4

Coupled Oscillations

The preceding chapters have shown in some detail how a single vibrating system will behave. Oscillators, however, rarely exist in complete isolation; wave motion owes its existence to neighbouring vibrating systems which are able to transmit their energy to each other.

Such energy transfer takes place, in general, because two oscillators share a common component, capacitance or stiffness, inductance or mass, or resistance. Resistance coupling inevitably brings energy loss and a rapid decay in the vibration, but coupling by either of the other two parameters consumes no power, and continuous energy transfer over many oscillators is possible. This is the basis of wave motion.

We shall investigate first a mechanical example of stiffness coupling between two pendulums. Two atoms set in a crystal lattice experience a mutual coupling force and would be amenable to a similar treatment. Then we investigate an example of mass, or inductive, coupling, and finally we consider the coupled motion of an extended array of oscillators which leads us naturally into a discussion on wave motion.

Stiffness (or Capacitance) Coupled Oscillators

Figure 4.1 shows two identical pendulums, each having a mass m suspended on a light rigid rod of length l. The masses are connected by a light spring of stiffness s whose natural length equals the distance between the masses when neither is displaced from equilibrium. The small oscillations we discuss are restricted to the plane of the paper.

If x and y are the respective displacements of the masses, then the equations of motion are

$$m\ddot{x} = -mg\frac{x}{l} - s(x - y)$$

and

$$m\ddot{y} = -mg\frac{y}{l} + s(x - y)$$

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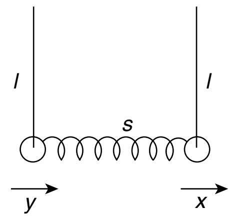


Figure 4.1 Two identical pendulums, each a light rigid rod of length l supporting a mass m and coupled by a weightless spring of stiffness s and of natural length equal to the separation of the masses at zero displacement

These represent the normal simple harmonic motion terms of each pendulum plus a coupling term $s(x - y)$ from the spring. We see that if $x > y$ the spring is extended beyond its normal length and will act against the acceleration of x but in favour of the acceleration of y.

Writing $\omega_0^2 = g/l$, where ω_0 is the natural vibration frequency of each pendulum, gives

$$\ddot{x} + \omega_0^2 x = -\frac{s}{m}(x - y) \tag{4.1}$$

$$\ddot{y} + \omega_0^2 y = -\frac{s}{m}(y - x) \tag{4.2}$$

Instead of solving these equations directly for x and y we are going to choose two new coordinates

$$X = x + y$$
$$Y = x - y$$

The importance of this approach will emerge as this chapter proceeds. Adding equations (4.1) and (4.2) gives

$$\ddot{x} + \ddot{y} + \omega_0^2(x + y) = 0$$

that is

$$\ddot{X} + \omega_0^2 X = 0$$

and subtracting (4.2) from (4.1) gives

$$\ddot{Y} + (\omega_0^2 + 2s/m)Y = 0$$

The motion of the coupled system is thus described in terms of the two coordinates X and Y, each of which has an equation of motion which is simple harmonic.

If $Y = 0$, $x = y$ at all times, so that the motion is completely described by the equation

$$\ddot{X} + \omega_0^2 X = 0$$

then the frequency of oscillation is the same as that of either pendulum in isolation and the stiffness of the coupling has no effect. This is because both pendulums are always swinging in phase (Figure 4.2a) and the light spring is always at its natural length.

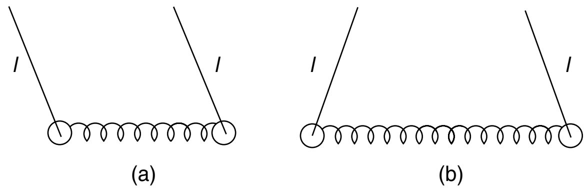


Figure 4.2 (a) The 'in phase' mode of vibration given by $\ddot{X} + \omega_0^2 X = 0$, where X is the normal coordinate $X = x + y$ and $\omega_0^2 = g/l$. (b) 'Out of phase' mode of vibration given by $\ddot{Y} + (\omega_0^2 + 2s/m)$ where Y is the normal coordinate $Y = x - y$

If $X = 0$, $x = -y$ at all times, so that the motion is completely described by

$$\ddot{Y} + (\omega_0^2 + 2s/m)Y = 0$$

The frequency of oscillation is greater because the pendulums are always out of phase (Figure 4.2b) so that the spring is either extended or compressed and the coupling is effective.

Normal Coordinates, Degrees of Freedom and Normal Modes of Vibration

The significance of choosing X and Y to describe the motion is that these parameters give a very simple illustration of normal coordinates.

- Normal coordinates are coordinates in which the equations of motion take the form of a set of linear differential equations with constant coefficients in which each equation contains *only one* dependent variable (our simple harmonic equations in X and Y).
- A vibration involving only one dependent variable X (or Y) is called a *normal mode of vibration* and has its own *normal frequency*. In such a *normal mode* all components of the system oscillate with the same *normal frequency*.
- The total energy of an undamped system may be expressed as a sum of the squares of the normal coordinates multiplied by constant coefficients and a sum of the squares of the first time derivatives of the coordinates multiplied by constant coefficients. The energy of a coupled system when the X and Y modes are both vibrating would then be expressed in terms of the squares of the velocities and displacements of X and Y.
- The importance of the normal modes of vibration is that they are entirely independent of each other. The energy associated with a normal mode is *never exchanged* with another mode; this is why we can add the energies of the separate modes to give the total energy. If only one mode vibrates the second mode of our system will always be at rest, acquiring no energy from the vibrating mode.
- Each independent way by which a system may acquire energy is called a *degree of freedom* to which is assigned its own particular normal coordinate. The number of such

different ways in which the system can take up energy defines its number of degrees of freedom and its number of normal coordinates. Each harmonic oscillator has two degrees of freedom, it may take up both potential energy (normal coordinate X) and kinetic energy (normal coordinate $\dot{X}$). In our two normal modes the energies may be written

$$E_X = a\dot{X}^2 + bX^2 \tag{4.3a}$$

and

$$E_Y = c\dot{Y}^2 + dY^2 \tag{4.3b}$$

where a, b, c and d are constant.

Our system of two coupled pendulums has, then, four degrees of freedom and four normal coordinates.

Any configuration of our coupled system may be represented by the super-position of the two normal modes

$$X = x + y = X_0 \cos(\omega_1 t + \phi_1)$$

and

$$Y = x - y = Y_0 \cos(\omega_2 t + \phi_2)$$

where X_0 and Y_0 are the normal mode amplitudes, whilst $\omega_1^2 = g/l$ and $\omega_2^2 = (g/l + 2s/m)$ are the normal mode frequencies. To simplify the discussion let us choose

$$X_0 = Y_0 = 2a$$

and put

$$\phi_1 = \phi_2 = 0$$

The pendulum displacements are then given by

$$x = \tfrac{1}{2}(X + Y) = a\cos\omega_1 t + a\cos\omega_2 t$$

and

$$y = \tfrac{1}{2}(X - Y) = a\cos\omega_1 t - a\cos\omega_2 t$$

with velocities

$$\dot{x} = -a\omega_1 \sin\omega_1 t - a\omega_2 \sin\omega_2 t$$

and

$$\dot{y} = -a\omega_1 \sin\omega_1 t + a\omega_2 \sin\omega_2 t$$

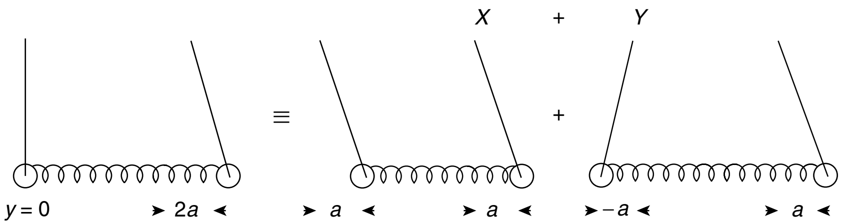


Figure 4.3 The displacement of one pendulum by an amount $2a$ is shown as the combination of the two normal coordinates $X + Y$

Now let us set the system in motion by displacing the right hand mass a distance $x = 2a$ and releasing both masses from rest so that $\dot{x} = \dot{y} = 0$ at time $t = 0$.

Figure 4.3 shows that our initial displacement $x = 2a$, $y = 0$ at $t = 0$ may be seen as a combination of the 'in phase' mode ($x = y = a$ so that $x + y = X_0 = 2a$) and of the 'out of phase' mode ($x = -y = a$ so that $Y_0 = 2a$). After release, the motion of the right hand pendulum is given by

$$\begin{aligned} x &= a\cos\omega_1 t + a\cos\omega_2 t \\ &= 2a\cos\frac{(\omega_2 - \omega_1)t}{2}\cos\frac{(\omega_1 + \omega_2)t}{2} \end{aligned}$$

and that of the left hand pendulum is given by

$$\begin{aligned} y &= a\cos\omega_1 t - a\cos\omega_2 t \\ &= -2a\sin\frac{(\omega_1 - \omega_2)t}{2}\sin\frac{(\omega_1 + \omega_2)t}{2} \\ &= 2a\sin\frac{(\omega_2 - \omega_1)t}{2}\sin\frac{(\omega_1 + \omega_2)t}{2} \end{aligned}$$

If we plot the behaviour of the individual masses by showing how x and y change with time (Figure 4.4), we see that after drawing the first mass aside a distance $2a$ and releasing it x follows a consinusoidal behaviour at a frequency which is the average of the two normal mode frequencies, but its amplitude varies cosinusoidally with a low frequency which is half the difference between the normal mode frequencies. On the other hand, y, which started at zero, vibrates sinusoidally with the average frequency but its amplitude builds up to $2a$ and then decays sinusoidally at the low frequency of half the difference between the normal mode frequencies. In short, the y displacement mass acquires all the energy of the x displacement mass which is stationary when y is vibrating with amplitude $2a$, but the energy is then returned to the mass originally displaced. This *complete* energy exchange is only possible when the masses are identical and the ratio $(\omega_1 + \omega_2)/(\omega_2 - \omega_1)$ is an integer, otherwise neither will ever be quite stationary. The slow variation of amplitude at half the normal mode frequency difference is the phenomenon of 'beats' which occurs between two oscillations of nearly equal frequencies. We shall discuss this further in the section on wave groups in Chapter 5.

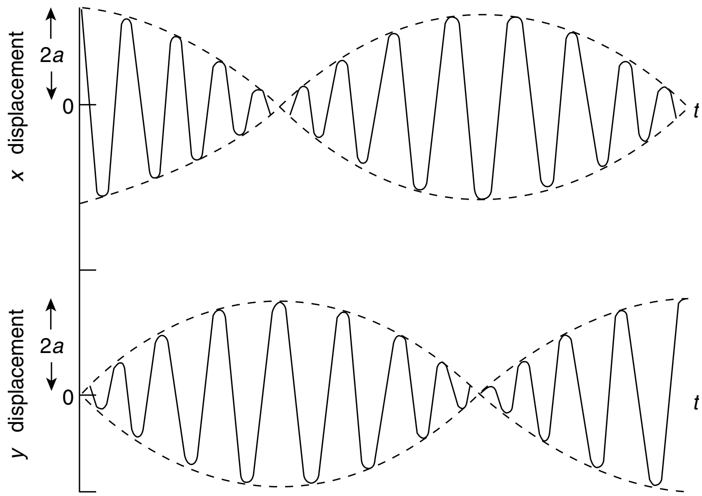


Figure 4.4 Behaviour with time of individual pendulums, showing complete energy exchange between the pendulums as x decreases from $2a$ to zero whilst y grows from zero to $2a$

The important point to recognize, however, is that although the *individual* pendulums may exchange energy, there is *no* energy exchange between the normal modes. Figure 4.3 showed the initial configuration $x = 2a$, $y = 0$, decomposed into the X and Y modes. The higher frequency of the Y mode ensures that after a number of oscillations the Y mode will have gained half a vibration (a phase of π rad) on the X mode; this is shown in Figure 4.5. The combination of the X and Y modes then gives y the value of $2a$ and $x = 0$, and the process is repeated. When Y gains another half vibration then x equals $2a$ again. The pendulums may exchange energy; the normal modes do not.

To reinforce the importance of normal modes and their coordinates let us return to equations (4.3a) and (4.3b). If we modify our normal coordinates to read

$$X_q = \left(\frac{m}{2}\right)^{1/2}(x+y) \quad \text{and} \quad Y_q = \left(\frac{m}{2}\right)^{1/2}(x-y)$$

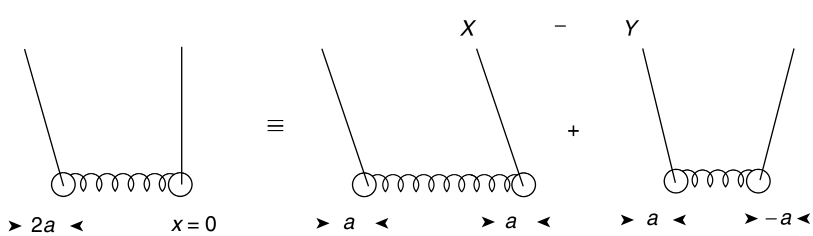


Figure 4.5 The faster vibration of the Y mode results in a phase gain of π rad over the X mode of vibration, to give $y = 2a$, which is shown here as a combination of the normal modes $X - Y$

then we find that the kinetic energy in those equations becomes

$$E_{\mathrm{k}} = T = a\dot{X}^2 + c\dot{Y}^2 = \frac{1}{2}\dot{X}_q^2 + \frac{1}{2}\dot{Y}_q^2 \tag{4.4a}$$

and the potential energy

$$\begin{aligned} V = bX^2 + dY^2 &= \frac{1}{2}\left(\frac{g}{l}\right)X_q^2 + \frac{1}{2}\left(\frac{g}{l} + \frac{2s}{m}\right)Y_q^2 \\ &= \frac{1}{2}\omega_0^2 X_q^2 + \frac{1}{2}\omega_s^2 Y_q^2, \end{aligned} \tag{4.4b}$$

where $\omega_0^2 = g/l$ and $\omega_s^2 = g/l + 2s/m$.
Note that the coefficients of X_q^2 and Y_q^2 depend only on the mode frequencies and that the properties of individual parts of the system are no longer explicit.

The total energy of the system is the sum of the energies of each separate excited mode for there are no cross products $X_q Y_q$ in the energy expression of our example, i.e.,

$$E = T + V = \left(\frac{1}{2}\dot{X}_q^2 + \frac{1}{2}\omega_0^2 X_q^2\right) + \left(\frac{1}{2}\dot{Y}_q^2 + \frac{1}{2}\omega_s^2 Y_q^2\right)$$

Atoms in polyatomic molecules behave as the masses of our pendulums; the normal modes of two triatomic molecules CO_2 and H_2O are shown with their frequencies in Figure 4.6. Normal modes and their vibrations will occur frequently throughout this book.

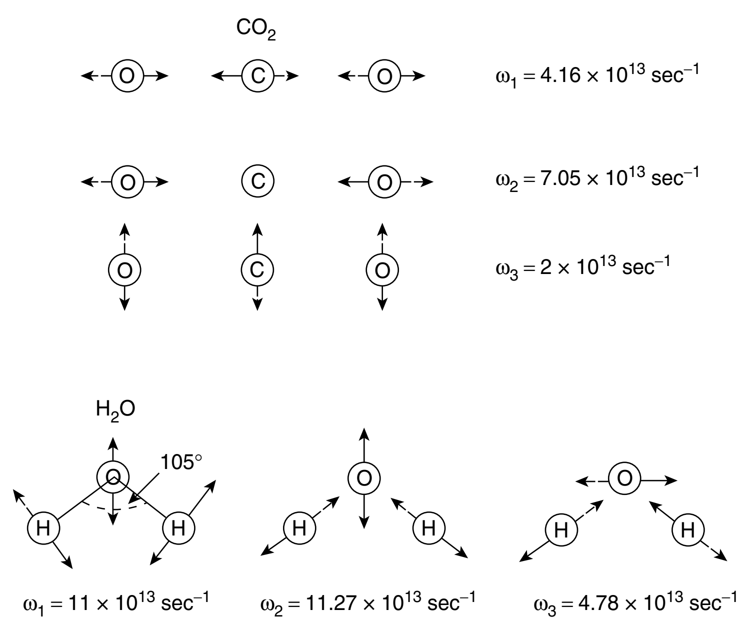


Figure 4.6 Normal modes of vibration for triatomic molecules CO_2 and H_2O

The General Method for Finding Normal Mode Frequencies, Matrices, Eigenvectors and Eigenvalues

We have just seen that when a coupled system oscillates in a *single* normal mode each component of the system will vibrate with frequency of that mode. This allows us to adopt a method which will always yield the values of the normal mode frequencies and the relative amplitudes of the individual oscillators at each frequency.

Suppose that our system of coupled pendulums in the last section oscillates in *only one* of its normal modes of frequency ω.

Then, in the equations of motion

$$m\ddot{x} + mg(x/l) + s(x - y) = 0$$

and

$$m\ddot{y} + mg(y/l) - s(x - y) = 0$$

If the pendulums start from test, we may assume the solutions

$$x = A\,\mathrm{e}^{\mathrm{i}\omega t}$$
$$y = B\,\mathrm{e}^{\mathrm{i}\omega t}$$

where A and B are the displacement amplitudes of x and y at the frequency ω. Using these solutions, the equations of motion become

$$\begin{aligned}
[-m\omega^2 A + (mg/l)A + s(A - B)]\,\mathrm{e}^{\mathrm{i}\omega t} &= 0 \\
[-m\omega^2 B + (mg/l)B - s(A - B)]\,\mathrm{e}^{\mathrm{i}\omega t} &= 0
\end{aligned} \tag{4.5}$$

The sum of these expressions gives

$$(A + B)(-m\omega^2 + mg/l) = 0$$

which is satisfied when $\omega^2 = g/l$, the first normal mode frequency. The difference between the expressions gives

$$(A - B)(-m\omega^2 + mg/l + 2s) = 0$$

which is satisfied when $\omega^2 = g/l + 2s/m$, the second normal mode frequency.

Inserting the value $\omega^2 = g/l$ in the pair of equations gives $A = B$ (the 'in phase' condition), whilst $\omega^2 = g/l + 2s/m$ gives $A = -B$ (the antiphase conditon).

These are the results we found in the previous section.

We may, however, by dividing through by $m\,\mathrm{e}^{\mathrm{i}\omega t}$, rewrite equation (4.5) in matrix form as

$$\begin{bmatrix} \omega_0^2 + \omega_s^2 & -\omega_s^2 \\ -\omega_s^2 & \omega_0^2 + \omega_s^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \omega^2 \begin{bmatrix} A \\ B \end{bmatrix} \tag{4.6}$$

where

$$\omega_0^2 = \frac{g}{l} \quad \text{and} \quad \omega_s^2 = \frac{s}{m}$$

This is called an *eigenvalue* equation. The value of ω^2 for which non-zero solutions exist are called the *eigenvalues* of the matrix. The column vector with components A and B is an *eigenvector* of the matrix.

Equation (4.6) may be written in the alternative form

$$\begin{bmatrix} (\omega_0^2 + \omega_s^2 - \omega^2) & -\omega_s^2 \\ -\omega_s^2 & (\omega_0^2 + \omega_s^2 - \omega^2) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0 \tag{4.7}$$

and these equations have a non-zero solution if and only if the determinant of the matrix vanishes; that is, if

$$(\omega_0^2 + \omega_s^2 - \omega^2)^2 - \omega_s^4 = 0$$

or

$$(\omega_0^2 + \omega_s^2 - \omega^2) = \pm\omega_s^2$$

i.e.

$$\omega_1^2 = \omega_0^2 \quad \text{or} \quad \omega_2^2 = \omega_0^2 + 2\,\omega_s^2$$

as we expect.

The solution $\omega_1^2 = \omega_0^2$ in equation (4.6) yields $A = B$ as previously and $\omega_2^2 = \omega_0^2 + 2\omega_s^2$ yields $A = -B$.

Because the system started from rest we have been able to assume solutions of the simple form

$$x = A\,e^{i\omega t}$$
$$y = B\,e^{i\omega t}$$

When the pendulums have an initial velocity at $t = 0$, the boundary conditions require solutions of the form

$$x = Ae^{i(\omega t + \alpha_x)}$$
$$y = Be^{i(\omega t + \alpha_y)}$$

where each normal mode frequency ω has its own particular value of the phase constant α. The number of adjustable constants then allows the solutions to satisfy the arbitrary values of the initial displacements and velocities of both pendulums.

(Problems 4.1, 4.2, 4.3, 4.4, 4.5, 4.6, 4.7, 4.8, 4.9, 4.10, 4.11)

Mass or Inductance Coupling

In a later chapter we shall discuss the propagation of voltage and current waves along a transmission line which may be considered as a series of coupled electrical oscillators having identical values of inductance and of capacitance. For the moment we shall consider the energy transfer between two electrical circuits which are inductively coupled.

A mutual inductance (shared mass) exists between two electrical circuits when the magnetic flux from the current flowing on one circuit threads the second circuit. Any change of flux induces a voltage in both circuits.

A transformer depends upon mutual inductance for its operation. The power source is connected to the transformer primary coil of n_p turns, over which is wound in the same sense a secondary coil of n_s turns. If unit current flowing in a single turn of the primary coil produces a magnetic flux ϕ, then the flux threading each primary turn (assuming no flux leakage outside the coil) is $n_p\phi$ and the total flux threading all n_p turns of the primary is

$$L_p = n_p^2\phi$$

where L_p is the self inductance of the primary coil. If unit current in a single turn of the secondary coil produces a flux ϕ, then the flux threading each secondary turn is $n_s\phi$ and the total flux threading the secondary coil is

$$L_s = n_s^2\phi,$$

where L_s is the self inductance of the secondary coil.

If all the flux lines from unit current in the primary thread all the turns of the secondary, then the total flux lines threading the secondary defines the *mutual inductance*

$$M = n_s(n_p\phi) = \sqrt{L_pL_s}$$

In practice, because of flux leakage outside the coils, $M < \sqrt{L_pL_s}$ and the ratio

$$\frac{M}{\sqrt{L_pL_s}} = k, \text{the } \textit{coefficient of coupling}.$$

If the primary current I_p varies with $e^{i\omega t}$, a change of I_p gives an induced voltage $-L_p\mathrm{d}I_p/\,\mathrm{d}t = -i\omega LI_p$ in the primary and an induced voltage $-M\,\mathrm{d}I_p/\mathrm{d}t = -i\omega MI_p$ in the secondary.

If we consider now the two resistance-free circuits of Figure 4.7, where L_1 and L_2 are coupled by flux linkage and allowed to oscillate at some frequency ω (the voltage and current frequency of both circuits), then the voltage equations are

$$i\omega L_1I_1 - i\frac{1}{\omega C_1}I_1 + i\omega MI_2 = 0 \tag{4.8}$$

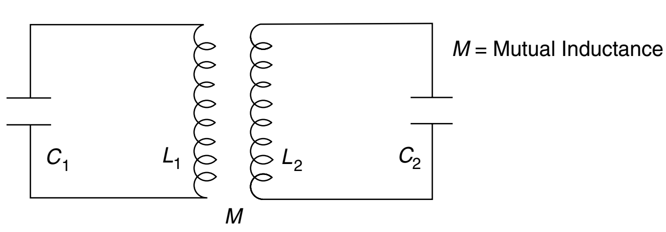


Figure 4.7 Inductively (mass) coupled *LC* circuits with mutual inductance *M*

and

$$\mathrm{i}\omega L_2 I_2 - \mathrm{i}\,\frac{1}{\omega C_2}\,I_2 + \mathrm{i}\omega M I_1 = 0 \tag{4.9}$$

where M is the mutual inductance.

Multiplying (4.8) by $\omega/\mathrm{i}L_1$ gives

$$\omega^2 I_1 - \frac{I_1}{L_1 C_1} + \frac{M}{L_1}\omega^2 I_2 = 0$$

and multiplying (4.9) by $\omega/\mathrm{i}L_2$ gives

$$\omega^2 I_2 - \frac{I_2}{L_2 C_2} + \frac{M}{L_2}\omega^2 I_1 = 0,$$

where the natural frequencies of the circuit $\omega_1^2 = 1/L_1 C_1$ and $\omega_2^2 = 1/L_2 C_2$ give

$$(\omega_1^2 - \omega^2) I_1 = \frac{M}{L_1}\,\omega^2 I_2 \tag{4.10}$$

and

$$(\omega_2^2 - \omega^2) I_2 = \frac{M}{L_2}\,\omega^2 I_1 \tag{4.11}$$

The product of equations (4.10) and (4.11) gives

$$(\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2) = \frac{M^2}{L_1 L_2}\,\omega^4 = k^2\omega^4, \tag{4.12}$$

where k is the coefficient of coupling.

Solving for ω gives the frequencies at which energy exchange between the circuits allows the circuits to resonate. If the circuits have equal natural frequencies $\omega_1 = \omega_2 = \omega_0$, say, then equation (4.12) becomes

$$(\omega_0^2 - \omega^2)^2 = k^2\omega^4$$

or

$$(\omega_0^2 - \omega^2) = \pm\, k\omega^2$$

that is

$$\omega = \pm\frac{\omega_0}{\sqrt{1 \pm k}}$$

The positive sign gives two frequencies

$$\omega' = \frac{\omega_0}{\sqrt{1+k}} \quad \text{and} \quad \omega'' = \frac{\omega_0}{\sqrt{1-k}}$$

at which, if we plot the current amplitude versus frequency, two maxima appear (Figure 4.8).

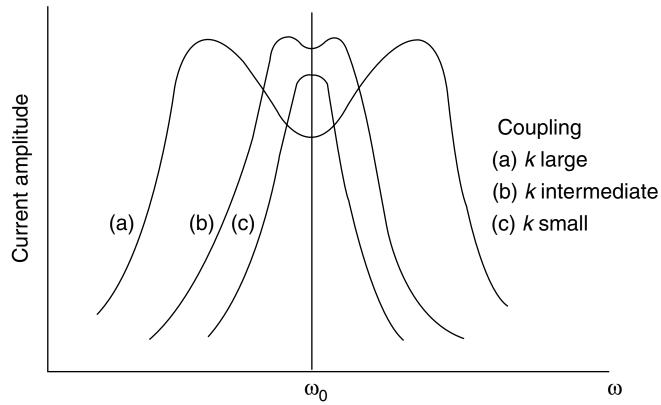


Figure 4.8 Variation of the current amplitude in each circuit near the resonant frequency. A small resistance prevents the amplitude at resonance from reaching infinite values but this has been ignored in the simple analysis. Flattening of the response curve maximum gives 'frequency band pass' coupling

In loose coupling k and M are small, and $\omega' \approx \omega'' \approx \omega_0$, so that both systems behave almost independently. In tight coupling the frequency difference $\omega'' - \omega'$ increases, the peak values of current are displaced and the dip between the peaks is more pronounced. In this simple analysis the effect of resistance has been ignored. In practice some resistance is always present to limit the amplitude maximum.

(Problems 4.12, 4.13, 4.14, 4.15, 4.16)

Coupled Oscillations of a Loaded String

As a final example involving a large number of coupled oscillators we shall consider a light string supporting n equal masses m spaced at equal distance a along its length. The string is fixed at both ends; it has a length $(n+1)a$ and a constant tension T exists at all points and all times in the string.

Small simple harmonic oscillations of the masses are allowed in only one plane and the problem is to find the frequencies of the normal modes and the displacement of each mass in a particular normal mode.

This problem was first treated by Lagrange, its particular interest being the use it makes of normal modes and the light it throws upon the wave motion and vibration of a continuous string to which it approximates as the linear separation and the magnitude of the masses are progressively reduced.

Figure 4.9 shows the displacement y_r of the r th mass together with those of its two neighbours. The equation of motion of this mass may be written by considering the components of the tension directed towards the equilibrium position. The r th mass is pulled *downwards* towards the equilibrium position by a force $T \sin\theta_1$, due to the tension

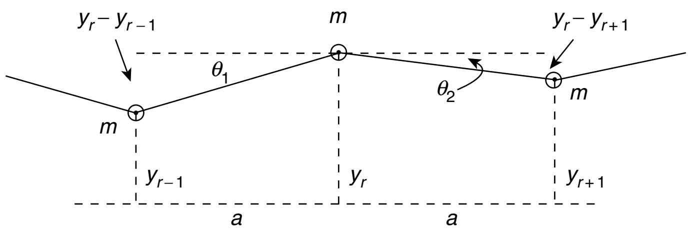


Figure 4.9 Displacements of three masses on a loaded string under tension T giving equation of motion $m\ddot{y}_r = T(y_{r+1} - 2y_r + y_{r-1})/a$

on its left and a force $T \sin\theta_2$ due to the tension on its right where

$$\sin\theta_1 = \frac{y_r - y_{r-1}}{a}$$

and

$$\sin\theta_2 = \frac{y_r - y_{r+1}}{a}$$

Hence the equation of motion is given by

$$\begin{aligned} m\frac{d^2 y_r}{dt^2} &= -T(\sin\theta_1 + \sin\theta_2) \\ &= -T\left(\frac{y_r - y_{r-1}}{a} + \frac{y_r - y_{r+1}}{a}\right) \end{aligned}$$

so

$$\frac{d^2 y_r}{dt^2} = \ddot{y}_r = \frac{T}{ma}(y_{r-1} - 2y_r + y_{r+1}) \tag{4.13}$$

If, in a normal mode of oscillation of frequency ω, the time variation of y_r is simple harmonic about the equilibrium axis, we may write the displacement of the r th mass in this mode as

$$y_r = A_r \, e^{i\omega t}$$

where A_r is the maximum displacement. Similarly $y_{r+1} = A_{r+1}\, e^{i\omega t}$ and $y_{r-1} = A_{r-1}\, e^{i\omega t}$. Using these values of y in the equation of motion gives

$$-\omega^2 A_r \, e^{i\omega t} = \frac{T}{ma}(A_{r-1} - 2A_r + A_{r+1})\, e^{i\omega t}$$

or

$$\boxed{-A_{r-1} + \left(2 - \frac{ma\omega^2}{T}\right) A_r - A_{r+1} = 0} \tag{4.14}$$

This is the fundamental equation.

The procedure now is to start with the first mass $r = 1$ and move along the string, writing out the set of similar equations as r assumes the values $r = 1, 2, 3, \ldots, n$ remembering that, because the ends are fixed

$$y_0 = A_0 = 0 \quad \text{and} \quad y_{n+1} = A_{n+1} = 0$$

Thus, when $r = 1$ the equation becomes

$$\left(2 - \frac{ma\omega^2}{T}\right)A_1 - A_2 = 0 \quad (A_0 = 0)$$

When $r = 2$ we have

$$-A_1 + \left(2 - \frac{ma\omega^2}{T}\right)A_2 - A_3 = 0$$

and when $r = n$ we have

$$-A_{n-1} + \left(2 - \frac{ma\omega^2}{T}\right)A_n = 0 \quad (A_{n+1} = 0)$$

Thus, we have a set of n equations which, when solved, will yield n different values of ω^2, each value of ω being the frequency of a normal mode, the number of normal modes being equal to the number of masses.

The formal solution of this set of n equations involves the theory of matrices. However, we may easily solve the simple cases for one or two masses on the string ($n = 1$ or 2) and, in additon, it is possible to show what the complete solution for n masses must be without using sophisticated mathematics.

First, when $n = 1$, one mass on a string of length $2a$, we need only the equation for $r = 1$ where the fixed ends of the string give $A_0 = A_2 = 0$.

Hence we have

$$\left(2 - \frac{ma\omega^2}{T}\right)A_1 = 0$$

giving

$$\omega^2 = \frac{2T}{ma}$$

a single allowed frequency of vibration (Figure 4.10a).

When $n = 2$, string length $3a$ (Figure 4.10b) we need the equations for both $r = 1$ and $r = 2$; that is

$$\left(2 - \frac{ma\omega^2}{T}\right)A_1 - A_2 = 0$$

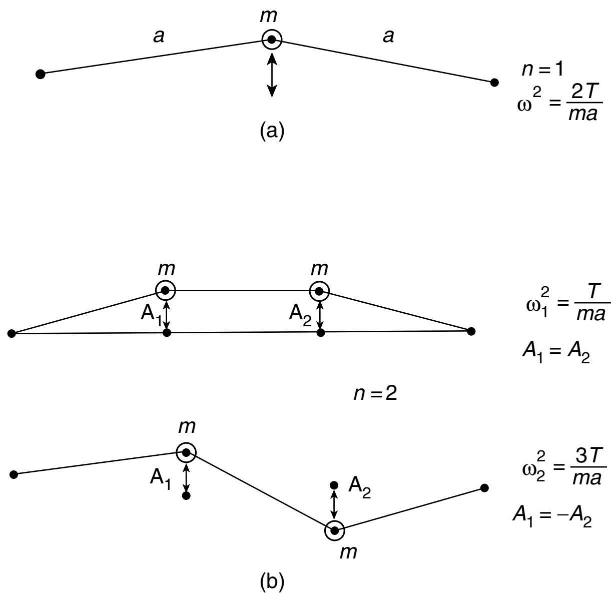


Figure 4.10 (a) Normal vibration of a single mass m on a string of length $2a$ at a frequency $\omega^2 = 2T/ma$. (b) Normal vibrations of two masses on a string of length $3a$ showing the loose coupled 'in phase' mode of frequency $\omega_1^2 = T/ma$ and the tighter coupled 'out of phase' mode of frequency $\omega_2^2 = 3T/ma$. The number of normal modes of vibration equals the number of masses

and

$$-A_1 + \left(2 - \frac{ma\omega^2}{T}\right) A_2 = 0 \quad (A_0 = A_3 = 0)$$

Eliminating A_1 or A_2 shows that these two equations may be solved (are consistent) when

$$\left(2 - \frac{ma\omega^2}{T}\right)^2 - 1 = 0$$

that is

$$\left(2 - \frac{ma\omega^2}{T} - 1\right)\left(2 - \frac{ma\omega^2}{T} + 1\right) = 0$$

Thus, there are two normal mode frequencies

$$\omega_1^2 = \frac{T}{ma} \quad \text{and} \quad \omega_2^2 = \frac{3T}{ma}$$

Using the values of ω_1 in the equations for $r = 1$ and $r = 2$ gives $A_1 = A_2$ the slow 'in phase' oscillation of Figure 4.10b, whereas ω_2 gives $A_1 = -A_2$ the faster 'anti-phase' oscillation resulting from the increased coupling.

To find the general solution for any value of n let us rewrite the equation

$$\boxed{-A_{r-1} + \left(2 - \frac{ma\omega^2}{T}\right) A_r - A_{r+1} = 0}$$

in the form

$$\frac{A_{r-1} + A_{r+1}}{A_r} = \frac{2\omega_0^2 - \omega^2}{\omega_0^2} \quad \text{where} \quad \omega_0^2 = \frac{T}{ma}$$

We see that for any particular *fixed* value of the normal mode frequency $\omega(\omega_j$ say) the right hand side of this equation is constant, independent of r, so the equation holds for all values of r. What values can we give to A_r which will satisfy this equation, meeting the boundary conditions $A_0 = A_{n+1} = 0$ at the end of the string?

Let us *assume* that we may express the amplitude of the rth mass at the frequency ω_j as

$$A_r = C\,\mathrm{e}^{\mathrm{i}r\theta}$$

where C is a constant and θ is some constant angle for a given value of ω_j. The left hand side of the equation then becomes

$$\begin{aligned}\frac{A_{r-1} + A_{r+1}}{A_r} &= \frac{C(\mathrm{e}^{\mathrm{i}(r-1)\theta} + \mathrm{e}^{\mathrm{i}(r+1)\theta})}{C\,\mathrm{e}^{\mathrm{i}r\theta}} = (\mathrm{e}^{-\mathrm{i}\theta} + \mathrm{e}^{\mathrm{i}\theta}) \\ &= 2\cos\theta\end{aligned}$$

which is constant and independent of r.

The value of θ_j (constant at ω_j) is easily found from the boundary conditions

$$A_0 = A_{n+1} = 0$$

which, using $\sin r\theta$ from $\mathrm{e}^{\mathrm{i}r\theta}$ gives

$$A_0 = C\sin r\theta = 0 \quad \text{(automatically at } r = 0)$$

and

$$A_{n+1} = C\sin(n+1)\theta = 0$$

when

$$(n+1)\,\theta_j = j\pi \quad \text{for} \quad j = 1, 2, \ldots, n$$

Hence

$$\theta_j = \frac{j\pi}{n+1}$$

and

$$A_r = C \sin r\theta_j = C \sin \frac{rj\pi}{n+1}$$

which is the amplitude of the rth mass at the fixed normal mode frequency ω_j.

To find the allowed values of ω_j we write

$$\frac{A_{r-1} + A_{r+1}}{A_r} = \frac{2\omega_0^2 - \omega_j^2}{\omega_0^2} = 2\cos\theta_j = 2\cos\frac{j\pi}{n+1}$$

giving

$$\omega_j^2 = 2\omega_0^2\left[1 - \cos\frac{j\pi}{n+1}\right] \tag{4.15}$$

where j may take the values $j = 1, 2, \ldots, n$ and $\omega_0^2 = T/ma$.

Note that there is a maximum frequency of oscillation $\omega_j = 2\omega_0$. This is called the 'cut off' frequency and such an upper frequency limit is characteristic of all oscillating systems composed of similar elements (the masses) repeated periodically throughout the structure of the system. We shall meet this in the next chapter as a feature of wave propagation in crystals.

To summarize, we have found the normal modes of oscillation of n coupled masses on the string to have frequencies given by

$$\omega_j^2 = \frac{2T}{ma}\left[1 - \cos\frac{j\pi}{n+1}\right] \quad (j = 1, 2, 3 \ldots n)$$

At each frequency ω_j the r th mass has an amplitude

$$A_r = C \sin\frac{rj\pi}{n+1}$$

where C is a constant.

(Problems 4.17, 4.18, 4.19, 4.20, 4.21, 4.22)

The Wave Equation

Finally, in this chapter, we show how the coupled vibrations in the periodic structure of our loaded string become waves in a continuous medium.

We found the equation of motion of the r th mass to be

$$\frac{\mathrm{d}^2 y_r}{\mathrm{d}t^2} = \frac{T}{ma}(y_{r+1} - 2y_r + y_{r-1}) \tag{4.13}$$

We know also that in a given normal mode all masses oscillate with the same mode frequency ω, so all y_r's have the same time dependence. However, as we see in Figure 4.10(b) where A_1 and A_2 are anti-phase, the transverse displacement y_r also depends upon the value of r; that is, the position of the r th mass on the string. In other words, y_r is a function of two independent variables, the time t and the location of r on the string.

If we use the separation $a \approx \delta x$ and let $\delta x \to 0$, the masses become closer and we can consider positions along the string in terms of a continuous variable x and any transverse displacement as $y(x, t)$, a function of both x and t.

The partial derivative notation $\partial y(x, t)/\partial t$ expresses the variation with time of $y(x, t)$ while x is kept constant.

The partial derivative $\partial y(x, t)/\partial x$ expresses the variation with x of $y(x, t)$ while the time t is kept constant. (Chapter 5 begins with an extended review of this process for students unfamiliar with this notation.)

In the same way, the second derivative $\partial^2 y(x, t)/\partial t^2$ continues to keep x constant and $\partial^2 y(x, t)/\partial x^2$ keeps t constant.

For example, if

$$y = \mathrm{e}^{\mathrm{i}(\omega t + kx)}$$

then

$$\frac{\partial y}{\partial t} = \mathrm{i}\omega\,\mathrm{e}^{\mathrm{i}(\omega t + kx)} = \mathrm{i}\omega y \quad \text{and} \quad \frac{\partial^2 y}{\partial t^2} = -\omega^2 y$$

while

$$\frac{\partial y}{\partial x} = \mathrm{i}k\,\mathrm{e}^{\mathrm{i}(\omega t + kx)} = \mathrm{i}ky \quad \text{and} \quad \frac{\partial^2 y}{\partial x^2} = -k^2 y$$

If we now locate the transverse displacement y_r at a position $x = x_r$ along the string, then the left hand side of equation (4.13) becomes

$$\frac{\partial^2 y_r}{\partial t^2} \to \frac{\partial^2 y}{\partial t^2},$$

where y is evaluated at $x = x_r$ and now, as $a = \delta x \to 0$, we may write $x_r = x, x_{r+1} = x + \delta x$ and $x_{r-1} = x - \delta x$ with $y_r(t) \to y(x, t), y_{r+1}(t) \to y(x + \delta x, t)$ and $y_{r-1}(t) \to y(x - \delta x, t)$.

Using a Taylor series expansion to express $y(x \pm \delta x, t)$ in terms of partial derivates of y with respect to x we have

$$y(x \pm \delta x, t) = y(x) \pm \delta x \frac{\partial y}{\partial x} + \frac{1}{2}(\pm\delta x)^2 \frac{\partial^2 y}{\partial x^2}$$

and equation (4.13) becomes after substitution

$$\begin{aligned}\frac{\partial^2 y}{\partial t^2} &= \frac{T}{m}\left(\frac{y_{r+1} - y_r}{a} - \frac{y_r - y_{r-1}}{a}\right)\\ &= \frac{T}{m}\left(\frac{\delta x \frac{\partial y}{\partial x} + \frac{1}{2}(\delta x)^2 \frac{\partial^2 y}{\partial x^2}}{\delta x} - \frac{\delta x \frac{\partial y}{\partial x} - \frac{1}{2}(\delta x)^2 \frac{\partial^2 y}{\partial x^2}}{\delta x}\right)\end{aligned}$$

so

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\frac{(\delta x)^2}{\delta x}\frac{\partial^2 y}{\partial x^2} = \frac{T}{m}\ \delta x\ \frac{\partial^2 y}{\partial x^2}$$

If we now write $m = \rho\,\delta x$ where ρ is the linear density (mass per unit length) of the string, the masses must $\longrightarrow 0$ as $\delta x \longrightarrow 0$ to avoid infinite mass density. Thus, we have

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 y}{\partial x^2}$$

This is the *Wave Equation*.

T/ρ has the dimensions of the square of a velocity, the velocity with which the waves; that is, the phase of oscillation, is propagated. The solution for y at any particular point along the string is always that of a harmonic oscillation.

(Problem 4.23)

Problem 4.1
Show that the choice of new normal coordinates X_q and Y_q expresses equations (4.3a) and (4.3b) as equations (4.4a) and (4.4b).

Problem 4.2
Express the total energy of Problem 4.1 in terms of the pendulum displacements x and y as

$$E = (E_{\text{kin}} + E_{\text{pot}})_x + (E_{\text{kin}} + E_{\text{pot}})_y + (E_{\text{pot}})_{xy},$$

where the brackets give the energy of each pendulum expressed in its own coordinates and $(E_{\text{pot}})_{xy}$ is the coupling or interchange energy involving the product of these coordinates.

Problem 4.3
Figures 4.3 and 4.5 show how the pendulum configurations $x = 2a, y = 0$ and $x = 0, y = 2a$ result from the superposition of the normal modes X and Y. Using the same initial conditions

$(x = 2a, y = 0, \dot{x} = \dot{y} = 0)$ draw similar sketches to show how X and Y superpose to produce $x = -2a, y = 0$ and $x = 0, y = -2a$.

Problem 4.4

In the figure two masses m_1 and m_2 are coupled by a spring of stiffness s and natural length l. If x is the extension of the spring show that equations of motion along the x axis are

$$m_1\ddot{x}_1 = sx$$

and

$$m_2\ddot{x}_2 = -sx$$

and combine these to show that the system oscillates with a frequency

$$\omega^2 = \frac{s}{\mu},$$

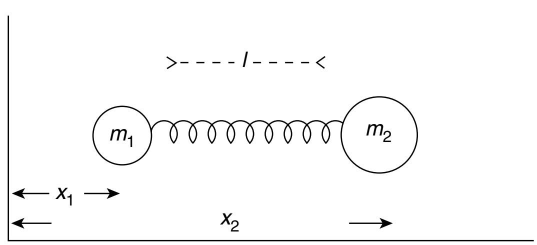
where

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

is called the reduced mass.

The figure now represents a diatomic molecule as a harmonic oscillator with an effective mass equal to its reduced mass. If a sodium chloride molecule has a natural vibration frequency $= 1.14 \times 10^{13}$ Hz (in the infrared region of the electromagnetic spectrum) show that the interatomic force constant $s = 120\,\mathrm{N\,m^{-1}}$ (this simple model gives a higher value for s than more refined methods which account for other interactions within the salt crystal lattice)

Mass of Na atom = 23 a.m.u.
Mass of Cl atom = 35 a.m.u.
1 a.m.u. = 1.67×10^{-27} kg

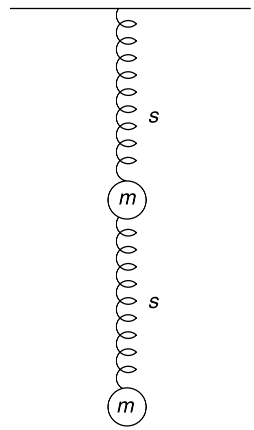


Problem 4.5

The equal masses in the figure oscillate in the vertical direction. Show that the frequencies of the normal modes of oscillation are given by

$$\omega^2 = (3 \pm \sqrt{5})\frac{s}{2m}$$

and that in the slower mode the ratio of the amplitude of the upper mass to that of the lower mass is $\frac{1}{2}(\sqrt{5} - 1)$ whilst in the faster mode this ratio is $-\frac{1}{2}(\sqrt{5} + 1)$.



In the calculations it is not necessary to consider gravitational forces because they play no part in the forces responsible for the oscillation.

Problem 4.6

In the coupled pendulums of Figure 4.3 let us write the modulated frequency $\omega_m = (\omega_2 - \omega_1)/2$ and the average frequency $\omega_a = (\omega_2 + \omega_1)/2$ and assume that the spring is so weak that it stores a negligible amount of energy. Let the modulated amplitude

$$2a \cos \omega_m t \quad \text{or} \quad 2a \sin \omega_m t$$

be constant over one cycle at the average frequency ω_a to show that the energies of the masses may be written

$$E_x = 2ma^2\omega_a^2 \cos^2 \omega_m t$$

and

$$E_y = 2ma^2\omega_a^2 \sin^2 \omega_m t$$

Show that the total energy E remains constant and that the energy difference at any time is

$$E_x - E_y = E \cos (\omega_2 - \omega_1)t$$

Prove that

$$E_x = \frac{E}{2}[1 + \cos (\omega_2 - \omega_1)t]$$

and

$$E_y = \frac{E}{2}[1 - \cos (\omega_2 - \omega_1)t]$$

to show that the constant total energy is completely exchanged between the two pendulums at the beat frequency $(\omega_2 - \omega_1)$.

Problem 4.7

When the masses of the coupled pendulums of Figure 4.1 are no longer equal the equations of motion become

$$m_1\ddot{x} = -m_1(g/l)x - s(x - y)$$

and

$$m_2\ddot{y} = -m_2(g/l)y + s(x - y)$$

Show that we may choose the normal coordinates

$$X = \frac{m_1 x + m_2 y}{m_1 + m_2}$$

with a normal mode frequency $\omega_1^2 = g/l$ and $Y = x - y$ with a normal mode frequency $\omega_2^2 = g/l + s(1/m_1 + 1/m_2)$.

Note that X is the coordinate of the centre of mass of the system whilst the effective mass in the Y mode is the reduced mass μ of the system where $1/\mu = 1/m_1 + 1/m_2$.

Problem 4.8

Let the system of Problem 4.7 be set in motion with the initial conditions $x = A, y = 0, \dot{x} = \dot{y} = 0$ at $t = 0$. Show that the normal mode amplitudes are $X_0 = (m_1/M)A$ and $Y_0 = A$ to yield

$$x = \frac{A}{M}(m_1 \cos \omega_1 t + m_2 \cos \omega_2 t)$$

and

$$y = A\frac{m_1}{M}(\cos \omega_1 t - \cos \omega_2 t),$$

where $M = m_1 + m_2$.

Express these displacements as

$$x = 2A \cos \omega_m t \cos \omega_a t + \frac{2A}{M}(m_1 - m_2) \sin \omega_m t \sin \omega_a t$$

and

$$y = 2A\frac{m_1}{M} \sin \omega_m t \sin \omega_a t,$$

where $\omega_m = (\omega_2 - \omega_1)/2$ and $\omega_a = (\omega_1 + \omega_2)/2$.

Problem 4.9

Apply the weak coupling conditions of Problem 4.6 to the system of Problem 4.8 to show that the energies

$$E_x = \frac{E}{M^2}[m_1^2 + m_2^2 + 2m_1 m_2 \cos(\omega_2 - \omega_1)t]$$

and

$$E_y = E\left(\frac{2m_1 m_2}{M^2}\right)[1 - \cos(\omega_2 - \omega_1)t]$$

Note that E_x varies between a maximum of E (at $t = 0$) and a minimum of $[(m_1 - m_2)/M]^2 E$, whilst E_y oscillates between a minimum of zero at $t = 0$ and a maximum of $4(m_1 m_2/M^2)E$ at the beat frequency of $(\omega_2 - \omega_1)$.

Problem 4.10

In the figure below the right hand pendulum of the coupled system is driven by the horizontal force $F_0 \cos \omega t$ as shown. If a small damping constant r is included the equations of motion may be written

$$m\ddot{x} = -\frac{mg}{l}x - r\dot{x} - s(x-y) + F_0 \cos \omega t$$

and

$$m\ddot{y} = -\frac{mg}{l}y - r\dot{y} + s(x-y)$$

Show that the equations of motion for the normal coordinates $X = x + y$ and $Y = x - y$ are those for damped oscillators driven by a force $F_0 \cos \omega t$.

Solve these equations for X and Y and, by neglecting the effect of r, show that

$$x \approx \frac{F_0}{2m} \cos \omega t \left[\frac{1}{\omega_1^2 - \omega^2} + \frac{1}{\omega_2^2 - \omega^2} \right]$$

and

$$y \approx \frac{F_0}{2m} \cos \omega t \left[\frac{1}{\omega_1^2 - \omega^2} - \frac{1}{\omega_2^2 - \omega^2} \right]$$

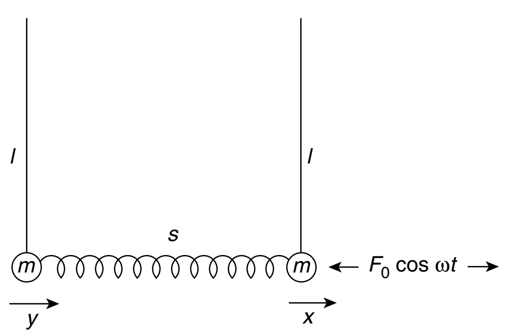
where

$$\omega_1^2 = \frac{g}{l} \quad \text{and} \quad \omega_2^2 = \frac{g}{l} + \frac{2s}{m}$$

Show that

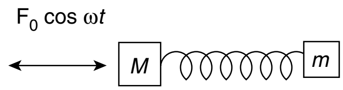
$$\frac{y}{x} \approx \frac{\omega_2^2 - \omega_1^2}{\omega_2^2 + \omega_1^2 - 2\omega^2}$$

and sketch the behaviour of the oscillator with frequency to show that outside the frequency range $\omega_2 - \omega_1$ the motion of y is attenuated.



Problem 4.11

The diagram shows an oscillatory force $F_o \cos \omega t$ acting on a mass M which is part of a simple harmonic system of stiffness k and is connected to a mass m by a spring of stiffness s. If all oscillations are along the x axis show that the condition for M to remain stationary is $\omega^2 = s/m$. (This is a simple version of small mass loading in engineering to quench undesirable oscillations.)



Problem 4.12

The figure below shows two identical LC circuits coupled by a common capacitance C with the directions of current flow indicated by arrows. The voltage equations are

$$V_1 - V_2 = L\frac{\mathrm{d}I_a}{\mathrm{d}t}$$

and

$$V_2 - V_3 = L\frac{\mathrm{d}I_b}{\mathrm{d}t}$$

whilst the currents are given by

$$\frac{\mathrm{d}q_1}{\mathrm{d}t} = -I_a \quad \frac{\mathrm{d}q_2}{\mathrm{d}t} = I_a - I_b$$

and

$$\frac{\mathrm{d}q_3}{\mathrm{d}t} = I_b$$

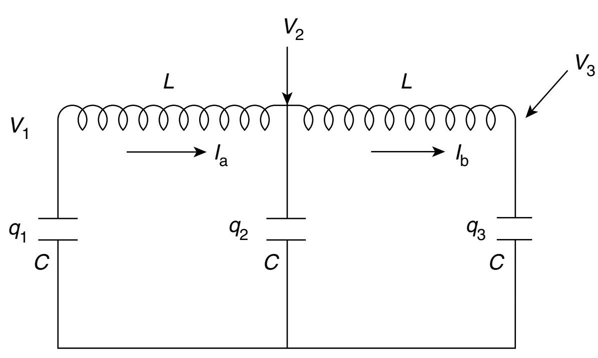
Solve the voltage equations for the normal coordinates $(I_a + I_b)$ and $(I_a - I_b)$ to show that the normal modes of oscillation are given by

$$I_a = I_b \quad \text{at} \quad \omega_1^2 = \frac{1}{LC}$$

and

$$I_a = -I_b \quad \text{at} \quad \omega_2^2 = \frac{3}{LC}$$

Note that when $I_a = I_b$ the coupling capacitance may be removed and $q_1 = -q_2$. When $I_a = -I_b$, $q_2 = -2q_1 = -2q_3$.



Problem 4.13

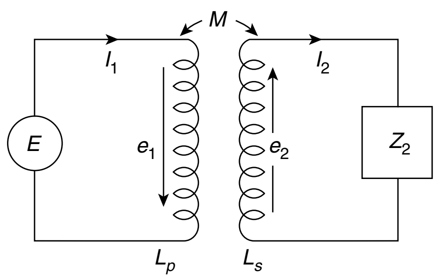
A generator of e.m.f. E is coupled to a load Z by means of an ideal transformer. From the diagram, Kirchhoff's Law gives

$$E = -e_1 = i\omega L_p I_1 - i\omega M I_2$$

and

$$I_2 Z_2 = e_2 = i\omega M I_1 - i\omega L_s I_2.$$

Show that E/I_1, the impedance of the whole system seen by the generator, is the sum of the primary impedance and a 'reflected impedance' from the secondary circuit of $\omega^2 M^2/Z_s$ where $Z_s = Z_2 + i\omega L_s$.



Problem 4.14

Show, for the perfect transformer of Problem 4.13, that the impedance seen by the generator consists of the primary impedance in parallel with an impedance $(n_p/n_s)^2 Z_2$, where n_p and n_s are the number of primary and secondary transformer coil turns respectively.

Problem 4.15

If the generator delivers maximum power when its load equals its own internal impedance show how an ideal transformer may be used as a device to match a load to a generator, e.g. a loudspeaker of a few ohms impedance to an amplifier output of $10^3\ \Omega$ impedance.

Problem 4.16

The two circuits in the diagram are coupled by a variable mutual inductance M and Kirchhoff's Law gives

$$Z_1 I_1 + Z_M I_2 = E$$

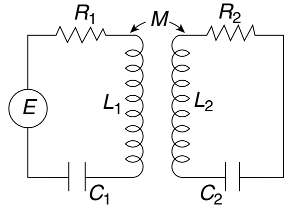
and

$$Z_M I_1 + Z_2 I_2 = 0,$$

where

$$Z_M = +i\omega M$$

M is varied at a resonant frequency where the reactance $X_1 = X_2 = 0$ to give a maximum value of I_2. Show that the condition for this maximum is $\omega M = \sqrt{R_1 R_2}$ and that this defines a 'critical coefficient of coupling' $k = (Q_1 Q_2)^{-1/2}$, where the Q's are the quality factors of the circuits.



Problem 4.17

Consider the case when the number of masses on the loaded string of this chapter is $n = 3$. Use equation (4.15) to show that the normal mode frequencies are given by

$$\omega_1^2 = (2 - \sqrt{2})\omega_0^2; \qquad \omega_2^2 = 2\omega_0^2$$

and

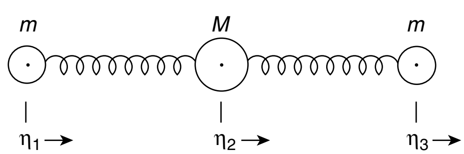
$$\omega_3^2 = (2 + \sqrt{2})\omega_0^2$$

Repeat the problem using equation (4.14) (with $\omega_0^2 = T/ma$) in the matrix method of equation (4.7), where the eigenvector components are A_{r-1}, A_r and A_{r+1}.

Problem 4.18

Show that the relative displacements of the masses in the modes of Problem 4.17 are $1 : \sqrt{2} : 1$, $1 : 0 : -1$, and $1 : -\sqrt{2} : 1$. Show by sketching these relative displacements that tighter coupling increases the mode frequency.

Problem 4.19



The figure represents a triatomic molecule with a heavy atom mass M bound to equal atoms of smaller mass m on either side. The binding is represented by springs of stiffness s and in equilibrium the atom centres are equally spaced along a straight line. Simple harmonic vibrations are considered only along this linear axis and are given by

$$\eta_J = \eta_J^0 \mathrm{e}^{\mathrm{i}\omega t}$$

where η_J is the displacement from equilibrium of the jth atom.

Set up the equation of motion for each atom and use the matrix method of equation (4.7) to show that the normal modes have frequencies

$$\omega_1^2 = 0, \omega_2^2 = \frac{s}{m} \quad \text{and} \quad \omega_3^2 = \frac{s(M + 2m)}{mM}$$

Describe the motion of the atoms in each normal mode.

Problem 4.20
Taking the maximum value of

$$\omega_J^2 = \frac{2T}{ma}\left(1 - \cos\frac{j\pi}{n+1}\right)$$

at $j = n$ as that produced by the strongest coupling, deduce the relative displacements of neighbouring masses and confirm your deduction by inserting your values in consecutive difference equations relating the displacements y_{r+1}, y_r and y_{r-1}. Why is your solution unlikely to satisfy the displacements of those masses near the ends of the string?

Problem 4.21
Expand the value of

$$\omega_J^2 = \frac{2T}{ma}\left(1 - \cos\frac{j\pi}{n+1}\right)$$

when $j \ll n$ in powers of $(j/n+1)$ to show that in the limit of very large values of n, a low frequency

$$\omega_J = \frac{j\pi}{l}\sqrt{\frac{T}{\rho}},$$

where $\rho = m/a$ and $l = (n+1)a$.

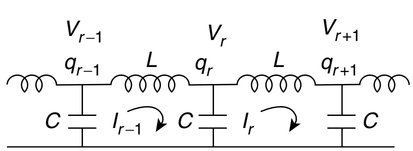
Problem 4.22
An electrical transmission line consists of equal inductances L and capacitances C arranged as shown. Using the equations

$$\frac{L\,\mathrm{d}I_{r-1}}{\mathrm{d}t} = V_{r-1} - V_r = \frac{q_{r-1} - q_r}{C}$$

and

$$I_{r-1} - I_r = \frac{\mathrm{d}q_r}{\mathrm{d}t},$$

show that an expression for I_r may be derived which is equivalent to that for y_r in the case of the mass-loaded string. (This acts as a low pass electric filter and has a cut-off frequency as in the case of the string. This cut-off frequency is a characteristic of wave propagation in periodic structures and electromagnetic wave guides.)



Problem 4.23
Show that

$$y = \mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{e}^{\mathrm{i}kx}$$

satisfies the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}, \quad \text{if} \quad \omega = ck$$

Summary of Important Results

In coupled systems each normal coordinate defines a degree of freedom, each degree of freedom defines a way in which a system may take up energy. The total energy of the system is the sum of the energies in its normal modes of oscillation because these remain separate and distinct, and energy is never exchanged between them.

A simple harmonic oscillator has two normal coordinates [velocity (or momentum) and displacement] and therefore two degrees of freedom, the first connected with kinetic energy, the second with potential energy.

n Equal Masses, Separation a, Coupled on a String under Constant Tension T

Equation of motion of the rth mass is

$$m\ddot{y}_r = (T/a)(y_{r-1} - 2y_r + y_{r+1})$$

which for $y_r = A_r\,\mathrm{e}^{\mathrm{i}\omega t}$ gives

$$-A_{r+1} + \left(\frac{2 - ma\omega^2}{T}\right)A_r - A_{r-1} = 0$$

There are n normal modes with frequencies ω_J given by

$$\omega_J^2 = \frac{2T}{ma}\left(1 - \cos\frac{j\pi}{n+1}\right)$$

In a normal mode of frequency ω_J the rth mass has an amplitude

$$A_r = C\sin\frac{rj\pi}{n+1}$$

where C is a constant.

Wave Equation

In the limit, as separation $a = \delta x \to 0$ equation of motion of the rth mass on a loaded string $m\ddot{y}_r = (T/a)(y_{r-1} - 2y_r + y_{r+1})$ becomes the wave equation

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 y}{\partial x^2} = c^2\,\frac{\partial^2 y}{\partial x^2}$$

where ρ is mass per unit length and c is the wave velocity.