Chapter 5

BASIC CLASSES OF INTEGRABLE FUNCTIONS

§ 5.1. Integration of Rational Functions

If the denominator $Q(x)$ of the *proper* rational fraction $\frac{P(x)}{Q(x)}$ can be represented in the following way:

$$Q(x)=(x-a)^k(x-b)^l\ldots(x^2+\alpha x+\beta)^r(x^2+\gamma x+\mu)^s\ldots,$$

where the binomials and trinomials are different and, furthermore, the trinomials have no real roots, then

$$\frac{P(x)}{Q(x)}=\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\ldots+\frac{A_k}{(x-a)^k}+$$
$$+\frac{B_1}{x-b}+\frac{B_2}{(x-b)^2}+\ldots+\frac{B_l}{(x-b)^l}+\ldots$$
$$\ldots+\frac{M_1x+N_1}{x^2+\alpha x+\beta}+\frac{M_2x+N_2}{(x^2+\alpha x+\beta)^2}+\ldots+\frac{M_rx+N_r}{(x^2+\alpha x+\beta)^r}+$$
$$+\frac{R_1x+L_1}{x^2+\gamma x+\mu}+\frac{R_2x+L_2}{(x^2+\gamma x+\mu)^2}+\ldots+\frac{R_sx+L_s}{(x^2+\gamma x+\mu)^s}+\ldots,$$

where
$A_1, A_2, \ldots, B_1, B_2, \ldots, M_1, N_1, M_2, N_2, \ldots, R_1, L_1, R_2, L_2, \ldots$ are some real constants to be determined. They are determined by reducing both sides of the above identity to integral form and then equating the coefficients at equal powers of x, which gives a system of linear equations with respect to the coefficients. (This method is called the *method of comparison of coefficients.)* A system of equations for the coefficients can also be obtained by substituting suitably chosen numerical values of x into both sides of the identity. (This method is called the *method of particular values*.) A successful combination of the indicated methods, prompted by experience, often allows us to simplify the process of finding the coefficients.

If the rational fraction $\frac{P(x)}{Q(x)}$ is *improper*, the integral part should first be singled out.

5.1.1.

$$I=\int \frac{15x^2-4x-81}{(x-3)(x+4)(x-1)}\,dx.$$

Solution. The integrand is a proper rational fraction. Since all roots of the denominator are real and simple, the integral will appear in the form of the sum of three simple fractions of the form

$$\frac{15x^2-4x-81}{(x-3)(x+4)(x-1)}=\frac{A}{x-3}+\frac{B}{x+4}+\frac{D}{x-1},$$

where A, B, D are the coefficients to be determined. Reducing the fractions to a common denominator and then rejecting it, we obtain the identity

$$15x^2-4x-81=A(x+4)(x-1)+B(x-3)(x-1)+ \\ +D(x-3)(x+4). \quad (*)$$

Comparing the coefficients at equal powers of x in both sides of the identity, we get a system of equations for determining the coefficients

$$A+B+D=15; \qquad 3A-4B+D=-4; \qquad -4A+3B-12D=-81.$$

Solving the system of equations we find $A=3$, $B=5$, $D=7$.

Hence,

$$I=3\int\frac{dx}{x-3}+5\int\frac{dx}{x+4}+7\int\frac{dx}{x-1}= \\ =3\ln|x-3|+5\ln|x+4|+7\ln|x-1|+C= \\ =\ln|(x-3)^3(x+4)^5(x-1)^7|+C.$$

Note. Let us use the same example to demonstrate the application of the method of particular values.

The identity (*) is true for any value of x. Therefore, setting three arbitrary particular values, we obtain three equations for determining the three undetermined coefficients. It is most convenient to choose the roots of the denominator as the values of x, since they nullify some factors. Putting $x=3$ in the identity (*), we get $A=3$; putting $x=-4$, we obtain $B=5$; and putting $x=1$, we get $D=7$.

5.1.2. $I=\int\frac{x^4\,dx}{(2+x)(x^2-1)}$.

5.1.3. $I=\int\frac{x^4-3x^2-3x-2}{x^3-x^2-2x}\,dx$.

Solution. Since the power of the numerator is higher than that of the denominator, i.e. the fraction is improper, we have to single out the integral part. Dividing the numerator by the denominator, we obtain

$$\frac{x^4-3x^2-3x-2}{x^3-x^2-2x}=x+1-\frac{x+2}{x(x^2-x-2)}.$$

Hence,

$$I=\int\frac{x^4-3x^2-3x-2}{x^3-x^2-2x}dx=\int(x+1)\,dx-\int\frac{(x+2)\,dx}{x(x-2)(x+1)}.$$

Expand the remaining proper fraction into simple ones:

$$\frac{x+2}{x(x-2)(x+1)}=\frac{A}{x}+\frac{B}{x-2}+\frac{D}{x+1}.$$

Hence

$$x+2=A(x-2)(x+1)+Bx(x+1)+Dx(x-2).$$

Substituting in turn the values $x_1=0$, $x_2=2$, $x_3=-1$ (the roots of the denominator) into both sides of the equality, we obtain

$$A=-1;\quad B=\frac{2}{3};\quad D=\frac{1}{3}.$$

And so

$$I=\int(x+1)\,dx+\int\frac{dx}{x}-\frac{2}{3}\int\frac{dx}{x-2}-\frac{1}{3}\int\frac{dx}{x+1}=$$
$$=\frac{x^2}{2}+x+\ln|x|-\frac{2}{3}\ln|x-2|-\frac{1}{3}\ln|x+1|+C.$$

5.1.4. $I=\int\frac{2x^2-3x+3}{x^3-2x^2+x}dx.$

Solution. Here the integrand is a proper rational fraction, whose denominator roots are real but some of them are multiple:

$$x^3-2x^2+x=x(x-1)^2.$$

Hence, the expansion into partial fractions has the form

$$\frac{2x^2-3x+3}{x^3-2x^2+x}=\frac{A}{x}+\frac{B}{(x-1)^2}+\frac{D}{x-1},$$

whence we get the identity:

$$2x^2-3x+3\equiv A(x-1)^2+Bx+Dx(x-1)=$$
$$=(A+D)x^2+(-2A-D+B)x+A. \qquad (*)$$

Equating the coefficients at equal powers of x we get a system of equations for determining the coefficients A, B, D:

$$A+D=2;\quad -2A-D+B=-3;\quad A=3.$$

Whence $A=3$; $B=2$; $D=-1$.
Thus,

$$I=3\int\frac{dx}{x}+2\int\frac{dx}{(x-1)^2}-\int\frac{dx}{x-1}=3\ln|x|-\frac{2}{x-1}-\ln|x-1|+C.$$

Note. The coefficients can be determined in a somewhat simpler way if in the identity (*) we put $x_1=0$; $x_2=1$ (the denominator roots), and x_3 equal to any arbitrary value.

At $x=0$ we get $3=A$; at $x=1$ we will have $2=B$; at $x=2$ we obtain $5=A+2B+2D$; $5=3+4+2D$; whence $D=-1$.

5.1.5. $I=\int \frac{x^3+1}{x(x-1)^3}\,dx.$

5.1.6. $I=\int \frac{x\,dx}{x^3+1}.$

Solution. Since $x^3+1=(x+1)(x^2-x+1)$ (the second factor is not expanded into real multipliers of the first power), the expansion of the given fraction will have the form

$$\frac{x}{x^3+1}=\frac{A}{x+1}+\frac{Bx+D}{x^2-x+1}.$$

Hence,

$$x=A(x^2-x+1)+(Bx+D)(x+1)=$$
$$=(A+B)x^2+(-A+B+D)x+(A+D).$$

Equating the coefficients at equal powers of x, we get

$$A=-\frac{1}{3};\quad B=\frac{1}{3};\quad D=\frac{1}{3}.$$

Thus,

$$I=-\frac{1}{3}\int\frac{dx}{x+1}+\frac{1}{3}\int\frac{x+1}{x^2-x+1}\,dx=-\frac{1}{3}\ln|x+1|+\frac{1}{3}I_1.$$

To calculate the integral

$$I_1=\int\frac{x+1}{x^2-x+1}\,dx$$

let us take the perfect square out of the denominator:

$$x^2-x+1=\left(x-\frac{1}{2}\right)^2+\frac{3}{4}$$

and make the substitution $x-\frac{1}{2}=t$. Then

$$I_1=\int\frac{t+\frac{1}{2}+1}{t^2+\frac{3}{4}}\,dt=\int\frac{t\,dt}{t^2+\frac{3}{4}}+\frac{3}{2}\int\frac{dt}{t^2+\frac{3}{4}}=$$
$$=\frac{1}{2}\ln\left(t^2+\frac{3}{4}\right)+\sqrt{3}\arctan\frac{2t}{\sqrt{3}}+C.$$

Returning to x, we obtain

$$I_1=\frac{1}{2}\ln(x^2-x+1)+\sqrt{3}\arctan\frac{2x-1}{\sqrt{3}}+C.$$

Thus,

$$I=\int \frac{x}{x^3+1}\,dx=$$

$$=-\frac{1}{3}\ln|x+1|+\frac{1}{6}\ln(x^2-x+1)+\frac{\sqrt{3}}{3}\arctan\frac{2x-1}{\sqrt{3}}+C.$$

5.1.7. $I=\int \frac{dx}{(x^2+1)(x^2+4)}$.

Solution. The denominator has two pairs of different conjugate complex roots, therefore

$$\frac{1}{(x^2+1)(x^2+4)}=\frac{Ax+B}{x^2+1}+\frac{Dx+E}{x^2+4},$$

hence

$$1=(Ax+B)(x^2+4)+(Dx+E)(x^2+1).$$

Here it is convenient to apply the method of particular values for determining the coefficients, since the complex roots of the denominator ($x=\pm i$ and $x=\pm 2i$) are sufficiently simple.

Putting $x=i$, we obtain

$$3B+3Ai=1,$$

whence $A=0$, $B=\frac{1}{3}$. Putting $x=2i$, we obtain $-3E-6Di=1$, whence $D=0$, $E=-\frac{1}{3}$. Thus,

$$\int \frac{dx}{(x^2+1)(x^2+4)}=\frac{1}{3}\int\frac{dx}{x^2+1}-\frac{1}{3}\int\frac{dx}{x^2+4}=$$

$$=\frac{1}{3}\arctan x-\frac{1}{6}\arctan\frac{x}{2}+C.$$

5.1.8. $I=\int \frac{(x+1)\,dx}{(x^2+x+2)(x^2+4x+5)}$.

5.1.9. $I=\int \frac{x^4+4x^3+11x^2+12x+8}{(x^2+2x+3)^2(x+1)}\,dx.$

Solution. Here we already have multiple complex roots. Expand the fraction into partial fractions:

$$\frac{x^4+4x^3+11x^2+12x+8}{(x^2+2x+3)^2(x+1)}=\frac{Ax+B}{(x^2+2x+3)^2}+\frac{Dx+E}{x^2+2x+3}+\frac{F}{x+1}.$$

Find the coefficients:

$$A=1;\quad B=-1;\quad D=0;\quad E=0;\quad F=1.$$

Hence,

$$I=\int \frac{x^4+4x^3+11x^2+12x+8}{(x^2+2x+3)^2(x+1)}\,dx=$$

$$=\int\frac{x-1}{(x^2+2x+3)^2}\,dx+\int\frac{dx}{x+1}=\ln|x+1|+I_1.$$

Calculate $I_1=\int \frac{x-1}{(x^2+2x+3)^2}\,dx$.

Since $x^2+2x+3=(x+1)^2+2$, let us make the substitution $x+1=t$. Then we obtain

$$I_1=\int \frac{t-2}{(t^2+2)^2}\,dt=\int \frac{t}{(t^2+2)^2}\,dt-2\int \frac{dt}{(t^2+2)^2}=-\frac{1}{2(t^2+2)}-2I_2.$$

The integral

$$I_2=\int \frac{dt}{(t^2+2)^2}$$

is calculated by the reduction formula (see Problem **4.4.1**):

$$I_2=\frac{1}{4}\frac{t}{t^2+2}+\frac{1}{4}\int \frac{dt}{t^2+2}=\frac{1}{4}\frac{t}{t^2+2}+\frac{1}{4\sqrt{2}}\arctan\frac{t}{\sqrt{2}}+C.$$

Thus

$$I_1=-\frac{1}{2(t^2+2)}-\frac{t}{2(t^2+2)}-\frac{1}{2\sqrt{2}}\arctan\frac{t}{\sqrt{2}}+C.$$

Returning to x, we obtain

$$I_1=-\frac{1}{2(x^2+2x+3)}-\frac{x+1}{2(x^2+2x+3)}-\frac{1}{2\sqrt{2}}\arctan\frac{x+1}{\sqrt{2}}+C.$$

We finally obtain

$$I=\int \frac{x^4+4x^3+11x^2+12x+8}{(x^2+2x+3)^2(x+1)}\,dx=$$
$$=\ln|x+1|-\frac{x+2}{2(x^2+2x+3)}-\frac{1}{2\sqrt{2}}\arctan\frac{x+1}{\sqrt{2}}+C.$$

Find the following integrals:

5.1.10. $\int \frac{5x^3+9x^2-22x-8}{x^3-4x}\,dx.$

5.1.11. $\int \frac{dx}{(x+1)(x+2)^2(x+3)^3}.$

5.1.12. $\int \frac{dx}{(x^2-4x+4)(x^2-4x+5)}.$

5.1.13. $\int \frac{dx}{(1+x)(1+x^2)(1+x^3)}.$

5.1.14. $\int \frac{x^3+3}{(x+1)(x^2+1)}\,dx.$

§ 5.2. Integration of Certain Irrational Expressions

Certain types of integrals of algebraic irrational expressions can be reduced to integrals of rational functions by an appropriate change of the variable. Such transformation of an integral is called its *rationalization*.

I. If the integrand is a rational function of fractional powers of an independent variable x, i. e. the function $R\left(x, x^{\frac{p_1}{q_1}}, \ldots, x^{\frac{p_k}{q_k}}\right)$, then the integral can be rationalized by the substitution $x=t^m$, where m is the least common multiple of the numbers $q_1, q_2, \ldots, q_k$.

II. If the integrand is a rational function of x and fractional powers of a linear fractional function of the form $\frac{ax+b}{cx+d}$, then rationalization of the integral is effected by the substitution $\frac{ax+b}{cx+d}=t^m$, where m has the same sense as above.

5.2.1. $I=\int \frac{x+\sqrt[3]{x^2}+\sqrt[6]{x}}{x\left(1+\sqrt[3]{x}\right)}\,dx.$

Solution. The least common multiple of the numbers 3 and 6 is 6, therefore we make the substitution:

$$x=t^6, \quad dx=6t^5\,dt,$$

whence

$$I=6\int \frac{(t^6+t^4+t)\,t^5}{t^6\,(1+t^2)}\,dt=6\int \frac{t^5+t^3+1}{1+t^2}\,dt=$$
$$=6\int t^3\,dt+6\int \frac{dt}{t^2+1}=\frac{3}{2}\,t^4+6\arctan t+C.$$

Returning to x, we obtain

$$I=\frac{3}{2}\,x^{\frac{2}{3}}+6\arctan\sqrt[6]{x}+C.$$

5.2.2. $I=\int \frac{\sqrt{x}+\sqrt[3]{x}}{\sqrt[4]{x^5}-\sqrt[6]{x^7}}\,dx.$

5.2.3. $I=\int \frac{(2x-3)^{\frac{1}{2}}\,dx}{(2x-3)^{\frac{1}{3}}+1}.$

Solution. The integrand is a rational function of $\sqrt[6]{2x-3}$, therefore we put $2x-3=t^6$, whence

$$dx=3t^5\,dt; \quad (2x-3)^{\frac{1}{2}}=t^3; \quad (2x-3)^{\frac{1}{3}}=t^2.$$

Hence,

$$I=\int \frac{3t^8}{t^2+1}\,dt=3\int (t^6-t^4+t^2-1)\,dt+3\int \frac{dt}{1+t^2}=$$
$$=3\,\frac{t^7}{7}-3\,\frac{t^5}{5}+3\,\frac{t^3}{3}-3t+3\arctan t+C.$$

Returning to x, we get

$$I=3\left[\frac{1}{7}(2x-3)^{\frac{7}{6}}-\frac{1}{5}(2x-3)^{\frac{5}{6}}\frac{1}{3}(2x-3)^{\frac{1}{2}}-\right.$$
$$\left.-(2x-3)^{\frac{1}{6}}+\arctan(2x-3)^{\frac{1}{6}}\right]+C.$$

5.2.4. $I=\int\frac{dx}{x\left(2+\sqrt[3]{\frac{x-1}{x}}\right)}$.

5.2.5. $I=\int\frac{2}{(2-x)^2}\sqrt[3]{\frac{2-x}{2+x}}\,dx$.

Solution. The integrand is a rational function of x and the expression $\sqrt[3]{\frac{2-x}{2+x}}$, therefore let us introduce the substitution

$$\sqrt[3]{\frac{2-x}{2+x}}=t;\quad \frac{2-x}{2+x}=t^3,$$

whence

$$x=\frac{2-2t^3}{1+t^3};\quad 2-x=\frac{4t^3}{1+t^3};\quad dx=\frac{-12t^2}{(1+t^3)^2}\,dt.$$

Hence

$$I=-\int\frac{2(1+t^3)^2\,t\cdot 12t^2}{16t^6(1+t^3)^2}\,dt=-\frac{3}{2}\int\frac{dt}{t^3}=\frac{3}{4t^2}+C.$$

Returning to x, we get

$$I=\frac{3}{4}\sqrt[3]{\left(\frac{2+x}{2-x}\right)^2}+C.$$

5.2.6. $I=\int\frac{dx}{\sqrt[4]{(x-1)^3(x+2)^5}}$.

Solution. Since

$$\sqrt[4]{(x-1)^3(x+2)^5}=(x-1)(x+2)\sqrt[4]{\frac{x+2}{x-1}},$$

the integrand is a rational function of x and $\sqrt[4]{\frac{x+2}{x-1}}$; therefore let us introduce the substitution:

$$\sqrt[4]{\frac{x+2}{x-1}}=t;\quad \frac{x+2}{x-1}=t^4,$$

whence

$$x=\frac{t^4+2}{t^4-1};\quad x-1=\frac{3}{t^4-1};\quad x+2=\frac{3t^4}{t^4-1};$$
$$dx=\frac{-12t^3}{(t^4-1)^2}\,dt.$$

Hence,

$$I=-\int \frac{(t^4-1)(t^4-1)\,12t^3\,dt}{3\cdot 3t^4 t\,(t^4-1)^2}=-\frac{4}{3}\int \frac{dt}{t^2}=\frac{4}{3t}+C.$$

Returning to x, we obtain

$$I=\frac{4}{3}\sqrt[4]{\frac{x-1}{x+2}}+C.$$

5.2.7. $\int \frac{dx}{(1-x)\sqrt{1-x^2}}$.

5.2.8. $\int \frac{dx}{\sqrt[3]{(x+1)^2(x-1)^4}}$.

5.2.9. $\int (x-2)\sqrt{\frac{1+x}{1-x}}\,dx$.

§ 5.3. Euler's Substitutions

Integrals of the form $\int R(x, \sqrt{ax^2+bx+c})\,dx$ are calculated with the aid of one of the three Euler substitutions:

(1) $\sqrt{ax^2+bx+c}=t\pm x\sqrt{a}$ if $a>0$;
(2) $\sqrt{ax^2+bx+c}=tx\pm\sqrt{c}$ if $c>0$;
(3) $\sqrt{ax^2+bx+c}=(x-\alpha)t$ if

$$ax^2+bx+c=a(x-\alpha)(x-\beta),$$

i.e. if α is a real root of the trinomial ax^2+bx+c.

5.3.1. $I=\int \frac{dx}{1+\sqrt{x^2+2x+2}}$.

Solution. Here $a=1>0$, therefore we make the substitution

$$\sqrt{x^2+2x+2}=t-x.$$

Squaring both sides of this equality and reducing the similar terms, we get

$$2x+2tx=t^2-2,$$

whence

$$x=\frac{t^2-2}{2(1+t)};\quad dx=\frac{t^2+2t+2}{2(1+t)^2}\,dt;$$

$$1+\sqrt{x^2+2x+2}=1+t-\frac{t^2-2}{2(1+t)}=\frac{t^2+4t+4}{2(1+t)}.$$

Substituting into the integral, we obtain

$$I=\int \frac{2(1+t)(t^2+2t+2)}{(t^2+4t+4)\,2(1+t)^2}\,dt=\int \frac{(t^2+2t+2)\,dt}{(1+t)(t+2)^2}.$$

Now let us expand the obtained proper rational fraction into partial fractions:

$$\frac{t^2+2t+2}{(t+1)(t+2)^2}=\frac{A}{t+1}+\frac{B}{t+2}+\frac{D}{(t+2)^2}.$$

Applying the method of undetermined coefficients we find: $A=1$, $B=0$, $D=-2$.

Hence,

$$\int\frac{t^2+2t+2}{(t+1)(t+2)^2}dt=\int\frac{dt}{t+1}-2\int\frac{dt}{(t+2)^2}=\ln|t+1|+\frac{2}{t+2}+C.$$

Returning to x, we get

$$I=\ln(x+1+\sqrt{x^2+2x+2})+\frac{2}{x+2+\sqrt{x^2+2x+2}}+C.$$

5.3.2. $I=\int\frac{dx}{x+\sqrt{x^2-x+1}}$.

Solution. Since here $c=1>0$, we can apply the second Euler substitution

$$\sqrt{x^2-x+1}=tx-1,$$

whence

$$(2t-1)x=(t^2-1)x^2;\quad x=\frac{2t-1}{t^2-1};$$

$$dx=-2\frac{t^2-t+1}{(t^2-1)^2}dt;\quad x+\sqrt{x^2-x+1}=\frac{t}{t-1}.$$

Substituting into I, we obtain an integral of a rational fraction:

$$\int\frac{dx}{x+\sqrt{x^2-x+1}}=\int\frac{-2t^2+2t-2}{t(t-1)(t+1)^2}dt,$$

$$\frac{-2t^2+2t-2}{t(t-1)(t+1)^2}=\frac{A}{t}+\frac{B}{t-1}+\frac{D}{(t+1)^2}+\frac{E}{t+1}.$$

By the method of undetermined coefficients we find

$$A=2;\quad B=-\frac{1}{2};\quad D=-3;\quad E=-\frac{3}{2}.$$

Hence

$$I=2\int\frac{dt}{t}-\frac{1}{2}\int\frac{dt}{t-1}-3\int\frac{dt}{(t+1)^2}-\frac{3}{2}\int\frac{dt}{t+1}=$$

$$=2\ln|t|-\frac{1}{2}\ln|t-1|+\frac{3}{t+1}-\frac{3}{2}\ln|t+1|+C,$$

where $t=\frac{\sqrt{x^2-x+1}+1}{x}$.

5.3.3. $I=\int\frac{dx}{(1+x)\sqrt{1+x-x^2}}$.

5.3.4. $I=\int \frac{x\,dx}{(\sqrt{7x-10-x^2})^3}$.

Solution. In this case $a<0$ and $c<0$ therefore neither the first, nor the second Euler substitution is applicable. But the quadratic trinomial $7x-10-x^2$ has real roots $\alpha=2$, $\beta=5$, therefore we use the third Euler substitution:

$$\sqrt{7x-10-x^2}=\sqrt{(x-2)(5-x)}=(x-2)\,t.$$

Whence

$$5-x=(x-2)\,t^2;$$

$$x=\frac{5+2t^2}{1+t^2};\quad dx=-\frac{6t\,dt}{(1+t^2)^2};$$

$$(x-2)\,t=\left(\frac{5+2t^2}{1+t^2}-2\right)t=\frac{3t}{1+t^2}.$$

Hence

$$I=-\frac{6}{27}\int\frac{5+2t^2}{t^2}dt=-\frac{2}{9}\int\left(\frac{5}{t^2}+2\right)dt=-\frac{2}{9}\left(-\frac{5}{t}+2t\right)+C,$$

where $t=\frac{\sqrt{7x-10-x^2}}{x-2}$.

Calculate the following integrals with the aid of one of the Euler substitutions:

5.3.5. $\int\frac{dx}{x-\sqrt{x^2+2x+4}}$.

5.3.6. $\int\frac{dx}{\sqrt{1-x^2}-1}$.

5.3.7. $\int\frac{dx}{\sqrt{(2x-x^2)^3}}$.

5.3.8. $\int\frac{(x+\sqrt{1+x^2})^{15}}{\sqrt{1+x^2}}dx$.

§ 5.4. Other Methods of Integrating Irrational Expressions

The Euler substitutions often lead to rather cumbersome calculations, therefore they should be applied only when it is difficult to find another method for calculating a given integral. For calculating many integrals of the form

$$\int R(x,\sqrt{ax^2+bx+c})\,dx,$$

simpler methods are used.

I. Integrals of the form

$$\mathrm{I}=\int\frac{Mx+N}{\sqrt{ax^2+bx+c}}\,dx$$

are reduced by the substitution $x+\frac{b}{2a}=t$ to the form

$$I=M_1\int\frac{t\,dt}{\sqrt{at^2+K}}+N_1\int\frac{dt}{\sqrt{at^2+K}},$$

where M_1, N_1, K are new coefficients.

The first integral is reduced to the integral of a power function, while the second, being a tabular one, is reduced to a logarithm (for $a>0$) or to an arc sine (for $a<0$, $K>0$).

II. Integrals of the form

$$\int\frac{P_m(x)}{\sqrt{ax^2+bx+c}}\,dx,$$

where $P_m(x)$ is a polynomial of degree m, are calculated by the reduction formula:

$$\int\frac{P_m(x)\,dx}{\sqrt{ax^2+bx+c}}=P_{m-1}(x)\sqrt{ax^2+bx+c}+K\int\frac{dx}{\sqrt{ax^2+bx+c}}, \quad (1)$$

where $P_{m-1}(x)$ is a polynomial of degree $m-1$, and K is some constant number.

The coefficients of the polynomial $P_{m-1}(x)$ and the constant number K are determined by the method of undetermined coefficients.

III. Integrals of the form

$$\int\frac{dx}{(x-a_1)^m\sqrt{ax^2+bx+c}}$$

are reduced to the preceding type by the substitution

$$x-a_1=\frac{1}{t}.$$

IV. For trigonometric and hyperbolic substitutions see § 5.7.

5.4.1. $I=\int\frac{(x+3)\,dx}{\sqrt{4x^2+4x-3}}$.

Solution. Make the substitution $2x+1=t$, whence

$$x=\frac{t-1}{2}, \qquad dx=\frac{1}{2}\,dt.$$

Hence,

$$I=\frac{1}{4}\int\frac{(t+5)\,dt}{\sqrt{t^2-4}}=\frac{1}{4}\sqrt{t^2-4}+\frac{5}{4}\ln|t+\sqrt{t^2-4}|+C.$$

Returning to x, we get

$$I=\frac{1}{4}\sqrt{4x^2+4x-3}+\frac{5}{4}\ln|2x+1+\sqrt{4x^2+4x-3}|+C.$$

5.4.2. $I=\int \frac{5x+4}{\sqrt{x^2+2x+5}}\,dx.$

5.4.3. $I=\int \frac{x^3-x-1}{\sqrt{x^2+2x+2}}\,dx.$

Solution. Here $P_m(x)=x^3-x-1$. Hence,

$$P_{m-1}(x)=Ax^2+Bx+D.$$

We seek the integral in the form

$$I=(Ax^2+Bx+D)\sqrt{x^2+2x+2}+K\int\frac{dx}{\sqrt{x^2+2x+2}}.$$

Differentiating this equality, we obtain

$$I'=\frac{x^3-x-1}{\sqrt{x^2+2x+2}}=$$
$$=(2Ax+B)\sqrt{x^2+2x+2}+(Ax^2+Bx+D)\frac{x+1}{\sqrt{x^2+2x+2}}+$$
$$+\frac{K}{\sqrt{x^2+2x+2}}.$$

Reduce to a common denominator and equate the numerators

$$x^3-x-1=(2Ax+B)(x^2+2x+2)+(Ax^2+Bx+D)(x+1)+K.$$

Equating the coefficients at equal powers of x, we get the following system of equations:

$$2A+A=1, \qquad B+4A+B+A=0;$$
$$2B+4A+D+B=-1; \qquad 2B+D+K=-1.$$

Solving the system, we obtain

$$A=\frac{1}{3};\quad B=-\frac{5}{6};\quad D=\frac{1}{6};\quad K=\frac{1}{2}.$$

Thus,

$$I=\left(\frac{1}{3}x^2-\frac{5}{6}x+\frac{1}{6}\right)\sqrt{x^2+2x+2}+\frac{1}{2}\int\frac{dx}{\sqrt{x^2+2x+2}},$$

where

$$I_1=\int\frac{dx}{\sqrt{x^2+2x+2}}=\int\frac{dx}{\sqrt{(x+1)^2+1}}=\ln\left(x+1+\sqrt{x^2+2x+2}\right)+C.$$

5.4.4. $I=\int\sqrt{4x^2-4x+3}\,dx.$

Solution. Transform the integral to the form

$$I=\int\frac{4x^2-4x+3}{\sqrt{4x^2-4x+3}}\,dx=(Ax+B)\sqrt{4x^2-4x+3}+K\int\frac{dx}{\sqrt{4x^2-4x+3}}.$$

Applying the method of undetermined coefficients, we get

$$I=\left(\frac{1}{2}x-\frac{1}{4}\right)\sqrt{4x^2-4x+3}+\int\frac{dx}{\sqrt{(2x-1)^2+2}}=$$
$$=\left(\frac{1}{2}x-\frac{1}{4}\right)\sqrt{4x^2-4x+3}+\frac{1}{2}\ln\left(2x-1+\sqrt{4x^2-4x+3}\right)+C.$$

5.4.5. $\int\frac{9x^3-3x^2+2}{\sqrt{3x^2-2x+1}}\,dx.$

5.4.6. $\int\sqrt{x^2+x+1}\,dx.$

5.4.7. $I=\int\frac{(x+4)\,dx}{(x-1)(x+2)^2\sqrt{x^2+x+1}}.$

Solution. Represent the given integral as follows:

$$\int\frac{(x+4)\,dx}{(x-1)(x+2)^2\sqrt{x^2+x+1}}=\int\frac{x+4}{(x-1)(x+2)^2}\cdot\frac{dx}{\sqrt{x^2+x+1}}.$$

Expand the fraction $\frac{x+4}{(x-1)(x+2)^2}$ into partial fractions

$$\frac{x+4}{(x-1)(x+2)^2}=\frac{A}{x-1}+\frac{B}{(x+2)^2}+\frac{D}{x+2}.$$

Find the coefficients

$$A=\frac{5}{9};\quad B=-\frac{2}{3};\quad D=-\frac{5}{9}.$$

Hence,

$$I=\int\left[\frac{5}{9(x-1)}-\frac{2}{3(x+2)^2}-\frac{5}{9(x+2)}\right]\cdot\frac{dx}{\sqrt{x^2+x+1}}=$$
$$=\frac{5}{9}\int\frac{dx}{(x-1)\sqrt{x^2+x+1}}-\frac{2}{3}\int\frac{dx}{(x+2)^2\sqrt{x^2+x+1}}-$$
$$-\frac{5}{9}\int\frac{dx}{(x+2)\sqrt{x^2+x+1}}.$$

The first integral is calculated by the substitution $x-1=\frac{1}{t}$, the second and the third by the substitution $x+2=\frac{1}{t}$.
We leave the solution to the reader.

5.4.8. $\int\frac{x^3-6x^2+11x-6}{\sqrt{x^2+4x+3}}\,dx.$

5.4.9. $\int\frac{3x^3+5x^2-7x+9}{\sqrt{2x^2+5x+7}}\,dx.$

5.4.10. $\int\frac{dx}{(x+1)^5\sqrt{x^2+2x}}.$

5.4.11. $\int \frac{x\,dx}{(x^2-3x+2)\sqrt{x^2-4x+3}}$.

5.4.12. $\int \frac{dx}{(x+1)^3\sqrt{x^2+3x+2}}$.

5.4.13. $\int \frac{(x^2-1)\,dx}{x\sqrt{1+3x^2+x^4}}$.

§ 5.5. Integration of a Binomial Differential

The integral $\int x^m (a+bx^n)^p\,dx$, where m, n, p are rational numbers, is expressed through elementary functions only in the following three cases:

Case I. p is an integer. Then, if $p > 0$, the integrand is expanded by the formula of the Newton binomial; but if $p < 0$, then we put $x = t^k$, where k is the common denominator of the fractions m and n.

Case II. $\frac{m+1}{n}$ is an integer. We put $a+bx^n = t^\alpha$, where α is the denominator of the fraction p.

Case III. $\frac{m+1}{n}+p$ is an integer. We put $a+bx^n = t^\alpha x^n$, where α is the denominator of the fraction p.

5.5.1. $I = \int \sqrt[3]{x}\,(2+\sqrt{x})^2\,dx$.

Solution. $I = \int x^{\frac{1}{3}}\left(2+x^{\frac{1}{2}}\right)^2 dx$. Here $p = 2$, i.e. an integer; hence, we have Case I.

$$I = \int x^{\frac{1}{3}}\left(x+4x^{\frac{1}{2}}+4\right)dx = \int \left(x^{\frac{4}{3}}+4x^{\frac{5}{6}}+4x^{\frac{1}{3}}\right)dx =$$

$$= \frac{3}{7}x^{\frac{7}{3}} + \frac{24}{11}x^{\frac{11}{6}} + 3x^{\frac{4}{3}} + C.$$

5.5.2. $I = \int x^{-\frac{2}{3}}\left(1+x^{\frac{2}{3}}\right)^{-1} dx$.

5.5.3. $I = \int \frac{\sqrt{1+\sqrt[3]{x}}}{\sqrt[3]{x^2}}\,dx$.

Solution. $I = \int x^{-\frac{2}{3}}\left(1+x^{\frac{1}{3}}\right)^{\frac{1}{2}} dx$.

Here $m = -\frac{2}{3}$; $n = \frac{1}{3}$; $p = \frac{1}{2}$; $\frac{m+1}{n} = \frac{\left(-\frac{2}{3}+1\right)}{\frac{1}{3}} = 1$, i.e. an integer.

We have Case II. Let us make the substitution

$$1+x^{\frac{1}{3}}=t^2; \quad \frac{1}{3}x^{-\frac{2}{3}}dx=2t\,dt.$$

Hence,

$$I=6\int t^2\,dt=2t^3+C=2\left(1+x^{\frac{1}{3}}\right)^{\frac{3}{2}}+C.$$

5.5.4. $I=\int x^{\frac{1}{3}}\left(2+x^{\frac{2}{3}}\right)^{\frac{1}{4}}dx.$

5.5.5. $I=\int x^5\left(1+x^2\right)^{\frac{2}{3}}dx.$

5.5.6. $I=\int x^{-11}\left(1+x^4\right)^{-\frac{1}{2}}dx.$

Solution. Here $p=-\frac{1}{2}$ is a fraction, $\frac{m+1}{n}=\frac{-11+1}{4}=-\frac{5}{2}$ also a fraction, but $\frac{m+1}{n}+p=-\frac{5}{2}-\frac{1}{2}=-3$ is an integer, i.e. we have Case III. We put $1+x^4=x^4t^2$. Hence

$$x=\frac{1}{(t^2-1)^{\frac{1}{4}}}; \quad dx=-\frac{t\,dt}{2\,(t^2-1)^{\frac{5}{4}}}.$$

Substituting these expressions into the integral, we obtain

$$I=-\frac{1}{2}\int (t^2-1)^{\frac{11}{4}}\left(\frac{t^2}{t^2-1}\right)^{-\frac{1}{2}}\frac{t\,dt}{(t^2-1)^{\frac{5}{4}}}=$$

$$=-\frac{1}{2}\int (t^2-1)^2\,dt=-\frac{t^5}{10}+\frac{t^3}{3}-\frac{t}{2}+C.$$

Returning to x, we get

$$I=-\frac{1}{10x^{10}}\sqrt{(1+x^4)^5}+\frac{1}{3x^6}\sqrt{(1+x^4)^3}-\frac{1}{2x^2}\sqrt{1+x^4}+C.$$

5.5.7. $\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}\,dx.$

5.5.8. $\int \frac{dx}{x\,(1+\sqrt[3]{x})^2}.$

5.5.9. $\int x^3\,(1+x^2)^{\frac{1}{2}}\,dx.$

5.5.10. $\int \frac{dx}{x^4\sqrt{1+x^2}}.$

5.5.11. $\int \sqrt[3]{x}\,\sqrt[7]{1+\sqrt[3]{x^4}}\,dx.$

5.5.12. $\int \frac{dx}{x^3\sqrt[5]{1+\frac{1}{x}}}.$

§ 5.6. Integration of Trigonometric and Hyperbolic Functions

I. Integrals of the form

$$I=\int \sin^m x\cos^n x\,dx,$$

where m and n are rational numbers, are reduced to the integral of the binomial differential

$$I=\int t^m(1-t^2)^{\frac{n-1}{2}}\,dt,\quad t=\sin x$$

and are, therefore, integrated in elementary functions only in the following three cases:

(1) n is odd $\left(\frac{n-1}{2}\text{ an integer}\right)$,

(2) m is odd $\left(\frac{m+1}{2}\text{ an integer}\right)$,

(3) $m+n$ is even $\left(\frac{m+1}{2}+\frac{n-1}{2}\text{ an integer}\right)$.

If n is an odd number, the substitution $\sin x=t$ is applied.

If m is an odd number, the substitution $\cos x=t$ is applied.

If the sum $m+n$ is an even number, use the substitution $\tan x=t$ (or $\cot x=t$).

In particular, this kind of substitution is convenient for integrals of the form

$$\int \tan^n x\,dx\ \left(\text{or }\int \cot^n x\,dx\right),$$

where n is a positive integer. But the last substitution is inconvenient if both m and n are positive numbers. If m and n are non-negative even numbers, then it appears more convenient to use the method of reducing the power with the aid of trigonometric transformations:

$$\cos^2 x=\frac{1}{2}(1+\cos 2x),\quad \sin^2 x=\frac{1}{2}(1-\cos 2x)$$

or $\sin x\cos x=\frac{1}{2}\sin 2x$.

5.6.1. $I=\int \frac{\sin^3 x}{\sqrt[3]{\cos^2 x}}\,dx.$

Solution. Here $m=3$ is an odd number. We put $\cos x=t$, $\sin x\,dx=-dt$, which gives

$$I=-\int(1-t^2)\,t^{-\frac{2}{3}}\,dt=-3t^{\frac{1}{3}}+\frac{3}{7}t^{\frac{7}{3}}+C=$$
$$=3\sqrt[3]{\cos x}\left(\frac{1}{7}\cos^2 x-1\right)+C.$$

5.6.2. $I=\int\frac{\cos^3 x}{\sin^6 x}\,dx.$

5.6.3. $I=\int\sin^4 x\cos^6 x\,dx.$

Solution. Here both m and n are positive even numbers. Let us use the method of reducing the power:

$$I=\frac{1}{16}\int(2\sin x\cos x)^4\cos^2 x\,dx=\frac{1}{32}\int\sin^4 2x\,(1+\cos 2x)\,dx=I_1+I_2.$$

The second of the obtained integrals is calculated by the substitution:

$$\sin 2x=t,\ \cos 2x\,dx=\frac{1}{2}\,dt,$$

$$I_2=\frac{1}{32}\int\sin^4 2x\cos 2x\,dx=\frac{1}{64}\int t^4\,dt=\frac{t^5}{320}+C=\frac{1}{320}\sin^5 2x+C.$$

We again apply to the first integral the method of reducing the power:

$$I_1=\frac{1}{32}\int\sin^4 2x\,dx=\frac{1}{128}\int(1-\cos 4x)^2\,dx=$$
$$=\frac{1}{128}\left(x-\frac{1}{2}\sin 4x\right)+\frac{1}{256}\int(1+\cos 8x)\,dx=$$
$$=\frac{3}{256}x-\frac{1}{256}\sin 4x+\frac{1}{2048}\sin 8x+C.$$

And so, finally,

$$I=\frac{3}{256}x-\frac{1}{256}\sin 4x+\frac{1}{2048}\sin 8x+\frac{1}{320}\sin^5 2x+C.$$

5.6.4. $I=\int\frac{\sin^2 x}{\cos^6 x}\,dx.$

Solution. Here both m and n are even numbers, but one of them is negative. Therefore, we put

$$\tan x=t;\quad \frac{1}{\cos^2 x}=1+t^2;\quad \frac{dx}{\cos^2 x}=dt.$$

Hence,

$$I=\int t^2(1+t^2)\,dt=\frac{t^3}{3}+\frac{t^5}{5}+C=\frac{\tan^3 x}{3}+\frac{\tan^5 x}{5}+C.$$

5.6.5. $I=\int \frac{\cos^4 x}{\sin^2 x}dx.$

Solution. Here we can put $\cot x=t$, but it is simpler to integrate by expansion:

$$I=\int \frac{(1-\sin^2 x)^2}{\sin^2 x}\,dx=\int\left(\frac{1}{\sin^2 x}-2+\sin^2 x\right)dx=$$

$$=-\cot x-2x+\frac{1}{2}\int(1-\cos 2x)\,dx=$$

$$=-\left(\cot x+\frac{\sin 2x}{4}+\frac{3x}{2}\right)+C.$$

5.6.6. $I=\int \frac{dx}{\cos^4 x}.$

5.6.7. $I=\int \frac{dx}{\sqrt[3]{\sin^{11} x\cos x}}.$

Solution. Here both exponents $\left(-\frac{11}{3}\text{ and }-\frac{1}{3}\right)$ are negative numbers and their sum $-\frac{11}{3}-\frac{1}{3}=-4$ is an even number, therefore we put

$$\tan x=t;\quad \frac{dx}{\cos^2 x}=dt.$$

$$I=\int\frac{dx}{\cos^4 x\sqrt[3]{\tan^{11} x}}=\int\frac{1+t^2}{\sqrt[3]{t^{11}}}\,dt=$$

$$=\int\left(t^{-\frac{11}{3}}+t^{-\frac{5}{3}}\right)dt=-\frac{3}{8}t^{-\frac{8}{3}}-\frac{3}{2}t^{-\frac{2}{3}}+C=$$

$$=-\frac{3(1+4\tan^2 x)}{8\tan^2 x\sqrt[3]{\tan^2 x}}+C.$$

5.6.8. Find the integrals of $\tan x$ and $\cot x$.

Solution.

$$\int\tan x\,dx=\int\frac{\sin x}{\cos x}dx=-\ln|\cos x|+C;$$

$$\int\cot x\,dx=\int\frac{\cos x}{\sin x}dx=\ln|\sin x|+C.$$

5.6.9. $I=\int\tan^7 x\,dx.$

Solution. We put $\tan x=t$, $x=\arctan t$; $dx=\frac{dt}{1+t^2}$. We get

$$I=\int t^7\frac{dt}{1+t^2}=\int\left(t^5-t^3+t-\frac{t}{1+t^2}\right)dt=$$

$$=\frac{t^6}{6}-\frac{t^4}{4}+\frac{t^2}{2}-\frac{1}{2}\ln(1+t^2)+C=$$

$$=\frac{1}{6}\tan^6 x-\frac{1}{4}\tan^4 x+\frac{1}{2}\tan^2 x+\ln|\cos x|+C.$$

5.6.10. (a) $I=\int \cot^6 x\,dx$; (b) $I=\int \tan^3 x\,dx$.

5.6.11. $I=\int \frac{\cos^4 x}{\sin^3 x}\,dx$.

Solution. Here $\sin x$ is raised to an odd power. Let us put

$$\cos x = t, \qquad -\sin x\,dx = dt.$$

We obtain an integral of a rational function.

$$I=\int \frac{\cos^4 x \sin x}{\sin^4 x}\,dx = -\int \frac{t^4}{(1-t^2)^2}\,dt.$$

Here, it is simpler to integrate by parts than to use the general methods of integration of rational functions (cf. Problem **4.4.1** (b)).

Let us put

$$u=t^3; \quad dv=\frac{t\,dt}{(1-t^2)^2}.$$

Then

$$du=3t^2\,dt; \quad v=\frac{1}{2(1-t^2)}.$$

Hence,

$$I=-\frac{t^3}{2(1-t^2)}+\frac{3}{2}\int \frac{t^2\,dt}{1-t^2}=$$

$$=-\frac{t^3}{2(1-t^2)}+\frac{3}{2}\int \frac{t^2-1+1}{1-t^2}\,dt=$$

$$=-\frac{t^3}{2(1-t^2)}-\frac{3}{2}t+\frac{3}{4}\ln\left|\frac{1+t}{1-t}\right|+C=$$

$$=-\frac{\cos^3 x}{2\sin^2 x}-\frac{3}{2}\cos x+\frac{3}{4}\ln\left|\frac{1+\cos x}{1-\cos x}\right|+C.$$

5.6.12. $I=\int \frac{\sin^4 x}{\cos x}\,dx$.

II. Integrals of the form $\int R(\sin x, \cos x)\,dx$ where R is a rational function of $\sin x$ and $\cos x$ are transformed into integrals of a rational function by the substitution:

$$\tan\left(\frac{x}{2}\right)=t \quad (-\pi < x < \pi).$$

This is so-called *universal* substitution. In this case

$$\sin x=\frac{2t}{1+t^2}; \qquad \cos x=\frac{1-t^2}{1+t^2};$$

$$x=2\arctan t; \qquad dx=\frac{2dt}{1+t^2}.$$

Sometimes instead of the substitution $\tan\frac{x}{2}=t$ it is more advantageous to make the substitution $\cot\frac{x}{2}=t$ $(0<x<2\pi)$.

Universal substitution often leads to very cumbersome calculations. Indicated below are the cases when the aim can be achieved with the aid of simpler substitutions:

(a) if the equality

$$R(-\sin x, \cos x)\equiv -R(\sin x, \cos x)$$

or

$$R(\sin x, -\cos x)\equiv -R(\sin x, \cos x)$$

is satisfied, then it is more advantageous to apply the substitution $\cos x=t$ to the former equality, and $\sin x=t$ to the latter;

(b) if the equality

$$R(-\sin x, -\cos x)\equiv R(\sin x, \cos x)$$

is fulfilled, then a better effect is gained by substituting $\tan x=t$ or $\cot x=t$.

The latter case is encountered, for example, in integrals of the form $\int R(\tan x)\,dx$.

5.6.13. $I=\int\frac{dx}{\sin x\,(2+\cos x-2\sin x)}$.

Solution. Let us put $\tan\frac{x}{2}=t$; then we have

$$I=\int\frac{\frac{2dt}{1+t^2}}{\frac{2t}{1+t^2}\left(2+\frac{1-t^2}{1+t^2}-\frac{4t}{1+t^2}\right)}=\int\frac{(1+t^2)\,dt}{t\,(t^2-4t+3)}.$$

Expand into simple fractions

$$\frac{1+t^2}{t\,(t-3)\,(t-1)}=\frac{A}{t}+\frac{B}{t-3}+\frac{D}{t-1}.$$

Find the coefficients

$$A=\frac{1}{3};\quad B=\frac{5}{3};\quad D=-1.$$

Hence

$$I=\frac{1}{3}\int\frac{dt}{t}+\frac{5}{3}\int\frac{dt}{t-3}-\int\frac{dt}{t-1}=$$

$$=\frac{1}{3}\ln|t|+\frac{5}{3}\ln|t-3|-\ln|t-1|+C=$$

$$=\frac{1}{3}\ln\left|\tan\frac{x}{2}\right|+\frac{5}{3}\ln\left|\tan\frac{x}{2}-3\right|-\ln\left|\tan\frac{x}{2}-1\right|+C.$$

5.6.14. $I=\int \frac{dx}{5+\sin x+3\cos x}$.

5.6.15. $I=\int \frac{dx}{\sin x\,(2\cos^2 x-1)}$.

Solution. If in the expression $\frac{1}{\sin x\,(2\cos^2 x-1)}$ we substitute $-\sin x$ for $\sin x$, then the fraction will change its sign. Hence, we take advantage of the substitution $t=\cos x$; $dt=-\sin x\,dx$. This gives

$$I=-\int \frac{dt}{(1-t^2)(2t^2-1)}.$$

Since

$$\frac{1}{(1-t^2)(1-2t^2)}=\frac{(2-2t^2)-(1-2t^2)}{(1-t^2)(1-2t^2)}=\frac{2}{1-2t^2}-\frac{1}{1-t^2},$$

then

$$I=2\int \frac{dt}{1-2t^2}-\int \frac{dt}{1-t^2}=\frac{1}{\sqrt{2}}\ln\left|\frac{1+t\sqrt{2}}{1-t\sqrt{2}}\right|-\frac{1}{2}\ln\left|\frac{1+t}{1-t}\right|+C=$$

$$=\frac{1}{\sqrt{2}}\ln\left|\frac{1+\sqrt{2}\cos x}{1-\sqrt{2}\cos x}\right|+\frac{1}{2}\ln\left|\frac{1-\cos x}{1+\cos x}\right|+C=$$

$$=\frac{1}{\sqrt{2}}\ln\left|\frac{1+\sqrt{2}\cos x}{1-\sqrt{2}\cos x}\right|+\ln\left|\tan\frac{x}{2}\right|+C.$$

5.6.16. $I=\int \frac{\sin^2 x\cos x}{\sin x+\cos x}\,dx$.

Solution. Since the integrand does not change sign when $\sin x$ and $\cos x$ do change their signs, we take advantage of the substitution

$$t=\tan x;\quad dt=\frac{dx}{\cos^2 x}.$$

Hence,

$$I=\int \frac{\tan^2 x\cdot\cos^4 x}{(\tan x+1)}\,\frac{dx}{\cos^2 x}=\int \frac{t^2\,dt}{(t+1)(t^2+1)^2}.$$

Expand into partial fractions

$$\frac{t^2}{(t+1)(t^2+1)^2}=\frac{A}{t+1}+\frac{Bt+D}{t^2+1}+\frac{Et+F}{(t^2+1)^2}.$$

Find the coefficients

$$A=\frac{1}{4};\quad B=-\frac{1}{4};\quad D=\frac{1}{4};\quad E=\frac{1}{2};\quad F=-\frac{1}{2}.$$

Hence,

$$I=\frac{1}{4}\int \frac{dt}{t+1}-\frac{1}{4}\int \frac{t-1}{t^2+1}\,dt+\frac{1}{2}\int \frac{t-1}{(t^2+1)^2}\,dt;$$

$$I=\frac{1}{4}\ln\frac{1+t}{\sqrt{1+t^2}}-\frac{1}{4}\cdot\frac{1+t}{1+t^2}+C=$$

$$=\frac{1}{4}\ln|\sin x+\cos x|-\frac{1}{4}\cos x(\sin x+\cos x)+C.$$

5.6.17. $I=\int\frac{2\tan x+3}{\sin^2 x+2\cos^2 x}dx.$

Solution. Dividing the numerator and denominator by $\cos^2 x$ and substituting $\tan x=t$; $\frac{dx}{\cos^2 x}=dt$, we obtain

$$I=\int\frac{2\tan x+3}{\sin^2 x+2\cos^2 x}dx=\int\frac{(2\tan x+3)\frac{dx}{\cos^2 x}}{\tan^2 x+2}=$$

$$=\int\frac{2t+3}{t^2+2}dt=\ln(t^2+2)+\frac{3}{\sqrt{2}}\arctan\frac{t}{\sqrt{2}}+C=$$

$$=\ln(\tan^2 x+2)+\frac{3}{\sqrt{2}}\arctan\frac{\tan x}{\sqrt{2}}+C.$$

5.6.18. $I=\int\frac{\sin x}{1+\sin x}dx.$

Solution. This integral, of course, can be evaluated with the aid of the universal substitution $\tan\frac{x}{2}=t$, but it is easier to get the desired result by resorting to the following transformation of the integrand:

$$\frac{\sin x}{1+\sin x}=\frac{\sin x(1-\sin x)}{(1+\sin x)(1-\sin x)}=\frac{\sin x(1-\sin x)}{\cos^2 x}=$$

$$=\frac{\sin x}{\cos^2 x}-\frac{\sin^2 x}{\cos^2 x}=\frac{\sin x}{\cos^2 x}-\tan^2 x.$$

Whence

$$I=\int\frac{\sin x}{\cos^2 x}dx-\int\sec^2 x\,dx+\int dx=\frac{1}{\cos x}-\tan x+x+C.$$

5.6.19. $I=\int\frac{1}{\cos^4 x\sin^2 x}dx.$

Solution. Here the substitution $\tan x=t$ can be applied, but it is simpler to transform the integrand. Replacing, in the numerator, unity by the trigonometric identity raised to the second power, we get

$$I=\int\frac{(\sin^2 x+\cos^2 x)^2}{\cos^4 x\sin^2 x}dx=\int\frac{\sin^4 x+2\sin^2 x\cos^2 x+\cos^4 x}{\cos^4 x\sin^2 x}dx=$$

$$=\int\frac{\sin^2 x}{\cos^4 x}dx+2\int\frac{dx}{\cos^2 x}+\int\frac{dx}{\sin^2 x}=\int\tan^2 x\frac{dx}{\cos^2 x}+2\tan x-\cot x=$$

$$=\frac{1}{3}\tan^3 x+2\tan x-\cot x+C.$$

III. **Integration of hyperbolic functions.** Functions rationally depending on hyperbolic functions are integrated in the same way as trigonometric functions.

Keep in mind the following basic formulas:

$$\cosh^2 x - \sinh^2 x = 1; \quad \sinh^2 x = \frac{1}{2}(\cosh 2x - 1);$$

$$\cosh^2 x = \frac{1}{2}(\cosh 2x + 1); \quad \sinh x \cosh x = \frac{1}{2}\sinh 2x.$$

If $\tanh \frac{x}{2} = t$, then $\sinh x = \frac{2t}{1-t^2}$; $\cosh x = \frac{1+t^2}{1-t^2}$;

$$x = 2 \operatorname{Artanh} t = \ln\left(\frac{1+t}{1-t}\right) \quad (-1 < t < 1); \ dx = \frac{2dt}{1-t^2}.$$

5.6.20. $I = \int \cosh^2 x\, dx.$

Solution.

$$I = \int \frac{1}{2}(\cosh 2x + 1)\ dx = \frac{1}{4}\sinh 2x + \frac{1}{2}x + C.$$

5.6.21. $I = \int \cosh^3 x\, dx.$

Solution. Since $\cosh x$ is raised to an odd power, we put $\sinh x = t$; $\cosh x\, dx = dt$. We obtain

$$I = \int \cosh^2 x \cosh x\, dx = \int (1 + t^2)\, dt = t + \frac{t^3}{3} + C =$$
$$= \sinh x + \frac{1}{3}\sinh^3 x + C.$$

5.6.22. Find the integrals:

(a) $\int \sinh^2 x \cosh^2 x\, dx$; (b) $\int \frac{dx}{\sinh x + 2\cosh x}$.

§ 5.7. *Integration of Certain Irrational Functions with the Aid of Trigonometric or Hyperbolic Substitutions*

Integration of functions rationally depending on x and $\sqrt{ax^2+bx+c}$ can be reduced to finding integrals of one of the following forms:

I. $\int R\,(t, \sqrt{p^2t^2+q^2})\, dt$;

II. $\int R\,(t, \sqrt{p^2t^2-q^2})\, dt$;

III. $\int R\,(t, \sqrt{q^2-p^2t^2})\, dt$, where $t=x+\frac{b}{2a}$; $ax^2+bx+c=\pm p^2t^2\pm q^2$ (singling out a perfect square).

Integrals of the forms I to III can be reduced to integrals of expressions rational with respect to sine or cosine (ordinary or hyperbolic) by means of the following substitutions:

I. $t=\frac{q}{p}\tan z$ or $t=\frac{q}{p}\sinh z$.

II. $t=\frac{q}{p}\sec z$ or $t=\frac{q}{p}\cosh z$.

III. $t=\frac{q}{p}\sin z$ or $t=\frac{q}{p}\tanh z$.

5.7.1. $I=\int \frac{dx}{\sqrt{(5+2x+x^2)^3}}$.

Solution. $5+2x+x^2=4+(x+1)^2$. Let us put $x+1=t$. Then

$$I=\int \frac{dx}{\sqrt{(5+2x+x^2)^3}}=\int \frac{dt}{(4+t^2)^3}.$$

We have obtained an integral of the form I. Let us introduce the substitution:

$$t=2\tan z;\ dt=\frac{2dz}{\cos^2 z};\ \sqrt{4+t^2}=2\sqrt{1+\tan^2 z}=\frac{2}{\cos z}.$$

We get

$$I=\frac{1}{4}\int \cos z\,dz=$$

$$=\frac{1}{4}\sin z+C=\frac{1}{4}\frac{\tan z}{\sqrt{1+\tan^2 z}}+C=\frac{1}{4}\frac{\frac{t}{2}}{\sqrt{1+\frac{t^2}{4}}}+C=$$

$$=\frac{x+1}{4\sqrt{5+2x+x^2}}+C.$$

5.7.2. $I=\int \frac{dx}{(x+1)^2\sqrt{x^2+2x+2}}$.

Solution. $x^2+2x+2=(x+1)^2+1$.

Let us put $x+1=t$; then

$$I=\int \frac{dt}{t^2\sqrt{t^2+1}}.$$

Again we have an integral of the form I. Make the substitution $t=\sinh z$. Then

$$dt=\cosh z\,dz;\quad \sqrt{t^2+1}=\sqrt{1+\sinh^2 z}=\cosh z.$$

Hence,

$$I=\int\frac{\cosh z\,dz}{\sinh^2 z\cosh z}=\int\frac{dz}{\sinh^2 z}=-\coth z+C=$$

$$=-\frac{\sqrt{1+\sinh^2 z}}{\sinh z}+C=-\frac{\sqrt{1+t^2}}{t}+C=-\frac{\sqrt{x^2+2x+2}}{x+1}+C.$$

5.7.3. $I=\int x^2\sqrt{x^2-1}\,dx.$

5.7.4. $I=\int\frac{\sqrt{x^2+1}}{x^2}\,dx.$

5.7.5. $I=\int\sqrt{(x^2-1)^3}\,dx.$

Solution. Perform the substitution:

$$x=\cosh t;\quad dx=\sinh t\,dt.$$

Hence

$$I=\int\sqrt{(\cosh^2 t-1)^3}\sinh t\,dt=\int\sinh^4 t\,dt=$$
$$=\int\left(\frac{\cosh 2t-1}{2}\right)^2 dt=$$
$$=\frac{1}{4}\int\cosh^2 2t\,dt-\frac{1}{2}\int\cosh 2t\,dt+\frac{1}{4}\int dt=$$
$$=\frac{1}{8}\int(\cosh 4t+1)\,dt-\frac{1}{4}\sinh 2t+\frac{1}{4}t=$$
$$=\frac{1}{32}\sinh 4t-\frac{1}{4}\sinh 2t+\frac{3}{8}t+C.$$

Let us return to x:

$$t=\operatorname{Arcosh} x=\ln(x+\sqrt{x^2-1});$$
$$\sinh 2t=2\sinh t\cosh t=2x\sqrt{x^2-1};$$
$$\sinh 4t=2\sinh 2t\cosh 2t=4x\sqrt{x^2-1}\,(2x^2-1).$$

Hence

$$I=\frac{1}{8}x(2x^2-1)\sqrt{x^2-1}-\frac{1}{2}x\sqrt{x^2-1}+\frac{3}{8}\ln(x+\sqrt{x^2-1})+C.$$

5.7.6. $I=\int\frac{dx}{(1+\sqrt{x})\sqrt{x-x^2}}.$

Solution. We make the substitution:

$$x=\sin^2 t;\quad dx=2\sin t\cos t\,dt$$

and get

$$I=\int \frac{2 \sin t \cos t \, dt}{(1+\sin t) \sqrt{\sin^2 t-\sin^4 t}}=\int \frac{2\, dt}{1+\sin t}=$$

$$=2 \int \frac{1-\sin t}{\cos^2 t} dt=2 \tan t-\frac{2}{\cos t}+C=$$

$$=\frac{2 \sqrt{x}}{\sqrt{1-x}}-\frac{2}{\sqrt{1-x}}+C=\frac{2(\sqrt{x}-1)}{\sqrt{1-x}}+C.$$

5.7.7. $I=\int \sqrt{3-2x-x^2}\, dx.$

5.7.8. $I=\int \frac{dx}{(x^2-2x+5)^{\frac{3}{2}}}.$

§ 5.8. Integration of Other Transcendental Functions

5.8.1. $I=\int \frac{\ln x}{x^2}\, dx.$

Solution. We integrate by parts, putting

$$u=\ln x; \quad dv=\frac{dx}{x^2};$$

$$du=\frac{dx}{x}; \quad v=-\frac{1}{x};$$

$$I=-\frac{\ln x}{x}+\int \frac{dx}{x^2}=-\frac{\ln x}{x}-\frac{1}{x}+C.$$

5.8.2. $I=\int \frac{\ln x\, dx}{\sqrt{1-x}}.$

5.8.3. $I=\int \frac{e^x\, dx}{(1+e^{2x})^2}.$

Solution. Let us put: $e^x=t$; $e^x\, dx=dt$. We get:

$$I=\int \frac{dt}{(1+t^2)^2}.$$

Apply the reduction formula (see Problem **4.4.1**):

$$I=I_2=\frac{t}{2(t^2+1)}+\frac{1}{2} \int \frac{dt}{1+t^2};$$

$$I=\frac{t}{2(t^2+1)}+\frac{1}{2} \arctan t+C=\frac{e^x}{2(1+e^{2x})}+\frac{1}{2} \arctan e^x+C.$$

5.8.4. $I=\int e^{-x} \ln (e^x+1)\, dx.$

Solution. We integrate by parts:

$$u=\ln(e^x+1); \quad dv=e^{-x}\,dx;$$

$$du=\frac{e^x}{1+e^x}\,dx; \quad v=-e^{-x};$$

$$I=-e^{-x}\ln(1+e^x)+\int\frac{dx}{1+e^x}=-e^{-x}\ln(1+e^x)+\int\frac{e^x+1-e^x}{1+e^x}\,dx=$$

$$=-e^{-x}\ln(1+e^x)+x-\ln(1+e^x)+C.$$

5.8.5. $I=\int\frac{e^{\alpha\arctan x}}{(1+x^2)^{\frac{3}{2}}}\,dx.$

5.8.6. $I=\int\frac{x\arctan x\,dx}{\sqrt{1+x^2}}.$

Solution. Integrating by parts, we get

$$u=\arctan x; \quad dv=\frac{x\,dx}{\sqrt{1+x^2}};$$

$$du=\frac{dx}{1+x^2}; \quad v=\sqrt{1+x^2};$$

$$I=\sqrt{1+x^2}\arctan x-\int\sqrt{1+x^2}\,\frac{dx}{1+x^2}=$$

$$=\sqrt{1+x^2}\arctan x-\ln(x+\sqrt{x^2+1})+C.$$

§ 5.9. Methods of Integration (List of Basic Forms of Integrals)

No.	Integral	Method of integration
1	$\int F[\varphi(x)]\varphi'(x)\,dx$	Substitution $\varphi(x)=t$
2	$\int f(x)\varphi'(x)\,dx$	Integration by parts $\int f(x)\varphi'(x)\,dx = f(x)\varphi(x) - \int \varphi(x) f'(x)\,dx.$ This method is applied, for example, to integrals of the form $\int p(x) f(x)\,dx$, where $p(x)$ is a polynomial, and $f(x)$ is one of the following functions: $e^{\alpha x}$; $\cos \alpha x$; $\sin \alpha x$; $\ln x$; $\arctan x$; $\arcsin x$, etc. and also to integrals of products of an exponential function by cosine or sine.
3	$\int f(x)\varphi^{(n)}(x)\,dx$	Reduced to integration of the product $f^{(n)}(x)\varphi(x)$ by the formula for multiple integration by parts $\int f(x)\varphi^{(n)}(x)\,dx = f(x)\varphi^{(n-1)}(x) - f'(x)\varphi^{(n-2)}(x) + f''(x)\varphi^{(n-3)}(x) - \ldots + (-1)^{n-1} f^{(n-1)}(x)\varphi(x) + (-1)^n \int f^{(n)}(x)\varphi(x)\,dx$
4	$\int e^{\alpha x} p_n(x)\,dx$, where $p_n(x)$ is a polynomial of degree n.	Applying the formula for multiple integration by parts (see above), we get $\int e^{\alpha x} p_n(x)\,dx = e^{\alpha x}\left[\frac{p_n(x)}{\alpha} - \frac{p_n'(x)}{\alpha^2} + \frac{p_n''(x)}{\alpha^3} - \ldots + (-1)^n \frac{p_n^{(n)}(x)}{\alpha^{n+1}}\right] + C$
5	$\int \frac{Mx+N}{x^2+px+q}\,dx$, $p^2-4q<0$	Substitution $x+\frac{p}{2}=t$

No.	Integral	Method of integration
6	$I_n=\int \frac{dx}{(x^2+1)^n}$	Reduction formula is used $I_n=\frac{x}{(2n-2)(x^2+1)^{n-1}}+\frac{2n-3}{2n-2}I_{n-1}$
7	$\int \frac{P(x)}{Q(x)}dx$, where $\frac{P(x)}{Q(x)}$ is a proper rational fraction $Q(x)=(x-x_1)^l(x-x_2)^m\ldots(x^2+px+q)^k\ldots$	Integrand is expressed in the form of a sum of partial fractions $\frac{P(x)}{Q(x)}=\frac{A_1}{(x-x_1)}+\frac{A_2}{(x-x_1)^2}+\ldots+\frac{A_l}{(x-x_1)^l}+\frac{B_1}{(x-x_2)}+\frac{B_2}{(x-x_2)^2}+\ldots+\frac{B_m}{(x-x_2)^m}+\ldots+\frac{M_1x+N_1}{x^2+px+q}+\frac{M_2x+N_2}{(x^2+px+q)^2}+\ldots+\frac{M_kx+N_k}{(x^2+px+q)^k}+\ldots$
8	$\int R\left(x, x^{\frac{m}{n}}, \ldots, x^{\frac{r}{s}}\right)dx$, where R is a rational function of its arguments.	Reduced to the integral of a rational fraction by the substitution $x=t^k$, where k is a common denominator of the fractions $\frac{m}{n}, \ldots, \frac{r}{s}$
9	$\int R\left[x, \left(\frac{ax+b}{cx+d}\right)^{\frac{1}{n}}\right]dx$, where R is a rational function of its arguments.	Reduced to the integral of a rational fraction by the substitution $\frac{ax+b}{cx+d}=t^n$
10	$\int \frac{Mx+N}{\sqrt{ax^2+bx+c}}dx$	By the substitution $x+\frac{b}{2a}=t$ the integral is reduced to a sum of two integrals: $\int \frac{Mx+N}{\sqrt{ax^2+bx+c}}dx=M_1\int\frac{tdt}{\sqrt{at^2+m}}+N_1\int\frac{dt}{\sqrt{at^2+m}}$. The first integral is reduced to the integral of a power function and the second one is a tabular integral.

No.	Integral	Method of integration
11	$\int R(x, \sqrt{ax^2+bx+c})\,dx$, where R is a rational function of x and $\sqrt{ax^2+bx+c}$	Reduced to an integral of rational fraction by the Euler substitutions: $\sqrt{ax^2+bx+c}=t \pm x\sqrt{a}$ $(a>0)$, $\sqrt{ax^2+bx+c}=tx \pm \sqrt{c}$ $(c>0)$, $\sqrt{ax^2+bx+c}=t(x-x_1)$ $(4ac-b^2<0)$, where x_1 is the root of the trinomial ax^2+ $+bx+c$. The indicated integral can also be evaluated by the trigonometric substitutions: $x+\frac{b}{2a}=\begin{cases}\frac{\sqrt{b^2-4ac}}{2a}\sin t\\ \frac{\sqrt{b^2-4ac}}{2a}\cos t\end{cases}$ $(a<0,$ $4ac-b^2<0)$ $x+\frac{b}{2a}=\begin{cases}\frac{\sqrt{b^2-4ac}}{2a}\sec t\\ \frac{\sqrt{b^2-4ac}}{2a}\operatorname{cosec} t\end{cases}$ $(a>0,$ $4ac-b^2<0)$ $x+\frac{b}{2a}=\begin{cases}\frac{\sqrt{4ac-b^2}}{2a}\tan t\\ \frac{\sqrt{4ac-b^2}}{2a}\cot t\end{cases}$ $(a>0,$ $4ac-b^2>0)$
12	$\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}\,dx$, where $P_n(x)$ is a polynomial of degree n.	Write the equality $\int \frac{P_n(x)\,dx}{\sqrt{ax^2+bx+c}}=Q_{n-1}(x)\sqrt{ax^2+bx+c}+$ $+k\int \frac{dx}{\sqrt{ax^2+bx+c}}$, where $Q_{n-1}(x)$ is a polynomial of degree $n-1$. Differentiating both parts of this equality and multiplying by $\sqrt{ax^2+bx+c}$, we get the identity $P_n(x) \equiv Q'_{n-1}(x)(ax^2+bx+c)+$ $+\frac{1}{2}Q_{n-1}(x)(2ax+b)+k$, which gives a system of $n+1$ linear equations for determining the coefficients of the polynomial $Q_{n-1}(x)$ and factor k.

No.	Integral	Method of integration
		And the integral $$\int \frac{dx}{\sqrt{ax^2+bx+c}}$$ is taken by the method considered in No. 10 ($M=0$; $N=1$).
13	$\int \frac{dx}{(x-x_1)^m \sqrt{ax^2+bx+c}}$	This integral is reduced to the above-considered integral by the substitution $$x-x_1=\frac{1}{t}$$
14	$\int x^m (a+bx^n)^p\, dx,$ where m, n, p are rational numbers (an integral of a binomial differential).	This integral is expressed through elementary functions only if one of the following conditions is fulfilled: (1) if p is an integer, (2) if $\frac{m+1}{n}$ is an integer, (3) if $\frac{m+1}{n}+p$ is an integer. *1st case* (a) if p is a positive integer, remove the brackets $(a+bx^n)^p$ according to the Newton binomial and calculate the integrals of powers; (b) if p is a negative integer, then the substitution $x=t^k$, where k is the common denominator of the fractions m and n, leads to the integral of a rational fraction; *2nd case* if $\frac{m+1}{n}$ is an integer, then the substitution $a+bx^n=t^k$ is applied, where k is the denominator of the fraction p; *3rd case* if $\frac{m+1}{n}+p$ is an integer, then the substitution $a+bx^n=x^n t^k$ is applied, where k is the denominator of the fraction p.
15	$\int R(\sin x,\ \cos x)\, dx$	Universal substitution $\tan\frac{x}{2}=t$. If $R(-\sin x,\ \cos x)=-R(\sin x,\ \cos x)$, then the substitution $\cos x=t$ is applied. If $R(\sin x,\ -\cos x)=-R(\sin x,\ \cos x)$, then the substitution $\sin x=t$ is applied. If $R(-\sin x,\ -\cos x)=R(\sin x,\ \cos x)$, then the substitution $\tan x=t$ is applied.

No.	Integral	Method of Integration
16	$\int R(\sinh x, \cosh x)\,dx$	The substitution $\tanh \frac{x}{2} = t$ is used. In this case $\sinh x = \frac{2t}{1-t^2}$; $\cosh x = \frac{1+t^2}{1-t^2}$; $dx = \frac{2dt}{1-t^2}$.
17	$\int \sin ax \sin bx\,dx$ $\int \sin ax \cos bx\,dx$ $\int \cos ax \cos bx\,dx$	Transform the product of trigonometric functions into a sum or difference, using one of the following formulas: $\sin ax \sin bx = \frac{1}{2}[\cos(a-b)x - \cos(a+b)x]$ $\cos ax \cos bx = \frac{1}{2}[\cos(a-b)x + \cos(a+b)x]$ $\sin ax \cos bx = \frac{1}{2}[\sin(a-b)x + \sin(a+b)x]$
18	$\int \sin^m x \cos^n x\,dx$, where m and n are integers.	If m is an odd positive number, then apply the substitution $\cos x = t$. If n is an odd positive number, apply the substitution $\sin x = t$. If $m+n$ is an even negative number, apply the substitution $\tan x = t$. If m and n are even non-negative numbers, use the formulas $\sin^2 x = \frac{1-\cos 2x}{2}$; $\cos^2 x = \frac{1+\cos 2x}{2}$
19	$\int \sin^p x \cos^q x\,dx$ $(0 < x < \pi/2)$, p and q—rational numbers.	Reduce to the integral of the binomial differential by the substitution $\sin x = t$ $\int \sin^p x \cos^q x\,dx = \int t^p (1-t^2)^{q-1}\,dt$ (see No. 14).
20	$\int R(e^{ax})\,dx$	Transform into an integral of a rational function by the substitution $e^{ax} = t$