

Fill Ups of Conic Sections

Q.1. The point of intersection of the tangents at the ends of the latus rectum of the parabola $y^2 = 4x$ is..... (1994 - 2 Marks)

Ans. $(-1, 0)$

Sol. Given parabola is $y^2 = 4x$; $a = 1$

Extremities of latus rectum are $(1, 2)$ and $(1, -2)$ tangent to $y^2 = 4x$ at $(1, 2)$ is $y^2 = 2(x + 1)$ i.e. $y = x + 1$...(1)

Similarly tangent at $(1, -2)$ is, $y = -x - 1$...(2)

Intersection pt. of these tangents can be obtained by solving (1) and (2), which is $(-1, 0)$.

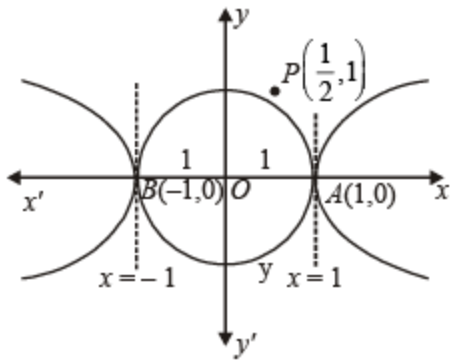
Q.2. An ellipse has eccentricity $1/2$ and one focus at the point $P\left(\frac{1}{2}, 1\right)$. Its one directrix is the common tangent, nearer to the point P, to the circle $x^2 + y^2 = 1$ and the hyperbola $x^2 - y^2 = 1$. The equation of the ellipse, in the standard form, is..... (1996 - 2 Marks)

Ans.
$$\frac{\left(x - \frac{1}{3}\right)^2}{\left(\frac{1}{3}\right)^2} + \frac{(y-1)^2}{\left(\frac{1}{2\sqrt{3}}\right)^2} = 1$$

Sol. Rough graph of $x^2 + y^2 = 1$ (circle) ...(1)

and $x^2 - y^2 = 1$ (hyperbola) ...(2)

is as shown below.



It is clear from graph that there are two common tangents to the curves (1) and (2) namely $x = 1$ and $x = -1$ out of which $x = 1$ is nearer to pt. P.

Hence directrix of required ellipse is $x - 1 = 0$ Also $e = 1/2$, focus $(1/2, 1)$ then equation of ellipse is given by

$$(x - 1/2)^2 + (y - 1)^2 = \frac{1}{4}(x - 1)^2$$

$$\Rightarrow \frac{(x - 1/3)^2}{(1/3)^2} + \frac{(y - 1)^2}{(1/2\sqrt{3})^2} = 1$$

which is the standard equation of the ellipse.

Subjective questions of Conic Sections

Q.1. Suppose that the normals drawn at three different points on the parabola $y^2 = 4x$ pass through the point (h, k) . Show that $h > 2$. (1981 - 4 Marks)

Ans. 2

Sol. The equation of a normal to the parabola $y^2 = 4ax$ in its slope form is given by

$$y = mx - 2am - am^3$$

$$\therefore \text{Eq. of normal to } y^2 = 4x, \text{ is } y = mx - 2m - m^3 \dots(1)$$

Since the normal drawn at three different points on the parabola pass through (h, k) , it must satisfy the equation (1)

$$\therefore k = mh - 2m - m^3 \Rightarrow m^3 - (h - 2)m + k = 0$$

This cubic eq. in m has three different roots say m_1, m_2, m_3

$$\therefore m_1 + m_2 + m_3 = 0 \dots(2)$$

$$m_1m_2 + m_2m_3 + m_3m_1 = -(h - 2) \dots(3)$$

Now, $(m_1 + m_2 + m_3)^2 = 0$ [Squaring (2)]

$$\Rightarrow m_1^2 + m_2^2 + m_3^2 = -2(m_1m_2 + m_2m_3 + m_3m_1) \Rightarrow m_1^2 + m_2^2 + m_3^2 = 2(h - 2) \text{ [Using (3)]}$$

Since LHS of this equation is the sum of perfect squares, therefore it is +ve

$$\therefore h - 2 > 0 \Rightarrow h > 2 \text{ Proved}$$

Q.2. A is a point on the parabola $y^2 = 4ax$. The normal at A cuts the parabola again at point B. If AB subtends a right angle at the vertex of the parabola. find the slope of AB. (1982 - 5 Marks)

Sol. Parabola $y^2 = 4ax$.

Let at any pt A equation of normal is $y = mx - 2am - am^3$ (1)

Combined equation of OA and OB can be obtained by making equation of parabola homogeneous with the help of normal.

∴ Combined eq. of OA and OB is

$$y^2 = 4ax \left(\frac{mx - y}{2am + am^3} \right)$$

[From eqn. (1) using $\frac{mx - y}{2am + am^3} = 1$]

$$y^2 = \frac{4x(mx - y)}{2m + m^3}$$

$\Rightarrow 4mx^2 - 4xy - (2m + m^3) y^2 = 0$ But angle between the lines represented by this pair is 90° .

$$\Rightarrow \text{coeff. of } x^2 + \text{coeff of } y^2 = 0 \Rightarrow 4m - 2m - m^3 = 0$$

$$\Rightarrow m^3 - 2m = 0 \Rightarrow m = 0, \sqrt{2}, -\sqrt{2}$$

But for $m = 0$ eq. of normal becomes $y = 0$ which does not intersect the parabola at any other point.

$$\therefore m = \pm \sqrt{2}$$

Q.3. Three normals are drawn from the point $(c, 0)$ to the curve $y^2 = x$. Show that c must be greater than $1/2$. One normal is always the x -axis. Find c for which the other two normals are perpendicular to each other. (1991 - 4 Marks)

Ans. $c = 3/4$

Sol. Given parabola is $y^2 = x$.

$$\text{Normal is } y = mx - \frac{m}{2} - \frac{m^3}{4}$$

As per question this normal passes through $(c, 0)$ therefore, we get

$$mc - \frac{m}{2} - \frac{m^3}{4} = 0 \dots (1)$$

$$\Rightarrow m \left[c - \frac{1}{2} - \frac{m^2}{4} \right] = 0 \Rightarrow m = 0 \text{ or } m^2 = 4 \left(c - \frac{1}{2} \right)$$

$m = 0$ shows normal is $y = 0$ i.e. x-axis is always a normal.

$$m^2 \geq 0 \Rightarrow 4 \left(c - \frac{1}{2} \right) \geq 0 \Rightarrow c \geq 1/2$$

At $c = \frac{1}{2}$ from (1) $m = 0$

\therefore for other real values of m , $c > 1/2$

Now for other two normals to be perpendicular to each other, we must have $m_1 \cdot m_2 = -1$

Or in other words, if m_1, m_2 are roots of $\frac{m^2}{4} + \frac{1}{2} - c = 0$, then product of roots $= -1$

$$\Rightarrow \frac{\left(\frac{1}{2} - c \right)}{1/4} = -1 \Rightarrow \frac{1}{2} - c = -\frac{1}{4} \Rightarrow c = 3/4$$

Q.4. Through the vertex O of parabola $y^2 = 4x$, chords OP and OQ are drawn at right angles to one another. Show that for all positions of P, PQ cuts the axis of the parabola at a fixed point. Also find the locus of the middle point of PQ. (1994 - 4 Marks)

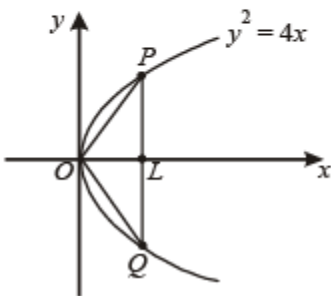
Ans. $y^2 = 2(x - 4)$

Sol. Let the equation of chord OP be $y = mx$.

Then eqn of chord OQ will be $y = -\frac{1}{m}x$ [$\because OQ \perp OP$]

P is pt. of intersection of $y = mx$ and $y^2 = 4x$.

Solving the two we get $P \left(\frac{4}{m^2}, \frac{4}{m} \right)$



Q is pt. of intersection of $y = -\frac{1}{m}x$ and $y^2 = 4x$.

Solving the two we get Q $(4m^2, -4m)$

Now eq. of PQ is

$$y + 4m = \frac{\frac{4}{m} + 4m}{\frac{4}{m^2} - 4m^2} (x - 4m^2)$$

$$\Rightarrow y + 4m = \frac{m}{1 - m^2} (x - 4m^2)$$

$$\Rightarrow (1 - m^2)y + 4m - 4m^3 = mx - 4m^3 \Rightarrow mx - (1 - m^2)y - 4m = 0$$

This line meets x-axis where $y = 0$ i.e. $x = 4$

$\Rightarrow OL = 4$, which is constant as independent of m . Again let (h, k) be the mid pt. of PQ, then

$$h = \frac{4m^2 + \frac{4}{m^2}}{2} \text{ and } k = \frac{\frac{4}{m} - 4m}{2}$$

$$\Rightarrow h = 2\left(m^2 + \frac{1}{m^2}\right) \text{ and } k = 2\left(\frac{1}{m} - m\right)$$

$$\Rightarrow h = 2\left[\left(\frac{1}{m} - m\right) + 2\right] \Rightarrow h = 2\left[\frac{k^2}{4} + 2\right]$$

$$\Rightarrow 2h = k^2 + 8 \Rightarrow k^2 = 2(h - 4)$$

\therefore Locus of (h, k) is $y^2 = 2(x - 4)$

Q.5. Show that the locus of a point that divides a chord of slope 2 of the parabola $y^2 = 4x$ internally in the ratio 1: 2 is a parabola. Find the vertex of this parabola. (1995 - 5 Marks)

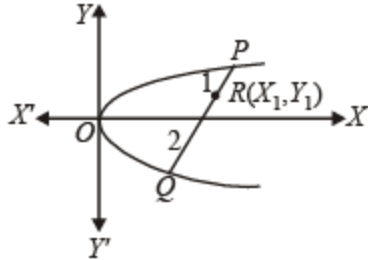
Ans. $(2/9, 8/9)$

Sol. Let P $(t_1^2, 2t_1)$ and Q $(t_2^2, 2t_2)$ be the ends of the chord PQ of the parabola

$$y^2 = 4x \quad \dots(1)$$

$$\therefore \text{Slope of chord PQ} = \frac{2t_2 - 2t_1}{t_2^2 - t_1^2} = 2$$

$$\Rightarrow t_2 + t_1 = 1 \dots (2)$$



If $R(x_1, y_1)$ is a point dividing PQ internally in the ratio 1 : 2, then

$$x_1 = \frac{1t_2^2 + 2t_1^2}{1+2}, \quad y_1 = \frac{1.2t_2 + 2.2t_1}{1+2}$$

$$\Rightarrow t_2^2 + 2t_1^2 = 3x_1 \dots (3)$$

$$\text{and } t_2 + 2t_1 = (3y_1) / 2 \dots (4)$$

From (2) and (4), we get

$$t_1 = \frac{3}{2}y_1 - 1, \quad t_2 = 2 - \frac{3}{2}y_1$$

Substituting in (3), we get

$$\left(2 - \frac{3}{2}y_1\right)^2 + 2\left(\frac{3}{2}y_1 - 1\right)^2 = 3x_1$$

$$\Rightarrow (9/4)y_1^2 - 4y_1 = x_1 - 2$$

$$\left(y_1 - \frac{8}{9}\right)^2 = \left(\frac{4}{9}\right)\left(x_1 - \frac{2}{9}\right)$$

\therefore Locus of the point $R(x_1, y_1)$ is $(y - 8/9)^2 = (4/9)(x - 2/9)$ which is a parabola having vertex at the point $(2/9, 8/9)$.

Q.6. Let 'd' be the perpendicular distance from the centre of the

ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to the tangent drawn at a point P on the ellipse. If F_1 and F_2 are the two foci of the ellipse, then show that

$$(PF_1 - PF_2)^2 = 4a^2 \left(1 - \frac{b^2}{a^2}\right) \quad \text{(1995 - 5 Marks)}$$

Ans.

Sol. Equation to the tangent at the point P ($a \cos \theta$, $b \sin \theta$) on $x^2/a^2 + y^2/b^2 = 1$ is

$$\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1 \quad \dots (1)$$

$\therefore d$ = perpendicular distance of (1) from the centre (0, 0) of the ellipse

$$= \frac{1}{\sqrt{\frac{1}{a^2} \cos^2 \theta + \frac{1}{b^2} \sin^2 \theta}} = \frac{(ab)}{\sqrt{b^2 \cos^2 \theta + a^2 \sin^2 \theta}}$$

$$\therefore 4a^2 \left(1 - \frac{b^2}{a^2}\right) = 4a^2 \left\{1 - \frac{b^2 \cos^2 \theta + a^2 \sin^2 \theta}{a^2}\right\}$$

$$= 4(a^2 - b^2) \cos^2 \theta = 4a^2 e^2 \cos^2 \theta \dots (2)$$

The coordinates of foci F_1 and F_2 are $F_1 = (ae, 0)$ and $F_2 = (-ae, 0)$

$$PF_1 = \sqrt{[(a \cos \theta - ae)^2 + (b \sin \theta)^2]}$$

$$= \sqrt{[a^2 (\cos \theta - e)^2 + (b \sin \theta)^2]}$$

$$= \sqrt{[a^2 (\cos \theta - e)^2 + a^2 (1 - e^2) \sin^2 \theta]}$$

$$= a \sqrt{[1 + e^2 (1 - \sin^2 \theta) - 2e \cos \theta]}$$

$$= a (1 - e \cos \theta) \text{ Similarly, } PF_2 = a (1 + e \cos \theta)$$

$$\therefore (PF_1 - PF_2)^2 = 4a^2 e^2 \cos^2 \theta \dots (3)$$

Hence from (2) and (3), we have

$$(PF_1 - PF_2)^2 = 4a^2 \left(1 - \frac{b^2}{a^2}\right)$$

Q.7. Points A, B and C lie on the parabola $y^2 = 4ax$. The tangents to the parabola at A, B and C, taken in pairs, intersect at points P, Q and R. Determine the ratio of the areas of the triangles ABC and PQR. (1996 - 3 Marks)

Ans. 2 : 1

Sol. Let the three points on the parabola $y^2 = 4ax$ be A (at_1^2 , $2at_1$), B(at_2^2 , $2at_2$) and C (at_3^2 , $2at_3$).

Then using the fact that equation of tangent to $y^2 = 4ax$ at (at^2 , $2at$) is $y = \frac{x}{t} + at$, we get equations of tangents at A, B and C as follows

$$y = \frac{x}{t_1} + at_1 \dots (1)$$

$$y = \frac{x}{t_2} + at_2 \dots (2)$$

$$y = \frac{x}{t_3} + at_3 \dots (3)$$

Solving the above equations pair wise we get the pts.

$$P (at_1t_2, a(t_1 + t_2))$$

$$Q (at_2t_3, a(t_2 + t_3))$$

$$R (at_3t_1, a(t_3 + t_1))$$

$$\text{Now, area of } \Delta ABC = \frac{1}{2} \begin{vmatrix} 1 & at_1^2 & 2at_1 \\ 1 & at_2^2 & 2at_2 \\ 1 & at_3^2 & 2at_3 \end{vmatrix}$$

$$= a^2 \begin{vmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_3 & t_3^2 \end{vmatrix}$$

$$= |a^2(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)| \dots (4)$$

$$\text{Also area of } \Delta PQR = \frac{1}{2} \begin{vmatrix} 1 & at_1t_2 & a(t_1 + t_2) \\ 1 & at_2t_3 & a(t_2 + t_3) \\ 1 & at_3t_1 & a(t_3 + t_1) \end{vmatrix}$$

$$= \frac{a^2}{2} \begin{vmatrix} 1 & t_1 t_2 & t_1 + t_2 \\ 1 & t_2 t_3 & t_2 + t_3 \\ 1 & t_3 t_1 & t_3 + t_1 \end{vmatrix}$$

$$= \frac{a^2}{2} \begin{vmatrix} 0 & (t_1 - t_3)t_2 & t_1 - t_3 \\ 0 & (t_2 - t_1)t_3 & t_2 - t_1 \\ 1 & t_3 t_1 & t_3 + t_1 \end{vmatrix}$$

$$[R_1 \rightarrow R_1 - R_3 \text{ and } R_2 \rightarrow R_2 - R_3]$$

Expanding along C_1 ,

$$= \left| \frac{a^2}{2} (t_1 - t_3)(t_2 - t_1)(t_2 - t_3) \right|$$

$$= \left| \frac{a^2}{2} (t_1 - t_2)(t_2 - t_3)(t_3 - t_1) \right| \dots (5)$$

From equations (4) and (5), we get

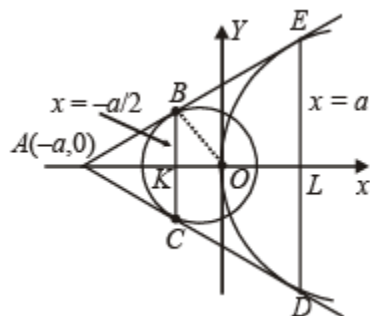
$$\frac{Ar(\Delta ABC)}{Ar(\Delta PQR)} = \frac{a^2 |(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)|}{\frac{a^2}{2} |(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)|} = \frac{2}{1}$$

\therefore The required ratio is 2 : 1

Q.8. From a point A common tangents are drawn to the circle $x^2 + y^2 = a^2/2$ and parabola $y^2 = 4ax$. Find the area of the quadrilateral formed by the common tangents, the chord of contact of the circle and the chord of contact of the parabola. (1996 - 2 Marks)

Ans.

Sol. This line will touch the circle $x^2 + y^2 = a^2/2$



$$\text{if } \frac{a}{m} = \pm \frac{a}{\sqrt{2}} \sqrt{m^2 + 1} \quad [c = \pm r\sqrt{1+m^2}]$$

$$\Rightarrow \frac{a^2}{m^2} = \frac{a^2}{2} (m^2 + 1)$$

$$\Rightarrow 2 = m^4 + m^2 \Rightarrow m^4 + m^2 - 2 = 0$$

$$\Rightarrow (m^2 + 2)(m^2 - 1) = 0 \Rightarrow m = 1, -1$$

Thus the two tangents (common one) are $y = x + a$ and $y = -x - a$

These two intersect each other at $(-a, 0)$ The chord of contact at A $(-a, 0)$ for the circle $x^2 + y^2 = a^2/2$ is

$(-a \cdot x) + 0 \cdot y = a^2/2$ i.e., $x = -a/2$ and the chord of contact at A $(-a, 0)$ for the parabola $y^2 = 4ax$ is $0 \cdot y = 2a(x - a)$ i.e., $x = a$

Note that DE is latus rectum of parabola $y^2 = 4ax$, therefore its length is $4a$.

Chords of contact are clearly parallel to each other, so req. quadrilateral is a trapezium.

Ar (trap BCDE) $\frac{1}{2} = BC + DE \times KL$

$$= \frac{1}{2} (a + 4a) \left(\frac{3a}{2} \right) = \frac{15a^2}{4}$$

Q.9. A tangent to the ellipse $x^2 + 4y^2 = 4$ meets the ellipse $x^2 + 2y^2 = 6$ at P and Q. Prove that the tangents at P and Q of the ellipse $x^2 + 2y^2 = 6$ are at right angles. (1997 - 5 Marks)

Ans.

Sol. The given ellipses are

$$\frac{x^2}{4} + \frac{y^2}{1} = 1 \dots (1)$$

$$\text{and } \frac{x^2}{6} + \frac{y^2}{3} = 1 \dots (2)$$

Then the equation of tangent to (1) at any point T $(2 \cos \theta, \sin \theta)$ is given by

$$\frac{x \cdot 2 \cos \theta}{4} + \frac{y \cdot \sin \theta}{1} = 1$$

$$\text{or } \frac{x \cos \theta}{2} + y \sin \theta = 1 \dots (3)$$

Let this tangent meet the ellipse (2) at P and Q.

Let the tangents drawn to ellipse (2) at P and Q meet each other at R (x_1, y_1)

Then PQ is chord of contact of ellipse (2) with respect to the pt R (x_1, y_1) and is given by

$$\frac{xx_1}{6} + \frac{yy_1}{3} = 1 \dots (4)$$

Clearly equations (3) and (4) represent the same lines and hence should be identical. Therefore comparing the coefficients, we get

$$\frac{\frac{\cos \theta}{2}}{\frac{x_1}{6}} = \frac{\frac{\sin \theta}{1}}{\frac{y_1}{3}} = \frac{1}{1}$$

$$\Rightarrow x_1 = 3 \cos \theta, y_1 = 3 \sin \theta \Rightarrow x_1^2 + y_1^2 = 9$$

$$\Rightarrow \text{Locus of } (x_1, y_1) \text{ is } x^2 + y^2 = 9$$

which is the director circle of the ellipse $\frac{x^2}{6} + \frac{y^2}{3} = 1$ and

Thus tangents at P and Q are at right \angle 's.

KEY CONCEPT : We know that the director circle is the locus of intersection point of the tangents which are at right \angle .

Q.10. The angle between a pair of tangents drawn from a point P to the parabola $y^2 = 4ax$ is 45° . Show that the locus of the point P is a hyperbola. (1998 - 8 Marks)

Ans.

Sol. Let P (e, f) be any point on the locus. Equation of pair of tangents from P (e, f) to the parabola $y^2 = 4ax$ is $[fy - 2a(x + e)]^2$

$$= (f^2 - 4ae)(y^2 - 4ax) \quad [T^2 = SS_1]$$

Here, a = coefficient of $x^2 = 4a^2 \dots(1)$

$2h$ = coefficient of $xy = -4af \dots(2)$

and b = coefficient of $y^2 = f^2 - (f^2 - 4ae) = 4ae \dots(3)$

If they include an angle 45° , then

$$\tan 45^\circ = \frac{2\sqrt{h^2 - ab}}{a + b}$$

$$\text{or}, (a + b)^2 = 4(h^2 - ab)$$

$$\text{or}, (4a^2 + 4ae)^2 = 4[4a^2f^2 - (4a^2)(4ae)]$$

$$\text{or}, (a + e)^2 = f^2 - 4ae \text{ or } e^2 + 6ae + a^2 - f^2 = 0$$

$$\text{or}, (e + 3a)^2 - f^2 = 8a^2$$

Hence the required locus is $(x + 3a)^2 - y^2 = 8a^2$, which is a hyperbola.

Q.11. Consider the family of circles $x^2 + y^2 = r^2$, $2 < r < 5$. If in the first quadrant, the common tangent to a circle of this family and the ellipse $4x^2 + 25y^2 = 100$ meets the co-ordinate axes at A and B, then find the equation of the locus of the mid-point of AB. (1999 - 10 Marks)

Ans. Sol. Let any point P on ellipse $4x^2 + 25y^2 = 100$ be $(5 \cos\theta, 2 \sin\theta)$. So equation of tangent to the ellipse at P will be

$$\frac{x \cos\theta}{5} + \frac{y \sin\theta}{2} = 1$$

Tangent (1) also touches the circle $x^2 + y^2 = r^2$, so its distance from origin must be r .

Tangent (2) intersects the coordinate axes at $A\left(\frac{5}{\cos\theta}, 0\right)$ and $B\left(0, \frac{2}{\sin\theta}\right)$ respectively.
Let M (h, k) be the

midpoint of line segment AB. Then by mid point formula

$$h = \frac{5}{2\cos\theta}, k = \frac{1}{\sin\theta} \Rightarrow \cos\theta = \frac{5}{2h}, \sin\theta = \frac{1}{k}$$

$$\Rightarrow \cos^2\theta + \sin^2\theta = \frac{25}{4h^2} + \frac{1}{k^2}$$

$$\text{Hence locus of } M(h, k) \text{ is } \frac{25}{x^2} + \frac{4}{y^2} = 4$$

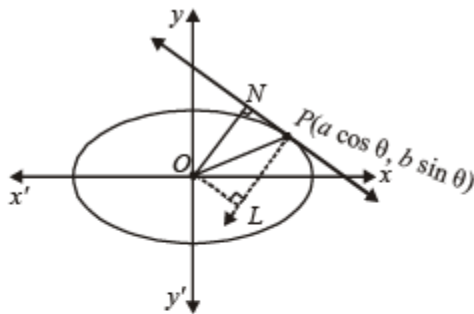
Locus is independent of r.

Q.12. Find the co-ordinates of all the points P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, for which the area of the triangle PON is maximum, where O denotes the origin and N, the foot of the perpendicular from O to the tangent at P. (1999 - 10 Marks)

Sol. The ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$..(1)

Since this ellipse is symmetrical in all four quadrants, either there exists no such P or four points, one in each quadrant.

Without loss of generality we can assume that $a > b$ and P lies in first quadrant.



Let P (a cosθ, b sinθ) then equation of tangent is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

$$\therefore ON = \frac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

$$\text{Equation of ON is } \frac{x}{b}\sin\theta - \frac{y}{a}\cos\theta = 0$$

Equation of normal at P is $ax \sec\theta - by \operatorname{cosec}\theta = a^2 - b^2$

$$\therefore OL = \frac{a^2 - b^2}{\sqrt{a^2 \sec^2 \theta + b^2 \csc^2 \theta}} = \frac{(a^2 - b^2) \sin \theta \cos \theta}{\sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}}$$

and NP = OL

$$\therefore NP = \frac{(a^2 - b^2) \sin \theta \cos \theta}{\sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}}$$

$$\therefore Z = \text{Area of OPN} = \frac{1}{2} \times ON \times NP$$

$$= \frac{1}{2} ab(a^2 - b^2) \frac{\sin \theta \cos \theta}{a^2 \sin^2 \theta + b^2 \cos^2 \theta}$$

$$\text{Let } u = \frac{a^2 \sin^2 \theta + b^2 \cos^2 \theta}{\sin \theta \cos \theta} = a^2 \tan \theta + b^2 \cot \theta$$

$$\frac{du}{d\theta} = a^2 \sec^2 \theta - b^2 \csc^2 \theta = 0 \Rightarrow \tan \theta = b/a$$

$$\left(\frac{d^2u}{d\theta^2} \right)_{\tan^{-1} b/a} > 0, \text{ u is minimum at } \theta = \tan^{-1} b/a$$

So Z is maximum at $\theta = \tan^{-1} b/a$

$$\therefore P \text{ is } \left(\frac{a^2}{\sqrt{a^2 + b^2}}, \frac{b^2}{\sqrt{a^2 + b^2}} \right)$$

By symmetry, we have four such points

$$\left(\pm \frac{a^2}{\sqrt{a^2 + b^2}}, \pm \frac{b^2}{\sqrt{a^2 + b^2}} \right)$$

Q.13. Let ABC be an equilateral triangle inscribed in the circle $x^2 + y^2 = a^2$. Suppose perpendiculars from A, B, C to the major axis of the ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, (a > b)$ meets the ellipse respectively, at P, Q, R. so that P, Q, R lie on the same side of the major axis as A, B, C respectively. Prove that the normals to the ellipse drawn at the points P, Q and R are concurrent. (2000 - 7 Marks)

Sol. Let A, B, C be the point on circle whose coordinates are $A = [a \cos \theta, a \sin \theta]$

$$B = \left[a \cos \left(\theta + \frac{2\pi}{3} \right), a \sin \left(\theta + \frac{2\pi}{3} \right) \right]$$

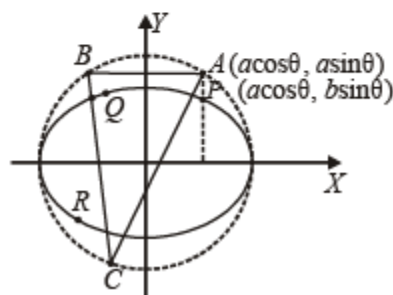
$$\text{and } C = \left[a \cos \left(\theta + \frac{4\pi}{3} \right), a \sin \left(\theta + \frac{4\pi}{3} \right) \right]$$

Further, $P [a \cos \theta, b \sin \theta]$ (Given)

$$Q = \left[a \cos \left(\theta + \frac{2\pi}{3} \right), b \sin \left(\theta + \frac{2\pi}{3} \right) \right]$$

$$\text{and } R = \left[a \cos \left(\theta + \frac{4\pi}{3} \right), b \sin \left(\theta + \frac{4\pi}{3} \right) \right]$$

It is given that P, Q, R are on the same side of x-axis as A, B, C.



So required normals to the ellipse are $ax \sec \theta - by \operatorname{cosec} \theta = a^2 - b^2 \dots (1)$

$$ax \sec \left(\theta + \frac{2\pi}{3} \right) - by \operatorname{cosec} \left(\theta + \frac{2\pi}{3} \right) = a^2 - b^2 \dots (2)$$

$$ax \sec \left(\theta + \frac{4\pi}{3} \right) - by \operatorname{cosec} \left(\theta + \frac{4\pi}{3} \right) = a^2 - b^2 \dots (3)$$

Now, above three normals are concurrent $\Rightarrow \Delta = 0$

$$\text{where } \Delta = \begin{vmatrix} \sec \theta & \operatorname{cosec} \theta & 1 \\ \sec \left(\theta + \frac{2\pi}{3} \right) & \operatorname{cosec} \left(\theta + \frac{2\pi}{3} \right) & 1 \\ \sec \left(\theta + \frac{4\pi}{3} \right) & \operatorname{cosec} \left(\theta + \frac{4\pi}{3} \right) & 1 \end{vmatrix}$$

Multiplying and dividing the different rows R_1, R_2 and R_3 by $\sin \theta \cos \theta$,

$$\sin\left(\theta + \frac{2\pi}{3}\right) \cos\left(\theta + \frac{2\pi}{3}\right)$$

and $\sin\left(\theta + \frac{4\pi}{3}\right) \cos\left(\theta + \frac{4\pi}{3}\right)$ respectively, we get

$$\Delta = \frac{1}{\sin \theta \cos \theta \sin\left(\theta + \frac{2\pi}{3}\right) \cos\left(\theta + \frac{2\pi}{3}\right)} \times$$

$$\sin\left(\theta + \frac{4\pi}{3}\right) \cos\left(\theta + \frac{4\pi}{3}\right)$$

$$\begin{vmatrix} \sin \theta & \cos \theta & \sin 2\theta \\ \sin\left(\theta + \frac{2\pi}{3}\right) & \cos\left(\theta + \frac{2\pi}{3}\right) & \sin\left(2\theta + \frac{4\pi}{3}\right) \\ \sin\left(\theta - \frac{2\pi}{3}\right) & \cos\left(\theta - \frac{2\pi}{3}\right) & \sin\left(2\theta - \frac{4\pi}{3}\right) \end{vmatrix} = 0$$

[Operating $R_2 \rightarrow R_2 + R_3$ and simplifying R_2 we get $R_2 \equiv R_1$]

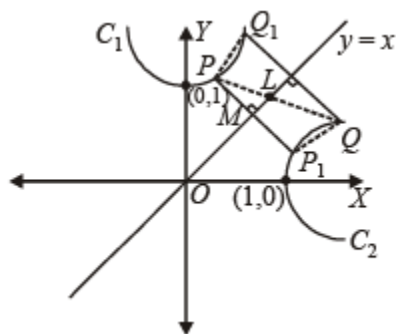
Q.14. Let C_1 and C_2 be respectively, the parabolas $x^2 = y - 1$ and $y^2 = x - 1$. Let P be any point on C_1 and Q be any point on C_2 . Let P_1 and Q_1 be the reflections of P and Q , respectively, with respect to the line $y = x$. Prove that P_1 lies on C_2 , Q_1 lies on C_1 and $PQ \geq \min\{PP_1, QQ_1\}$. Hence or otherwise determine points P_0 and Q_0 on the parabolas C_1 and C_2 respectively such that $P_0Q_0 \leq PQ$ for all pairs of points (P, Q) with P on C_1 and Q on C_2 . (2000 - 10 Marks)

Ans. Sol. Given that $C_1 : x^2 = y - 1$; $C_2 : y^2 = x - 1$

Let $P (x_1, x_1^2 + 1)$ on C_1 and $Q (y_2^2 + 1, y_2^2)$ on C_2 .

Now the reflection of pt P in the line $y = x$ can be obtained by interchanging the values of abscissa and ordinate.

Thus reflection of pt. $P (x_1, x_1^2 + 1)$ is $P_1 (x_1^2 + 1, x_1)$ and reflection of pt. $Q (y_2^2 + 1, y_2^2)$ is $Q_1 (y_2^2, y_2^2 + 1)$



It can be seen clearly that P_1 lies on C_2 and Q_1 on C_1 .

Now PP_1 and QQ_1 both are perpendicular to mirror line $y = x$.

Also M is mid pt. of PP_1 ($Q_1 P_1$ is mirror image of P in $y = x$)

$$PM = \frac{1}{2} PP_1$$

In rt $\triangle PML$,

$$PL > PM \Rightarrow PL > \frac{1}{2} PP_1 \dots (i)$$

Similarly,

$$LQ > \frac{1}{2} QQ_1 \dots (ii)$$

Adding (i) and (ii) we get

$$PL + LQ > \frac{1}{2} (PP_1 + QQ_1)$$

$$\Rightarrow PQ > \frac{1}{2} (PP_1 + QQ_1)$$

$\Rightarrow PQ$ is more than the mean of PP_1 and QQ_1

$\Rightarrow PQ \geq \min (PP_1, QQ_1)$ Let $\min (PP_1, QQ_1) = PP_1$

$$\text{then } PQ^2 \geq PP_1^2 = (x_1^2 + 1 - x_1)^2 + (x_1^2 + 1 - x_1)^2$$

$$= 2(x_1^2 + 1 - x_1)^2 = f(x_1)$$

$$\Rightarrow f'(x_1) = 4(x_1^2 + 1 - x_1)(2x_1 - 1)$$

$$= 4 \left(\left(x_1 - \frac{1}{2} \right)^2 + \frac{3}{4} \right) (2x_1 - 1)$$

$$\therefore f'(x_1) = 0 \text{ when } x_1 = \frac{1}{2}$$

$$\text{Also } f'(x_1) < 0 \text{ if } x_1 < \frac{1}{2} \text{ and } f'(x_1) > 0 \text{ if } x_1 > \frac{1}{2}$$

$$\Rightarrow f(x_1) \text{ is min when } x_1 = \frac{1}{2}$$

Thus if at $x_1 = \frac{1}{2}$ pt P is P_O on C_1

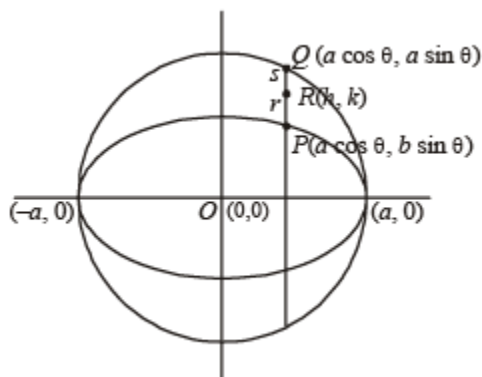
$$P_0 \left(\frac{1}{2}, \left(\frac{1}{2} \right)^2 + 1 \right) = \left(\frac{1}{2}, \frac{5}{4} \right)$$

Similarly Q_O on C_2 will be image of P_O with respect to $y = x$

$$\therefore Q_0 \left(\frac{5}{4}, \frac{1}{2} \right)$$

Q.15. Let P be a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, 0 < b < a$. Let the line parallel to y-axis passing through P meet the circle $x^2 + y^2 = a^2$ at the point Q such that P and Q are on the same side of x-axis. For two positive real numbers r and s, find the locus of the point R on PQ such that $PR : RQ = r : s$ as P varies over the ellipse. (2001 - 4 Marks)

Sol. Let the co-ordinates of P be $(a \cos \theta, b \sin \theta)$ then coordinates of Q are $(a \cos \theta, a \sin \theta)$



As R (h, k) divides PQ in the ratio r : s, then

$$h = \frac{s(a \cos \theta) + r(a \cos \theta)}{(r+s)} = a \cos \theta$$

$$\Rightarrow \cos \theta = \frac{h}{a}$$

$$k = \frac{s(b \sin \theta) + r(a \sin \theta)}{(r+s)} = \frac{\sin \theta (bs + ar)}{(r+s)}$$

$$\Rightarrow \sin \theta = \frac{k(r+s)}{(bs+ar)} \because \cos^2 \theta + \sin^2 \theta = 1$$

$$\therefore \frac{h^2}{a^2} + \frac{k^2(r+s)^2}{(bs+ar)^2} = 1$$

Hence locus of R is : $\frac{x^2}{a^2} + \frac{y^2(r+s)^2}{(bs+ar)^2} = 1$ which is equation of an ellipse.

Q.16. Prove that, in an ellipse, the perpendicular from a focus upon any tangent and the line joining the centre of the ellipse to the point of contact meet on the corresponding directrix. (2002 - 5 Marks)

Ans. Sol.

Let the ellipse be : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and O be the centre.

Tangent at P (x_1, y_1) is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0$ whose

slope = $-\frac{b^2 x_1}{a^2 y_1}$, Focus is S ($ae, 0$).

Equation of the line perpendicular to tangent at P is

$$y = \frac{a^2 y_1}{b^2 x_1} (x - ae) \dots (1)$$

Equation of OP is $y = \frac{y_1}{x_1} x \dots (2)$

(1) and (2) intersect, $\frac{y_1}{x_1} x = \frac{a^2 y_1}{b^2 x_1} (x - ae)$

$$\Rightarrow x(a^2 - b^2) = a^3 e \Rightarrow x \cdot a^2 e^2 = a^3 e$$

$$\Rightarrow x = a/e$$

Which is the corresponding directrix.

Q.17. Normals are drawn from the point P with slopes m_1, m_2, m_3 to the parabola $y^2 = 4x$. If locus of P with $m_1 m_2 = a$ is a part of the parabola itself then find a. (2003 - 4 Marks)

Ans. $a = 2$

Sol. Let P be the pt. (h, k). Then eqn of normal to parabola $y^2 = 4x$ from point (h, k), if m is the slope of normal, is $y = mx - 2m - m^3$

As it passes through (h, k), therefore $mh - k - 2m - m^3 = 0$ or, $m^3 + (2 - h) + k = 0$... (1)

Which is cubic in m, giving three values of m say m_1, m_2 and m_3 .

Then $m_1 m_2 m_3 = -k$ (from eqn) but given that $m_1 m_2 = a$

$$\therefore \text{We get } m_3 = -\frac{k}{a}$$

But m^3 must satisfy eqⁿ (1)

$$\therefore \frac{-k^3}{a^3} + (2-h)\left(\frac{-k}{a}\right) + k = 0$$

$$\Rightarrow k^2 - 2a^2 - ha^2 - a^3 = 0$$

\therefore Locus of P (h, k) is $y^2 = a^2x + (a^3 - 2a^2)$

But ATQ, locus of P is a part of parabola $y^2 = 4x$, therefore comparing the two, we get $a^2 = 4$ and $a^3 - 2a^2 = 0 \Rightarrow a = 2$

Q.18. Tangent is drawn to parabola $y^2 - 2y - 4x + 5 = 0$ at a point P which cuts the directrix at the point Q. A point R is such that it divides QP externally in the ratio $1/2 : 1$. Find the locus of point R. (2004 - 4 Marks)

Ans. $(x - 1)(y - 1)^2 + 4 = 0$

Sol. The given eqⁿ of parabola is $y^2 - 2y - 4x + 5 = 0$... (1)

$$\Rightarrow (y - 1)^2 = 4(x - 1)$$

Any parametric point on this parabola is P ($t^2 + 1, 2t + 1$)

Differentiating (1) w.r. to x, we get

$$2y \frac{dy}{dx} - 2 \frac{dy}{dx} - 4 = 0 \Rightarrow \frac{dy}{dx} = \frac{2}{y-1}$$

∴ Slope of tangent to (1) at point P ($t^2 + 1, 2t + 1$) is

$$m = \frac{2}{2t} = \frac{1}{t}$$

∴ Eqn of tangent at P ($t^2 + 1, 2t + 1$) is

$$y - (2t + 1) = \frac{1}{t}(x - t^2 - 1)$$

$$\Rightarrow yt - 2t^2 - t = x - t^2 = 1$$

$$\Rightarrow x - yt + (t^2 + t - 1) = 0 \dots (2)$$

Now directrix of given parabola is $(x - 1) = -1 \Rightarrow x = 0$

Tangent (2) meets directrix at $Q\left(0, \frac{t^2 + t - 1}{t}\right)$

Let pt. R be (h, k)

ATQ, R divides the line joining QP in the ratio $\frac{1}{2}:1$ i.e., $1:2$ externally.

$$\therefore (h, k) = \left(\frac{1(1+t^2)-0}{-1}, \frac{t+2t^2-2t^2-2t+2}{-t} \right)$$

$$\Rightarrow h = -(1 + t^2) \text{ and } k = \frac{t-2}{t}$$

$$\Rightarrow t^2 = -1 - h \text{ and } t = \frac{2}{1-k}$$

Eliminating t, we get $\left(\frac{2}{1-k}\right)^2 = -1-h$

$$\Rightarrow 4 = -(1-k)^2(1-h) \Rightarrow (h-1)(k-1)^2 + 4 = 0$$

∴ Locus of R (h, k) is $(x-1)(y-1)^2 + 4 = 0$

Q.19. Tangents are drawn from any point on the hyperbola

$\frac{x^2}{9} - \frac{y^2}{4} = 1$ to the circle $x^2 + y^2 = 9$. Find the locus of mid-point of the chord of contact. (2005 - 4 Marks)

Sol. Any pt on the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ is $(3 \sec\theta, 2 \tan\theta)$ Then, equation of chord of contact to the circle $x^2 + y^2 = 9$,

with respect to the pt. $(3 \sec\theta, 2 \tan\theta)$ is $(3 \sec\theta) x + (2 \tan\theta) y = 9 \dots(i)$

If (h, k) be the mid point of chord of contact then equation of chord of contact will be $hx + ky - 9 = h^2 + k^2 - 9$ ($T = S_1$)

or, $hx + ky = h^2 + k^2 \dots(ii)$

But equations (i) and (ii) represent the same st. line and hence should be identical, therefore, we get

$$\frac{3\sec\theta}{h} = \frac{2\tan\theta}{k} = \frac{9}{h^2 + k^2}$$

$$\Rightarrow \sec\theta = \frac{3h}{h^2 + k^2}, \tan\theta = \frac{9k}{2(h^2 + k^2)}$$

$$\sec^2\theta - \tan^2\theta = 1$$

$$\Rightarrow \frac{9h^2}{(h^2 + k^2)^2} - \frac{81k^2}{4(h^2 + k^2)^2} = 1$$

$$\Rightarrow 4h^2 - 9k^2 = \frac{4}{9}(h^2 + k^2)^2$$

$$\text{or, } \frac{h^2}{9} - \frac{k^2}{4} = \left(\frac{h^2 + k^2}{9}\right)^2$$

$$\therefore \text{Locus of } (h, k) \text{ is } \frac{x^2}{9} - \frac{y^2}{4} = \left(\frac{x^2 + y^2}{9}\right)^2$$

Q.20. Find the equation of the common tangent in 1st quadrant to the circle $x^2 + y^2 = 16$ and the ellipse

$\frac{x^2}{25} + \frac{y^2}{4} = 1$ Also find the length of the intercept of the tangent between the coordinate axes. (2005 - 4 Marks)

Sol. Let the common tangent to circle $x^2 + y^2 = 16$ and ellipse $x^2/25 + y^2/4 = 1$ be

$$y = mx + \sqrt{25m^2 + 4} \text{ ..(i)}$$

As it is tangent to circle $x^2 + y^2 = 16$, we should have

$$\frac{\sqrt{25m^2 + 4}}{\sqrt{m^2 + 1}} = 4$$

[Using : length of perpendicular from (0,0) to (1) = 4]

$$\Rightarrow 25m^2 + 4 = 16m^2 + 16 \Rightarrow 9m^2 = 12$$

$$\Rightarrow m = \frac{-2}{\sqrt{3}}$$

[Leaving + ve sign to consider tangent in I quadrant] \therefore Equation of common tangent is

$$y = -\frac{2}{\sqrt{3}}x + \sqrt{25 \cdot \frac{4}{3} + 4} \Rightarrow y = -\frac{2}{\sqrt{3}}x + 4\sqrt{\frac{7}{3}}$$

This tangent meets the axes at $A(2\sqrt{7}, 0)$ and $B\left(0, 4\sqrt{\frac{7}{3}}\right)$

\therefore Length of intercepted portion of tangent between axes

$$= AB = \sqrt{(2\sqrt{7})^2 + \left(4\sqrt{\frac{7}{3}}\right)^2} = 14/\sqrt{3}$$

Match the following of Conic Sections

DIRECTIONS : Each question contains statements given in two columns, which have to be matched. The statements in Column-I are labelled A, B, C and D, while the statements in Column-II are labelled p, q, r, s and t. Any given statement in Column-I can have correct matching with ONE OR MORE statement(s) in Column-II. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example :

	p	q	r	s	t
A	<input checked="" type="radio"/>	<input checked="" type="radio"/>	<input type="radio"/>	<input checked="" type="radio"/>	<input checked="" type="radio"/>
B	<input type="radio"/>	<input checked="" type="radio"/>	<input checked="" type="radio"/>	<input type="radio"/>	<input type="radio"/>
C	<input checked="" type="radio"/>	<input checked="" type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>
D	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input checked="" type="radio"/>	<input type="radio"/>

If the correct matches are A-p, s and t; B-q and r; C-p and q; and D-s then the correct darkening of bubbles will look like the given.

Q.1. Match the following : (3, 0) is the pt. from which three normals are drawn to the parabola $y^2 = 4x$ which meet the parabola in the points P, Q and R. Then (2006 - 6M)

Column I	Column II
(A) Area of ΔPQR	(p) 2
(B) Radius of circumcircle of ΔPQR	(q) $5/2$
(C) Centroid of ΔPQR	(r) $(5/2, 0)$
(D) Circumcentre of ΔPQR	(s) $(2/3, 0)$

Ans. (A)-(p), (B)-(q), (C)-(s), (D)-(r)

Sol. Let $y = mx - 2m - m^3$ be the equation of normal to $y^2 = 4x$.

As it passes through (3, 0), we get $m = 0, 1, -1$

Then three points on parabola are given by $(m^2, -2m)$ for $m = 0, 1, -1$

\therefore P (0, 0), Q (1, 2), R (1, -2)

$$\therefore \text{Area of } \Delta PQR = \frac{1}{2} \times \begin{vmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & -2 & 1 \end{vmatrix} = 2 \text{ sq. units}$$

Radius of circum-circle,

$$R = \frac{abc}{4\Delta} = \frac{\sqrt{5} \times \sqrt{5} \times 4}{4 \times 2} = \frac{5}{2} \quad \text{NOTE THIS STEP}$$

(where, a, b, c are the sides of ΔPQR)

$$\text{Centroid of } \Delta PQR = \left(\frac{2}{3}, 0\right)$$

$$\text{Circumcentre} = \left(\frac{5}{2}, 0\right)$$

Thus, (A) - (p); (B) - (q); (C) - (s); (D) - (r)

DIRECTIONS : Each question contains statements given in two columns, which have to be matched. The statements in Column-I are labelled A, B, C and D, while the statements in Column-II are labelled p, q, r, s and t. Any given statement in Column-I can have correct matching with **ONE OR MORE** statement(s) in Column-II. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example :

	p	q	r	s	t
A	<input checked="" type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input checked="" type="radio"/>	<input checked="" type="radio"/>
B	<input type="radio"/>	<input checked="" type="radio"/>	<input checked="" type="radio"/>	<input type="radio"/>	<input type="radio"/>
C	<input checked="" type="radio"/>	<input checked="" type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>
D	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input checked="" type="radio"/>	<input type="radio"/>

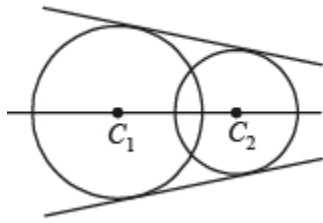
If the correct matches are A-p, s and t; B-q and r; C-p and q; and D-s then the correct darkening of bubbles will look like the given.

2. Match the statements in Column I with the properties in Column II and indicate your answer by darkening the appropriate bubbles in the 4×4 matrix given in the ORS. (2007 -6 marks)

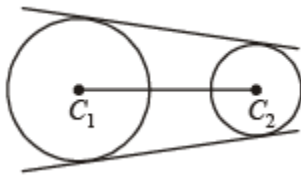
Column I	Column II
(A) Two intersecting circles	(p) have a common tangent
(B) Two mutually external circles	(q) have a common normal
(C) Two circles, one strictly inside the other	(r) do not have a common tangent
(D) Two branches of a hyperbola	(s) do not have a common normal

Ans. (A)-p, q; (B)-p, q ; (C)-q, r ; (D)-q, r

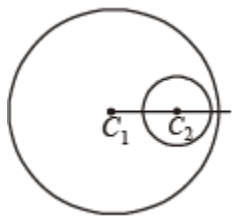
Sol. (A) - p, q



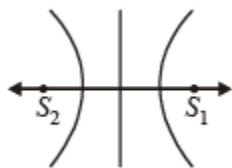
It is clear from the figure that two intersecting circles have a common tangent and a common normal joining the centres (B) - p, q



(C) - q, r Two circle when one is completely inside the other have a common normal C_1C_2 but no common tangent.



(D) - q, r Two branches of hyperbola have no common tangent but have a common normal joining S_1S_2



Matrix Match (A) - p, q; (B) - p, q; (C) - q, r; (D) - q, r

DIRECTIONS : Each question contains statements given in two columns, which have to be matched. The statements in Column-I are labelled A, B, C and D, while the statements in Column-II are labelled p, q, r, s and t. Any given statement in Column-I can have correct matching with ONE OR MORE statement(s) in Column-II. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example :

	p	q	r	s	t
A	<input checked="" type="radio"/>	<input checked="" type="radio"/>	<input type="radio"/>	<input checked="" type="radio"/>	<input checked="" type="radio"/>
B	<input type="radio"/>	<input checked="" type="radio"/>	<input checked="" type="radio"/>	<input type="radio"/>	<input type="radio"/>
C	<input checked="" type="radio"/>	<input checked="" type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>
D	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input checked="" type="radio"/>	<input type="radio"/>

If the correct matches are A-p, s and t; B-q and r; C-p and q; and D-s then the correct darkening of bubbles will look like the given.

3. Match the conics in Column I with the statements/expressions in Column II. (2009)

	Column II
	(p) The locus of the point (h, k) for which the line $hx + ky = 1$ touches the circle $x^2 + y^2 = 4$
Column I	(q) Points z in the complex plane satisfying
(A) Circle	$ z + 2 - z - 2 = \pm 3$
(B) Parabola	(r) Points of the conic have parametric representation
(C) Ellipse	$x = \sqrt{3} \left(\frac{1-t^2}{1+t^2} \right), \quad y = \frac{2t}{1+t^2}$
	(s) The eccentricity of the conic lies in the interval $1 \leq e < \infty$
(D) Hyperbola	(t) Points z in the complex plane satisfying
	$\operatorname{Re} (z+1)^2 = z ^2 + 1$

Ans. (A)-p; (B)-s, t ; (C)-r ; (D)-q, s

Sol. (p) As the line $hx + ky = 1$, touches the circle $x^2 + y^2 = 4$

\therefore Length of perpendicular from centre $(0, 0)$ of circle to line = radius of the circle

$$\Rightarrow \frac{1}{\sqrt{h^2 + k^2}} = 2 \Rightarrow h^2 + k^2 = \frac{1}{4}$$

$$\Rightarrow \frac{1}{\sqrt{h^2+k^2}} = 2 \Rightarrow h^2+k^2 = \frac{1}{4}$$

\therefore Locus of (h, k) is $x^2+y^2 = \frac{1}{4}$, which is a circle.

(q) We know that if $|z-z_1| - |z-z_2| = k$

where $|k| < |z_1-z_2|$ then z traces a hyperbola.

Here $|z+2| - |z-2| = \pm 3$

\therefore Locus of z is a hyperbola.

(r) We have $x = \sqrt{3} \left(\frac{1-t^2}{1+t^2} \right)$, $y = \frac{2t}{1+t^2}$

$$\Rightarrow \frac{x}{\sqrt{3}} = \frac{1-t^2}{1+t^2} \text{ and } y = \frac{2t}{1+t^2}$$

On squaring and adding, we get

$$\frac{x^2}{3} + y^2 = \frac{(1-t^2)^2 + 4t^2}{(1+t^2)^2} = 1 \text{ or } \frac{x^2}{3} + \frac{y^2}{1} = 1$$

which is the equation of an ellipse.

(s) We know eccentricity for a parabola = 1

for an ellipse < 1

for a hyperbola > 1

\therefore The conics whose eccentricity lies in $1 \leq e < \infty$ are parabola and hyperbola.

(t) Let $z = x + iy$ then

$$\operatorname{Re} [(x+1) + iy]^2 = x^2 + y^2 + 1$$

$$\Rightarrow (x+1)^2 - y^2 = x^2 + y^2 + 1 \Rightarrow y^2 = x, \text{ which is a parabola.}$$

DIRECTIONS (Q. 4) : Following question has matching lists. The codes for the list have choices (a), (b), (c) and (d) out of which ONLY ONE is correct.

Q. 4. A line $L : y = mx + 3$ meets y - axis at $E(0, 3)$ and the arc of the parabola $y^2 = 16x$, $0 \leq y \leq 6$ at the point $F(x_0, y_0)$. The tangent to the parabola at $F(x_0, y_0)$ intersects the y -axis at $G(0, y_1)$. The slope m of the line L is chosen such that the area of the triangle EFG has a local maximum. (JEE Adv. 2013) Match List I with List II and select the correct answer using the code given below the lists :

List I		List II							
P.	$m =$	1.	$\frac{1}{2}$						
Q.	Maximum area of $\triangle EFG$ is	2.	4						
R.	$y_0 =$	3.	2						
S.	$y_1 =$	4.	1						
Codes:									
	P	Q	R	S		P	Q	R	S
(a)	4	1	2	3	(b)	3	4	1	2
(c)	1	3	2	4	(d)	1	3	4	2

Ans. (a)

Sol. (a) Equation of tangent to $y^2 = 16x$ at $F(x_0, y_0)$ $yy_0 = 8(x + x_0)$

$$\Rightarrow G\left(0, \frac{8x_0}{y_0}\right)$$

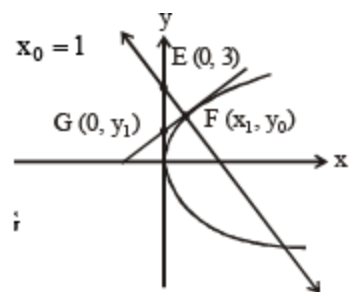
$$\text{Area of } \triangle EFG = \frac{1}{2} \times (3 - y_1) \times x_0$$

$$A = \frac{1}{2} x_0 \left(3 - \frac{8x_0}{y_0} \right)$$

$$A = \frac{1}{2} \times \frac{y_0^2}{16} \left(3 - \frac{y_0}{2} \right) = \frac{1}{32} \left(3y_0^2 - \frac{y_0^3}{2} \right)$$

$$\frac{dA}{dy_0} = \frac{1}{32} \left(6y_0 - \frac{3y_0^2}{2} \right)$$

$$\frac{dA}{dy_0} = 0 \Rightarrow y_0 = 4 \Rightarrow$$



$$\therefore y_1 = \frac{8 \times 1}{4} = 2$$

Also $y_0 = mx_0 + 3$

$\therefore 4 = m + 3$ or $m = 1$ maximum area of $\triangle EFG$

$$= \frac{1}{32} \left[3 \times 4^2 - \frac{4^3}{2} \right]$$

$$= \frac{1}{32} [48 - 32] = \frac{1}{2}$$

$\therefore (P) \rightarrow (4), (Q) \rightarrow (1), (R) \rightarrow (2), (S) \rightarrow (3)$

Integar Type ques of Conic Sections

1.

The line $2x + y = 1$ is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

If this line passes through the point of intersection of the nearest directrix and the x-axis, then the eccentricity of the hyperbola is (2010)

Ans. (2)

Sol. Intersection point of nearest directrix $x = \frac{a}{e}$ and x-axis

is $\left(\frac{a}{e}, 0\right)$

As $2x + y = 1$ passes through $\left(\frac{a}{e}, 0\right)$

$$\therefore \frac{2a}{e} = 1 \Rightarrow a = \frac{e}{2}$$

Also $y = -2x + 1$ is a tangent to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$\therefore 1 = a^2(-2)^2 - b^2 \Rightarrow 4a^2 - b^2 = 1$$

$$\Rightarrow 4a^2 - a^2e^2 - 1 = 1 \Rightarrow 4 \times \frac{e^2}{4} - \frac{e^2}{4}(e^2 - 1) = 1$$

$$\Rightarrow 4e^2 - e^4 + e^2 = 4 \Rightarrow e^4 - 5e^2 + 4 = 0$$

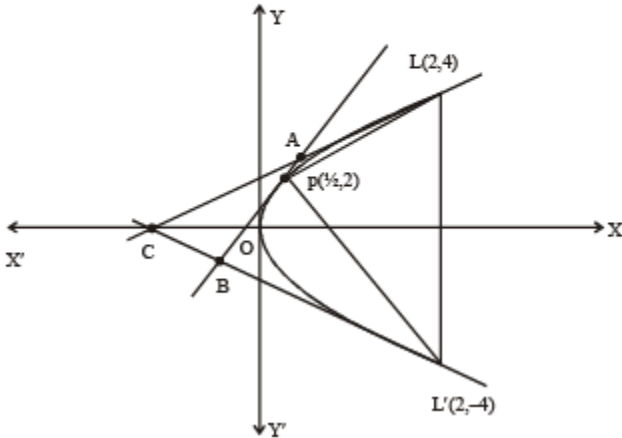
$$\Rightarrow e^2 = 4 \text{ as } e > 1 \text{ for hyperbola.} \Rightarrow e = 2$$

2. Consider the parabola $y^2 = 8x$. Let Δ_1 be the area of the triangle formed by the end points of its latus rectum and the point $P\left(\frac{1}{2}, 2\right)$ on the parabola and Δ_2 be

the area of the triangle formed by drawing tangents at P and at the end points of the latus rectum. Then $\frac{\Delta_1}{\Delta_2}$ is (2011)

Ans. (2)

Sol. $\Delta_1 = \text{Area of } \triangle PLL' = \frac{1}{2} \times 8 \times \frac{3}{2} = 6$



Equation of AB, $y = 2x + 1$ Equation of AC, $y = x + 2$ Equation of BC, $-y = x + 2$
Solving above equations we get A (1, 3), B (-1, -1), C (-2, 0)

$$\therefore \Delta_2 = \frac{1}{2} \begin{vmatrix} 1 & 3 & 1 \\ -1 & -1 & 1 \\ -2 & 0 & 1 \end{vmatrix} = 3 \therefore \frac{\Delta_1}{\Delta_2} = 2$$

3. Let S be the focus of the parabola $y^2 = 8x$ and let PQ be the common chord of the circle $x^2 + y^2 - 2x - 4y = 0$ and the given parabola. The area of the triangle PQS is (2012)

Ans. (4)

Sol. We observe both parabola $y^2 = 8x$ and circle $x^2 + y^2 - 2x - 4y = 0$ pass through origin

\therefore One end of common chord PQ is origin. Say P(0, 0)

Let Q be the point $(2t^2, 4t)$, then it will satisfy the equation of circle.

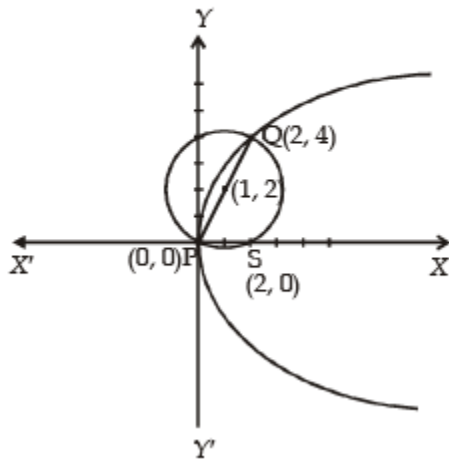
$$\therefore 4t^4 + 16t^2 - 4t^2 - 16t = 0 \Rightarrow t^4 + 3t^2 - 4t = 0 \Rightarrow t(t^3 + 3t - 4) = 0$$

$$\Rightarrow t(t-1)(t^2+t-4) = 0 \Rightarrow t = 0 \text{ or } 1$$

For $t = 0$, we get point P, therefore $t = 1$ gives point Q as $(2, 4)$.

We also observe here that $P(0, 0)$ and $Q(2, 4)$ are end points of diameter of the given circle and focus of the parabola is the point $S(2, 0)$.

$$\therefore \text{Area of } \triangle PQS = \frac{1}{2} \times PS \times QS = \frac{1}{2} \times 2 \times 4 = 4 \text{ sq. units}$$



4. A vertical line passing through the point $(h, 0)$ intersects the

ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$

at the points P and Q. Let the tangents to the ellipse at P and Q meet at the point

R. If $\Delta(h) = \text{area of the triangle } PQR$, $\Delta_1 = \max_{1/2 \leq h \leq 1} \Delta(h)$ and $\Delta_2 = \min_{1/2 \leq h \leq 1} \Delta(h)$, then

$$\frac{8}{\sqrt{5}} \Delta_1 - 8\Delta_2 = \text{(JEE Adv. 2013)}$$

(a) $g(x)$ is continuous but not differentiable at a

(b) $g(x)$ is differentiable on \mathbb{R}

(c) $g(x)$ is continuous but not differentiable at b

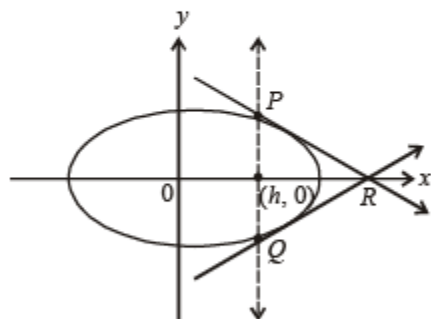
(d) $g(x)$ is continuous and differentiable at either (a) or (b) but not both

Ans. (9)

Sol. Vertical line $x = h$, meets the ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$ at

$$P\left(h, \frac{\sqrt{3}}{2}\sqrt{4-h^2}\right) \text{ and } Q\left(h, -\frac{\sqrt{3}}{2}\sqrt{4-h^2}\right)$$

By symmetry, tangents at P and Q will meet each other at x -axis.



Tangent at P is $\frac{xh}{4} + \frac{y\sqrt{3}}{6}\sqrt{4-h^2} = 1$

which meets x -axis at $R\left(\frac{4}{h}, 0\right)$

$$\text{Area of } \Delta PQR = \frac{1}{2} \times \sqrt{3}\sqrt{4-h^2} \times \left(\frac{4}{h} - h\right)$$

$$\text{i.e., } \Delta(h) = \frac{\sqrt{3}(4-h^2)^{3/2}}{2h}$$

$\therefore \Delta(h)$ is a decreasing function.

$$\therefore \frac{1}{2} \leq h \leq 1 \Rightarrow \Delta_{\max} = \Delta\left(\frac{1}{2}\right) \text{ and } \Delta_{\min} = \Delta(1)$$

$$\therefore \Delta_1 = \frac{\sqrt{3} \left(4 - \frac{1}{4}\right)^{3/2}}{\frac{1}{2}} = \frac{45}{8} \sqrt{5}$$

$$\Delta_2 = \frac{\sqrt{3} \cdot 3\sqrt{3}}{2 \cdot 1} = \frac{9}{2} \therefore \frac{8}{\sqrt{5}} \Delta_1 - 8\Delta_2 = 45 - 36 = 9$$

5. If the normals of the parabola $y^2 = 4x$ drawn at the end points of its latus rectum are tangents to the circle $(x - 3)^2 + (y + 2)^2 = r^2$, then the value of r^2 is (JEE Adv. 2015)

Ans. (2)

Sol. End points of latus rectum of $y^2 = 4x$ are $(1, +2)$

Equation of normal to $y^2 = 4x$ at $(1, 2)$ is $y - 2 = -1(x - 1)$

or $x + y - 3 = 0$

As it is tangent to circle $(x - 3)^2 + (y + 2)^2 = r^2$

$$\therefore \left| \frac{3 + (-2) - 3}{\sqrt{2}} \right| = r \Rightarrow r^2 = 2$$

6. Let the curve C be the mirror image of the parabola $y^2 = 4x$ with respect to the line $x + y + 4 = 0$. If A and B are the points of intersection of C with the line $y = -5$, then the distance between A and B is (JEE Adv. 2015)

Ans. (4)

Sol. Let $(t^2, 2t)$ be any point on $y^2 = 4x$. Let (h, k) be the image of $(t^2, 2t)$ in the line $x + y + 4 = 0$. Then

$$\frac{h - t^2}{1} = \frac{k - 2t}{1} = \frac{-2(t^2 + 2t + 4)}{2}$$

$$\Rightarrow h = -(2t + 4) \text{ and } k = -(t^2 + 4)$$

For its intersection with, $y = -5$, we have $-(t^2 + 4) = -5 \Rightarrow t = +1$

$\therefore A(-6, -5)$ and $B(-2, -5) \therefore AB = 4$.

7. Suppose that the foci of the ellipse

$\frac{x^2}{9} + \frac{y^2}{5} = 1$ are $(f_1, 0)$ and $(f_2, 0)$ where $f_1 > 0$ and $f_2 < 0$. Let P_1 and P_2 be two parabolas with a common vertex at $(0, 0)$ and with foci at $(f_1, 0)$ and $(2f_2, 0)$, respectively. Let T_1 be a tangent to P_1 which passes through $(2f_2, 0)$ and T_2 be a tangent to P_2 which passes through $(f_1, 0)$. If m_1 is the slope of T_1 and m_2 is the slope of T_2 , then the value of

$\left(\frac{1}{m_1^2} + m_2^2 \right)$ is (JEE Adv. 2015)

Ans. (4)

Sol. Ellipse : $\frac{x^2}{9} + \frac{y^2}{5} = 1$

$$\Rightarrow a = 3, b = \sqrt{5} \text{ and } e = \frac{2}{3}$$

$$\therefore f_1 = 2 \text{ and } f_2 = -2 \quad P_1 : y^2 = 8x \text{ and } P_2 : y^2 = -16x$$

$$T_1 : y = m_1 x + \frac{2}{m_1}$$

It passes through $(-4, 0)$,

$$0 = -4m_1 + \frac{2}{m_1} \Rightarrow m_1^2 = \frac{1}{2}$$

$$T_2 : y = m_2 x - \frac{4}{m_2}$$

It passes through $(2, 0)$

$$0 = 2m_2 - \frac{4}{m_2} \Rightarrow m_2^2 = 2$$

$$\therefore \frac{1}{m_1^2} + m_2^2 = 4$$