (\tan^{-1} 3x)^3 = 3 \tan^{-1} x \tan^{-1} 2x \tan^{-1} 3x$

$\Rightarrow x = 0$

82. a. $2 \cos^{-1} x = \cot^{-1}\left(\frac{2x^2 - 1}{2x\sqrt{1 - x^2}}\right)$

Put $x = \cos \theta$: LHS $= 2\theta$; $0 \le \theta \le \pi$ and $-1 \le x \le 1$ (i)

R.H.S. $= \cot^{-1}\left(\frac{\cos 2\theta}{2 \cos \theta |\sin \theta|}\right) = \cot^{-1}(\cot 2\theta) = 2\theta$ if $0 < 2\theta < \pi$ (ii)

From Eqs. (i) and (ii), we get $0 < \theta < \pi/2$

$\therefore x \in (0, 1)$

83. d. $\frac{x}{\sqrt{1 + x^2}} = \frac{x + 1}{\sqrt{(x + 1)^2 + 1}}$

$\Rightarrow x^2[(x+1)^2+1]=(x+1)^2[(x^2+1)]$

$\Rightarrow x^2(x+1)^2+x^2=x^2(x+1)^2+(x+1)^2$

$\Rightarrow x^2=(x+1)^2 \Rightarrow x+1=x$ not possible as $x \to \infty$

$\Rightarrow x+1=-x \Rightarrow x=-1/2$ which is also not possible as for this L.H.S. <0 but R.H.S. >0

84. c. Let $\tan^{-1}(x)=\theta \Rightarrow x=\tan\theta \Rightarrow \cos\theta=x \Rightarrow \dfrac{1}{\sqrt{1+x^2}}=x$

$$\Rightarrow x^2(1+x^2)=1 \Rightarrow x^2=\frac{-1\pm\sqrt{5}}{2} \Rightarrow x^2=\frac{\sqrt{5}-1}{2} \Rightarrow \frac{x^2}{2}=\frac{\sqrt{5}-1}{4}$$

$$\text{Now } \cos^{-1}\left(\frac{\sqrt{5}-1}{4}\right)=\cos^{-1}\left(\sin\frac{\pi}{10}\right)=\cos^{-1}\left(\cos\frac{2\pi}{5}\right)=\frac{2\pi}{5}=\frac{2\pi}{5}$$

85. b. $\sin\cos^{-1}(\cos(\tan^{-1}x))=p$

For $x \in R$ $\tan^{-1}x \in (-\pi/2, \pi/2)$

$\cos(\tan^{-1}x) \in (0,1]$

$\cos^{-1}\cos(\tan^{-1}x) \in [0, \pi/2)$

$\sin(\cos^{-1}(\cos(\tan^{-1}x))) \in [0, 1)$

86. a. $\sin^{-1}x-\cos^{-1}x=\sin^{1}(3x-2)$

$$\Rightarrow \frac{\pi}{2}-\cos^{-1}x-\cos^{-1}x=\frac{\pi}{2}-\cos^{-1}(3x-2)$$

$\Rightarrow 2\cos^{-1}x=\cos^{-1}(3x-2)$. Also $x \in [-1, 1]$

$\Rightarrow \cos^{-1}(2x^2-1)=\cos^{-1}(3x-2)$ and $(3x-2) \in [-1,1]$, i.e., $-1 \le 3x-2 \le 1$

$\Rightarrow 2x^2-1=3x-2$; hence, $x \in \left[\dfrac{1}{3}, 1\right]$

$\Rightarrow 2x^2-3x+1=0 \Rightarrow x=1$ or $\dfrac{1}{2}$

87. c. $\sin^{-1}\sqrt{1-x^2}+\cos^{-1}x=\cot^{-1}\dfrac{\sqrt{1-x^2}}{x}-\sin^{-1}x$

or $\dfrac{\pi}{2}+\sin^{-1}\sqrt{1-x^2}=\cot^{-1}\dfrac{\sqrt{1-x^2}}{x}$

$\tan^{-1}\dfrac{\sqrt{1-x^2}}{x}+\sin^{-1}\sqrt{1-x^2}=0$

$\Rightarrow x \in [-1, 0) \cup \{1\}$

88. c. Here $|\cos x| = \sin^{-1}(\sin x)$

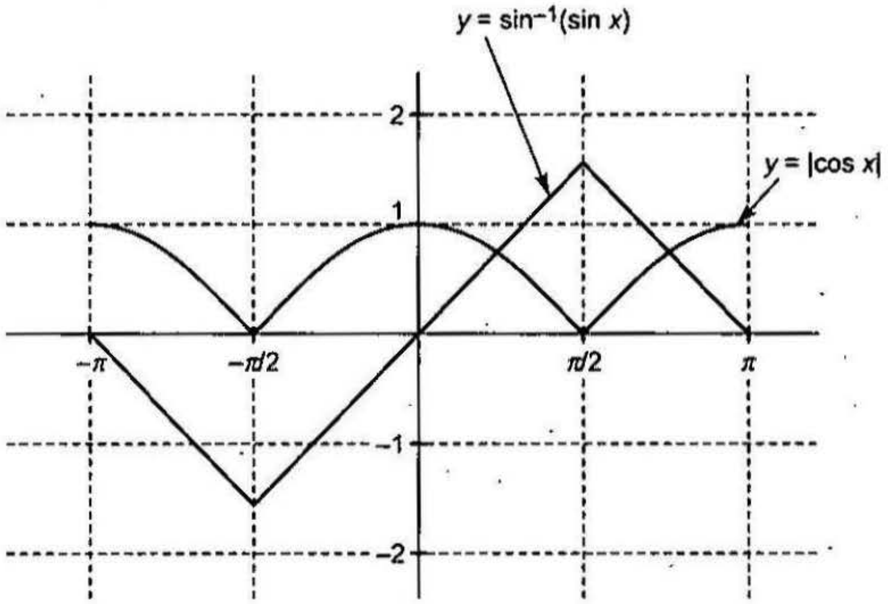


Fig. 4.29

From the graph, number of solutions is 2

89.b. $3\cos^{-1}x - \pi x - \dfrac{\pi}{2} = 0 \Rightarrow \cos^{-1}x = \dfrac{\pi x}{3} + \dfrac{\pi}{6}$

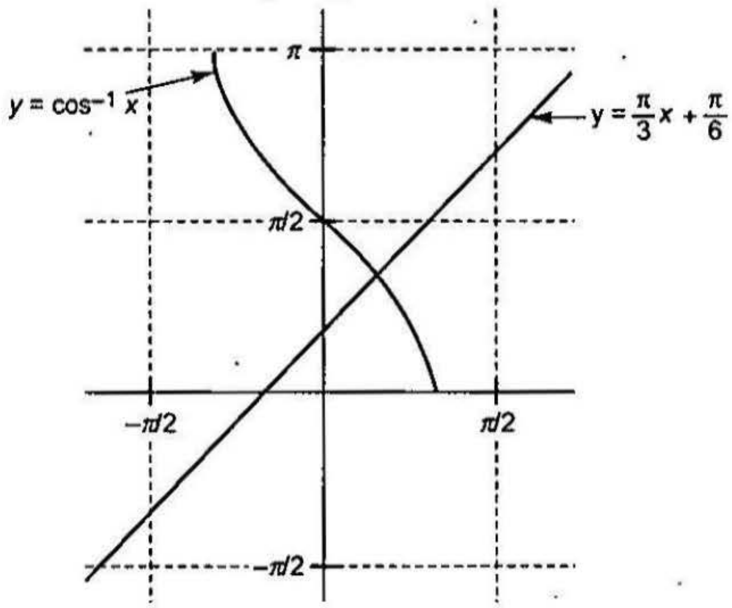


Fig. 4.30

90. b. We have

$$\left|\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right| < \frac{\pi}{3} \quad \Rightarrow -\frac{\pi}{3} < \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) < \frac{\pi}{3}$$

$$\Rightarrow 0 \le \cos^{-1}\frac{1-x^2}{1+x^2} < \frac{\pi}{3} \quad \Rightarrow \frac{1}{2} < \frac{1-x^2}{1+x^2} \le 1$$

$$\Rightarrow 1 + x^2 < 2(1-x^2) \le 2(1+x^2) \quad \Rightarrow 0 \le x^2 < \frac{1}{3} \Rightarrow -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$$

91.b. $\sin^{-1}x+\sin^{-1}y=\frac{\pi}{2} \Rightarrow \sin^{-1}x=\frac{\pi}{2}-\sin^{-1}y \Rightarrow \sin^{-1}x=\sin^{-1}\sqrt{1-y^2} \Rightarrow x^2+y^2=1$

$$\Rightarrow \frac{1+x^4+y^4}{x^2-x^2y^2+y^2}=\frac{1+(x^2+y^2)^2-2x^2y^2}{1-x^2y^2}=\frac{1+1-2x^2y^2}{1-x^2y^2}=2$$

92.d. $\sin^{-1}(x^2-4x+6)+\cos^{-1}(x^2-4x+6)=\sin^{-1}((x-2)^2+2)+\cos^{-1}((x-2)^2+2)$

$(x-2)^2+2\geq 2$, for which $\sin^{-1}x$ and $\cos^{-1}x$ are not defined.

93. a. $\frac{\pi}{2}-\cos^{-1}\cos\left(\frac{2(x^2+5|x|+3)-2}{x^2+5|x|+3}\right)=\cot\cot^{-1}\left(\frac{2}{9|x|}-2\right)+\frac{\pi}{2}$

$\frac{\pi}{2}-2+\frac{2}{x^2+5|x|+3}=\frac{2}{9|x|}-2+\frac{\pi}{2}$

$\Rightarrow |x|^2-4|x|+3=0$

$|x|=1, 3 \Rightarrow x=\pm 1, \pm 3$

94. c. $2\tan^{-1}(\operatorname{cosec}\tan^{-1}x-\tan\cot^{-1}x)=2\tan^{-1}[\operatorname{cosec}\tan^{-1}x-\tan\cot^{-1}x]$

$$=2\tan^{-1}\left[\operatorname{cosec}\left\{\operatorname{cosec}^{-1}\frac{\sqrt{1+x^2}}{x}\right\}-\tan^{-1}\left\{\tan^{-1}\left(\frac{1}{x}\right)\right\}\right]$$

$$=2\tan^{-1}\left[\sqrt{\frac{1+x^2}{x}}-\frac{1}{x}\right]=2\tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$$

$$=2\tan^{-1}\left[\frac{\sec\theta-1}{\tan\theta}\right] \quad \text{[putting } x=\tan\theta]$$

$$=2\tan^{-1}\left[\frac{1-\cos\theta}{\sin\theta}\right]=2\tan^{-1}\left[\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right]$$

$$=2\tan^{-1}\tan\frac{\theta}{2}=2\times\frac{\theta}{2}=\theta=\tan^{-1}x$$

Multiple Correct Answers Type

1. **b, c, d.**

$6x^2+11x+3=0$

$\Rightarrow \ (2x+3)(3x+1)=0$

$\Rightarrow \ x=-3/2, -1/3$

For $x=-3/2$, $\cos^{-1}x$ is not defined as domain of $\cos^{-1}x$ is $[-1, 1]$ and for $x=-1/3$, $\operatorname{cosec}^{-1}x$ is not defined as domain of $\operatorname{cosec}^{-1}x$ is $R-(-1, 1)$. However, $\cot^{-1}x$ is defined for both of these values as domain of $\cot^{-1}x$ is R.

2. **a, b, c.**

Let $\tan^{-1}(-2)=\theta \Rightarrow \tan\theta=-2 \Rightarrow \theta=(-\pi/2, 0)$

$\Rightarrow 2\theta=(-\pi, 0)$

$$\cos(-2\theta)=\cos 2\theta=\frac{1-\tan^2\theta}{1+\tan^2\theta}=\frac{-3}{5}$$

$$\Rightarrow -2\theta=\cos^{-1}\left(\frac{-3}{5}\right)=\pi-\cos^{-1}\frac{3}{5}$$

$$\Rightarrow 2\theta=-\pi+\cos^{-1}\frac{3}{5}=-\pi+\tan^{-1}\frac{4}{3}=-\pi+\cot^{-1}\frac{3}{4}=-\pi+\frac{\pi}{2}-\tan^{-1}\frac{3}{4}=-\frac{\pi}{2}-\tan^{-1}\frac{3}{4}$$

$$=-\frac{\pi}{2}+\tan^{-1}\left(-\frac{3}{4}\right)$$

3. **a, b, d.**

$\tan^{-1}(x-1)+\tan^{-1}(x)+\tan^{-1}(x+1)=\tan^{-1}3x$

$\Rightarrow \tan^{-1}(x-1)+\tan^{-1}(x)=\tan^{-1}3x-\tan^{-1}(x+1)$

$$\Rightarrow \tan^{-1}\left[\frac{(x-1)+x}{1-(x-1)(x)}\right]=\tan^{-1}\left[\frac{3x-(x+1)}{1+3x(x+1)}\right]$$

$$\Rightarrow \frac{2x-1}{1-x^2+x}=\frac{2x-1}{1+3x^2+3x}$$

$\Rightarrow (1-x^2+x)(2x-1)=(1+3x^2+3x)(2x-1)$

$$\Rightarrow x=0, \pm\frac{1}{2}$$

4. **b.**

We know that $\sin^{-1}x$ is defined for $x\in[-1, 1]$ and $\sec^{-1}x$ is defined for $x\in(-\infty, -1]\cup[1, \infty)$.
Hence, the given function is defined for $x\in\{-1, 1\}$.
Therefore, $f(1)=\pi/2, f(-1)=\pi/2$.

5. **a, c, d.**

$(\sin^{-1}x+\sin^{-1}w)(\sin^{-1}y+\sin^{-1}z)=\pi^2$

$\Rightarrow \sin^{-1}x+\sin^{-1}w=\sin^{-1}y+\sin^{-1}z=\pi$

or $\sin^{-1}x+\sin^{-1}w=\sin^{-1}y+\sin^{-1}z=-\pi$

$\Rightarrow x=y=z=w=1$ or $x=y=z=w=-1$

Hence, the maximum value of $\begin{vmatrix} x^{N_1} & y^{N_2} \\ z^{N_3} & w^{N_4} \end{vmatrix}=\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}=2$ and minimum value $\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix}=-2$.

Also, there are 16 different determinants as each place value is either 1 or -1.

6. **a, b, c, d.**

$$\text{Since } |\tan^{-1}x|=\begin{cases} \tan^{-1}x, \text{ if } x\ge 0 \\ -\tan^{-1}x, \text{ if } x<0 \end{cases}$$

$\Rightarrow |\tan^{-1}x|=\tan^{-1}|x| \ \forall x\in R$

$\Rightarrow \tan|\tan^{-1}x|=\tan\tan^{-1}|x|=|x|$

Also $|\cot^{-1} x| = \cot^{-1} x;\ \forall x \in R$
$\Rightarrow \cot |\cot^{-1} x| = x,\ \forall x \in R$

$$\tan^{-1} |\tan x| = \begin{cases} x, & \text{if } \tan x > 0 \\ -x, & \text{if } \tan x < 0 \end{cases}$$

$$\sin |\sin^{-1} x| = \begin{cases} x, & x \in [0, 1] \\ -x, & x \in [-1, 0) \end{cases}$$

7. **a, c.**
Domain of $f(x) = \log_e \cos^{-1} x$ is $x \in [-1, 1)$
$\therefore \quad [\alpha] = -1$ or 0

8. **a, b, c.**

(a) $\sin\left(\tan^{-1} 3 + \tan^{-1} \frac{1}{3}\right) = \sin \frac{\pi}{2} = 1$

(b) $\cos\left(\frac{\pi}{2} - \sin^{-1} \frac{3}{4}\right) = \cos\left(\cos^{-1} \frac{3}{4}\right) = \frac{3}{4}$

(c) $\sin\left(\frac{1}{4} \sin^{-1} \frac{\sqrt{63}}{8}\right)$

Let $\sin^{-1} \frac{\sqrt{63}}{8} = \theta$

$\Rightarrow \quad \sin \theta = \frac{\sqrt{63}}{8} \Rightarrow \cos \theta = \frac{1}{8}$

We have $\cos \frac{\theta}{2} = \sqrt{\frac{1 + \cos \theta}{2}} = \frac{3}{4}$

$\Rightarrow \quad \sin \frac{\theta}{4} = \sqrt{\frac{1 - \cos \frac{\theta}{2}}{2}} = \frac{1}{2\sqrt{2}}$

Now, $\log_2 \sin\left(\frac{1}{4} \sin^{-1} \frac{\sqrt{63}}{8}\right) = \log_2 \frac{1}{2\sqrt{2}} = -\frac{3}{2}$

(d) $\cos^{-1} \frac{\sqrt{5}}{3} = \theta \Rightarrow \cos \theta = \frac{\sqrt{5}}{3}$

$\therefore \quad \tan \frac{\theta}{2} = \frac{3 - \sqrt{5}}{2}$ which is irrational.

9. **c, d.**

$xy < 0 \Rightarrow \quad x + \frac{1}{x} \geq 2, y + \frac{1}{y} \leq -2$

or $x + \frac{1}{x} \leq -2, y + \frac{1}{y} \geq 2$

$$x+\frac{1}{x}\geq 2 \Rightarrow \sec^{-1}\left(x+\frac{1}{x}\right)\in\left[\frac{\pi}{3},\frac{\pi}{2}\right)$$

$$y+\frac{1}{y}\leq -2 \Rightarrow \sec^{-1}\left(y+\frac{1}{y}\right)\in\left(\frac{\pi}{2},\frac{2\pi}{3}\right] \Rightarrow z\in\left(\frac{5\pi}{6},\frac{7\pi}{6}\right)$$

10. a, d.

Let $f(x)=(\sin^{-1}x)^2+(\cos^{-1}x)^2$

$$=(\sin^{-1}x+\cos^{-1}x)^2-2\sin^{-1}x\cos^{-1}x$$

$$=\frac{\pi^2}{4}-2\sin^{-1}x\left[\frac{\pi}{2}-\sin^{-1}x\right]$$

$$=\frac{\pi^2}{4}-\pi\sin^{-1}x+2(\sin^{-1}x)^2$$

$$=2\left[\left(\sin^{-1}x\right)^2-\frac{\pi}{2}\sin^{-1}x+\frac{\pi^2}{8}\right]$$

$$=2\left(\sin^{-1}x-\frac{\pi}{4}\right)^2+2\left[\frac{\pi^2}{16}\right]$$

Now, $-\frac{\pi}{2}\leq\sin^{-1}x\leq\frac{\pi}{2}$

$$\Rightarrow -\frac{3\pi}{4}\leq\sin^{-1}x-\frac{\pi}{4}\leq\frac{\pi}{4}$$

$$\Rightarrow 0\leq\left(\sin^{-1}x-\frac{\pi}{4}\right)^2\leq\frac{9\pi^2}{16}$$

$$\Rightarrow 0\leq 2\left(\sin^{-1}x-\frac{\pi}{4}\right)^2\leq\frac{9\pi^2}{8}$$

$$\Rightarrow \frac{\pi^2}{8}\leq\left(\sin^{-1}x-\frac{\pi}{4}\right)^2+\frac{\pi^2}{8}\leq\frac{5\pi^2}{4}$$

11. a, d.

For the given equation $0\leq x, y\leq 1$.

Also, $\sin^{-1}x+\sin^{-1}y=\frac{\pi}{2}$

$$\Rightarrow \sin^{-1}x=\cos^{-1}y=\sin^{-1}\sqrt{1-y^2}$$

$$\Rightarrow x=\sqrt{1-y^2} \quad \Rightarrow x^2+y^2=1 \qquad \text{(i)}$$

Again, $\sin 2x=\cos 2y$

$$\Rightarrow \cos\left(\frac{\pi}{2}-2x\right)=\cos 2y$$

$$\Rightarrow \frac{\pi}{2}-2x=2n\pi\pm 2y, \text{ where } n\in I$$

$$\Rightarrow x\pm y=\frac{\pi}{4}-n\pi \qquad \text{(ii)}$$

From Eqs. (i) and (ii), we get

$$x = \frac{\pi}{8} + \sqrt{\frac{1}{2} - \frac{\pi^2}{64}} \text{ and } y = \sqrt{\frac{1}{2} - \frac{\pi^2}{64}} - \frac{\pi}{8}$$

12. a, c.

We have

$$\cot^{-1}\left(\frac{n^2 - 10n + 21.6}{\pi}\right) > \frac{\pi}{6}$$

$$\Rightarrow \frac{n^2 - 10n + 21.6}{\pi} < \cot\frac{\pi}{6} \qquad \text{(as } \cot x \text{ is decreasing for } 0 < x < \pi)$$

$$\Rightarrow n^2 - 10n + 21.6 < \pi\sqrt{3}$$

$$\Rightarrow n^2 - 10n + 25 + 21.6 - 25 < \pi\sqrt{3}$$

$$\Rightarrow (n-5)^2 < \pi\sqrt{3} + 3.4$$

$$\Rightarrow -\sqrt{\sqrt{3}\pi + 3.4} < n - 5 < \sqrt{\sqrt{3}\pi + 3.4}$$

$$\Rightarrow 5 - \sqrt{\sqrt{3}\pi + 3.4} < n < 5 + \sqrt{\sqrt{3}\pi + 3.4} \qquad \text{(i)}$$

Since $\sqrt{3\pi} = 5.5$ nearly, $\sqrt{\sqrt{3}\pi + 3.4} \sim \sqrt{8.9} \sim 2.9$

$$\Rightarrow 2.1 < n < 7.9$$

$$\therefore n = 3, 4, 5, 6, 7 \qquad \{\text{as } n \in N\}$$

13. a, b, d.

Let t_r denote the rth term of the series 3, 7, 13, 21, ... and

$$\begin{array}{l} S = 3 + 7 + 13 + 21 + \cdots + t_n \\ -S = \quad 3 + 7 + 13 + \cdots + t_{n-1} + t_n \\ \hline 0 = 3 + 4 + 6 + 8 + \cdots + 2n - t_n \end{array}$$

$$\Rightarrow t_n = 3 + 4 + 6 + \cdots + 2n = 1 + 2 \times \frac{1}{2}n(n+1) = n^2 + n + 1$$

Let $T_r = \cot^{-1}(r^2 + r + 1) = \tan^{-1}\left(\frac{1}{r^2 + r + 1}\right) = \tan^{-1}\left(\frac{r + 1 - r}{1 + r(r+1)}\right) = \tan^{-1}(r+1) - \tan^{-1} r$

Thus, the sum of the first n terms of the given series is

$$\sum_{r=1}^{n}\left[\tan^{-1}(r+1) - \tan^{-1} r\right] = \tan^{-1}(n+1) - \tan^{-1}(1)$$

$$= \tan^{-1}\left[\frac{n + 1 - n}{1 + 1(n+1)}\right] = \tan^{-1}\left(\frac{n}{n+2}\right) = \tan^{-1}\left(\frac{1}{1 + \frac{2}{n}}\right)$$

$$\Rightarrow S_\infty = \lim_{n\to\infty} \tan^{-1}\left(\frac{1}{1 + \frac{2}{n}}\right) = \frac{\pi}{4}, \quad S_{10} = \tan^{-1}\frac{10}{12} = \tan^{-1}\frac{5}{6}$$

$$S_6 = \tan^{-1}\frac{3}{4} = \sin^{-1}\frac{3}{5}$$

$$S_{20} = \tan^{-1}\frac{10}{11} = \cot^{-1} 1.1$$

14. b.

$f(x) = \sin^{-1}|\sin kx| + \cos^{-1}(\cos kx)$
Let $g(x) = \sin^{-1}|\sin x| + \cos^{-1}(\cos x)$

$$g(x) = \begin{cases} 2x, & 0 \le x \le \frac{\pi}{2} \\ \pi, & \frac{\pi}{2} < x \le \frac{3\pi}{2} \\ 4\pi - 2x, & \frac{3\pi}{2} < x \le 2\pi \end{cases}$$

$g(x)$ is periodic with period 2π and is constant in the continuous interval $\left[2n\pi + \frac{\pi}{2}, 2n\pi + \frac{3\pi}{2}\right]$ (where $n \in I$) and $f(x) = g(kx)$.

So, $f(x)$ is constant in the interval $\left[\frac{2n\pi}{k} + \frac{\pi}{2k}, \frac{2n\pi}{k} + \frac{3\pi}{2k}\right]$

$\Rightarrow \quad \frac{\pi}{4} = \frac{3\pi}{2k} - \frac{\pi}{2k} \Rightarrow \frac{\pi}{k} = \frac{\pi}{4} \Rightarrow k = 4$

15. a, c.

Given equation is $x^2 + 2x\sin(\cos^{-1}y) + 1 = 0$. Since x is real, $D \ge 0$

$\therefore \quad 4(\sin(\cos^{-1}y))^2 - 4 \ge 0$

$\Rightarrow \quad (\sin(\cos^{-1}y))^2 \ge 1$

$\Rightarrow \quad \sin(\cos^{-1}y) = \pm 1$

$\Rightarrow \quad \cos^{-1}y = \pm\frac{\pi}{2} \Rightarrow y = 0$

Putting value of y in the original equation, we have $x^2 + 2x + 1 = 0 \Rightarrow x = -1$.
Hence, the equation has only one solution.

16. b, c.

$1 \le \frac{\pi}{\cos^{-1}x} < \infty \quad \Rightarrow \quad 2 \le 2^{\frac{\pi}{\cos^{-1}x}} < \infty$

Hence, 2 should lie between or on the roots of $t^2 - \left(a + \frac{1}{2}\right)t - a^2 = 0$ where $t = 2^{\pi/\cos^{-1}x}$

$\Rightarrow \quad f(2) \le 0 \quad \Rightarrow \quad a^2 + 2a - 3 \ge 0 \Rightarrow a \in (-\infty, -3] \cup [1, \infty)$

17. a, c.

The given relation is possible when $a - \frac{a^2}{3} + \frac{a^3}{9} + \cdots = 1 + b + b^2 + \cdots$

Also, $-1 \le a - \frac{a^2}{3} + \frac{a^3}{9} + \cdots \le 1$ and $-1 \le 1 + b + b^2 + \cdots \le 1$

$\Rightarrow \quad |b| < 1 \Rightarrow |a| < 3$ and $\frac{a}{1 + \frac{a}{3}} = \frac{1}{1 - b}$

$$= \tan^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{1-\frac{x}{y}}{1+\frac{x}{y}}\right) = \tan^{-1}\left(\frac{x}{y}\right) + \tan^{-1}1 - \tan^{-1}\frac{x}{y} = \frac{\pi}{4}$$

Now, in Eq. (i), putting $\frac{x}{y} = \frac{3}{4}$, we get

$$\tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \frac{\pi}{4}$$

Hence, both the statements are correct and statement 2 is the correct explanation of statement 1.

9. a. See theory

Linked Comprehension Type

For Problems 1 – 3

1. d, 2. b, 3. c

Sol.

$\sin^{-1}x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\cos^{-1}y \in [0, \pi]$

$\sec^{-1}z \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ $\quad \Rightarrow \quad \sin^{-1}x + \cos^{-1}y + \sec^{-1}z \le \frac{\pi}{2} + \pi + \pi = \frac{5\pi}{2}$

Also, $t^2 - \sqrt{2\pi}\, t + 3\pi$

$$= t^2 - 2\sqrt{\frac{\pi}{2}}\, t + \frac{\pi}{2} - \frac{\pi}{2} + 3\pi = \left(t - \sqrt{\frac{\pi}{2}}\right)^2 + \frac{5\pi}{2} \ge \frac{5\pi}{2}$$

The given inequation exists if equality holds, i.e.,

$$\text{L.H.S.} = \text{R.H.S.} = \frac{5\pi}{2}$$

$\Rightarrow \quad x = 1, y = -1, z = -1$ and $t = \sqrt{\frac{\pi}{2}}$ $\quad \Rightarrow \quad \cos^{-1}(\cos 5t^2) = \cos^{-1}\left(\cos\left(\frac{5\pi}{2}\right)\right) = \frac{\pi}{2}$

$\cos^{-1}(\min\{x, y, z\}) = \cos^{-1}(-1) = \pi$

For Problems 4 – 6

4. b, 5. a, 6. b

Sol.

Given $ax + b(\sec(\tan^{-1} x)) = c$ and $ay + b(\sec(\tan^{-1} y)) = c$

Let $\tan^{-1}x = \alpha$ and $\tan^{-1}y = \beta$, then the given relations are

$a \tan \alpha + b \sec \alpha = c$ and $a \tan \beta + b \sec \beta = c$

From these two relations, we can conclude that equation $a \tan \theta + b \sec\theta = c$ has roots α and β.

$a \tan \theta + b \sec \theta = c$

$\Rightarrow \quad b \sec \theta = c - a \tan \theta$

$\Rightarrow \quad b^2 \sec^2\theta = c^2 - 2\, ac \tan \theta + a^2 \tan^2\theta$

$\Rightarrow \quad b^2 + b^2 \tan^2\theta = c^2 - 2\, ac \tan\theta + a^2 \tan^2\theta$

$\Rightarrow \quad (a^2 - b^2)\tan^2\theta - 2\, ac \tan\theta + c^2 - b^2 = 0$

Therefore, sum of the roots, $\tan\alpha + \tan\beta = x + y = \dfrac{2ac}{a^2 - b^2}$

and the product of roots, $\tan\alpha\tan\beta = xy = \dfrac{c^2 - b^2}{a^2 - b^2}$

and $\dfrac{x+y}{1-xy} = \dfrac{\dfrac{2ac}{a^2-b^2}}{1-\dfrac{c^2-b^2}{a^2-b^2}} = \dfrac{2ac}{a^2-c^2}$

For Problems 7 – 9

7. b, 8. d, 9. c

Sol.

Let $\cos^{-1}x = a \Rightarrow a \in [0, \pi]$
and $\sin^{-1}y = b \Rightarrow b \in [-\pi/2, \pi/2]$

We have $a + b^2 = \dfrac{p\pi^2}{4}$ (i)

and $ab^2 = \dfrac{\pi^4}{16}$ (ii)

Since $b^2 \in [0, \pi^2/4]$, we get $a + b^2 \in [0, \pi + \pi^2/4]$

So, from Eq. (i) we get $0 \le \dfrac{p\pi^2}{4} \le \pi + \dfrac{\pi^2}{4}$

i.e., $0 \le p \le \dfrac{4}{\pi} + 1$

Since $p \in Z$, so $p = 0$, 1 or 2.
Substituting the value of b^2 from Eq. (i) in Eq. (ii), we get

$a\left(\dfrac{p\pi^2}{4} - a\right) = \dfrac{\pi^2}{16} \quad \Rightarrow \quad 16a^2 - 4p\,\pi^2 a + \pi^4 = 0$ (iii)

since $a \in R \Rightarrow D \ge 0$
i.e., $16p^2\pi^4 - 64\pi^4 \ge 0 \quad \Rightarrow \quad p^2 \ge 4 \Rightarrow p \ge 2 \quad \Rightarrow \quad p = 2$
Substituting $p = 2$ in Eq. (iii), we get
$16a^2 - 8\pi^2 a + \pi^4 = 0$.

$\Rightarrow \quad (4a - \pi^2)^2 = 0 \quad \Rightarrow \quad a = \dfrac{\pi^2}{4} = \cos^{-1}x \quad \Rightarrow \quad x = \cos\dfrac{\pi^2}{4}$

From Eq. (ii), we get $\dfrac{\pi^2}{4}\; b^2 = \dfrac{\pi^4}{16} \quad \Rightarrow \quad b = \pm\dfrac{\pi}{2} = \sin^{-1}y \quad \Rightarrow \quad y = \pm 1$

For Problems 10 – 12

10. c, 11. a, 12. d

Sol.

Let $\cos^{-1}x = \theta \Rightarrow x = \cos\theta$, where $\theta \in [0, \pi]$
$\cos^{-1}(4x^3 - 3x) = \cos^{-1}(4\cos^3\theta - 3\cos\theta) = \cos^{-1}(\cos 3\theta) = \cos^{-1}(\cos\alpha)$

$\Rightarrow \quad \dfrac{3a}{a+3} = \dfrac{1}{1-b}$, there are infinitely many solutions

$\Rightarrow \quad 3a - 3ab = a + 3 \qquad \Rightarrow \quad 2a - 3ab = 3$

$\Rightarrow \quad b = \dfrac{2a-3}{3a}$ and $a = \dfrac{3}{2-3b}$

18. a, b.

We know that

if $|x| \le 1$, then $2\tan^{-1} x = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

if $x > 1$, $2\tan^{-1} x = \pi - \sin^{-1} \dfrac{2x}{1+x^2}$

if $x < -1$, $2\tan^{-1} x = -\pi - \sin^{-1} \dfrac{2x}{1+x^2}$

Hence, the required values are $x < -1$ or $x > 1$.

19. a, d.

Case 1: If $0 \le x \le \dfrac{1}{2}$, then

$\cos^{-1}\left(\dfrac{x}{2} + \dfrac{1}{2}\sqrt{3-3x^2}\right) = \cos^{-1}\left(x\dfrac{1}{2} + \sqrt{1-x^2}\,\dfrac{\sqrt{3}}{2}\right) = \cos^{-1} x - \cos^{-1}\dfrac{1}{2}$

Case2: If $\dfrac{1}{2} \le x \le 1$, then

$\cos^{-1}\left(\dfrac{x}{2} + \dfrac{1}{2}\sqrt{3-3x^2}\right) = \cos^{-1}\dfrac{1}{2} - \cos^{-1} x$

20. a, b, c.

a. $\cos(\tan^{-1}(\tan(4-\pi))) = \cos(4-\pi) = \cos(\pi-4) = -\cos 4 > 0$
b. $\sin(\cot^{-1}(\cot(4-\pi))) = \sin(4-\pi) = -\sin 4 > 0$ (as $\sin 4 < 0$)
c. $\tan(\cos^{-1}(\cos(2\pi-5))) = \tan(2\pi-5) = -\tan 5 > 0$ (as $\tan 5 < 0$)
d. $\cot(\sin^{-1}(\sin(\pi-4)) = \cot(\pi-4) = -\cot 4 < 0$

21. a, b. $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi \Rightarrow \sin^{-1} + \sin^{-1} y + \sin^{-1} z = \pi/2$

$\Rightarrow \cos^{-1} x + \cos^{-1} y = \cos^{-1}(-z) \Rightarrow xy - \sqrt{1-x^2}\sqrt{1-y^2} = -z \Rightarrow x^2 + y^2 + z^2 + 2xyz = 1$

22.a, b.

a. $\cos(\tan^{-1}(\tan(4-\pi))) = \cos(4-\pi) = \cos(\pi-4) = -\cos 4 > 0$
b. $\sin(\cot^{-1}(\cot(4-\pi))) = \sin(4-\pi) = -\sin 4 > 0$ (as $\sin 4 < 0$)
c. $\tan(\cos^{-1}(\cos(2\pi-5))) = \tan(2\pi-5) = -\tan 5 > 0$ (as $\tan 5 < 0$)
d. $\cot(\sin^{-1}(\sin(\pi-4)) = \cot(\pi-4) = -\cot 4 < 0$

23.b, c, d.

$\cos\left(-\dfrac{14\pi}{5}\right) = \cos\dfrac{14\pi}{5} = \cos\dfrac{4\pi}{5}$

Hence, $\cos\dfrac{1}{2}\cos^{-1}\left(\cos\dfrac{4\pi}{5}\right) = \cos\dfrac{2\pi}{5}$

Reasoning Type

1. a. Statement 2 is correct, from which we can say $\cot^{-1}x + \cos^{-1}2x = -\pi$ is not possible. Hence, both the statements are correct and statement 2 is the correct explanation of statement 1.

2. d. Obviously, statement 2 is correct, but when $x \in [-1, 1]$ we have $\tan^{-1}x \in [-\pi/4, \pi/4]$.

It implies that $\frac{\pi}{2} + \tan^{-1}x \in [\pi/4, 3\pi/4]$.

Hence, statement 2 is true but statement 1 is false.

3. c. $\text{cosec}^{-1}x > \sec^{-1}x$

$$\Rightarrow \quad \text{cosec}^{-1}x > \frac{\pi}{2} - \text{cosec}^{-1}x$$

$$\Rightarrow \quad \text{cosec}^{-1}x > \frac{\pi}{4}$$

$$\Rightarrow \quad 1 \le x < \sqrt{2} \text{ and } \left(\frac{1}{2} + \frac{1}{\sqrt{2}}\right) \in [1, \sqrt{2})$$

But statement 2 is false.

4. a. $\sin^{-1}x = \tan^{-1}\frac{x}{\sqrt{1-x^2}} > \tan^{-1}x > \tan^{-1}y$ $\quad \left[\because x > y, \frac{x}{\sqrt{1-x^2}} > x\right]$

Therefore, statement 2 is true.

Now, $e < \pi \quad \Rightarrow \quad \frac{1}{\sqrt{e}} > \frac{1}{\sqrt{\pi}}$

By statement 2, we have

$$\sin^{-1}\left(\frac{1}{\sqrt{e}}\right) > \tan^{-1}\left(\frac{1}{\sqrt{e}}\right) > \tan^{-1}\left(\frac{1}{\sqrt{\pi}}\right)$$

Therefore, statement 1 is true.

5. d. $30 - 9\pi \in [0, \pi]$ is true but it is not principal value of $\cos^{-1}(\cos 30)$ as $\cos^{-1}(\cos 30) = \cos^{-1}(\cos(9\pi + (30 - 9\pi))) = \cos^{-1}(-\cos(30 - 9\pi)) = \pi - (30 - 9\pi) = 10\pi - 30$.

Hence, statement 2 is true but statement 1 is false.

6. a. $f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \pi - 2\tan^{-1}x, x \ge 1$

$$\Rightarrow \quad f'(x) = -\frac{2}{1+x^2} \Rightarrow f'(2) = -\frac{2}{5}$$

Thus statement 1 is true, statement 2 is true and statement 2 is the correct explanation of statement 1.

7. b. We know that $\tan^{-1}x$ and $\cot^{-1}x$ have domain R, also $\tan x$ and $\cot x$ are unbounded functions. On the other hand; $\sec x$ is an unbounded function, but its range is $R - (-1, 1)$, and not R.

8. a. For $x > 0, y > 0$,

$$\tan^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y-x}{y+x}\right) \qquad \text{(i)}$$

where $\alpha = 3\theta \in [0, 3\pi]$
Refer the graph of $y = \cos^{-1}(\cos\alpha)$, $\alpha \in [0, 3\pi]$.

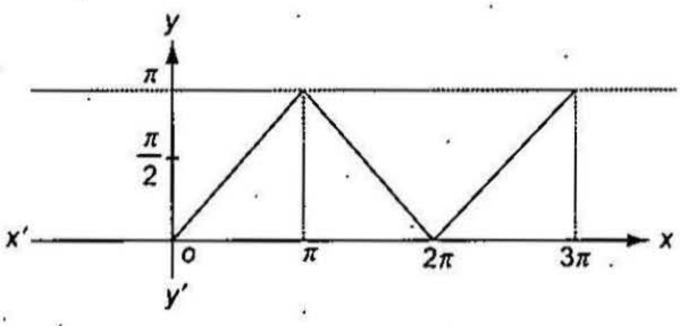


Fig. 4.31

From the graph,
$\cos^{-1}(4x^3 - 3x) = \cos^{-1}(\cos\alpha)$

$$= \begin{cases} \alpha, & 0 \le \alpha < \pi \\ 2\pi - \alpha, & \pi \le \alpha \le 2\pi \\ \alpha - 2\pi, & 2\pi < \alpha \le 3\pi \end{cases}$$

$$= \begin{cases} 3\cos^{-1}x, & 0 \le 3\cos^{-1}x < \pi \\ 2\pi - 3\cos^{-1}x, & \pi \le 3\cos^{-1}x \le 2\pi \\ 3\cos^{-1}x - 2\pi, & 2\pi < 3\cos^{-1}x \le 3\pi \end{cases}$$

$$= \begin{cases} 3\cos^{-1}x, & 0 \le \cos^{-1}x < (\pi/3) \\ 2\pi - 3\cos^{-1}x, & (\pi/3) \le \cos^{-1}x \le (2\pi/3) \\ 3\cos^{-1}x - 2\pi, & (2\pi/3) < 3\cos^{-1}x \le \pi \end{cases}$$

$$= \begin{cases} 3\cos^{-1}x, & (1/2) < x \le 1 \\ 2\pi - 3\cos^{-1}x, & (-1/2) \le x \le (1/2) \\ 3\cos^{-1}x - 2\pi, & -1 \le x < -(1/2) \end{cases}$$

$$= \begin{cases} 3\cos^{-1}x - 2\pi, & -1 \le x < (-1/2) \\ 2\pi - 3\cos^{-1}x, & (-1/2) \le x \le (1/2) \\ 3\cos^{-1}x, & (1/2) < x \le 1 \end{cases}$$

For $x \in \left[-1, -\frac{1}{2}\right)$, $\cos^{-1}(4x^3 - 3x) = 3\cos^{-1}x - 2\pi$

$\Rightarrow \quad a = -2\pi$ and $b = 3 \Rightarrow a + b\pi = \pi$

For $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, $\cos^{-1}(4x^3 - 3x) = 2\pi - 3\cos^{-1}x$

$\Rightarrow \quad a = 2\pi$ and $b = -3 \quad \Rightarrow \quad \sin^{-1}\left(\sin\frac{a}{b}\right) = \sin^{-1}\left(\sin\frac{2\pi}{-3}\right)$

$$= \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$$

For $x \in \left(\frac{1}{2}, 1\right]$, $\cos^{-1}(4x^3 - 3x) = 3\cos^{-1}x \Rightarrow \quad a = 0$ and $b = 3$

$\therefore \quad \lim_{y\to a} b\cos(y) = \lim_{y\to 0} 3\cos(y) = 3$

Matrix-Match Type

1. a → p; b → q, s; c → r, s; d → r, s

a. $\cos^{-1}(4x^3 - 3x) = 3\cos^{-1}x$

$0 \le \cos^{-1}(4x^3 - 3x) \le \pi$

$\Rightarrow \quad 0 \le 3\cos^{-1}x \le \pi \quad \Rightarrow \quad 0 \le \cos^{-1}x \le (\pi/3) \quad \Rightarrow \quad (1/2) \le x \le 1$

b. $\sin^{-1}(3x - 4x^3) = 3\sin^{-1}x$

$(-\pi/2) \le \sin^{-1}(3x - 4x^3) \le (\pi/2)$

$\Rightarrow \quad (-\pi/2) \le 3\sin^{-1}x \le (\pi/2)$

$\Rightarrow \quad (-\pi/6) \le \sin^{-1}x \le (\pi/6)$

$\Rightarrow \quad (-1/2) \le x \le (1/2)$

c. $\cos^{-1}(4x^3 - 3x) = 3\sin^{-1}x$

$0 \le \cos^{-1}(3x - 4x^3) \le \pi$

$\Rightarrow \quad 0 \le 3\sin^{-1}x \le \pi$

$\Rightarrow \quad 0 \le \sin^{-1}x \le \pi/3$

$\Rightarrow \quad 0 \le x \le (\sqrt{3}/2)$

d. $\sin^{-1}(3x - 4x^3) = 3\cos^{-1}x$

$(-\pi/2) \le \sin^{-1}(3x - 4x^3) \le (\pi/2)$

$\Rightarrow \quad (-\pi/2) \le 3\cos^{-1}x \le (\pi/2)$

$\Rightarrow \quad (-\pi/6) \le \cos^{-1}x \le (\pi/6)$

$\Rightarrow \quad 0 \le \cos^{-1}x \le (\pi/6)$

$\Rightarrow \quad 0 \le x \le (\sqrt{3}/2)$

2. a. → q, r, s; b → q; c → r, s; d → p.

a. $(\sin^{-1}x)^2 + (\sin^{-1}y)^2 = \dfrac{\pi^2}{2}$

$\Rightarrow \quad (\sin^{-1}x)^2 = (\sin^{-1}y)^2 = \dfrac{\pi^2}{4}$

$\Rightarrow \quad \sin^{-1}x = \pm\dfrac{\pi}{2}, \sin^{-1}y = \pm\dfrac{\pi}{2}$

$\Rightarrow \quad x = \pm 1$ and $y = \pm 1$

$\Rightarrow \quad x^3 + y^3 = -2, 0, 2$

b. $(\cos^{-1}x)^2 + (\cos^{-1}y)^2 = 2\pi^2$

$\Rightarrow \quad (\cos^{-1}x)^2 = (\cos^{-1}y)^2 = \pi$

$\Rightarrow \quad x = y = -1$

$\Rightarrow \quad x^5 + y^5 = -2$

c. $(\sin^{-1} x)^2 (\cos^{-1} y)^2 = \frac{\pi^4}{4}$

$\Rightarrow \quad (\sin^{-1} x)^2 = \frac{\pi^2}{4}$ and $(\cos^{-1} y)^2 = \pi^2$

$\Rightarrow \quad (\sin^{-1} x) = \pm\frac{\pi}{2}$ and $(\cos^{-1} y) = \pi$

$\Rightarrow \quad x = \pm 1$ and $y = -1$

$\Rightarrow \quad -|x - y| = 0, 2$

d. $|\sin^{-1} x - \sin^{-1} y| = \pi$

$\Rightarrow \quad \sin^{-1} x = -\frac{\pi}{2}$ and $\sin^{-1} y = \frac{\pi}{2}$

or $\sin^{-1} x = \frac{\pi}{2}$ and $\sin^{-1} y = -\frac{\pi}{2}$

$\Rightarrow \quad x^y = 1^{(-1)}$ or $(-1)^1 = 1$ or -1

3. a → p, q; b → q; c → q, r, s; d → p, r

Refer the graphs of $y = \sin^{-1}(\sin x)$, $y = \cos^{-1}(\cos x)$, $y = \tan^{-1}(\tan x)$ and $y = \cot^{-1}(\cot x)$.

4. a → q; b → s; c → p; d → r

a. $\sin^{-1}\frac{4}{5} = \tan^{-1}\frac{4}{3}$

$2\tan^{-1}\frac{1}{3} = \tan^{-1}\frac{3}{4} = \cot^{-1}\frac{4}{3}$

and $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$

b. $\tan^{-1}\frac{12}{5} + \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{63}{16} = \pi + \tan^{-1}\frac{48+15}{20-36} + \tan^{-1}\frac{63}{16}$

$$= \pi - \tan^{-1}\frac{63}{16} + \tan^{-1}\frac{63}{16} = \pi$$

c. $A = \tan^{-1}\frac{x\sqrt{3}}{2\lambda - x}$ and $B = \tan^{-1}\left(\frac{2x - \lambda}{\lambda\sqrt{3}}\right)$

Now, $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

$$= \frac{\frac{x\sqrt{3}}{2\lambda - x} - \frac{2x - \lambda}{\lambda\sqrt{3}}}{1 + \frac{x\sqrt{3}}{2\lambda - x}\,\frac{2x - \lambda}{\lambda\sqrt{3}}}$$

$$= \frac{3x\lambda + (2x - \lambda)(x - 2\lambda)}{\sqrt{3}\left[\lambda(2\lambda - x) + x(2x - \lambda)\right]}$$

$$= \frac{1}{\sqrt{3}}\left[\frac{2x^2 - 2\lambda x + 2\lambda^2}{2x^2 - 2\lambda x + 2\lambda^2}\right] = \frac{1}{\sqrt{3}} = \tan 30^\circ$$

$\therefore \quad A - B = 30^\circ$

d. $\tan^{-1}\frac{1}{7}+2\tan^{-1}\frac{1}{3}=\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{2\frac{1}{3}}{1-\frac{1}{9}}=\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{3}{4}$

$$=\tan^{-1}\frac{\frac{1}{7}+\frac{3}{4}}{1-\frac{1}{7}\frac{3}{4}}=\tan^{-1}1=\pi/4$$

5. **a → s; b → p; c → q; d → r**

a. $f(x)=\sin^{-1}x+\cos^{-1}x+\cot^{-1}x$

$=\frac{\pi}{2}+\cot^{-1}x, x\in[-1,1]$

For $x\in[-1,1]$, $\cot^{-1}x\in\left[\frac{\pi}{4},\frac{3\pi}{4}\right]$ $\Rightarrow$ $\frac{\pi}{2}+\cot^{-1}x\in\left[\frac{3\pi}{4},\frac{5\pi}{4}\right]$

b. $f(x)=\cot^{-1}x+\tan^{-1}x+\text{cosec}^{-1}x$

$=\frac{\pi}{2}+\text{cosec}^{-1}x$, where $x\in(-\infty,-1]\cup[1,\infty)$

Now $\text{cosec}^{-1}x\in\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$ $\Rightarrow$ $\frac{\pi}{2}+\text{cosec}^{-1}x\in\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$

c. $f(x)=\cot^{-1}x+\tan^{-1}x+\cos^{-1}x$

$=\frac{\pi}{2}+\cos^{-1}x$, where $x\in[-1,1]$ $\Rightarrow$ $\frac{\pi}{2}+\cos^{-1}x\in\left[\frac{\pi}{2},\frac{3\pi}{2}\right]$

d. $\sec^{-1}x+\text{cosec}^{-1}x+\sin^{-1}x$, where $x\in\{-1,1\}$

$=\frac{\pi}{2}+\sin^{-1}x$, where $x\in\{-1,1\}$

Hence, $f(x)\in\{0,\pi\}$.

6. **a → q; b → r; c → p, r; d → q, r, s**

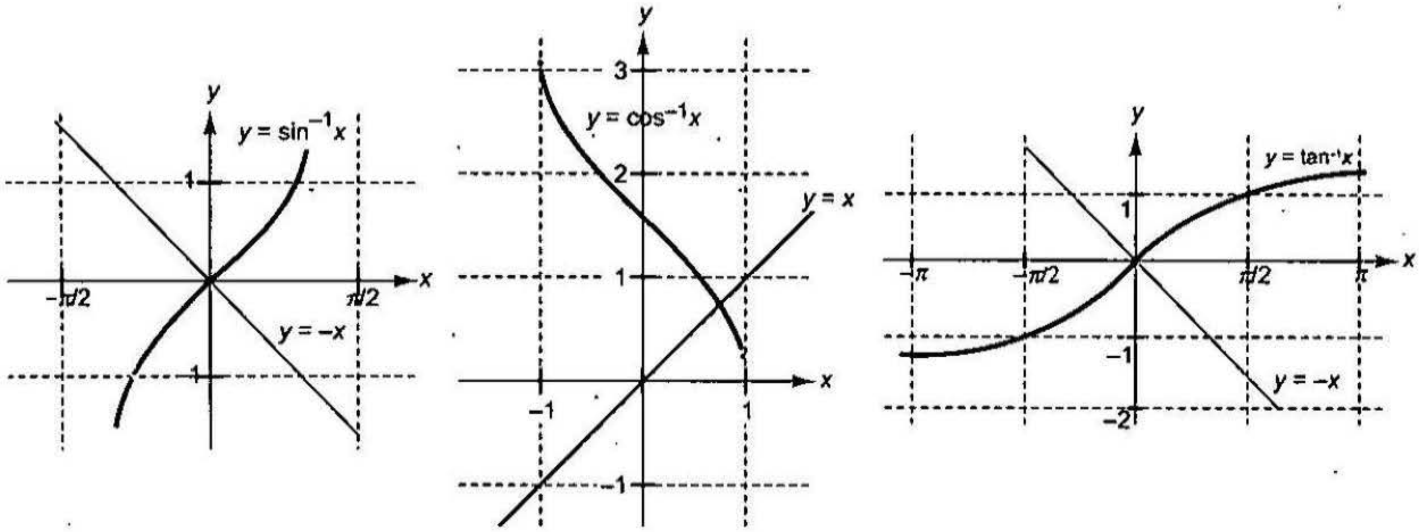


Fig. 4.32

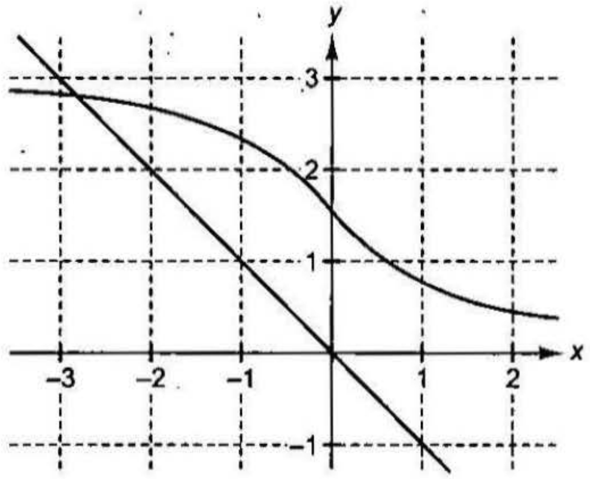


Fig. 4.33

Refer graph for solution.

Integer Type

1. (5) $(\cot^{-1}x)(\tan^{-1}x) + \left(2-\frac{\pi}{2}\right)\cot^{-1}x - 3\tan^{-1}x - 3\left(2-\frac{\pi}{2}\right) > 0$

$\Rightarrow \cot^{-1}x > 0$

$\Rightarrow (\cot^{-1}x - 3)(2 - \cot^{-1}x) > 0$

$\Rightarrow (\cot^{-1}x - 3)(\cot^{-1}x - 2) < 0$

$\Rightarrow 2 < \cot^{-1}x < 3$

$\Rightarrow \cot 3 < x < \cot 2$ [as $\cot^{-1}x$ is a decreasing function]

$\Rightarrow$ Hence, $x \in (\cot 3, \cot 2)$

$\Rightarrow \cot^{-1}a + \cot^{-1}b = \cot^{-1}(\cot 3) + \cot^{-1}(\cot 2) = 5$

2. (2) Since $\sin^{-1}$ is defined for $[-1, 1]$

$\therefore a = 0$

$\therefore x = \sin^{-1}1 + \cos^{-1}1 - \tan^{-1}1 = \frac{\pi}{4}$

$\Rightarrow \sec^2 x = 2$

3. (6) Let $\tan^{-1}u = \alpha \Rightarrow \tan\alpha = u$

$\tan^{-1}v = \beta \Rightarrow \tan\beta = v$

$\tan^{-1}w = \gamma \Rightarrow \tan\gamma = w$

$$\tan(\alpha+\beta+\gamma) = \frac{s_1 - s_3}{1 - s_2} = \frac{0-(-11)}{1-(-10)} = \frac{11}{11} = 1$$

$\therefore \alpha+\beta+\gamma = \tan^{-1}(1) = \frac{\pi}{4}$

$\Rightarrow 3\operatorname{cosec}^2(\tan^{-1}u + \tan^{-1}v + \tan^{-1}w) = 6$

4. (3) $\sin^{-1}\left(x^2 - \frac{x^4}{3} + \frac{x^6}{9} - \cdots\right) + \cos^{-1}\left(x^4 - \frac{x^8}{3} + \frac{x^{12}}{9}\cdots\right) = \frac{\pi}{2}$

$\Rightarrow \left(x^2 - \frac{x^4}{3} + \frac{x^6}{9}\ldots\ldots\right) = \left(x^4 - \frac{x^8}{3} + \frac{x^{12}}{9}\ldots\ldots\right)$

$$\Rightarrow \frac{x^2}{1+\frac{x^2}{3}}=\frac{x^4}{1+\frac{x^4}{3}}$$

$$\Rightarrow \frac{3}{3+x^2}=\frac{3x^2}{3+x^4} \text{ or } x=0$$

$$\Rightarrow 9+3x^4=9x^2+3x^4 \text{ or } x=0$$

$$\Rightarrow x^2=1 \Rightarrow x=0,\ 1 \text{ or } -1$$

Therefore, the number of values is equal to 3.

5. (7) $f(x)=\sqrt{3\cos^{-1}(4x)-\pi}$ is defined

If $\cos^{-1}4x \geq \frac{\pi}{3} \Rightarrow 4x \leq \frac{1}{2} \Rightarrow x \leq \frac{1}{8}$ (i)

Also, $-1 \leq 4x \leq 1 \Rightarrow \frac{-1}{4} \leq x \leq \frac{1}{4}$ (ii)

Therefore, from Eqs. (i) and (ii), we have domain: $x \in \left[\frac{-1}{4}, \frac{1}{8}\right]$

$\Rightarrow 4a+64b=7$

6. (9) $1+\sin(\cos^{-1}x)+\sin^2(\cos^{-1}x)+\cdots\infty=2$

$$\Rightarrow \frac{1}{1-\sin(\cos^{-1}x)}=2$$

$$\Rightarrow \frac{1}{2}=1-\sin(\cos^{-1}x)$$

$$\Rightarrow \sin(\cos^{-1}x)=\frac{1}{2} \quad \Rightarrow \quad \cos^{-1}x=\frac{\pi}{6}$$

$$\Rightarrow x=\frac{\sqrt{3}}{2} \quad \Rightarrow \quad 12x^2=9$$

7. (9) $\tan^{-1}\left(x+\frac{3}{x}\right)-\tan^{-1}\left(x-\frac{3}{x}\right)=\tan^{-1}\frac{6}{x}$

$$\Rightarrow \tan^{-1}\left(\frac{\left(x+\frac{3}{x}\right)-\left(x-\frac{3}{x}\right)}{1+\left(x+\frac{3}{x}\right)\left(x-\frac{3}{x}\right)}\right)=\tan^{-1}\frac{6}{x}$$

$$\Rightarrow x^2-\frac{9}{x^2}=0 \quad \Rightarrow \quad x^4=9$$

8. (4) $f(x)=\sin^{-1}x+2\tan^{-1}x+(x+2)^2-3$

Domain of $f(x)$ is $[-1, 1]$.

Also $f(x)$ is an increasing function in the domain

$$\therefore p = f_{\min.}(x) = f(-1) = -\frac{\pi}{2} + 2\left(\frac{-\pi}{4}\right) + 1 - 3 = -\pi - 2$$

and $q = f_{\max.}(x) = f(1) = \frac{\pi}{2} + 2\left(\frac{\pi}{4}\right) + 9 - 6 = \pi + 6.$

Therefore, the range of $f(x)$ is $[-\pi - 2, \pi + 6]$.

Hence, $(p + q) = 4$.

9. (6) $T_n = \tan^{-1}\left(\frac{n+1-1}{1+(n+1)1}\right)$

$$= \tan^{-1}(n+1) - \tan^{-1}(n)$$

Hence, $S_n = \tan^{-1}(n+1) - \tan^{-1} 1$

$$= \tan^{-1}\left(\frac{n+1-1}{1+(n+1)\cdot 1}\right) = \left(\tan^{-1}\frac{n}{n+2}\right) = \frac{1}{2}\cos^{-1}\left(\frac{24}{145}\right)$$

$$\Rightarrow 2\left(\tan^{-1}\frac{n}{n+2}\right) = \cos^{-1}\left(\frac{24}{145}\right)$$

$$\Rightarrow \cos^{-1}\left(\frac{2(n+1)}{n^2+2n+2}\right) = \cos^{-1}\left(\frac{24}{145}\right)$$

$$\Rightarrow \left(\frac{2(n+1)}{n^2+2n+2}\right) = \left(\frac{24}{145}\right)$$

$$\Rightarrow 12(n+1)^2 - 145(n+1) + 12 = 0$$

$$\Rightarrow ((n+1) - 12)(12(n+1) - 1) = 0$$

$$\Rightarrow n + 1 = 12 \Rightarrow n = 11$$

10. (1) We have $g(x) = \sin^{-1}(\cos x) = \frac{\pi}{2} - \cos^{-1}(\cos x)$

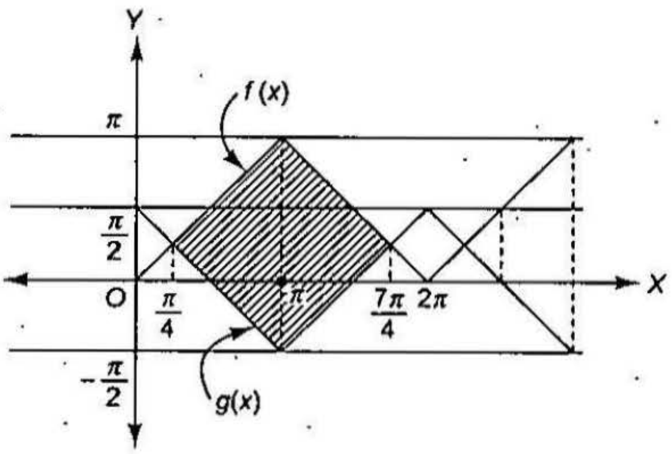


Fig. 4.34

Both the curves bound the regions of same area

in $\left[\frac{\pi}{4}, \frac{7\pi}{4}\right]$, $\left[\frac{9\pi}{4}, \frac{15\pi}{4}\right]$ and so on

Therefore, the required area = area of shaded square = $\frac{9\pi^2}{8} = \frac{a\pi^2}{b}$

$\therefore$ $a = 9$ and $b = 8 \Rightarrow a - b = 1$

11. (3) We must have $x(x+3) \geq 0$

$\Rightarrow x \geq 0$ or $x \leq -3$ (i)

Also, $-1 \leq x^2 + 3x + 1 \leq 1$

$\Rightarrow x(x+3) \leq 0 \Rightarrow -3 \leq x \leq 0$ (ii)

From Eqs. (i) and (ii), we get $x = \{0, -3\}$

Hence, required sum is 3.

12. (1) Given expression is defined only for $x = 1$ and -1

$\therefore f(1) = 1$ and $f(-1) = (1+\pi)(1+\pi) = (1+\pi)^2$

Hence, the least value is 1.

13. (3) $\cos^{-1}(x) + \cos^{-1}(2x) + \cos^{-1}(3x) = \pi$

$\Rightarrow \cos^{-1}(2x) + \cos^{-1}(3x) = \pi - \cos^{-1}(x) = \cos^{-1}(-x)$

$\Rightarrow \cos^{-1}[(2x)(3x) - \sqrt{1-4x^2}\sqrt{1-9x^2}\,] = \cos^{-1}(-x)$

$\Rightarrow 6x^2 - \sqrt{1-4x^2}\sqrt{1-9x^2} = -x$

$\Rightarrow (6x^2 + x)^2 = (1-4x^2)(1-9x^2)$

$\Rightarrow x^2 + 12x^3 = 1 - 13x^2$

$\Rightarrow 12x^3 + 14x^2 - 1 = 0$

$\Rightarrow a = 12;\ b = 14; c = 0$

$\Rightarrow b - a - c = 14 - 12 + 1 = 3$

14. (1) $\tan^{-1}(3x) + \tan^{-1}(5x) = \tan^{-1}(7x) + \tan^{-1}(2x)$

$\Rightarrow \tan^{-1}(3x) - \tan^{-1}(2x) = \tan^{-1}(7x) - \tan^{-1}(5x)$

$\Rightarrow \tan^{-1}\left(\frac{3x-2x}{1+6x^2}\right) = \tan^{-1}\left(\frac{7x-5x}{1+35x^2}\right)$

$\Rightarrow \frac{x}{1+6x^2} = \frac{2x}{1+35x^2}$

$\Rightarrow x = 0$ or $1 + 35x^2 = 2 + 12x^2$

$\Rightarrow x = 0$ or $x = \frac{1}{\sqrt{23}}$ or $-\frac{1}{\sqrt{23}}$

Archives

Subjective

1. $\cos(2\cos^{-1}x + \sin^{-1}x) = \cos\left(\cos^{-1}x + \frac{\pi}{2}\right)$

$$= -\sin(\cos^{-1}x) = -\sin\left(\sin^{-1}\sqrt{1-x^2}\right) = -\sqrt{1-x^2}$$

At $x = \frac{1}{5}$, value $= -\sqrt{1-\frac{1}{25}} = -\sqrt{\frac{24}{25}} = -\frac{2\sqrt{6}}{5}$

2. L.H.S. $= \cos(\tan^{-1}(\sin(\cot^{-1}x)))$

$$= \cos\left(\tan^{-1}\left(\sin\left(\sin^{-1}\frac{1}{\sqrt{1+x^2}}\right)\right)\right) \text{ if } x > 0$$

and $\cos\left(\tan^{-1}\left(\sin\left(\pi - \sin^{-1}\frac{1}{\sqrt{1+x^2}}\right)\right)\right)$ if $x < 0$

In each case, L.H.S. $= \cos\left(\tan^{-1}\frac{1}{\sqrt{1+x^2}}\right) = \cos\left(\cos^{-1}\sqrt{\frac{1+x^2}{2+x^2}}\right) = \sqrt{\frac{x^2+1}{x^2+2}}$

Objective

Fill in the blanks

1. $\theta = \tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}} + \tan^{-1}\sqrt{\frac{b(a+b+c)}{ca}} + \tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}$

$$\sqrt{\frac{a(a+b+c)}{bc}}\sqrt{\frac{b(a+b+c)}{ca}} = \frac{a+b+c}{c} = 1 + \frac{b}{c} + \frac{a}{c} > 1$$

$$\Rightarrow \quad \theta = \pi + \tan^{-1}\frac{\sqrt{\frac{a(a+b+c)}{bc}} + \sqrt{\frac{b(a+b+c)}{ca}}}{1 - \sqrt{\frac{a(a+b+c)}{bc}} + \sqrt{\frac{b(a+b+c)}{ca}}} + \tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}$$

$$= \pi + \tan^{-1}\frac{\sqrt{\frac{a+b+c}{c}}\left(\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}}\right)}{1 - \frac{a+b+c}{c}} + \tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}$$

$$= \pi + \tan^{-1}\left(-\sqrt{\frac{c(a+b+c)}{ab}}\right) + \tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}} = \pi \quad \Rightarrow \quad \tan\theta = 0$$

2. $\tan\left(2\tan^{-1}\frac{1}{5} - \frac{\pi}{4}\right) = \tan\left(\tan^{-1}\left(\frac{2/5}{1-(1/5)^2}\right) - \tan^{-1}(1)\right)$

$$= \tan\left(\tan^{-1}\left(\frac{5}{12}\right) - \tan^{-1}(1)\right)$$

$$= \tan\left(\tan^{-1}\left(\frac{(5/12)-1}{1+(5/12)}\right)\right)$$

$$= \tan(\tan^{-1}(-7/17)) = -7/17$$

3. We have,

$$A = 2\tan^{-1}(2\sqrt{2}-1)$$

$$= 2\tan^{-1}(2\times 1.414 - 1)$$

$$= 2\tan^{-1}(1.828) > 2\tan^{-1}\sqrt{3} = 2\pi/3 \quad \Rightarrow \quad A > (2\pi/3) \qquad \text{(i)}$$

Also, $B = 3\sin^{-1}(1/3) + \sin^{-1}(3/5) = \sin^{-1}\left[3\times\frac{1}{3} - 4\times\frac{1}{27}\right] + \sin^{-1}(3/5)$

$$= \sin^{-1}\left(\frac{23}{27}\right) + \sin^{-1}(0.6) = \sin^{-1}(0.852) + \sin^{-1}(0.6) < \sin^{-1}(\sqrt{3}/2) + \sin^{-1}(\sqrt{3}/2) = 2\pi/3$$

$$\Rightarrow \quad B < (2\pi/3) \qquad \text{(ii)}$$

From Eqs. (i) and (ii), we conclude $A > B$.

Multiple choice questions with one correct answer

1. d. $\tan\left(\cos^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\left(\frac{2}{3}\right)\right) = \tan\left(\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{2}{3}\right) = \tan\left(\tan^{-1}\left(\frac{(3/4)+(2/3)}{1-(3/4)\times(2/3)}\right)\right)$

$$= \frac{17}{12}\times\frac{12}{6} = \frac{17}{6}$$

2. e. The principal value of $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ = principal value of $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \pi/3$

3. d. $\tan\left(\cos^{-1}\frac{1}{5\sqrt{2}} - \sin^{-1}\frac{4}{\sqrt{17}}\right) = \tan[\tan^{-1}7 - \tan^{-1}4] = \tan\left(\tan^{-1}\left(\frac{3}{29}\right)\right) = \frac{3}{29}$

4. c. $\tan^{-1}\sqrt{[x(x+1)]} = (\pi/2) - \sin^{-1}\sqrt{(x^2+x+1)} = \cos^{-1}\sqrt{x^2+x+1} = \tan^{-1}\frac{\sqrt{-x^2-x}}{\sqrt{x^2+x+1}}$

$$\Rightarrow \quad \sqrt{x(x+1)} = \frac{\sqrt{-x^2-x}}{\sqrt{x^2+x+1}} \Rightarrow \quad x = 0, -1 \text{ are the only real solutions.}$$

5. b. $\sin^{-1}\left(x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right) + \cos^{-1}\left(x^2 - \frac{x^4}{2} + \frac{x^6}{4} + \cdots\right) = \frac{\pi}{2}$

$$\Rightarrow \quad \cos^{-1}\left(x^2 - \frac{x^4}{2} + \frac{x^6}{4}\cdots\right) = \frac{\pi}{2} - \sin^{-1}\left(x - \frac{x^2}{2} + \frac{x^3}{4}\cdots\right)$$

$$= \cos^{-1}\left(x - \frac{x^2}{2} + \frac{x^3}{4}\cdots\right)$$

$\Rightarrow \quad x^2 - \frac{x^4}{2} + \frac{x^6}{4} \cdots = x - \frac{x^2}{2} + \frac{x^3}{4} \cdots$

We have G. P. of infinite terms on both sides.

$\therefore \quad \frac{x}{1-\left(-\frac{x}{2}\right)} = \frac{x^2}{1-\left(\frac{-x^2}{2}\right)} \quad \Rightarrow \quad \frac{2x}{2+x} = \frac{2x^2}{2+x^2} \quad \Rightarrow \quad 2x + x^3 = 2x^2 + x^3$

$\Rightarrow \quad x(x-1) = 0 \quad \Rightarrow \quad x = 0, 1 \text{ but } 0 < |x| < \sqrt{2} \quad \Rightarrow \quad x = 1$

6. a. For $f(x) = \sqrt{\sin^{-1}(2x) + \frac{\pi}{6}}$ to be defined and real $\sin^{-1} 2x + (\pi/6) \geq 0$

$\Rightarrow \quad \sin^{-1} 2x \geq -\frac{\pi}{6}$ (i)

But we know that $-\pi/2 \leq \sin^{-1} 2x \leq \pi/2$ (ii)

Combining Eqs. (i) and (ii), we get

$-\pi/6 \leq \sin^{-1} 2x \leq \pi/2$

$\Rightarrow \quad \sin(-\pi/6) \leq 2x \leq \sin(\pi/2) \quad \Rightarrow \quad -1/2 \leq 2x \leq 1$

$\Rightarrow \quad -1/4 \leq x \leq 1/2$

$\therefore \quad D_f = \left[-\frac{1}{4}, \frac{1}{2}\right]$

7. d. $\sin\left[\cot^{-1}(x+1)\right] = \sin\left(\sin^{-1} \frac{1}{\sqrt{x^2+2x+2}}\right) = \frac{1}{\sqrt{x^2+2x+2}}$

$\cos\left(\tan^{-1} x\right) = \cos\left(\cos^{-1} \frac{1}{\sqrt{1+x^2}}\right) = \frac{1}{\sqrt{1+x^2}}$

Thus, $\frac{1}{\sqrt{x^2+2x+2}} = \frac{1}{\sqrt{1+x^2}} \quad \Rightarrow \quad x^2 + 2x + 2 = 1 + x^2 \Rightarrow x = -\frac{1}{2}$

Match the following type

1. a → p; b → q; c → p; d → s

$\sin^{-1}(ax) + \cos^{-1} y + \cos^{-1}(bxy) = \frac{\pi}{2}$

$\Rightarrow \quad \cos^{-1} y + \cos^{-1}(bxy) = \frac{\pi}{2} - \sin^{-1}(ax) = \cos^{-1}(ax)$

Let $\cos^{-1} y = \alpha$, $\cos^{-1}(bxy) = \beta$, $\cos^{-1}(ax) = \gamma$

$\Rightarrow \quad y = \cos\alpha,\ bxy = \cos\beta,\ ax = \cos\gamma$

Therefore, we get $\alpha + \beta = \gamma$. $\Rightarrow \quad \cos(\gamma - \alpha) = bxy \quad \Rightarrow \quad \cos\gamma\cos\alpha + \sin\gamma\sin\alpha = bxy$

$\Rightarrow \quad (a-b)xy = -\sin\alpha\sin\gamma \quad \Rightarrow \quad (a-b)^2 x^2 y^2 = \sin^2\alpha\sin^2\gamma = (1-\cos^2\alpha)(1-\cos^2\gamma)$

$\Rightarrow \quad (a-b)^2 x^2y^2 = (1-a^2x^2)(1-y^2)$ (i)

a. For $a = 1$, $b = 0$, Eq. (i) reduces to

$x^2 y^2 = (1-x^2)(1-y^2) \Rightarrow x^2 + y^2 = 1$

b. For $a = 1$, $b = 1$, Eq. (i) becomes $(1-x^2)(1-y^2) = 0$

$\Rightarrow \quad (x^2-1)(y^2-1) = 0$

c. For $a = 1$, $b = 2$, Eq. (i) reduces to

$x^2y^2 = (1-x^2)(1-y^2)$

$\Rightarrow \quad x^2 + y^2 = 1$

d. For $a = 2$, $b = 2$, Eq. (i) reduces to $0 = (1-4x^2)(1-y^2)$

$\Rightarrow \quad (4x^2-1)(y^2-1) = 0$