Chapter - Matrices

Topic-1: Order of Matrices, Types of Matrices, Addition & Subtraction of Matrices, Scalar Multiplication of Matrices, Multiplication of Matrices

1 MCQs with One Correct Answer

1. If $M = \begin{pmatrix} \frac{5}{2} & \frac{3}{2} \\ -\frac{3}{2} & -\frac{1}{2} \end{pmatrix}$, then which of the following matrices is equal to M^{2022}? **[Adv. 2022]**

(a) $\begin{pmatrix} 3034 & 3033 \\ -3033 & -3032 \end{pmatrix}$ (b) $\begin{pmatrix} 3034 & -3033 \\ 3033 & -3032 \end{pmatrix}$

(c) $\begin{pmatrix} 3033 & -3032 \\ -3032 & -3031 \end{pmatrix}$ (d) $\begin{pmatrix} 3032 & 3031 \\ -3031 & -3030 \end{pmatrix}$

2. How many 3×3 matrices M with entries from {0, 1, 2} are there, for which the sum of the diagonal entries of $M^T M$ is 5? **[Adv. 2017]**

(a) 126 (b) 198 (c) 162 (d) 135

3. Let $P = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix}$ and I be the identity matrix of order 3.

If $Q = [q_{ij}]$ is a matrix such that $P^{50} - Q = I$, then $\frac{q_{31} + q_{32}}{q_{21}}$ equals **[Adv. 2016]**

(a) 52 (b) 103 (c) 201 (d) 205

4. If A and B are square matrices of equal degree, then which one is correct among the followings? **[1995S]**

(a) $A+B=B+A$ (b) $A+B=A-B$

(c) $A-B=B-A$ (d) $AB=BA$

2 Integer Value Answer/ Non-Negative Integer

5. Let β be a real number. Consider the matrix

$$A = \begin{pmatrix} \beta & 0 & 1 \\ 2 & 1 & -2 \\ 3 & 1 & -2 \end{pmatrix}.$$

If $A^7 - (\beta - 1)A^6 - \beta A^5$ is a singular matrix, then the value of 9β is ________. **[Adv. 2022]**

6. Let $z = \frac{-1+\sqrt{3}i}{2}$, where $i = \sqrt{-1}$, and r, s ∈ {1, 2, 3}. Let

$P = \begin{bmatrix} (-z)^r & z^{2s} \\ z^{2s} & z^r \end{bmatrix}$ and I be the identity matrix of order 2.

Then the total number of ordered pairs (r, s) for which $P^2 = -I$ is **[Adv. 2016]**

7. Let M be a 3×3 matrix satisfying

$M\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$, $M\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, and $M\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}$. Then the sum of the diagonal entries of M is **[2011]**

6 MCQs with One or More than One Correct Answer

8. Let ω be a complex cube root of unity with ω ≠ 1 and $P = [p_{ij}]$ be a $n \times n$ matrix with $p_{ij} = \omega^{i+j}$. Then $p^2 \neq 0$, when n = **[Adv. 2013]**

(a) 57 (b) 55 (c) 58 (d) 56

Topic-2: Transpose of Matrices, Symmetric & Skew Symmetric Matrices, Inverse of a Matrix by Elementary Row Operations

1 MCQs with One Correct Answer

1. If P is a 3×3 matrix such that $P^T = 2P + I$, where P^T is the transpose of P and I is the 3×3 identity matrix, then there exists a column matrix $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ such that **[2012]**

(a) $PX = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (b) $PX = X$

(c) $PX = 2X$ (d) $PX = -X$

2. If $P = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $Q = PAP^T$ and $X = P^T Q^{2005} P$ then X is equal to **[2005S]**

(a) $\begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 4+2005\sqrt{3} & 6015 \\ 2005 & 4-2005\sqrt{3} \end{bmatrix}$

(c) $\frac{1}{4}\begin{bmatrix} 2+\sqrt{3} & 1 \\ -1 & 2-\sqrt{3} \end{bmatrix}$

(d) $\frac{1}{4}\begin{bmatrix} 2005 & 2-\sqrt{3} \\ 2+\sqrt{3} & 2005 \end{bmatrix}$

2 Integer Value Answer/ Non-Negative Integer

3. Let $R = \left\{ \begin{pmatrix} a & 3 & b \\ c & 2 & d \\ 0 & 5 & 0 \end{pmatrix} : a, b, c, d \in \{0, 3, 5, 7, 11, 13, 17, 19\} \right\}$.

Then the number of invertible matrices in R is **[Adv. 2023]**

6 MCQs with One or More than One Correct Answer

4. Let X and Y be two arbitrary, 3×3, non-zero, skew-symmetric matrices and Z be an arbitrary 3×3, non zero, symmetric matrix. Then which of the following matrices is (are) skew symmetric? **[Adv. 2015]**

(a) $Y^3Z^4 - Z^4Y^3$ (b) $X^{44} + Y^{44}$

(c) $X^4Z^3 - Z^3X^4$ (d) $X^{23} + Y^{23}$

7 Match the Following

5. Match the Statements/Expressions in **Column I** with the Statements / Expressions in **Column II** and indicate your answer by darkening the appropriate bubbles in the 4×4 matrix given in the ORS. **[2008]**

Column I	Column II
(A) The minimum value of $\frac{x^2+2x+4}{x+2}$ is	(p) 0
(B) Let A and B be 3×3 matrices of real numbers, where A is symmetric, B is skew-symmetric, and $(A+B)(A-B) = (A-B)(A+B)$. If $(AB)^t = (-1)^k AB$, where $(AB)^t$ is the transpose of the matrix AB, then the possible values of k are	(q) 1
(C) Let $a = \log_3 \log_3 2$. An integer k satisfying $1 < 2^{(-k+3^{-a})} < 2$, must be less than	(r) 2
(D) If $\sin\theta = \cos\phi$, then the possible values of $\frac{1}{\pi}\left(\theta \pm \phi - \frac{\pi}{2}\right)$ are	(s) 3

Answer Key

Topic-1 : Order of Matrices, Types of Matrices, Addition & Subtraction of Matrices, Scalar Multiplication of Matrices, Multiplication of Matrices

1. (a) **2.** (b) **3.** (b) **4.** (a) **5.** (3) **6.** (1) **7.** (9) **8.** (b, c, d)

Topic-2 : Transpose of Matrices, Symmetric & Skew Symmetric Matrices, Inverse of a Matrix by Elementary Row Operations

1. (d) **2.** (a) **3.** (3780) **4.** (c, d) **5.** $A \to r, B \to q, s; C \to r, s; D \to p, r$

Hints & Solutions

Topic-1: Order of Matrices, Types of Matrices, Addition & Subtraction of Matrices, Scalar Multiplication of Matrices, Multiplication of Matrices

1. (a) Given that $M = \begin{pmatrix} \frac{5}{2} & \frac{3}{2} \\ -\frac{3}{2} & -\frac{1}{2} \end{pmatrix} \Rightarrow M^2 = \begin{pmatrix} 4 & 3 \\ -3 & -2 \end{pmatrix}$

$$\Rightarrow M^3 = \begin{pmatrix} \frac{11}{2} & \frac{9}{2} \\ -\frac{9}{2} & -\frac{7}{2} \end{pmatrix} \Rightarrow M^4 = \begin{pmatrix} 7 & 6 \\ -6 & -5 \end{pmatrix} \text{ and soon}$$

$$\Rightarrow M^n = \begin{pmatrix} \frac{3n}{2}+1 & \frac{3n}{2} \\ -\frac{3n}{2} & -\frac{3n}{2}+1 \end{pmatrix}$$

$$\text{Now, } M^n = \begin{pmatrix} \frac{3\times 2022}{2}+1 & \frac{3\times 2022}{2} \\ -\frac{3\times 2022}{2} & -\frac{3\times 2022}{2}+1 \end{pmatrix}$$

$= \begin{pmatrix} 3034 & 3033 \\ -3033 & -3032 \end{pmatrix}$ $\therefore$ option (a) is correct

2. (b) Let $M = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}$ where $a_i \in \{0, 1, 2\}$

Then $M^TM = \begin{bmatrix} a_1 & a_4 & a_7 \\ a_2 & a_5 & a_8 \\ a_3 & a_6 & a_9 \end{bmatrix}\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}$

Sum of the diagonal entries in $M^TM = 5$

$\Rightarrow$ $(a_1^2 + a_4^2 + a_7^2) + (a_2^2 + a_5^2 + a_8^2) + (a_3^2 + a_6^2 + a_9^2) = 5$, which is possible when

Case I: Five a_i's are 1 and four a_i's are zero

Which can be done in ${}^9C_4$ ways

$$= \frac{9\times 8\times 7\times 6}{4\times 3\times 2\times 1} = 126$$

Case II: One a_i is 1, one a_i is 2 and rest seven a_i's are zero

It can be done in ${}^9C_1 \times {}^8C_1 = 9 \times 8 = 72$ ways

$\therefore$ Total no. of ways = 126 + 72 = 198.

3. (b) $P = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix} = I + \begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 16 & 4 & 0 \end{bmatrix} = I + A$

where $A = \begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 16 & 4 & 0 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 16 & 0 & 0 \end{bmatrix}$

and $A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\therefore$ $A^n = O, \forall n \geq 3$

Now $P^{50} = (I + A)^{50} = {}^{50}C_0\, I^{50} + {}^{50}C_1\, I^{49} A + {}^{50}C_2\, I^{48} A^2 + O$

$= I + 50A + 25 \times 49 A^2.$

$\therefore$ $Q = P^{50} - I = 50A + 25 \times 49 A^2.$

$\Rightarrow q_{21} = 50 \times 4 = 200,\ q_{31} = 50 \times 16 + 25 \times 49 \times 16 = 20400$

and $q_{32} = 50 \times 4 = 200$

$$\therefore \quad \frac{q_{31}+q_{32}}{q_{21}} = \frac{20600}{200} = 103$$

4. (a) Since A and B are square matrices of the same degree, therefore matrices A and B can be added or subtracted or multiplied. By algebra of matrices the only correct option is $A + B = B + A$.

5. (3) $A = \begin{pmatrix} \beta & 0 & 1 \\ 2 & 1 & -2 \\ 3 & 1 & -2 \end{pmatrix}$, $\therefore |A| = -1$

Given that $A^7 - (\beta - 1)A^6 - \beta A^5$ is a singular matrix

$|A^5|\,|A^2 - (\beta - 1)A - \beta I| = 0$

$\Rightarrow$ $|A^5|\,|(A + I)(A - \beta I)| = 0$

$\therefore$ $|A^5|\,|A + I|\,|A - \beta I| = 0$ $[\because |A^5| = |A|^5]$

As $|A| \neq 0$; $|A + I|$ or $|A - \beta I| = 0$

$$\Rightarrow \quad |A+I| = \begin{vmatrix} \beta+1 & 0 & 1 \\ 2 & 2 & -2 \\ 3 & 1 & -1 \end{vmatrix} = 0$$

$-1 = 0$ (Rejected) $\{|A + 1| \neq 0\}$

$$\therefore \quad |A - \beta I| = \begin{vmatrix} 0 & 0 & 1 \\ 2 & 1-\beta & -2 \\ 3 & 1 & -2-\beta \end{vmatrix} = 0$$

$\Rightarrow$ $2 - 3(1 - \beta) = 0 \Rightarrow 2 - 3 + 3\beta = 0$

$\Rightarrow$ $\beta = \frac{1}{3}$ $\therefore 9\beta = 3$

6. (1) $z = \dfrac{-1+i\sqrt{3}}{2} \Rightarrow z^3 = 1$ and $1 + z + z^2 = 0$

$$P^2 = \begin{bmatrix} (-z)^r & z^{2s} \\ z^{2s} & z^r \end{bmatrix}\begin{bmatrix} (-z)^r & z^{2s} \\ z^{2s} & z^r \end{bmatrix}$$

$$= \begin{bmatrix} z^{2r}+z^{4s} & z^{2s}\left((-z)^r+z^r\right) \\ z^{2s}\left((-z)^r+z^r\right) & z^{4s}+z^{2r} \end{bmatrix}$$

For $P^2 = -I$, we should have

$z^{2r} + z^{4s} = -1$ and $z^{2s}((-z)^r + z^r) = 0$

$\Rightarrow z^{2r} + z^{4s} + 1 = 0$ and $(-z)^r + z^r = 0$

$\Rightarrow$ r is odd and s = r but not a multiple of 3,

which is possible when s = r = 1 $\therefore$ only one pair is there.

7. **(9)**

Let $M = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ then $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$

$\Rightarrow b_1 = -1, b_2 = 2, b_3 = 3$

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$\Rightarrow a_1 - b_1 = 1, a_2 - b_2 = 1, a_3 - b_3 = -1 \Rightarrow a_1 = 0, a_2 = 3, a_3 = 2$

and $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}$

$\Rightarrow a_3 + b_3 + c_3 = 12 \Rightarrow c_3 = 7$

$\therefore$ Sum of diagonal elements $= a_1 + b_2 + c_3 = 0 + 2 + 7 = 9$

8. **(b, c, d)**

For $n = 3$, $P = \begin{bmatrix} w^2 & w^3 & w^4 \\ w^3 & w^4 & w^5 \\ w^4 & w^5 & w^6 \end{bmatrix}$ and $P^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

It shows $P^2 = 0$ if n is a multiple of 3.

So for $P^2 \neq 0$, n should not be a multiple of 3, hence n can take values 55, 58 and 56

Topic-2: Transpose of Matrices, Symmetric & Skew Symmetric Matrices, Inverse of a Matrix by Elementary Row Operations

1. **(d)** $P^T = 2P + I$

$\Rightarrow P = 2P^T + I \Rightarrow P = 2(2P + I) + I$

$\Rightarrow P = 4P + 3I \Rightarrow P + I = 0 \Rightarrow PX + X = 0 \Rightarrow PX = -X$

2. **(a)** Given : $P = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$,

$Q = PAP^T$ and $X = P^T Q^{2005} P$

Now $Q = PAP^T \Rightarrow Q^2 = (PAP^T)(PAP^T)$

$= PA(P^T P)AP^T = PA(IA)P^T = PA^2P^T$

Proceeding in the same way, $Q^{2005} = PA^{2005}P^T$

Now, $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

Proceeding in the same way, $A^{2005} = \begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$

Now, $X = P^T Q^{2005} P = P^T(PA^{2005}P^T)P$

$= (P^T P)A^{2005}(P^T P) = IA^{2005}I = A^{2005} = \begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$

3. **(3780)**

Let us calculate when |R| = 0

Case -I : ad = bc = 0

If ad = 0 (When none of a & d is 0)

Total $= 8^2 - 1 = 15$ ways

Similarly bc = 0 $\Rightarrow$ 15 ways

$\therefore$ 15 × 15 = 225 ways of ad = bc = 0

Case-II : ad = bc ≠ 0

either a = d = b = c OR a ≠ d, b ≠ d but ad = bc

${}^7C_1 = 7$ ways or ${}^7C_2 \times 2 \times 2 = 84$ ways

Total = 7 + 84 = 91 ways

$\therefore$ |R| = 0 in 225 + 91 = 316 ways

$|R| \neq 0$ in $8^4 - 316 = 3780$.

4. **(c, d)** $X' = -X, Y' = -Y, Z' = Z$

$(Y^3Z^4 - Z^4Y^3)' = (Z^4)'(Y^3)' - (Y^3)'(Z^4)'$

$= (Z')^4(Y')^3 - (Y')^3(Z')^4$

$= -Z^4Y^3 + Y^3Z^4 = Y^3Z^4 - Z^4Y^3$

Therefore $(Y^3Z^4 - Z^4Y^3)$ is a symmetric matrix.

Similarly $X^{44} + Y^{44}$ is a symmetric matrix and $X^4Z^3 - Z^3X^4$ and $X^{23} + Y^{23}$ are skew symmetric matrices.

5. **A → r, B → q, s; C → r, s; D → p, r**

(A) Let $y = \frac{x^2+2x+4}{x+2} \Rightarrow \frac{dy}{dx} = \frac{x^2+4x}{(x+2)^2}$

Now put $\frac{dy}{dx} = 0 \Rightarrow \frac{x^2+4x}{(x+2)^2} = 0 \Rightarrow x = 0, -4$

Now $\frac{d^2y}{dx^2} = \frac{8}{(x+2)^3}$ At $x = 0$, $\frac{d^2y}{dx^2} = 1 > 0$

$\therefore$ y is min when $x = 0$

$\therefore$ minimum value of y i.e., $\frac{x^2+2x+4}{x+2}$ is 2.

$\therefore$ (A) → (r)

(B) Since A is symmetric and B is skew symmetric matrix,

$\therefore$ $A^t = A$ and $B^t = -B$ ...(i)

Now $(A + B)(A - B) = (A - B)(A + B)$

$\Rightarrow A^2 - AB + BA - B^2 = A^2 + AB - BA - B^2$

$\Rightarrow 2BA = 2AB \Rightarrow AB = BA$ ...(ii)

Also $(AB)^t = (-1)^k AB$

$\Rightarrow (BA)^t = (-1)^k AB$ (using (ii))

$\Rightarrow A^t B^t = (-1)^k AB$

$\Rightarrow -AB = (-1)^k AB$ [using (i)]

$\Rightarrow k$ should be an odd number

$\therefore$ (B) $\rightarrow$ (q), (s)

(C) $a = \log_3 \log_3 2 \Rightarrow \log_3 2 = 3^a$

$$\Rightarrow \frac{1}{\log_2 3} = 3^a \Rightarrow \log_2 3 = 3^{-a}$$

$\therefore \quad 3 = 2^{(3^{-a})}$ ...(i)

Now $1 < 2^{(-k+3^{-a})} < 2 \Rightarrow 1 < 2^{-k}.2^{3^{-a}} < 2$

$\Rightarrow 1 < 2^{-k}.3 < 2$ (using (i))

$$\Rightarrow \frac{1}{3}.< 2^{-k} < \frac{2}{3} \Rightarrow \frac{3}{2} < 2^k < 3 \Rightarrow k = 1$$

$\therefore k$ is less than 2 and 3

$\therefore$ (C) $\rightarrow$ (r), (s).

(D) $\sin\theta = \cos\phi \Rightarrow \cos\left(\frac{\pi}{2} - \theta\right) = \cos\phi$

$$\Rightarrow \frac{\pi}{2} - \theta = 2n\pi \pm \phi,\ n \in Z \Rightarrow \theta \pm \phi - \frac{\pi}{2} = -2n\pi$$

$$\Rightarrow \frac{1}{\pi}\left(\theta \pm \phi - \frac{\pi}{2}\right) = -2n$$

For n = 0, –1; possible values of $\frac{1}{\pi}\left(\theta \pm \phi - \frac{\pi}{2}\right)$ are 0 and 2.

$\therefore$ D$\rightarrow$ (p) ,(r).