Definite Integrals

Q.1. Evaluate : $0 \int^{9} f(x)\, dx$, where f(x) is defined by f(x) = sin x , x ε [0, π/2] ; f(x) = 1 , x ε [π/2, 5] and $f(x) = e^{x-5}$, x ε [5, 9] .

Solution : 1

$0\int 9\, f(x)\, dx = (0, \pi/2) \int \sin x\, dx + \pi/2 \int^{5} 1.dx + 5\int^{9} ex^{-5}\, dx$

$= [-\cos x]0\pi/2 + [x]\pi/2^{5}) + [ex^{-5}]59$

$= [1] + [5 - \pi/2] + [e^{4} - ^{1}]$

$= [e^{4} + 5 - \pi/2]$

Q.2. Evaluate : $0 \int^{\pi/4} \log(1 + \tan x)dx$.

Solution : 2

$I = {}^{\pi/4}0\int \log(1 + \tan x)dx = 0\int^{\pi/4} \log \{1 + \tan(\pi/4 - x)\}dx$

[As, $0\int af(x)\, dx = 0\int af(a - x)dx$]

Therefore, $I = 0\int^{\pi/4}\log [1 + (\tan^{\pi/4} - \tan x)/(1 + \tan^{\pi/4}.\tan x)]\, dx$

$= 0\int^{\pi/4}\log [1 + (1 - \tan x)/(1 + \tan x)]dx$

$= 0\int^{\pi/4}\log [2/(1 + \tan x)]dx$

$= 0\int^{\pi/4}\log 2\, dx - 0\int^{\pi/4}\log (1 + \tan x)dx$

$= 0\int^{\pi/4}\log 2\, dx - I$

Or, $2 I = 0\int z^{\pi/4}\log 2\, dx = [x\log 2]0\pi/4$

Or, 2 I = π/4 log 2 Therefore, I = π/8 log 2.

Q.3. Prove that $_0\int^{2\pi}$ xcos x/(1 + cos x) dx = $2\pi^2$.

Solution : 3

I = $_0\int^{2\pi}$ x cos x/(1 + cos x) dx ------------------ (1)

= $_0\int^{2\pi}$(2π – x) cos (2π – x)/[1 + cos (2π – x)] dx

= $_0\int^{2\pi}$(2π – x) cos x/(1 + cos x) ---------------- (2)

Adding (1) and (2) we get,

I + I = $_0\int^{2\pi}$(x cos x + 2π cos x – x cos x)/(1 + cos x)dx

2I = $_0\int^{2\pi}$(2π cos x)/(1 + cosx) dx

= 2π $_0\int^{2\pi}$ cos x(1 – cos x)/[(1 + cos x)(1 – cos x)]dx

= 2π $_0\int^{2\pi}$ (cos x – $\cos^2$ x)/sin2 x dx

= 2π $_0\int^{2\pi}$ [(cos x/$\sin^2$ x) – (cos2 x/$\sin^2$ x)] dx

= 2π $_0\int^{2\pi}$ (cot x cosec x – cot2 x) dx

= 2π $_0\int^{2\pi}$ (cot x cosec x – cosec^2 x + 1) dx

= 2π [– cosec x + cot x + x]02π

= 2π [– cosec 2π + cot 2π + 2π – (– cosec 0 + cot 0 + 0)]

= 2π[2π] = $4\pi^2$

Therefore, I = $2\pi^2$.

Q.4. Prove that $_0\int^{\pi/2}$ [3 sin θ + 4 cos θ]/[sin θ + cos θ] dθ = 7π/4 .

Solution : 4

Let I = $_0\int^{\pi/2}$ [3sinθ + 4cos θ]/[sin θ + cos θ] dθ ----------------- (1)

= $_0\int^{\pi/2}$ [3sin (π/2 –θ) + 4cos (π/2 – θ)]/[sin (π/2 – θ) + cos (π – θ)] dθ

[As, $_a\int^b$ f(x) dx

= $_a\int^b$ f(a + b – x) dx]

= $_0\int^{\pi/2}$[3cos θ + 4sin θ]/[cos θ + sin θ] dθ ----------------- (2)

Adding (1) and (2) , 2I = $_0\int^{\pi/2}$7[sin θ + cos θ]/[sin θ + cos θ] dθ

=7$_0\int^{\pi/2}$dθ = 7[θ]$_0^{\pi/2}$ = 7×(π/2)

Or, I = 7π/2 .

Q.5. Evaluate : $_0\int^{\pi/4}$ (tan x + cot x)–1dx.

Solution : 5

We have $_0\int^{\pi/4}$(tan x + cot x)–1dx

= $_0\int^{\pi/4}$ [1/(tan x + cot x)]dx

= $_0\int^{\pi/4}$ [1/{(sin2 x + cos2 x)/sin x cos x}]dx

= 1/2$_0\int^{\pi/4}$ sin 2x dx = 1/4 [– cos 2x]$_0^{\pi/4}$

= 1/4 [– cos (2×π/4) + cos (2×0)]

= 1/4 [– cos π/2 + cos 0º]

= 1/4 [0 + 1] = 1/4 .

Q.6. Evaluate : $_0\int^{\pi/4}$ [2 cos 2x/(1 + sin 2x)]dx.

Solution : 6

Let I = a $\int^{\pi/4}$[2 cos 2x/(1 + sin 2x)]dx

Put 1 + sin 2x = t => 2 cos 2x dx = dt ; when x = 0, t = 1 + 0 = 1

and when x = π/4, t = 1 + sin π/2 = 1 + 1 = 2.

Therefore, I = 1∫2 dt/t = [log t]12 = log 2 – log 1

= log 2 – 0 = log 2.

Q.7. Evaluate : 0$\int^{1}$x tan $^{-1}$ x dx.

Solution : 7

Let I = 0$\int^{1}$x tan $^{-1}$ x dx

= [tan $^{-1}$ x .∫x dx – ∫{1/(1 + x2)}.1/2 x2.dx]0^{1}

= [tan $^{-1}$ x.(1/2)x2 – 1/2 ∫{x2/(1 + x2)}.dx]0^{1}

= [1/2 x2 tan $^{-1}$ x – 1/2 ∫dx + 1/2 ∫dx/(1 + x2)]0^{1}

= [1/2 x2 tan $^{-1}$ – 1/2 x + 1/2 tan $^{-1}$ x]0^{1}

= 1/2[(1 + x2) tan $^{-1}$ x – x]0^{1}

= 1/2 [(1 + 1) tan $^{-1}$ (1) – 1 – (1 + 0) tan^{-1} (0) – 0]

= 1/2 [2 tan $^{-1}$ (1) $^{-1}$] = 1/2 [2 ×(π/4) $^{-1}$]

= π/4 – 1/2 .

Q.8. Evaluate : 0$\int^{\pi/2}$ sin 2x log tan x dx.

Solution : 8

We are given, I = 0$\int^{\pi/2}$ sin 2x log tan x dx ---------------- (1) Or,

$I = {}_0\int^{\pi/2} \log \tan(\pi/2 - x) \sin 2(\pi/2 - x)\, dx$

$= {}_0\int^{\pi/2} \log \cot x \sin(\pi - 2x) dx$

$= {}_0\int^{\pi/2} \log \cot x \sin 2x dx$ ------------------ (2)

adding (1) and (2) we get

$2 I = {}_0\int^{\pi/2} \sin 2x[\log \tan x + \cot x]\, dx$

$= {}_0\int^{\pi/2} \sin 2x \log(\tan x . \cot x) dx$

$= {}_0\int^{\pi/2} \sin 2x \log(1) dx = 0$ [As, log(1) = 0]

Therefore, I = 0 .

Q.9. Evaluate : $0\int^{\pi/2} \{\sqrt{(\sec x)}\}/[\sqrt{(\sec x)} + \sqrt{(\mathrm{cosec}\ x)}]dx$.

Solution : 9

Let $I = a \int^{\pi/2} [\{\sqrt{(\sec x)}\}/\{\sqrt{(\sec x)} + \sqrt{(\mathrm{cosec}\ x)}\}]\, dx$ --------------- (1)

Then $I = a \int^{\pi/2} [\sqrt{\{\sec(\pi/2 - x)\}}]/[\sqrt{\{\sec(\pi/2 - x)\}} + \sqrt{\{\mathrm{cosec}(\pi/2 - x)\}}]dx$

$= a \int^{\pi/2} \{\{\sqrt{(\mathrm{cosec}\ x)}\}/\{\sqrt{(\mathrm{cosec}\ x)} + \sqrt{(\sec x)}\}]\, dx$ -------------- (2)

adding (1) and (2), we get

$2I = {}_0\int^{\pi/2} [\{\sqrt{(\sec x)}\}/\{\sqrt{(\sec x)} + \sqrt{(\mathrm{cosec}\ x)}\}]dx +$

$a\int^{\pi/2}[\{\sqrt{(\mathrm{cosec}\ x)}\}/\{\sqrt{(\mathrm{cosec}\ x)} + \sqrt{(\sec x)}\}]dx$

$= a \int^{\pi/2}[\{\sqrt{(\sec x)} + \sqrt{(\mathrm{cosec}\ x)}\}/\{\sqrt{(\sec x)} + \sqrt{(\mathrm{cosec}\ x)}\}]dx$

$= a \int^{\pi/2} dx = [x]0\pi/2 = \pi/2$

Therefore, $I = \pi/4$.

Q.10. Evaluate : ${}_a\int^b (\log x/x)dx$.

Solution : 10

We are given,

$_a\int^b$(log x/x) dx

Let I = ∫ (log x)(1/x)dx

= log x ∫(1/x)dx – ∫(1/x) log x dx

= log x . log x – I

Or, 2I = $(\log x)^2$

Or, I = $[(\log x)^2]/2$

Therefore, a∫b(log x/x)dx = $[\{(\log x)^2\}/2]$ab = $\{(\log b)^2\}/2 - \{(\log a)^2\}/2$

= 1/2 (log b + log a)(log b – log a) 1/2 log (ab)log (a/b).

Q.11. Evaluate : $0\int^{1/2} \sin^{-1} x/(1 - x^2)^{3/2}dx$.

Solution : 11

We have, I = a $\int^{1/2}$ sin –1 x/(1 – x2)$^{3/2}$dx

Put sin –1x = t => x = sin t and dx = cos t dt

Therefore, I = a $\int^{1/2}$sin –1x/(1 – x^2)$^{3/2}$dx

= a $\int^{\pi/6}$(t/cos3t).(cost)dt,

[x = 0, sint = 0 => t = 0; x = 1/2, sint = 1/2 => t= π/6]

= a $\int^{\pi/6}$ (t/cos2 t)dt = 0∫π/6 t $\sec^2$ tdt

= [t∫ sec2 t dt – ∫{d/dt(t).∫$\sec^2$ t dt}]0π/6

= [t. tan t – ∫tan t dt]$0^{\pi/6}$ = [t. tan t – log sec t]$0^{\pi/6}$

$= [\pi/6 \tan \pi/6 - \log \sec^{\pi/6}] - [0 - \log \sec 0]$

$= [\pi/6 (1/\sqrt{3}) - \log (2/\sqrt{3})] - [0 - 0]$

$= \pi/(6\sqrt{3}) - \log (2/\sqrt{3})$.

Q.12. Evaluate $_0\int^{\pi/2} \log(\tan x)dx$.

Solution : 12

We have, $I = {}_a\int^{\pi/2} \log(\tan x)dx = {}_a\int^{\pi/2} \log[\tan(\pi/2 - x)]dx$

$= {}_a\int^{\pi/2} \log(\cot x)dx$

Therefore, $2I = {}_a\int^{\pi/2} [\log(\tan x) + \log(\cot x)]dx$

$= {}_a\int^{\pi/2} \log(\tan x.\cot x)dx = {}_a\int^{\pi/2} \log 1.dx = 0$

Therefore, $I = 0$.