Trigonometric Functions

EXERCISE 3.1 [PAGE 75]

Exercise 3.1 | Q 1.1 | Page 75

Find the principal solution of the following equation:

cosθ = 1/2

Solution:

We know that, $\cos\ \dfrac{\pi}{3} = \dfrac{1}{2}$ and cos(2π - θ) = cosθ

$$\therefore \cos\ \frac{\pi}{3} = \cos\left(2\pi - \frac{\pi}{3}\right) = \cos\ \frac{5\pi}{3}$$

$$\therefore \cos\ \frac{\pi}{3} = \cos\ \frac{5\pi}{3} = \frac{1}{2}, \text{ where}$$

$$0 < \frac{\pi}{3} < 2\pi \text{ and } 0 < \frac{5\pi}{3} < 2\pi$$

$$\therefore \cos\theta = \frac{1}{2} \text{gives} \cos\theta = \cos\ \frac{\pi}{3} = \cos\ \frac{5\pi}{3}$$

$$\therefore \theta = \frac{\pi}{3} \text{ and } \theta = \frac{5\pi}{3}$$

Hence, the required principal solutions are

$$\theta = \frac{\pi}{3} \text{ and } \theta = \frac{5\pi}{3}.$$

Exercise 3.1 | Q 1.2 | Page 75

Find the principal solution of the following equation:

Secθ = 2/ √3

Solution:

$$\theta = \frac{\pi}{6} \text{ and } \theta = \frac{11\pi}{6}$$

Solution is not available.

Exercise 3.1 | Q 1.3 | Page 75

Find the principal solution of the following equation :

cotθ = √3

Solution:

The given equation is cot θ = √3 which is same as tan θ = $\frac{1}{\sqrt{3}}$.

We know that,

$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$ and tan(π + θ) = tan θ

$$\therefore \tan\frac{\pi}{6} = \tan\left(\pi + \frac{\pi}{6}\right) = \tan\frac{7\pi}{6}$$

$\therefore \tan\frac{\pi}{6} = \tan\frac{7\pi}{6} = \frac{1}{\sqrt{3}}$, where

$$0 < \frac{\pi}{6} < 2\pi \text{ and } 0 < \frac{7\pi}{6} < 2\pi$$

∴ cot θ = √3, i.e. tan θ = $\frac{1}{\sqrt{3}}$ gives

tan θ = $\tan\frac{\pi}{6} = \tan\frac{7\pi}{6}$

∴ θ = $\frac{\pi}{6}$ and $\theta = \frac{7\pi}{6}$

Hence, the required principal solution are

$$\theta = \frac{\pi}{6} \text{ and } \theta = \frac{7\pi}{6}.$$

Exercise 3.1 | Q 1.4 | Page 75

Find the principal solution of the following equation:

cotθ = 0

Solution:

$$\theta = \frac{\pi}{2} \text{ and } \theta = \frac{3\pi}{2}$$

Solution is not available

Exercise 3.1 | Q 2.1 | Page 75

Find the principal solution of the following equation:

sin θ = - 1/2

Solution:

We now that,

$$\sin \frac{\pi}{6} = \frac{1}{2} \text{ and } \sin(\pi + \theta) = -\sin\theta,$$

$\sin(2\pi - \theta) = -\sin\theta$.

$$\therefore \sin\left(\pi + \frac{\pi}{6}\right) = -\frac{\sin\pi}{6} = -\frac{1}{2}$$

$$\text{and} \sin\left(2\pi - \frac{\pi}{6}\right) = -\sin \frac{\pi}{6} = -\frac{1}{2}$$

$$\therefore \sin \frac{7\pi}{6} = \sin \frac{11\pi}{6} = -\frac{1}{2}, \text{ where}$$

$$0 < \frac{7\pi}{6} < 2\pi \text{ and } 0 < \frac{11\pi}{6} < 2\pi$$

$\therefore \sin\theta = -\frac{1}{2}$ gives,

$\sin\theta = \sin\frac{7\pi}{6} = \sin\frac{11\pi}{6}$

$\therefore \theta = \frac{7\pi}{6}$ and $\theta = \frac{11\pi}{6}$

Hence, the required principal solutions are

$\theta = \frac{7\pi}{6}$ and $\theta = \frac{11\pi}{6}$.

Exercise 3.1 | Q 2.2 | Page 75

Find the principal solution of the following equation:

$\tan\theta = -1$

Solution:

We know that,

$\tan\frac{\pi}{4} = 1$ and $\tan(\pi - \theta) = -\tan\theta$,

$\tan(2\pi - \theta) = -\tan\theta$

$\therefore \tan\left(\pi - \frac{\pi}{4}\right) = -\tan\frac{\pi}{4} = -1$

and $\tan\left(2\pi - \frac{\pi}{4}\right) = -\tan\frac{\pi}{4} = -1$

$\therefore \tan\frac{3\pi}{4} = \tan\frac{7\pi}{4} = -1$, where

$0 < \frac{3\pi}{4} < 2\pi$ and $0 < \frac{7\pi}{4} < 2\pi$

$\therefore$ tan θ = – 1 gives,

$$\tan\theta = \tan\frac{3\pi}{4} = \tan\frac{7\pi}{4}$$

$$\therefore \theta = \frac{3\pi}{4} \text{ and } \theta = \frac{7\pi}{4}$$

Hence, the required principal solutions are

$$\theta = \frac{3\pi}{4} \text{ and } \theta = \frac{7\pi}{4}.$$

Exercise 3.1 | Q 2.3 | Page 75

Find the principal solution of the following equation:

$\sqrt{3}\text{cosec}\theta + 2 = 0$

Solution:

$$\theta = \frac{4\pi}{3} \text{ and } \theta = \frac{5\pi}{3}.$$

The solution is not available.

Exercise 3.1 | Q 3.1 | Page 75

Find the general solution of the following equation:

$\sin\theta = 1/2$.

Solution:

The general solution of sin θ = sin α is

$\theta = n\pi + (-1)^n\alpha, n \in Z$

Now,

$$\sin\theta = \frac{1}{2} = \sin\frac{\pi}{6} \quad \ldots\left[\because \sin\frac{\pi}{6} = \frac{1}{2}\right]$$

$\therefore$ the required general solution is $\theta = n\pi + (-1)^n\frac{\pi}{6}, n \in Z$.

Exercise 3.1 | Q 3.2 | Page 75

Find the general solution of the following equation :

cosθ = √38/2

Solution:

The general solution of cos θ = cos α is

$\theta = 2n\pi \pm \alpha, n \in Z$

Now,

$$\cos\theta = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6} \quad \ldots\left[\because \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}\right]$$

∴ the required general solution is

$$\theta = 2n\pi \pm \frac{\pi}{6}, n \in Z.$$

Exercise 3.1 | Q 3.3 | Page 75

Find the general solution of the following equation:

tan θ = 1/√3

Solution:

The general solution of tan θ = tan α is

$\theta = n\pi + \alpha, n \in Z$

Now,

$$\tan\theta = \frac{1}{\sqrt{3}} = \tan\frac{\pi}{6} \quad \ldots\left[\because \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}\right]$$

∴ the required general solution is

$$\theta = n\pi + \frac{\pi}{6}, n \in Z.$$

Exercise 3.1 | Q 3.4 | Page 75

Find the general solution of the following equation:

cot θ = 0.

Solution:

The general solution of tan θ = tan α is

θ = nπ + α, n ∈ Z

Now, cot θ = 0

∴ tan θ does not exist

∴ tan θ = $\tan \frac{\pi}{2}$ ...$\left[\because \tan \frac{\pi}{2}\right.$ does not exist]

∴ the required general solution is

θ = nπ + $\frac{\pi}{2}$, n ∈ Z.

Exercise 3.1 | Q 4.1 | Page 75

Find the general solution of the following equation:

sec θ = √2.

Solution:

The general solution of cos θ = cos α is

θ = 2nπ ± α, n ∈ Z.

Now,

sec θ = √2

∴ cos θ = $\frac{1}{\sqrt{2}}$

∴ cos θ = cos $\frac{\pi}{4}$ ...$\left[\because \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}\right]$

∴ the required general solution is

θ = 2nπ ± $\frac{\pi}{4}$, n ∈ Z.

Exercise 3.1 | Q 4.2 | Page 75

Find the general solution of the following equation:

cosec θ = - √2.

Solution: The general solution of sin θ = sin α is

$\theta = n\pi + (-1)^n\alpha$, n ∈ Z.

Now,

Cosec θ = - √2

$\therefore \sin\theta = -\frac{1}{\sqrt{2}}$

$\therefore \sin\theta = -\sin\frac{\pi}{4}$ ... $\left[\because \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}\right]$

$\therefore \sin\theta = \sin\left(\pi + \frac{\pi}{4}\right)$ ...[∵ sin(π + θ) = – sin θ]

$\therefore \sin\theta = \sin\frac{5\pi}{4}$

∴ the required general solution is

$\theta = n\pi + (-1)^n\left(\frac{5\pi}{4}\right)$, n ∈ Z.

Exercise 3.1 | Q 4.3 | Page 75

Find the general solution of the following equation:

tan θ = - 1

Solution:

The general solution of tan θ = tan α is

θ = nπ + α, n ∈ Z.

Now, tan θ = – 1

$\therefore \tan\theta = -\tan\frac{\pi}{4}$ ..... $\left[\because \tan\frac{\pi}{4} = 1\right]$

$\therefore \tan\theta = \tan\left(\pi - \frac{\pi}{4}\right)$ ...[∵ tan(π - θ) = – tan θ]

$\therefore \tan\theta = \tan\dfrac{3\pi}{4}$

$\therefore$ the required general solution is

$\theta = n\pi + \dfrac{3\pi}{4}$, $n \in Z$.

Exercise 3.1 | Q 5.1 | Page 75

Find the general solution of the following equation:

sin 2θ = 1/2

Solution:

The general solution of $\sin\theta = \sin\alpha$ is

$\theta = n\pi + (-1)^n\alpha$, $n \in Z$.

Now,

$\sin 2\theta = \dfrac{1}{2}$

$\therefore \sin 2\theta = \sin\dfrac{\pi}{6}$ $\quad\ldots\left[\because \sin\dfrac{\pi}{6} = \dfrac{1}{2}\right]$

$\therefore$ the required general solution is given by

$2\theta = n\pi + (-1)^n\left(\dfrac{\pi}{6}\right)$, $n \in Z$.

i.e. $\theta = \dfrac{n\pi}{2} + (-1)^n\left(\dfrac{\pi}{12}\right)$, $n \in Z$.

Exercise 3.1 | Q 5.2 | Page 75

Find the general solution of the following equation:

tan 2θ/3 = √3.

Solution:

The general solution of tan θ = tan α is

θ = nπ + α, n ∈ Z

Now,

$\tan \frac{2\theta}{3}$ = √3.

$$\therefore \tan \frac{2\theta}{3} = \tan \frac{\pi}{3} \quad ...\left[\because \tan \frac{\pi}{3} = \sqrt{3}\right]$$

∴ the required general solution is given by

$\frac{2\theta}{3} = n\pi + \frac{\pi}{3}$, n ∈ Z.

i.e. θ = $\frac{3n\pi}{2} + \frac{\pi}{2}$, n ∈ Z.

Exercise 3.1 | Q 5.3 | Page 75

Find the general solution of the following equation:

cot 4θ = – 1

Solution: The general solution of tan θ = tan α is
θ = nπ + α, n ∈ Z
Now,

cot 4θ = – 1

∴ tan 4θ = – 1

∴ tan 4θ = $-\tan \frac{\pi}{4} \quad ...\left[\because \tan \frac{\pi}{4} = 1\right]$

∴ tan 4θ = $\tan\left(\pi - \frac{\pi}{4}\right)$ ...[∵ tan(π – θ) = – tan θ]

∴ tan 4θ = $\tan \frac{3\pi}{4}$

∴ the required general solution is given by

$$4\theta = n\pi + \frac{3\pi}{4}, n \in Z$$

i.e. $$\theta = \frac{n\pi}{4} + \frac{3\pi}{16}, n \in Z.$$

Exercise 3.1 | Q 6.1 | Page 75

Find the general solution of the following equation:

$4\cos^2\theta = 3$.

Solution:

The general solution of $\cos^2\theta = \cos^2\alpha$ is

$\theta = n\pi \pm \alpha, n \in Z$.

Now, $4\cos^2\theta = 3$

$$\therefore \cos^2\theta = \frac{3}{4} = \left(\frac{\sqrt{3}}{2}\right)^2$$

$$\therefore \cos^2\theta = \left(\cos\frac{\pi}{6}\right)^2 \quad \ldots\left[\because \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}\right]$$

$$\therefore \cos^2\theta = \cos^2\frac{\pi}{6}$$

$\therefore$ the required general solution is given by

$$\theta = n\pi \pm \frac{\pi}{6}, n \in Z.$$

Exercise 3.1 | Q 6.2 | Page 75

Find the general solution of the following equation:

$4\sin^2\theta = 1$.

Solution:

The general solution of $\sin^2\theta = \sin^2\alpha$ is

$\theta = n\pi \pm \alpha, n \in Z$.

Now, $4 \sin^2\theta = 1$

$$\therefore \sin^2\theta = \frac{1}{4} = \left(\frac{1}{2}\right)^2$$

$$\therefore \sin^2\theta = \left(\sin \frac{\pi}{6}\right)^2 \quad \ldots \left[\because \sin \frac{\pi}{6} = \frac{1}{2}\right]$$

$$\therefore \sin^2\theta = \sin^2 \frac{\pi}{6}$$

$\therefore$ the required general solution is $\theta = n\pi \pm \frac{\pi}{6}, n \in Z$.

Exercise 3.1 | Q 6.3 | Page 75

Find the general solution of the following equation:

$\cos 4\theta = \cos 2\theta$

Solution: The general solution of $\cos \theta = \cos \alpha$ is
$\theta = 2n\pi \pm \alpha, n \in Z$.

$\therefore$ the general solution of $\cos 4\theta = \cos 2\theta$ is given by

$4\theta = 2n\pi \pm 2\theta, n \in Z$

Taking positive sign, we get

$4\theta = 2n\pi + 2\theta, n \in Z$

$\therefore 2\theta = 2n\pi, n \in Z$

$\therefore \theta = n\pi, n \in Z$

Taking negative sign, we get

$4\theta = 2n\pi - 2\theta, n \in Z$

$\therefore 6\theta = 2n\pi, n \in Z$

$\therefore \theta = \frac{n\pi}{3}, n \in Z$

Hence, the required general solution is

$\theta = \frac{n\pi}{3}, n \in Z$ or $\theta = n\pi, n \in Z$.

Alternative Method:

$\cos 4\theta = \cos 2\theta$

$\therefore \cos 4\theta - \cos 2\theta = 0$

$$\therefore -2\sin\left(\frac{4\theta + 2\theta}{2}\right).\sin\left(\frac{4\theta - 2\theta}{2}\right) = 0$$

$\therefore \sin 3\theta. \sin \theta = 0$

$\therefore$ either $\sin 3\theta = 0$ or $\sin \theta = 0$

The general solution of $\sin \theta = 0$ is $\theta = n\pi, n \in Z$.

$\therefore$ the required general solution is given by

$3\theta = n\pi, n \in Z$ or $\theta = n\pi, n \in Z$

i.e. $\theta = n\pi/3, n \in Z$ or $\theta = n\pi, n \in Z$.

Exercise 3.1 | Q 7.1 | Page 75

Find the general solution of the following equation:

$\sin \theta = \tan \theta$.

Solution:

$\sin \theta = \tan \theta$

$\therefore \sin\theta = \frac{\sin\theta}{\cos\theta}$

$\therefore \sin\theta \cos\theta = \sin\theta$
$\therefore \sin\theta \cos\theta - \sin\theta = 0$
$\therefore \sin\theta (\cos\theta - 1) = 0$
$\therefore$ either $\sin\theta = 0$ or $\cos\theta - 1 = 0$
$\therefore$ either $\sin\theta = 0$ or $\cos\theta = 1$
$\therefore$ either $\sin\theta = 0$ or $\cos\theta = \cos 0$ ...[$\because \cos 0 = 1$]

The general solution of $\sin\theta = 0$ is $\theta = n\pi$, $n \in Z$ and $\cos\theta = \cos\alpha$ is $\theta = 2n\pi \pm \alpha$, where $n \in Z$.

$\therefore$ the required general solution is given by
$\theta = n\pi$, $n \in Z$ or $\theta = 2n\pi \pm 0$, $n \in Z$
$\therefore \theta = n\pi$, $n \in Z$ or $\theta = 2n\pi$, $n \in Z$.

Exercise 3.1 | Q 7.2 | Page 75

Find the general solution of the following equation:

$\tan^3\theta = 3\tan\theta$.

Solution: $\tan^3\theta = 3\tan\theta$

$\therefore \tan^3\theta - 3\tan\theta = 0$

$\therefore \tan\theta (\tan^2\theta - 3) = 0$

$\therefore$ either $\tan\theta = 0$ or $\tan^2\theta - 3 = 0$

$\therefore$ either $\tan\theta = 0$ or $\tan^2\theta = 3$

$\therefore$ either $\tan\theta = 0$ or $\tan^2\theta = (\sqrt{3})^2$

$\therefore$ either $\tan\theta = 0$ or $\tan^2\theta = \left(\tan\frac{\pi}{3}\right)^2$ ...$\left[\because \tan\frac{\pi}{3} = \sqrt{3}\right]$

$\therefore$ either $\tan\theta = 0$ or $\tan^2\theta = \tan^2\frac{\pi}{3}$

The general solution of

$\tan\theta = 0$ is $\theta = n\pi$, $n \in Z$ and

$\tan^2\theta = \tan^2\alpha$ is $\theta = n\pi \pm \alpha$, $n \in Z$.

$\therefore$ the required general solution is given by
$\theta = n\pi$, $n \in Z$ or $\theta = n\pi \pm \frac{\pi}{3}$, $n \in Z$.

Exercise 3.1 | Q 7.3 | Page 75

Find the general solution of the following equation:

cos θ + sin θ = 1.

Solution:

$\cos\theta + \sin\theta = 1$

Dividing both sides by $\sqrt{(1)^2 + (1)^2} = \sqrt{2}$, we get

$$\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta = \frac{1}{\sqrt{2}}$$

$$\therefore \cos\frac{\pi}{4}\cos\theta + \sin\frac{\pi}{4}\sin\theta = \cos\frac{\pi}{4}$$

$$\therefore \cos\left(\theta - \frac{\pi}{4}\right) = \cos\frac{\pi}{4} \quad ...(1)$$

The general solution of

$\cos\theta = \cos\alpha$ is $\theta = 2n\pi \pm \alpha$, $n \in Z$.

$\therefore$ the general solution of (1) is given by

$$\theta - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4}, n \in Z$$

Taking positive sign, we get

$$\theta - \frac{\pi}{4} = 2n\pi + \frac{\pi}{4}, n \in Z$$

$$\therefore \theta = 2n\pi + \frac{\pi}{2}, n \in Z$$

Taking negative sign, we get,

$$\theta - \frac{\pi}{4} = 2n\pi - \frac{\pi}{4}, n \in Z$$

$$\therefore \theta = 2n\pi, n \in Z$$

$\therefore$ the required general solution is

$\theta = 2n\pi + \frac{\pi}{2}$, $n \in Z$ or $\theta = 2n\pi$, $n \in Z$.

Alternative Method:

$\cos\theta + \sin\theta = 1$

$\therefore \sin\theta = 1 - \cos\theta$

$$\therefore 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\sin^2\frac{\theta}{2}$$

$$\therefore 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} - 2\sin^2\frac{\theta}{2} = 0$$

$$\therefore 2\sin\frac{\theta}{2}\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right) = 0$$

$$\therefore 2\sin\frac{\theta}{2} = 0 \text{ or } \cos\frac{\theta}{2} - \sin\frac{\theta}{2} = 0$$

$$\therefore \sin\frac{\theta}{2} = 0 \text{ or } \sin\frac{\theta}{2} = \cos\frac{\theta}{2}$$

$$\therefore \sin\frac{\theta}{2} = 0 \text{ or } \tan\frac{\theta}{2} = 1 \quad \ldots\left[\because \cos\frac{\theta}{2} \neq 0\right]$$

$$\therefore \sin\frac{\theta}{2} = 0 \text{ or } \tan\frac{\theta}{2} = \tan\frac{\pi}{4} \quad \ldots\left[\because \tan\frac{\pi}{4} = 1\right]$$

The general solution of $\sin\theta = 0$ is $\theta = n\pi$, $n \in Z$ and $\tan\theta = \tan\alpha$ is $\theta = n\pi + \alpha$, $n \in Z$.

$\therefore$ the required general solution is

$$\frac{\theta}{2} = n\pi, n \in Z \text{ or } \frac{\theta}{2} = n\pi + \frac{\pi}{4}, n \in Z$$

i.e. $\theta = 2n\pi$, $n \in Z$ or $\theta = 2n\pi + \frac{\pi}{2}, n \in Z$.

Exercise 3.1 | Q 8.1 | Page 75

State whether the following equation have solution or not?

$\cos 2\theta = -1$

Solution: $\cos 2\theta = -1$

Since $-1 \leq \cos\theta \leq 1$ for any θ,

$\cos 2\theta = -1$ has solution.

Exercise 3.1 | Q 8.2 | Page 75

State whether the following equation has a solution or not?

$\cos^2\theta = -1$.

Solution: $\cos^2\theta = -1$

This is not possible because $\cos^2\theta \geq 0$ for any θ.

$\therefore \cos^2\theta = -1$ does not have any solution.

Exercise 3.1 | Q 8.3 | Page 75

State whether the following equation has a solution or not?

$2\sin\theta = 3$

Solution: $2\sin\theta = 3$

$\therefore \sin\theta = 3/2$

This is not possible because $-1 \leq \sin\theta \leq 1$ for any θ.
$\therefore 2\sin\theta = 3$ does not have any solution.

Exercise 3.1 | Q 8.4 | Page 75

State whether the following equation have solution or not?

$3\tan\theta = 5$

Solution: $3\tan\theta = 5$

$\therefore \tan\theta = 5/3$

This is possible because $\tan\theta$ is any real number.

$\therefore$ $3\tan\theta = 5$ has solution.

EXERCISE 3.2 [PAGE 88]

Exercise 3.2 | Q 1.1 | Page 88

Find the Cartesian co-ordinates of the point whose polar co-ordinates are :

$\left(\sqrt{2}, \frac{\pi}{4}\right)$

Solution:

Here, r = $\sqrt{2}$ and $\theta = \frac{\pi}{4}$

Let the cartesian coordinates be (x,y)

Then, x = r cos θ = $\sqrt{2}\cos\ \frac{\pi}{4} = \sqrt{2}\left(\frac{1}{\sqrt{2}}\right)$ = 1

y = r sin θ = $\sqrt{2}\sin\ \frac{\pi}{4} = \sqrt{2}\left(\frac{1}{\sqrt{2}}\right)$ = 1

∴ the cartesian coordinates of the given point are (1, 1).

Exercise 3.2 | Q 1.2 | Page 88

Find the Cartesian co-ordinates of the point whose polar co-ordinates are :
(4, π/2)

Solution:

The cartesian coordinates of the given point are (0, 4).

Solution is not available.

Exercise 3.2 | Q 1.3 | Page 88

Find the Cartesian co-ordinates of the point whose polar co-ordinates are:

$\left(\frac{3}{4}, \frac{3\pi}{4}\right)$

Solution:

Here, $r = \frac{3}{4}$ and $\theta = \frac{3\pi}{4}$

Let the cartesian coordinates be (x, y)

Then,

$$x = r\cos\theta = \frac{3}{4}\cos\ \frac{3\pi}{4} = \frac{3}{4}\cos\left(\pi - \frac{\pi}{4}\right)$$

$$= -\frac{3}{4}\cos\ \frac{\pi}{4} = -\frac{3}{4} \times \frac{1}{\sqrt{2}} = -\ \frac{3}{4\sqrt{2}}$$

$$y = r\sin\theta = \frac{3}{4}\sin\ \frac{3\pi}{4} = \frac{3}{4}\sin\left(\pi - \frac{\pi}{4}\right)$$

$$= \frac{3}{4}\sin\ \frac{\pi}{4} = \frac{3}{4} \times \frac{1}{\sqrt{2}} = \frac{3}{4\sqrt{2}}$$

$\therefore$ The cartesian coordinates of the given point are $\left(-\frac{3}{4\sqrt{2}}, \frac{3}{4\sqrt{2}}\right)$.

Exercise 3.2 | Q 1.4 | Page 88

Find the Cartesian co-ordinates of the point whose polar co-ordinates are:

$$\left(\frac{1}{2}, \frac{7\pi}{3}\right)$$

Solution:

Here, $r = \frac{1}{2}$ and $\theta = \frac{7\pi}{3}$

Let the cartesian coordinates be (x, y)

Then,

$$x = r\cos\theta = \frac{1}{2}\cos\ \frac{7\pi}{3} = \frac{1}{2}\cos\left(2\pi + \frac{\pi}{3}\right)$$

$$= \frac{1}{2}\cos\ \frac{\pi}{3} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

$$y = r\sin\theta = \frac{1}{2}\sin\ \frac{7\pi}{3} = \frac{1}{2}\sin\left(2\pi + \frac{\pi}{3}\right)$$

$$= \frac{1}{2}\sin\ \frac{\pi}{3} = \frac{1}{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}$$

$\therefore$ The cartesian coordinates of the given point are $\left(\frac{1}{4}, \frac{\sqrt{3}}{4}\right)$

Exercise 3.2 | Q 2.1 | Page 88

Find the polar co-ordinates of the point whose Cartesian co-ordinates are.
($\sqrt{2}$, $\sqrt{2}$)

Solution:

Here $x = \sqrt{2}$ and $y = \sqrt{2}$

$\therefore$ the point lies in the first quadrant.

Let the polar coordinates be (r, θ)

Then, $r^2 = x^2 + y^2 = (\sqrt{2})^2 + (\sqrt{2})^2 = 2 + 2 = 4$

$\therefore r = 2$ ...[$\because r > 0$]

$\cos\theta = \dfrac{x}{r} = \dfrac{\sqrt{2}}{2} = \dfrac{1}{\sqrt{2}}$

and

$\sin\theta = \dfrac{y}{r} = \dfrac{\sqrt{2}}{2} = \dfrac{1}{\sqrt{2}}$

Since the point lies in the first quadrant and

$0 \le \theta < 2\pi$, $\tan\theta = 1 = \tan\dfrac{\pi}{4}$

$\therefore \theta = \dfrac{\pi}{4}$

$\therefore$ the polar coordinates of the given point are $\left(2, \dfrac{\pi}{4}\right)$.

Exercise 3.2 | Q 2.2 | Page 88

Find the polar co-ordinates of the point whose Cartesian co-ordinates are.

$\left(0, \dfrac{1}{2}\right)$

Solution: Here x = 0 and y = 2

∴ the point lies on the positive side of Y-axis.

Let the polar coordinates be (r, θ)

Then, $r^2 = x^2 + y^2$

$= (0)^2 + \left(\frac{1}{2}\right)^2$

$= 0 + \frac{1}{4}$

$= \frac{1}{4}$

$\therefore r = \frac{1}{2}$ ...[∵ r > 0]

$\cos\theta = \frac{x}{r} = \frac{0}{\frac{1}{2}} = 0$

and

$\sin\theta = \frac{y}{r} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$

Since, the point lies on the positive side of Y-axis and

$0 \le \theta < 2\pi$

$\cos\theta = 0 = \cos\frac{\pi}{2}$ and $\sin\theta = 1 = \sin\frac{\pi}{2}$

$\therefore \theta = \frac{\pi}{2}$

∴ the polar coordinates of the given point are $\left(\frac{1}{2}, \frac{\pi}{2}\right)$.

Exercise 3.2 | Q 2.3 | Page 88

Find the polar co-ordinates of the point whose Cartesian co-ordinates are.

(1, - √3)

Solution: Here x = 1 and y = - √3

∴ the point lies in the fourth quadrant.

Let the polar coordinates be (r, θ).

Then $r^2 = x^2 + y^2 = (1)^2 + (-\sqrt{3})^2 = 1 + 3 = 4$

∴ r = 2 ...[∵ r > 0]

$$\cos\theta = \frac{x}{r} = \frac{1}{2}$$

$$\text{and} \quad \sin\theta = \frac{y}{r} = -\frac{\sqrt{3}}{2}$$

∴ tan θ = - √3

Since, the point lies in the fourth quadrant and 0 ≤ θ < 2π.

$\tan\theta = -\sqrt{3} = -\tan\frac{\pi}{3}$

$= \tan\frac{5\pi}{3}$

∴ $\theta = \frac{5\pi}{3}$

∴ The polar coordinates of the given point are $\left(2, \frac{5\pi}{3}\right)$.

Exercise 3.2 | Q 2.4 | Page 88

Find the polar co-ordinates of the point whose Cartesian co-ordinates are.

$\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$.

Solution: The polar coordinates of the given point are (3, π/3).

Solution is not available.

Exercise 3.2 | Q 3 | Page 88

In ΔABC, if ∠A = 45°, ∠B = 60° then find the ratio of its sides.

Solution: By the sine rule,

$$\frac{a}{\sin \mathrm{A}} = \frac{b}{\sin \mathrm{B}} = \frac{c}{\sin \mathrm{C}}$$

$$\therefore \frac{a}{b} = \frac{\sin \mathrm{A}}{\sin \mathrm{B}} \text{ and } \frac{b}{c} = \frac{\sin \mathrm{B}}{\sin \mathrm{C}}$$

$\therefore$ a : b : c = sinA : sinB : sinC

Given $\angle A = 45°$ and $\angle B = 60°$

$\because \angle A + \angle B + \angle C = 180°$

$\therefore 45° + 60° + \angle C = 180°$

$\therefore \angle C = 180° - 105° = 75°$

Now, $\sin A = \sin 45° = \frac{1}{\sqrt{2}}$

$\sin B = \sin 60° = \frac{\sqrt{3}}{2}$

and $\sin C = \sin 75° = \sin(45° + 30°)$

$= \sin 45° \cos 30° + \cos 45° \sin 30°$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}$$

$$= \frac{\sqrt{3}}{2(\sqrt{2})} + \frac{1}{2(\sqrt{2})}$$

$$= \frac{\sqrt{3}+1}{2(\sqrt{2})}$$

$\therefore$ the ratio of the sides of ΔABC

= a : b : c

= sinA : sinB : sinC

$$= \frac{1}{\sqrt{2}} : \frac{\sqrt{3}}{2} : \frac{\sqrt{3}+1}{2\sqrt{2}}$$

∴ a : b : c = 2: √6: (√3 + 1).

Exercise 3.2 | Q 4 | Page 88

In Δ ABC, prove that $\sin\left(\frac{B - C}{2}\right) = \left(\frac{b - c}{2}\right) \cos \frac{A}{2}$.

Solution:

By the sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

∴ a = k sinA, b = k sinB, c = k sinC

$$\text{R.H.S.} = \left(\frac{b - c}{a}\right) \cos \frac{A}{2}$$

$$= \left(\frac{k \sin B - k \sin C}{k \sin A}\right) \cos \frac{A}{2}$$

$$= \left(\frac{\sin B - \sin C}{\sin A}\right) \cos \frac{A}{2}$$

$$= \frac{2 \cos\left(\frac{B+C}{2}\right).\sin\left(\frac{B-C}{2}\right)}{2 \sin \frac{A}{2}.\cos \frac{A}{2}}.\cos \frac{A}{2}$$

$$= \frac{\cos\left(\frac{B+C}{2}\right).\sin\left(\frac{B-C}{2}\right)}{\sin \frac{A}{2}}$$

$$= \frac{\cos\left(\frac{\pi}{2} - \frac{A}{2}\right).\sin\left(\frac{B-c}{2}\right)}{\sin \frac{A}{2}} \quad \text{...[} \because A + B + C = \pi \text{]}$$

$$= \frac{\sin \frac{A}{2}.\sin\left(\frac{B-C}{2}\right)}{\frac{\sin A}{2}}$$

$= \sin\left(\frac{B - C}{2}\right)$

= L.H.S.

Exercise 3.2 | Q 5 | Page 88

With the usual notations prove that $2\left\{a\sin^2\frac{C}{2} + c\sin^2\frac{A}{2}\right\}$ = a − b + c.

Solution:

L.H.S. $= 2\left\{a\sin^2\frac{C}{2} + c\sin^2\frac{A}{2}\right\}$

$= a\left(2\sin^2\frac{C}{2}\right) + c\left(2\sin^2\frac{A}{2}\right)$

= a(1 − cos C) + c(1 − cos A)

$= a\left[1 - \frac{a^2 + b^2 - c^2}{2ab}\right] + c\left[1 - \frac{b^2 + c^2 - a^2}{2bc}\right]$ ...[By cosine rule]

$= a\left[\frac{2ab - a^2 - b^2 + c^2}{2ab}\right] + c\left[\frac{2bc - b^2 - c^2 + a^2}{2bc}\right]$

$= \frac{2ab - a^2 - b^2 + c^2}{2b} + \frac{2bc - b^2 - c^2 + a^2}{2b}$.

$= \frac{2ab - a^2 - b^2 + c^2 + 2bc - b^2 - c^2 + a^2}{2b}$.

$= \frac{2ab - 2b^2 + 2bc}{2b}$

= a − b + c

= R.H.S.

Exercise 3.2 | Q 6 | Page 88

In Δ ABC, prove that $a^3 \sin(B - C) + b^3\sin(C - A) + c^3\sin(A - B) = 0$

Solution: By the sine rule,

By the sine rule,

$$\frac{a}{\sin A} - \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$\therefore a = k \sin A,\ b = k \sin B,\ c = k \sin C$

L.H.S. $= a^3\sin(B - C) + b^3\sin(C - A) + c^3\sin(A - B)$

$= a^3(\sin B \cos C - \cos B \sin C) + b^3(\sin C \cos A - \cos C \sin A) + c^3(\sin A \cos B - \cos A \sin B)$

$$= a^3\left(\frac{b}{k}\cos C - \frac{c}{k}\cos B\right) + b^3\left(\frac{c}{k}\cos A - \frac{a}{k}\cos C\right) + c^3\left(\frac{a}{k}\cos B - \frac{b}{k}\cos A\right)$$

$$= \frac{1}{k}\left[a^3 b\cos C - a^3 c\cos B + b^3 c\cos A - b^3 a\cos C + c^3 a\cos B - c^3 b\cos A\right]$$

$$= \frac{1}{k}\left[a^3 b\left(\frac{a^2+b^2-c^2}{2ab}\right) - a^3 c\left(\frac{c^2+a^2-b^2}{2ca}\right) + b^3 c\left(\frac{b^2+c^2-a^2}{2bc}\right) - ab^3\left(\frac{a^2+b^2-c^2}{2ab}\right) + ac^3\left(\frac{c^2+a^2-b^2}{2ca}\right) - bc^3\left(\frac{b^2+c^2-a^2}{2bc}\right)\right]$$ ...[By cosine rule]

$$= \frac{1}{2k}\left[a^2(a^2+b^2-c^2) - a^2(a^2+c^2-b^2) + b^2(b^2+c^2-a^2) - b^2(a^2+b^2-c^2) + c^2(c^2+a^2-b^2) - c^2(b^2+c^2-a^2)\right]$$

$$= \frac{1}{2k}\left[a^4 + a^2b^2 - a^2c^2 - a^4 - a^2c^2 + a^2b^2 + b^4 + b^2c^2 - a^2b^2 - a^2b^2 - b^4 + b^2c^2 + c^4 + a^2c^2 - b^2c^2 - b^2c^2 - c^4 + a^2c^2\right]$$

$$= \frac{1}{2k}(0)$$

$= 0$

= R.H.S.

Exercise 3.2 | Q 7 | Page 88

In ΔABC, if cot A, cot B, cot C are in A.P. then show that a^2, b^2, c^2 are also in A.P.

Solution:

By the sine rule,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k$$

$\therefore \sin A = ka,\ \sin B = kb,\ \sin C = kc$...(1)

Now, cot A, cot B, cot C are in A.P.

$\therefore$ cot C – cot B = cot B – cot A

$\therefore$ cot A + cot C = 2cot B

$\therefore \dfrac{\cos A}{\sin A} + \dfrac{\cos C}{\sin C}$ = 2cot B

$\therefore \dfrac{\sin C \cos A + \sin A \cos C}{\sin A.\, \sin C}$ = 2cotB

$\therefore \dfrac{\sin(A + C)}{\sin A.\, \sin C}$ = 2cot B

$\therefore \dfrac{\sin(\pi - B)}{\sin A.\, \sin C}$ = 2cot B ...[$\because$ A + B + C = π]

$$\therefore \frac{\sin B}{\sin A.\, \sin C} = \frac{2\cos B}{\sin B}$$

$$\therefore \frac{k^2 b^2}{(ka)(kc)} = 2\left(\frac{a^2 + c^2 - b^2}{2ac}\right)$$

$$\therefore \frac{b^2}{ac} = \frac{a^2 + c^2 - b^2}{ac}$$

$\therefore b^2 = a^2 + c^2 - b^2$

$\therefore 2b^2 = a^2 + c^2$

Hence, a^2 b^2, c^2 are in A.P.

Exercise 3.2 | Q 8 | Page 88

In ΔABC, if a cos A = b cos B then prove that the triangle is either a right angled or an isosceles traingle.

Solution: Using the sine rule,

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$ = k a = k sin A and b = k sin B

∴ a cos A = b cos B gives

k sinA cosA = k sinB cosB

∴ 2sinA cosA = 2sinB cosB

∴ sin 2A = sin 2B

∴ sin2A – sin2B = 0

∴ 2cos(A + B).sin(A – B) = 0

∴ 2cos(π – C).sin(A – B) = 0 ...[∵ A + B + C = π]

∴ - 2cosC. sin(A – B) = 0

∴ cosC = 0 OR sin(A – B) = 0

∴ C = 90° OR A – B = 0

∴ C = 90° OR A = B

∴ the triangle is either rightangled or an isosceles triangle.

Exercise 3.2 | Q 9 | Page 88

With usual notations prove that 2(bc cosA + ac cosB + ab cosC) = $a^2 + b^2 + c^2$.

Solution:

L.H.S. = 2(bc cosA + ac cosB + ab cosC)

= 2bc cosA + 2ac cosB + 2ab cosC

$$= 2bc\left(\frac{b^2+c^2-a^2}{2bc}\right) + 2ac\left(\frac{c^2+a^2-b^2}{2ca}\right) + 2ab\left(\frac{a^2+b^2-c^2}{2ab}\right) \quad \text{...[By cosine rule]}$$

$= b^2 + c^2 - a^2 + c^2 + a^2 - b^2 + a^2 + b^2 - c^2$

$= a^2 + b^2 + c^2$

= R.H.S.

Exercise 3.2 | Q 10.1 | Page 88

In ΔABC, if a = 18, b = 24, c = 30 then find the values of cosA

Solution: Given: a = 18, b = 24 and c = 30

∴ 2s = a + b + c

= 18 + 24 + 30

= 72

∴ s = 36

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{(24)^2 + (30)^2 - (18)^2}{2(24)(30)}$$

$$= \frac{576 + 900 - 324}{1440}$$

$$= \frac{1152}{1440}$$

$$= \frac{4}{5}.$$

Exercise 3.2 | Q 10.2 | Page 88

In ΔABC, if a = 18, b = 24, c = 30 then find the values of sin A/2.

Solution: Given: a = 18, b = 24 and c = 30

∴ 2s = a + b + c

= 18 + 24 + 30

= 72

∴ s = 36

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$= \sqrt{\frac{(36-24)(36-30)}{(24)(30)}}$$

$$= \sqrt{\frac{12 \times 6}{24 \times 30}}$$

$= \sqrt{\dfrac{1}{10}}$

$= \dfrac{1}{\sqrt{10}}.$

Exercise 3.2 | Q 10.3 | Page 88

In ΔABC, if a = 18, b = 24, c = 30 then find the values of cos A/2

Solution: Given: a = 18, b = 24 and c = 30

∴ 2s = a + b + c

= 18 + 24 + 30

= 72

∴ s = 36

$\cos\ \dfrac{A}{2} = \sqrt{\dfrac{s(s-a)}{bc}}$

$= \sqrt{\dfrac{36(36-18)}{(24)(30)}}$

$= \sqrt{\dfrac{36 \times 18}{24 \times 30}}$

$= \sqrt{\dfrac{9}{10}}$

$= \dfrac{3}{\sqrt{10}}.$

Exercise 3.2 | Q 10.4 | Page 88

In ΔABC, if a = 18, b = 24, c = 30 then find the values of tan A/2

Solution: Given : a = 18, b = 24 and c = 30

∴ 2s = a + b + c

= 18 + 24 + 30

= 72

∴ s = 36

$$\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}$$

$$= \frac{\frac{1}{\sqrt{10}}}{\frac{3}{\sqrt{10}}}$$

$$= \frac{1}{3}.$$

Exercise 3.2 | Q 10.5 | Page 88

In ΔABC, if a = 18, b = 24, c = 30 then find the values of A(ΔABC)

Solution:

Given: a = 18, b = 24 and c = 30

∴ 2s = a + b + c

= 18 + 24 + 30

= 72

∴ s = 36

$$A(\Delta ABC) = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{36(36-18)(36-24)(36-30)}$$

$$= \sqrt{36 \times 18 \times 12 \times 6}$$

$$= \sqrt{36 \times 18 \times 4 \times 18}$$

= 6 x 18 x 2

= 216 sq units.

Exercise 3.2 | Q 10.6 | Page 88

In ΔABC, if a = 18, b = 24, c = 30 then find the values of sinA

Solution: Given : a = 18, b = 24 and c = 30

∴ 2s = a + b + c

= 18 + 24 + 30

= 72

∴ s = 36

$$216 = \frac{1}{2}(24)(30)\ \sin A$$

$$\therefore \sin A = \frac{216}{12 \times 30}$$

$$= \frac{216}{360}$$

$$= \frac{3}{5}.$$

Exercise 3.2 | Q 11 | Page 88

In ΔABC prove that (b+c-a)tan A/2=(c+a-b)tan B/2=(a+b-c)tan C/2.

Solution:

$$(b + c - a)\tan\frac{A}{2}$$

$$= (a + b + c - 2a).\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$= (2s - 2a).\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$= 2\sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \quad ....(1)$$

$(c+a-b)\tan\dfrac{B}{2}$

$= (a+b+c-2b).\sqrt{\dfrac{(s-a)(s-c)}{s(s-b)}}$

$= (2s-2b).\sqrt{\dfrac{(s-a)(s-c)}{s(s-b)}}$

$= 2\sqrt{\dfrac{(s-a)(s-b)(s-c)}{s}}$ ...(2)

$(a+b-c)\tan\dfrac{C}{2}$

$= (a+b+c-2c).\sqrt{\dfrac{(s-a)(s-b)}{s(s-c)}}$

$= (2s-2c).\sqrt{\dfrac{(s-a)(s-b)}{s(s-c)}}$

$= 2\sqrt{\dfrac{(s-a)(s-b)(s-c)}{s}}$ ...(3)

From (1), (2) an (3), we get

$$(b+c-a)\tan\frac{A}{2} = (c+a-b)\tan\frac{B}{2} = (a+b-c)\tan\frac{C}{2}.$$

Exercise 3.2 | Q 12 | Page 88

In ΔABC prove that $\sin\dfrac{A}{2}.\sin\dfrac{B}{2}.\sin\dfrac{C}{2} = \dfrac{[A(\Delta ABC)]^2}{abcs}$

Solution:

L.H.S.

$= \sin\frac{A}{2} . \sin\frac{B}{2} . \sin\frac{C}{2}$

$$= \sqrt{\frac{(s-b)(s-c)}{bc}} . \sqrt{\frac{(s-a)(s-c)}{ac}} . \sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$= \sqrt{\frac{(s-a)^2(s-b)^2(s-c)^2}{a^2b^2c^2}}$$

$$= \frac{(s-a)(s-b)(s-c)}{abc}$$

$$= \frac{s(s-a)(s-b)(s-c)}{abcs}$$

$$= \frac{([A(\Delta ABC)]^2}{abcs} \quad ...\left[\because A(\Delta ABC) = \sqrt{s(s-a)(s-b)(s-c)}\right]$$

= R.H.S.

EXERCISE 3.3 [PAGES 102 - 103]

Exercise 3.3 | Q 1.1 | Page 102

Find the principal value of the following: $\sin^{-1}(1/2)$

Solution:

The principal value branch of $\sin^{-1} x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Let $\sin^{-1}\left(\frac{1}{2}\right) = \alpha$, where $\frac{-\pi}{2} \leq \alpha \leq \frac{\pi}{2}$

$\therefore \sin\alpha = \frac{1}{2} = \sin\frac{\pi}{6}$

$\therefore \alpha = \frac{\pi}{6} \quad ...\left[\because -\frac{\pi}{2} \leq \frac{\pi}{6} \leq \frac{\pi}{2}\right]$

$\therefore$ the principal value of $\sin^{-1}\left(\frac{1}{2}\right)$ is $\frac{\pi}{6}$.

Exercise 3.3 | Q 1.2 | Page 102

Find the principal value of the following: $\operatorname{cosec}^{-1}(2)$

Solution:

The principal value branch of $\operatorname{cosec}^{-1}x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$.

Let $\operatorname{cosec}^{-1}(2) = \alpha$, where $\frac{-\pi}{2} \leq \alpha \leq \frac{\pi}{2}$, $\alpha \neq 0$.

$\therefore \operatorname{cosec} \alpha = 2 = \operatorname{cosec} \frac{\pi}{6}$

$\therefore \alpha = \frac{\pi}{6}$ $\quad ...\left[\because -\frac{\pi}{2} \leq \frac{\pi}{6} \leq \frac{\pi}{2}\right]$

$\therefore$ the principal value of $\operatorname{cosec}^{-1}(2)$ is $\frac{\pi}{6}$.

Exercise 3.3 | Q 1.3 | Page 102

Find the principal value of the following: $\tan^{-1}(-1)$

Solution:

The principal value branch of $\tan^{-1}x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Let $\tan^{-1}(-1) = \alpha$, where $\frac{-\pi}{2} \leq \alpha \leq \frac{\pi}{2}$

$\therefore \tan \alpha = -1 = -\tan \frac{\pi}{4}$

$\therefore \tan \alpha = \tan\left(-\frac{\pi}{4}\right)$ $\quad ...[\because \tan(-\theta) = -\tan \theta]$

$\therefore \alpha = -\frac{\pi}{4}$ $\quad ...\left[\because -\frac{\pi}{2} \leq -\frac{\pi}{4} \leq \frac{\pi}{2}\right]$

$\therefore$ the principal value of $\tan^{-1}(-1)$ is $-\frac{\pi}{4}$.

Exercise 3.3 | Q 1.4 | Page 102

Find the principal value of the following: $\tan^{-1}(-\sqrt{3})$

Solution:

The principal value branch of $\tan^{-1}x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Let $\tan^{-1}(-\sqrt{3}) = \alpha$, where $\frac{-\pi}{2} \leq \alpha \leq \frac{\pi}{2}$

$\therefore \tan\alpha = -\sqrt{3} = -\tan\frac{\pi}{3}$

$\therefore \tan\alpha = \tan\left(-\frac{\pi}{3}\right)$ ...[$\because \tan(-\theta) = -\tan\theta$]

$\therefore \alpha = -\frac{\pi}{3}$ ...$\left[\because -\frac{\pi}{2} < \frac{-\pi}{3} < \frac{\pi}{2}\right]$

$\therefore$ the principal value of $\tan^{-1}(-\sqrt{3})$ is $-\frac{\pi}{3}$.

Exercise 3.3 | Q 1.5 | Page 102

Find the principal value of the following: $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

Solution:

The principal value branch of $\sin^{-1}x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Let $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \alpha$, where $\frac{-\pi}{2} \leq \alpha \leq \frac{\pi}{2}$

$\therefore \sin\alpha = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4}$

$\therefore \alpha = \frac{\pi}{4}$ ...$\left[\because -\frac{\pi}{2} \leq \frac{\pi}{4} \leq \frac{\pi}{2}\right]$

$\therefore$ the principal value of $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ is $\frac{\pi}{4}$.

Exercise 3.3 | Q 1.6 | Page 102

Find the principal value of the following: $\cos^{-1}\left(-\frac{1}{2}\right)$

Solution:

The principal value branch of $\cos^{-1}x$ [0, π].

Let $\cos^{-1}\left(-\frac{1}{2}\right) = \alpha$, where $0 \le \alpha \le \pi$

$\therefore \cos\alpha = -\frac{1}{2} = -\cos\frac{\pi}{3}$

$\therefore \cos\alpha = \cos\left(\pi - \frac{\pi}{3}\right)$ ...[$\because \cos(\pi - \theta) = -\cos\theta$]

$\therefore \cos\alpha = \cos\frac{2\pi}{3}$

$\therefore \alpha = \frac{2\pi}{3}$ ... $\left[\because 0 \le \frac{2\pi}{3} \le \pi\right]$

$\therefore$ the principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$ is $\frac{2\pi}{3}$.

Exercise 3.3 | Q 2.1 | Page 102

Evaluate the following:

$$\tan^{-1}(1) + \cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(\frac{1}{2}\right)$$

Solution:

Let $\tan^{-1}(1) = \alpha$, where $\frac{-\pi}{2} < \alpha < \frac{\pi}{2}$

$\therefore \tan\alpha = 1 = \tan\frac{\pi}{4}$

$\therefore \alpha = \frac{\pi}{4}$ ...$\left[\because \frac{-\pi}{2} < \frac{\pi}{4} < \frac{\pi}{2}\right]$

$\therefore \tan^{-1}(1) = \frac{\pi}{4}$ ...(1)

Let $\cos^{-1}\left(\frac{1}{2}\right) = \beta$, where $0 \le \beta \le \pi$

$\therefore \cos\beta = \frac{1}{2} = \cos\frac{\pi}{3}$

$\therefore \beta = \frac{\pi}{3}$ ...$\left[\because 0 < \frac{\pi}{3} < \pi\right]$

$\therefore \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$ ...(2)

$\therefore \sin\gamma = \frac{1}{2} = \sin\frac{\pi}{6}$

$\therefore \gamma = \frac{\pi}{6}$ ...$\left[\because \frac{-\pi}{2} \le \frac{\pi}{6} \le \frac{\pi}{2}\right]$

$\therefore \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$ ...(3)

$\therefore \tan^{-1}(1) + \cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(\frac{1}{2}\right)$

$= \frac{\pi}{4} + \frac{\pi}{3} + \frac{\pi}{6}$ ...[By (1), (2) and (3)]

$= \frac{3\pi + 4\pi + 2\pi}{12}$

$= \dfrac{9\pi}{12}$

$= \dfrac{3\pi}{4}$.

Exercise 3.3 | Q 2.2 | Page 102

Evaluate the following:

$$\cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$$

Solution:

Let $\cos^{-1}\left(\frac{1}{2}\right)$ = α, where 0 ≤ α ≤ π

∴ cos α = $\dfrac{1}{2} = \cos\ \dfrac{\pi}{3}$

∴ α = $\dfrac{\pi}{3}$ $\quad\quad ...\left[\because 0 < \dfrac{\pi}{3} < \pi\right]$

∴ $\cos^{-1}\left(\dfrac{1}{2}\right) = \dfrac{\pi}{3}$ $\quad\quad$ ...(1)

Let $\sin^{-1}\left(\dfrac{1}{2}\right) = \beta, \text{where} \dfrac{-\pi}{2} \le \beta \le \dfrac{\pi}{2}$

∴ sin β = $\dfrac{1}{2} = \sin\ \dfrac{\pi}{6}$

∴ β = $\dfrac{\pi}{6}$ $\quad\quad ...\left[\because \dfrac{-\pi}{2} \le \dfrac{\pi}{6} \le \dfrac{\pi}{2}\right]$

$$\therefore \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6} \qquad ...(2)$$

$$\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} \text{ and } \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$\therefore \cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$$

$$= \frac{\pi}{3} + 2\left(\frac{\pi}{6}\right)$$

$$= \frac{\pi}{3} + \frac{\pi}{3}$$

$$= \frac{2\pi}{3}.$$

Exercise 3.3 | Q 2.3 | Page 102

Evaluate the following:

$$\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$$

Solution:

Let $\tan^{-1}\left(\sqrt{3}\right) = \alpha$, where $\frac{-\pi}{2} < \alpha < \frac{\pi}{2}$

$\therefore \tan \alpha = \sqrt{3} = \tan \frac{\pi}{3}$

$$\therefore \alpha = \frac{\pi}{3} \qquad ...\left[\because \frac{-\pi}{2} < \frac{\pi}{3} < \frac{\pi}{2}\right]$$

$$\therefore \tan^{-1}(\sqrt{3}) = \frac{\pi}{3} \qquad ...(1)$$

Let $\sec^{-1}(-2) = \beta$, where $0 \le \beta \le \pi$, $\beta \ne \frac{\pi}{2}$

$\therefore \sec \beta = -2 = -\sec \frac{\pi}{3}$

$\therefore$ sec β = $\sec\left(\pi - \frac{\pi}{3}\right)$ ...[∵ sec(π – θ) = – secθ]

$\therefore$ sec β = $\sec\ \frac{2\pi}{3}$

$\therefore$ β = $\frac{2\pi}{3}$ $\quad ...\left[\because 0 \leq \frac{2\pi}{3} \leq \pi\right]$

$\therefore$ $\sec^{-1}(-2)$ = $\frac{2\pi}{3}$ ...(2)

$\therefore \tan^{-1}(\sqrt{3}) - \sec^{-1}(-2)$

= $\frac{\pi}{3} - \frac{2\pi}{3}$ ...[By (1) and (2)]

= $-\frac{\pi}{3}$.

Exercise 3.3 | Q 2.4 | Page 103

Evaluate the following:

$\operatorname{cosec}^{-1}\left(-\sqrt{2}\right) + \cot^{-1}\left(\sqrt{3}\right)$

Solution:

Let $\operatorname{cosec}^{-1}\left(-\sqrt{2}\right) = \alpha$, where $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0$

$\therefore$ cosec α = $-\sqrt{2} = -\operatorname{cosec}\ \frac{\pi}{4}$

$\therefore$ cosec α = $\operatorname{cosec}\left(-\frac{\pi}{4}\right)$ ...[∵ cosec (– θ) = – cosec θ]]

$\therefore$ α = $-\frac{\pi}{4}$ $\quad ...\left[\because \frac{-\pi}{2} \leq \frac{-\pi}{4} \leq \frac{\pi}{2}\right]$

$\therefore \operatorname{cosec}^{-1}\left(-\sqrt{2}\right) = -\frac{\pi}{4}$ ...(1)

Let $\cot^{-1}\left(\sqrt{3}\right)$ = β, where 0 < β < π

∴ cot β = $\sqrt{3} = \cot\ \frac{\pi}{6}$

∴ β = $\frac{\pi}{6}$ $\quad\ldots\left[\because 0 < \frac{\pi}{6} < \pi\right]$

$$\therefore \cot^{-1}\left(\sqrt{3}\right) = \frac{\pi}{6} \quad \ldots(2)$$

$$\therefore \operatorname{cosec}^{-1}\left(-\sqrt{2}\right) + \cot^{-1}\left(\sqrt{3}\right)$$

= $-\frac{\pi}{4} + \frac{\pi}{6}$ ...[By (1) and (2)]

= $\frac{-3\pi + 2\pi}{12}$

= $-\frac{\pi}{12}$.

Exercise 3.3 | Q 3.1 | Page 103

Prove the following:

$$\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - 3\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = -\frac{3\pi}{4}$$

Solution:

Let $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \alpha$, where $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$

∴ sin α = $\frac{1}{\sqrt{2}} = \sin\ \frac{\pi}{4}$

∴ α = $\frac{\pi}{4}$ $\quad\ldots\left[\because -\frac{\pi}{2} \le \frac{\pi}{4} \le \frac{\pi}{2}\right]$

$\therefore \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$ ...(1)

Let $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \beta$, where $-\frac{\pi}{2} \le \beta \le \frac{\pi}{2}$

$\therefore \sin \beta = \frac{\sqrt{3}}{2} = \sin \frac{\pi}{3}$

$\therefore \beta = \frac{\pi}{3}$ $\quad ...\left[\because -\frac{\pi}{2} \le \frac{\pi}{3} \le \frac{\pi}{2}\right]$

$\therefore \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$ ...(2)

L.H.S. = $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - 3\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

= $\frac{\pi}{4} - 3\left(\frac{\pi}{3}\right)$ ...[By (1) and (2)]

= $\frac{\pi}{4} - \pi$

= $-\frac{3\pi}{4}$

= R.H.S.

Exercise 3.3 | Q 3.2 | Page 103

Prove the following:

$$\sin^{-1}\left(-\frac{1}{2}\right) + \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \cos^{-1}\left(-\frac{1}{2}\right)$$

Solution:

Let $\sin^{-1}\left(-\frac{1}{2}\right) = \alpha$, where $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$

$\therefore \sin\alpha = -\frac{1}{2} = -\sin\frac{\pi}{6}$

$\therefore \sin\alpha = \sin\left(-\frac{\pi}{6}\right)$ ...[$\because \sin(-\theta) = -\sin\theta$]

$\therefore \alpha = -\frac{\pi}{6}$ ...$\left[\because -\frac{\pi}{2} \leq -\frac{\pi}{6} \leq \frac{\pi}{2}\right]$

$\therefore \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$ ...(1)

Let $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \beta$, where $0 \leq \beta \leq \pi$

$\therefore \cos\beta = -\frac{\sqrt{3}}{2} = -\cos\frac{\pi}{6}$

$\therefore \cos\beta = \cos\left(\pi - \frac{\pi}{6}\right)$ ...[$\because \cos(\pi - \theta) = -\cos\theta$]

$\therefore \cos\beta = \cos\frac{5\pi}{6}$

$\therefore \beta = \frac{5\pi}{6}$ ...$\left[\because 0 \leq \frac{5\pi}{6} \leq \pi\right]$

$\therefore \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$ ...(2)

Let $\cos^{-1}\left(-\frac{1}{2}\right) = Y$, where $0 \leq Y \leq \pi$

$\therefore \cos Y = -\frac{1}{2} = -\cos\frac{\pi}{3}$

$\therefore$ cos Y = $\cos\left(\pi - \frac{\pi}{3}\right)$ ...[$\because$ cos(π – θ) = – cos θ]

$\therefore$ cos Y = $\cos\ \frac{2\pi}{3}$

$\therefore$ Y = $\frac{2\pi}{3}$ $\quad ...\left[\because 0 \le \frac{2\pi}{3} \le \pi\right]$

$\therefore \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$ ...(3)

L.H.S. = $\sin^{-1}\left(-\frac{1}{2}\right) + \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

= $-\frac{\pi}{6} + \frac{5\pi}{6}$ ...[By (1) and (2)]

= $\frac{4\pi}{6} = \frac{2\pi}{3}$

= $\frac{4\pi}{6} = \frac{2\pi}{3}$

= $\cos^{-1}\left(-\frac{1}{2}\right)$ ...[By (3)]

= R.H.S.

Exercise 3.3 | Q 3.3 | Page 103

Prove the following:

$$\sin^{-1}\left(\frac{3}{5}\right) + \cos^{-1}\left(\frac{12}{13}\right) = \sin^{-1}\left(\frac{56}{65}\right)$$

Solution:

Let $\sin^{-1}\left(\frac{3}{5}\right) = x, \cos^{-1}\left(\frac{12}{13}\right) = y$ and $\sin^{-1}\left(\frac{56}{65}\right)$ = z.

Then $\sin x = \frac{3}{5}$, where $0 < x < \frac{\pi}{2}$

cos y = $\frac{12}{13}$, where$0 < y < \frac{\pi}{2}$

and sin z = $\frac{56}{65}$, where $0 < z < \frac{\pi}{2}$

∴ cos x > 0, sin y > 0

Now, cos x = $\sqrt{1 - \sin^2 x}$

= $\sqrt{1 - \frac{9}{25}}$

= $\sqrt{\frac{16}{25}} = \frac{4}{5}$

and sin y = $\sqrt{1 - \cos^2 y}$

= $\sqrt{1 - \frac{144}{169}}$

= $\sqrt{\frac{25}{169}} = \frac{5}{13}$

We have to prove, that, x + y = z

Now, sin(x + y) = sin x cos y + cos x sin y

= $\left(\frac{3}{5}\right)\left(\frac{12}{13}\right) + \left(\frac{4}{5}\right)\left(\frac{5}{13}\right)$

= $\frac{36}{65} + \frac{20}{65} = \frac{56}{65}$

∴ sin(x + y) = sinz

∴ x + y = z

Hence, $\sin^{-1}\left(\frac{3}{5}\right)+\cos^{-1}\left(\frac{12}{13}\right)=\sin^{-1}\left(\frac{56}{65}\right)$.

Exercise 3.3 | Q 3.4 | Page 103

Prove the following:

$$\cos^{-1}\left(\frac{3}{5}\right)+\cos^{-1}\left(\frac{4}{5}\right)=\frac{\pi}{2}$$

Solution:

Let $\cos^{-1}\left(\frac{3}{5}\right)$ = x

∴ cos x = $\frac{3}{5}$, where $0<x<\frac{\pi}{2}$

∴ sin x > 0

Now,

sin x = $\sqrt{1-\cos^2 x}$

= $\sqrt{1-\frac{9}{25}}$

= $\sqrt{\frac{16}{25}}$

= $\frac{4}{5}$

∴ x = $\sin^{-1}\left(\frac{4}{5}\right)$

$$\therefore \cos^{-1}\left(\frac{3}{5}\right) = \sin^{-1}\left(\frac{4}{5}\right) \quad ...(1)$$

L.H.S. = $\cos^{-1}\left(\frac{3}{5}\right) + \cos^{-1}\left(\frac{4}{5}\right)$

$= \sin^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{4}{5}\right)$ ...[By (1)]

$= \frac{\pi}{2} \qquad \ldots\left[\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right]$

= R.H.S.

Exercise 3.3 | Q 3.5 | Page 103

Prove the following:

$$\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{4}$$

Solution:

L.H.S. = $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)$

$= \tan^{-1}\left[\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}\right]$

$= \tan^{-1}\left(\frac{3+2}{6-1}\right)$

$= \tan^{-1}(1)$

$= \tan^{-1}\left(\tan\ \frac{\pi}{4}\right)$

$= \frac{\pi}{4}$

= R.H.S.

Exercise 3.3 | Q 3.6 | Page 103

Prove the following:

$$2\tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}\left(\frac{3}{4}\right)$$

Solution:

L.H.S. = $2\tan^{-1}\left(\frac{1}{3}\right)$

$= \tan^{-1}\left[\frac{2\left(\frac{1}{3}\right)}{1-\left(\frac{1}{3}\right)^2}\right] \ldots \left[\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$

= $\tan^{-1}\left[\frac{\left(\frac{2}{3}\right)}{1-\frac{1}{9}}\right]$

= $\tan^{-1}\left(\frac{2}{3}\times\frac{9}{8}\right)$

= $\tan^{-1}\left(\frac{3}{4}\right)$

= R.H.S.

Alternative Method:

L.H.S. = $2\tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{3}\right)$

$$= \tan^{-1}\left[\frac{\frac{1}{3}+\frac{1}{3}}{1-\frac{1}{3}\times\frac{1}{3}}\right]$$

$$= \tan^{-1}\left(\frac{3+3}{9-1}\right)$$

$$= \tan^{-1}\left(\frac{6}{8}\right)$$

$$= \tan^{-1}\left(\frac{3}{4}\right)$$

= R.H.S.

Exercise 3.3 | Q 3.7 | Page 103

Prove the following:

$$\tan^{-1}\left[\frac{\cos\theta+\sin\theta}{\cos\theta - \sin\theta}\right] = \frac{\pi}{4}+\theta, \quad \text{if} \quad \theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$$

Solution:

$$\text{L.H.S.} = \tan^{-1}\left[\frac{\cos\theta+\sin\theta}{\cos\theta - \sin\theta}\right]$$

$$= \tan^{-1}\left[\frac{\frac{\cos\theta}{\cos\theta}+\frac{\sin\theta}{\cos\theta}}{\frac{\cos\theta}{\cos\theta}-\frac{\sin\theta}{\cos\theta}}\right]$$

$$= \tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)$$

$$= \tan^{-1}\left[\frac{\tan\frac{\pi}{4}+\tan\theta}{1-\tan\frac{\pi}{4}\tan\theta}\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\theta\right)\right]$$

$$= \frac{\pi}{4}+\theta \qquad \ldots[\because \tan^{-1}(\tan\theta) = \theta]$$

= R.H.S.

Exercise 3.3 | Q 3.8 | Page 103

Prove the following:

$$\tan^{-1}\left[\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right] = \frac{\theta}{2}, \text{ if } \theta \in (-\pi, \pi).$$

Solution:

$$\frac{1-\cos\theta}{1+\cos\theta} = \frac{2\sin^2\left(\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\theta}{2}\right)}$$

$$= \tan^2\left(\frac{\theta}{2}\right)$$

$$\therefore \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \sqrt{\tan^2\left(\frac{\theta}{2}\right)}$$

$$= \tan\left(\frac{\theta}{2}\right)$$

$$\therefore \text{L.H.S.} = \tan^{-1}\left[\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\theta}{2}\right)\right]$$

$$= \frac{\theta}{2} \qquad \ldots[\because \tan^{-1}(\tan\theta) = \theta]$$

= R.H.S.

MISCELLANEOUS EXERCISE 3 [PAGES 106 - 108]

Miscellaneous exercise 3 | Q 1.01 | Page 106

Select the correct option from the given alternatives:

The principal solutions of equation sin θ = $-\frac{1}{2}$ are

Options

$\frac{5\pi}{6}, \frac{\pi}{6}$

$\frac{7\pi}{6}, \frac{11\pi}{6}$

$\frac{\pi}{6}, \frac{7\pi}{6}$

$\frac{7\pi}{6}, \frac{\pi}{3}$

Solution:

The principal solutions of equation sin θ = $-\frac{1}{2}$ are $\frac{7\pi}{6}, \frac{11\pi}{6}$.

Miscellaneous exercise 3 | Q 1.02 | Page 106

Select the correct option from the given alternatives:

The principal solutions of equation cot θ = $\sqrt{3}$ are

Options

$\frac{\pi}{6}, \frac{7\pi}{6}$

$\frac{\pi}{6}, \frac{5\pi}{6}$

$\frac{\pi}{6}, \frac{8\pi}{6}$

$\frac{7\pi}{6}, \frac{\pi}{3}$

Solution:

The principal solutions of equation cot θ = $\sqrt{3}$ are $\frac{\pi}{6}, \frac{7\pi}{6}$

Miscellaneous exercise 3 | Q 1.03 | Page 106

Select the correct option from the given alternatives:

The general solution of sec x = $\sqrt{2}$ is

Options

$2n\pi \pm \frac{\pi}{4}, n \in Z$

$2n\pi \pm \frac{\pi}{2}, n \in Z$

$n\pi \pm \frac{\pi}{2}, n \in Z$

$2n\pi \pm \frac{\pi}{3}, n \in Z$

Solution:

The general solution of sec x = √2 is $2n\pi \pm \frac{\pi}{4}, n \in Z$.

Miscellaneous exercise 3 | Q 1.04 | Page 106

Select the correct option from the given alternatives:

If cos pθ = cos qθ, p ≠ q, then,

Options

θ = $\frac{2n\pi}{p \pm q}$

θ = 2nπ

θ - 2nπ ± p

θ = nπ ± q

Solution:

If cos pθ = cos qθ, p ≠ q, then, $\theta = \dfrac{2n\pi}{p \pm q}$

Miscellaneous exercise 3 | Q 1.05 | Page 106

Select the correct option from the given alternatives:

If polar coordinates of a point are $\left(2, \dfrac{\pi}{4}\right)$, then its cartesian coordinates are

Options

$\left(2, \sqrt{2}\right)$

$\left(\sqrt{2}, 2\right)$

(2, 2)

$\left(\sqrt{2}, \sqrt{2}\right)$

Solution:

If polar coordinates of a point are $\left(2, \dfrac{\pi}{4}\right)$, then its cartesian coordinates are **(√2, √2)**.

Miscellaneous exercise 3 | Q 1.06 | Page 106

Select the correct option from the given alternatives:

If $\sqrt{3}$cos x - sin x = 1, then general value of x is

Options

$$2\text{n}\pi \pm \frac{\pi}{3}$$

$$2\text{n}\pi \pm \frac{\pi}{6}$$

$$2\text{n}\pi \pm \frac{\pi}{3} - \frac{\pi}{6}$$

$$\text{n}\pi + (-1)^{\text{n}} \frac{\pi}{3}$$

Solution:

If √3 cos x - sin x = 1, then general value of x is $2\text{n}\pi \pm \frac{\pi}{3} - \frac{\pi}{6}$

Miscellaneous exercise 3 | Q 1.07 | Page 107

Select the correct option from the given alternatives:

In Δ ABC if ∠A = 45°, ∠B = 60°, then the ratio of its sides are

Options

$$2 : \sqrt{6} : \sqrt{3} + 1$$

$$\sqrt{2} : 2 : \sqrt{3} + 1$$

$$2\sqrt{2} : \sqrt{2} : \sqrt{3}$$

$$2 : 2\sqrt{2} : \sqrt{3} + 1$$

Solution: In Δ ABC if ∠A = 45°, ∠B = 60°, then the ratio of its sides are **2: √6: √3 + 1**.

Miscellaneous exercise 3 | Q 1.08 | Page 107

Select the correct option from the given alternatives:

In ΔABC if $c^2 + a^2 - b^2 = ac$, then ∠B = ____

Options

$\frac{\pi}{4}$

$\frac{\pi}{3}$

$\frac{\pi}{2}$

$\frac{\pi}{6}$

Solution:

In ΔABC if $c^2 + a^2 - b^2 = ac$, then ∠B = $\frac{\pi}{3}$

Miscellaneous exercise 3 | Q 1.09 | Page 107

Select the correct option from the given alternatives:

In ΔABC, ac cos B - bc cos A = _______

1. **$a^2 - b^2$**
2. $b^2 - c^2$
3. $c^2 - a^2$
4. $a^2 - b^2 - c^2$

Solution: In ΔABC, ac cos B - bc cos A = **$a^2 - b^2$.**

Miscellaneous exercise 3 | Q 1.1 | Page 107

Select the correct option from the given alternatives:

If in a triangle, the angles are in A.P. and b: c = √3: √2, then A is equal to

1. 30°
2. 60°
3. **75°**
4. 45°

Solution: If in a triangle, the angles are in A.P. and b: c = √3: √2, then A is equal to **75°.**

Miscellaneous exercise 3 | Q 1.11 | Page 107

Select the correct option from the given alternatives:

$$\cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \_\_\_\_\_\_\_\_.$$

Options

$\frac{7\pi}{6}$

$\frac{5\pi}{6}$

$\frac{\pi}{6}$

$\frac{3\pi}{2}$

Solution:

$$\cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \frac{5\pi}{6}.$$

Miscellaneous exercise 3 | Q 1.12 | Page 107

Select the correct option from the given alternatives:

The value of cot ($\tan^{-1}2x + \cot^{-1}2x$) is

1. **0**

2. 2x
3. π + 2x
4. π - 2x

Solution: The value of cot ($\tan^{-1}2x + \cot^{-1}2x$) is 0.

Miscellaneous exercise 3 | Q 1.13 | Page 107

Select the correct option from the given alternatives:

The principal value of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ is

Options

$\left(-\frac{2\pi}{3}\right)$

$\frac{4\pi}{3}$

$\frac{5\pi}{3}$

$-\frac{\pi}{3}$

Solution:

The principal value of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ is $-\frac{\pi}{3}$.

Miscellaneous exercise 3 | Q 1.14 | Page 107

Select the correct option from the given alternatives:

If $\sin^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \sin^{-1}\alpha$, then α = _____

1. **63/65**
2. 62/65
3. 61/65
4. 60/65

Solution:

$$If \sin^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \sin^{-1}\alpha, \text{then} \alpha = \frac{63}{65}.$$

Miscellaneous exercise 3 | Q 1.15 | Page 107

Select the correct option from the given alternatives:

If $\tan^{-1}(2x) + \tan^{-1}(3x) = \pi/4$, then x = _____

1. - 1
2. **16**
3. 26
4. 32

Solution:

If $\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}, \text{then x} = \frac{1}{6}$

Miscellaneous exercise 3 | Q 1.16 | Page 108

Select the correct option from the given alternatives:

$$2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \_\_\_\_$$

Options

$\tan^{-1}\left(\frac{4}{5}\right)$

$\frac{\pi}{2}$

1

$\frac{\pi}{4}$

Solution:

$$2\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)=\frac{\pi}{4}.$$

Miscellaneous exercise 3 | Q 1.17 | Page 108

Select the correct option from the given alternatives:

$\tan\left(2\tan^{-1}\left(\frac{1}{5}\right)-\frac{\pi}{4}\right)$ = _____

Options

$\frac{17}{7}$

$-\frac{17}{7}$

$\frac{7}{17}$

$-\frac{7}{17}$

Solution:

$$\tan\left(2\tan^{-1}\left(\frac{1}{5}\right)-\frac{\pi}{4}\right)=-\frac{7}{17}.$$

Miscellaneous exercise 3 | Q 1.18 | Page 108

Select the correct option from the given alternatives:

The principal value branch of $\sec^{-1}x$ is

Options

$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

$[0, \pi] - \left\{\frac{\pi}{2}\right\}$

(0, π)

$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Solution:

The principal value branch of $\sec^{-1}x$ is $[0, \pi] - \left\{\frac{\pi}{2}\right\}$

Miscellaneous exercise 3 | Q 1.19 | Page 108

Select the correct option from the given alternatives:

$\cos\left[\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{2}\right]$ = _____

Options

$\frac{1}{\sqrt{2}}$

$\frac{\sqrt{3}}{2}$

$\frac{1}{2}$

$\frac{\pi}{4}$

Solution:

$$\cos\left[\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{2}\right]=\frac{1}{\sqrt{2}}$$

Miscellaneous exercise 3 | Q 1.2 | Page 108

Select the correct option from the given alternatives:

If tan θ + tan 2θ + tan 3θ = tan θ.tan 2θ. tan 3θ, then the general value of the θ is

1. nπ
2. **nπ/6**
3. nπ±π/4
4. nπ/2

Solution: If tan θ + tan 2θ + tan 3θ = tan θ.tan 2θ. tan 3θ, then the general value of the θ is nπ/6

Miscellaneous exercise 3 | Q 1.21 | Page 108

Select the correct option from the given alternatives:

In any ΔABC, if acos B = bcos A, then the triangle is

1. equilateral triangle
2. **isosceles triangle**
3. scalene
4. right-angled

Solution: In any ΔABC, if acos B = bcos A, then the triangle is **isosceles triangle.**

MISCELLANEOUS EXERCISE 3 [PAGES 108 - 111]

Miscellaneous exercise 3 | Q 1.1 | Page 108

Find the principal solutions of the following equation:

sin 2θ = - 1/2

Solution:

sin 2θ = $-\frac{1}{2}$

Since, θ ∈ (0, 2π), 2θ ∈ (0, 4π)

$$\sin 2\theta = -\frac{1}{2} = -\sin\frac{\pi}{6} = \sin\left(\pi + \frac{\pi}{6}\right) = \sin\left(2\pi - \frac{\pi}{6}\right)$$

$= \sin\left(3\pi + \frac{\pi}{6}\right) = \sin\left(4\pi - \frac{\pi}{6}\right)$ ....[∵ sin (π + θ) = sin(2π - θ) = sin(3π + θ) = sin(4π - θ) = - sin θ]

$$\therefore \sin 2\theta = \sin\frac{7\pi}{6} = \sin\frac{11\pi}{6} = \sin\frac{19\pi}{6} = \sin\frac{23\pi}{6}$$

$$\therefore 2\theta = \frac{7\pi}{6} \text{ or } 2\theta = \frac{11\pi}{6} \text{ or } 2\theta = \frac{19\pi}{6} \text{ or } 2\theta = \frac{23\pi}{6}$$

$$\therefore \theta = \frac{7\pi}{12} \text{ or } \theta = \frac{11\pi}{12} \text{ or } \theta = \frac{19\pi}{12} \text{ or } \theta = \frac{23\pi}{12}$$

Hence, the required principal solutions are

$\left\{\frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}\right\}$.

Miscellaneous exercise 3 | Q 1.2 | Page 108

Find the principal solutions of the following equation:

tan 3θ = - 1

Solution:

tan 3θ = - 1

Since, θ ∈ (0, 2π), 3θ ∈ (0, 6π)

$$\tan 3\theta = -1 = -\tan\frac{\pi}{4} = \tan\left(\pi - \frac{\pi}{4}\right)$$

$$= \tan\left(2\pi - \frac{\pi}{4}\right) = \tan\left(3\pi - \frac{\pi}{4}\right)$$

$$= \tan\left(4\pi - \frac{\pi}{4}\right) = \tan\left(5\pi - \frac{\pi}{4}\right)$$

$= \tan\left(6\pi - \frac{\pi}{4}\right)$ ......[∵ tan (π - θ) = tan(2π - θ) = tan(3π - θ) = tan(4π - θ) = tan (5π - θ) = tan (6π - θ) = - tan θ]

tan 3θ = - 1

Since, θ ∈ (0, 2π), 3θ ∈ (0, 6π)

$$\tan 3\theta = -1 = -\tan\frac{\pi}{4} = \tan\left(\pi - \frac{\pi}{4}\right)$$

$$= \tan\left(2\pi - \frac{\pi}{4}\right) = \tan\left(3\pi - \frac{\pi}{4}\right)$$

$$= \tan\left(4\pi - \frac{\pi}{4}\right) = \tan\left(5\pi - \frac{\pi}{4}\right)$$

Miscellaneous exercise 3 | Q 1.3 | Page 108

Find the principal solutions of the following equation:

cot θ = 0

Solution:

cot θ = 0

Since θ ∈ (0, 2π)

∴ cot θ = 0 = $\cot\frac{\pi}{2} = \cot\left(\pi + \frac{\pi}{2}\right)$ ... [∴ cot (π + θ) = cot θ]

∴ cot θ = $\cot\frac{\pi}{2} = \cot\frac{3\pi}{2}$

$\therefore \theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$

Hence, the required principal solutions are $\left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$.

Miscellaneous exercise 3 | Q 2.1 | Page 108

Find the principal solutions of the following equation:

sin 2θ = - 1/√2.

Solution:

$$\left\{\frac{5\pi}{8}, \frac{7\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}\right\}$$

Miscellaneous exercise 3 | Q 2.2 | Page 108

Find the principal solutions of the following equation:

tan 5θ = -1

Solution:

$$\left\{\frac{3\pi}{20}, \frac{7\pi}{20}, \frac{11\pi}{20}, \frac{15\pi}{20}, \frac{19\pi}{20}, \frac{23\pi}{20}, \frac{27\pi}{20}, \frac{31\pi}{20}, \frac{35\pi}{20}, \frac{39\pi}{20}\right\}$$

Miscellaneous exercise 3 | Q 2.3 | Page 108

Find the principal solutions of the following equation:

cot 2θ = 0.

Solution:

$$\left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}.$$

Miscellaneous exercise 3 | Q 3.1 | Page 109

State whether the following equation has a solution or not?

cos 2θ = 1/3

Solution:

$\cos 2\theta = \frac{1}{3}$

since $\frac{1}{3} \leq \cos\theta \leq 1$ for any θ

$\cos 2\theta = \frac{1}{3}$ has solution.

Miscellaneous exercise 3 | Q 3.2 | Page 109

State whether the following equation has a solution or not?

$\cos^2\theta = -1$.

Solution: $\cos^2\theta = -1$

This is not possible because $\cos^2\theta \geq 0$ for any θ.

∴ $\cos^2\theta = -1$ does not have any solution.

Miscellaneous exercise 3 | Q 3.3 | Page 109

State whether the following equation has a solution or not?

$2\sin\theta = 3$

Solution: $2\sin\theta = 3$

∴ $\sin\theta = 3/2$

This is not possible because $-1 \leq \sin\theta \leq 1$ for any θ.
∴ $2\sin\theta = 3$ does not have any solution.

Miscellaneous exercise 3 | Q 3.4 | Page 109

State whether the following equation has a solution or not?

$3\sin\theta = 5$.

Solution: ∴ $\sin\theta = 5/3$

This is not possible because $-1 \leq \sin\theta \leq 1$ for any θ.

∴ $3\sin\theta = 5$ does not have any solution.

Miscellaneous exercise 3 | Q 4.1 | Page 109

Find the general solutions of the following equation:

$\tan\theta = -\sqrt{3}$

Solution:

The general solution of tan θ = tan α is

$\theta = n\pi + \alpha, n \in Z.$

Now, $\tan\theta = -\sqrt{3}$

$\therefore \tan\theta = -\tan\frac{\pi}{3}$ .... $\left[\because \tan\frac{\pi}{3} = \sqrt{3}\right]$

$\therefore \tan\theta = \tan\left(\pi - \frac{\pi}{3}\right)$ ...$[\because \tan(\pi - \theta) = -\tan\theta]$

$\therefore \tan\theta = \tan\frac{2\pi}{3}$

∴ the required general solution is

$\therefore \theta = n\pi + \frac{2\pi}{3}, n \in Z$

Miscellaneous exercise 3 | Q 4.2 | Page 109

Find the general solutions of the following equation:

$\tan^2\theta=3$

Solution: The general solution of $\tan^2\theta = \tan^2\alpha$ is $\theta = n\pi \pm \alpha, n \in Z$.

Now, $\tan^2\theta = 3 = \left(\sqrt{3}\right)^2$

$\therefore \tan^2\theta = \left(\tan\frac{\pi}{3}\right)^2$ .... $\left[\because \tan\frac{\pi}{3} = \sqrt{3}\right]$

$\therefore \tan^2\theta = \tan^2\frac{\pi}{3}$

∴ the required general solution is

$\therefore \theta = n\pi \pm \frac{\pi}{3}, n \in Z.$

Miscellaneous exercise 3 | Q 4.3 | Page 109

Find the general solutions of the following equation:

sin θ - cos θ = 1

Solution: sin θ - cos θ = 1

cos θ - sin θ = - 1

Dividing both sides by $\sqrt{(1)^2 + (-1)^2} = \sqrt{2}$, we get

$$\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta = -\frac{1}{\sqrt{2}}$$

$$\therefore \cos\frac{\pi}{4}\cos\theta - \sin\frac{\pi}{4}\sin\theta = -\cos\frac{\pi}{4}$$

$$\therefore \cos\left(\theta - \frac{\pi}{4}\right) = \cos\left(\pi - \frac{\pi}{4}\right) \quad .... [\because \cos(\pi - \theta) = -\cos\theta]$$

$$\therefore \cos\left(\theta - \frac{\pi}{4}\right) = \cos\frac{3\pi}{4} \quad ...(1)$$

The general solution of cos θ = cos α is

θ = 2nπ ± α, n ∈ Z

∴ the general solution of (1) is given by

$$\theta - \frac{\pi}{4} = 2\text{n}\pi \pm \frac{3\pi}{4}, \text{n} \in \mathbf{Z}$$

Taking positive sign, we get

$$\theta - \frac{\pi}{4} = 2\text{n}\pi + \frac{3\pi}{4}, \text{n} \in \mathbf{Z}$$

$\therefore \theta = 2n\pi + \pi = (2n + 1)\pi, n \in Z$

Taking negative sign, we get

$$\theta - \frac{\pi}{4} = 2n\pi - \frac{3\pi}{4}, n \in Z$$

$\therefore \theta = 2n\pi - \frac{\pi}{2}, n \in Z$

$\therefore$ the required general solution is

$\theta = (2n + 1)\pi$, $n \in Z$ or $\theta = 2n\pi - \frac{\pi}{2}, n \in Z$

Miscellaneous exercise 3 | Q 4.4 | Page 109

Find the general solutions of the following equation:

$\sin^2 \theta - \cos^2 \theta = 1$

Solution: $\sin^2 \theta - \cos^2 \theta = 1$

$\therefore \cos^2 \theta - \sin^2 \theta = -1$

$\therefore \cos 2\theta = \cos \pi$ .....(1)

The general solution of $\cos \theta = \cos \alpha$ is

$\theta = 2n\pi \pm \alpha, n \in Z$.

$\therefore$ the general solution of (1) is given by

$2\theta = 2n\pi \pm \pi, n \in Z$.

$\therefore \theta = n\pi \pm \pi/2, n \in Z$

Miscellaneous exercise 3 | Q 5 | Page 109

In Δ ABC, prove that $\cos\left(\frac{A-B}{2}\right) = \left(\frac{a+b}{c}\right)\sin\frac{C}{2}$.

Solution:

By the sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

∴ a = k sin A, b = k sin B, c = k sin C

RHS = $\left(\frac{a+b}{c}\right) \sin \frac{C}{2}$

$$= \left(\frac{k \sin A + k \sin B}{k \sin C}\right) \sin \frac{C}{2}$$

$$= \left(\frac{\sin A + \sin B}{\sin C}\right) \sin \frac{C}{2}$$

$$= \frac{2 \sin\left(\frac{A+B}{2}\right). \cos \frac{A-B}{2}}{2\sin \frac{C}{2} . \cos \frac{C}{2}} . \sin \frac{C}{2}$$

$$= \frac{\sin \frac{A+B}{2} . \cos \frac{A-B}{2}}{\cos \frac{C}{2}}$$

$$= \frac{\sin\left(\frac{\pi}{2} - \frac{C}{2}\right). \cos \frac{A-B}{2}}{\cos \frac{C}{2}} \quad ...[\because A + B + C = \pi]$$

$$= \frac{\cos \frac{C}{2} . \cos \frac{A-B}{2}}{\cos \frac{C}{2}}$$

$$= \cos \left(\frac{A-B}{2}\right)$$

= LHS

Miscellaneous exercise 3 | Q 6 | Page 109

With the usual notations, prove that $\frac{\sin(A-B)}{\sin(A+B)} = \frac{a^2-b^2}{c^2}$

Solution:

By the sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$\therefore$ a = k sin A, b = k sin B, c = k sin C

RHS = $\frac{a^2-b^2}{c^2} = \frac{k^2\sin^2 A - k^2\sin^2 B}{k^2\sin^2 C}$

$$= \frac{\sin^2 A - \sin^2 B}{\sin^2 C}$$

$$= \frac{(\sin A + \sin B)(\sin A - \sin B)}{[\sin\{\pi - (A+B)\}]^2}$$ .....[$\because$ A + B + C = π]

$$= \frac{2\sin\left(\frac{A+B}{2}\right).\cos\left(\frac{A-B}{2}\right) \times 2\cos\left(\frac{A+B}{2}\right).\sin\left(\frac{A-B}{2}\right)}{\sin^2(A+B)}$$

$$= \frac{2\sin\left(\frac{A+B}{2}\right).\cos\left(\frac{A+B}{2}\right) \times 2\sin\left(\frac{A-B}{2}\right).\cos\left(\frac{A-B}{2}\right)}{\sin^2(A+B)}$$

$$= \frac{\sin(A+B).\sin(A-B)}{\sin^2(A+B)}$$

$$= \frac{\sin(A - B)}{\sin(A+B)}$$ = LHS

Miscellaneous exercise 3 | Q 7 | Page 109

In ΔABC, prove that $(a - b)^2\cos^2\frac{C}{2} + (a+b)^2\sin^2\frac{C}{2} = c^2$

Solution:

LHS = $(a - b)^2 \cos^2 \frac{C}{2} + (a + b)^2 \sin^2 \frac{C}{2} = c^2$

$$= \left(a^2 + b^2 - 2ab\right) \cos^2 \frac{C}{2} + \left(a^2 + b^2 + 2ab\right) \sin^2 \frac{C}{2}$$

$$= \left(a^2 + b^2\right) \cos^2 \frac{C}{2} - 2ab \cos^2 \frac{C}{2} + \left(a^2 + b^2\right) \sin^2 \frac{C}{2} + 2ab \sin^2 \frac{C}{2}$$

$$= \left(a^2 + b^2\right) \left(\cos^2 \frac{C}{2} + \sin^2 \frac{C}{2}\right) - 2ab \left(\cos^2 \frac{C}{2} - \sin^2 \frac{C}{2}\right)$$

= $a^2 + b^2 - 2ab \cos C$

= c^2 = RHS

Miscellaneous exercise 3 | Q 8 | Page 109

In Δ ABC, if cos A = sin B - cos C then show that it is a right-angled triangle.

Solution:

cos A = sin B - cos C

∴ cos A + cos C = sin B

$$\therefore 2 \cos\left(\frac{A + C}{2}\right) . \cos\left(\frac{A - C}{2}\right) = \sin B$$

$$\therefore 2 \cos\left(\frac{\pi}{2} - \frac{B}{2}\right) . \cos\left(\frac{A - C}{2}\right) = \sin B \text{ .....}[\because A + B + C = \pi]$$

$$\therefore 2 \sin \frac{B}{2} . \cos\left(\frac{A - C}{2}\right) = 2 \sin \frac{B}{2} . \cos \frac{B}{2}$$

$$\therefore \cos \left(\frac{A - C}{2}\right) = \cos \frac{B}{2}$$

$$\therefore \frac{A - C}{2} = \frac{B}{2}$$

∴ A - C = B

$\therefore$ A = B + C

$\therefore$ A + B + C = 180° gives

$\therefore$ A + A = 180°

$\therefore$ 2A = 180°

$\therefore$ A = 90°

$\therefore$ Δ ABC is a right angled triangle.

Miscellaneous exercise 3 | Q 9 | Page 109

If $\dfrac{\sin A}{\sin C} = \dfrac{\sin(A - B)}{\sin(B - C)}$, then show that a^2, b^2, c^2 are in A.P.

Solution: By sine rule,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k$$

$\therefore$ sin A = ka, sin B = kb, sin C = kc

Now, $\dfrac{\sin A}{\sin C} = \dfrac{\sin(A - B)}{\sin(B - C)}$

$\therefore$ sin A . sin (B - C) = sin C. sin (A - B)

$\therefore$ sin [π - (B + C)]. sin (B - C)

= sin [π - (A + B)]. sin(A - B) .....[$\because$ A + B + C = π]

$\therefore$ sin (B + C). sin (B - C) = sin (A + B). sin (A - B)

$\therefore$ $\sin^2 B - \sin^2 C = \sin^2 A - \sin^2 B$

$\therefore$ $2 \sin^2 B = \sin^2 A + \sin^2 C$

$\therefore$ $2k^2b^2 = k^2a^2 + k^2c^2$

$\therefore$ $2b^2 = a^2 + c^2$

Hence, a^2, b^2, c^2 are in A.P.

Miscellaneous exercise 3 | Q 10 | Page 109

Solve the triangle in which a = (√3+1), b = (√3-1) and ∠C = 60°.

Solution:

Given: a = $\left(\sqrt{3}+1\right)$, b = $\left(\sqrt{3}-1\right)$ and $\angle$C = 60°

By cosine rule,

$c^2 = a^2 + b^2 - 2ab\cos C$

$$= \left(\sqrt{3}+1\right)^2 + \left(\sqrt{3}-1\right)^2 - 2\left(\sqrt{3}+1\right)\left(\sqrt{3}-1\right)\cos 60^\circ$$

$$= 3 + 1 + 2\sqrt{3} + 3 + 1 - 2\sqrt{3} - 2(3 - 1)\left(\frac{1}{2}\right)$$

$= 8 - 2 = 6$

$\therefore c = \sqrt{6}$ ....[$\because c > 0$]

By sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\therefore \frac{\sqrt{3}+1}{\sin A} = \frac{\sqrt{3}-1}{\sin B} = \frac{\sqrt{6}}{\sin 60^\circ}$$

$$\therefore \frac{\sqrt{3}+1}{\sin A} = \frac{\sqrt{3}-1}{\sin B} = \frac{\sqrt{6}}{\sqrt{3}/2} = 2\sqrt{2}$$

$$\therefore \sin A = \frac{\sqrt{3}+1}{2\sqrt{2}} \text{ and } \sin B = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$\therefore \sin A = \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} \text{ and } \sin B = \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$$

$$\therefore \sin A = \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{2} \times \frac{1}{\sqrt{2}}$$

$\therefore$ and sin B = $\dfrac{\sqrt{3}}{2} \times \dfrac{1}{\sqrt{2}} - \dfrac{1}{2} \times \dfrac{1}{\sqrt{2}}$

∴ sin A = sin 60° cos 45° + cos 60° sin 45° and
sin B = sin 60° cos 45° - cos 60° sin 45°

∴ sin A = sin (60° + 45°) = sin 105°

and sin B = sin (60° - 45°) = sin 15°

∴ A = 105° and B = 15°

Hence, A = 105°, B = 15° and C = √6 units

Miscellaneous exercise 3 | Q 11.1 | Page 109

In any Δ ABC, prove the following:

a sin A - b sin B = c sin (A - B)

Solution:

By sine rule,

$$\frac{\mathrm{a}}{\sin \mathrm{A}} = \frac{\mathrm{b}}{\sin \mathrm{B}} = \frac{\mathrm{c}}{\sin \mathrm{C}} = \mathrm{k}$$

∴ a = k sin A, b = k sin B, c = k sin C

LHS = a sin A - b sin B

= k sin A. sin A - k sin B. sin B

= k ($\sin^2$ A - $\sin^2$ B)

= k (sin A + sin B)(sin A - sin B)

$$= \mathrm{k} \times 2\sin\left(\frac{\mathrm{A}+\mathrm{B}}{2}\right).\cos\left(\frac{\mathrm{A}-\mathrm{B}}{2}\right) \times 2\cos\left(\frac{\mathrm{A}+\mathrm{B}}{2}\right).\sin\left(\frac{\mathrm{A}-\mathrm{B}}{2}\right)$$

$$= \mathrm{k} \times 2\sin\left(\frac{\mathrm{A}+\mathrm{B}}{2}\right).\cos\left(\frac{\mathrm{A}+\mathrm{B}}{2}\right) \times 2\sin\left(\frac{\mathrm{A}-\mathrm{B}}{2}\right).\cos\left(\frac{\mathrm{A}-\mathrm{B}}{2}\right)$$

= k × sin (A + B) × sin (A - B)

= k sin (π - C). sin (A - B) … [∴ A + B + C = π]

= k sin C. sin (A - B)

= c sin (A - B)

= RHS.

Miscellaneous exercise 3 | Q 11.2 | Page 109

In any Δ ABC, prove the following:

$$\frac{c - b\cos A}{b - c\cos A} = \frac{\cos B}{\cos C}$$

Solution:

$$\text{LHS} = \frac{c - b\cos A}{b - c\cos A}$$

$$= \frac{c - b\left(\frac{b^2+c^2-a^2}{2bc}\right)}{b - c\left(\frac{b^2+c^2-a^2}{2bc}\right)}$$

$$= \frac{c - \left(\frac{b^2+c^2-a^2}{2c}\right)}{b - \left(\frac{b^2+c^2-a^2}{2b}\right)}$$

$$= \frac{\frac{2c^2-b^2-c^2+a^2}{2c}}{\frac{2b^2-b^2-c^2+a^2}{2b}}$$

$$= \frac{\left(\frac{c^2+a^2-b^2}{2c}\right)}{\left(\frac{a^2+b^2-c^2}{2b}\right)}$$

$$= \frac{\left(\frac{c^2+a^2-b^2}{2ca}\right)}{\left(\frac{a^2+b^2-c^2}{2ab}\right)}$$

$$= \frac{\cos B}{\cos C}$$

= RHS.

Miscellaneous exercise 3 | Q 11.3 | Page 109

In any Δ ABC, prove the following:

$a^2 \sin (B - C) = (b^2 - c^2) \sin A$.

Solution:

By sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$\therefore$ a = k sin A, b = k sin B, c = k sin C

RHS = $(b^2 - c^2) \sin A$

$= (k^2 \sin^2 B - k^2 \sin^2 C)\sin A$

$= k^2 (\sin^2 B - \sin^2 C) \sin A$

$= k^2 (\sin B + \sin C)(\sin B - \sin C) \sin A$

$$= k^2 \times 2\sin\left(\frac{B + C}{2}\right).\cos\left(\frac{B - C}{2}\right) \times 2\cos\left(\frac{B + C}{2}\right).\sin\left(\frac{B - C}{2}\right) \times \sin A$$

$$= k^2 \times 2\sin\left(\frac{B + C}{2}\right).\cos\left(\frac{B + C}{2}\right) \times 2\sin\left(\frac{B - C}{2}\right).\cos\left(\frac{B - C}{2}\right) \times \sin A$$

$= k^2 \times \sin(B + C) \times \sin (B - C) \times \sin A$

$= k^2 . \sin (\pi - A). \sin (B - C). \sin A$ ....[$\because A + B + C = \pi$]

$= k^2. \sin A. \sin (B - C). \sin A$

$= (k \sin A)^2. \sin(B - C)$

$= a^2 \sin (B - C)$

= LHS

Miscellaneous exercise 3 | Q 11.4 | Page 109

In any Δ ABC, prove the following:

$ac \cos B - bc \cos A = a^2 - b^2$

Solution: LHS = $ac \cos B - bc \cos A = a^2 - b^2$

LHS = $ac \cos B - bc \cos A = a^2 - b^2$

$$= ac\left(\frac{c^2 + a^2 - b^2}{2ca}\right) - bc\left(\frac{b^2 + c^2 - a^2}{2bc}\right)$$

$$= \frac{1}{2}\left(c^2 + a^2 - b^2\right) - \frac{1}{2}\left(b^2 + c^2 - a^2\right)$$

$$= \frac{1}{2}\left(c^2 + a^2 - b^2 - b^2 - c^2 + a^2\right)$$

$$= \frac{1}{2}\left(2a^2 - 2b^2\right)$$

$$= a^2 - b^2$$

= RHS

Miscellaneous exercise 3 | Q 11.5 | Page 109

In any Δ ABC, prove the following:

$$\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2 + b^2 + c^2}{2abc}$$

Solution:

$$\text{LHS} = \frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$$

$$= \frac{\left(\frac{b^2+c^2-a^2}{2bc}\right)}{a} + \frac{\left(\frac{c^2+a^2-b^2}{2ca}\right)}{b} + \frac{\left(\frac{a^2+b^2-c^2}{2ab}\right)}{c}$$

$$= \frac{b^2+c^2-a^2}{2abc} + \frac{c^2+a^2-b^2}{2abc} + \frac{a^2+b^2-c^2}{2abc}$$

$$= \frac{b^2+c^2-a^2+c^2+a^2-b^2+a^2+b^2-c^2}{2abc}$$

$$= \frac{a^2+b^2+c^2}{2abc}$$

= RHS

Miscellaneous exercise 3 | Q 11.6 | Page 109

In any Δ ABC, prove the following:

$$\frac{\cos 2A}{a^2} - \frac{\cos 2B}{b^2} = \frac{1}{a^2} - \frac{1}{b^2}$$

Solution:

By sine rule,

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\therefore \frac{\sin^2 A}{a^2} = \frac{\sin^2 B}{b^2} \quad \ldots(1)$$

$$\text{LHS} = \frac{\cos 2A}{a^2} - \frac{\cos 2B}{b^2}$$

$$= \frac{1-2\sin^2 A}{a^2} - \frac{1-2\sin^2 B}{b^2}$$

$$= \frac{1}{a^2} - \frac{2\sin^2 A}{a^2} - \frac{1}{b^2} + \frac{2\sin^2 B}{b^2}$$

$$= \frac{1}{a^2} - \frac{1}{b^2} - 2\left(\frac{\sin^2 A}{a^2} - \frac{\sin^2 B}{b^2}\right)$$

$$= \frac{1}{a^2} - \frac{1}{b^2} - 2\left(\frac{\sin^2 B}{b^2} - \frac{\sin^2 B}{b^2}\right)$$ .....[By (1)]

$$= \frac{1}{a^2} - \frac{1}{b^2} - 2 \times 0$$

$$= \frac{1}{a^2} - \frac{1}{b^2}$$

= RHS

Miscellaneous exercise 3 | Q 11.7 | Page 109

In any Δ ABC, prove the following:

$$\frac{b - c}{a} = \frac{\tan \frac{B}{2} - \tan \frac{C}{2}}{\tan \frac{B}{2} + \tan \frac{C}{2}}$$

Solution:

By sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

∴ a = k sin A, b = k sin B, c = k sin C

LHS = $\frac{b - c}{a}$

$$= \frac{k \sin B - k \sin C}{k \sin A}$$

$$= \frac{\sin B - \sin C}{\sin A}$$

$$= \frac{\sin B - \sin C}{\sin\{\pi - (B + C)\}} \quad .... [\because A + B + C = \pi]$$

$$= \frac{\sin B - \sin C}{\sin(B + C)}$$

$$= \frac{2\cos\left(\frac{B+C}{2}\right).\sin\left(\frac{B-C}{2}\right)}{2\sin\left(\frac{B+C}{2}\right).\cos\left(\frac{B+C}{2}\right)}$$

$$= \frac{\sin\ \frac{B-C}{2}}{\sin\ \frac{B+C}{2}}$$

$$= \frac{\sin\left(\frac{B}{2} - \frac{C}{2}\right)}{\sin\left(\frac{B}{2} + \frac{C}{2}\right)}$$

$$= \frac{\sin\ \frac{B}{2}\cos\ \frac{C}{2} - \cos\ \frac{B}{2}\sin\ \frac{C}{2}}{\sin\ \frac{B}{2}\cos\ \frac{C}{2} + \cos\ \frac{B}{2}\ \sin\ \frac{C}{2}}$$

$$= \frac{\frac{\sin\ \frac{B}{2}\cos\ \frac{C}{2}}{\cos\ \frac{B}{2}\cos\ \frac{C}{2}} - \frac{\cos\ \frac{B}{2}\sin\ \frac{C}{2}}{\cos\ \frac{B}{2}\cos\ \frac{C}{2}}}{\frac{\sin\ \frac{B}{2}\cos\ \frac{C}{2}}{\cos\ \frac{B}{2}\cos\ \frac{C}{2}} + \frac{\cos\ \frac{B}{2}\sin\ \frac{C}{2}}{\cos\ \frac{B}{2}\cos\ \frac{C}{2}}}$$

$$= \frac{\frac{\sin\ \frac{B}{2}}{\cos\ \frac{B}{2}} - \frac{\sin\ \frac{C}{2}}{\cos\ \frac{C}{2}}}{\frac{\sin\ \frac{B}{2}}{\cos\ \frac{C}{2}} + \frac{\sin\ \frac{C}{2}}{\cos\ \frac{C}{2}}}$$

$$= \frac{\tan\ \frac{B}{2} - \tan\ \frac{C}{2}}{\tan\ \frac{B}{2} + \tan\ \frac{C}{2}}$$

= RHS.

Miscellaneous exercise 3 | Q 12 | Page 109

In Δ ABC, if a, b, c are in A.P., then show that cot A/2,cot B/2,cot C/2 are also in A.P.

Solution: a, b, c are in A.P.

∴ 2b = a + c   ....(1)

Now,

$$\cot\frac{A}{2}+\cot\frac{C}{2}$$

$$=\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}}+\frac{\cos\frac{C}{2}}{\sin\frac{C}{2}}$$

$$=\frac{\cos\frac{A}{2}.\sin\frac{C}{2}+\sin\frac{A}{2}.\cos\frac{C}{2}}{\sin\frac{A}{2}.\sin\frac{C}{2}}$$

$$=\frac{\sin\left(\frac{A}{2}+\frac{C}{2}\right)}{\sin\frac{A}{2}.\sin\frac{C}{2}}$$

$$=\frac{\sin\left(\frac{\pi}{2}-\frac{B}{2}\right)}{\sqrt{\frac{(s-b)(s-c)}{bc}}.\sqrt{\frac{(s-a)(s-b)}{ab}}} \quad .....[\because A+B+C=\pi]$$

$$=\frac{\cos\frac{B}{2}}{\left(\frac{s-b}{b}\right).\sqrt{\frac{(s-c)(s-a)}{ca}}}$$

$$=\frac{b\cos\frac{B}{2}}{(s-b).\sin\frac{B}{2}}$$

$$=\frac{b}{s-b}.\cot\frac{B}{2}$$

$$=\frac{b}{\left(\frac{a+b+c}{2}-b\right)}.\cot\frac{B}{2} \quad ....[\because 2s=a+b+c]$$

$$=\left(\frac{2b}{a+c-b}\right).\cot\frac{B}{2}$$

$$=\frac{2b}{(2b-b)}.\cot\frac{B}{2} \quad ....[\text{By } (1)]$$

$$= \frac{2b}{b} . \cot \frac{B}{2}$$

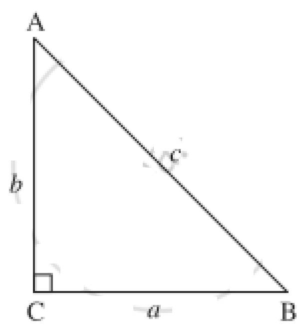
$$\therefore \cot \frac{A}{2} + \cot \frac{C}{2} = 2 \cot \frac{B}{2}$$

Hence, $\cot \frac{A}{2}, \cot \frac{B}{2}, \cot \frac{C}{2}$ are in A.P.

Miscellaneous exercise 3 | Q 13 | Page 109

In Δ ABC, if ∠C = 90°, then prove that sin (A - B) = $\frac{a^2 - b^2}{a^2 + b^2}$

Solution:



In Δ ABC, if ∠C = 90°

$\therefore c^2 = a^2 + b^2$ ....(1)

By sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin 90^\circ}$$

$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = c \quad .....[\because \sin 90^\circ = 1]$$

$\therefore$ sin A = $\frac{\mathrm{a}}{\mathrm{c}}$ and $\sin \mathrm{B} = \frac{\mathrm{b}}{\mathrm{c}}$ ....(2)

LHS = sin (A - B)

= sin A cos B - cos A sin B

$= \frac{\mathrm{a}}{\mathrm{c}} \cos \mathrm{B} - \frac{\mathrm{b}}{\mathrm{c}} \cos \mathrm{A}$ ....[By (2)]

$$= \frac{\mathrm{a}}{\mathrm{c}}\left(\frac{\mathrm{c}^2 + \mathrm{a}^2 - \mathrm{b}^2}{2\mathrm{ca}}\right) - \frac{\mathrm{b}}{\mathrm{c}}\left(\frac{\mathrm{b}^2 + \mathrm{c}^2 - \mathrm{a}^2}{2\mathrm{bc}}\right)$$

$$= \frac{\mathrm{c}^2 + \mathrm{a}^2 - \mathrm{b}^2}{2\mathrm{c}^2} - \frac{\mathrm{b}^2 + \mathrm{c}^2 - \mathrm{a}^2}{2\mathrm{c}^2}$$

$$= \frac{\mathrm{c}^2 + \mathrm{a}^2 - \mathrm{b}^2 - \mathrm{b}^2 - \mathrm{c}^2 + \mathrm{a}^2}{2\mathrm{c}^2}$$

$$= \frac{2\mathrm{a}^2 - 2\mathrm{b}^2}{2\mathrm{c}^2}$$

$$= \frac{\mathrm{a}^2 - \mathrm{b}^2}{\mathrm{c}^2}$$

$= \frac{\mathrm{a}^2 - \mathrm{b}^2}{\mathrm{a}^2 + \mathrm{b}^2}$ ...[By (1)]

= RHS.

Miscellaneous exercise 3 | Q 14 | Page 110

In Δ ABC, if $\frac{\cos \mathrm{A}}{\mathrm{a}} = \frac{\cos \mathrm{B}}{\mathrm{b}}$, then show that it is an isosceles triangle.

Solution:

Given: $\frac{\cos A}{a} = \frac{\cos B}{b}$ ....(1)

By sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = k$$

$\therefore$ a = k sin A, b = k sin B

$\therefore$ (1) gives,

$$\frac{\cos A}{k \sin A} = \frac{\cos B}{k \sin B}$$

$$\therefore \frac{\cos A}{\sin A} = \frac{\cos B}{\sin B}$$

$\therefore$ sin A cos B = cos A sin B

$\therefore$ sin A cos B - cos A sin B = 0

$\therefore$ sin (A - B) = 0 = sin 0

$\therefore$ A - B = 0

$\therefore$ A = B

$\therefore$ the triangle is an isosceles triangle.

Miscellaneous exercise 3 | Q 15 | Page 110

In Δ ABC, if $\sin^2 A + \sin^2 B = \sin^2 C$, then show that the triangle is a right-angled triangle.

Solution:

By sine rule,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k$$

$\therefore \sin A = ka, \sin B = kb, \sin C = kc$

$\therefore \sin^2 A + \sin^2 B = \sin^2 C$

$\therefore k^2a^2 + k^2b^2 = k^2c^2$

$\therefore a^2 + b^2 = c^2$

$\therefore$ Δ ABC is a rightangled triangle, rightangled at C.

Miscellaneous exercise 3 | Q 16 | Page 110

In Δ ABC, prove that $a^2 (\cos^2 B - \cos^2 C) + b^2 (\cos^2 C - \cos^2 A) + c^2 (\cos^2 A - \cos^2 B) = 0$.

Solution:

By sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$\therefore a = k \sin A, b = k \sin B, c = k \sin C$

LHS = $a^2\left(\cos^2 B - \cos^2 C\right) + b^2\left(\cos^2 C - \cos^2 A\right) + c^2\left(\cos^2 A - \cos^2 B\right)$

$= k^2 \sin^2 A\left[\left(1 - \sin^2 B\right) - \left(1 - \sin^2 C\right)\right] + k^2 \sin^2 B\left[\left(1 - \sin^2 C\right) - \left(1 - \sin^2 A\right)\right] + k^2 \sin^2 C\left[\left(1 - \sin^2 A\right) - \left(1 - \sin^2 B\right)\right]$

$= k^2 \sin^2 A\left(\sin^2 C - \sin^2 B\right) + k^2 \sin^2 B\left(\sin^2 A - \sin^2 C\right) + k^2 \sin^2 C\left(\sin^2 B - \sin^2 A\right)$

$= k^2\left(\sin^2 A \sin^2 C - \sin^2 A \sin^2 B + \sin^2 A \sin^2 B - \sin^2 B \sin^2 C + \sin^2 B \sin^2 C - \sin^2 A \sin^2 C\right)$

$= k^2(0)$

= 0

= RHS.

Miscellaneous exercise 3 | Q 17 | Page 110

With the usual notations, show that

$(c^2 - a^2 + b^2) \tan A = (a^2 - b^2 + c^2) \tan B = (b^2 - c^2 + a^2) \tan C$

Solution:

By sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$\therefore$ a = k sin A, b = k sin B, c = k sin C

Now,

$$(c^2 - a^2 + b^2) \tan A = (c^2 - a^2 + b^2) . \frac{\sin A}{\cos A}$$

$$= (c^2 + b^2 - a^2) \times \frac{ka}{\left(\frac{c^2+b^2-a^2}{2bc}\right)}$$

$$= (c^2 + b^2 - a^2) \times \frac{2kabc}{c^2 + b^2 - a^2}$$

$= 2\ kabc$ ......(1)

$$(a^2 - b^2 + c^2) \tan B = (a^2 - b^2 + c^2) . \frac{\sin B}{\cos B}$$

$$= (a^2 + c^2 - b^2) \times \frac{kb}{\left(\frac{a^2+c^2-b^2}{2ac}\right)}$$

$$= (a^2 + c^2 - b^2) \times \frac{2kabc}{a^2 + c^2 - b^2}$$

$= 2kabc$ ....(2)

$$= \left(a^2 + c^2 - b^2\right) \times \frac{kb}{\left(\frac{a^2+c^2-b^2}{2ac}\right)}$$

$$= \left(a^2 + c^2 - b^2\right) \times \frac{2kabc}{a^2 + c^2 - b^2}$$

= 2kabc   ....(2)

$(b^2 - c^2 + a^2)$ tan C = $(b^2 - c^2 + a^2)$. $\frac{\sin C}{\cos C}$

$$= \left(a^2 + b^2 - c^2\right) \times \frac{kc}{\left(\frac{a^2+b^2-c^2}{2ab}\right)}$$

$$= \left(a^2 + b^2 - c^2\right) \times \frac{2kabc}{a^2 + b^2 - c^2}$$

= 2kabc   .....(3)

From (1), (2) and (3), we get

$(c^2 - a^2 + b^2)$ tan A = $(a^2 - b^2 + c^2)$ tan B = $(b^2 - c^2 + a^2)$ tan C

Miscellaneous exercise 3 | Q 18 | Page 110

In Δ ABC, if a $\cos^2 \frac{C}{2} + c \cos^2 \frac{A}{2} = \frac{3b}{2}$, then prove that a, b, c are in A.P.

Solution:

$$a \cos^2 \frac{C}{2} + c \cos^2 \frac{A}{2} = \frac{3b}{2}$$

$$\therefore a\left(\frac{1 + \cos C}{2}\right) + c\left(\frac{1 + \cos A}{2}\right) = \frac{3b}{2}$$

$$\therefore \frac{1}{2}(a + a \cos C + c + c \cos A) = \frac{3b}{2}$$

$\therefore$ a + c + (a cos C + c cos A) = 3b

$\therefore$ a + c + b = 3b .....[$\because$ a cos C + c cos A = b]

$\therefore$ a + c = 2b

Hence, a, b, c are in A.P.

Miscellaneous exercise 3 | Q 19 | Page 110

Show that $2\sin^{-1}\left(\frac{3}{5}\right) = \tan^{-1}\left(\frac{24}{7}\right)$

Solution:

Let $2\sin^{-1}\left(\frac{3}{5}\right)$ = x

Then sin x = $\frac{3}{5}$, where $0 < x < \frac{\pi}{2}$

$\therefore$ cos x > 0

Now, cos x = $\sqrt{1-\sin^2 x} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$

$\therefore \tan x = \frac{\sin x}{\cos x} = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$

$\therefore$ x = $\tan^{-1}\left(\frac{3}{4}\right)$

$\therefore \sin^{-1}\left(\frac{3}{5}\right) = \tan^{-1}\left(\frac{3}{4}\right)$

Now, LHS = $2\sin^{-1}\left(\frac{3}{5}\right) = 2\tan^{-1}\left(\frac{3}{4}\right)$

$$= \tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{3}{4}\right)$$

$$= \tan^{-1}\left[\frac{\frac{3}{4}+\frac{3}{4}}{1-\frac{3}{4}\times\frac{3}{4}}\right] = \tan^{-1}\left[\frac{12+12}{16-9}\right]$$

$$= \tan^{-1}\left(\frac{24}{7}\right) = \text{RHS}$$

Alternative Method:

$$\text{LHS} = 2\sin^{-1}\left(\frac{3}{5}\right) = 2\tan^{-1}\left(\frac{3}{4}\right)$$

$$= \tan^{-1}\left[\frac{2\left(\frac{3}{4}\right)}{1-\left(\frac{3}{4}\right)^2}\right] \quad \ldots.. \left[\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$= \tan^{-1}\left[\frac{\frac{3}{2}}{1-\left(\frac{9}{16}\right)}\right]$$

$$= \tan^{-1}\left(\frac{3}{2}\times\frac{16}{7}\right)$$

$$= \tan^{-1}\left(\frac{24}{7}\right)$$

= RHS

Miscellaneous exercise 3 | Q 20 | Page 110

Show that

$$\tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}.$$

Solution:

LHS = $\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{8}\right)$

$$=\tan^{-1}\left[\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}\times\frac{1}{7}}\right]+\tan^{-1}\left[\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}\times\frac{1}{8}}\right]$$

$$=\tan^{-1}\left(\frac{7+5}{35-1}\right)+\tan^{-1}\left(\frac{8+3}{24-1}\right)$$

$$=\tan^{-1}\left(\frac{12}{34}\right)+\tan^{-1}\left(\frac{11}{23}\right)$$

$$=\tan^{-1}\left(\frac{6}{17}\right)+\tan^{-1}\left(\frac{11}{23}\right)$$

$$=\tan^{-1}\left[\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}\times\frac{11}{23}}\right]$$

$$=\tan^{-1}\left(\frac{138+187}{391-66}\right)=\tan^{-1}\left(\frac{325}{325}\right)$$

$$=\tan^{-1}(1)=\tan^{-1}\left(\tan\ \frac{\pi}{4}\right)$$

$$=\frac{\pi}{4}$$

= RHS.

Miscellaneous exercise 3 | Q 21 | Page 110

Prove that $\tan^{-1}\sqrt{x}=\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$, if x ∈ [0, 1]

Solution:

Let $\tan^{-1}\sqrt{x}$ = y

∴ tan y = $\sqrt{x}$

∴ x = $\tan^2 y$

Now,

RHS = $\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$

$= \frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2 y}{1+\tan^2 y}\right)$

$= \frac{1}{2}\cos^{-1}(\cos 2y)$

$= \frac{1}{2}(2y) = y$

$= \tan^{-1}\sqrt{x}$

= LHS.

Miscellaneous exercise 3 | Q 22 | Page 110

Show that $\frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) = \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$

Solution:

We have to show that

$$\frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) = \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$

i.e. to show that,

$$\frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)+\frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)=\frac{9\pi}{8}$$

Let $\sin^{-1}\left(\frac{1}{3}\right)$ = x

∴ sin x = $\frac{1}{3}$, where $0<x<\frac{\pi}{3}$

∴ cos x > 0

Now, $\cos x=\sqrt{1-\sin^2 x}=\sqrt{1-\frac{1}{9}}=\sqrt{\frac{8}{9}}=\left(\frac{2\sqrt{2}}{3}\right)$

∴ x = $\cos^{-1}\left(\frac{2\sqrt{2}}{3}\right)$

∴ $\sin^{-1}\left(\frac{1}{3}\right)=\cos^{-1}\left(\frac{2\sqrt{2}}{3}\right)$ ....(1)

∴ LHS = $\frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)+\frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$

= $\frac{9}{4}\left[\sin^{-1}\left(\frac{1}{3}\right)+\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)\right]$

$=\frac{9}{4}\left[\cos^{-1}\left(\frac{2\sqrt{2}}{3}\right)+\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)\right]$ ...[By (1)]

$=\frac{9}{4}\left(\frac{\pi}{2}\right)\ \ldots.\ \left[\because \sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}\right]$

$=\frac{9\pi}{8}$

= RHS.

Miscellaneous exercise 3 | Q 23 | Page 110

Show that $\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x$, for $-\frac{1}{\sqrt{2}}\leq x\leq 1$

Solution:

LHS = $\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$

Put x = cos θ

∴ θ = $\cos^{-1}x$

∴ LHS = $\tan^{-1}\left(\frac{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}\right)$

$$=\tan^{-1}\left[\frac{\sqrt{2\cos^2\left(\frac{\theta}{2}\right)}-\sqrt{2\sin^2\left(\frac{\theta}{2}\right)}}{\sqrt{2\cos^2\left(\frac{\theta}{2}\right)}+\sqrt{2\sin^2\left(\frac{\theta}{2}\right)}}\right]$$

$$=\tan^{-1}\left[\frac{\sqrt{2}\cos\left(\frac{\theta}{2}\right)-\sqrt{2}\sin\left(\frac{\theta}{2}\right)}{\sqrt{2}\cos\left(\frac{\theta}{2}\right)+\sqrt{2}\sin\left(\frac{\theta}{2}\right)}\right]$$

$$=\tan^{-1}\left[\frac{\frac{\sqrt{2}\cos\left(\frac{\theta}{2}\right)}{\sqrt{2}\cos\left(\frac{\theta}{2}\right)}-\frac{\sqrt{2}\sin\left(\frac{\theta}{2}\right)}{\sqrt{2}\cos\left(\frac{\theta}{2}\right)}}{\frac{\sqrt{2}\cos\left(\frac{\theta}{2}\right)}{\sqrt{2}\cos\left(\frac{\theta}{2}\right)}+\frac{\sqrt{2}\sin\left(\frac{\theta}{2}\right)}{\sqrt{2}\cos\left(\frac{\theta}{2}\right)}}\right]$$

$$=\tan^{-1}\left[\frac{1-\tan\left(\frac{\theta}{2}\right)}{1+\tan\left(\frac{\theta}{2}\right)}\right]$$

$$=\tan^{-1}\left[\frac{\tan\ \frac{\pi}{4}-\tan\left(\frac{\theta}{2}\right)}{1+\tan\ \frac{\pi}{4}.\tan\left(\frac{\theta}{2}\right)}\right]\ \ldots..\left[\because \tan\ \frac{\pi}{4}=1\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right]$$

$$= \frac{\pi}{4} - \frac{\theta}{2}$$

$$= \frac{\pi}{4} - \frac{1}{2}\cos^{-1} x \quad .....[\because \theta = \cos^{-1} x]$$

= RHS.

Miscellaneous exercise 3 | Q 24 | Page 110

If $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1} x\right) = 1$, then find the value of x.

Solution:

$$\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1} x\right) = 1$$

$$\therefore \sin^{-1}\frac{1}{5} + \cos^{-1} x = \sin^{-1}(1)$$

$$\therefore \sin^{-1}\frac{1}{5} + \cos^{-1} x = \sin^{-1}\left(\sin\frac{\pi}{2}\right)$$

$$\therefore \sin^{-1}\frac{1}{5} + \cos^{-1} x = \frac{\pi}{2}$$

$$\therefore x = \frac{1}{5} \quad .....\left[\because \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}\right]$$

Miscellaneous exercise 3 | Q 25 | Page 110

If $\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$, find the value of x.

Solution:

$$\tan^{-1}\left(\frac{x-1}{x-2}\right)+\tan^{-1}\left(\frac{x+1}{x+2}\right)=\frac{\pi}{4}$$

$$\therefore \tan^{-1}\left[\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)}\right]=\frac{\pi}{4}$$

$$\therefore \frac{(x-1)(x+2)+(x+1)(x-2)}{(x-2)(x+2)-(x-1)(x+1)}=\tan\frac{\pi}{4}$$

$$\therefore \frac{\left(x^2+x-2\right)+\left(x^2-x-2\right)}{\left(x^2-4\right)-\left(x^2-1\right)}=1$$

$$\therefore \frac{x^2+x-2+x^2-x-2}{x^2-4-x^2+1}=1$$

$$\therefore \frac{2x^2-4}{-3}=1$$

$\therefore 2x^2 - 4 = - 3$

$\therefore 2x^2 = 1$

$\therefore x^2 = \dfrac{1}{2}$

$\therefore x = \pm\dfrac{1}{\sqrt{2}}$.

Miscellaneous exercise 3 | Q 26 | Page 110

If 2 $\tan^{-1}(\cos x) = \tan^{-1}(2 \operatorname{cosec} x)$, then find the value of x.

Solution:

$2 \tan^{-1}(\cos x) = \tan^{-1}(2 \operatorname{cosec} x)$

$\therefore \tan^{-1}\left[\frac{2\cos x}{1-\cos^2 x}\right] = \tan^{-1}(2 \operatorname{cosec} x) \quad \ldots\left[\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$

$\therefore \frac{2\cos x}{1-\cos^2 x} = 2 \operatorname{cosec} x$

$\therefore \frac{2\cos x}{\sin^2 x} = \frac{2}{\sin x}$

$\therefore \cos x = \sin x$

$\therefore x = \frac{\pi}{4} \quad \ldots .\left[\because \sin\frac{\pi}{4} = \cos\frac{\pi}{4}\right]$

Miscellaneous exercise 3 | Q 27 | Page 110

Solve: $\tan^{-1}\left(\frac{1 - x}{1+x}\right) = \frac{1}{2}\left(\tan^{-1}x\right)$, for $x > 0$.

Solution:

$\tan^{-1}\left(\frac{1 - x}{1+x}\right) = \frac{1}{2}\left(\tan^{-1}x\right)$

$\therefore 2\tan^{-1}\left(\frac{1 - x}{1+x}\right) = \left(\tan^{-1}x\right)$

$\therefore \tan^{-1}\left[\frac{2\left(\frac{1 - x}{1+x}\right)}{1-\left(\frac{1 - x}{1+x}\right)^2}\right] = \tan^{-1}x \quad \ldots .\left[\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$

$\therefore \frac{2\left(\frac{1 - x}{1+x}\right)(1+x)^2}{(1+x)^2-(1-x)^2} = x$

$\therefore \frac{2(1-x)(1+x)}{(1+2x+x^2)-(1-2x+x^2)} = x$

$\therefore \frac{2\left(1-x^2\right)}{1+2x+x^2-1+2x-x^2} = x$

$\therefore \frac{2-2x^2}{4x} = x$

$\therefore 2 - 2x^2 = 4x^2$

$\therefore 6x^2 = 2$

$\therefore x^2 = \frac{1}{3}$

$\therefore x = \frac{1}{\sqrt{3}}$ . ....[$\because x > 0$]

Miscellaneous exercise 3 | Q 28 | Page 110

If $\sin^{-1}(1 - x) - 2 \sin^{-1}x = \pi/2$, then find the value of x.

Solution:

$\sin^{-1}(1 - x) - 2 \sin^{-1}x = \frac{\pi}{2}$

$\therefore \sin^{-1}(1 - x) = \frac{\pi}{2} + 2\sin^{-1} x$

$\therefore 1 - x = \sin\left(\frac{\pi}{2} + 2\sin^{-1} x\right)$

$\therefore 1 - x = \cos(2 \sin^{-1}x)$ ....$\left[\because \sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta\right]$

$\therefore 1 - x = 1 - 2[\sin(\sin^{-1} x)]^2$ ....[$\because \cos 2\theta = 1 - 2 \sin^2\theta$]

$\therefore 1 - x = 1 - 2x^2$

$\therefore 2x^2 - x = 0$

$\therefore x(2x - 1) = 0$

$\therefore x = 0$ or $x = \frac{1}{2}$

When $x = \frac{1}{2}$

$$\text{LHS} = \sin^{-1}\left(1 - \frac{1}{2}\right) - 2\sin^{-1}\left(\frac{1}{2}\right)$$

$$= \sin^{-1}\left(\frac{1}{2}\right) - 2\sin^{-1}\left(\frac{1}{2}\right)$$

$$= -\sin^{-1}\left(\frac{1}{2}\right)$$

$$= -\sin^{-1}\left(\sin\ \frac{\pi}{6}\right)$$

$$= -\frac{\pi}{6} \neq \frac{\pi}{2}$$

$$\therefore\ x \neq \frac{1}{2}$$

Hence, x = 0.

Miscellaneous exercise 3 | Q 29 | Page 110

If $\tan^{-1}2x + \tan^{-1}3x = \pi/4$, then find the value of x.

Solution:

$$\tan^{-1}2x + \tan^{-1}3x = \frac{\pi}{4}$$

$$\therefore\ \tan^{-1}\left(\frac{2x + 3x}{1 - 2x \times 3x}\right) = \frac{\pi}{4}, \text{ where } 2x > 0,\ 3x > 0$$

$$\therefore\ \frac{5x}{1 - 6x^2} = \tan\ \frac{\pi}{4} = 1$$

$\therefore\ 5x = 1 - 6x^2$

$\therefore\ 6x^2 + 5x - 1 = 0$

$\therefore\ 6x^2 + 6x - x - 1 = 0$

$\therefore\ 6x(x + 1) - 1(x + 1) = 0$

∴ (x + 1)(6x - 1) = 0

∴ x = -1 or x = 1/6

But x > 0 ∴ x ≠ - 1

Hence, x = 1/6

Miscellaneous exercise 3 | Q 30 | Page 110

Show that $\tan^{-1}\frac{1}{2}-\tan^{-1}\frac{1}{4}=\tan^{-1}\frac{2}{9}$.

Solution:

LHS = $\tan^{-1}\frac{1}{2}-\tan^{-1}\frac{1}{4}$

$=\tan^{-1}\left[\frac{\frac{1}{2}-\frac{1}{4}}{1+\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)}\right]$

$=\tan^{-1}\left(\frac{4-2}{8+1}\right)$

$=\tan^{-1}\left(\frac{2}{9}\right)$ = RHS.

Miscellaneous exercise 3 | Q 31 | Page 110

Show that $\cot^{-1}\frac{1}{3}-\tan^{-1}\frac{1}{3}=\cot^{-1}\frac{3}{4}$.

Solution:

LHS = $\cot^{-1}\frac{1}{3}-\tan^{-1}\frac{1}{3}$

$=\tan^{-1}3-\tan^{-1}\frac{1}{3}$ .... $\left[\because \cot^{-1}x=\tan^{-1}\left(\frac{1}{x}\right)\right]$

$=\tan^{-1}\left[\frac{3-\frac{1}{3}}{1+3\left(\frac{1}{3}\right)}\right]$

$$= \tan^{-1}\left[\frac{\frac{8}{3}}{1+1}\right]$$

$$= \tan^{-1}\left(\frac{4}{3}\right)$$

$$= \cot^{-1}\left(\frac{3}{4}\right) \quad .... \left[\tan^{-1} x = \cot^{-1}\left(\frac{1}{x}\right)\right]$$

= RHS.

Miscellaneous exercise 3 | Q 32 | Page 110

Show that $\tan^{-1} \frac{1}{2} = \frac{1}{3} \tan^{-1} \frac{11}{2}$

Solution:

We have to show that

$$\tan^{-1} \frac{1}{2} = \frac{1}{3} \tan^{-1} \frac{11}{2}$$

i.e. to show that $3 \tan^{-1} \frac{1}{2} = \tan^{-1} \frac{11}{2}$

LHS = $3 \tan^{-1} \frac{1}{2}$

$$= 2 \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{2}$$

$$= \tan^{-1}\left[\frac{2 \times \frac{1}{2}}{1 - \left(\frac{1}{2}\right)^2}\right] + \tan^{-1} \frac{1}{2} \quad ..... \left[\because 2 \tan^{-1} x = \tan^{-1}\left(\frac{2x}{1 - x^2}\right)\right]$$

$$= \tan^{-1}\left[\frac{1}{\frac{3}{4}}\right] + \tan^{-1} \frac{1}{2}$$

$$= \tan^{-1} \frac{4}{3} + \tan^{-1} \frac{1}{2}$$

$$= \tan^{-1} \left[\frac{\frac{4}{3} + \frac{1}{2}}{1 - \frac{4}{3} \times \frac{1}{2}} \right]$$

$$= \tan^{-1} \left(\frac{8+3}{6-4} \right)$$

$= \tan^{-1} \left(\frac{11}{2} \right)$ = RHS

Miscellaneous exercise 3 | Q 33 | Page 111

Show that $\cos^{-1} \frac{\sqrt{3}}{2} + 2\sin^{-1} \frac{\sqrt{3}}{2} = \frac{5\pi}{6}$.

Solution:

LHS = $\cos^{-1} \frac{\sqrt{3}}{2} + 2\sin^{-1} \frac{\sqrt{3}}{2}$

$$= \cos^{-1}\left(\cos \frac{\pi}{6}\right) + 2\sin^{-1} \left(\sin \frac{\pi}{3}\right) \ldots.. \left[\because \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2} = \sin \frac{\pi}{3}\right]$$

$$= \frac{\pi}{6} + 2\left(\frac{\pi}{3}\right) \ldots.. \left[\because \sin^{-1}(\sin \mathrm{x}) = \mathrm{x},\ \cos^{-1}(\cos x) = \mathrm{x}\right]$$

$$= \frac{\pi}{6} + \frac{2\pi}{3}$$

$= \frac{5\pi}{6}$ = RHS.

Miscellaneous exercise 3 | Q 34 | Page 111

Show that $2\cot^{-1} \frac{3}{2} + \sec^{-1} \frac{13}{12} = \frac{\pi}{2}$

Solution:

$$2\cot^{-1}\frac{3}{2} = 2\tan^{-1}\frac{2}{3} \;\ldots.\left[\because \cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right)\right]$$

$$= \tan^{-1}\left[\frac{2\times\frac{2}{3}}{1-\left(\frac{2}{3}\right)^2}\right] \ldots.\left[\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$= \tan^{-1}\left[\frac{\frac{4}{3}}{1-\frac{4}{9}}\right]$$

$$= \tan^{-1}\left(\frac{4}{3}\times\frac{9}{5}\right) = \tan^{-1}\frac{12}{5} \quad \ldots.(1)$$

Let $\sec^{-1}\frac{13}{12} = \alpha$

Then, $\sec\alpha = \frac{13}{12}$, where $0 < \alpha < \frac{\pi}{2}$

$\therefore \tan\alpha > 0$

Now, $\tan\alpha = \sqrt{\sec^2\alpha - 1}$

$$= \sqrt{\frac{169}{144} - 1} = \sqrt{\frac{25}{144}} = \frac{5}{12}$$

$$\therefore \alpha = \tan^{-1}\frac{5}{12} = \cot^{-1}\frac{12}{5} \;\ldots.\left[\because \tan^{-1}x = \cot^{-1}\left(\frac{1}{x}\right)\right]$$

$$\therefore \sec^{-1}\frac{13}{12} = \cot^{-1}\frac{12}{5} \quad \ldots..(2)$$

Now,

LHS = $2\cot^{-1}\frac{3}{2} + \sec^{-1}\frac{13}{12}$

$= \tan^{-1}\frac{12}{5} + \cot^{-1}\frac{12}{5}$ ...[By (1) and (2)]

$$= \frac{\pi}{2} \quad \ldots.. \left[\because \tan^{-1} x + \cot^{-1} x - \frac{\pi}{2}\right]$$

= RHS.

Miscellaneous exercise 3 | Q 35.1 | Page 111

Prove the following:

$\cos^{-1} x = \tan^{-1}\left(\frac{\sqrt{1 - x^2}}{x}\right)$, if x > 0

Solution:

Let $\cos^{-1} x = \alpha$

Then, $\cos \alpha = x$, where $0 < \alpha < \pi$

Since, $x > 0$, $0 < \alpha < \frac{\pi}{2}$

$\therefore \sin \alpha > 0, \cos \alpha > 0$

Now, $\tan^{-1}\left(\frac{\sqrt{1 - x^2}}{x}\right) = \tan^{-1}\left(\frac{\sqrt{1 - \cos^2 \alpha}}{\cos \alpha}\right)$

$$= \tan^{-1}\left(\frac{\sqrt{\sin^2 \alpha}}{\cos \alpha}\right)$$

$$= \tan^{-1}(\tan \alpha)$$

$$= \alpha = \cos^{-1} x$$

Hence, $\cos^{-1} x = \tan^{-1}\left(\frac{\sqrt{1 - x^2}}{x}\right)$, if x > 0

Miscellaneous exercise 3 | Q 35.2 | Page 111

Prove the following:

$$\cos^{-1}x = \pi + \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$$, if x < 0

Solution:

Let $\cos^{-1} x = \alpha$

Then, $\cos \alpha = x$, where $0 < \alpha < \pi$

Since, $x < 0$, $\frac{\pi}{2} < \alpha < \pi$

Now, $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \tan^{-1}\left(\frac{\sqrt{1-\cos^2\alpha}}{\cos\alpha}\right)$

$= \tan^{-1}(\tan \alpha)$ ....(1)

But $\frac{\pi}{2} < \alpha < \pi$, therefore inverse of tangent does not exist.

Consider, $\frac{\pi}{2} - \pi < \alpha - \pi < \pi - \pi$,

$$\therefore -\frac{\pi}{2} < \alpha - \pi < 0$$

and $\tan(\alpha - \pi) = \tan[-(\pi - \alpha)]$

$= -\tan(\pi - \alpha)$ .....$[\because \tan(-\theta) = -\tan\theta]$

$= -(-\tan\alpha) = \tan\alpha$

$\therefore$ from (1), we get

$$\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \tan^{-1}[\tan(\alpha - \pi)]$$

$$= \alpha - \pi \quad ......[\because \tan^{-1}(\tan x) = x]$$

$$= \cos^{-1} x - \pi$$

$$\therefore \cos^{-1} x = \pi + \tan^{-1}\left(\frac{\sqrt{1 - x^2}}{x}\right), \text{ if } x < 0$$

Miscellaneous exercise 3 | Q 36 | Page 111

If | x | < 1, then prove that

$$2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1 - x^2}\right) = \sin^{-1}\left(\frac{2x}{1 + x^2}\right) = \cos^{-1}\left(\frac{1 - x^2}{1 + x^2}\right)$$

Solution:

Let $\tan^{-1}x = y$

Then, $x = \tan y$

Now, $\tan^{-1}\left(\frac{2x}{1 - x^2}\right) = \tan^{-1}\left(\frac{2\tan y}{1 - \tan^2 y}\right)$

$$= \tan^{-1}(\tan 2y)$$

$= 2y$

$= 2\tan^{-1}x$ ......(1)

$$\sin^{-1}\left(\frac{2x}{1 + x^2}\right) = \sin^{-1}\left(\frac{2\tan y}{1 + \tan^2 y}\right)$$

$$= \sin^{-1}(\sin 2y)$$

$= 2y$

$= 2\tan^{-1}x$ ......(2)

$$\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \cos^{-1}\left(\frac{1-\tan^2 y}{1+\tan^2 y}\right)$$

$$= \cos^{-1}(\cos 2y)$$

= 2y

$$= 2\tan^{-1} x \qquad \text{......(3)}$$

From (1), (2) and (3), we get

$$2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

Miscellaneous exercise 3 | Q 37 | Page 111

If x, y, z are positive, then prove that

$$\tan^{-1}\left(\frac{x - y}{1+xy}\right) + \tan^{-1}\left(\frac{y - z}{1+yz}\right) + \tan^{-1}\left(\frac{z - x}{1+zx}\right) = 0$$

Solution:

LHS = $\tan^{-1}\left(\frac{x - y}{1+xy}\right) + \tan^{-1}\left(\frac{y - z}{1+yz}\right) + \tan^{-1}\left(\frac{z - x}{1+zx}\right)$

$= \tan^{-1} x - \tan^{-1} y + \tan^{-1} y - \tan^{-1} z + \tan^{-1} z - \tan^{-1} x$ ....[$\because x > 0, y > 0, z > 0$]

= 0

= RHS

Miscellaneous exercise 3 | Q 38 | Page 111

If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \frac{\pi}{2}$, then show that xy + yz + zx = 1

Solution:

$$\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \frac{\pi}{2}$$

$$\therefore \tan^{-1}\left(\frac{x+y}{1-xy}\right) + \tan^{-1}z = \frac{\pi}{2}$$

$$\therefore \tan^{-1}\left[\frac{\frac{x+y}{1-xy}+z}{1-\left(\frac{x+y}{1-xy}\right)z}\right] = \frac{\pi}{2}$$

$$\therefore \tan^{-1}\left[\frac{x+y+z\text{ - }xyz}{1-xy-xz-yz}\right] = \frac{\pi}{2}$$

$$\therefore \frac{x+y+z\text{ - }xyz}{1-xy-yz-zx} = \tan\frac{\pi}{2}, \text{ which does not exist}$$

$\therefore$ 1 - xy - yz - zx = 0

$\therefore$ xy + yz + zx = 1

Miscellaneous exercise 3 | Q 39 | Page 111

If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = 3\pi$, then show that $x^2 + y^2 + z^2 + 2xyz = 1$.

Solution: $0 \le \cos^{-1}x \le \pi$ and

$\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = 3\pi$

$\therefore \cos^{-1}x = \pi$, $\cos^{-1}y = \pi$ and $\cos^{-1}z = \pi$

$\therefore x = y = z = \cos\pi = -1$

$\therefore x^2 + y^2 + z^2 + 2xyz$

$= (-1)^2 + (-1)^2 + (-1)^2 + 2(-1)(-1)(-1)$

$= 1 + 1 + 1 - 2$

$= 3 - 2$

$= 1.$