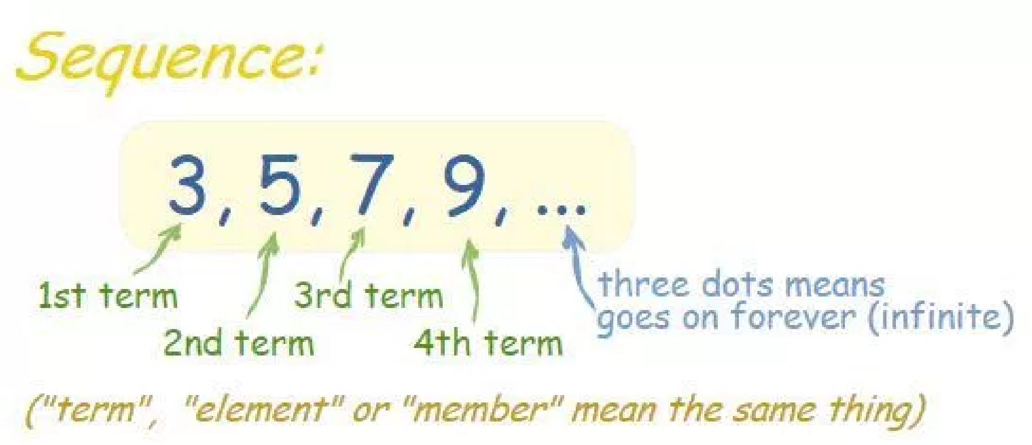
14. Sequences, Series and Partial Sums

Sequences

In the previous lesson, we learned about pattern of numbers. In this lesson we discuss about Sequences.

A sequence is an ordered list of numbers.



The sum of the terms of a sequence is called a **series**.

- Each number of a sequence is called a term (or element) of the sequence.
- A finite sequence contains a finite number of terms (you can count them). 1, 4, 7, 10, 13
- An infinite sequence contains an infinite number of terms (you cannot count them). 1, 4, 7, 10, 13, . . .
- The terms of a sequence are referred to in the subscripted form shown below, where the natural number subscript refers to the location (position) of the term in the sequence.

$$\underset{a_1}{1},\ \underset{a_2}{4},\ \underset{a_3}{7},\ \underset{a_4}{10},\ \underset{a_5}{13},\ \underset{a_6}{16},\ \ldots$$

(If you study computer programming languages such as C, C++, and Java,
you will find that the first position in their arrays (sequences) start with a subscript of zero.)

The general form of a sequence is represented:

- The domain of a sequence consists of the counting numbers 1, 2, 3, 4, … and the range consists of the terms of the sequence.
- The terms in a sequence may, or may not, have a pattern, or a related formula.
- For some sequences, the terms are simply random.

Let’s examine some sequences that have patterns:

Sequences often possess a definite pattern that is used to arrive at the sequence’s terms.

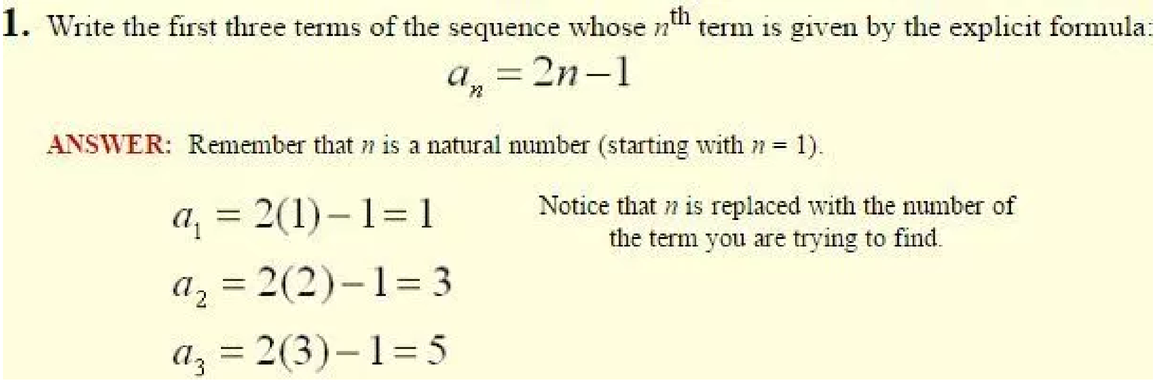
It is often possible to express such patterns as a formula. In the sequence shown at the left, an explicit formula may be:

$$a_n = 12n$$

where n represents the term's position in the sequence.

$$\overset{a_1}{\underset{12(1)}{12}},\ \overset{a_2}{\underset{12(2)}{24}},\ \overset{a_3}{\underset{12(3)}{36}},\ \ldots$$

Examples:



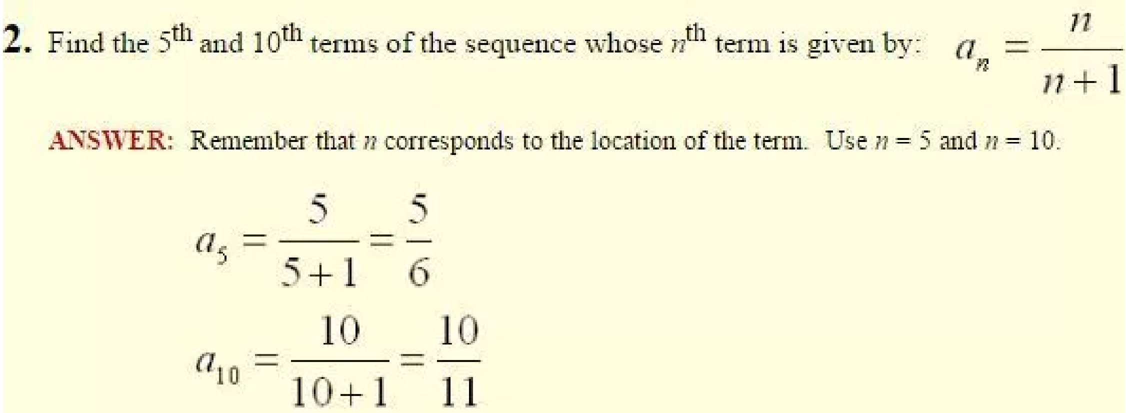
1. Write the first three terms of the sequence whose n^{th} term is given by the explicit formula:

$$a_n = 2n - 1$$

ANSWER: Remember that n is a natural number (starting with $n = 1$).

$$a_1 = 2(1) - 1 = 1$$
$$a_2 = 2(2) - 1 = 3$$
$$a_3 = 2(3) - 1 = 5$$

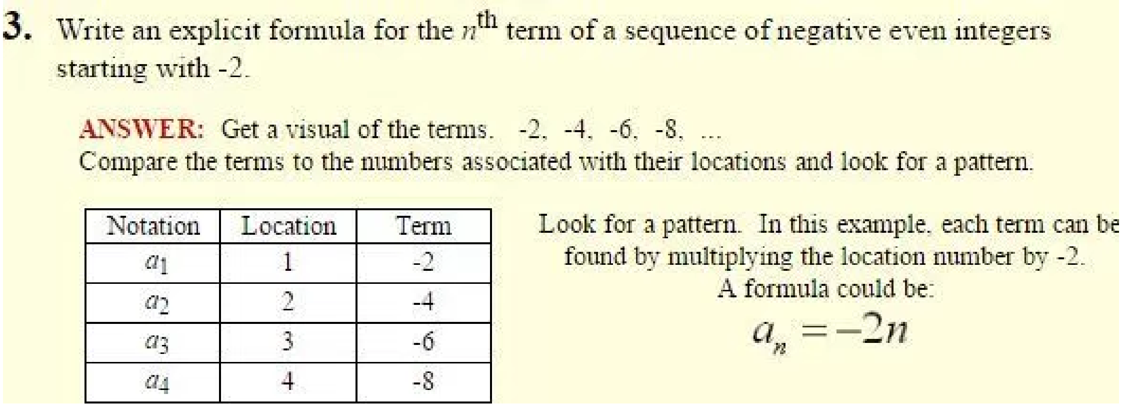
Notice that n is replaced with the number of the term you are trying to find.



2. Find the 5^{th} and 10^{th} terms of the sequence whose n^{th} term is given by: $a_n = \frac{n}{n+1}$

ANSWER: Remember that n corresponds to the location of the term. Use $n = 5$ and $n = 10$.

$$a_5 = \frac{5}{5+1} = \frac{5}{6}$$
$$a_{10} = \frac{10}{10+1} = \frac{10}{11}$$



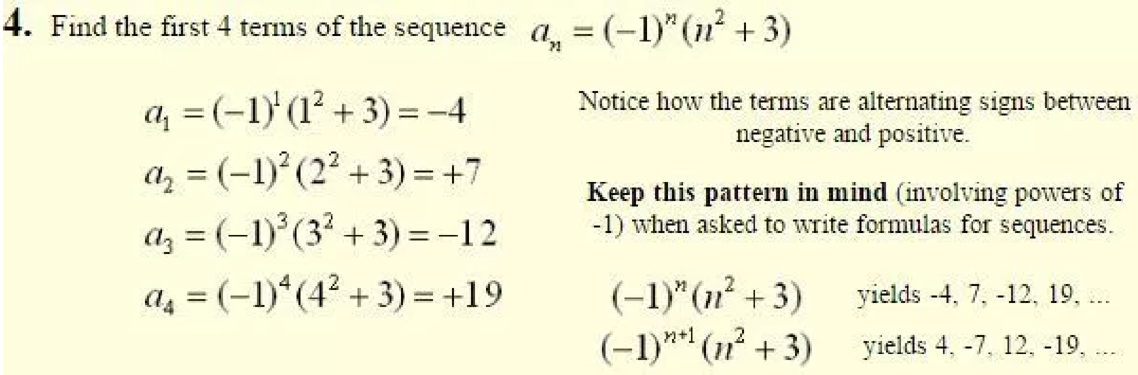
3. Write an explicit formula for the n^{th} term of a sequence of negative even integers starting with -2.

ANSWER: Get a visual of the terms. -2, -4, -6, -8, ...
Compare the terms to the numbers associated with their locations and look for a pattern.

Notation	Location	Term
a_1	1	-2
a_2	2	-4
a_3	3	-6
a_4	4	-8

Look for a pattern. In this example, each term can be found by multiplying the location number by -2.
A formula could be:

$$a_n = -2n$$



4. Find the first 4 terms of the sequence $a_n = (-1)^n(n^2 + 3)$

$$a_1 = (-1)^1(1^2 + 3) = -4$$
$$a_2 = (-1)^2(2^2 + 3) = +7$$
$$a_3 = (-1)^3(3^2 + 3) = -12$$
$$a_4 = (-1)^4(4^2 + 3) = +19$$

Notice how the terms are alternating signs between negative and positive.

Keep this pattern in mind (involving powers of -1) when asked to write formulas for sequences.

$(-1)^n(n^2 + 3)$ yields -4, 7, -12, 19, ...
$(-1)^{n+1}(n^2 + 3)$ yields 4, -7, 12, -19, ...

Sigma Notation and Series

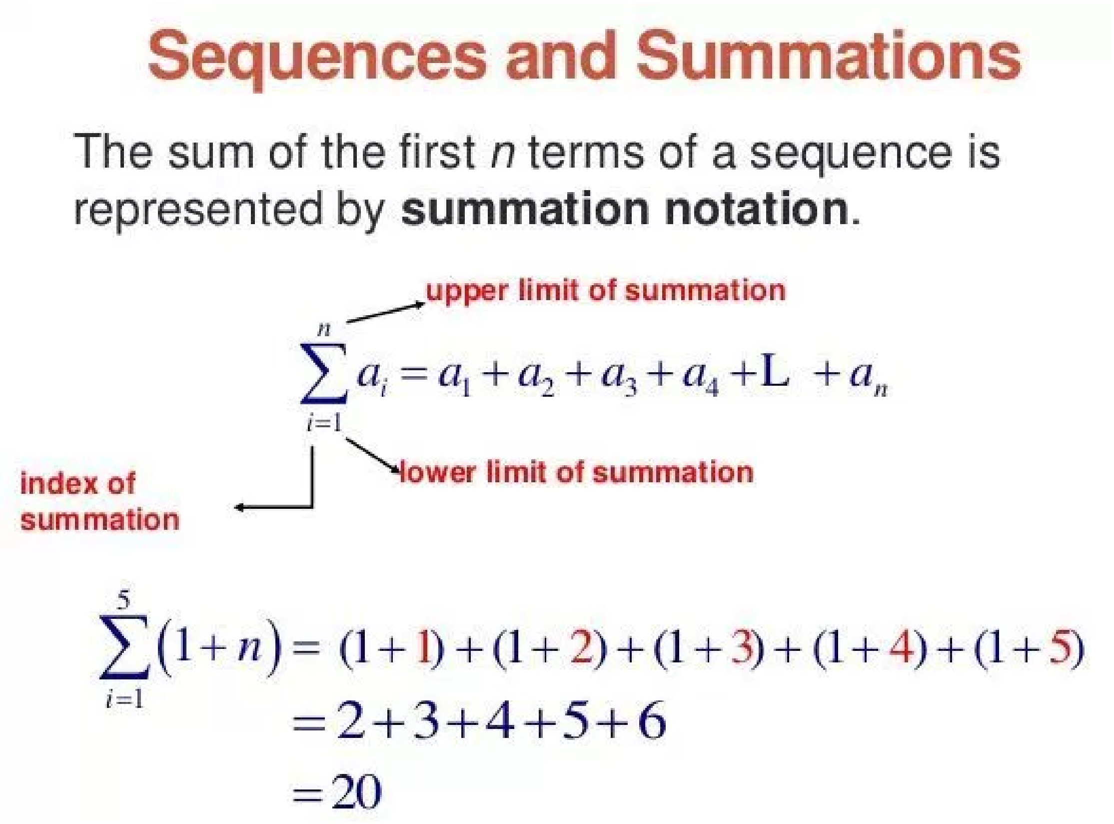
Consider the sequence 10.20, 11.40, 12.10, 13.40 where each term represents the amount of money you earned as interest on your savings account for each of four years.

The sum of the terms, 10.2 + 11.4 + 12.1 + 13.4, represents the total interest you earned in the four year period. Such a sequence summation is called a series and is designated by Sn where n represents the number of terms of the sequence being added.

Sn is often called an nth **partial sum**, since it can represent the sum of a certain “part” of a sequence.

$S_1 = 10.2$	$S_1 = a_1$
$S_2 = 10.2 + 11.4 = 21.6$	$S_2 = a_1 + a_2$
$S_3 = 10.2 + 11.4 + 12.1 = 32.7$	$S_3 = a_1 + a_2 + a_3$
$S_4 = 10.2 + 11.4 + 12.1 + 13.4 = 46.1$	$S_4 = a_1 + a_2 + a_3 + a_4$
	$S_n = a_1 + a_2 + a_3 + a_4 + \ldots + a_n$

A series can be represented in a compact form, called summation notation, or sigma notation.
The Greek capital letter sigma, , is used to indicate a sum.



Sequences and Summations

The sum of the first *n* terms of a sequence is represented by **summation notation.**

upper limit of summation

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + a_4 + \mathrm{L} \ + a_n$$

lower limit of summation

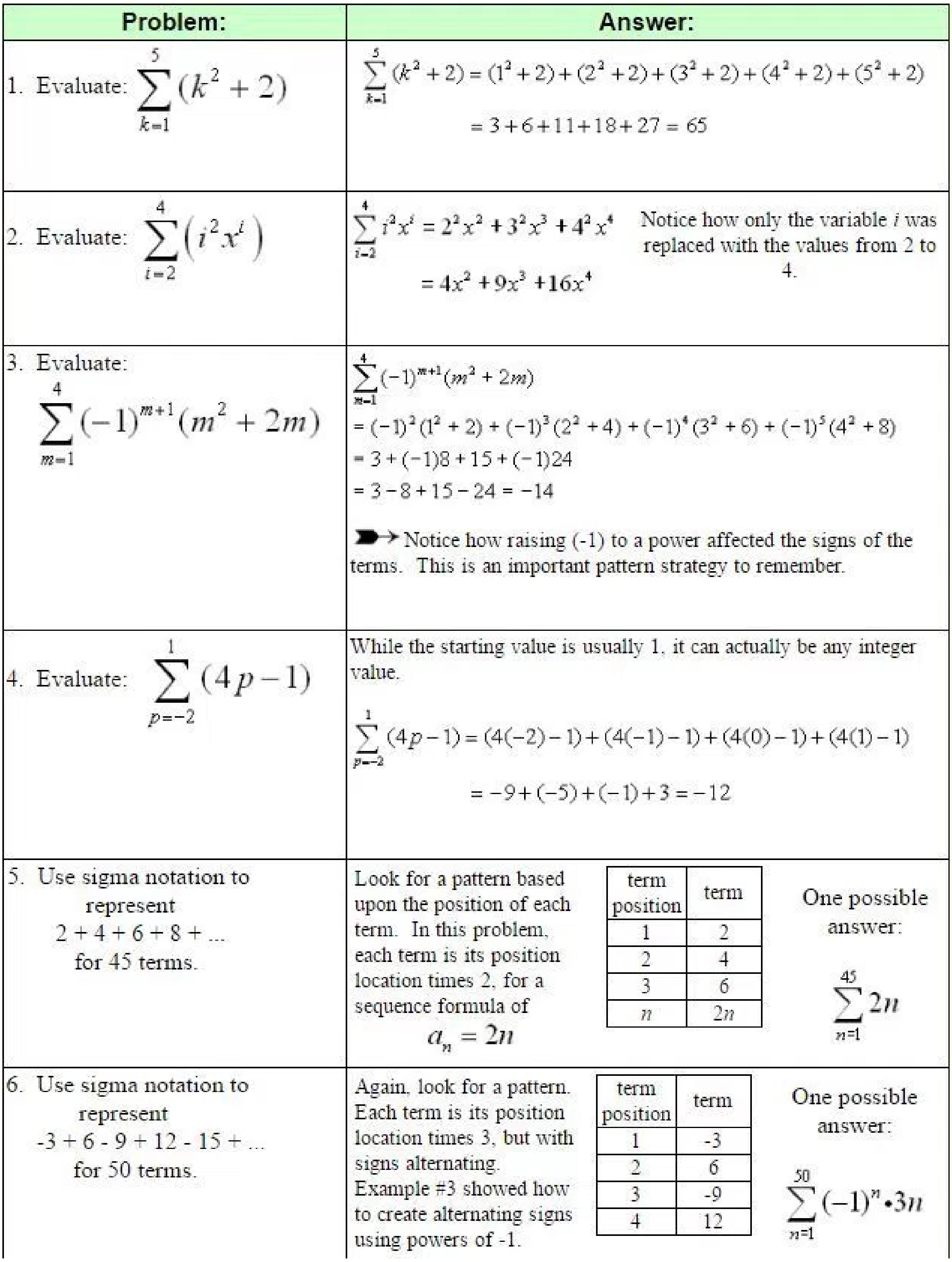
index of summation

$$\sum_{i=1}^{5} (1+n) = (1+1) + (1+2) + (1+3) + (1+4) + (1+5)$$

$$= 2+3+4+5+6$$

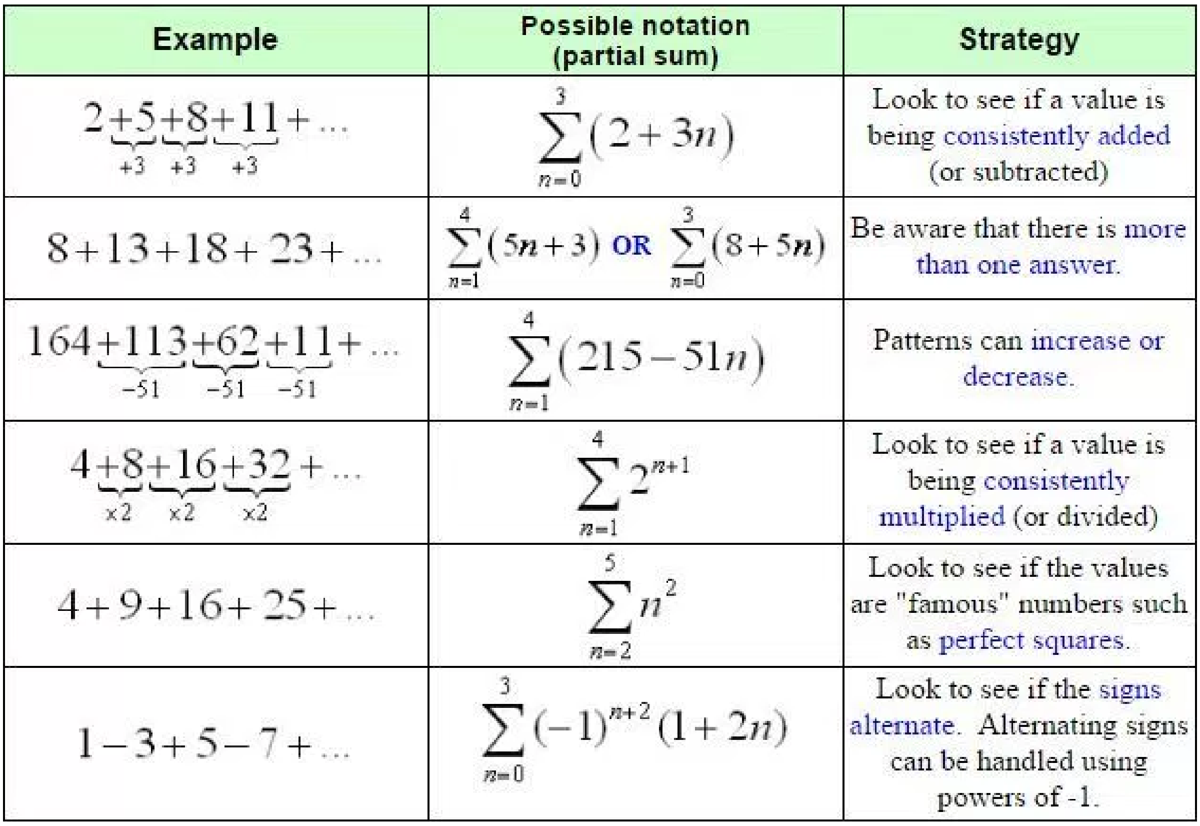
$$= 20$$

Examples:



<table>
<tr><th>Problem:</th><th>Answer:</th></tr>
<tr><td>1. Evaluate: $\sum_{k=1}^{5}(k^2+2)$</td><td>$\sum_{k=1}^{5}(k^2+2)=(1^2+2)+(2^2+2)+(3^2+2)+(4^2+2)+(5^2+2)$
$=3+6+11+18+27=65$</td></tr>
<tr><td>2. Evaluate: $\sum_{i=2}^{4}\left(i^2x^i\right)$</td><td>$\sum_{i=2}^{4} i^2x^i=2^2x^2+3^2x^3+4^2x^4$
$=4x^2+9x^3+16x^4$
Notice how only the variable *i* was replaced with the values from 2 to 4.</td></tr>
<tr><td>3. Evaluate:
$\sum_{m=1}^{4}(-1)^{m+1}(m^2+2m)$</td><td>$\sum_{m=1}^{4}(-1)^{m+1}(m^2+2m)$
$=(-1)^2(1^2+2)+(-1)^3(2^2+4)+(-1)^4(3^2+6)+(-1)^5(4^2+8)$
$=3+(-1)8+15+(-1)24$
$=3-8+15-24=-14$
➔ Notice how raising (-1) to a power affected the signs of the terms. This is an important pattern strategy to remember.</td></tr>
<tr><td>4. Evaluate: $\sum_{p=-2}^{1}(4p-1)$</td><td>While the starting value is usually 1, it can actually be any integer value.
$\sum_{p=-2}^{1}(4p-1)=(4(-2)-1)+(4(-1)-1)+(4(0)-1)+(4(1)-1)$
$=-9+(-5)+(-1)+3=-12$</td></tr>
<tr><td>5. Use sigma notation to represent
2 + 4 + 6 + 8 + ...
for 45 terms.</td><td>Look for a pattern based upon the position of each term. In this problem, each term is its position location times 2, for a sequence formula of
$a_n=2n$
<table><tr><th>term position</th><th>term</th></tr><tr><td>1</td><td>2</td></tr><tr><td>2</td><td>4</td></tr><tr><td>3</td><td>6</td></tr><tr><td>n</td><td>$2n$</td></tr></table>One possible answer:
$\sum_{n=1}^{45}2n$</td></tr>
<tr><td>6. Use sigma notation to represent
-3 + 6 - 9 + 12 - 15 + ...
for 50 terms.</td><td>Again, look for a pattern. Each term is its position location times 3, but with signs alternating. Example #3 showed how to create alternating signs using powers of -1.
<table><tr><th>term position</th><th>term</th></tr><tr><td>1</td><td>-3</td></tr><tr><td>2</td><td>6</td></tr><tr><td>3</td><td>-9</td></tr><tr><td>4</td><td>12</td></tr></table>One possible answer:
$\sum_{n=1}^{50}(-1)^n\cdot 3n$</td></tr>
</table>

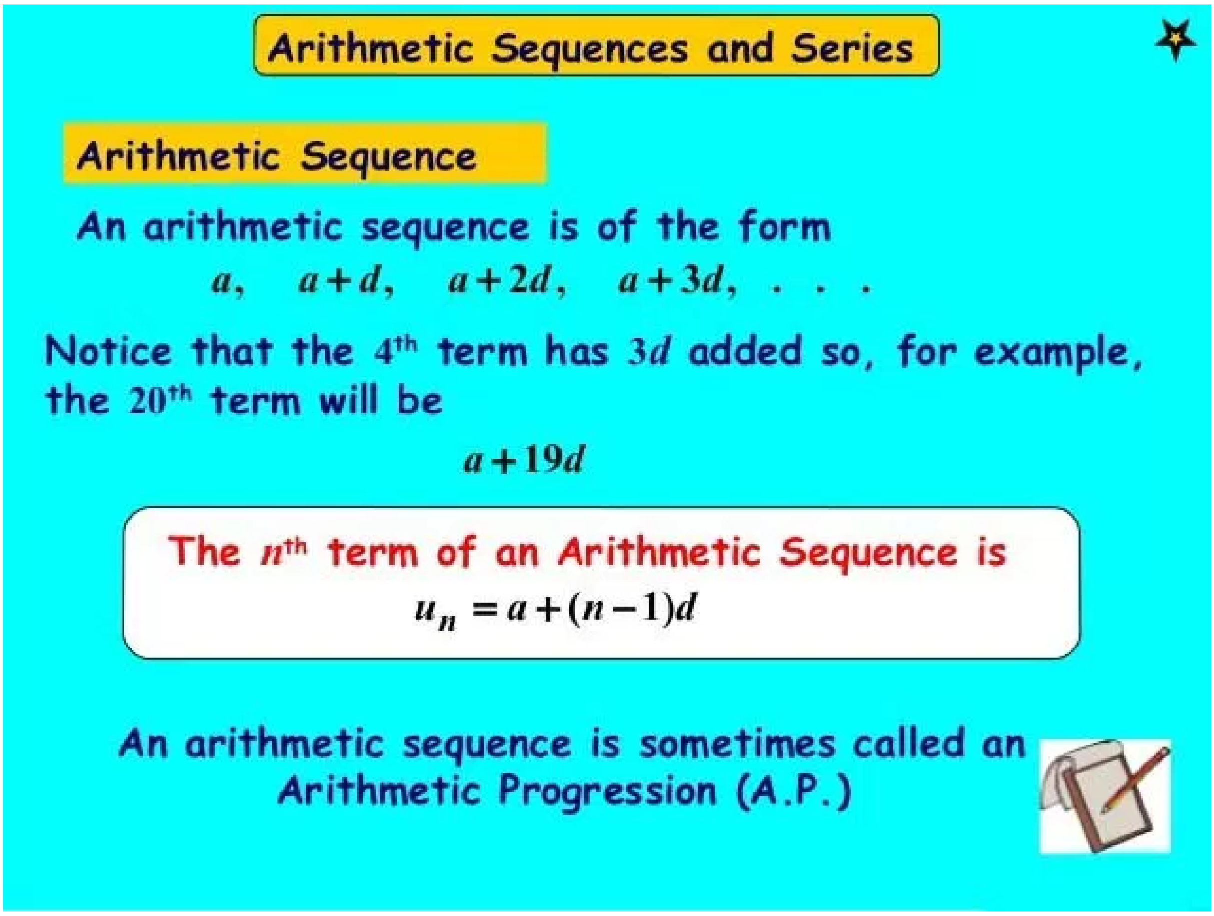
Strategies to remember when trying to find an expression for a sequence (series):



Example	Possible notation (partial sum)	Strategy
$2\underbrace{+5}_{+3}\underbrace{+8}_{+3}\underbrace{+11}_{+3}+\ldots$	$\sum_{n=0}^{3}(2+3n)$	Look to see if a value is being consistently added (or subtracted)
$8+13+18+23+\ldots$	$\sum_{n=1}^{4}(5n+3)$ OR $\sum_{n=0}^{3}(8+5n)$	Be aware that there is more than one answer.
$164\underbrace{+113}_{-51}\underbrace{+62}_{-51}\underbrace{+11}_{-51}+\ldots$	$\sum_{n=1}^{4}(215-51n)$	Patterns can increase or decrease.
$4\underbrace{+8}_{\times 2}\underbrace{+16}_{\times 2}\underbrace{+32}_{\times 2}+\ldots$	$\sum_{n=1}^{4}2^{n+1}$	Look to see if a value is being consistently multiplied (or divided)
$4+9+16+25+\ldots$	$\sum_{n=2}^{5}n^2$	Look to see if the values are "famous" numbers such as perfect squares.
$1-3+5-7+\ldots$	$\sum_{n=0}^{3}(-1)^{n+2}(1+2n)$	Look to see if the signs alternate. Alternating signs can be handled using powers of -1.

Arithmetic Sequences and Series

A sequence is an ordered list of numbers.
The sum of the terms of a sequence is called a **series**.



Arithmetic Sequences and Series

Arithmetic Sequence

An arithmetic sequence is of the form

$$a, \quad a+d, \quad a+2d, \quad a+3d, \quad . \; . \; .$$

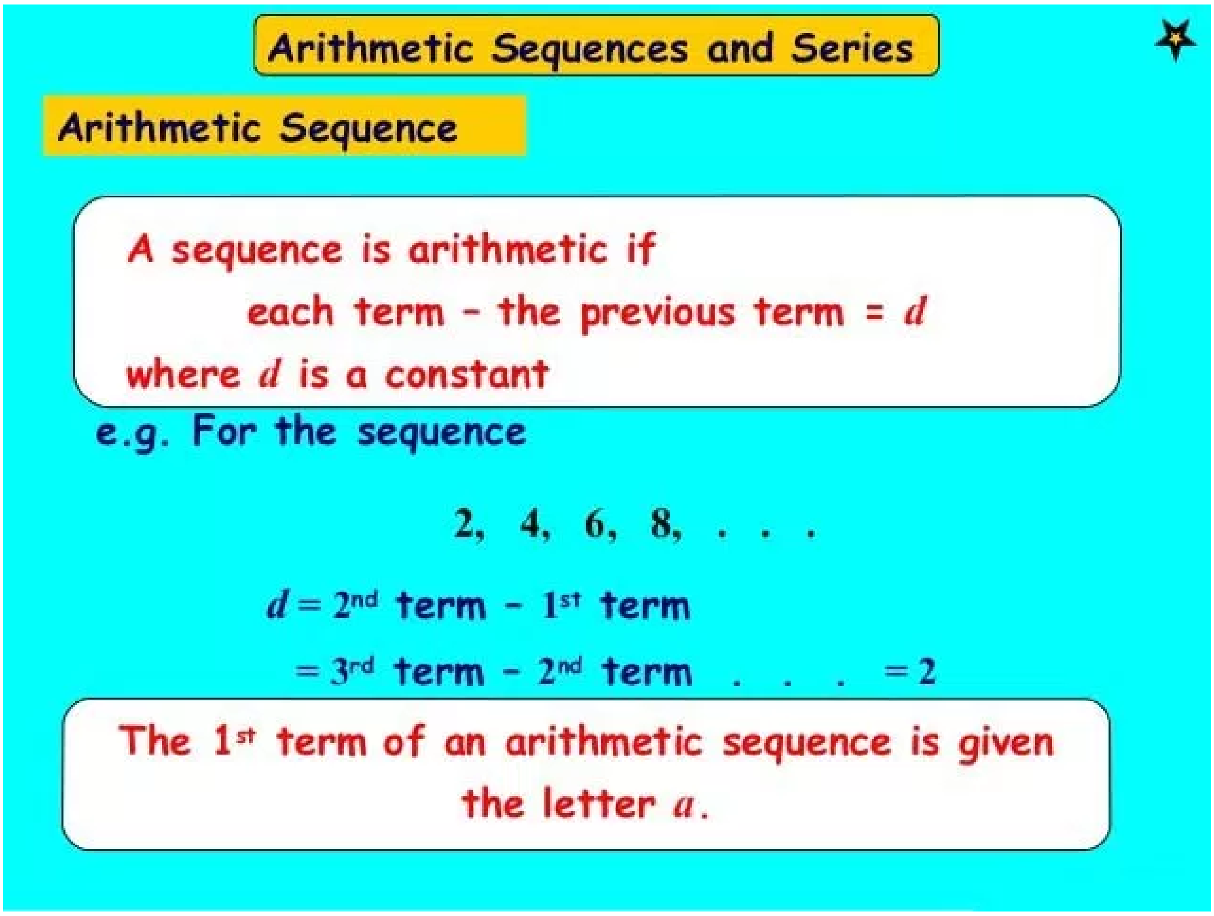
Notice that the 4th term has $3d$ added so, for example, the 20^{th} term will be

$$a+19d$$

The n^{th} term of an Arithmetic Sequence is

$$u_n = a+(n-1)d$$

An arithmetic sequence is sometimes called an Arithmetic Progression (A.P.)



Arithmetic Sequences and Series

Arithmetic Sequence

A sequence is arithmetic if

each term – the previous term = d

where d is a constant

e.g. For the sequence

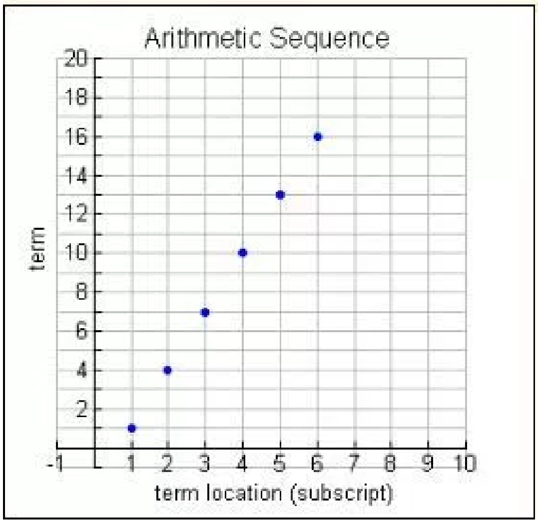
$$2, \quad 4, \quad 6, \quad 8, \quad . \quad . \quad .$$

$$d = 2^{nd} \text{ term} - 1^{st} \text{ term}$$

$$= 3^{rd} \text{ term} - 2^{nd} \text{ term} \quad . \quad . \quad . \quad = 2$$

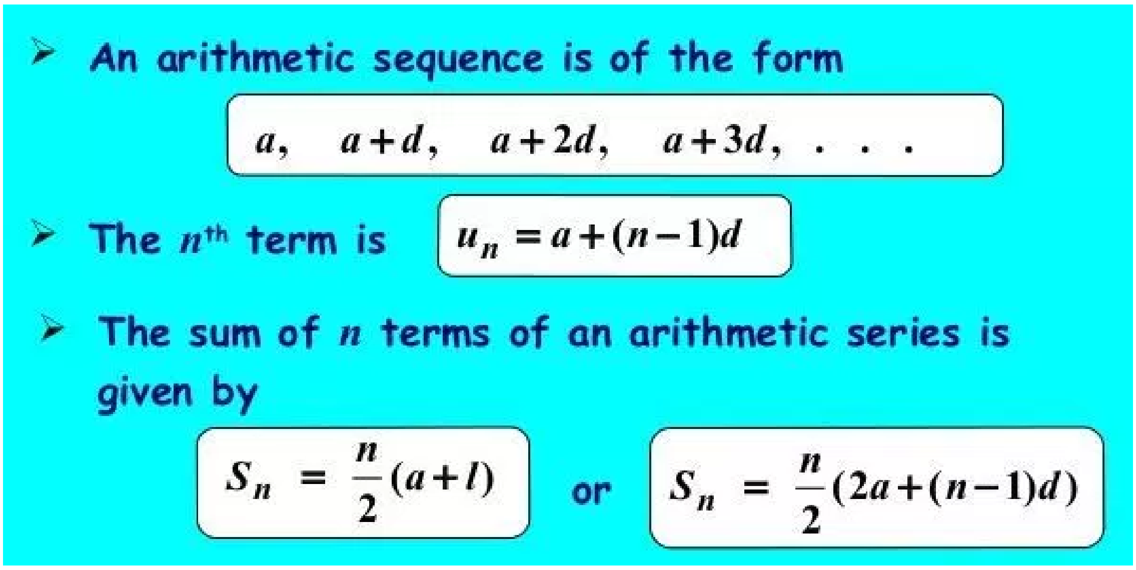
The 1st term of an arithmetic sequence is given the letter a.

While some sequences are simply random values, other sequences have a definite pattern that is used to arrive at the sequence's terms.
Two such sequences are the arithmetic and geometric sequences. Let's investigate the arithmetic sequence.



If a sequence of values follows a pattern of adding a fixed amount from one term to the next, it is referred to as an arithmetic sequence. The number added to each term is constant (always the same).

The fixed amount is called the common difference, d, referring to the fact that the difference between two successive terms yields the constant value that was added. To find the common difference, subtract the first term from the second term.
Notice the linear nature of the scatter plot of the terms of an arithmetic sequence. The domain consists of the counting numbers 1, 2, 3, 4, … and the range consists of the terms of the sequence. While the x value increases by a constant value of one, the y value increases by a constant value of 3 (for this graph).



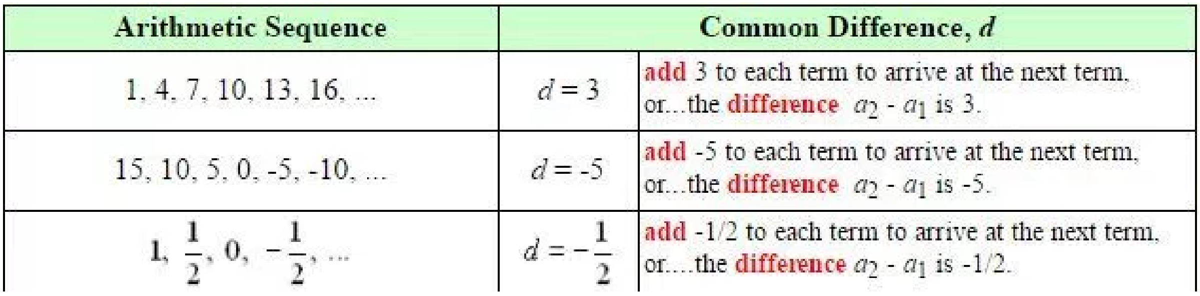
- An arithmetic sequence is of the form

$$a,\quad a+d,\quad a+2d,\quad a+3d,\quad . \; . \; .$$

- The n^{th} term is $u_n = a+(n-1)d$
- The sum of n terms of an arithmetic series is given by

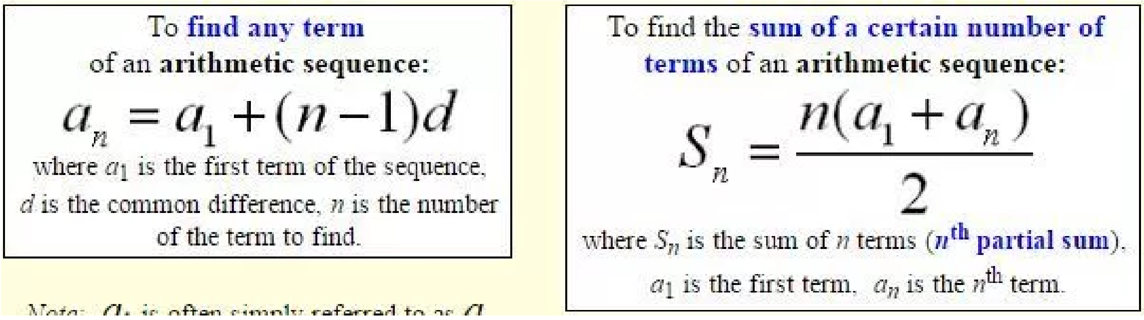
$$S_n = \frac{n}{2}(a+l) \quad \text{or} \quad S_n = \frac{n}{2}(2a+(n-1)d)$$

Examples:



Arithmetic Sequence	Common Difference, d	
1, 4, 7, 10, 13, 16, ...	$d = 3$	add 3 to each term to arrive at the next term, or...the difference a_2 - a_1 is 3.
15, 10, 5, 0, -5, -10, ...	$d = -5$	add -5 to each term to arrive at the next term, or...the difference a_2 - a_1 is -5.
$1, \frac{1}{2}, 0, -\frac{1}{2}, \ldots$	$d = -\frac{1}{2}$	add -1/2 to each term to arrive at the next term, or....the difference a_2 - a_1 is -1/2.

Formulas used with arithmetic sequences and arithmetic series:



To find any term of an arithmetic sequence:

$$a_n = a_1 + (n-1)d$$

where a_1 is the first term of the sequence, d is the common difference, n is the number of the term to find.

Note: a_1 is often simply referred to as a

To find the sum of a certain number of terms of an arithmetic sequence:

$$S_n = \frac{n(a_1 + a_n)}{2}$$

where S_n is the sum of n terms (n^{th} partial sum), a_1 is the first term, a_n is the n^{th} term.

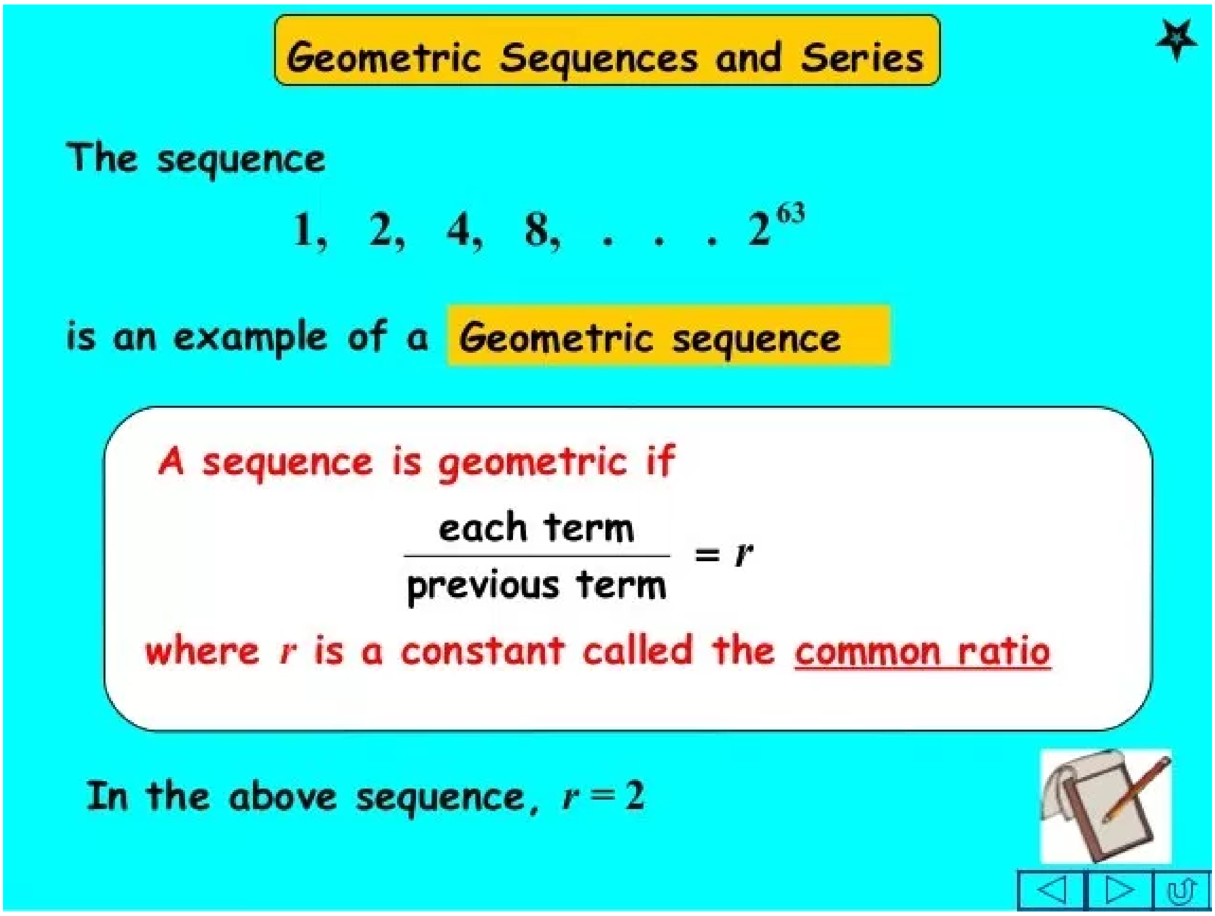
Examples:

Question	Answer
1. Find the common difference for this arithmetic sequence 5, 9, 13, 17 ...	**1.** The common difference, *d*, can be found by subtracting the first term from the second term, which in this problem yields **4**. Checking shows that 4 is the difference between all of the entries.
2. Find the common difference for the arithmetic sequence whose formula is $a_n = 6n + 3$	**2.** The formula indicates that 6 is the value being added (with increasing multiples) as the terms increase. A listing of the terms will also show what is happening in the sequence (start with $n = 1$). 9, 15, 21, 27, 33, ... The list shows the common difference to be 6.
3. Find the 10th term of the sequence 3, 5, 7, 9, ...	**3.** $n = 10$; $a_1 = 3$, $d = 2$ $a_n = a_1 + (n-1)d$ $a_{10} = 3 + (10-1)2$ $a_{10} = 21$ The tenth term is 21.
4. Find a_7 for an arithmetic sequence where $a_1 = 3x$ and $d = -x$.	**4.** $n = 7$; $a_1 = 3x$, $d = -x$ $a_n = a_1 + (n-1)d$ $a_7 = 3x + (7-1)(-x)$ $a_7 = 3x + 6(-x) = -3x$
5. Find t_{15} for an arithmetic sequence where $t_3 = -4 + 5i$ and $t_6 = -13 + 11i$ Using high subscript - low subscript + 1 will count the number of terms.	**5.** Notice the change of labeling from *a* to *t*. The letter used in labeling is of no importance. Get a visual image of this problem $\underbrace{\quad}_{t_1}, \underbrace{\quad}_{t_2}, \boxed{\underbrace{-4+5i}_{t_3}, \underbrace{\quad}_{t_4}, \underbrace{\quad}_{t_5}, \underbrace{-13+11i}_{t_6}}$ Using the third term as the "first" term, find the common difference from these known terms. $a_n = a_1 + (n-1)d$ $t_6 = t_3 + (4-1)d$ $-13 + 11i = -4 + 5i + (4-1)d$ $-13 + 11i = -4 + 5i + 3d$ $-9 + 6i = 3d$ $-3 + 2i = d$ Now, from t_3 to t_{15} is 13 terms. $t_{15} = -4 + 5i + (13-1)(-3 + 2i) = -4 + 5i - 36 + 24i$ $= -40 + 29i$
6. Find a formula for the sequence 1, 3, 5, 7, ...	**6.** A formula will relate the subscript number of each term to the actual value of the term. $a_n = 2n - 1$ Substituting $n = 1$, gives 1. Substituting $n = 2$, gives 3, and so on.
7. Find the 25th term of the sequence -7, -4, -1, 2, ...	**7.** $n = 25$; $a_1 = -7$, $d = 3$ $a_n = a_1 + (n-1)d$ $a_{25} = -7 + (25-1)3$ $a_{25} = 65$

8. Find the sum of the first 12 positive even integers. Notice how BOTH formulas work together.	**8. The word "sum" indicates the need for the sum formula.** *positive even integers:* 2, 4, 6, 8, ... $n = 12;\ a_1 = 2, d = 2$ We are missing a_{12}, for the sum formula, so we use the "any term" formula to find it. $a_n = a_1 + (n-1)d$ $a_{12} = 2 + (12-1)2$ $a_{12} = 24$ Now, let's find the sum: $S_{12} = \frac{12(2+24)}{2} = 156$
9. Insert 3 arithmetic means between 7 and 23. *Note:* An **arithmetic mean** is the term between any two terms of an arithmetic sequence. It is simply the average (mean) of the given terms.	**9.** While there are several solution methods, we will use our arithmetic sequence formulas. Draw a picture to better understand the situation. 7, ____, ____, ____, 23 This set of terms will be an arithmetic sequence. We know the first term, a_1, the last term, a_n, but not the common difference, d. *This question makes NO mention of "sum", so avoid that formula.* Find the common difference: $a_n = a_1 + (n-1)d$ $23 = 7 + (5-1)d$ $23 = 7 + 4d$ $16 = 4d$ $4 = d$ Now, insert the terms using d. 7, 11, 15, 19, 23
10. Find the number of terms in the sequence 7, 10, 13, ..., 55. n must be an integer!	**10.** $a_1 = 7, a_n = 55,\ d = 3$. We need to find n. *This question makes NO mention of "sum", so avoid that formula.* $a_n = a_1 + (n-1)d$ $55 = 7 + (n-1)3$ $55 = 7 + 3n - 3$ $55 = 4 + 3n$ $51 = 3n$ $17 = n$ **When solving for n, be sure your answer is a positive integer. There is no such thing as a fractional number of terms in a sequence!**
11. A theater has 60 seats in the first row, 68 seats in the second row, 76 seats in the third row, and so on in the same increasing pattern. If the theater has 20 rows of seats, how many seats are in the theater?	**11.** The seating pattern is forming an arithmetic sequence. 60, 68, 76, ... We wish to find "the sum" of all of the seats. $n = 20,\ a_1 = 60,\ d = 8$ and we need a_{20} for the sum. $a_n = a_1 + (n-1)d$ $a_{20} = 60 + (20-1)8 = 212$ Now, use the sum formula: $S_n = \frac{n(a_1 + a_n)}{2}$ $S_{20} = \frac{20(60+212)}{2} = 2720$ There are 2720 seats.

Geometric Sequences and Series

A sequence is an ordered list of numbers.
The sum of the terms of a sequence is called a series.



Geometric Sequences and Series

The sequence

$$1,\quad 2,\quad 4,\quad 8,\quad .\ .\ .\ 2^{63}$$

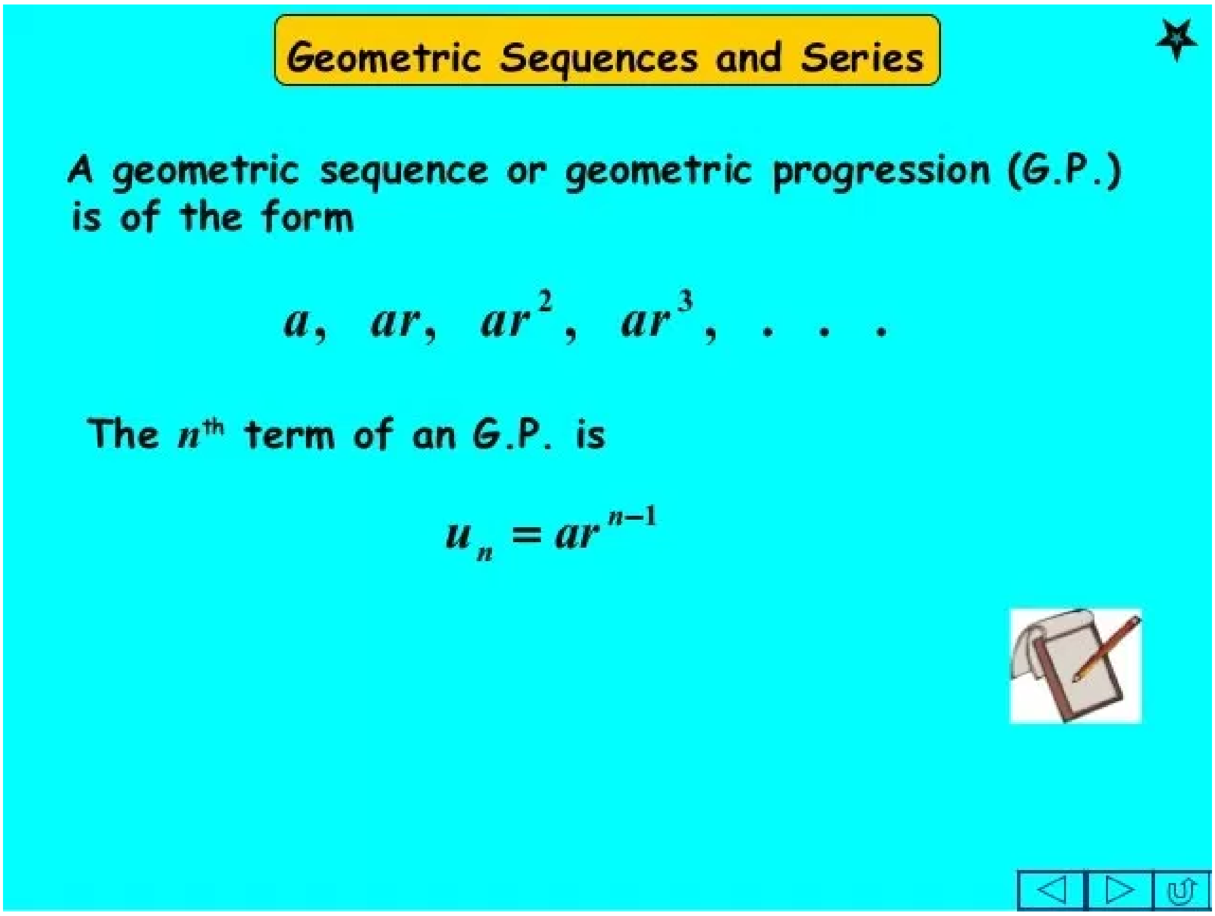
is an example of a **Geometric sequence**

A sequence is geometric if

$$\frac{\text{each term}}{\text{previous term}} = r$$

where r is a constant called the common ratio

In the above sequence, $r = 2$



Geometric Sequences and Series

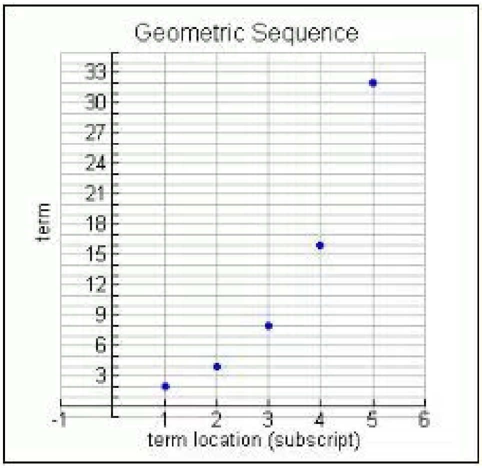
A geometric sequence or geometric progression (G.P.) is of the form

$$a,\quad ar,\quad ar^2,\quad ar^3,\quad .\ .\ .$$

The n^{th} term of an G.P. is

$$u_n = ar^{n-1}$$

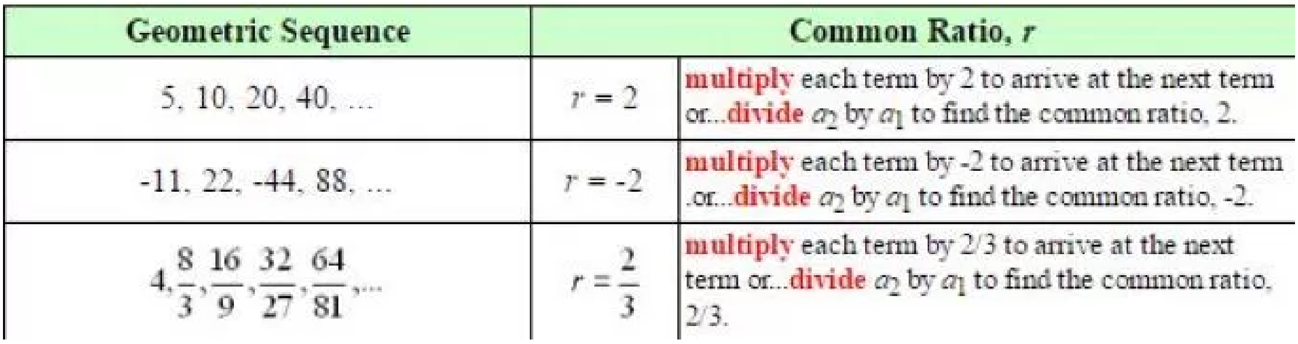
While some sequences are simply random values, other sequences have a definite pattern that is used to arrive at the sequence's terms.
Two such sequences are the arithmetic and geometric sequences. Let's investigate the geometric sequence.



If a sequence of values follows a pattern of multiplying a fixed amount (not zero) times each term to arrive at the following term, it is referred to as a geometric sequence. The number multiplied each time is constant (always the same).

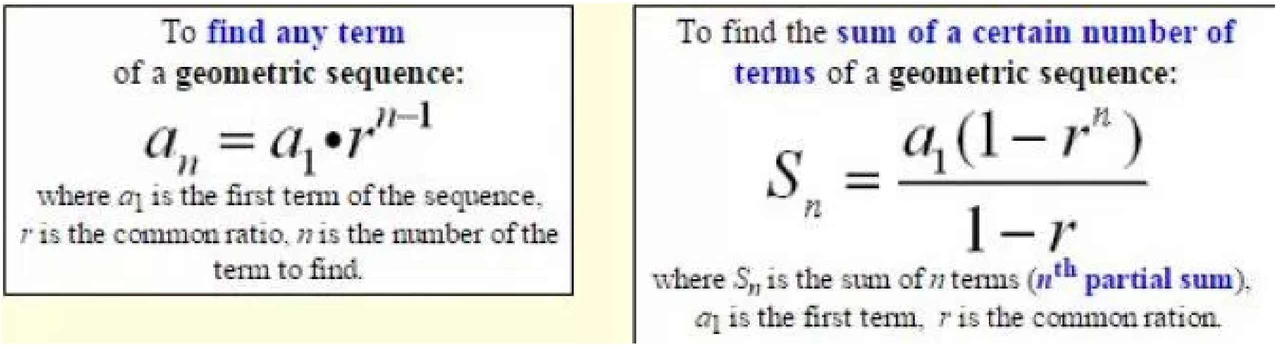
The fixed amount multiplied is called the common ratio, r, referring to the fact that the ratio (fraction) of the second term to the first term yields this common multiple. To find the common ratio, divide the second term by the first term.

Notice the non-linear nature of the scatter plot of the terms of a geometric sequence. The domain consists of the counting numbers 1, 2, 3, 4, … and the range consists of the terms of the sequence. While the x value increases by a constant value of one, the y value increases by multiples of two (for this graph).
Examples:



Geometric Sequence	Common Ratio, r	
5, 10, 20, 40, ...	$r = 2$	multiply each term by 2 to arrive at the next term or...divide a_2 by a_1 to find the common ratio, 2.
-11, 22, -44, 88, ...	$r = -2$	multiply each term by -2 to arrive at the next term .or...divide a_2 by a_1 to find the common ratio, -2.
$4, \frac{8}{3}, \frac{16}{9}, \frac{32}{27}, \frac{64}{81}, ...$	$r = \frac{2}{3}$	multiply each term by 2/3 to arrive at the next term or...divide a_2 by a_1 to find the common ratio, 2/3.

Formulas used with geometric sequences and geometric series:



To find any term of a geometric sequence:

$$a_n = a_1 \bullet r^{n-1}$$

where a_1 is the first term of the sequence, r is the common ratio, n is the number of the term to find.

To find the sum of a certain number of terms of a geometric sequence:

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

where S_n is the sum of n terms (n^{th} partial sum), a_1 is the first term, r is the common ration.

Examples:

Question	Answer
1. Find the common ratio for the sequence $6,\ -3,\ \frac{3}{2},\ -\frac{3}{4},\ \ldots$	**1**. The common ratio, r, can be found by dividing the second term by the first term, which in this problem yields **-1/2.** Checking shows that multiplying each entry by -1/2 yields the next entry.
2. Find the common ratio for the sequence given by the formula $a_n = 5(3)^{n-1}$	**2.** The formula indicates that 3 is the common ratio by its position in the formula. A listing of the terms will also show what is happening in the sequence (start with $n = 1$). 5, 15, 45, 135, ... The list also shows the common ratio to be 3.
3. Find the 7th term of the sequence 2, 6, 18, 54, ...	**3**. $n = 7;\ a_1 = 2, r = 3$ $a_n = a_1 \bullet r^{n-1}$ $a_7 = 2 \bullet 3^{7-1} = 1458$ The seventh term is 1458.
4. Find the 11th term of the sequence $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \ldots$	**4.** $n = 11;\ a_1 = 1, r = -1/2$ $a_{11} = 1 \bullet \left(-\frac{1}{2}\right)^{11-1} = \frac{1}{1024}$
5. Find a_8 for the sequence 0.5, 3.5, 24.5, 171.5, ...	**5.** $n = 8;\ a_1 = 0.5,\ r = 7$ $a_n = a_1 \bullet r^{n-1}$ $a_8 = 0.5 \bullet 7^{8-1} = 411{,}771.5$
6. Evaluate using a formula: $\sum_{k=1}^{5} 3^k$	**6.** Examine the summation $\sum_{k=1}^{5} 3^k = 3^1 + 3^2 + 3^3 + 3^4 + 3^5$ This is a geometric series with a common ratio of 3. $n = 5;\ a_1 = 3, r = 3$ $S_5 = \frac{3(1-3^5)}{1-3} = \frac{-726}{-2} = 363$
7. Find the sum of the first 8 terms of the sequence -5, 15, -45, 135, ...	**7.** The word "sum" indicates a need for the sum formula. $n = 8;\ a_1 = -5, r = -3$ $S_8 = \frac{-5(1-(-3)^8)}{1-(-3)}$ $S_8 = \frac{-5(1-6561)}{4} = \frac{32800}{4} = 8200$
8. The third term of a geometric sequence is 3 and the sixth term is 1/9. Find the first term.	**8.** Think of the sequence as "starting with" 3, until you find the common ratio. $\_\_, \_\_, \boxed{3, \_\_, \_\_, \frac{1}{9}}$ *For this modified sequence:* $a_1 = 3,\ a_4 = 1/9,$ $n = 4$ $a_n = a_1 \bullet r^{n-1}$ $\frac{1}{9} = 3 \bullet r^{4-1}$ $\frac{1}{27} = r^3$ $\frac{1}{3} = r$ Now, work backward multiplying by 3 (or dividing by 1/3) to find the actual first term. $a_1 = 27$
9. A ball is dropped from a height of 8 feet. The ball bounces to 80% of its previous height with each bounce. How high (*to the nearest tenth of a foot*) does the ball bounce on the fifth bounce?	**9.** Set up a model drawing for each "bounce". 6.4, 5.12, ___, ___, ___ The common ratio is 0.8. $a_n = a_1 \bullet r^{n-1}$ $a_n = 6.4 \bullet (.8)^{5-1} = 2.62144$ Answer: 2.6 feet