CHAPTER 6

Monotonocity and Maxima–Minima of Functions

MONOTONOCITY: INTRODUCTION

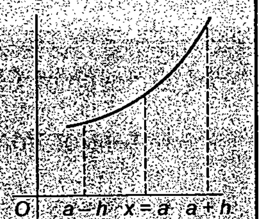
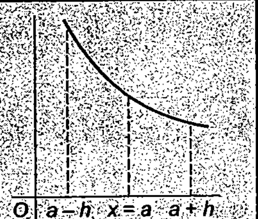
The most useful element taken into consideration among the total post mortem activities of functions is their monotonic behaviour.

Functions are said to be monotonic if they are either increasing or decreasing in their entire domain, e.g., $f(x)=e^x$; $f(x)=\log_e x$ and $f(x)=2x+3$ are some of the examples of functions which are increasing, whereas $f(x)=-x^3$; $f(x)=e^{-x}$ and $f(x)=\cot^{-1}x$ are some of the examples of the functions which are decreasing.

Functions which are increasing as well as decreasing in their domain are said to be non-monotonic, e.g.,

$f(x)=\sin x$; $f(x)=ax^2+bx+c$ and $f(x)=|x|$; however, in the interval $\left[0, \frac{\pi}{2}\right]$, $f(x)=\sin x$ will be said to be increasing.

Monotonocity of a Function at a Point

A function is said to be monotonically increasing at $x=a$ if $f(x)$ satisfies $\left.\begin{matrix} f(a+h)>f(a) \\ f(a-h)<f(a) \end{matrix}\right]$ for a small positive h. Small positive h means no discontinuity in f between $a-h$ and a and a and $a+h$.	 Fig. 6.1(a)
A function is said to be monotonically decreasing at $x=a$ if $f(x)$ satisfies $\left.\begin{matrix} f(a+h)<f(a) \\ f(a-h)>f(a) \end{matrix}\right]$ for a small positive h.	 Fig. 6.1(b)

It should be noted that we can talk of monotonocity of $f(x)$ at $x=a$ only if $x=a$ lies in the domain of f, without any consideration of continuity or differentiability of $f(x)$ at $x=a$.

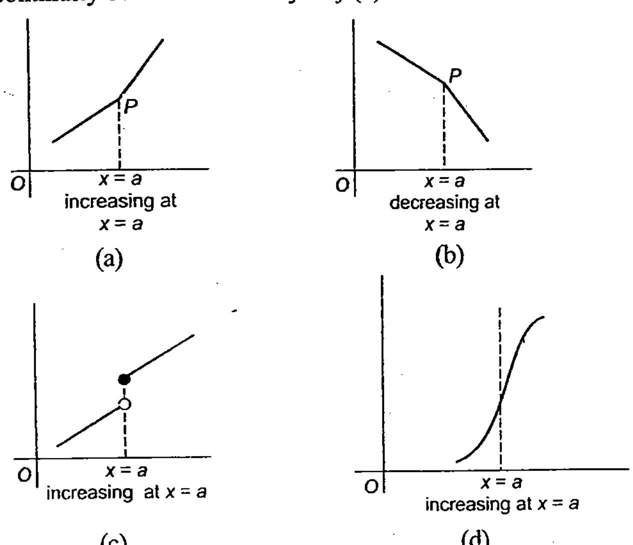


Fig. 6.2

Example 6.1 For each of the following graph, comment whether $f(x)$ is increasing or decreasing or neither increasing nor decreasing at $x=a$.

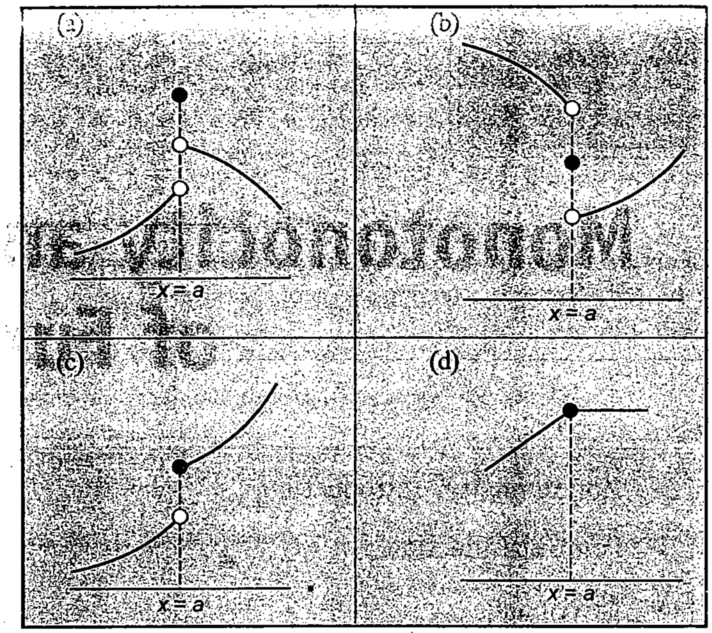


Fig. 6.3

Sol.

a. Neither monotonically increasing nor decreasing as $f(a-h)<f(a)$ and $f(a+h)<f(a)$

b. Monotonically decreasing as $f(a-h)>f(a)>f(a+h)$

c. Monotonically increasing as $f(a-h)<f(a)<f(a+h)$

d. Neither monotonically increasing nor decreasing as $f(a-h)<f(a)$ but $f(a+h)=f(a)$

Example 6.2 Find the complete set of values of λ, for which the function $f(x)=\begin{cases} x+1, & x<1 \\ \lambda, & x=1 \\ x^2-x+3, & x>1 \end{cases}$ is strictly increasing at $x=1$.

Sol. Let $g(x)=x+1$, where $x<1$, then $g(x)$ is strictly increasing. Let $h(x)=x^2-x+3$, where $x>1$, $h(x)$ is also strictly increasing. ($\because h'(x)=2x-1>0\ \forall x>1$).

Since $f(x)$ is an increasing function

$\therefore \lim_{x\to1^-}(x+1)\le\lambda\le\lim_{x\to1^+}(x^2-x+3) \Rightarrow 2\le\lambda\le3$

Monotonocity in an Interval

Let I be an open interval contained in the domain of a real-valued function f. Then f is said to be

(i) increasing on I if $x_1<x_2$ in $I \Rightarrow f(x_1)\le f(x_2)$ for all $x_1, x_2\in I$.

(ii) strictly increasing on I if $x_1<x_2$ in $I \Rightarrow f(x_1)<f(x_2)$ for all $x_1, x_2\in I$.

(iii) decreasing on I if $x_1<x_2$ in $I \Rightarrow f(x_1)\ge f(x_2)$ for all $x_1, x_2\in I$.

(iv) strictly decreasing on I if $x_1<x_2$ in $I \Rightarrow f(x_1)>f(x_2)$ for all $x_1, x_2\in I$.

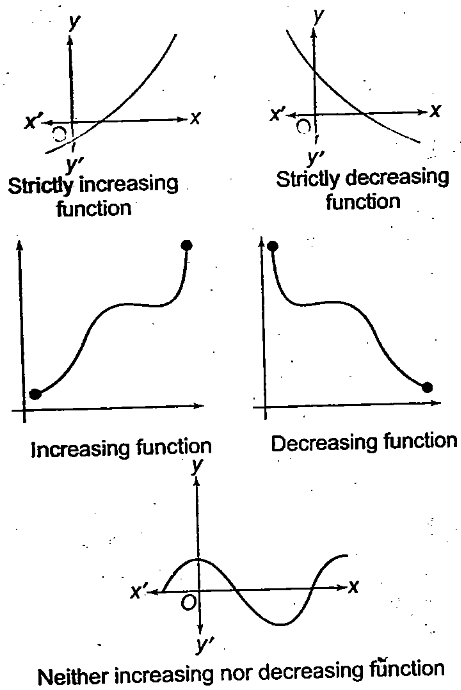


Fig. 6.4

It should however be noted that $\frac{dy}{dx}$ at some point may be equal to zero but $f(x)$ may still be increasing at $x = a$. Consider $f(x) = x^3$ which is increasing at $x = 0$ although $f'(x) = 0$.

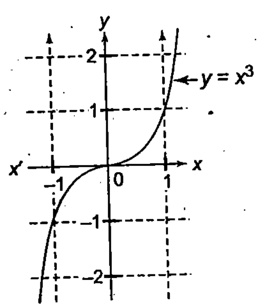


Fig. 6.5

This is because $f(0 + h) > f(0)$ and $f(0 - h) < f(0)$. At all such points where $\frac{dy}{dx} = 0$ but y is still increasing or decreasing are known as point of inflection, which indicate the change of concavity of the curve.

Example 6.3 Prove that the following functions are increasing for the given intervals

a. $f(x) = e^x + \sin x, x \in R^+$

b. $f(x) = \sin x + \tan x - 2x, x \in (0, \pi/2)$

c. $f(x) = \sec x - \operatorname{cosec} x, x \in (0, \pi/2)$

Sol. a. $f(x) = e^x + \sin x, x \in R^+$

$\Rightarrow f'(x) = e^x + \cos x$

Clearly, $f'(x) > 0 \ \forall x \in R^+$ (as $e^x > 1, x \in R^+$ and $-1 \le \cos x \le 1, x \in R^+$)

Hence, $f(x)$ is strictly increasing.

b. $f(x) = \sin x + \tan x - 2x, x \in (0, \pi/2)$

$\Rightarrow f'(x) = \cos x + \sec^2 x - 2$

as $\cos x > \cos^2 x, x \in (0, \pi/2)$

$\Rightarrow f'(x) > \cos^2 x + \sec^2 x - 2$

$= (\cos x - \sec x)^2 > 0, x \in (0, \pi/2)$

Hence, $f(x)$ is strictly increasing in $(0, \pi/2)$.

c. $f(x) = \sec x - \operatorname{cosec} x, x \in (0, \pi/2)$

$\Rightarrow f'(x) = \sec x \tan x + \operatorname{cosec} x \cot x > 0 \ \forall x \in (0, \pi/2)$

Thus, $f(x)$ is increasing in $(0, \pi/2)$.

Example 6.4 Find the least value of k for which the function $x^2 + kx + 1$ is an increasing function in the interval $1 < x < 2$.

Sol. $f(x) = x^2 + kx + 1$

For $f(x)$ to be increasing, $f'(x) > 0$

$\Rightarrow \frac{d}{dx}\left(x^2 + kx + 1\right) > 0$

$\Rightarrow 2x + k > 0 \Rightarrow k > -2x$

For $x \in (1, 2)$, the least value of k is -2.

Example 6.5 If $f : [0, \infty[\to R$ is the function defined by $f(x) = \frac{e^{x^2} - e^{-x^2}}{e^{x^2} + e^{-x^2}}$, then check whether $f(x)$ is injective or not.

Sol. $y = f(x) = \frac{e^{x^2} - e^{-x^2}}{e^{x^2} + e^{-x^2}}$

$= \frac{e^{2x^2} - 1}{e^{2x^2} + 1} = 1 - \frac{2}{e^{2x^2} + 1}$

Now for $x \in [0, \infty)$, x^2 is increasing.

$\Rightarrow 2x^2$ is increasing

$\Rightarrow e^{2x^2}$ is increasing

$\Rightarrow e^{2x^2} + 1$ is increasing

$\Rightarrow \frac{2}{e^{2x^2} + 1}$ is decreasing

$\Rightarrow \frac{-2}{e^{2x^2} + 1}$ is increasing

$\Rightarrow 1 - \frac{2}{e^{2x^2} + 1}$ is increasing

$\Rightarrow f(x)$ is monotonous.

Hence, $f(x)$ is one-one (injective)

Alternative method

$$f'(x) = \frac{e^{2x^2} 4x(e^{2x^2} + 1) - e^{2x^2} 4x(e^{2x^2} - 1)}{(e^{2x^2} + 1)^2}$$

$$= \frac{4x\, e^{2x^2}}{(e^{2x^2} + 1)^2} \ge 0 \ \forall x \in [0, \infty)$$

Hence, $f(x)$ is increasing.

Example 6.6 Let $f(x)$ and $g(x)$ be two continuous functions defined from $R \to R$, such that $f(x_1) > f(x_2)$ and $g(x_1) < g(x_2), \forall x_1 > x_2$, then find the solution set of $f(g(\alpha^2 - 2\alpha)) > f(g(3\alpha - 4))$.

Sol. Obviously, f is increasing and g is decreasing in R.

Hence, $f(g(\alpha^2 - 2\alpha)) > f(g(3\alpha - 4))$

$\Rightarrow g(a^2 - 2\alpha) > g(3\alpha - 4)$ $\quad$ ($\because f$ is increasing)

$\Rightarrow \quad \alpha^2-2\alpha<3\alpha-4$ as g is decreasing

$\Rightarrow \quad \alpha^2-5\alpha+4<0$

$\Rightarrow \quad (\alpha-1)(\alpha-4)<0$

$\Rightarrow \quad \alpha \in (1,4)$

Example 6.7 Prove that the function $f(x)=\log_e(x^2+1)-e^{-x}+1$ is strictly increasing $\forall\ x \in R$.

Sol. $f(x)=\log_e(x^2+1)-e^{-x}+1$

$$\Rightarrow f'(x)=\frac{2x}{1+x^2}+e^{-x}$$

$$\Rightarrow f'(x)=e^{-x}+\frac{2}{x+\frac{1}{x}}$$

for $x<0, -1<\dfrac{2}{x+\frac{1}{x}}<0$ and $e^{-x}>1$

hence, $e^{-x}+\dfrac{2x}{1+x^2}>0$

$\Rightarrow f(x)$ is a strictly increasing function $\forall\ x \in R$.

Example 6.8 Prove that $f(x) = x - \sin x$ is an increasing function.

Sol. $f(x)=x-\sin x$

$\Rightarrow \quad f'(x)=1-\cos x$

Now, $f'(x)>0$ everywhere except at $x=0, \pm 2\pi, \pm 4\pi$, etc., but all these points are discrete and do not form an interval. Hence, we can conclude that $f(x)$ is monotonically increasing for $x \in R$. In fact, we can also see it graphically.

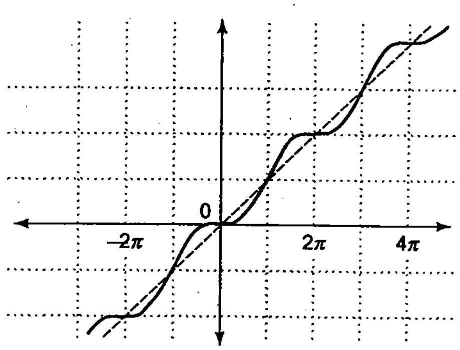


Fig. 6.6

Example 6.9 Find the values of p if $f(x) = \cos x - 2px$ is invertible.

Sol. For $f(x)=\cos x-2px$ to be invertible, it must be monotonic, i.e., either always increasing or always decreasing.

$f(x)$ will be monotonically decreasing if $f'(x)\le 0$

$\Rightarrow \quad f'(x)=-\sin x-2p\le 0$ for all x

$\Rightarrow \quad p\ge -\dfrac{1}{2}\sin x$ for all x

$\Rightarrow \quad p\ge \dfrac{1}{2}$ $\qquad [\because -1\le \sin x\le 1] \qquad (1)$

$f(x)$ will be monotonically increasing if $f'(x)\ge 0$.

$\Rightarrow \quad f'(x)=-\sin x-2p\ge 0$ for all x

$\Rightarrow \quad p\le -\dfrac{1}{2}\sin x$ for all x

$\Rightarrow \quad p\le -\dfrac{1}{2}$ $\qquad [\because -1\le \sin x\le 1] \qquad (2)$

From equations (1) and (2), $|p|\ge \dfrac{1}{2}$.

Example 6.10 Find the values of a if $f(x) = 2e^x - ae^{-x} + (2a+1)x-3$ is increasing for all values of x.

Sol. $f'(x)=2e^x+ae^{-x}+2a+1$

$$=e^{-x}(2e^{2x}+(2a+1)e^x+a)$$

$$=2e^{-x}\left(e^{2x}+\left(a+\frac{1}{2}\right)e^x+\frac{a}{2}\right)$$

$$=2e^{-x}(e^x+a)\left(e^x+\frac{1}{2}\right)$$

For $f(x)$ to be increasing, $f'(x)\ge 0\ \forall\, x\in R$.

$\Rightarrow \quad e^x+a\ge 0\ \forall\, x\in R \Rightarrow a\ge 0$.

Example 6.11 Is every invertible function monotonic ?

Sol. Consider the following function which is invertible but not monotonic.

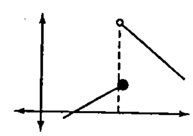


Fig. 6.7

Example 6.12 If $fogoh(x)$ is an increasing function, then which of the following is not possible?

(i) $f(x)$, $g(x)$ and $h(x)$ are increasing

(ii) $f(x)$ and $g(x)$ are decreasing and $h(x)$ is increasing

(iii) $f(x)$, $g(x)$ and $h(x)$ are decreasing

Sol. $fogoh(x)$ is increasing, then obviously $f(x)$, $g(x)$ and $h(x)$ can be increasing functions.

Also, $f(x)$ and $g(x)$ are decreasing and $h(x)$ is increasing.

$\Rightarrow$ for $x_2>x_1$

$h(x_2)>h(x_1)$

$\Rightarrow \quad goh(x_2)<goh(x_1)$

$\Rightarrow \quad fogoh(x_2)>fogoh(x_1)$

$\Rightarrow \quad fogoh(x)$ is increasing.

If all $f(x)$, $g(x)$ and $h(x)$ are decreasing, then for $x_2>x_1$, $fogoh(x_2)<fogoh(x_1)$, hence $fogoh(x)$ is decreasing.

Example 6.13 Let $f:[0,\infty)\to[0,\infty)$ and $g:[0,\infty)\to[0,\infty)$ be non-increasing and non-decreasing functions and $h(x) = g(f(x))$. If f and g are differentiable functions, $h(x)=g(f(x))$. If f and g are differentiable for all points in their respective domains and $h(0) = 0$. Then, show $h(x)$ is always identically zero.

Sol. Here, $h(x)=g(f(x))$, since $g(x)\in[0,\infty)$

$h(x)\ge 0, \forall\, x\in$ domain

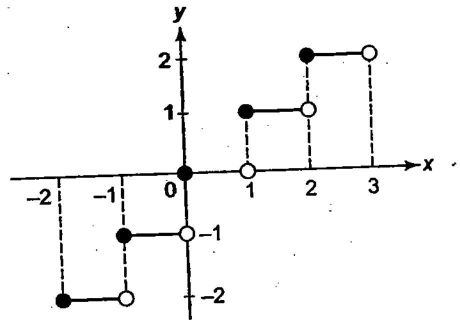
Also, $h'(x)=g'(f(x)).f'(x)\le 0$ as $g'(x)\ge 0$

and $h(x)\le 0\ \forall\, x\in$ domain as $h(0)=0$.

Hence, $h(x)=0\ \forall\, x\in$ domain.

Example 6.14 $f(x) = [x]$ is a step-up function. Is it a monotonically increasing function for $x \in R$?

Sol. No, $f(x) = [x]$ is not monotonically increasing for $x \in R$ rather, it is a non-decreasing function as illustrated in Fig. 6.8.



Graph of $y = [x]$

Fig. 6.8

Separating the Intervals of Monotonocity

Example 6.15 Separate the intervals of monotonocity of the following functions:

a. $f(x) = 3x^4 - 8x^3 - 6x^2 + 24x + 7$

b. $f(x) = -\sin^3 x + 3\sin^2 x + 5, x \in [-\pi/2, \pi/2]$

c. $f(x) = (2^x - 1)(2^x - 2)^2$

Sol.

a. $f(x) = 3x^4 - 8x^3 - 6x^2 + 24x + 7$

$f'(x) = 12x^3 - 24x^2 - 12x + 24$

$= 12(x^3 - 2x^2 - x + 2)$

$= 12(x-1)(x-2)(x+1)$

Now, $f'(x) = 0$ when $x = -1$, 1 and 2.

Hence, critical points are -1, 1 and 2.

The sign scheme of the derivative is given in Fig. 6.9.

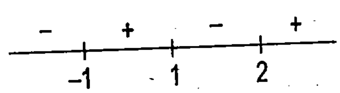


Fig. 6.9

Hence, the function increases in the interval $(-1, 1) \cup (2, \infty)$ and decreases in the interval $(-\infty, -1) \cup (1, 2)$.

b. $f(x) = -\sin^3 x + 3\sin^2 x + 5, x \in (-\pi/2, \pi/2)$

$\Rightarrow + f'(x) = -3\sin^2 x \cos x + 6 \sin x \cos x$

$= 3 \sin x \cos x (2 - \sin x)$

As $\cos x > 0$ and $2 - \sin x > 0 \ \forall x \in (-\pi/2, \pi/2)$

and $\sin x > 0 \ \forall x \in (0, \pi/2)$, $\sin x < 0 \ \forall x \in (-\pi/2, 0)$

$f'(x) > 0, x \in (0, \pi/2)$ and $f'(x) < 0, x \in (-\pi/2, 0)$

$\Rightarrow$ $f(x)$ is increasing in $(0, \pi/2)$ and decreasing in $(-\pi/2, 0)$.

c. $f(x) = (2^x - 1)(2^x - 2)^2$

$\Rightarrow f'(x) = 2^x \log 2 (2^x - 2)^2 + 2(2^x - 2)\log 2 (2^x - 1)$

$= 2^x \log 2 (2^x - 2)[(2^x - 2) + 2(2^x - 1)]$

$= 2^x \log 2 (2^x - 2)[3 \times 2^x - 4]$

$2^x - 2 = 0 \Rightarrow x = 1$

$3 \times 2^x - 4 \Rightarrow x = \log_2(4/3)$

The sign scheme of $f'(x)$ is given in Fig. 6.10.

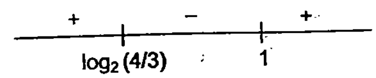


Fig. 6.10

Thus, $f(x)$ is increasing in $(-\infty, \log_2(4/3)) \cup (1, \infty)$ and decreasing in $(\log_2(4/3), 1)$.

Example 6.16 Find the interval of monotonocity of the function $f(x) = |x-1|/x^2$.

Sol. $f(x) = \frac{|x-1|}{x^2} = \begin{cases} \frac{1-x}{x^2}, & x < 1, x \neq 0 \\ \frac{x-1}{x^2}, & x > 1 \end{cases}$

Clearly $f(x)$ is continuous for all $x \in R$ except at $x = 0$.

$f'(x) = \begin{cases} \frac{x-2}{x^3}, & x < 1, x \pm 0 \\ \frac{2-x}{x^3}, & x > 1 \end{cases}$

$f'(x) > 0 \Rightarrow x < 0$ or $1 < x < 2$

$f'(x) < 0 \Rightarrow 0 < x < 1$ or $x > 2$

Hence, $f(x)$ is increasing in $(-\infty, 0) \cup (1, 2)$ and decreasing in $(0, 1) \cup (2, \infty)$.

Example 6.17 Find the intervals of decrease and increase for the function $f(x) = \cos\left(\frac{\pi}{x}\right)$.

Sol. $f(x) = \cos\left(\frac{\pi}{x}\right)$. The function is defined for all x, where $x \neq 0$

$$f'(x) = -\sin\left(\frac{\pi}{x}\right)\pi\left(-\frac{1}{x^2}\right) = \frac{\pi}{x^2}\sin\left(\frac{\pi}{x}\right) \quad (1)$$

$\therefore$ f is differentiable for all x, $(x \neq 0)$.

Here, sign of $f'(x)$ is same as that of $\sin\left(\frac{\pi}{x}\right)$.

Thus, $f'(x)$ is positive if $\sin\left(\frac{\pi}{x}\right) > 0$ and $f'(x)$ is negative if $\sin\left(\frac{\pi}{x}\right) < 0$

or $\sin\left(\frac{\pi}{x}\right) > 0$, if $2k\pi < \frac{\pi}{x} < (2k+1)\pi, k \in Z$

and $\sin\left(\frac{\pi}{x}\right) < 0$, if $(2k+1)\pi < \frac{\pi}{x} < (2k+2)\pi, k \in Z$.

Hence, the function f is increasing in the interval $\left(\frac{1}{2k+1}, \frac{1}{2k}\right)$ and decreasing in the interval $\left(\frac{1}{2k+2}, \frac{1}{2k+1}\right)$ (k being a non-negative integer).

Example 6.18 A function $y = f(x)$ is represented parametrically as follows:

$x = \phi(t) = t^5 - 5t^3 - 20t + 7$

$y = \psi(t) = 4t^3 - 3t^2 - 18t + 3$

$-2 < t < 2$

Find the intervals of monotonocity.

Sol. We have $x=\phi(t)=t^5-5t^3-20t+7$

$y=\psi(t)=4t^3-3t^2-18t+3$

$\therefore \dfrac{dx}{dt}=\phi'(t)=5t^4-15t^2-20=5(t^2-4)(t^2+1)$

$\dfrac{dy}{dt}=\psi'(t)=12t^2-6t-18=6(t+1)(2t-3)$

$\dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}=\dfrac{6(t+1)(2t-3)}{5(t-2)(t+2)(t^2+1)}$

The sign scheme of $\dfrac{dy}{dx}$ is given in Fig. 6.11.

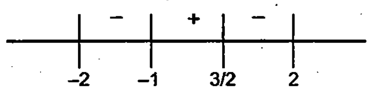


Fig. 6.11

$y=f(x)$ increases if $\dfrac{dy}{dx}>0$

$\Rightarrow t\in(-1, 3/2)$

$y=f(x)$ decreases if $\dfrac{dy}{dx}<0$

$\Rightarrow t\in(-2,-1)\cup(3/2,2)$

Example 6.19 Let $g(x)=f(x)+f(1-x)$ and $f''(x)>0$, $\forall x\in(0,1)$. Find the intervals of increase and decrease of $g(x)$.

Sol. We have $g(x)=f(x)+f(1-x)$, then

$g'(x)=f'(x)-f'(1-x)$ (1)

We are given that $f''(x)>0, \forall x\in(0,1)$.

It means that $f'(x)$ would be increasing on (0, 1) which leads to two cases.

Case I: Let $g(x)$ is increasing,

$\Rightarrow f'(x)-f'(1-x)>0$

$\Rightarrow f'(x)>f'(1-x)$

$\Rightarrow x>1-x$ (as f' is increasing)

$\Rightarrow \dfrac{1}{2}<x<1$

$\Rightarrow g(x)$ is increasing in $\left(\dfrac{1}{2},1\right)$.

Case II: Let $g(x)$ is decreasing,

$\Rightarrow f'(x)-f'(1-x)<0$

$\Rightarrow f'(x)<f'(1-x)$

$\Rightarrow x<1-x$ (as f' is increasing)

$\Rightarrow 0<x<\dfrac{1}{2}$

$\Rightarrow g(x)$ is decreasing in $\left(0,\dfrac{1}{2}\right)$.

Example 6.20 Find the number of solution of the equation $3\tan x+x^3=2$ in $\left(0,\dfrac{\pi}{4}\right)$.

Sol. Let $f(x)=3\tan x+x^3-2$.

Then $f'(x)=3\sec^2x+3x^2>0$, hence $f(x)$ increases.

Also, $f(0)=-2$ and $f\left(\dfrac{\pi}{4}\right)>0$.

So, by intermediate value theorem, $f(c)=2$ for some c in $\left(0,\dfrac{\pi}{4}\right)$.

Hence, $f(x)=0$ has only one root

Concept Application Exercise 6.1

1. Prove that the following functions are strictly increasing:
   a. $f(x)=\cot^{-1}x+x$
   b. $f(x)=\log(1+x)-\dfrac{2x}{2+x}$
2. Separate the intervals of monotonocity for the following functions:
   a. $f(x)=-2x^3-9x^2-12x+1$
   b. $f(x)=x^2e^{-x}$
   c. $f(x)=\sin x+\cos x, x\in(0,2\pi)$
   d. $f(x)=3\cos^4x+10\cos^3x+6\cos^2x-3, x\in[0,\pi]$
3. Discuss monotonocity of $f(x)=\dfrac{x}{\sin x}$ and $g(x)=\dfrac{x}{\tan x}$, where $0<x\le 1$.
4. Discuss monotonocity of $y=f(x)$ which is given by $x=\dfrac{1}{1+t^2}$ and $y=\dfrac{1}{t(1+t^2)}$, $t>0$.
5. Find the value of a for which the function $(a+2)x^3-3ax^2+9ax-1$ decreases monotonically for all real x.
6. Find the value of a in order that $f(x)=\sqrt{3}\sin x-\cos x-2ax+b$ decreases for all real values of x.
7. Discuss the monotonocity of function $f(x)=2\log|x-1|-x^2+2x+3$.
8. Let $g(x)=f(\log x)+f(2-\log x)$ and $f''(x)<0$, $\forall x\in(0,3)$. Then find the interval in which $g(x)$ increases.

POINT OF INFLECTION

For continuous function $f(x)$, if $f''(x_0)=0$ or $f''(x_0)$ does not exist at points where $f'(x_0)$ exists and if $f''(x)$ changes sign when passing through $x=x_0$, then x_0 is called the point of inflection. At the point of inflection, the curve changes its concavity, i.e.,

a. If $f''(x)<0$, $x\in(a,b)$, then the curve $y=f(x)$ is concave downward in (a,b).

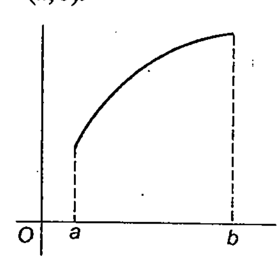


Fig. 6.12

b. If $f''(x) > 0, x \in (a, b)$, then the curve $y = f(x)$ is concave upward in (a, b).

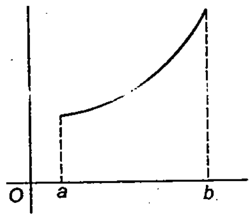


Fig. 6.13

Consider function $f(x) = x^3$. At $x = 0, f'(x) = 0$. Also, $f''(x) = 0$ at $x = 0$. Such a point is called the point of inflection. Here the 2nd derivative is zero.

Consider the function $f(x)$ whose graph is given in Fig. 6.14.

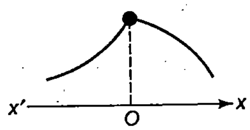


Fig. 6.14

Here $f(x)$ is non-differentiable at $x = c$, but curve changes its concavity. Hence, $x = c$ is the point of inflection.

Example 6.21 Find the points of inflection for

a. $f(x) = \sin x$

b. $f(x) = 3x^4 - 4x^3$

c. $f(x) = x^{1/3}$

Sol. **a.** $f(x) = \sin x$

$\Rightarrow \quad f'(x) = \cos x$

$\Rightarrow \quad f''(x) = -\sin x$

$f''(x) = 0 \Rightarrow x = n\pi, n \in Z$

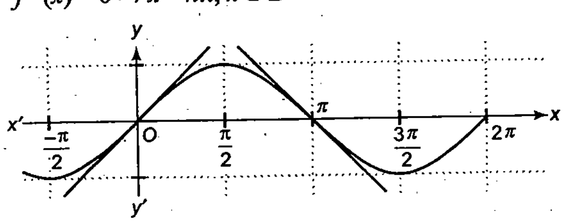


Fig. 6.15

b. $f(x) = 3x^4 - 4x^3$

$\Rightarrow \quad f'(x) = 12x^3 - 12x^2$

$\Rightarrow \quad f''(x) = 36x^2 - 24x$

Now $f''(x) = 0 \Rightarrow x = 0$ and $\frac{2}{3}$ are the points of inflection.

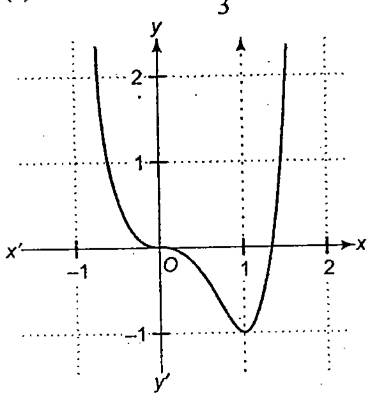


Fig. 6.16

c. $f(x) = x^{1/3} \Rightarrow f'(x) = \dfrac{1}{3x^{2/3}}$

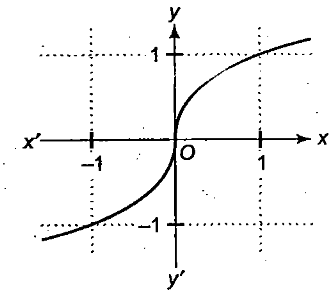


Fig. 6.17

$f(x)$ is non-differentiable at $x = 0$, but curve changes its concavity, hence $x = 0$ is the point of inflection.

Inequalities Using Monotonocity

Example 6.22 Prove that $\ln(1 + x) < x$ for $x > 0$.

Sol. Let us assume $f(x) = \ln(1 + x) - x$.

Investigating the behaviour of $f(x)$, i.e.,

$$f'(x) = \frac{1}{1+x} - 1 = \frac{-x}{1+x}$$

In the domain of $f(x)$, $f'(x) > 0$ for $x \in (-1, 0)$ and $f'(x) < 0 \ \forall x \in (0, \infty)$.

Hence, for $x > 0$, $f(x)$ is decreasing.

Moreover, $f(0) = 0$. Hence, further $f(x) < 0 \Rightarrow \ln(1 + x) - x < 0$

$\Rightarrow \quad \ln(1 + x) < x$

Example 6.23 Show that $0 < x \sin x - \frac{1}{2}\sin^2 x < \frac{(\pi-1)}{2}$, $\forall x \in \left(0, \frac{\pi}{2}\right)$.

Sol. Let $f(x) = x \sin x - \frac{1}{2}\sin^2 x$

$\Rightarrow \quad f'(x) = x \cos x + \sin x - \sin x \cos x$

$= \sin x(1 - \cos x) + x \cos x$

For $x \in \left(0, \frac{\pi}{2}\right)$, $\sin x > 0$, $1 - \cos x > 0$, $\cos x > 0$

$\Rightarrow \quad f'(x) > 0 \ \forall x \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow \quad f(x)$ is strictly increasing in $\left(0, \frac{\pi}{2}\right)$

$\Rightarrow \quad$ The range of $f(x)$ is $\left(\lim_{x\to 0} f(x), \lim_{x\to \pi/2} f(x)\right)$

$\equiv \left(0, \frac{\pi-1}{2}\right)$

$\Rightarrow \quad 0 < x \sin x - \frac{1}{2}\sin^2 x < \frac{\pi - 1}{2}$

Example 6.24 If $a, b > 0$ and $0 < p < 1$, then prove that $(a + b)^p < a^p + b^p$.

Sol. Let $f(x) = (1 + x)^p - 1 - x^p, x > 0$

$\Rightarrow \quad f'(x) = p(1 + x)^{p-1} - p x^{p-1} = p\{(1 + x)^{p-1} - x^{p-1}\}$ (1)

Now, $1 + x > x$

$\Rightarrow \quad (1+x)^{1-p} > x^{1-p} \qquad (\because 1-p>0)$

$\Rightarrow \quad \frac{1}{(1+x)^{p-1}} > \frac{1}{x^{p-1}} \quad \Rightarrow \quad (1+x)^{p-1} < x^{p-1}$

$\Rightarrow \quad (1+x)^{p-1} - x^{p-1} < 0$ (2)

From equations (1) and (2), we get $f'(x) < 0$

$\Rightarrow$ $f(x)$ is a decreasing function.

Now, $f(0)=0$

$\because \quad x>0 \quad \therefore f(x)<f(0)$

$\Rightarrow \quad (1+x)^p - 1 - x^p < 0 \quad \Rightarrow \quad (1+x)^p < 1 + x^p$

Put $x = \frac{a}{b}$, hence $(a+b)^p < a^p + b^p$.

Example 6.25 Prove that $|\cos\alpha - \cos\beta| \le |\alpha - \beta|$.

Sol. Here, first we have to select an appropriate function.

Let $f(x) = x + \cos x$

$\Rightarrow \quad f'(x) = 1 - \sin x \ge 0 \qquad (\because -1 \le \sin x \le 1)$

Hence, $f(x)$ is a monotonically increasing function.

For $\alpha \ge \beta \quad \Rightarrow \quad f(\alpha) \ge f(\beta)$

or $\alpha + \cos\alpha \ge \beta + \cos\beta$ or $\cos\beta - \cos\alpha \le \alpha - \beta$

$\Rightarrow \quad -(\cos\alpha - \cos\beta) \le (\alpha - \beta)$ (1)

and for $\alpha \le \beta$

$\Rightarrow \quad f(\alpha) \le f(\beta)$ or $\alpha + \cos\alpha \le \beta + \cos\beta$

or $\cos\alpha - \cos\beta \le (-\alpha + \beta)$ (2)

Combining equations (1) and (2), we get $|\cos\alpha - \cos\beta| \le |\alpha - \beta|$.

Example 6.26 For $0 < x < \frac{\pi}{2}$, prove that $\cos(\sin x) > \sin(\cos x)$.

Sol. Let $f(x) = x - \sin x \Rightarrow f'(x) = 1 - \cos x > 0 \left(\because 0 < x < \frac{\pi}{2}\right)$

Hence, $f(x)$ is an increasing function in $x \in \left(0, \frac{\pi}{2}\right)$.

$\because \quad x > 0$, then $f(x) > f(0)$ or $x - \sin x > 0$

$\Rightarrow \quad x > \sin x$ (1)

Again, $0 < x < \frac{\pi}{2}$, therefore $0 < \cos x < 1$.

$\cos x > \sin(\cos x)$ [From (1)] (2)

Now, in $\left(0, \frac{\pi}{2}\right)$, $\cos x$ is monotonically decreasing.

$\Rightarrow \quad \cos x < \cos(\sin x)$ [From (1)] (3)

From equations (2) and (3), we get

$\sin(\cos x) < \cos x < \cos(\sin x)$

Hence, $\sin(\cos x) < \cos(\sin x)$.

Concept Application Exercise 6.2

1. Show that $\frac{x}{(1+x)} < \ln(1+x)$ for $x > 0$.
2. For $0 < x \le \frac{\pi}{2}$, show that $x - \frac{x^3}{6} < \sin x < x$.
3. Show that $\tan^{-1} x > \frac{x}{1+\frac{x^2}{3}}$, if $x \in (0, \infty)$.
4. Prove that $f(x) = \frac{\sin x}{x}$ is monotonically decreasing in $[0, \pi/2]$.

   Hence, prove that $\frac{2x}{\pi} < \sin x < x$ for $x \in (0, \pi/2)$.

EXTREMUM

Introduction

The notion of optimizing functions is one of the most useful applications of calculus used in almost every sphere of life including geometry, business, trade, industries, economics, medicines and even at home. In this section, we shall see how calculus defines the notion of maxima and minima and distinguishes it from the greatest and least value, or global maxima and global minima of a function.

Critical points of a function

Critical point of a function of a real variable is any value in the domain where either the function is not differentiable or its derivative is 0.

Basic Theorem of Maxima and Minima

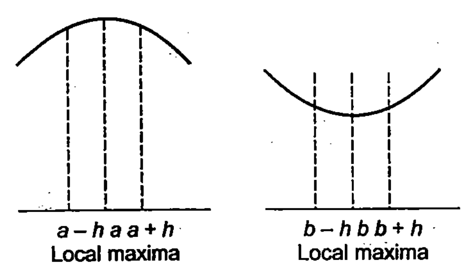


Fig. 6.18

A function $f(x)$ is said to have a maximum at $x = a$ if $f(a)$ is greater than every other value assumed by $f(x)$ in the immediate neighbourhood of $x = a$.

Symbolically, $\left.\begin{matrix} f(a) > f(a+h) \\ f(a) > f(a-h) \end{matrix}\right] \Rightarrow x = a$ gives maxima for a sufficiently small positive h.

Similarly, a function $f(x)$ is said to have a minimum value at $x = b$ if $f(b)$ attains the least value than every other value assumed by $f(x)$ in the immediate neighbourhood at $x = b$.

Symbolically, if $\left.\begin{matrix} f(b) < f(b+h) \\ f(b) < f(b-h) \end{matrix}\right] \Rightarrow x = b$ gives minima for a sufficiently small positive h.

Note:

- *The maximum and minimum values of a function are also known as local/relative maxima or local/relative minima as these are the greatest and least values of the function relative to some neighbourhood of the point in question.*
- *The term 'extremum' or (extremal) or 'turning value' is used for both maximum and minimum values.*
- *A maximum (minimum) value of a function may not be the greatest (least) value in a finite interval.*
- *A function can have several maximum and minimum values, and a minimum value may even be greater than a maximum value.*

- *The maximum and minimum values of a continuous function occur alternately, and between two consecutive maximum values there is a minimum value and vice versa.*

Example 6.27 The function $f(x) = (x^2 - 4)^n (x^2 - x + 1)$, $n \in N$ assumes a local minimum value at $x = 2$, then find the possible values of n.

Sol. $f(x) = (x^2 - 4)^n (x^2 - x + 1)$

$f(2) = 0,$

Now, $x^2 - x + 1 > 0$ for $\forall x$

$$f(2^+) = \lim_{x \to 2^+} (x^2 - 4)^n (x^2 - x + 1)$$
$$= 3 \lim_{h \to 0} ((h+2)^2 - 4)^n$$
$$= 3 \lim_{h \to 0} (4h + h^2)^n$$
$$> 0$$
$$f(2^-) = \lim_{x \to 2^-} (x^2 - 4)^n (x^2 - x + 1)$$
$$= 3 \lim_{h \to 0} ((h-2)^2 - 4)^n$$
$$= 3 \lim_{h \to 0} (h^2 - 4h)^n$$
$$= 3 \times (\text{very small negative value})^n$$

For $x = 0$ to be a point of minima, we must have $f(2^-) > 0$ for which n must be an even integer.

TESTS FOR LOCAL MAXIMUM/MINIMUM

When $f(x)$ is Differentiable at $x = a$

First-order Derivative Test in Ascertaining the Maxima or Minima

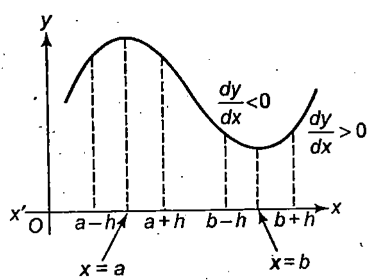


Fig. 6.19

Consider the interval $(a - h, a)$. For this interval, we find $f(x)$, i.e., increasing $\frac{dy}{dx} > 0$. Similarly, for the interval $(a, a + h)$, we find $f(x)$ is decreasing, i.e., $\frac{dy}{dx} < 0$. Hence, at the point $x = a$ (maxima); $\frac{dy}{dx} = 0$.

Similarly, $\frac{dy}{dx} = 0$ at $x = b$ which is the point of minima. Hence, $\frac{dy}{dx} = 0$ is the necessary condition for maxima or minima.

These points, where $\frac{dy}{dx}$ vanishes, are known as stationary points as instantaneous rate of change of function momentarily ceases at these points.

Hence, if $\left.\begin{array}{l} f'(a-h) > 0 \\ f'(a+h) < 0 \end{array}\right] \Rightarrow x = a$ is a point of local maxima,

where $f'(a) = 0$. It means that $f'(x)$ should change its sign from positive to negative.

Similarly, $\left.\begin{array}{l} f'(b-h) < 0 \\ f'(b+h) > 0 \end{array}\right] \Rightarrow x = b$ is a point of local minima,

where $f'(b) = 0$. It means that $f'(x)$ should change its sign from negative to positive.

However, if $f'(x)$ does not change sign, i.e., has the same sign in a certain complete neighbourhood of c, then $f(x)$ is either increasing or decreasing throughout this neighbourhood implying that $f(c)$ is not an extreme value of f, e.g., $f(x) = x^3$ at $x = 0$.

Second-Order Derivative Test in Ascertaining the Maxima or Minima

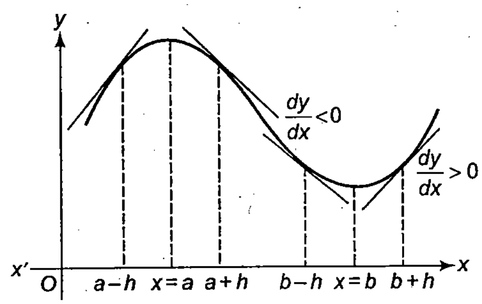


Fig. 6.20

As shown in the Fig. 6.20, it is clear that as x increases from $a - h$ to $a + h$, the function $\frac{dy}{dx}$ continuously decreases, i.e., positive for $x < a$, zero at $x = a$ and negative for $x > a$. Hence, $\frac{dy}{dx}$ itself is a decreasing function.

Therefore, $\frac{d^2y}{dx^2} < 0$ in $(a - h, a + h)$.

Hence, at local maxima, $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$.

Similarly, at local minima, $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$.

However, if $\frac{d^2y}{dx^2} = 0$, then the test fails. In this case, f can still have a maxima or minima or point of inflection (neither maxima nor minima). In this case, revert back to the first-order derivative check for ascertaning the maxima or minima.

nth Derivative Test

It is nothing but the general version of the second-order derivative test. It says that if $f'(a) = f''(a) = f'''(a) = \cdots f^n(a) = 0$ and $f^{n+1}(a) \neq 0$ (all derivatives of the function up to order n vanish and $(n+1)$th order derivative does not vanish at $x = a$), then $f(x)$ would have a local maximum or minimum at $x = a$ iff n is an odd natural number and that $x = a$ would be a point of local maxima if $f^{n+1}(a) < 0$, and would be a point of local minima if $f^{n+1}(a) > 0$. However, if n is even, then f has neither a maxima nor a minima at $x = a$.

Example 6.28 The function $y = \dfrac{ax+b}{(x-1)(x-4)}$ has turning point at $P(2,-1)$. Then find the value of a and b.

Sol. $y = \dfrac{ax+b}{(x-1)(x-4)} = \dfrac{ax+b}{x^2-5x+4}$ has turning point at $P(2,-1)$

$\Rightarrow$ $P(2,-1)$ lies on the curve $\Rightarrow 2a+b=2$ (1)

Also, $\dfrac{dy}{dx}=0$ at $P(2,-1)$

Now, $\dfrac{dy}{dx} = \dfrac{a(x^2-5x+4)-(2x-5)(ax+b)}{(x^2-5x+4)^2}$

At $P(2,-1)$, $\dfrac{dy}{dx} = \dfrac{-2a+2a+b}{4} = 0$

$\Rightarrow\ b=0 \ \Rightarrow\ a=1.$ [from equation (1)]

Example 6.29 Let $f: [a, b] \to R$ be a function such that for $c \in (a, b)$, $f'(c) = f''(c) = f'''(c) = f^{iv}(c) = f^{v}(c) = 0$, then

a. f has a local extremum at $x = c$

b. f has neither local maximum nor minimum at $x = c$

c. f is necessarily a constant function

d. it is difficult to say whether (*a*) or (*b*).

Sol. d. For $f(x) = x^6$ and $f(x) = x^7$, $f'(0) = f''(0) = f'''(0) = f^{iv}(0) = f^{v}(0) = 0$

$x = 0$ is point of minima for $f(x) = x^6$

But $x = 0$ is not point of maxima/minima for $f(x) = x^7$

Hence, it is difficult to say anything.

Example 6.30 Discuss the extremum of $f(x) = 40/(3x^4 + 8x^3 - 18x^2 + 60)$.

Sol. $f(x) = \dfrac{40}{3x^4+8x^3-18x^2+60}$

$$\Rightarrow f'(x) = -\frac{40(12x^3+24x^2-36x)}{(3x^4+8x^3-18x^2+60)^2}$$

$$= -\frac{12x(x^2+2x-3)}{(3x^4+8x^3-18x^2+60)^2}$$

$$= \frac{-12x(x-1)(x+3)}{(3x^4+8x^3-18x^2+60)^2}$$

The sign scheme of $f'(x)$ is given in Fig. 6.21

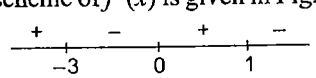


Fig. 6.21

Hence, $x = -3$ and $x = 1$ are the points of maxima and $x = 0$ is the point of minima.

Example 6.31 Discuss the extremum of $f(x) = \sin x\,(1 + \cos x)$, $x \in (0, \pi/2)$.

Sol. Let $f(x) = \sin x\,(1 + \cos x)$

$\Rightarrow\ f'(x) = \cos 2x + \cos x$

and $f''(x) = -2\sin 2x - \sin x = -(2\sin 2x + \sin x)$

For maximum or minimum value of $f(x)$, $f'(x) = 0$.

$\Rightarrow\ \cos 2x + \cos x = 0$

$\Rightarrow\ \cos x = -\cos 2x$

$\Rightarrow\ \cos x = \cos(\pi \pm 2x)$

$\therefore\ x = \pi \pm 2x$ or $x = \dfrac{\pi}{3}$

Now, $f''\left(\dfrac{\pi}{3}\right) = -2\sin\dfrac{2\pi}{3} - \sin\dfrac{\pi}{3}$

$$= -2\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\frac{3\sqrt{3}}{2} = -\text{ve}$$

Hence, $f(x)$ has maxima at $x = \dfrac{\pi}{3}$.

Example 6.32 Discuss the maxima/minima of the function

$$f(x) = \frac{4\sin x - 2x - x\cos x}{2+\cos x},\ 0 < x < 2\pi.$$

Sol. $y = f(x) = \dfrac{4\sin x - 2x - x\cos x}{2+\cos x},\ 0<x<2\pi$

$$= \frac{4\sin x}{2+\cos x} - x \quad (1)$$

$$f'(x) = \frac{(2+\cos x)4\cos x + 4\sin^2 x}{(2+\cos x)^2} - 1$$

$$= \frac{\cos x(4-\cos x)}{(2+\cos x)^2}$$

$f'(x) = 0$ at $\cos x = 0$, i.e., at $x = \pi/2, 3\pi/2$

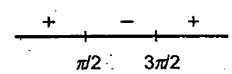


Fig. 6.22

Hence, $f(x)$ is an increasing function in $(0, \pi/2) \cup (3\pi/2, 2\pi)$ and decreasing function in $(\pi/2, 3\pi/2)$. Also $x = (\pi/2)$ is the point of maxima and $x = (3\pi/2)$ is the point of minima

Example 6.33 Discuss the extremum of $f(x) = x^2 + \dfrac{1}{x^2}$.

Sol. $f(x) = x^2 + \dfrac{1}{x^2}$

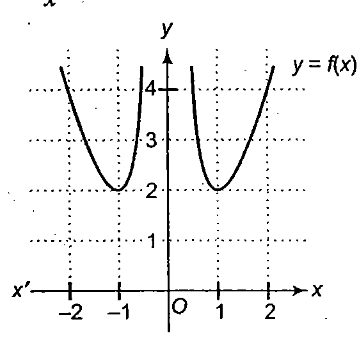


Fig. 6.23

$f'(x) = 2x - \dfrac{2}{x^3}$

Let $f'(x) = 0 \Rightarrow x^4 = 1 \Rightarrow x = \pm 1$

Also, $f''(x) = 2 + \dfrac{6}{x^4} > 0$ for all $x \neq 0$

$\Rightarrow$ Both the points $x = 1$ and $x = -1$ are the points of minima.

Note:
Here two consecutive points of extrema are minima, this is because f(x) is discontinuous at x = 0. However, discontinuous function can also have two consecutive points of extrema of which one is maxima and the other minima, e.g. for $f(x) = x + \frac{1}{x}$. For continuous function, consecutive points of extrema are maxima and minima.

Example 6.34 Find the maximum value of $f(x) = \left(\frac{1}{x}\right)^x$.

Sol. $f(x) = \left(\frac{1}{x}\right)^x \Rightarrow f'(x) = \left(\frac{1}{x}\right)^x \left(\log\frac{1}{x} - 1\right)$

$f'(x) = 0 \Rightarrow \log\frac{1}{x} = 1 \Rightarrow \frac{1}{x} = e \Rightarrow x = \frac{1}{e}$

Also, for $x < 1/e$, $f'(x)$ is positive and for $x > 1/e$, $f'(x)$ is negative.

Hence, $x = 1/e$ is point of maxima.

Therefore, the maximum value of function is $e^{1/e}$.

Also, $\lim\limits_{x\to 0}\left(\frac{1}{x}\right)^x = e^{\lim\limits_{x\to 0} x\log\left(\frac{1}{x}\right)} = e^{-\lim\limits_{x\to 0} x\log x} = e^0 = 1$

$\lim\limits_{x\to\infty}\left(\frac{1}{x}\right)^x = 0$

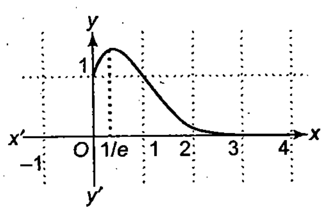


Fig. 6.24

Example 6.35 Let $f(x) = \frac{a}{x} + x^2$. If it has a maximum at $x = -3$, then find the value of a.

Sol. $f'(x) = -\frac{a}{x^2} + 2x$

For $f'(x) = 0$, $x^3 = \frac{a}{2}$

For $x = -3$, $a = -54$

Now, $f''(x) = \frac{2a}{x^3} + 2 \Rightarrow f''(-3) = \frac{-54}{(-3)^3} + 2 = 0$

Hence, $f(x)$ cannot have maxima at $x = -3$.

Example 6.36 a. Discuss the extrema of $f(x) = \frac{x}{1 + x\tan x}$, $x \in \left(0, \frac{\pi}{2}\right)$.

b. Discuss the extremum of $f(x) = a\sec x + b\operatorname{cosec} x$, $0 < a < b$.

Sol. a. $f'(x) = \frac{1 - x^2\sec^2 x}{(1 + x\tan x)^2} = \frac{\sec^2 x(\cos x + x)(\cos x - x)}{(1 + x\tan x)^2}$

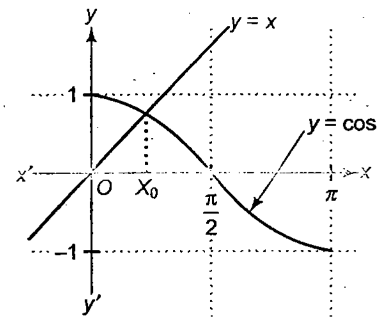


Fig. 6.25

Clearly $f'(x_0) = 0$

and $f'(x) > 0\ \forall\, x \in (0, x_0)$

$f'(x) < 0\ \forall\, x \in (x_0, \pi/2)$.

Thus, $x = x_0$ is the only point of maxima for $y = f(x)$.

b. $f(x) = a\sec x + b\operatorname{cosec} x$, $0 < a < b$.

$f'(x) = a\sec x\tan x - b\operatorname{cosec} x\cot x$

Let $f'(x) = 0 \Rightarrow a\frac{\sin x}{\cos^2 x} = b\frac{\cos x}{\sin^2 x}$

$\Rightarrow \tan^3 x = b/a$

$\Rightarrow x = \tan^{-1}\left(\frac{b}{a}\right)^{1/3}\quad a, b > 0$

$\Rightarrow x = \tan^{-1}\left(\frac{b}{a}\right)^{1/3} > 0$

$\Rightarrow$ x lies in either the first or third quadrant for extremum.

Case I: $0 < x < \pi/2$

$\lim\limits_{x\to 0}(a\sec x + b\operatorname{cosec} x) \to \infty$

$\lim\limits_{x\to\pi/2}(a\sec x + b\operatorname{cosec}) \to \infty$

Also $f(x)$ is +ve for this value of x.

Hence, only one point of extremum is the point of minima.

and $\tan x = \left(\frac{b}{a}\right)^{1/3}$

$\Rightarrow \cos x = \frac{a^{1/3}}{\sqrt{a^{2/3} + b^{2/3}}}$, $\sin x = \frac{b^{1/3}}{\sqrt{a^{2/3} + b^{2/3}}}$

$\Rightarrow$ Minimum value of $f = \frac{a\sqrt{a^{2/3} + b^{2/3}}}{a^{1/3}} + \frac{b\sqrt{a^{2/3} + b^{2/3}}}{b^{1/3}}$

$= (a^{2/3} + b^{2/3})^{3/2}$

Case II: $\pi < x < 3\pi/2$

$\lim\limits_{x\to\pi}(a\sec x + b\operatorname{cosec} x) \to -\infty$

$\lim\limits_{x\to 3\pi/2}(a\sec x + b\operatorname{cosec} x) \to -\infty$

Also $f(x)$ is –ve for this values of x

Hence, only one point of extremum is the point of maxima.

$\Rightarrow$ Maximum value $f_{max} = -(a^{2/3} + b^{2/3})^{3/2}$

Example 6.37 Find the range of the function

$$f(x)=\frac{x^4+x^2+5}{(x^2+1)^2}.$$

Sol.

$$f(x)=\frac{x^4+x^2+5}{x^4+2x^2+1}=\frac{(x^4+2x^2+1)+4-x^2}{(x^4+2x^2+1)}=1+\frac{4-x^2}{(x^4+2x^2+1)}$$

Let $g(x)=\dfrac{4-x^2}{(x^2+1)^2}$

$g'(x)=0$

$\Rightarrow g'(x)=(x^2+1)^2\cdot(-2x)-(4-x^2)\cdot 2(x^2+1)\cdot 2x=0$

$\Rightarrow (x^2+1)2x[-(x^2+1)-2(4-x^2)]=0$

$\Rightarrow x=0$ or $x^2=9 \Rightarrow x=3$ or -2

$g(3)=-\dfrac{1}{20}=g(-3)$ $\quad(\because g(x)$ is even)

$f(3)=1-\dfrac{1}{20}=\dfrac{19}{20}.$

Also, $\lim\limits_{x\to\pm\infty} f(x)=1$ and $f(0)=5$

Hence, range is $\left[\dfrac{19}{20},5\right]$

When $F(x)$ is not Differentiable at $x = a$

Case 1: When $f(x)$ is continuous at $x=a$ and $f'(a-h)$ and $f'(a+h)$ exist, and are non-zero, then $f(x)$ has a local maximum or minimum at $x=a$ if $f'(a-h)$ and $f'(a+h)$ are of opposite signs.

If $f'(a-h)>0$ and $f'(a+h)<0$, then $x=a$ will be the point of local maximum.

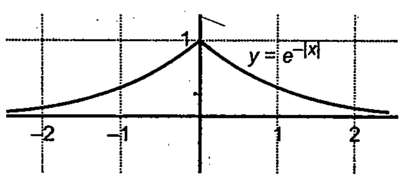


Fig. 6.26

If $f'(a-h)<0$ and $f'(a+h)>0$, then $x=a$ will be the point of local minimum.

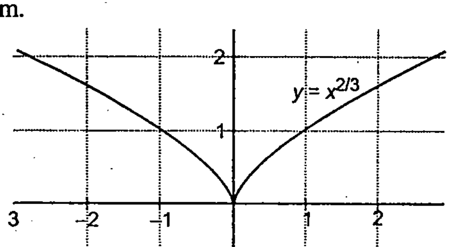


Fig. 6.27

Case 2: When $f(x)$ is continuous and $f'(a-h)$ and $f'(a+h)$ exist but one of them is zero, then we can infer the information about the existence of local maxima/minima from the basic definition of local maxima/minima.

Case 3: If $f(x)$ is not continuous at $x=a$, then compare the values of $f(x)$ at the neighbouring points of $x=a$.

It is advisable to draw the graph of the function in the vicinity of the point $x=a$, because the graph would give us the clear picture about the existence of local maxima/minima at $x=a$.

Consider the following cases:

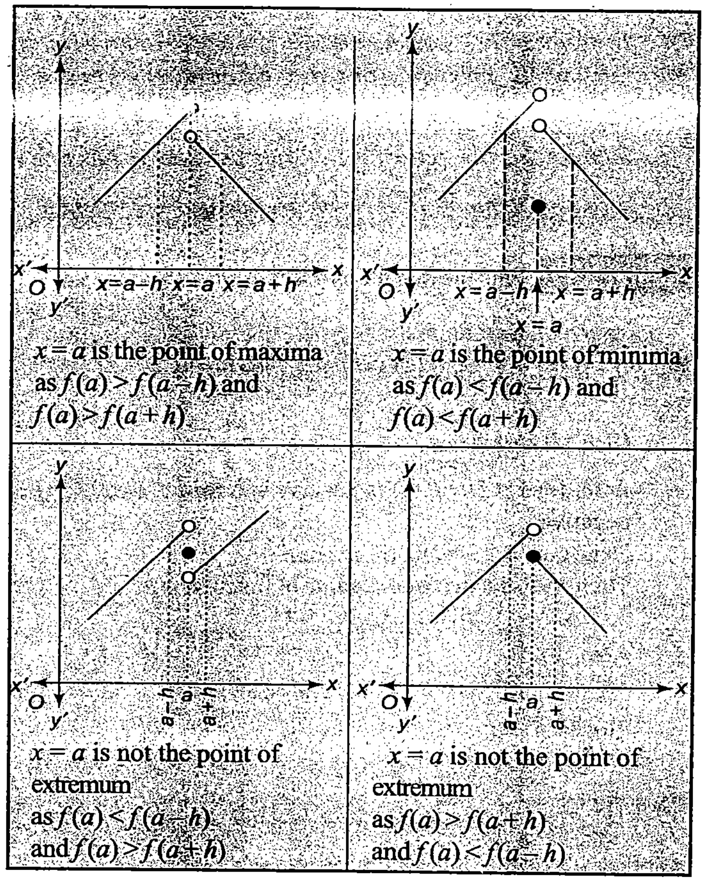


Fig. 6.28

Example 6.38 If $f(x)=\begin{cases}x^2, & x\le 0\\ 2\sin x, & x>0\end{cases}$, investigate the function at $x=0$ for maxima/minima.

Sol. Analysing the graph of $f(x)$, we get $x=0$ as the point of minima.

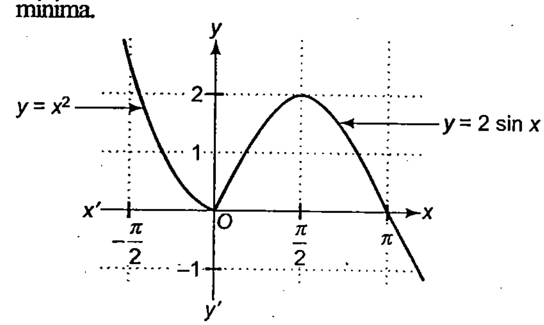


Fig. 6.29

Also, derivative changes sign from –ve to +ve and $f(x)$ is continuous at $x=0$, hence $x=0$ is the point of minima.

Note:

We cannot say that the change of sign of derivative helps to determine minima because if the function was given as

$$f(x)=\begin{cases}x^2, & x<0\\ 2, & x=0\\ 2\sin x, & x>0\end{cases}$$

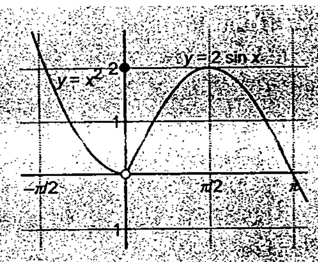


Fig. 6.30

$$\Rightarrow f'(x)=\begin{cases}2x, & x<0\\ \text{non-diff}, & x=0\\ 2\cos x, & x>0\end{cases}$$

Here also the derivative is changing sign in the same manner but the point $x = 0$ is the point of maxima as $f(0^-) < f(0)$ and $f(0^+) < f(0)$.

This type of problem happens particularly with discontinuous functions.

Example 6.39 Let $f(x)=\begin{cases}x^3+x^2+10x, & x<0\\ -3\sin x, & x\geq 0\end{cases}$. Investigate $x=0$ for local maxima/minima.

Sol. Clearly $f(x)$ is continuous at $x=0$ as $f(0)=f(0^-)=f(0^+)=0$.

$$f'(0^-)=\lim_{h\to 0}\frac{f(-h)-f(0)}{-h}$$

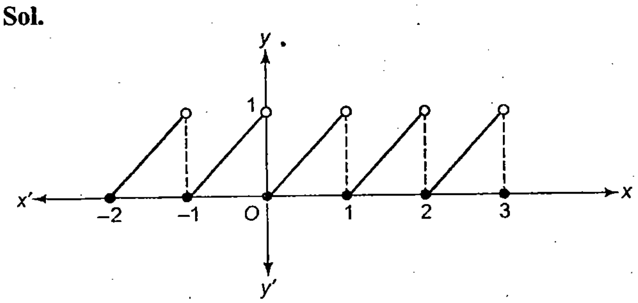
$$=\lim_{h\to 0}\frac{-h^3+h^2-10h-0}{-h}=10$$

But $f'(0^+)=\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\frac{-3\sinh}{h}=-3$

Since $f'(0^-)>0$ and $f'(0^+)<0$, $x=0$ is the point of local maxima.

Example 6.40 Test $f(x)=\{x\}$ for the existence of a local maximum and minimum at $x = 1$, where $\{\cdot\}$ represents fractional part function.

Sol.



Graph of $y = \{x\}$

Fig. 6.31

Clearly $x = 1$ is the point of discontinuity of $f(x) = \{x\}$ as $f(1)=0$, $f(1-0)=1$ and $f(1+0)=0$.

Now $f(1-h)>0$ and $f(1+h)>0$, i.e., the value of the function at $x = 1$ is less than the values of the function at the neighbouring points. Thus, $x = 1$ is the point of minimum.

Example 6.41 $f(x)=\begin{cases}\cos\frac{\pi x}{2}, & x>0\\ x+a, & x\leq 0\end{cases}$. Find the values of a if $x=0$ is a point of maxima.

Sol. Clearly, $f(x)$ increases before $x = 0$ and decreases after $x=0$.

$f(0)=a$.

For $x=0$ to be the point of local maxima,

$f(0)\geq \lim_{x\to 0+} f(x)$

$\Rightarrow f(0)\geq \lim_{x\to 0+}\cos\left(\frac{\pi x}{2}\right)$

$\Rightarrow a\geq 1$

Graphical method

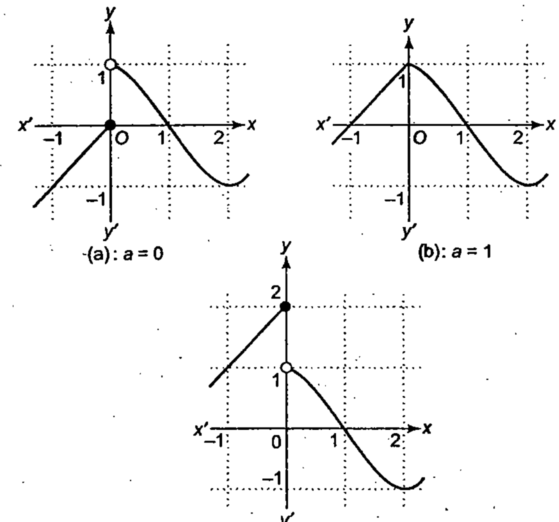


Fig. 6.32

For $a=0$, $x=0$ is not the point of extrema.

The graph of $y=x+a$ must shift at least 1 unit upward for $x=0$ to be the point of maxima.

Hence, $a\geq 1$.

Example 6.42 The function $f(x)=|ax-b|+c|x|\ \forall x\in(-\infty,\infty)$, where $a>0$, $b>0$, $c>0$. Find the condition if $f(x)$ attains the minimum value only at one point.

Sol. $f(x)=\begin{cases}b-(a+c)x, & x<0\\ b+(c-a)x, & 0\leq x<\frac{b}{a}\\ (a+c)x+b, & x\geq\frac{b}{a}\end{cases}$

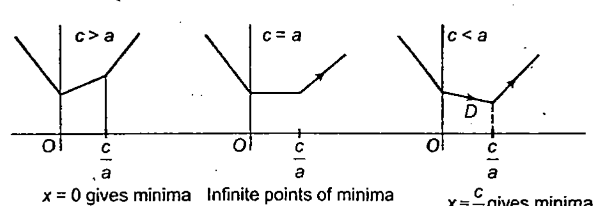


Fig. 6.33

Figure 6.33 clearly indicates that for exactly one point of minima, $a\neq c$.

Example 6.43 Discuss the extremum of $f(x)=2x+3x^{2/3}$.

Sol. $f(x)=2x+3x^{2/3}$

$f'(x)=2+3\times\frac{2}{3}x^{-1/3}=2(1+x^{-1/3})$

Let $f'(x)=0$

$\Rightarrow\ x^{1/3}+1=0\ \Rightarrow\ x=-1$

$\Rightarrow\ f''(x)=-\frac{2}{3}x^{-\frac{4}{3}}$

and $f''(-1)=-\frac{2}{3}(-1)^{-4/3}=-\frac{2}{3}<0$

$\Rightarrow\ x=-1$ is the point of maxima.

Also, $f(x)$ is non-differentiable at $x=0$.

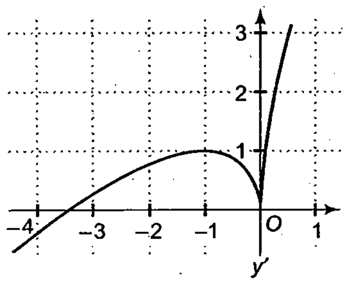


Fig. 6.34

From the graph, $x=0$ is the point of local minima.

CONCEPT OF GLOBAL MAXIMUM/MINIMUM

Let $y=f(x)$ be a given function with domain D. Let $[a, b]\subseteq D$. Global maximum/minimum of $f(x)$ in $[a, b]$ is basically the greatest/least value of $f(x)$ in $[a, b]$.

Global maximum and minimum in $[a, b]$ would occur at the critical point of $f(x)$ within $[a, b]$ or at the endpoints of the interval.

Global Maximum/Minimum in $[a, b]$

In order to find the global maximum and minimum of $f(x)$ in $[a, b]$, find all the critical points of $f(x)$ in (a, b). Let $c_1, c_2, \ldots, c_n$ be the different critical points. Find the value of the function at these critical points.

Let $f(c_1), f(c_2), \ldots, f(c_n)$ be the values of the function at critical points.

Say, $M_1=\max\{f(a), f(c_1), f(c_2),\ldots, f(c_n), f(b)\}$

and $M_2=\min\{f(a), f(c_1), f(c_2), \ldots, f(c_n), f(b)\}$,

then M_1 is the greatest value of $f(x)$ in $[a, b]$ and M_2 is the least value of $f(x)$ in $[a, b]$.

Global Maximum/Minimum in (a, b)

The method for obtaining the greatest and least values of $f(x)$ in (a, b) is almost same as the method used for obtaining the greatest and least values in $[a, b]$, however with a caution.

Let $y=f(x)$ be a function and $c_1, c_2, \ldots, c_n$ be the different critical points of the function in (a, b).

Let $M_1=\max.\{f(c_1), f(c_2), f(c_3), \ldots, f(c_n)\}$

and $M_2=\min.\{f(c_1), f(c_2), f(c_3), \ldots, f(c_n)\}$.

Now, if $\lim\limits_{\substack{x\to a+0\\(\text{or } x\to b-0)}} f(x)>M_1$ or $<M_2$, $f(x)$ will not have global maximum (or global minimum) in (a, b).

This means that if the limiting values at the endpoints are greater than M_1 or less than M_2, then $f(x)$ will not have global maximum/minimum in (a, b).

On the other hand, if $M_1>\lim\limits_{\substack{x\to a+0\\(\text{and } x\to b-0)}} f(x)$ and $M_2<\lim\limits_{\substack{x\to a+0\\(\text{and } x\to b-0)}} f(x)$,

then M_1 and M_2 will, respectively, be the global maximum and global minimum of $f(x)$ in (a, b).

Consider the following cases:

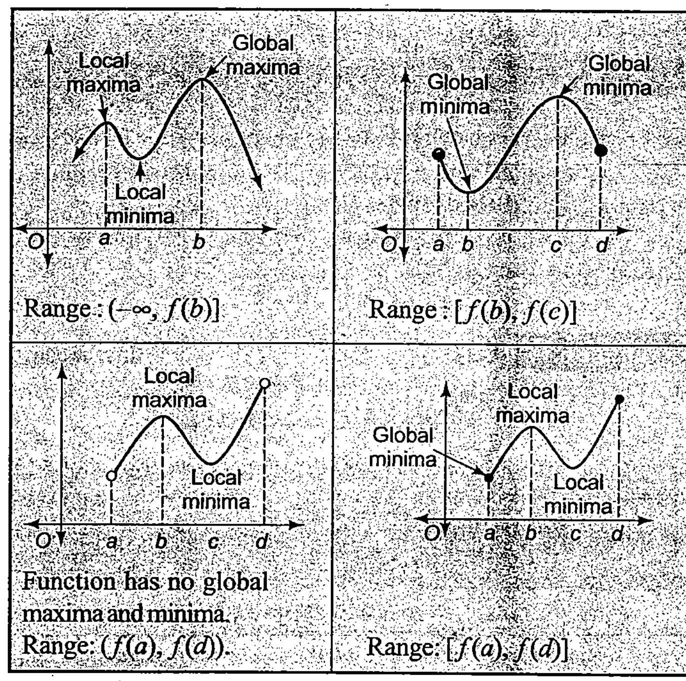


Fig. 6.35

Example 6.44 Let $f(x)=2x^3-9x^2+12x+6$. Discuss the global maxima and minima of $f(x)$ in $[0, 2]$ and $(1, 3)$ and hence, find the range of $f(x)$ for corresponding intervals.

Sol. $f(x)=2x^3-9x^2+12x+6$

$\Rightarrow\ f'(x)=6x^2-18x+12=6(x^2-3x+2)=6(x-1)(x-2)$

Clearly, the critical point of $f(x)$ in $[0, 2]$ is $x=1$.

Now, $f(0)=6, f(1)=11, f(2)=10$

Thus, $x=0$ is the point of minimum of $f(x)$ in $[0, 2]$ and $x=1$ is the point of global maximum.

Hence, range is $[6, 11]$.

For $x\in(1, 3)$, clearly $x=2$ is the only critical point in $(1, 3)$.

$f(2)=10.\ \lim\limits_{x\to 1^+} f(x)=11$ and $\lim\limits_{x\to 3^-} f(x)=15$

Thus, $x=2$ is the point of global minimum in $(1, 3)$ and the global maximum in $(1, 3)$ does not exist.

Hence, range is $[10, 15)$.

Example 6.45 Discuss the global maxima and global minima of

$$f(x)=\tan^{-1}x-\log_e x \text{ in } \left[\frac{1}{\sqrt{3}}, \sqrt{3}\right].$$

Sol. $f(x)=\tan^{-1}x-\ln x$

$$\Rightarrow f'(x)=\frac{1}{1+x^2}-\frac{1}{x}=-\frac{(x^2+1-x)}{x(x^2+1)}<0\ \forall x\in\left[\frac{1}{\sqrt{3}},\sqrt{3}\right]$$

Hence, $f_{min}=f(\sqrt{3})=\tan^{-1}\sqrt{3}-\ln\sqrt{3}=\frac{\pi}{3}-\ln\sqrt{3}$

$$f_{max}=f\left(\frac{1}{\sqrt{3}}\right)=\tan^{-1}\frac{1}{\sqrt{3}}-\ln\sqrt{3}=\frac{\pi}{6}-\ln\frac{1}{\sqrt{3}}$$

Example 6.46 Find the range of the function

$$f(x)=2\sqrt{x-2}+\sqrt{4-x}.$$

Sol. Clearly, domain of the function is [2, 4].

Now, $f'(x)=\frac{1}{\sqrt{x-2}}-\frac{1}{2\sqrt{4-x}}$

$f'(x)=0$

$\Rightarrow \sqrt{x-2}=2\sqrt{4-x}$

$\Rightarrow x-2=16-4x$

$\Rightarrow x=\frac{18}{5}$

Now, $f(2)=\sqrt{2}$, $f\left(\frac{18}{5}\right)=2\sqrt{\frac{18}{5}-2}+\sqrt{4-\frac{18}{5}}=\sqrt{10}$,

$f(4)=2\sqrt{2}$

Hence range of the function is $[\sqrt{2},\sqrt{10}]$.

Also, here $x=(18/5)$ is the point of global maxima.

Concept Application Exercise 6.3

1. Discuss the extremum of $f(x)=2x^3-3x^2-12x+5$ for $x\in[-2,4]$ and find the range of $f(x)$ for the given interval.
2. Discuss the extremum of $f(x)=1+2\sin x+3\cos^2 x$, $0\le x\le 2\pi/3$.
3. Discuss the extremum of $f(x)=\sin x+\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x, 0\le x\le\pi$.
4. Discuss the extremum of $f(\theta)=\sin^p\theta\cos^q\theta, p, q>0$, $0<\theta<\pi/2$.
5. Find the maximum and minimum values of the function $y=\log_e(3x^4-2x^3-6x^2+6x+1), \forall x\in(0,2)$. Given that $(3x^4-2x^3-6x^2+6x+1)>0\ \forall x\in(0,2)$.
6. Let $f(x)=-\sin^3 x+3\sin^2 x+5$ on $[0,\pi/2]$. Find the local maximum and local minimum of $f(x)$.
7. Discuss the extremum of $f(x)=\frac{1}{3}\left(x+\frac{1}{x}\right)$.
8. Discuss the extremum of $f(x)=x(x^2-4)^{-1/3}$.
9. Let $f(x)=\begin{cases}x^3-x^2+10x-5, & x\le 1\\ -2x+\log_2(b^2-2), & x>1\end{cases}$ Find the values of b for which $f(x)$ has the greatest value at $x=1$.
10. Let $f(x)$ be defined as $f(x)=\begin{cases}\tan^{-1}\alpha-5x^2, & 0<x<1\\ -6x, & x\ge 1\end{cases}$. If $f(x)$ has a maximum at $x=1$, then find the values of α.
11. Discuss the extremum of $f(x)=\begin{cases}|x^2-2|, & -1\le x<\sqrt{3}\\ \frac{x}{\sqrt{3}}, & \sqrt{3}\le x<2\sqrt{3}\\ 3-x, & 2\sqrt{3}\le x\le 4\end{cases}$.
12. Find the minimum value of $|x|+\left|x+\frac{1}{2}\right|+|x-3|+\left|x-\frac{5}{2}\right|$.
13. Discuss the extremum of $f(x)=\begin{cases}1+\sin x, & x<0\\ x^2-x+1, & x\ge 0\end{cases}$ at $x=0$.
14. Discuss the maxima and minima of the function $f(x)=x^{2/3}-x^{4/3}$. Draw the graph of $y=f(x)$ and find the range of $f(x)$.
15. The curve $f(x)=\frac{x^2+ax+6}{x-10}$ has a stationary point at (4, 1). Find the values of a and b. Also show that $f(x)$ has point of maxima at this point.

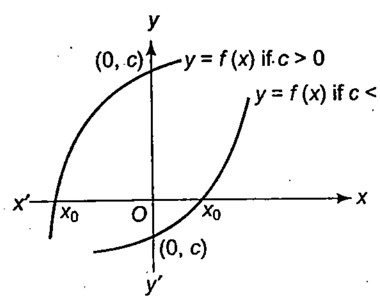
NATURE OF ROOTS OF CUBIC POLYNOMIALS

Let $f(x)=x^3+ax^2+bx+c$ be the given cubic polynomial, and $f(x)=0$ be the corresponding cubic equation, where $a, b, c\in R$.

Now, $f'(x)=3x^2+2ax+b$

Let $D=4a^2-12b=4(a^2-3b)$ be the discriminant of the equation $f'(x)=0$.

- If $D<0\Rightarrow f'(x)>0\ \forall x\in R$. That means $f(x)$ would be an increasing function of x. Also, $\lim_{x\to-\infty}f(x)=-\infty$ and $\lim_{x\to+\infty}f(x)=\infty$. Thus, the graph of $f(x)$ would look like Fig. 6.36. It is clear that graph of $y=f(x)$ would cut the x-axis only once. That means we would have just one real root, (say x_0). Clearly $x_0>0$ if $c<0$, and $x_0<0$ if $c>0$.



Graph of f(x)

Fig. 6.36

- If $D>0, f'(x)=0$ would have two real roots (say x_1 and x_2, let $x_1<x_2$)

  $\Rightarrow f'(x)=3(x-x_1)(x-x_2)$

  Clearly, $f'(x)<0, x\in(x_1,x_2)$ and $f'(x)>0, x\in(-\infty,x_1)\cup(x_2,\infty)$

  That means $f(x)$ would increase in $(-\infty,x_1)$ and (x_2,∞), and would decrease in (x_1,x_2). Hence, $x=x_1$ would be a point of local maxima and $x=x_2$ would be a point of local minima.

  Thus, the graph of $y=f(x)$ could have these five possibilities [Figs. 6.37 (a–e)].

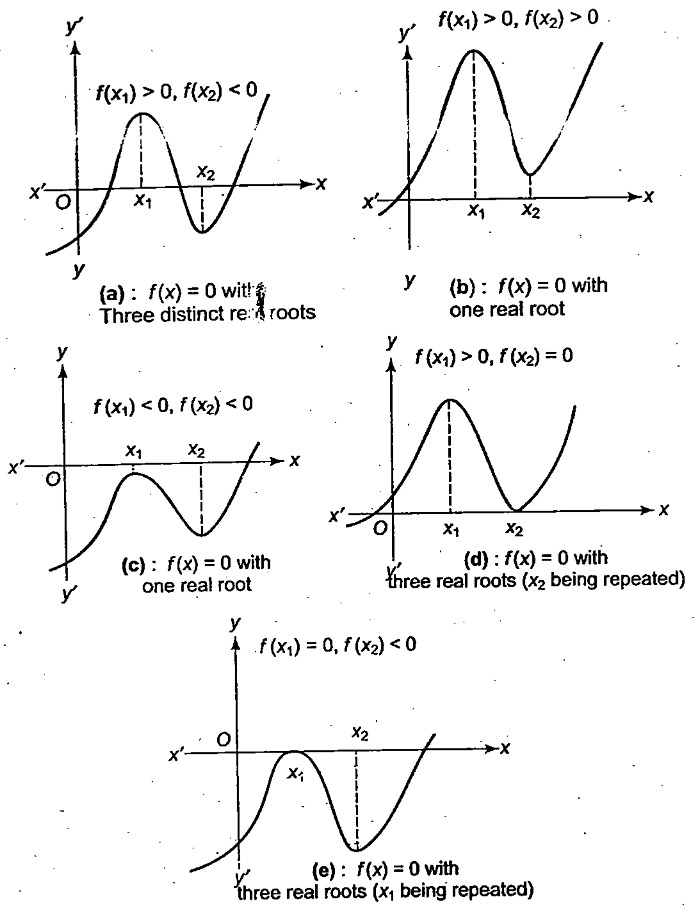


Fig. 6.37

Clearly, in Fig. 6.37(a), we have three real and distinct roots. In Figs. 6.37 (b) and (c), we have just one real root and in Figs. 6.37 (d) and (e), we have three real roots but one of them would be repeated.

a. If $f(x_1)f(x_2) > 0, f(x) = 0$ would have just one real root.
b. If $f(x_1)f(x_2) < 0, f(x) = 0$ would have three real and distinct roots.
c. If $f(x_1)f(x_2) = 0, f(x) = 0$ would have three real roots but one of the roots would be repeated.

- If $D = 0, f'(x) = 3(x - x_1)^2$ where x_1 is the root of $f'(x) = 0$

$\Rightarrow \quad f(x) = (x - x_1)^3 + k$

Now, if $k = 0$, then $f(x) = 0$ has three equal real roots and if $k \neq 0$, then $f(x) = 0$ has one real root.

Example 6.47 Find the value of a if $x^3 - 3x + a = 0$ has three distinct real roots.

Sol. Let $f(x) = x^3 - 3x + a$

Let $f'(x) = 0$

$\Rightarrow \quad 3x^2 - 3 = 0 \quad \Rightarrow \quad x = \pm 1$

For three distinct real roots, $f(1)f(-1) < 0$

$\Rightarrow \quad (1 - 3 + a)(-1 + 3 + a) < 0$

$\Rightarrow \quad (a + 2)(a - 2) < 0$

$\Rightarrow \quad -2 < a < 2$

Example 6.48 Prove that there exist exactly two non-similar isosceles triangles ABC such that $\tan A + \tan B + \tan C = 100$.

Sol. Let $A = B$, then $2A + C = 180°$ and $2 \tan A + \tan C = 100$

Now $2A + C = 180° \Rightarrow \tan 2A = -\tan C$ (1)

Also $2 \tan A + \tan C = 100$

$\Rightarrow \quad 2 \tan A - 100 = -\tan C$ (2)

From equations (1) and (2), $2 \tan A - 100 = \dfrac{2 \tan A}{1 - \tan^2 A}$

Let $\tan A = x$, then $\dfrac{2x}{1 - x^2} = 2x - 100$

$\Rightarrow \quad x^3 - 50x^2 + 50 = 0$

Let $f(x) = x^3 - 50x^2 + 50$. Then $f'(x) = 3x^2 - 100x$. Thus, $f'(x) = 0$ has roots $0, \dfrac{100}{3}$. Also, $f(0)\ f\left(\dfrac{100}{3}\right) < 0$. Thus, $f(x) = 0$ has exactly three distinct real roots. Therefore, $\tan A$ and hence A has three distinct values but one of them will be obtuse angle. Hence, there exist exactly two non-similar isosceles triangles.

Example 6.49 If t be a real number satisfying the equation $2t^3 - 9t^2 + 30 - a = 0$, then find the values of the parameter a for which the equation $x + \dfrac{1}{x} = t$ gives six real and distinct values of x.

Sol. We have $2t^3 - 9t^2 + 30 - a = 0$

Any real root t_0 of this equation gives two real and distinct values of x if $|t_0| > 2$.

Thus, we need to find the condition for the equation in t to have three real and distinct roots, none of which lies in $[-2, 2]$.

Let $f(t) = 2t^3 - 9t^2 + 30 - a$

$f'(t) = 6t^2 - 18t = 0 \Rightarrow t = 0, 3$.

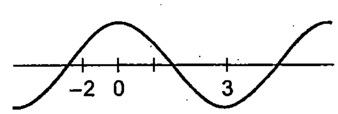


Fig. 6.38

So, the equation $f(t) = 0$ has three real and distinct roots if $f(0)f(3) < 0$.

$\Rightarrow \quad (30 - a)(54 - 81 + 30 - a) < 0 \Rightarrow (30 - a)(3 - a) < 0$

$\Rightarrow \quad (a - 3)(a - 30) < 0 \Rightarrow a \in (3, 30)$ (1)

Also, none of the roots lies in $[-2, 2]$ if $f(-2) > 0$ and $f(2) > 0$

$-16 - 36 + 30 - a > 0$ and $16 - 36 + 30 - a > 0$

$-22 - a > 0$ and $10 - a > 0 \Rightarrow a + 22 < 0$ and $a - 10 < 0$

$\Rightarrow \quad a < -22$ and $a < 10$

$\Rightarrow \quad a < -22$ (2)

From equations (1) and (2), no real value of a exists.

APPLICATION OF EXTREMUM

Drawing the Graph of the Rational Functions

Following tips are useful for drawing the graphs of the rational functions:

1. Examine the point of intersection of $y = f(x)$ with x-axis and y-axis.
2. Examine whether denominator has a root or not. If no, then graph is continuous and f is non-monotonic.

e.g., $f(x)=\dfrac{x}{x^2+x+1}$, $f(x)=\dfrac{x^2+x-2}{x^2+x+1}$.

If denominator has roots, then $f(x)$ is discontinuous. Such functions can be monotonic/non-monotonic.

e.g., $f(x)=\dfrac{x^2-x}{x^2-3x-4}$

3. If numerator and denominator have a common factor (say $x=a$), then $y=f(x)$ has removable discontinuity at $x=a$,

e.g., $f(x)=\dfrac{x^2-x}{x^2-3x+2}=\dfrac{x(x-1)}{(x-1)(x-2)}=\dfrac{x}{x-2}, x\neq 1$

Functions of type linear/linear represent rectangular hyperbola excluding the point of discontinuity and will always be monotonic.

4. Compute $\dfrac{dy}{dx}$ and find the intervals where $f(x)$ is increasing or decreasing and also where it has horizontal tangent.
5. At the point of discontinuity (say $x=a$) check the limiting values $\lim\limits_{x\to a^+} f(x)$ and $\lim\limits_{x\to a^-} f(x)$ to find whether f approaches to ∞ or $-\infty$.

Illustrations

Example 6.50 Draw the graph of $f(x)=\dfrac{x^2-x+1}{x^2+x+1}$.

Sol. The given function is continuous for all $x\in R$.

$$f'(x)=\frac{2(x^2-1)}{(x^2+x+1)^2}$$

$f'(x)=0 \Rightarrow x=\pm 1$

The sign scheme of $f'(x)$ is given in Fig. 6.39.

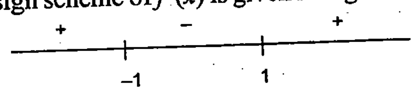


Fig. 6.39

From the sign scheme, $x=1$ is the point of minima and $x=-1$ is the point of maxima.

Also, $f(1)=\dfrac{1}{3}$ and $f(-1)=3, f(0)=1$

Further $\lim\limits_{x\to\pm\infty}\dfrac{x^2-x+1}{x^2+x+1}=1$

From the above information, graph of $y=f(x)$ is

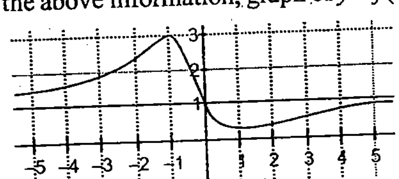


Fig. 6.40

Example 6.51 Draw the graph of $f(x)=\dfrac{x^2-5x+6}{x^2-x}$.

Sol. $f(x)=\dfrac{x^2-5x+6}{x^2-x}=\dfrac{(x-2)(x-3)}{x(x-1)}$

i. The function is discontinuous when $x^2-x=0$ or at $x=0$ and $x=1$
ii. Also, $y=f(x)$ intersects the x-axis when $x^2-5x+6=0$ or at $x=2$ and $x=3$
iii. $\lim\limits_{x\to\pm\infty}\dfrac{x^2-5x+6}{x(x-1)}=1$
iv. $\lim\limits_{x\to 1^+}\dfrac{(x-2)(x-3)}{x(x-1)}=+\infty$ and $\lim\limits_{x\to 1^-}\dfrac{(x-2)(x-3)}{x(x-1)}=-\infty$

$\lim\limits_{x\to 0^+}\dfrac{(x-2)(x-3)}{x(x-1)}=-\infty$ and $\lim\limits_{x\to 0^-}\dfrac{(x-2)(x-3)}{x(x-1)}=+\infty$

v. $f'(x)=\dfrac{(2x-5)(x^2-x)-(2x-1)(x^2-5x+6)}{(x^2-x)^2}$

$=2\dfrac{2x^2-6x+3}{(x^2-x)^2}$

$f'(x)=0 \Rightarrow 2x^2-6x+3=0$ or $x=\dfrac{3\pm\sqrt{3}}{2}$

Clearly $x=\dfrac{3+\sqrt{3}}{2}$ is the point of minima and $x=\dfrac{3-\sqrt{3}}{2}$ is the point of maxima.

From the above information, graph of $y=f(x)$ is:

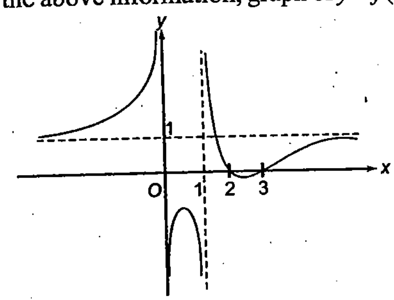


Fig. 6.41

Example 6.52 Draw the graph of $f(x)=\left|\dfrac{x^2-2}{x^2-1}\right|$.

Sol. Consider the function $g(x)=\dfrac{x^2-2}{x^2-1}$

i. $g(x)$ is even function, hence graph is symmetrical about the y-axis.
ii. $g(x)$ is discontinuous at $x=\pm 1$.
iii. $y=g(x)$ intersects the x-axis at $x=\pm\sqrt{2}$.
iv. $\lim\limits_{x\to\pm\infty}\dfrac{x^2-2}{x^2-1}=1$
v. $\lim\limits_{x\to 1^+}\dfrac{x^2-2}{x^2-1}=-\infty$ and $\lim\limits_{x\to 1^-}\dfrac{x^2-2}{x^2-1}=\infty$

$\lim\limits_{x\to -1^+}\dfrac{x^2-2}{x^2-1}=\infty$ and $\lim\limits_{x\to -1^-}\dfrac{x^2-2}{x^2-1}=-\infty$

Hence, the graph of $y=g(x)$ is as follows:

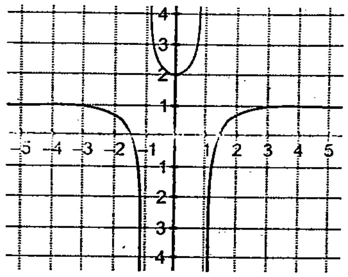


Fig. 6.42

Then the graph of $y=f(x)=|g(x)|$ or $y=f(x)=$ is as follows:

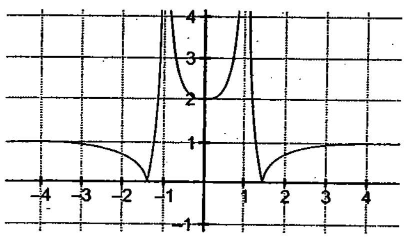


Fig. 6.43

Optimization

Example 6.53 Find two positive numbers x and y such that $x+\ y=60$ and x^3y is maximum.

Sol. $x+y=60$

$\Rightarrow\quad y=60-x$

$\Rightarrow\quad x^3y=(60-x)x^3$

Let $f(x)=(60-x)x^3; x\in(0,60)$

$f'(x)=3x^2(60-x)-x^3=0$

$\Rightarrow\quad x=45\ (\because x\neq 0)$

$f'(45^+)<0$ and $f'(45^-)>0$

Hence, local maxima is at $x=45$.

So, $x=45$ and $y=15$.

Example 6.54 Two towns A and B are 60 km apart. A school is to be built to serve 150 students in town A and 50 students in town B. If the total distance to be travelled by 200 students is to be as small as possible, then the school should be built at

a. town B b. 45 km from town A

c. town A d. 45 km from town B

Sol. Given that $AB=60$

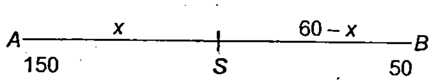


Fig. 6.44

Let the school be at a distance x from A (with 150 students), then the distance travelled by 200 students is $D=150x+50(60-x)=100x+3000$

D will be least and equal to 3000 if $x=0$, i.e., school is built at A.

Example 6.55 Assuming the petrol burnt (per hour) in driving a motor boat varies as the cube of its velocity, show that the most economical speed when going against the current of c miles per hour is $(3c/2)$ miles per hour.

Sol. Let the speed of the motor boat be v mph.

$\Rightarrow$ Velocity of the boat relative to the current $=(v-c)$ mph.

If s miles is the distance covered, then the time taken to cover this distance is $t=s/(v-c)$ hours.

Since, the petrol burnt $=kv^3$ per hour.

where k is a constant

$\Rightarrow\quad z=$ total amount of petrol burnt for a distance of s miles $=kv^3s/(v-c)$

$$\Rightarrow\quad \frac{dz}{dv}=\frac{2ksv^2(v-3c/2)}{(v-c)^2}$$

For maximum or minimum of z, $dz/dv=0\Rightarrow v=3c/2$.

If v is little less or little greater than $3c/2$, then the sign of dz/dv changes from –ve to + ve. Hence, z is minimum when $v=3c/2$ mph.

Since, minima is the only extreme value, z is least at $v=3c/2$, i.e., the most economical speed is $3c/2$ mph.

Plane Geometry

Example 6.56 Rectangles are inscribed inside a semi-circle of radius r. Find the rectangle with maximum area.

Sol. Let us choose coordinate system with origin as the centre of circle.

Area, $A=xy$

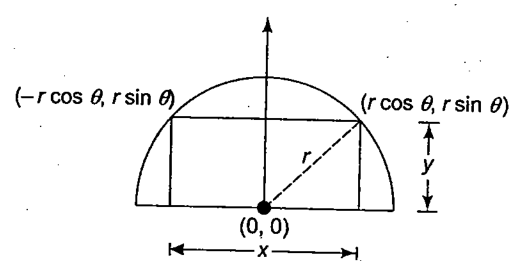


Fig. 6.45

$$\Rightarrow\quad A=2(r\cos\theta)(r\sin\theta),\ \theta\in\left(0,\ \frac{\pi}{2}\right)$$

$\Rightarrow\quad A=r^2\sin 2\theta$

A is maximum when $\sin 2\theta=1\Rightarrow 2\theta=\pi/2$

$\Rightarrow\quad \theta=\pi/4$

$\Rightarrow$ Sides of the rectangle are $2r\cos(\pi/4)=\sqrt{2}\,r$ and $r\sin(\pi/4)=r/\sqrt{2}$

Example 6.57 A running track of 440 ft is to be laid out enclosing a football field, the shape of which is a rectangle with a semi-circle at each end. If the area of the rectangular portion is to be maximum, then find the lengths of its sides.

Sol.

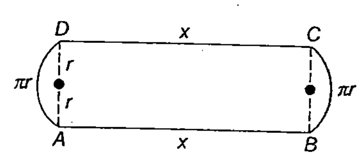


Fig. 6.46

Perimeter = 440 ft.

$\Rightarrow\ 2x+\pi r+\pi r=440$ or $2x+2\pi r=440$

A = Area of the rectangular portion $= x\,2r$.

$\Rightarrow A = x\dfrac{(440-2x)}{\pi} = \dfrac{1}{\pi}(440x - 2x^2)$

Let $\dfrac{dA}{dx} = \dfrac{1}{\pi}(440-4x) = 0$

$\Rightarrow x = 110$ for which $\dfrac{d^2A}{dx^2} < 0$

$\Rightarrow A$ is maximum when $x = 110$

$\Rightarrow 2r = \dfrac{440-2x}{\pi} = \dfrac{440-220}{22/7} = 70$

$\Rightarrow r = 35$ ft and $x = 110$ ft

Example 6.58 If the sum of the lengths of the hypotenuse and another side of a right-angled triangle is given, show that the area of the triangle is maximum when the angle between these sides is $\pi/3$.

Sol.

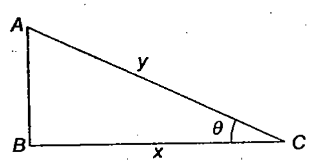


Fig. 6.47

Let ABC be a right-angled triangle in which side $BC = x$ (say) and hypotenuse $AC = y$ (say). Given $x + y = k$ (const.)

$\Rightarrow y = k - x$

Now, the area of the triangle ABC is given by

$A = \dfrac{1}{2}BC\,AB = \dfrac{1}{2}x\sqrt{(y^2-x^2)} = \dfrac{1}{2}x\sqrt{[(k-x)^2 - x^2]}$

Let $u = A^2 = \dfrac{1}{4}x^2(k^2 - 2kx)$

$\Rightarrow du/dx = \dfrac{1}{2}k(kx - 3x^2)$ and $d^2u/dx^2 = \dfrac{1}{2}k(k-6x)$

For maximum or minimum of u, $du/dx = 0 \Rightarrow x = k/3$ $(\because x \neq 0)$

When $x = k/3$, $d^2u/dx^2 = \dfrac{1}{2}k(k - 6\times\dfrac{1}{3}k) = -\dfrac{1}{2}k^2$ (–ve)

$\Rightarrow u$, i.e., A is maximum when $x = k/3$ and when $y = k - x = 2k/3$.

Now, $\cos\theta = BC/AC = x/y = 1/2 \Rightarrow \theta = \pi/3$.

Hence, the required angle is $\pi/3$.

Coordinate Geometry

Example 6.59 The tangent to the parabola $y = x^2$ has been drawn so that the abscissa x_0 of the point of tangency belong to the interval [1, 2]. Find x_0 for which the triangle is to be bounded by the tangent, the axis of ordinates, and the straight line $y = x_0^2$ has the greatest area.

Sol.

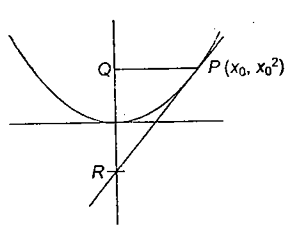


Fig. 6.48

$y = x^2 \Rightarrow dy/dx = 2x$

$\Rightarrow$ Equation of the tangent at (x_0, x_0^2) is $y - x_0^2 = 2x_0(x - x_0)$.

It meets y-axis in $R(0, -x_0^2)$. Q is $(0, x_0^2)$

$\Rightarrow Z$ = area of the triangle PQR

$= \dfrac{1}{2}2x_0^2\,x_0 = x_0^3,\ 1 \le x_0 \le 2$

$\dfrac{dZ}{dx_0} = 3x_0^2 > 0$ in $1 \le x_0 \le 2$

$\Rightarrow Z$ is an increasing function in [1, 2].

Hence, Z, i.e., the area of ΔPQR is greatest at $x_0 = 2$.

Example 6.60 Find the point (α, β) on the ellipse $4x^2 + 3y^2 = 12$, in the first quadrant, so that the area enclosed by the lines $y = x$, $y = \beta$, $x = \alpha$ and the x-axis is maximum.

Sol. Equation of the ellipse is $x^2/3 + y^2/4 = 1$.

Let point P be $(\sqrt{3}\cos\theta, 2\sin\theta)$, $\theta \in (0, \pi/2)$

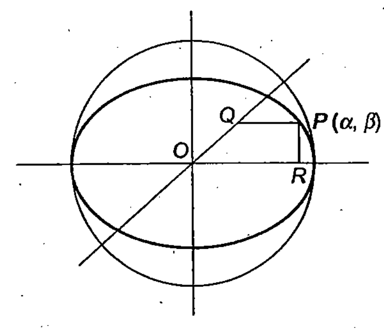


Fig. 6.49

Clearly, line PQ is $y = 2\sin\theta$,

line PR is $x = \sqrt{3}\cos\theta$ and OQ is $y = x$, and

Q is $(2\sin\theta, 2\sin\theta)$.

Z = area of the region $PQORP$ (trapezium)

$= \dfrac{1}{2}(OR + PQ)PR$

$= \dfrac{1}{2}(\sqrt{3}\cos\theta + (\sqrt{3}\cos\theta - 2\sin\theta))2\sin\theta$

$= \dfrac{1}{2}(2\sqrt{3}\cos\theta\sin\theta - 2\sin^2\theta)$

$= \dfrac{1}{2}(\sqrt{3}\sin 2\theta + \cos 2\theta - 1)$

$= \cos\left(2\theta - \dfrac{\pi}{3}\right) - \dfrac{1}{2}$

which is maximum when $\cos\left(2\theta - \dfrac{\pi}{3}\right) = 1$ or $2\theta - \dfrac{\pi}{3} = 0$ or

$\theta = \dfrac{\pi}{6}$

Hence, point P be (3/2, 1).

Example 6.61 LL' is the latus rectum of the parabola $y^2 = 4ax$ and PP' is a double ordinate drawn between the vertex and the latus rectum. Show that the area of the trapezium $PP'LL'$ is maximum when the distance PP' from the vertex is $a/9$.

Sol. Let $LL' = 4a$ be the latus rectum of the parabola $y^2 = 4ax$ and let $(at^2, 2at)$ be the coordinates of the point P.

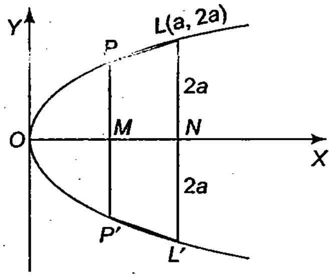


Fig. 6.50

Here PP' is the double ordinate of the parabola.

$\Rightarrow$ $OM = at^2 \Rightarrow MN = ON - OM = a - at^2$ and $PP' = 2PM = 4at$.

Now, area of the trapezium $PP'L'L$

$$= A = \frac{1}{2}(PP' + LL') \times MN$$

$$A = \frac{1}{2}(4at + 4a)(a - at^2) = 2a^2(-t^3 - t^2 + t + 1)$$

$\Rightarrow$ $dA/dt = 2a^2(-3t^2 - 2t + 1)$ and $d^2A/dt^2 = 2a^2(-6t - 2)$

For maximum or minimum of A, $dA/dt = 0$

$\Rightarrow$ $-2a^2(3t - 1)(t + 1) = 0$

$\Rightarrow$ $t = -1, 1/3$ when $t = -1$, $d^2A/dt^2 = 8a^2$ (+ve)

$\Rightarrow$ A is minimum when $t = -1$

And when $t = 1/3$, $d^2A/dt^2 = -8a^2$, (–ve)

$\Rightarrow$ A is maximum when $t = 1/3$ (only point of maxima)

$\Rightarrow$ For the area of the trapezium $PP'LL'$ to be maximum, distance of PP' from vertex $O = OM = at^2 = a(1/3)^2 = a/9$.

Example 6.62 Find the points on the curve $5x^2 - 8xy + 5y^2 = 4$ whose distance from the origin is maximum or minimum.

Sol. Let (r, θ) be the polar coordinates of any point P on the curve where r is the distance of the point from the origin.

$\Rightarrow$ $r^2 [5(\cos^2\theta + \sin^2\theta) - 8 \sin\theta \cos\theta] = 4$

$$\Rightarrow \quad r^2 = \frac{4}{5 - 4\sin 2\theta}$$

r^2 is maximum when $5 - 4 \sin 2\theta$ is minimum $= 5 - 4 = 1$ (when $\sin 2\theta = 1$)

$\Rightarrow$ $2\theta = 90° \Rightarrow \theta = 45° \Rightarrow r = \pm 2, \theta = 45°$ (1)

Again r^2 is minimum when $5 - 4 \sin 2\theta$ is maximum.

$$= 5 + 4 = 9 \text{ when } \sin 2\theta = -1 \Rightarrow 2\theta = \frac{3\pi}{2} \Rightarrow \theta = \frac{3\pi}{4}$$

$$\Rightarrow \quad r = \pm\frac{2}{3}, \theta = \frac{3\pi}{4} \qquad (2)$$

Hence, the points are $(r \cos\theta, r \sin\theta)$ where r and θ are given by equations (1) and (2).

Thus, we get four points $(\sqrt{2}, \sqrt{2})$, $(-\sqrt{2}, -\sqrt{2})$,

$$\left(\frac{\sqrt{2}}{3}, -\frac{\sqrt{2}}{3}\right) \text{ and } \left(-\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}\right).$$

Solid Geometry

Useful Formulae of Mensuration

- Volume of a cuboid $= lbh$, where l, b, h are length, breadth, and height, respectively.
- Surface area of a cuboid $= 2(lb + bh + hl)$
- Volume of a prism = area of the base × height
- Volume of a pyramid $= \frac{1}{3}$ (area of the base) × (height)
- Volume of a cone $= \frac{1}{3}\pi r^2 h$
- Curved surface of a cylinder $= 2\pi rh$
- Total surface of a cylinder $= 2\pi rh + 2\pi r^2$
- Volume of a sphere $= \frac{4}{3}\pi r^3$
- Surface area of a sphere $= 4\pi r^2$
- Area of a circular sector $= \frac{1}{2}r^2\theta$, when θ is in radians

Example 6.63 A sheet of area 40 m² is used to make an open tank with square base. Find the dimensions of the base such that the volume of this tank is maximum.

Sol. Let the length of base be x m and height be y m.

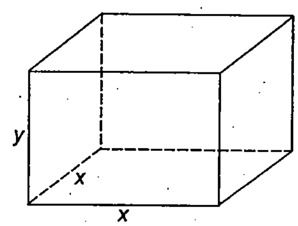


Fig. 6.51

Volume $V = x^2y$

Again x and y are related to the surface area of this tank which is equal to 40 m².

$\Rightarrow$ $x^2 + 4xy = 40$

$$y = \frac{40 - x^2}{4x}, x \in (0, \sqrt{40})$$

$$\Rightarrow \quad V(x) = x^2\left(\frac{40 - x^2}{4x}\right) = \frac{40x - x^3}{4}$$

Maximizing volume,

$$V'(x) = \frac{40 - 3x^2}{4} = 0 \Rightarrow x = \sqrt{\frac{40}{3}} \text{ m}$$

$$\text{and } V''(x) = -\frac{3x}{2} \Rightarrow V''\left(\sqrt{\frac{40}{3}}\right) < 0$$

$$\Rightarrow \quad \text{volume is maximum at } x = \sqrt{\frac{40}{3}} \text{ m}$$

Example 6.64 The lateral edge of a regular hexagonal pyramid is 1 cm. If the volume is maximum, then find its height.

Sol.

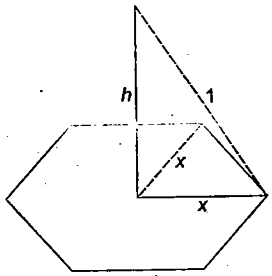


Fig. 6.52

$x^2 + h^2 = 1$;

Volume, $V = \frac{1}{3} \times 6 \times \frac{\sqrt{3}}{4} x^2 h = \frac{\sqrt{3}}{2} h(1-h^2)$

For $V'(h) = 0 \Rightarrow h = \frac{1}{\sqrt{3}} \quad \Rightarrow \quad V_{max} = 1/3$

Example 6.65 Find the height of the cylinder of maximum volume that can be inscribed in a sphere of radius a.

Sol.

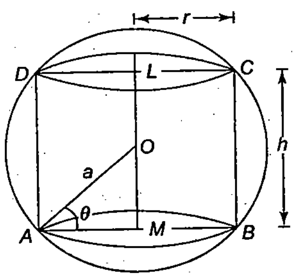


Fig. 6.53

If a be the radius of sphere and h the height of cylinder, then from Fig. 6.53,

$r^2 + (h^2/4) = a^2 \Rightarrow h^2 = 4(a^2 - r^2)$

Now, $V = \pi r^2 h = \pi\left(a^2 - \frac{1}{4}h^2\right)h = \pi\left(a^2 h - \frac{1}{4}h^3\right)$

$\Rightarrow \quad \frac{dV}{dh} = \pi\left(a^2 - \frac{3}{4}h^2\right) = 0$ for maximum or minimum

This gives $h = (2/\sqrt{3})\, a$ for which $d^2V/dh^2 = -6h/4 < 0$

Hence, V is maximum when $h = 2a/\sqrt{3}$.

Example 6.66 A right-circular cylinder is inscribed in a given cone. Find the dimensions of the cylinder such that its volume is maximum.

Sol. Let x be the radius of cylinder and y be its height.

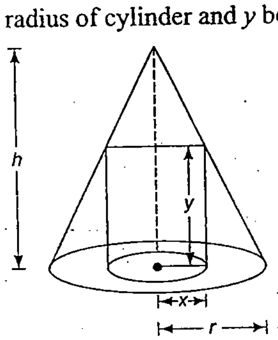


Fig. 6.54

Volume $V = \pi x^2 y$

x, y can be related by using similar triangles

$\frac{y}{r-x} = \frac{h}{r}$

$\Rightarrow \quad y = \frac{h}{r}(r-x)$

$\Rightarrow \quad V(x) = \pi x^2 \frac{h}{r}(r-x), x \in (0, r)$

$\Rightarrow \quad V(x) = \frac{\pi h}{r}(rx^2 - x^3)$

$\Rightarrow \quad V'(x) = \frac{\pi h}{r} x(2r - 3x)$

$V'(x) = 0 \Rightarrow x = \frac{2r}{3}$

Also, $V''(x) = \frac{\pi h}{r}(2r - 6x)$

$\Rightarrow \quad V''\left(\frac{2r}{3}\right) < 0$

Thus, volume is maximum when $x = \frac{2r}{3}$ and $y = \frac{h}{3}$.

Concept Application Exercise 6.4

1. A private telephone company serving a small community makes a profit of ₹12.00 per subscriber, if it has 725 subscribers. It decides to reduce the rate by a fixed sum for each subscriber over 725, thereby reducing the profit by 1 paise per subscriber. Thus, there will be profit of ₹11.99 on each of the 726 subscribers. ₹11.98 on each 727 subscribers, etc. What is the number of subscribers which will give the company the maximum profit?
2. The lateral edge of a regular rectangular pyramid is a cm long. The lateral edge makes an angle α with the plane of the base. Find the value of α for which the volume of the pyramid is greatest.
3. A figure is bounded by the curves $y = x^2 + 1, y = 0, x = 0$ and $x = 1$. At what point (a, b), a tangent should be drawn to the curve $y = x^2 + 1$ for it to cut off a trapezium of the greatest area from the figure.
4. Prove that the cone of the greatest volume which can be inscribed in a given sphere has an altitude equal to 2/3rd the diameter of the sphere.
5. Find the point at which the slope of the tangent of the function $f(x) = e^x \cos x$ attains minima, when $x \in [0, 2\pi]$.
6. An electric light is placed directly over the centre of a circular plot of lawn 100 m in diameter. Assuming that the intensity of light varies directly as the sine of the angle at which it strikes an illuminated surface and inversely as the square of its distance from its surface. How should the light be hung in order that the intensity may be as great as possible at the circumference of the plot?

EXERCISES

Subjective Type

Solutions on page 6.38

1. Find the values of x where $f(x) = \sin(\ln x) - \cos(\ln x)$ is strictly increasing.
2. Let $f(x) = x^3 + ax^2 + bx + 5\sin^2 x$ be an increasing function on the set R. Then find the condition on a and b.
3. Find the possible values of a such that $f(x) = e^{2x} - (a+1)e^x + 2x$ is monotonically increasing for $x \in R$.
4. Prove that for any two numbers x_1 and x_2,
$$\frac{e^{2x_1} + e^{x_2}}{3} > e^{\frac{2x_1 + x_2}{3}}.$$
5. If $0 < x_1 < x_2 < x_3 < \pi$, then prove that $\sin\left(\frac{x_1 + x_2 + x_3}{3}\right) > \frac{\sin x_1 + \sin x_2 + \sin x_3}{3}$. Hence or otherwise prove that if A, B, C are angles of a triangle, then the maximum value of $\sin A + \sin B + \sin C$ is $\frac{3\sqrt{3}}{2}$.
6. Discuss the monotonocity of $Q(x)$, where $Q(x) = 2f\left(\frac{x^2}{2}\right) + f(6 - x^2)\ \forall x \in R$. It is given that $f''(x) > 0\ \forall x \in R$. Find also the points of maxima and minima of $Q(x)$.
7. Prove that
$$\left(\tan^{-1}\frac{1}{e}\right)^2 + \frac{2e}{\sqrt{(e^2+1)}} < (\tan^{-1} e)^2 + \frac{2}{\sqrt{(e^2+1)}}.$$
8. Prove that $\sin^2\theta < \theta \sin(\sin\theta)$ for $0 < \theta < \frac{\pi}{2}$.
9. Let $f(x) = x^3 - 3x^2 + 6\ \forall x \in R$
$$\text{and } g(x) = \begin{cases} \max : f(t); x+1 \le t \le x+2, -3 \le x \le 0 \\ 1 - x \text{ for } x \ge 0 \end{cases}$$
Test continuity of $g(x)$ for $x \in [-3, 1]$.
10. If f is a real function such that $f(x) > 0$, $f'(x)$ is continuous for all real x and $axf'(x) \ge 2\sqrt{f(x)} - 2af(x)$, $(ax \ne 2)$, show that $\sqrt{f(x)} \ge \frac{\sqrt{f(1)}}{x}$, $x \ge 1$.
11. The lower corner of a leaf in a book is folded over so as to reach the inner edge of the page. Show that the fraction of the width folded over when the area of the folded part is minimum is 2/3.
12. From a fixed point A on the circumference of a circle of radius r, the perpendicular AY falls on the tangent at P. Find the maximum area of the triangle APY.
13. For what values of a, the function
$$f(x) = \left(\frac{\sqrt{a+4}}{1-a} - 1\right)x^5 - 3x + \log(5) \text{ decreases for all real } x.$$
14. Find the greatest value of $f(x) = \frac{1}{2ax - x^2 - 5a^2}$ in $[-3, 5]$ depending upon the parameter a.
15. P and Q are two points on a circle of centre C and radius α. The angle PCQ being 2θ, find the value of $\sin\theta$ when the radius of the circle inscribed in the triangle CPQ is maximum.
16. If $f(x) = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) - \log(x^2 + x + 1) + (\lambda^2 - 5\lambda + 3)x + 10$ is a decreasing function for all $x \in R$, find the permissible values of λ.
17. Discuss the number of roots of the equation $e(k - x\log x) = 1$, for different values of k.
18. Prove that $\sin 1 > \cos(\sin 1)$. Also show that the equation $\sin(\cos(\sin x)) = \cos(\sin(\cos x))$ has only one solution in $\left[0, \frac{\pi}{2}\right]$.
19. Let $f: R \to R$ be a twice differentiable function such that $f(x + \pi) = f(x)$ and $f''(x) + f(x) \ge 0$ for all $x \in R$. Show that $f(x) \ge 0$ for all $x \in R$.
20. Show that $5x \le 8\sin x - \sin 2x \le 6x$ for $0 \le x \le \frac{\pi}{3}$.
21. Let $f(x)$, $x \ge 0$, be a non-negative continuous function. If $f'(x)\cos x \le f(x)\sin x$, $\forall x \ge 0$, then find $f\left(\frac{5\pi}{3}\right)$.

Objective Type

Solutions on page 6.42

Each question has four choices a, b, c and d, out of which *only one* is correct.

1. If $f(x) = kx^3 - 9x^2 + 9x + 3$ is monotonically increasing in R, then
   a. $k < 3$　　b. $k \le 2$
   c. $k \ge 3$　　d. None of these
2. If the function $f(x) = \frac{K\sin x + 2\cos x}{\sin x + \cos x}$ is strictly increasing for all values of x, then
   a. $K < 1$　　b. $K > 1$　　c. $K < 2$　　d. $K > 2$
3. Let $f: R \to R$ be a function such that $f(x) = ax + 3\sin x + 4\cos x$. Then $f(x)$ is invertible if
   a. $a \in (-5, 5)$　　b. $a \in (-\infty, 5)$
   c. $a \in (-5, +\infty)$　　d. None of these
4. Let $g(x) = 2f\left(\frac{x}{2}\right) + f(2 - x)$ and $f''(x) < 0\ \forall x \in (0, 2)$. Then $g(x)$ increases in
   a. (1/2, 2)　　b. (4/3, 2)
   c. (0, 2)　　d. (0, 4/3)
5. On which of the following intervals is the function $x^{100} + \sin x - 1$ decreasing?
   a. $(0, \pi/2)$　　b. $(0, 1)$
   c. $(\pi/2, \pi)$　　d. None of these

6. A function is matched below against an interval where it is supposed to be increasing. Which of the following parts is incorrectly matched?

Interval	Function
a. $[2, \infty)$	$2x^3 - 3x^2 - 12x + 6$
b. $(-\infty, \infty)$	$x^3 - 3x^2 + 3x + 3$
c. $(-\infty, -4]$	$x^3 + 6x^2 + 6$
d. $\left(-\infty, \frac{1}{3}\right]$	$3x^2 - 2x + 1$

7. The function $f(x) = \tan^{-1}(\sin x + \cos x)$ is an increasing function in

a. $\left(-\frac{\pi}{2}, \frac{\pi}{4}\right)$ b. $\left(0, \frac{\pi}{2}\right)$
c. $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ d. $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

8. The function x^x decreases in the interval

a. $(0, e)$ b. $(0, 1)$
c. $\left(0, \frac{1}{e}\right)$ d. None of these

9. The function $f(x) = \sum_{K=1}^{5} (x - K)^2$ assumes the minimum value of x given by

a. 5 b. $\frac{5}{2}$ c. 3 d. 2

10. Which of the following statements is always true?

a. If $f(x)$ is increasing, then $f^{-1}(x)$ is decreasing.
b. If $f(x)$ is increasing, then $\frac{1}{f(x)}$ is also increasing.
c. If f and g are positive function and f is increasing and g is decreasing, then f/g is a decreasing function.
d. If f and g are positive function and f is decreasing and g is increasing, then f/g is a decreasing function.

11. Let $f: R \to R$ be a differentiable function for all values of x and has the property that $f(x)$ and $f'(x)$ have opposite signs for all values of x. Then,

a. $f(x)$ is an increasing function
b. $f(x)$ is a decreasing function
c. $f^2(x)$ is a decreasing function
d. $|f(x)|$ is an increasing function

12. Let $f(x) = x\sqrt{4ax - x^2}$, $(a > 0)$. Then $f(x)$ is

a. increasing in $(0, 3a)$, decreasing in $(3a, 4a)$
b. increasing in $(a, 4a)$, decreasing in $(5a, \infty)$
c. increasing in $(0, 4a)$
d. None of these

13. Let $f: R \to R$ be a differentiable function $\forall x \in R$. If the tangent drawn to the curve at any point $x \in (a, b)$ always lies below the curve, then

a. $f'(x) > 0, f''(x) < 0 \ \forall x \in (a, b)$
b. $f'(x) < 0, f''(x) < 0 \ \forall x \in (a, b)$
c. $f'(x) > 0, f''(x) > 0 \ \forall x \in (a, b)$
d. None of these

14. If $f'(x) = |x| - \{x\}$ where $\{x\}$ denotes the fractional part of x, then $f(x)$ is decreasing in

a. $\left(-\frac{1}{2}, 0\right)$ b. $\left(-\frac{1}{2}, 2\right)$
c. $\left(-\frac{1}{2}, 2\right)$ d. $\left(\frac{1}{2}, \infty\right)$

15. Function $f(x) = |x| - |x - 1|$ is monotonically increasing when

a. $x < 0$ b. $x > 1$
c. $x < 1$ d. $0 < x < 1$

16. Let f be continuous and differentiable function such that $f(x)$ and $f'(x)$ have opposite signs everywhere. Then

a. f is increasing
b. f is decreasing
c. $|f|$ is non-monotonic
d. $|f|$ is decreasing

17. If the function $f(x)$ increases in the interval (a, b), and $\phi(x) = [f(x)]^2$, then

a. $\phi(x)$ increases in (a, b)
b. $\phi(x)$ decreases in (a, b)
c. We cannot say that $\phi(x)$ increases or decreases in (a, b)
d. None of these

18. If $\phi(x)$ is a polynomial function and $\phi'(x) > \phi(x), \forall x \geq 1$ and $\phi(1) = 0$, then

a. $\phi(x) \geq 0, \forall x \geq 1$ b. $\phi(x) < 0, \forall x \geq 1$
c. $\phi(x) = 0, \forall x \geq 1$ d. None of these

19. Which of the following statements is true for the function

$$f(x) = \begin{cases} \sqrt{x}, & x \geq 1 \\ x^3, & 0 \leq x \leq 1 \\ \frac{x^3}{3} - 4x, & x < 0 \end{cases}$$

a. It is monotonic increasing $\forall x \in R$.
b. $f'(x)$ fails to exist for three distinct real values of x.
c. $f'(x)$ changes its sign twice as x varies from $-\infty$ to ∞.
d. The function attains its extreme values at x_1 and x_2, such that $x_1 x_2 > 0$.

20. If $f''(x) > 0, \forall x \in R, f'(3) = 0$ and $g(x) = f(\tan^2 x - 2\tan x + 4)$, $0 < x < \frac{\pi}{2}$, then $g(x)$ is increasing in

a. $\left(0, \frac{\pi}{4}\right)$ b. $\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$
c. $\left(0, \frac{\pi}{3}\right)$ d. $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

21. Let $f(x)$ be a function such that $f'(x) = \log_{1/3}[\log_3(\sin x + a)]$. If $f(x)$ is decreasing for all real values of x, then

a. $a \in (1, 4)$ b. $a \in (4, \infty)$
c. $a \in (2, 3)$ d. $a \in (2, \infty)$

22. If $f(x) = x + \sin x$; $g(x) = e^{-x}$; $u = \sqrt{c+1} - \sqrt{c}$; $v = \sqrt{c} - \sqrt{c-1}$; $(c > 1)$, then

a. $fog(u) < fog(v)$ b. $gof(u) < gof(v)$
c. $gof(u) > gof(v)$ d. $fog(u) < fog(v)$

23. The length of the largest continuous interval in which the function $f(x) = 4x - \tan 2x$ is monotonic is
   a. $\pi/2$ b. $\pi/4$ c. $\pi/8$ d. $\pi/16$

24. $f(x) = (x-1)\,|(x-2)(x-3)|$, then '$f$' decreases in
   a. $\left(2-\frac{1}{\sqrt{3}}, 2\right)$ b. $\left(2, 2+\frac{1}{\sqrt{3}}\right)$
   c. $\left(2+\frac{1}{\sqrt{3}}, 4\right)$ d. $(3, \infty)$

25. The number of solutions of the equation $x^3 + 2x^2 + 5x + 2\cos x = 0$ in $[0, 2\pi]$ is
   a. one b. two c. three d. zero

26. $f(x) = (x-2)\,|x-3|$ is monotonically increasing in
   a. $(-\infty, 5/2) \cup (3, \infty)$ b. $(5/2, \infty)$
   c. $(2, \infty)$ d. $(-\infty, 3)$

27. $f(x) = (x-8)^4 (x-9)^5, 0 \le x \le 10$, monotonically decreases in
   a. $\left(\frac{76}{9}, 10\right]$ b. $\left(8, \frac{76}{9}\right)$
   c. $[0, 8)$ d. $\left(\frac{76}{9}, 10\right]$

28. If $f(x) = x^3 + 4x^2 + \lambda x + 1$ is a monotonically decreasing function of x in the largest possible interval $(-2, -2/3)$, then
   a. $\lambda = 4$ b. $\lambda = 2$
   c. $\lambda = -1$ d. λ has no real value

29. $f(x) = |x \log_e x|$ monotonically decreases in
   a. $(0, 1/e)$ b. $(1/e, 1)$
   c. $(1, \infty)$ d. $(1/e, \infty)$

30. Given that $f'(x) > g'(x)$ for all real x, and $f(0) = g(0)$, then $f(x) < g(x)$ for all x belongs to
   a. $(0, \infty)$ b. $(-\infty, 0)$
   c. $(-\infty, \infty)$ d. None of these

31. A function $g(x)$ is defined as $g(x) = \frac{1}{4} f(2x^2 - 1) + \frac{1}{2} f(1 - x^2)$ and $f'(x)$ is an increasing function, then $g(x)$ is increasing in the interval
   a. $(-1, 1)$ b. $\left(-\sqrt{\frac{2}{3}}, 0\right) \cup \left(\sqrt{\frac{2}{3}}, \infty\right)$
   c. $\left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right)$ d. None of these

32. Let $f(x)$ be a function defined as below:
   $f(x) = \sin(x^2 - 3x), x \le 0$; and $6x + 5x^2, x > 0$
   Then at $x = 0, f(x)$
   a. has a local maximum b. has a local minimum
   c. is discontinuous d. None of these

33. The greatest value of $f(x) = \cos(xe^{[x]} + 7x^2 - 3x)$, $x \in [-1, \infty)$ is (where $[\cdot]$ represents the greatest integer function)
   a. -1 b. 1
   c. 0 d. None of these

34. If $f(x) = x^5 - 5x^4 + 5x^3 - 10$ has local maximum and minimum at $x = p$ and $x = q$, respectively, then $(p, q) =$
   a. $(0, 1)$ b. $(1, 3)$
   c. $(1, 0)$ d. None of these

35. The maximum value of $(\log x)/x$ is
   a. 1 b. $2/e$
   c. e d. $1/e$

36. If a function $f(x)$ has $f'(a) = 0$ and $f''(a) = 0$, then
   a. $x = a$ is a maximum for $f(x)$
   b. $x = a$ is a minimum for $f(x)$
   c. it is difficult to say a and b
   d. $f(x)$ is necessarily a constant function

37. The minimum value of $2^{(x^2-3)^3+27}$ is
   a. 2^{27} b. 2
   c. 1 d. None of these

38. The number of real roots of the equation $e^{x-1} + x - 2 = 0$ is
   a. 1 b. 2 c. 3 d. 4

39. The greatest value of the function $f(x) = \dfrac{\sin 2x}{\sin\left(x + \frac{\pi}{4}\right)}$ on the interval $\left(0, \frac{\pi}{2}\right)$ is
   a. $\frac{1}{\sqrt{2}}$ b. $\sqrt{2}$
   c. 1 d. $-\sqrt{2}$

40. The function $f(x) = (4\sin^2 x - 1)^n (x^2 - x + 1), n \in N$, has a local minimum at $x = \frac{\pi}{6}$, then
   a. n is any even number
   b. n is an odd number
   c. n is odd prime number
   d. n is any natural number

41. All possible values of x for which the function $f(x) = x \ln x - x + 1$ is positive is
   a. $(1, \infty)$ b. $(1/e, \infty)$
   c. $[e, \infty)$ d. $(0, 1) \cup (1, \infty)$

42. The greatest value of $f(x) = (x+1)^{1/3} - (x-1)^{1/3}$ on $[0, 1]$ is
   a. 1 b. 2 c. 3 d. $\frac{1}{3}$

43. If the function $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$, where $a > 0$, attains its maximum and minimum at p and q, respectively such that $p^2 = q$, then a equals to
   a. 1 b. 2 c. $\frac{1}{2}$ d. 3

44. The real number x when added to its inverse gives the minimum value of the sum at x equals to
   a. 1 b. -1
   c. -2 d. 2

45. The function $f(x) = \frac{x}{2} + \frac{2}{x}$ has a local minimum at
   a. $x = 2$ b. $x = -2$
   c. $x = 0$ d. $x = 1$

46. The maximum value of the function $f(x) = \sin\left(x+\frac{\pi}{6}\right) + \cos\left(x+\frac{\pi}{6}\right)$ in the interval $\left(0, \frac{\pi}{2}\right)$ occurs at

a. $\frac{\pi}{12}$ b. $\frac{\pi}{6}$
c. $\frac{\pi}{4}$ d. $\frac{\pi}{3}$

47. Let $f(x)=\begin{cases} x+2, & -1\le x<0 \\ 1, & x=0 \\ \frac{x}{2}, & 0<x\le 1 \end{cases}$

Then on [–1, 1], this function has
a. a minimum
b. a maximum
c. either a maximum or a minimum
d. neither a maximum nor a minimum

48. The maximum value of $f(x)=\frac{x}{1+4x+x^2}$ is

a. $-\frac{1}{4}$ b. $-\frac{1}{3}$
c. $\frac{1}{6}$ d. $\frac{1}{5}$

49. The maximum slope of the curve $y=-x^3+3x^2+9x-27$ is
a. 0 b. 12
c. 16 d. 32

50. Let $f(x)=\cos \pi x+10x+3x^2+x^3, -2\le x\le 3$. The absolute minimum value of $f(x)$ is
a. 0 b. –15
c. $3-2\pi$ d. None of these

51. The minimum value of $e^{(2x^2-2x+1)\sin^2 x}$ is
a. e b. $1/e$ c. 1 d. 0

52. The maximum value of $x^4e^{-x^2}$ is
a. e^2 b. e^{-2} c. $12e^{-2}$ d. $4e^{-2}$

53. If $a^2x^4+b^2y^4=c^6$, then the maximum value of xy is

a. $\frac{c^2}{\sqrt{ab}}$ b. $\frac{c^3}{ab}$ c. $\frac{c^3}{\sqrt{2ab}}$ d. $\frac{c^3}{2ab}$

54. The global maximum value of $f(x) = \log_{10}(4x^3-12x^2+11x-3), x\in[2, 3]$ is

a. $-\frac{3}{2}\log_{10}3$ b. $1+\log_{10}3$
c. $\log_{10}3$ d. $\frac{3}{2}\log_{10}3$

55. The least natural number a for which $x+ax^{-2}>2, \forall \in (0, \infty)$ is
a. 1 b. 2
c. 5 d. None of these

56. A function f is defined by $f(x)=|x|^m|x-1|^n, \forall x\in R$. The local maximum value of the function is $(m, n\in N)$
a. 1 b. $m^n n^m$
c. $\frac{m^m n^n}{(m+n)^{m+n}}$ d. $\frac{(mn)^{mn}}{(m+n)^{m+n}}$

57. $f(x)=\begin{cases} 4x-x^3+\ln(a^2-3a+3), & 0\le x<3 \\ x-18, & x\ge 3 \end{cases}$

Complete set of values of a such that $f(x)$ as a local minima at $x=3$ is
a. $[-1,2]$ b. $(-\infty,1)\cup(2,\infty)$
c. $[1,2]$ d. $(-\infty,-1)\cup(2,\infty)$

58. Let the function $f(x)$ be defined as follows

$$f(x)=\begin{cases} x^3+x^2-10x, & -1\le x<0 \\ \cos x, & 0\le x<\pi/2 \\ 1+\sin x, & \pi/2\le x\le\pi \end{cases}$$

Then $f(x)$ has
a. a local minimum at $x=\pi/2$
b. a global maximum at $x=\pi/2$
c. an absolute minimum at $x=-1$
d. an absolute maximum at $x=\pi$

59. A differentiable function $f(x)$ has a relative minimum at $x=0$, then the function $y=f(x)+ax+b$ has a relative minimum at $x=0$ for
a. all a and all b b. all b if $a=0$
c. all $b>0$ d. all $a>0$

60. If $f(x)=4x^3-x^2-2x+1$ and

$$g(x)=\begin{cases} \min\{f(t): 0\le t\le x\}; & 0\le x\le 1 \\ 3-x; & 1<x\le 2 \end{cases}$$

then $g\left(\frac{1}{4}\right)+g\left(\frac{3}{4}\right)+g\left(\frac{5}{4}\right)$ has the value equal to
a. 7/4 b. 9/4 c. 13/4 d. 5/2

61. The set of value(s) of a for which the function $f(x)=\frac{ax^3}{3}+(a+2)x^2+(a-1)x+2$ possesses a negative point of inflection is
a. $(-\infty,-2)\cup(0,\infty)$ b. $\{-4/5\}$
c. $(-2,0)$ d. empty set

62. Suppose that f is a polynomial of degree 3 and that $f''(x)\ne 0$ at any of the stationary point. Then
a. f has exactly one stationary point
b. f must have no stationary point
c. f must have exactly two stationary points
d. f has either zero or two stationary point

63. The maximum value of the function $f(x) = \frac{(1+x)^{0.6}}{1+x^{0.6}}$ in the interval [0, 1] is
 a. $2^{0.4}$ b. $2^{-0.4}$
 c. 1 d. $2^{0.6}$

64. $f: R \to R, f(x)$ is differentiable such that $f(f(x)) = k(x^5 + x), (k \neq 0)$, then $f(x)$ is always
 a. increasing
 b. decreasing
 c. either increasing or decreasing
 d. non-monotonic

65. Consider the function $f: (-\infty, \infty) \to (-\infty, \infty)$ defined by $f(x) = \frac{x^2 - a}{x^2 + a}$, $a > 0$. Which of the following is not true?
 a. Maximum value of f is not attained even though f is bounded.
 b. $f(x)$ is increasing on $(0, \infty)$ and has minimum at $x = 0$.
 c. $f(x)$ is decreasing on $(-\infty, 0)$ and has minimum at $x = 0$.
 d. $f(x)$ is increasing on $(-\infty, \infty)$ and has neither a local maximum nor a local minimum at $x = 0$.

66. If $f: R \to R$ and $g: R \to R$ are two functions such that $f(x) + f''(x) = -x\, g(x) f'(x)$ and $g(x) > 0\ \forall x \in R$, then the functions $f^2(x) + (f'(x))^2$ has
 a. a maxima at $x = 0$
 b. a minima at $x = 0$
 c. a point of inflexion at $x = 0$
 d. None of these

67. Let $h(x) = x^{m/n}$ for $x \in R$, where m and n are odd numbers and $0 < m < n$, then $y = h(x)$ has
 a. no local extremums b. one local maximum
 c. one local minimum d. None of these

68. $f(x) = 4 \tan x - \tan^2 x + \tan^3 x, x \neq n\pi + \frac{\pi}{2}$
 a. is monotonically increasing
 b. is monotonically decreasing
 c. has a point of maxima
 d. has a point of minima

69. If for a function $f(x), f'(a) = 0, f''(a) = 0, f'''(a) > 0$, then at $x = a, f(x)$ is
 a. minimum b. maximum
 c. not an extreme point d. extreme point

70. The function $f(x) = x(x+4)\, e^{-x/2}$ has its local maxima at $x = a$, then
 a. $a = 2\sqrt{2}$ b. $a = 1 - \sqrt{3}$
 c. $a = -1 + \sqrt{3}$ d. $a = -4$

71. If $f(x) = \begin{cases} \sin^{-1}(\sin x), & x > 0 \\ \frac{\pi}{2}, & x = 0 \\ \cos^{-1}(\cos x), & x < 0 \end{cases}$, then
 a. $x = 0$ is a point of maxima
 b. $x = 0$ is a point of minima
 c. $x = 0$ is a point of intersection
 d. None of these

72. $f(x) = \begin{cases} 2 - |x^2 + 5x + 6|, & x \neq -2 \\ a^2 + 1, & x = -2 \end{cases}$, then the range of a, so that $f(x)$ has maxima at $x = -2$, is
 a. $|a| \geq 1$ b. $|a| < 1$
 c. $a > 1$ d. $a < 1$

73. If $A > 0, B > 0$ and $A + B = \frac{\pi}{3}$, then the maximum value of $\tan A \tan B$ is
 a. $\frac{1}{\sqrt{3}}$ b. $\frac{1}{3}$
 c. 3 d. $\sqrt{3}$

74. If the function $f(x) = \frac{t + 3x - x^2}{x - 4}$, where t is a parameter that has a minimum and maximum, then the range of values of t is
 a. (0, 4) b. $(0, \infty)$
 c. $(-\infty, 4)$ d. $(4, \infty)$

75. The value of a for which the function $f(x) = a \sin x + (1/3)\sin 3x$ has an extremum at $x = \pi/3$ is
 a. 1 b. –1
 c. 0 d. 2

76. The least value of a, for which the equation $\frac{4}{\sin x} + \frac{1}{1 - \sin x} = a$ has at least one solution in the interval $(0, \pi/2)$, is
 a. 9 b. 4
 c. 8 d. 1

77. The largest term in the sequence $a_n = \frac{n^2}{n^3 + 200}$ is given by
 a. $\frac{529}{49}$ b. $\frac{8}{89}$
 c. $\frac{49}{543}$ d. None of these

78. The number of values of k for which the equation $x^3 - 3x + k = 0$ has two distinct roots lying in the interval (0, 1) is
 a. three b. two
 c. infinitely many d. zero

79. Consider the function $f(x) = x\cos x - \sin x$, then identify the statement which is correct.
  a. f is neither odd nor even
  b. f is monotonic decreasing at $x = 0$
  c. f has a maxima at $x = \pi$
  d. f has a minima at $x = -\pi$

80. Let $f(x) = ax^3 + bx^2 + cx + 1$ have extrema at $x = \alpha, \beta$ such that $\alpha\beta < 0$ and $f(\alpha)\ f(\beta) < 0$. Then the equation $f(x) = 0$ has
  a. three equal real roots
  b. one negative root if $f(\alpha) < 0$ and $f(\beta) > 0$
  c. one positive root if $f(\alpha) > 0$ and $f(\beta) < 0$
  d. None of these

81. A factory D is to be connected by a road with a straight railway line on which a town A is situated. The distance DB of the factory to the railway line is $5\sqrt{3}$ km. Length AB of the railway line is 20 km. Freight charges on the road are twice the charges on the railway. The point P ($AP < AB$) on the railway line should the road DP be connected so as to ensure minimum freight charges from the factory to the town is
  a. $BP = 5$ km b. $AP = 5$ km
  c. $BP = 7.5$ km d. None of these

82. The volume of the greatest cylinder which can be inscribed in a cone of height 30 cm and semi-vertical angle 30° is
  a. $4000\pi/3$ cubic cm b. $400\pi/3$ cubic cm
  c. $4000\pi/\sqrt{3}$ cubic cm d. None of these

83. A rectangle of the greatest area is inscribed in a trapezium $ABCD$. One of whose non-parallel sides AB is perpendicular to the base, so that one of the rectangle's side lies on the larger base of the trapezium. The base of trapezium are 6 and 10 cm and AB is 8 cm long. Then the maximum area of the rectangle is
  a. 24 sq. cm b. 48 sq. cm
  c. 36 sq. cm d. None of these

84. A bell tent consists of a conical portion above a cylindrical portion near the ground. For a given volume and a circular base of a given radius, the amount of the canvas used is a minimum when the semi-vertical angle of the cone is
  a. $\cos^{-1}2/3$. b. $\sin^{-1}2/3$
  c. $\cos^{-1}1/3$ d. None of these

85. A rectangle is inscribed in an equilateral triangle of side length $2a$ units. The maximum area of this rectangle can be
  a. $\sqrt{3}a^2$ b. $\frac{\sqrt{3}a^2}{4}$
  c. a^2 d. $\frac{\sqrt{3}a^2}{2}$

86. Tangents are drawn to $x^2 + y^2 = 16$ from the point $P(0, h)$. These tangents meet the x–axis at A and B. If the area of triangle PAB is minimum, then
  a. $h = 12\sqrt{2}$ b. $h = 6\sqrt{2}$
  c. $h = 8\sqrt{2}$ d. $h = 4\sqrt{2}$

87. The largest area of a trapezium inscribed in a semi-circle of radius R, if the lower base is on the diameter, is
  a. $\frac{3\sqrt{3}}{4}R^2$ b. $\frac{\sqrt{3}}{2}R^2$
  c. $\frac{3\sqrt{3}}{8}R^2$ d. R^2

88. In a ΔABC, $\angle B = 90°$ and $b + a = 4$. The area of the triangle is maximum when $\angle C$ is
  a. $\pi/4$ b. $\pi/6$
  c. $\pi/3$ d. None of these

89. The three sides of a trapezium are equal, each being 8 cm. The area of the trapezium, when it is maximum, is
  a. $24\sqrt{3}$ sq. cm b. $48\sqrt{3}$ sq. cm
  c. $72\sqrt{3}$ sq. cm d. None of these

90. The fuel charges for running a train are proportional to the square of the speed generated in km per hour, and the cost is ₹48 at 16 km per hour. If the fixed charges amount to ₹300 per hour, the most economical speed is
  a. 60 kmph b. 40 kmph
  c. 48 kmph d. 36 kmph

91. A cylindrical gas container is closed at the top and open at the bottom, if the iron plate of the top is 5/4 times as thick as the plate forming the cylindrical sides, the ratio of the radius to the height of the cylinder using minimum material for the same capacity is
  a. 3 : 4 b. 5 : 6
  c. 4 : 5 d. None of these

92. The least perimeter of an isosceles triangle in which a circle of radius r can be inscribed is
  a. $4\sqrt{3}r$ b. $2\sqrt{3}r$ c. $6\sqrt{3}r$ d. $8\sqrt{3}r$

93. A given right cone has a volume p, and the largest right circular cylinder that can be inscribed in the cone has a volume q. Then $p : q$ is
  a. 9 : 4 b. 8 : 3
  c. 7 : 2 d. None of these

94. A wire of length a is cut into two parts which are bent, respectively, in the form of a square and a circle. The least value of the sum of the areas so formed is
  a. $\frac{a^2}{\pi + 4}$ b. $\frac{a}{\pi + 4}$
  c. $\frac{a}{4(\pi + 4)}$ d. $\frac{a^2}{4(\pi + 4)}$

95. A box, constructed from a rectangular metal sheet, is 21 cm by 16 cm by cutting equal squares of sides x from the corners of the sheet and then turning up the projected portions. The value of x so that volume of the box is maximum is
  a. 1 b. 2 c. 3 d. 4

96. The vertices of a triangle are $(0, 0)$, $(x, \cos x)$ and $(\sin^3 x, 0)$ where $0 < x < \frac{\pi}{2}$. The maximum area for such a triangle in sq. units is

a. $\frac{3\sqrt{3}}{32}$ b. $\frac{\sqrt{3}}{32}$

c. $\frac{4}{32}$ d. $\frac{6\sqrt{3}}{32}$

Multiple Correct Answers Type

Solutions on page 6.54

Each question has four choices a, b, c and d, out of which *one or more* answers are correct.

1. Let $f(x) = \begin{cases} x^2 + 3x, & -1 \le x < 0 \\ -\sin x, & 0 \le x < \pi/2 \\ -1 - \cos x, & \frac{\pi}{2} \le x \le \pi \end{cases}$ then
   a. $f(x)$ has global minimum value -2
   b. global maximum value occurs at $x = 0$
   c. global maximum value occurs at $x = \pi$
   d. $x = \pi/2$ is point of local minima
2. Let $f(x) = x^4 - 4x^3 + 6x^2 - 4x + 1$, then
   a. f increases on $[1, \infty)$
   b. f decreases on $[1, \infty)$
   c. f has a minimum at $x = 1$
   d. f has neither maximum nor minimum
3. Let $f(x) = 2x - \sin x$ and $g(x) = \sqrt[3]{x}$, then
   a. range of gof is R
   b. gof is one-one
   c. both f and g are one-one
   d. both f and g are onto
4. If $f(x) = 2x + \cot^{-1} x + \log(\sqrt{1+x^2} - x)$, then $f(x)$
   a. increases in $[0, \infty)$
   b. decreases in $[0, \infty)$
   c. neither increases nor decreases in $[0, \infty)$
   d. increases in $(-\infty, \infty)$
5. Let $f(x) = |x^2 - 3x - 4|, -1 \le x \le 4$, then
   a. $f(x)$ is monotonically increasing in $[-1, 3/2]$
   b. $f(x)$ is monotonically decreasing in $(3/2, 4]$
   c. the maximum value of $f(x)$ is $\frac{25}{4}$
   d. the minimum value of $f(x)$ is 0
6. If $f(x) = \int_0^x \frac{\sin t}{t} dt,\ x > 0$, then
   a. $f(x)$ has a local maxima at $x = n\pi\ (n = 2k, k \in I^+)$
   b. $f(x)$ has a local minima at $x = n\pi\ (n = 2k, k \in I^+)$
   c. $f(x)$ has neither maxima nor minima at $x = n\pi\ (n \in I^+)$
   d. $f(x)$ has local maxima at $x = n\pi\ (n = 2k - 1, k \in I^+)$
7. The values of parameter a for which the point of minimum of the function $f(x) = 1 + a^2x - x^3$ satisfies the inequality $\frac{x^2+x+2}{x^2+5x+6} < 0$ are
   a. $(2\sqrt{3}, 3\sqrt{3})$ b. $(-3\sqrt{3}, -2\sqrt{3})$
   c. $(-2\sqrt{3}, 3\sqrt{3})$ d. $(-3\sqrt{2}, 2\sqrt{3})$
8. Let $f(x) = ax^2 - b|x|$, where a and b are constants. Then at $x = 0, f(x)$ has
   a. a maxima whenever $a > 0, b > 0$
   b. a maxima whenever $a > 0, b < 0$
   c. minima whenever $a > 0, b < 0$
   d. neither a maxima nor a minima whenever $a > 0, b < 0$
9. The function $y = \frac{2x-1}{x-2}\ (x \ne 2)$
   a. is its own inverse
   b. decreases at all values of x in the domain
   c. has a graph entirely above the x-axis
   d. is unbounded
10. Let $g'(x) > 0$ and $f'(x) < 0, \forall x \in R$, then
    a. $(f(x+1)) > g(f(x-1))$
    b. $f(g(x-1)) > f(g(x+1))$
    c. $g(f(x+1)) < g(f(x-1))$
    d. $g(g(x+1)) < g(g(x-1))$
11. If $f(x) = x^3 - x^2 + 100x + 2002$, then
    a. $f(1000) > f(1001)$ b. $f\left(\frac{1}{2000}\right) > f\left(\frac{1}{2001}\right)$
    c. $f(x-1) > f(x-2)$ d. $f(2x-3) > f(2x)$
12. If $f'(x) = g(x)(x-a)^2$ where $g(a) \ne 0$ and g is continuous at $x = a$, then
    a. f is increasing in the neighbourhood of a if $g(a) > 0$
    b. f is increasing in the neighbourhood of a if $g(a) < 0$
    c. f is decreasing in the neighbourhood of a if $g(a) > 0$
    d. f is decreasing in the neighbourhood of a if $g(a) < 0$
13. The value of a for which the function $f(x) = (4a-3)(x + \log_e 5) + 2(a-7)\cot\frac{x}{2}\sin^2\frac{x}{2}$ does not possess critical points is
    a. $(-\infty, -4/3)$ b. $(-\infty, -1)$
    c. $[1, \infty)$ d. $(2, \infty)$
14. Let $f(x) = (x-1)^4 (x-2)^n, n \in N$. Then $f(x)$ has
    a. a maximum at $x = 1$ if n is odd
    b. a maximum at $x = 1$ if n is even
    c. a minimum at $x = 1$ if n is even
    d. a minima at $x = 2$ if n is even
15. Let $f(x) = \sin x + ax + b$, then which of the following is/are true.
    a. $f(x) = 0$ has only one real root which is positive if $a > 1, b < 0$
    b. $f(x) = 0$ has only one real root which is negative if $a > 1, b > 0$
    c. $f(x) = 0$ has only one real root which is negative if $a < -1, b < 0$
    d. None of these
16. The function $\frac{\sin(x+a)}{\sin(x+b)}$ has no maxima or minima if
    a. $b - a = n\pi, n \in I$ b. $b - a = (2n+1)\pi, n \in I$
    c. $b - a = 2n\pi, n \in I$ d. None of these
17. If composite function $f_1(f_2(f_3(\ldots(f_n(x)))))$ n times is an increasing function and if r of f_i s are decreasing function while rest are increasing, then maximum value of $r(n-r)$ is

a. $\frac{n^2-1}{4}$, when n is an even number

b. $\frac{n^2}{4}$, when n is an odd number

c. $\frac{n^2-1}{4}$, when n is an odd number

d. $\frac{n^2}{4}$, when n is an even number

18. Let $f(x)=\begin{cases} \frac{(x-1)(6x-1)}{2x-1}, & \text{if } x \neq \frac{1}{2} \\ 0, & \text{if } x=\frac{1}{2} \end{cases}$

then at $x=\frac{1}{2}$, which of the following is/are not true?

a. f has a local maxima
b. f has a local minima
c. f has an inflection point
d. f has a removable discontinuity

19. In which of the following graphs is $x = c$ the point of inflection?

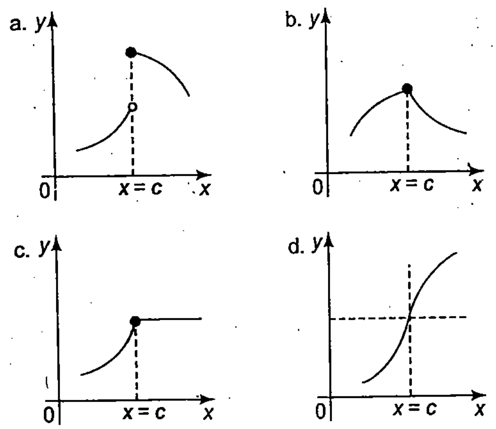


Fig. 6.55

20. Let $f(x)$ be an increasing function defined on $(0, \infty)$. If $f(2a^2+a+1)>f(3a^2-4a+1)$, then the possible integers in the range of a is/are

a. 1 b. 2 c. 3 d. 4

21. If $f(x)=(\sin^2 x-1)^n$, then $x=\frac{\pi}{2}$ is a point of

a. local maximum, if n is odd
b. local minimum, if n is odd
c. local maximum, if n is even
d. local minimum, if n is even

22. For the cubic function $f(x)=2x^3+9x^2+12x+1$, which one of the following statement/statements hold good?

a. $f(x)$ is non-monotonic
b. $f(x)$ increases in $(-\infty,-2)\cup(-1,\infty)$ and decreases in $(-2,-1)$
c. $f: R\to R$ is bijective
d. Inflection point occurs at $x=-3/2$

23. Let $f(x)=a_5x^5+a_4x^4+a_3x^3+a_2x^2+a_1x$, where a_i's are real and $f(x)=0$ has a positive root α_0. Then

a. $f'(x)=0$ has a root α_1 such that $0<\alpha_1<\alpha_0$
b. $f'(x)=0$ has at least two real roots
c. $f''(x)=0$ has at least one real root
d. None of these

24. If $f(x)$ and $g(x)$ are two positive and increasing functions, then which of the following is not always true?

a. $[f(x)]^{g(x)}$ is always increasing
b. if $[f(x)]^{g(x)}$ is decreasing, then $f(x)<1$
c. if $[f(x)]^{g(x)}$ is increasing, then $f(x)>1$
d. if $f(x)\not> 1$, then $[f(x)]^{g(x)}$ is increasing

25. An extremum of the function $f(x)=\frac{2-x}{\pi}\cos\pi(x+3)+\frac{1}{\pi^2}\sin\pi(x+3)$, $0<x<4$ occurs at

a. $x=1$ b. $x=2$
c. $x=3$ d. $x=\pi$

26. For the function $f(x)=x^4(12\log_e x-7)$

a. the point $(1,-7)$ is the point of inflection
b. $x=e^{1/3}$ is the point of minima
c. the graph is concave downwards in $(0, 1)$
d. the graph is concave upwards in $(1, \infty)$

27. Let $f(x)=\log(2x-x^2)+\sin\frac{\pi x}{2}$. Then which of the following is/are true?

a. graph of f is symmetrical about the line $x=1$
b. maximum value of f is 1
c. absolute minimum value of f does not exist
d. None of these

28. Which of the following hold(s) good for the function $f(x)=2x-3x^{2/3}$?

a. $f(x)$ has two points of extremum
b. $f(x)$ is concave upward for $\forall\ x\in R$
c. $f(x)$ is non-differentiable function
d. $f(x)$ is continuous function

29. For the function $f(x)=\frac{e^x}{1+e^x}$, which of the following hold good?

a. f is monotonic in its entire domain
c. maximum of f is not attained even though f is bounded
c. f has a point of inflection.
d. f has one asymptote

30. Which of the following is true about point of extremum $x=a$ of function $y=f(x)$?

a. at $x=a$, function $y=f(x)$ may be discontinuous
b. at $x=a$, function $y=f(x)$ may be continuous but non-differentiable
c. at $x=a$, function $y=f(x)$ may have point of inflection
d. None of these

31. Which of the following function has point of extremum at $x=0$?

a. $f(x)=e^{-|x|}$
b. $f(x)=\sin|x|$

c. $f(x)=\begin{cases} x^2+4x+3, x<0 \\ -x, x\geq 0 \end{cases}$

d. $f(x)=\begin{cases} |x|, & x<0 \\ \{x\}, & 0\leq x<1 \end{cases}$

(where $\{x\}$ represents fractional part function).

32. Which of the following function/functions has/have point of inflection?

a. $f(x)=x^{6/7}$ b. $f(x)=x^6$
c. $f(x)=\cos x+2x$ d. $f(x)=x|x|$

33. The function $f(x)=x^2+\dfrac{\lambda}{x}$ has a

a. minimum at $x=2$ if $\lambda=16$
b. maximum at $x=2$ if $\lambda=16$
c. maximum for no real value of λ
d. point of inflection at $x=1$ if $\lambda=-1$

34. The function $f(x)=x^{1/3}(x-1)$

a. has two inflection points
b. has one point of extremum
c. is non-differentiable
d. Range of $f(x)$ is $[-3\times 2^{-8/3}, \infty)$

Reasoning Type

Solutions on page 6.59

Each question has four choices a, b, c, and d, out of which *only one* is correct. Each question contains STATEMENT 1 and STATEMENT 2.

a. if both the statements are TRUE and STATEMENT 2 is the correct explanation of STATEMENT 1.
b. if both the statements are TRUE but STATEMENT 2 is NOT the correct explanation of STATEMENT 1.
c. if STATEMENT 1 is TRUE and STATEMENT 2 is FALSE.
d. if STATEMENT 1 is FALSE and STATEMENT 2 is TRUE.

1. **Statement 1:** Both sin x and cos x are decreasing functions in $\left(\dfrac{\pi}{2},\pi\right)$.
Statement 2: If a differentiable function decreases in an interval (a, b), then its derivative also decreases in (a, b).

2. **Statement 1:** $\alpha^{\beta}>\beta^{\alpha}$, for $2.91<\alpha<\beta$.
Statement 2: $f(x)=\dfrac{\log_e x}{x}$ is a decreasing function for $x>e$.

3. **Statement 1:** $f(x)=|x-1|+|x-2|+|x-3|$ has point of minima at $x=3$.
Statement 2: $f(x)$ is non-differentiable at $x=3$.

4. **Statement 1:** The function $f(x)=x\ln x$ is increasing in $(1/e,\infty)$.
Statement 2: If both $f(x)$ and $g(x)$ are increasing in (a, b) then $f(x)\,g(x)$ must be increasing in (a, b).

5. Let $f: R\to R$ is differentiable and strictly increasing function throughout its domain.
Statement 1: If $|f(x)|$ is also strictly increasing function, then $f(x)=0$ has no real roots.
Statement 2: When $x\to\infty$ or $\to-\infty, f(x)\to 0$, but cannot be equal to zero.

6. **Statement 1:** Let $f(x)=5-4(x-2)^{2/3}$, then at $x=2$ the function $f(x)$ attains neither the least value nor the greatest value.
Statement 2: At $x=2$, first derivative does not exist.

7. **Statement 1:** $f(x)=x+\cos x$ is increasing for $\forall x\in R$.
Statement 2: If $f(x)$ is increasing, then $f'(x)$ may vanish at some finite number of points.

8. **Statement 1:** Both $f(x)=2\cos x+3\sin x$ and $g(x)=\sin^{-1}\dfrac{x}{\sqrt{13}}-\tan^{-1}\dfrac{3}{2}$ are increasing for $x\in(0,\pi/2)$.
Statement 2: If $f(x)$ is increasing then its inverse is also increasing.

9. **Statement 1:** $f(x)=\dfrac{x^3}{3}+\dfrac{ax^2}{2}+x+5$ has positive point of maxima for $a<-2$.
Statement 2: $x^2+ax+1=0$ has both roots positive for $a<-2$.

10. **Statement 1:** For all $a, b\in R$ the function $f(x)=3x^4-4x^3+6x^2+ax+b$ has exactly one extremum.
Statement 2: If a cubic function is monotonic, then its graph cuts the x-axis only once.

11. **Statement 1:** The value of $\left[\lim\limits_{x\to 0+}\dfrac{\sin x\tan x}{x^2}\right]$ is 1, where [.] denotes the greatest integer function.
Statement 2: For $\left(0,\dfrac{\pi}{2}\right)$, $\sin x<x<\tan x$.

12. **Statement 1:** Let $f(x)=\sin(\cos x)$ in $\left[0,\dfrac{\pi}{2}\right]$, then $f(x)$ is decreasing in $\left[0,\dfrac{\pi}{2}\right]$.
Statement 2: $\cos x$ is a decreasing function $\forall x\in\left[0,\dfrac{\pi}{2}\right]$.

13. Let $f(x)=(x^3-6x^2+12x-8)\,e^x$.
Statement 1: $f(x)$ is neither maximum nor minimum at $x=2$.
Statement 2: If a function $x=2$ is a point of inflection, then it is not a point of extremum.

14. **Statement 1:** The function $f(x)=x^4-8x^3+22x^2-24x+21$ is decreasing for every $x\in(-\infty,1)\cup(2,3)$.
Statement 2: $f(x)$ is increasing for $x\in(1,2)\cup(3,\infty)$ and has no point of inflection.

15. **Statement 1:** If $f(0)=0, f'(x)=\ln(x+\sqrt{1+x^2})$, then $f(x)$ is positive for all $x\in R_0$.
Statements 2: $f(x)$ is increasing for $x>0$ and decreasing for $x<0$.

Linked Comprehension Type

Solutions on page 6.60

Based upon each paragraph, three multiple choice questions have to be answered. Each question has four choices a, b, c and d, out of which *only one* is correct.

For Problems 1–2

$f(x)=\sin^{-1}x+x^2-3x+\dfrac{x^3}{3}, x\in[0,1]$

1. Which of the following is true about $f(x)$?
   a. $f(x)$ has a point of maxima
   b. $f(x)$ has a point of minima
   c. $f(x)$ is increasing
   d. $f(x)$ is decreasing

2. Which of the following is true for $x\in[0,1]$?
   a. $\sin^{-1}x+x^2-\dfrac{x(9-x^2)}{3}\le 0$
   b. $\sin^{-1}x+x^2-\dfrac{x(9-x^2)}{3}\ge 0$
   c. $\sin^{-1}x+x^2-\dfrac{x(9-x^2)}{3}\le 1$
   d. $\sin^{-1}x+x^2-\dfrac{x(9-x^2)}{3}\ge 1$

For Problems 3–4

Let $f'(\sin x)<0$ and $f''(\sin x)>0\ \forall x\in\left(0,\dfrac{\pi}{2}\right)$ and $g(x)=f(\sin x)+f(\cos x)$.

3. Which of the following is true?
   a. g' is increasing
   b. g' is decreasing
   c. g' has a point of minima
   d. g' has a point of maxima

4. Which of the following is true?
   a. $g(x)$ is decreasing in $\left(\dfrac{\pi}{4},\dfrac{\pi}{2}\right)$
   b. $g(x)$ increasing in $\left(0,\dfrac{\pi}{4}\right)$
   c. $g(x)$ is monotonically increasing
   d. None of these

For Problems 5–8

Consider function $f(x)=\begin{cases}-x^2+4x+a, & x\le 3\\ ax+b, & 3<x<4\\ -\dfrac{b}{4}x+6, & x\ge 4\end{cases}$

(For questions 6 to 8 consider $f(x)$ as a continuous function).

5. Which of the following is true?
   a. $f(x)$ is discontinuous function for any value of a and b
   b. $f(x)$ is continuous for finite number of values of a and b
   c. $f(x)$ cannot be differentiable for any value of a and b
   d. $f(x)$ is continuous for infinite values of a and b

6. If $x=3$ is the only point of minima in its neighbourhood and $x=4$ is neither a point of maxima nor a point of minima, then which of the following can be true?
   a. $a>0, b<0$
   b. $a<0, b<0$
   c. $a>0, b\in R$
   d. None of these

7. If $x=4$ is the only point of maxima in its neighbourhood but $x=3$ is neither a point of maxima nor a point of minima, then which of the following can be true?
   a. $a<0, b>0$
   b. $a>0, b<0$
   c. $a>0, b>0$
   d. Not possible

8. If $x=3$ is a point of minima and $x=4$ is a point of maxima, then which of the following is true?
   a. $a<0, b>0$
   b. $a>0, b<0$
   c. $a>0, b>0$
   d. Not possible

For Problems 9–10

If $\phi(x)$ is a differentiable real-valued function satisfying $\phi'(x)+2\phi(x)\le 1$, then it can be adjusted as $e^{2x}\phi'(x)+2e^{2x}\phi(x)\le e^{2x}$ or $\dfrac{d}{dx}\left(e^{2x}\phi(x)-\dfrac{e^{2x}}{2}\right)\le 0$ or $\dfrac{d}{dx}e^{2x}\left(\phi(x)-\dfrac{1}{2}\right)\le 0$.

Here e^{2x} is called integrating factor which helps in creating single differential coefficient as shown above. Answer the following questions:

9. If $P(1)=0$ and $\dfrac{dP(x)}{dx}>P(x)$ for all $x\ge 1$, then
   a. $P(x)>0\ \forall x>1$
   b. $P(x)$ is a constant function
   c. $P(x)<0\ \forall x>1$
   d. None of these

10. If $H(x_0)=0$ for some $x=x_0$ and $\dfrac{d}{dx}H(x)>2cxH(x)$ for all $x\ge x_0$, where $c>0$, then
    a. $H(x)=0$ has root for $x>x_0$
    b. $H(x)=0$ has no roots for $x>x_0$
    c. $H(x)$ is a constant function
    d. None of these

For Problems 11–13

Let $h(x)=f(x)-a(f(x))^2+a(f(x))^3$ for every real number x.

11. $h(x)$ increases as $f(x)$ increases for all real values of x if
    a. $a\in(0,3)$
    b. $a\in(-2,2)$
    c. $[3,\infty)$
    d. None of these

12. $h(x)$ increases as $f(x)$ decreases for all real values of x if
    a. $a\in(0,3)$
    b. $a\in(-2,2)$
    c. $(3.\infty)$
    d. None of these

13. If $f(x)$ is strictly increasing function, then $h(x)$ is non-monotonic function given
    a. $a\in(0,3)$
    b. $a\in(-2,2)$
    c. $(3.\infty)$
    d. $a\in(-\infty,0)\cup(3,\infty)$

For Problems 14–16

$f(x)=x^3-9x^2+24x+c=0$ has three real and distinct roots α, β and γ.

14. Possible values of c are
    a. $(-20,-16)$
    b. $(-20,-18)$
    c. $(-18,-16)$
    d. None of these

15. If $[\alpha]+[\beta]+[\gamma]=8$, then the values of c, where $[\cdot]$ represents the greatest integer function, are
    a. $(-20,-16)$
    b. $(-20,-18)$
    c. $(-18,-16)$
    d. None of these

16. If $[\alpha]+[\beta]+[\gamma]=7$, then the values of c, where $[\cdot]$ represents the greatest integer function, are
    a. $(-20,-16)$
    b. $(-20,-18)$
    c. $(-18,-16)$
    d. None of these

For Problems 17–21

Consider the graph of $y=g(x)=f'(x)$, given that $f(c)=0$, where $y=f(x)$ is a polynomial function.

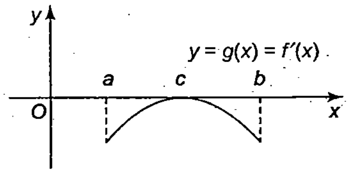


Fig. 6.56

17. The graph of $y=f(x)$ will intersect the x-axis
    a. twice
    b. once
    c. never
    d. None of these
18. The equation $f(x)=0$, $a \le x \le b$ has
    a. four real roots
    b. no real roots
    c. two distinct real roots.
    d. at least three repeated roots
19. The graph of $y=f(x)$, $a \le x \le b$, has
    a. two points of inflection
    b. one point of inflection
    c. no point of inflection
    d. none of these
20. The function $y=f(x)$, $a < x < b$, has
    a. exactly one local maxima
    b. one local minima and one maxima
    c. exactly one local minima
    d. none of these
21. The equation $f''(x)=0$
    a. has no real roots
    b. has at least one real root
    c. has at least two distinct real roots
    d. None of these

For Problems 22–24

Let $f(x)=4x^2-4ax+a^2-2a+2$ and the global minimum value of $f(x)$ for $x \in [0, 2]$ is equal to 3.

22. The number of values of a for which the global minimum value equal to 3 for $x \in [0, 2]$ occurs at the end point of interval $[0, 2]$ is
    a. 1
    b. 2
    c. 3
    d. 0
23. The number of values of a for which the global minimum value equal to 3 for $x \in [0, 2]$ occurs for the value of x lying in $(0, 2)$ is
    a. 1
    b. 2
    c. 3
    d. 0
24. The values of a for which $f(x)$ is monotonic for $x \in [0, 2]$ are
    a. $a \le 0$ or $a \ge 4$
    b. $0 \le a \le 4$
    c. $a > 0$
    d. None of these

For Problems 25–27

Let $f(x)=x^3-3(7-a)x^2-3(9-a^2)x+2$.

25. The values of parameter a if $f(x)$ has a negative point of local minimum are
    a. ϕ
    b. $(-3, 3)$
    c. $\left(-\infty, \frac{58}{14}\right)$
    d. None of these
26. The values of parameter a if $f(x)$ has a positive point of local maxima are
    a. ϕ
    b. $(-\infty, -3) \cup (\frac{58}{14}, \infty)$
    c. $\left(-\infty, \frac{58}{14}\right)$
    d. None of these
27. The values of parameter a if $f(x)$ has points of extrema which are opposite in sign are
    a. ϕ
    b. $(-3, 3)$
    c. $\left(-\infty, \frac{58}{14}\right)$
    d. None of these

For Problems 28–30

Consider the function $f(x)=\left(1+\frac{1}{x}\right)^x$.

28. The domain of $f(x)$ is
    a. $(-1, 0) \cup (0, \infty)$
    b. $R-\{0\}$
    c. $(-\infty, -1) \cup (0, \infty)$
    d. $(0, \infty)$
29. The function $f(x)$
    a. has a maxima but no minima
    b. has a minima but no maxima
    c. has exactly one maxima and one minima
    d. is monotonic
30. The range of the function $f(x)$ is
    a. $(0, \infty)$
    b. $(-\infty, e)$
    c. $(1, \infty)$
    d. $(1, e) \cup (e, \infty)$

For Problems 31–33

Consider the function $f(x)=x+\cos x - a$

31. Which of the following is not true about $y=f(x)$?
    a. it is an increasing function
    b. it is a monotonic function
    c. it has infinite points of inflections
    d. None of these
32. Values of a for which $f(x)=0$ has exactly one positive root.
    a. $(0, 1)$
    b. $(-\infty, 1)$
    c. $(-1, 1)$
    d. $(1, \infty)$
33. Values of a for which $f(x)=0$ has exactly one negative root.
    a. $(0, 1)$
    b. $(-\infty, 1)$
    c. $(-1, 1)$
    d. $(1, \infty)$

For Problems 34–36

Consider the function $f(x)=3x^4+4x^3-12x^2$

34. $y=f(x)$ increases in the interval
    a. $(-1, 0) \cup (2, \infty)$
    b. $(-\infty, 0) \cup (1, 2)$
    c. $(-2, 0) \cup (1, \infty)$
    d. None of these
35. The range of the function $y=f(x)$ is
    a. $(-\infty, \infty)$
    b. $[-32, \infty)$
    c. $[0, \infty)$
    d. None of these
36. The range of values of a for which $f(x) = a$ has no real roots is
    a. $(4, \infty)$
    b. $(10, \infty)$
    c. $(20, \infty)$
    d. none of these

For Problems 37–39

Consider the function $f: R \to R, f(x)=\dfrac{x^2-6x+4}{x^2+2x+4}$

37. $f(x)$ is
    a. unbounded function
    b. one-one function
    c. onto function
    d. None of these

38. Which of the following is not true about $f(x)$?
  a. $f(x)$ has two points of extremum
  b. $f(x)$ has only one asymptote
  c. $f(x)$ is differentiable for all $x \in R$
  d. None of these

39. Range of $f(x)$ is
  a. $\left(-\infty, -\frac{2}{3}\right] \cup [2, \infty)$ b. $\left[-\frac{1}{3}, 5\right]$
  c. $(-\infty, 2] \cup \left[\frac{7}{3}, \infty\right)$ d. None of these

For Problems 40–42

Consider a polynomial $y = P(x)$ of the least degree passing through $A(-1, 1)$ and whose graph has two points of inflection $B(1, 2)$ and C with abscissa 0 at which the curve is inclined to the positive axis of abscissa at an angle of $\sec^{-1}\sqrt{2}$.

40. The value of $P(-1)$ is
  a. −1   b. 0
  c. 1   d. 2

41. The value of $P(0)$ is
  a. 1   b. 0
  c. $\frac{3}{4}$ d. $\frac{1}{2}$

42. The equation $P'(x) = 0$ has
  a. three distinct real roots
  b. one real roots
  c. three real roots such that one root is repeated
  d. none of these

For Problems 43–45

Let $f(x)$ be a real-valued continuous function on R defined as $f(x) = x^2 e^{-|x|}$

43. The values of k for which the equation $x^2 e^{-|x|} = k$ has four real roots
  a. $0 < k < e$ b. $0 < k < \frac{8}{e^2}$
  c. $0 < k < \frac{4}{e^2}$ d. None of these

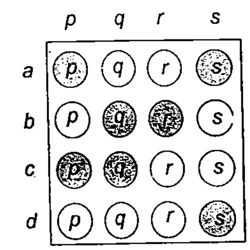
44. Which of the following is not true?
  a. $y = f(x)$ has two points of maxima
  b. $y = f(x)$ has only one asymptote
  c. $f'(x) = 0$ has three real roots
  d. none of these

45. Number of points of inflection for $y = f(x)$ is
  a. 1   b. 2   c. 3   d. 4

Matrix-Match Type

Solutions on page 6.64

Each question contains statements given in two columns which have to be matched. Statements a, b, c and d in column I have to be matched with statements p, q, r and s in column II. If the correct match are a-p, a-s, b-q, b-r, c-p, c-q and *d*-s, then the correctly bubbled 4 × 4 matrix should be as follows:



1. Consider function $f(x) = x^4 - 14x^2 + 24x - 3$.

Column I: equation $f(x) + p = 0$ has	Column II
a. two negative real roots	p. for $p \geq 120$
b. two real roots of opposite sign	q. for $-8 \leq p \leq -5$
c. four real roots	r. for $3 < p < 120$
d. no real roots	s. for $p < -8$ or $-5 < p < 3$

2.

Column I	Column II
a. $f(x) = x^2 \log x$	p. $f(x)$ has one point of minima
b. $f(x) = x \log_e x$	q. $f(x)$ has one point of maxima
c. $f(x) = \frac{\log x}{x}$	r. $f(x)$ increases in $(0, e)$
d. $f(x) = x^{-x}$	s. $f(x)$ decreases in $(0, 1/e)$

3. Let $f(x) = (x-1)^m (2-x)^n$; $m, n \in N$ and $m, n > 2$

Column I	Column II
a. Both $x = 1$ and $x = 2$ are the points of minima if	p. m is even
b. $x = 1$ is a point of minima and $x = 2$ is a point of inflection if	q. m is odd
c. $x = 2$ is a point of minima and $x = 1$ is a point of inflection if	r. n is even
d. Both $x = 1$ and $x = 2$ are the points of inflection if	s. n is odd

4. The function $f(x) = \sqrt{(ax^3 + bx^2 + cx + d)}$ has its non-zero local minimum and maximum values at $x = -2$ and $x = 2$, respectively. If a is a root of $x^2 - x - 6 = 0$, then match the following:

Column I	Column II
a. The value/values of a	p. $= 0$
b. The value/values of b	q. $= 24$
c. The value/values of c	r. ≥ 32
d. The value/values of d	s. -2

5.

Column I	Column II
a. $f(x)=\sin x-x^2+1$	p. has point of minima
b. $f(x)=x\log_e x-x+e^{-x}$	q. has point of maxima
c. $f(x)=-x^3+2x^2-3x+1$	r. is always increasing
d. $f(x)=\cos\pi x+10x+3x^2+x^3$	s. is always decreasing

6.

Column I	Column II
a. $f(x)=\|2x-1\|+\|2x-3\|$	p. has no points of extrema
b. $f(x)=2\sin x-x$	q. has one point of maxima
c. $f(x)=\|x-1\|+\|2x-3\|$	r. has one point of minima
d. $f(x)=\|x\|-\|2x-3\|$	s. has infinite points of minima

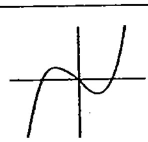
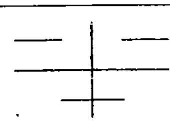
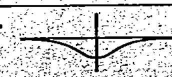
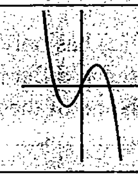
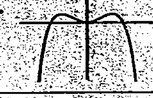
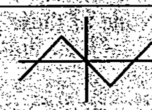
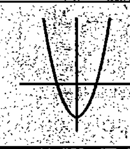
7.

Column I	Column II
a. $f(x)=(x-1)^3(x-2)^5$	p. has points of maxima
b. $f(x)=3\sin x+4\cos x-5x$	q. has point of minima
c. $f(x)=\begin{cases}\sin\frac{\pi x}{2}, 0<x\le 1\\ x^2-4x+4, 1<x<2\end{cases}$	r. has point of inflection
d. $f(x)=(x-1)^{3/5}$	s. has no point of extrema

8.

Column I	Column II
a. At $x=1, f(x)=\begin{cases}\log x, x<1\\ 2x-x^2, x\ge 1\end{cases}$	p. is increasing
b. At $x=2, f(x)=\begin{cases}x-1, x<2\\ 0, \quad x=2\\ \sin x, x>2\end{cases}$	q. is decreasing
c. At $x=0, f(x)=\begin{cases}2x+3, \ x<0\\ 5, \quad x=0\\ x^2+7, \ x>0\end{cases}$	r. has point of maxima
d. At $x=0, f(x)=\begin{cases}e^{-x} \quad x<0\\ 0, \quad x=0\\ -\cos x, x>0\end{cases}$	s. has point of minima

9.

Column I: Graph of $y=f(x)$	Column II: Graph of $y=f'(x)$
a. 	p. 
b. 	q. 
c. 	r. 
d. 	s. 

Integer Type

Solutions on page 6.66

1. If α is an integer satisfying $|a|\le 4-|[x]|$, where x is a real number for which $2x\tan^{-1}x$ is greater than or equal to $\ln(1+x^2)$, then the number of maximum possible values of a (where $[\cdot]$ represents the greatest integer function)
2. From a given solid cone of height H, another inverted cone is carved whose height is h such that its volume is maximum. Then the ratio H/h is
3. Let $f(x)=\begin{cases}\left|x^3+x^2+3x+\sin x\right|\left(3+\sin\frac{1}{x}\right), & x\ne 0,\\ 0 & , \quad x=0\end{cases}$ then number of points where $f(x)$ attains its minimum value is
4. Let $f(x)$ be a cubic polynomial which has local maximum at $x=-1$ and $f'(x)$ has a local minimum at $x=1$. If $f(-1)=10$ and $f(3)=-22$, then one fourth of the distance between its two horizontal tangents is
5. Consider $P(x)$ be a polynomial of degree 5 having extremum at $x=-1$, 1 and $\lim\limits_{x\to 0}\left(\frac{P(x)}{x^3}-2\right)=4$. Then the value of $[P(1)]$ is (where $[\cdot]$ represents greatest integer function)
6. If m is the minimum value of $f(x,y)=x^2-4x+y^2+6y$ when x and y are subjected to the restrictions $0\le x\le 1$ and $0\le y\le 1$, then the value of $|m|$ is
7. For a cubic function $y=f(x), f''(x)=4x$ at each point (x,y) on it and it crosses the x-axis at $(-2, 0)$ at an angle of $45°$ with positive direction of the x-axis. Then the value of $\left|\frac{f(1)}{5}\right|$ is
8. Number of integral values of b for which the equation $\frac{x^3}{3}-x=b$ has 3 distinct solutions is
9. Let $f(x)=\begin{cases}x+2, x<-1\\ x^2, -1\le x<1\\ (x-2)^2, x\ge 1\end{cases}$, then number of times $f'(x)$ changes its sign in $(-\infty,\infty)$ is

10. The number of non-zero integral values of 'a' for which the function $f(x) = x^4 + ax^3 + \frac{3x^2}{2} + 1$ is concave upward along the entire real line is

11. Let $f(x) = \begin{cases} x^{3/5} & \text{if } x \le 1 \\ -(x-2)^3 & \text{if } x > 1 \end{cases}$, then the number of critical points on the graph of the function is

12. A right triangle is drawn in a semicircle of radius $\frac{1}{2}$ with one of its legs along the diameter. If the maximum area of the triangle is M, then the value of $32\sqrt{3}M$ is

13. A rectangle with one side lying along the x-axis is to be inscribed in the closed region of the xy plane bounded by the lines $y = 0$, $y = 3x$ and $y = 30 - 2x$. If M is the largest area of such a rectangle, then the value of $\frac{2M}{27}$ is

14. The least integral value of x where $f(x) = \log_{1/2}(x^2 - 2x - 3)$ is monotonically decreasing is

15. The least area of a circle circumscribing any right triangle of area $\frac{9}{\pi}$ is

16. Let $f(x) = \begin{cases} |x^2 - 3x| + a, & 0 \le x < \frac{3}{2} \\ -2x + 3, & x \ge \frac{3}{2} \end{cases}$. If $f(x)$ has a local maxima at $x = \frac{3}{2}$, then greatest value of $|4a|$ is

Archives

Solutions on page 6.68

Subjective

1. Prove that the minimum value of $\frac{(a+x)(b+x)}{(c+x)}$ $a, b > c$, $x > -c$ is $\left(\sqrt{a-c} + \sqrt{b-c}\right)^2$. **(IIT-JEE, 1979)**
2. Let x and y be two real variable such that $x > 0$ and $xy = 1$. Find the minimum value of $x + y$. **(IIT-JEE, 1981)**
3. Use the function $f(x) = x^{1/x}$, $x > 0$, to determine the bigger of the two numbers e^{π} and π^e. **(IIT-JEE, 1981)**
4. Find the shortest distance of the point $(0, c)$ from the parabola $y = x^2$, where $0 \le c \le 5$. **(IIT-JEE, 1982)**
5. If $ax^2 + \frac{b}{x} \ge c$ for all positive x where $a > 0$ and $b > 0$, show that $27ab^2 \ge 4c^3$. **(IIT-JEE, 1982)**
6. Show that $1 + x \ln\left(x + \sqrt{x^2+1}\right) \ge \sqrt{1+x^2}$ for all $x \ge 0$. **(IIT-JEE, 1983)**
7. A swimmer S is in the sea at a distance d km from the closest point A on a straight shore. The house of the swimmer is on the shore at distance L km from A. He can swim at a speed of u km per hour and walk at a speed of v km per hour, $u < v$. At what point on the shore should he land so that he reaches his house in the shortest possible time? **(IIT-JEE, 1983)**
8. Find the co-ordinates of the point on the curve $y = \frac{x}{1+x^2}$ where the tangent to the curve has the greatest slope. **(IIT-JEE, 1994)**
9. Let $f(x) = \sin^3 x + \lambda \sin^2 x, -\frac{\pi}{2} < x < \frac{\pi}{2}$. Find the intervals in which λ should lie in order that $f(x)$ has exactly one minimum and exactly one maximum. **(IIT-JEE, 1985)**
10. Let $A(p^2, -p)$, $B(q^2, q)$, $C(r^2, -r)$ be the vertices of the triangle ABC. A parallelogram $AFDE$ is drawn with D, E and F on the line segments BC, CA and AB, respectively. Using calculus show that the maximum area of such a parallelogram is $\frac{1}{2}(p+q)(q+r)(p-r)$. **(IIT-JEE, 1986)**
11. Find the point on the curve $4x^2 + a^2y^2 = 4a^2$, $4 < a^2 < 8$ that is farthest from the point $(0, -2)$. **(IIT-JEE, 1987)**
12. Investigate for the maxima and minima of the function $f(x) = \int_1^x \left[2(t-1)(t-2)^3 + 3(t-1)^2(t-2)^2\right] dt$. **(IIT-JEE, 1988)**
13. Find the maxima and minima of the function $y = x(x-1)^2$, $0 \le x \le 2$. **(IIT-JEE, 1989)**
14. Show that $2\sin x + \tan x \ge 3x$, where $0 \le x < \frac{\pi}{2}$. **(IIT-JEE, 1990)**
15. A point P is given on the circumference of a circle of radius r. Chords QR are parallel to the tangent at P. Determine the maximum possible area of the triangle PQR. **(IIT-JEE, 1990)**
16. A window of perimeter P (including the base of the arch) is in the form of a rectangle surrounded by a semi-circle. The semi-circular portion is fitted with the coloured glass while the rectangular part is fitted with the clear glass that transmits three times as much light per square meter as the coloured glass does. What is the ratio for the sides of the rectangle so that the window transmits the maximum light? **(IIT-JEE, 1991)**
17. A cubic function $f(x)$ vanishes at $x = -2$ and has relative minimum/maximum at $x = -1$ and $x = \frac{1}{3}$ if $\int_{-1}^{1} f(x)\,dx = \frac{14}{3}$. Find the cubic function $f(x)$. **(IIT-JEE, 1992)**
18. Let $f(x) = \begin{cases} -x^3 + \frac{(b^3 - b^2 + b - 1)}{(b^2 + 3b + 2)}, & 0 \le x < 1 \\ 2x - 3, & 1 \le x \le 3 \end{cases}$. Find all the possible real values of b such that $f(x)$ has the smallest value at $x = 1$. **(IIT-JEE, 1993)**

19. The circle $x^2+y^2=1$ cuts the x-axis at P and Q. Another circle with centre at Q and variable radius intersects the first circle at R above the x-axis and the line segment PQ at S. Find the maximum area of the triangle QSR. **(IIT-JEE, 1994)**

20. Let (h, k) be a fixed point, where $h>0, k>0$. A straight line passing through this point cuts the positive direction of the co-ordinates axes at the points P and Q. Find the minimum area of the triangle OPQ, O being the origin. **(IIT-JEE, 1995)**

21. Determine the points of maxima and minima of the function $f(x)=\frac{1}{8}\log_e x-bx+x^2, x>0$, where $b\ge 0$ is a constant. **(IIT-JEE, 1996)**

22. Let $f(x)=\begin{cases} xe^{ax}, & x\le 0 \\ x+ax^2-x^3, & x>0 \end{cases}$ where a is a positive constant. Find the interval in which $f'(x)$ is increasing. **(IIT-JEE, 1993)**

23. Suppose $f(x)$ is a function satisfying the following conditions:
   a. $f(0)=2, f(1)=1$
   b. f has a minimum value at $x=5/2$,
   c. For all x, $f'(x)=\begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2ax+2b+1 & 2ax+b \end{vmatrix}$

   where a, b are some constants. Determine the constants a, b, and the function $f(x)$.

24. Let $-1\le p\le 1$. Show that the equation $4x^3-3x-p=0$ has a unique root in the interval $[1/2, 1]$ and identify it. **(IIT-JEE, 2001)**

25. Using the relation $2(1-\cos x)<x^2, x\ne 0$ or otherwise, prove that $\sin(\tan x)\ge x, \forall x\in\left[0,\frac{\pi}{4}\right]$. **(IIT-JEE, 2003)**

26. If $P(1)=0$ and $\frac{dP(x)}{dx}>P(x)$ for all $x\ge 1$, then prove $P(x)>0$ for all $x>1$. **(IIT-JEE, 2004)**

27. Prove that for $x\in\left[0,\frac{\pi}{2}\right]$, $\sin x+2x\ge\frac{3x(x+1)}{\pi}$. Explain the identity, if any, used in the proof. **(IIT-JEE, 2004)**

28. If $P(x)$ be a polynomial of degree 3 satisfying $p(-1)=10$, $p(1)=-6$ and $p(x)$ has maxima at $x=-1$ and $p'(x)$ has minima at $x=1$. Find the distance between the local maxima and local minima of the curve. **(IIT-JEE, 2005)**

Objective

Fill in the blanks

1. The larger of $\cos(\ln\theta)$ and $\ln(\cos\theta)$ if $e^{-\pi/2}<\theta<\frac{\pi}{2}$ is ________. **(IIT-JEE, 1983)**

2. The function $y=2x^2-\ln|x|$ is monotonically increasing for values of $x\ (\ne 0)$ satisfying the inequalities ________ and monotonically decreasing for values of x satisfying the inequalities ________. **(IIT-JEE, 1983)**

3. The set of all for which $\log_e(1+x)\le x$ is equal to ________. **(IIT-JEE, 1987)**

4. The maximum value of the function $f(x)=2x^3-15x^2+36x-48$ on the set $A=\{x\,|\,x^2+20\le 9x\}$ is ________. **(IIT-JEE, 2009)**

5. If $f(x)=x^{3/2}(3x-10), x\ge 0$, then $f(x)$ is increasing in ________. **(IIT-JEE, 2011)**

True or false

1. If $x-r$ is a factor of the polynomial $f(x)=a_0x^n+a_1x^{n-1}+\cdots+a_n$ repeated m times $(1<m\le n)$, then r is a root of $f'(x)=0$ repeated m times. **(IIT-JEE, 1984)**

2. For $0<a<x$, the minimum value of the function $\log_a x+\log_x a$ is 2. **(IIT-JEE, 1984)**

Multiple choice question with one correct answer

1. AB is a diameter of a circle and C is any point on the circumference of the circle. Then
   a. the area of ΔABC is maximum when it is isosceles
   b. the area of ΔABC is minimum when it is isosceles
   c. the perimeter of ΔABC is minimum when it is isosceles
   d. none of these **(IIT-JEE, 1983)**

2. If $f(x)=a\log|x|+bx^2+x$ has its extremum values at $x=-1$ and $x=2$, then
   a. $a=2, b=-1$ b. $a=2, b=-1/2$
   c. $a=-2, b=1/2$ d. None of these **(IIT-JEE, 1983)**

3. The function $f(x)=\frac{\ln(\pi+x)}{\ln(e+x)}$ is
   a. increasing in $(0,\infty)$
   b. decreasing in $(0,\infty)$
   c. increasing in $(0,\pi/e)$, decreasing in $(\pi/e,\infty)$
   d. decreasing in $0,\pi/e)$, increasing in $(\pi/e,\infty)$ **(IIT-JEE, 1995)**

4. In the interval $[0, 1]$, the function $x^{25}(1-x)^{75}$ takes its maximum value at the point **(IIT-JEE, 1995)**
   a. 0 b. $\frac{1}{4}$ c. $\frac{1}{2}$ d. $\frac{1}{3}$

5. If $f(x)=\frac{x}{\sin x}$ and $g(x)=\frac{x}{\tan x}$, where $0<x\le 1$, then in this interval
   a. both $f(x)$ and $g(x)$ are increasing function
   b. both $f(x)$ and $g(x)$ are decreasing function
   c. $f(x)$ is an increasing function
   d. $g(x)$ is an increasing function **(IIT-JEE, 1997)**

6. The function $f(x)=\sin^4 x+\cos^4 x$ increases if
   a. $0<x<\pi/8$ b. $\pi/4<x<3\pi/8$
   c. $3\pi/8<x<5\pi/8$ d. $5\pi/8<x<3\pi/4$ **(IIT-JEE, 1999)**

7. Consider the following statements in S and R
   S: Both $\sin x$ and $\cos x$ are decreasing functions in the interval $\left(\frac{\pi}{2},\pi\right)$.

R: If a differentiable function decreases in an interval (a, b), then its derivative also decreases in (a, b), which of the following is true?
 a. Both S and R are wrong
 b. Both S and R are correct, but R is not the correct explanation of S
 c. S is correct and R is the correct explanation for S
 d. S is correct and R is wrong **(IIT-JEE, 2000)**

8. Let $f(x) = \int e^x (x-1)(x-2)\,dx$. Then f decreases in the interval
 a. $(-\infty, -2)$ b. $(-2, -1)$
 c. $(1, 2)$ d. $(2, +\infty)$ **(IIT-JEE, 2000)**

9. Let $f(x) = \begin{cases} |x|, & \text{for } 0 < |x| \le 2 \\ 1, & \text{for } x = 0 \end{cases}$ then at $x = 0$, f has
 a. a local maximum b. no local maximum
 c. a local minimum d. no extremum **(IIT-JEE, 2000)**

10. For all $x \in (0, 1)$
 a. $e^x < 1 + x$ b. $\log_e(1 + x) < x$
 c. $\sin x > x$ d. $\log_e x > x$ **(IIT-JEE, 2001)**

11. If $f(x) = xe^{x(x-1)}$, then $f(x)$ is
 a. increasing on $[-1/2, 1]$ b. decreasing on R
 c. increasing on R d. decreasing on $[-1/2, 1]$

12. Let $f(x) = (1 + b^2)x^2 + 2bx + 1$ and let $m(b)$ be the minimum value of $f(x)$. As b varies, the range of $m(b)$ is
 a. $[0, 1]$ b. $(0, 1/2]$
 c. $[1/2, 1]$ d. $(0, 1]$ **(IIT-JEE, 2001)**

13. The length of the longest interval in which the function $3\sin x - 4\sin^3 x$ is increasing is
 a. $\frac{\pi}{3}$ b. $\frac{\pi}{2}$ c. $\frac{3\pi}{2}$ d. π **(IIT-JEE, 2002)**

14. Tangent is drawn to ellipse $\frac{x^2}{27} + y^2 = 1$ at $(3\sqrt{3}\cos\theta, \sin\theta)$ (where $\theta \in (0, \pi/2)$). Then the value of θ, such that sum of intercepts on axes made by this tangent is minimum, is
 a. $\pi/3$ b. $\pi/6$
 c. $\pi/8$ d. $\pi/4$ **(IIT-JEE, 2003)**

15. If $f(x) = x^3 + bx^2 + cx + d$ and $0 < b^2 < c$, then in $(-\infty, \infty)$
 a. $f(x)$ is a strictly increasing function
 b. $f(x)$ has local maxima
 c. $f(x)$ is a strictly decreasing function
 d. $f(x)$ is bounded **(IIT-JEE, 2004)**

16. The function defined by $f(x) = (x + 2)e^{-x}$ is
 a. decreasing for all x
 b. decreasing in $(-\infty, -1)$ and increasing in $(-1, \infty)$
 c. increasing for all x
 d. decreasing in $(-1, \infty)$ and increasing in $(-\infty, -1)$ **(IIT-JEE, 1994)**

Multiple choice question with one or more than one correct answer

1. Let $P(x) = a_0 + a_1x^2 + a_2x^4 + \cdots + a_nx^{2n}$ be a polynomial in a real variable x with $0 < a_0 < a_1 < a_2 < \cdots < a_n$. The function $P(x)$ has
 a. neither a maximum nor a minimum
 b. only one maximum
 c. only one minimum
 d. only one maximum and only one minimum
 e. none of these **(IIT-JEE, 1986)**

2. The smallest positive root of the equation $\tan x - x = 0$ lies in
 a. $\left(0, \frac{\pi}{2}\right)$ b. $\left(\frac{\pi}{2}, \pi\right)$ c. $\left(\pi, \frac{3\pi}{2}\right)$ d. $\left(\frac{3\pi}{2}, 2\pi\right)$
 e. None of these **(IIT-JEE, 1987)**

3. Let f and g be increasing and decreasing functions, respectively from $[0, \infty]$ to $[0, \infty]$. Let $h(x) = f(g(x))$. If $h(0) = 0$, then $h(x) - h(1)$ is
 a. always zero b. always negative
 c. always positive d. strictly increasing
 e. none of these **(IIT-JEE, 1987)**

4. If $f(x) = \begin{cases} 3x^2 + 12x - 1, & -1 \le x \le 2 \\ 37 - x, & 2 < x \le 3 \end{cases}$, then
 a. $f(x)$ is increasing in $[-1, 2]$
 b. $f(x)$ is continuous on $[-1, 3]$
 c. $f'(2)$ does not exist
 d. $f(x)$ has the maximum value at $x = 2$ **(IIT-JEE, 1998)**

5. Let $h(x) = f(x) - (f(x))^2 + (f(x))^3$ for every real number x, then
 a. h is increasing whenever f is increasing
 b. h is increasing whenever f is decreasing
 c. h is decreasing whenever f is decreasing
 d. nothing can be said in general **(IIT-JEE, 1998)**

6. If $f(x) = \frac{x^2 - 1}{x^2 + 1}$, for every real number x, then the minimum value of f
 a. does not exist because f is unbounded
 b. is not attained even through f is bounded
 c. is equal to 1
 d. is equal to -1 **(IIT-JEE, 1998)**

7. The number of values of x where the function $f(x) = \cos x + \cos(\sqrt{2}x)$ attains its maximum is
 a. 0 b. 1
 c. 2 d. infinite **(IIT-JEE, 1998)**

8. The function $f(x) = \int_{-1}^{x} t(e^t - 1)(t-1)(t-2)^3(t-3)^5\,dt$ has a local minimum at $x =$ **(IIT-JEE, 1993)**
 a. 0 b. 1 c. 2 d. 3

9. $f(x)$ is cubic polynomial with $f(2) = 18$ and $f(1) = -1$. Also $f(x)$ has local maxima at $x = -1$ and $f'(x)$ has local minima at $x = 0$, then
 a. the distance between $(-1, 2)$ and $(a\ f(a))$, where $x = a$ is the point of local minima is $2\sqrt{5}$
 b. $f(x)$ is increasing for $x \in [1, 2\sqrt{5}]$
 c. $f(x)$ has local minima at $x = 1$
 d. the value of $f(0) = 15$ **(IIT-JEE, 2006)**

10. $f(x) = \begin{cases} c^x, & 0 \le x \le 1 \\ 2 - c^{x-1}, & 1 < x \le 2 \\ x - c, & 2 < x \le 3 \end{cases}$ and $g(x) = \int_0^x f(t)\,dt$, $x \in [1, 3]$

then $g(x)$ has

a. local maxima at $x=1+\ln 2$ and local minima at $x=c$
b. local maxima at $x=1$ and local minima at $x=2$
c. no local maxima
d. no local minima **(IIT-JEE, 2006)**

Match the column type

1. Match the statements/expressions in Column I with the open intervals in Column II. **(IIT-JEE, 2009)**

Column I	**Column II**
a. Interval contained in the domain of definition of non-zero solutions of the differential equation $(x-3)^2 y' + y = 0$	p. $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
b. Interval containing the value of the integral $\int_1^5 (x-1)(x-2)(x-3)(x-4)(x-5)\,dx$	q. $\left(0, \frac{\pi}{2}\right)$
c. Interval in which at least one of the points locus maximum of $\cos^2 x + \sin x$ lies	r. $\left(\frac{\pi}{8}, \frac{5\pi}{4}\right)$
d. Interval in which $\tan^{-1}(\sin x + \cos x)$ is increasing	s. $\left(0, \frac{\pi}{8}\right)$
	t. $(-\pi, \pi)$

Integer type

1. Let f be a function defined on R (the set of all real numbers) such that $f'(x) = 2010(x-2009)(x-2010)^2(x-2011)^3(x-2012)^4$, for all $x \in R$. If g is a function defined on R with values in the interval $(0, \infty)$ such that $f(x) = \ln(g(x))$, for all $x \in R$, then the number of points in R at which g has a local maximum is **(IITJEE 2010)**

ANSWERS AND SOLUTIONS

Subjective Type

1. Since $f(x) = \sin(\ln x) - \cos(\ln x), x > 0$

$$= \sqrt{2}\sin\left(\ln x - \frac{\pi}{4}\right)$$

$$\therefore\ f'(x) = \frac{\sqrt{2}}{x}\cos\left(\ln x - \frac{\pi}{4}\right)$$

$$= \frac{\sqrt{2}}{x}\sin\left(\frac{\pi}{2} + \ln x - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{x}\sin\left(\frac{\pi}{4} + \ln x\right) > 0\ (\because x > 0)$$

or $\sin\left(\frac{\pi}{4} + \ln x\right) > 0$

or $2n\pi < \frac{\pi}{4} + \ln x < (2n+1)\pi,\ n \in I$

or $2n\pi - \frac{\pi}{4} < \ln x < 2n\pi + \frac{3\pi}{4},\ n \in I$

$\Rightarrow\ e^{2n\pi - \frac{\pi}{4}} < x < e^{2n\pi + \frac{3\pi}{4}},\ n \in I$

Therefore, $f(x)$ is strictly increasing when $x \in \left(e^{2n\pi - \pi/4}, e^{2n\pi + 3\pi/4}\right)\ n \in I.$

2. $f(x) = x^3 + ax^2 + bx + 5\sin^2 x$ is increasing on R

$\Rightarrow\ f'(x) > 0$ for all $x \in R$

$\Rightarrow\ 3x^2 + 2ax + b + 5\sin 2x > 0$ for all $x \in R$

$\Rightarrow\ 3x^2 + 2ax + (b-5) > 0$ for all $x \in R$

$\Rightarrow\ (2a)^2 - 4 \times 3 \times (b-5) < 0$

$\Rightarrow\ a^2 - 3b + 15 < 0$

3. $f(x) = e^{2x} - (a+1)e^x + 2x$

$\Rightarrow\ f'(x) = 2e^{2x} - (a+1)e^x + 2$

Now, $2e^{2x} - (a+1)e^x + 2 \ge 0$ for all $x \in R$

$\Rightarrow\ 2\left(e^x + \frac{1}{e^x}\right) - (a+1) \ge 0$ for all $x \in R$

$\Rightarrow\ (a+1) \le 2\left(e^x + \frac{1}{e^x}\right)$ for all $x \in R$

$\Rightarrow\ \frac{a+1}{2} \le 4\ \left(\because e^x + \frac{1}{e^x}\text{ has minimum value } 2\right)$

4. Assume $f(x) = e^x$ and let x_1 and x_2 be two points on the curve $y = e^x$.

Let R be another point which divides P and Q in the ratio $1:2$. y-coordinate of point R is $\frac{e^{2x_1} + e^{x_2}}{3}$ and y-co-ordinate of point S is $e^{\frac{2x_1 + x_2}{3}}$

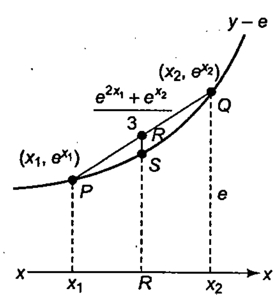


Fig. 6.57

Since $f(x) = e^x$ is always concave upward, hence point R will always be above point S.

$$\Rightarrow\ \frac{e^{2x_1} + e^{x_2}}{3} > e^{\frac{2x_1 + x_2}{3}}$$

5. Let point A, B, C form a triangle. y-co-ordinate of centroid G is $\frac{\sin x_1 + \sin x_2 + \sin x_3}{3}$ and y-co-ordinate of point F is $\sin\left(\frac{x_1 + x_2 + x_3}{3}\right)$.

From Fig. 6.58, $FD > GD$.

Hence, $\sin\left(\frac{x_1 + x_2 + x_3}{3}\right) > \frac{\sin x_1 + \sin x_2 + \sin x_3}{3}$

If $A + B + C = \pi$, then

$$\sin\left(\frac{A + B + C}{3}\right) > \frac{\sin A + \sin B + \sin C}{3}$$

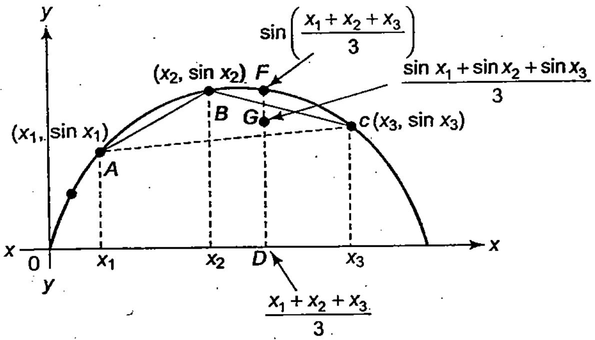


Fig. 6.58

$$\Rightarrow \sin\frac{\pi}{3} > \frac{\sin A+\sin B+\sin C}{3}$$

$$\Rightarrow \frac{3\sqrt{3}}{2} > \sin A+\sin B+\sin C$$

$$\Rightarrow \text{maximum value of } (\sin A+\sin B+\sin C) = \frac{3\sqrt{3}}{2}$$

6. Given $Q(x) = 2f\left(\frac{x^2}{2}\right) + f(6-x^2)$

$$\therefore \quad Q'(x) = 2xf'\left(\frac{x^2}{2}\right) - 2xf'(6-x^2)$$

$$= 2x\left\{f'\left(\frac{x^2}{2}\right) - f'(6-x^2)\right\}$$

But given that $f''(x) > 0 \Rightarrow f'(x)$ is increasing for all $x \in R$.

Case I: Let $\frac{x^2}{2} > (6-x^2) \Rightarrow x^2 > 4$

$\therefore \quad x \in (-\infty, -2) \cup (2, \infty)$

$$\Rightarrow f\left(\frac{x^2}{2}\right) > f(6-x^2)$$

$$\Rightarrow f\left(\frac{x^2}{2}\right) - f(6-x^2) > 0$$

If $x > 0$, then $Q'(x) > 0 \Rightarrow x \in (2, \infty)$
and if $x > 0$, then $Q'(x) < 0 \Rightarrow x \in (-\infty, -2)$.

Case II: Let $\frac{x^2}{2} < (6-x^2) \Rightarrow x^2 < 4 \Rightarrow x \in (-2, 2)$

$$\Rightarrow f'\left(\frac{x^2}{2}\right) < f'(6-x^2) \Rightarrow f'\left(\frac{x^2}{2}\right) - f'(6-x^2) < 0$$

If $x > 0$, then $Q'(x) < 0 \Rightarrow x \in (0, 2)$
and if $x < 0$, then $Q'(x) > 0 \Rightarrow x \in (-2, 0)$
Combining both cases, $Q(x)$ is increasing in $x \in (-2, 0) \cup (2, \infty)$, and $Q(x)$ is decreasing in $x \in (-\infty, -2) \cup (0, 2)$.

7. We have to prove

$$\left(\tan^{-1}\frac{1}{e}\right)^2 + \frac{2e}{\sqrt{(e^2+1)}} < (\tan^{-1} e)^2 + \frac{2}{\sqrt{(e^2+1)}}$$

$$\text{or } \left(\tan^{-1}\frac{1}{e}\right)^2 + \frac{2}{\sqrt{\left(\left(\frac{1}{e}\right)^2+1\right)}} < (\tan^{-1} e)^2 + \frac{2}{\sqrt{(e^2+1)}}$$

Now, let $f(x) = (\tan^{-1} x)^2 + \frac{2}{\sqrt{(x^2+1)}}$ (1)

$$\therefore \quad f'(x) = \frac{2\tan^{-1} x}{(1+x^2)} - \frac{2x}{(x^2+1)^{3/2}}$$

$$= \frac{2}{(1+x^2)}\left\{\tan^{-1} x - \frac{x}{\sqrt{(x^2+1)}}\right\} \quad (2)$$

To find sign of $f'(x)$ we consider

$$g(x) = \tan^{-1} x - \frac{x}{\sqrt{(x^2+1)}}$$

$$\therefore \quad g'(x) = \frac{1}{(1+x^2)}\left\{1 - \frac{1}{\sqrt{(x^2+1)}}\right\} > 0$$

$\Rightarrow \quad g'(x) > 0$
$\therefore \quad g(x)$ is an increasing function $\Rightarrow f'(x) > 0$ {From (2)}
$\Rightarrow \quad f(x)$ is an increasing function

$\because \quad \frac{1}{e} < e \qquad \therefore f\left(\frac{1}{e}\right) < f(e)$

$$\Rightarrow \left(\tan^{-1}\frac{1}{e}\right)^2 + \frac{2}{\sqrt{\left(\frac{1}{e^2}+1\right)}} < (\tan^{-1} e)^2 + \frac{2}{\sqrt{(e^2+1)}}$$

{From equation (1)}

Hence, $\left(\tan^{-1}\frac{1}{e}\right)^2 + \frac{2e}{\sqrt{(e^2+1)}} < (\tan^{-1} e)^2 + \frac{2}{\sqrt{(e^2+1)}}$.

8. In this problem, first we have to select an appropriate function.
Now by observation, given inequality can be set as $\frac{\sin(\sin\theta)}{\sin\theta} > \frac{\sin\theta}{\theta}$. This clearly gives indication that one has to study the function $f(x) = \frac{\sin x}{x}$.

$\Rightarrow f'(x) = \frac{(x\cos x - \sin x)}{x^2} = \frac{\cos x(x-\tan x)}{x^2} < 0$ (as in first quadrant $x < \tan x$)
$\Rightarrow \quad f(x)$ is a decreasing function
Now, $\sin\theta < \theta$ for $0 < \theta < \frac{\pi}{2}$

$$\Rightarrow f(\sin\theta) > f(\theta) \Rightarrow \frac{\sin(\sin\theta)}{\sin\theta} > \frac{\sin\theta}{\theta} \quad \text{\{From (1)\}}$$

Hence, $\sin^2\theta < \theta\sin(\sin\theta)$ for $0 < \theta < \frac{\pi}{2}$.

9. $f(x) = x^3 - 3x^2 + 6$
If $f'(x) = 3x^2 - 6x = 0$ then $x = 0, 2$ are the critical points of $f(x)$
$x = 0$ is a point of local maxima and $x = 2$ is a point of local minima.
Clearly, $f(x)$ is increasing in $(-\infty, 0)$ and $(2, \infty)$ and decreasing in $(0, 2)$.

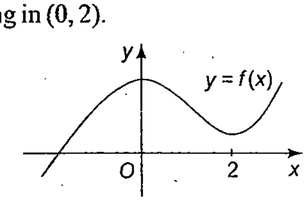


Fig. 6.59

Case 1: $x+2\le 0 \Rightarrow x\le -2$

$\Rightarrow\; g(x)=f(x+2), -3\le x\le -2$

Case 2: $x+1<0$ and $0<x+2<2$

$x<-1$ and $-2<x<0$

i.e., $-2<x<-1$

$g(x)=f(0)$

Case 3: $0\le x+1, x+2\le 2$

$\Rightarrow\; -1\le x\le 0, g(x)=f(x+1)$

$$\Rightarrow\; g(x)=\begin{cases} f(x+2), & -3\le x<-2 \\ f(0), & -2\le x<-1 \\ f(x+1), & -1\le x<0 \\ 1-x, & x\ge 0 \end{cases}$$

Hence, $g(x)$ is continuous in the interval [–3, 1].

10. We have to prove that $\sqrt{f(x)}\ge \dfrac{\sqrt{f(1)}}{x}$, $x\ge 1$ or $x\sqrt{f(x)}\ge 1\sqrt{f(1)}$.

This suggests that we have to consider function $x\sqrt{f(x)}$.

Now, given that $axf'(x)\ge 2\sqrt{f(x)}-2af(x)$, dividing both sides by $2\sqrt{f(x)}$ we have

$$ax\frac{f'(x)}{2\sqrt{f(x)}}+a\sqrt{f(x)}-1\ge 0$$

$$\Rightarrow\; \frac{d}{dx}\left(ax\sqrt{f(x)}-x\right)\ge 0$$

$\Rightarrow$ Hence, $ax\sqrt{f(x)}-x$ is an increasing function.

$\Rightarrow\; x\ge 1$ then $f(x)\ge f(1)$

$\Rightarrow\; ax\sqrt{f(x)}-x\ge a\sqrt{f(1)}-1$

$\Rightarrow\; ax\sqrt{f(x)}\ge a\sqrt{f(1)}+x-1\ge a\sqrt{f(1)}$ (as $x\ge 1$)

$$\Rightarrow\; \sqrt{f(x)}\ge \frac{\sqrt{f(1)}}{x}.$$

11. Let the corner A of the leaf $ABCD$ be folded over to A' which is on the inner edge BC of the page.

Let $AP=x$ and $AB=a$, $\therefore\; BP=a-x$

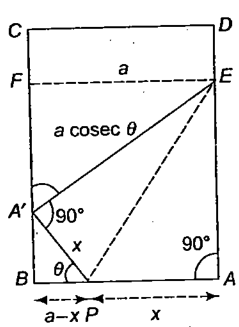


Fig. 6.60

If $\angle A'PB=\theta$, then $\angle EA'F=\theta$. EF is parallel to AB, then $EF=a$.

In $\Delta A'BP$, $\cos\theta=BP/PA'=(a-x)/x$

In $\Delta A'FE$, $A'E = EF \operatorname{cosec}\theta = a/\sqrt{(1-\cos^2\theta}$

$= a/\sqrt{[1-((a-x)/x)^2]}$

$$=\frac{ax}{\sqrt{[x^2-(a-x)^2]}}=\frac{ax}{\sqrt{(2ax-a^2)}}=AE$$

The triangle APE is folded to $A'PE$

$\Rightarrow$ Area of folded part = Area of $\Delta PAE = \dfrac{1}{2}AP.AE$

$$=\frac{1}{2}x\frac{ax}{\sqrt{(2ax-a^2)}}=A \text{ (say) or } A^2=\frac{a^2x^4}{4(2ax-a^2)}$$

$$=\frac{a^2/4}{(2a/x^3-a^2/x^4)}=\frac{a^2/4}{y} \quad \text{where } y=(2a/x^3)-a^2/x^4,$$

$0<x<a$.

Obviously, A^2 (i.e., A) is minimum when y is maximum.

Now, $y=2a/x^3-a^2/x^4 \Rightarrow \dfrac{dy}{dx}=-\dfrac{6a}{x^4}+\dfrac{4a^2}{x^5}$ and

$$\frac{d^2y}{dx^2}=\frac{24a}{x^5}-\frac{20a^2}{x^6}.$$

For maximum or minimum of y, $\dfrac{dy}{dx}=-\dfrac{6a}{x^4}+\dfrac{4a^2}{x^5}=0$

$\Rightarrow x=2a/3$

When $x=2a/3$, $\dfrac{d^2y}{dx^2}=\dfrac{4a}{x^5}\left(6-\dfrac{5a}{x}\right)$

$$=4a\left(\frac{3}{2a}\right)^5\left(6-5a\frac{3}{2a}\right)=-4a\left(\frac{3}{2a}\right)^5\frac{3}{2}<0.$$

$\Rightarrow$ y is maximum, i.e., A is minimum when $x=2a/3$ which is the only critical point (least). Hence, the folded area is minimum when 2/3 of the width of the page is folded over.

12.

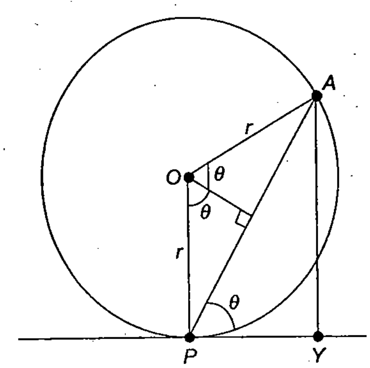


Fig. 6.61

From Fig. 6.61, $AP=2r\sin\theta$

$PY=2r\sin\theta\cos\theta=r\sin 2\theta$

$\Rightarrow\; AY=2r\sin\theta\sin\theta$

$\Rightarrow\; \Delta=$ area of $\Delta APY=\dfrac{1}{2}PY\cdot AY$

or $\Delta=r^2\sin^2\theta\sin 2\theta, 0<\theta<\pi/2$ (1)

$d\Delta/d\theta=r^2[\sin^2 2\theta+2\cos 2\theta\sin^2\theta]$

$=r^2[4\sin^2\theta\cos^2\theta+2\sin^2\theta\cos 2\theta]$

$=2r^2\sin^2\theta[4\cos^2\theta-1]$

$d\Delta/d\theta=0\Rightarrow \sin\theta=0, \cos\theta=\pm 1/2$

Therefore, the only critical point within $(0,\pi/2)$ is $\pi/3$.

Clearly $\left(\frac{d\Delta}{d\theta}\right)_{(\pi/3-h)} > 0$ and $\left(\frac{d\Delta}{d\theta}\right)_{(\pi/3+h)} < 0$

$\Rightarrow$ Δ is maximum at $\theta = \pi/3$. Being the only extrema, area is also greatest at $\pi/3$.

$\Rightarrow$ The greatest area of such triangle $= (3\sqrt{3}r^2)/8$.

13. $f(x) = \left(\frac{\sqrt{a+4}}{1-a} - 1\right)x^5 - 3x + \log 5$.

$$\Rightarrow f'(x) = 5\left(\frac{\sqrt{a+4}}{1-a} - 1\right)x^4 - 3 < 0 \text{ for all } x \in R.$$

as $f(x)$ decreases for real x.

$$\therefore \left(\frac{\sqrt{a+4}}{1-a} - 1\right)x^4 < 3/5 \text{ for all } x \in R$$

$$\therefore \frac{\sqrt{a+4}}{1-a} - 1 \le 0. \qquad (1)$$

For $a > 1$, (1) is satisfied.
For $-4 \le a < 1$,

$\sqrt{(a+4)} \le 1 - a$

$\Rightarrow a + 4 \le a^2 - 2a + 1$

$\Rightarrow a^2 - 3a - 3 \ge 0$

$$\Rightarrow a \le \frac{3-\sqrt{21}}{2} \quad \text{or} \quad a \ge \frac{3+\sqrt{21}}{2}$$

But $-4 \le a < 1$

$$\Rightarrow -4 \le a \le \frac{3-\sqrt{21}}{2}$$

Hence, $-4 \le a \le \frac{3-\sqrt{21}}{2}$ or $a > 1$.

14. $f(x)\ \frac{1}{2ax - x^2 - 5a^2} = \frac{1}{-4a^2 - (x-a)^2}$

Clearly $f(x)$ is continuous $\forall\, x \in R$.

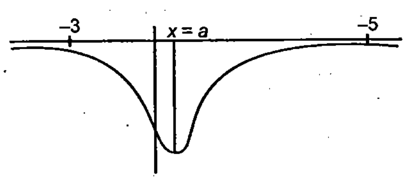


Fig. 6.62

The graph is symmetrical about the line $x = a$.
If $a = 1$ (mid point of $x = -3$ and $x = 5$), greatest value is $f(5) = f(-3)$

If $a < 1$, $f_{\max}(x) = f(5) = \frac{-1}{5(a^2 - 2a + 5)}$

and if $a > 1$, $f_{\max}(x) = f(-3) = \frac{-1}{5a^2 + 6a + 9}$

15. We know that $r = \frac{\Delta}{s}$ where Δ = Area of triangle CPQ and s = semiperimeter of ΔCPQ.

$$\Rightarrow r = \frac{\alpha^2 \sin 2\theta}{2s} = \frac{\alpha^2 \sin 2\theta}{2\alpha + 2\alpha \sin\theta} = \frac{\alpha}{2}\cdot\frac{\sin 2\theta}{1+\sin\theta}$$

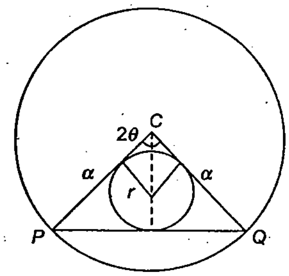


Fig. 6.63

Now, for $f(\theta) = \frac{\sin 2\theta}{1+\sin\theta}$

$$f'(\theta) = \frac{(1+\sin\theta)\,2\cos 2\theta - \sin 2\theta\cdot\cos\theta}{(1+\sin\theta)^2} = 0$$

$\Rightarrow 2(1+\sin\theta)(1 - 2\sin^2\theta) - 2\sin\theta\,(1 - \sin^2\theta) = 0$

$\Rightarrow 2(1 - 2\sin^2\theta) = 2\sin\theta\,(1 - \sin\theta)$

$\Rightarrow 1 - 2\sin^2\theta = \sin\theta - \sin^2\theta$

$\Rightarrow \sin^2\theta + \sin\theta - 1 = 0$

$$\Rightarrow \sin\theta = \frac{\sqrt{5}-1}{2}$$

16. $f'(x) = \frac{2}{\sqrt{3}}\frac{1}{1+\left(\frac{2x+1}{\sqrt{3}}\right)^2}\cdot\frac{2}{\sqrt{3}} - \frac{2x+1}{x^2+x+1} + (\lambda^2 - 5\lambda + 3) \le 0$

$$= \frac{-2x}{x^2+x+1} + (\lambda^2 - 5\lambda + 3) \le 0$$

$$\Rightarrow (\lambda^2 - 5\lambda + 3) \le \frac{2x}{x^2+x+1} \qquad (1)$$

Now, let $y = \frac{2x}{x^2+x+1} = \frac{2}{x+1+\frac{1}{x}}$

putting $x = 1$ and $x = -1$; $y = \frac{2}{3}, y = -2$

So range of $y \in \left[-2, \frac{2}{3}\right]$

From (1) $\Rightarrow$ $\lambda^2 - 5\lambda + 3 < -2$

$\Rightarrow \lambda^2 - 5\lambda + 5 < 0$

$$\Rightarrow \left(\lambda - \frac{5-\sqrt{5}}{2}\right)\left(\lambda - \frac{5+\sqrt{5}}{2}\right) \le 0$$

$$\Rightarrow \lambda \in \left[\frac{5-\sqrt{5}}{2}, \frac{5+\sqrt{5}}{2}\right]$$

17. $e(k - x\log x) = 1 \qquad (1)$

$$\Rightarrow k - \frac{1}{e} = x\ln x$$

Then equation (1) has solution where graphs of $y = x\ln x$ and $y = k - \frac{1}{e}$ intersect.

Now, consider the function $f(x) = x \log_e x$

$f'(x) = 1 + \log_e x$

$f'(x) = 0 \Rightarrow x = 1/e,$

$f''(x) = 1/x \Rightarrow f''(1/e) = e > 0$

$\Rightarrow x = 1/e$ is the point of minima.

Also, $\lim_{x\to 0} x \log_e x = \lim_{x\to 0} \frac{\log_e x}{\frac{1}{x}} = \lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = -\lim_{x\to 0} x = 0$

Hence, the graph of $f(x) = x \log_e x$ is as follows:

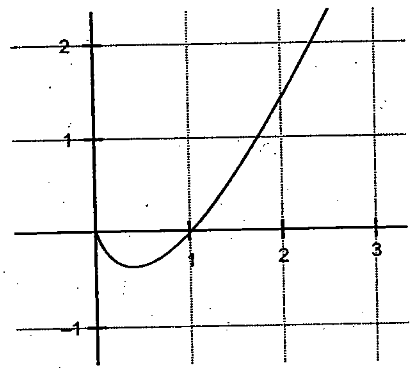


Fig. 6.64

$f(1/e) = -1/e$

Hence, equation $k - \frac{1}{e} = x \log_e x$ has two distinct roots.

if $-\frac{1}{e} < k - \frac{1}{e} < 0 \Rightarrow 0 < k < \frac{1}{e}$

Equation has no roots if $k - \frac{1}{e} < -\frac{1}{e} \Rightarrow k < 0$

Equation has one root if $k - \frac{1}{e} = -\frac{1}{e}$ or $k - \frac{1}{e} \geq 0 \Rightarrow k = 0$

or $k \geq \frac{1}{e}$

18. $\sin 1 > \cos(\sin 1)$

if $\cos\left(\frac{\pi}{2} - 1\right) > \cos(\sin 1)$

if $\frac{\pi}{2} - 1 < \sin 1$

if $\sin 1 > \left(\frac{\pi - 2}{2}\right)$ (1)

and $\sin 1 > \sin\frac{\pi}{4} > \frac{1}{\sqrt{2}}$

Hence (1) is true $\Rightarrow \sin 1 > \cos(\sin 1)$

Now, let $f(x) = \sin(\cos(\sin x)) - \cos(\sin(\cos x))$

$\Rightarrow f'(x) = \cos(\cos(\sin x))\sin(\sin x)(-\cos x) - \sin(\sin x(\cos x))\cos(\cos x)\sin x$

$\Rightarrow f'(x) < 0 \quad \forall x \in \left[0, \frac{\pi}{2}\right]$

$\Rightarrow f(x)$ is decreasing in $\left[0, \frac{\pi}{2}\right]$

and $f(0) = \sin 1 - \cos(\sin 1) > 0$

$f\left(\frac{\pi}{2}\right) = \sin(\cos(1)) - 1 < 0$

Since, $f(0)$ is positive and $f\left(\frac{\pi}{2}\right)$ is negative.

$f(x) = 0$ has one solution in $\left[0, \frac{\pi}{2}\right]$.

19. For a real number $a \in R$ we define a function $g: R \to R$ by $g(x) = f'(x+a)\sin x - f(x+a)\cos x$, then $\forall x \in [0, \pi]$, we have

$g'(x) = \sin x\,(f(x+a) + f''(x+a)) \geq 0$

and therefore g is a non-decreasing function. Hence, for every $a \in R$

$0 \leq g(\pi) - g(0) = f(\pi + a) + f(a) = 2f(a)$

$\Rightarrow f(a) \geq 0$

20. Let $f(x) = 8\sin x - \sin 2x$

$f'(x) = 8\cos x - 2\cos 2x$

$f''(x) = -8\sin x + 4\sin 2x = -8\sin x(1 - \cos x)$

for these we see that $f'(0) = 6$,

$f'\left(\frac{\pi}{3}\right) = 5, f(0) = 0, f''(x) < 0$ in $\left[0, \frac{\pi}{3}\right]$

Therefore $5 \leq f'(x) \leq 6$ in $\left[0, \frac{\pi}{3}\right]$

Integrating from 0 to x, gives

$\Rightarrow 5x \leq f(x) \leq 6x$ in $\left[0, \frac{\pi}{3}\right]$

21. Given $f(x) \geq 0\ \forall x \geq 0$ (1)

and $f'(x)\cos x - f(x)\sin x \leq 0$

$\Rightarrow (f(x)\cos x)' \leq 0$ (2)

Let $g(x) = f(x)\cos x$

$g'(x) \leq 0$

$\Rightarrow g(x)$ is a decreasing function. (from (2))

$\Rightarrow g\left(\frac{\pi}{2}\right) \geq g\left(\frac{5\pi}{3}\right)$

$\Rightarrow g\left(\frac{5\pi}{3}\right) \leq 0$

$\Rightarrow f\left(\frac{5\pi}{3}\right) \leq 0$ (3)

From equations (1) and (3),

$f\left(\frac{5\pi}{3}\right) = 0$

Objective Type

1. c. $f'(x) = 3kx^2 - 18x + 9 = 3[kx^2 - 6x + 3] \geq 0, \forall x \in R$

$\Rightarrow D = b^2 - 4ac \leq 0, k > 0$, i.e., $36 - 12k \leq 0$

$\Rightarrow k \geq 3.$

2. d. Since $f(x) = \dfrac{K\sin x+2\cos x}{\sin x+\cos x}$ is increasing for all x,

therefore $f'(x) > 0$ for all x

$\Rightarrow \dfrac{K-2}{(\sin x+\cos x)^2} > 0$ for all x

$\Rightarrow K-2>0 \Rightarrow K>2$

3. d. $f'(x) = a + 3\cos x - 4\sin x$

$= a + 5\cos(x+\alpha)$, where $\cos\alpha = \dfrac{3}{5}$

For invertible, $f(x)$ must be monotonic

$\Rightarrow f'(x) \ge 0\ \forall x$ or $f'(x) \le 0\ \forall x$

$\Rightarrow a + 5\cos(x+\alpha) \ge 0$ or $a + 5\cos(x+\alpha) \le 0$

$\Rightarrow a \ge -5\cos(x+\alpha)$ or $a \le -5\cos(x+\alpha)$

$\Rightarrow a \ge 5$ or $a \le -5$

4. d. We have $g'(x) = f'\left(\dfrac{x}{2}\right) - f'(2-x)$

Given $f''(x) < 0\ \forall x \in (0, 2)$

So, $f'(x)$ is decreasing on $(0, 2)$

Let $\dfrac{x}{2} > 2 - x \Rightarrow f'\left(\dfrac{x}{2}\right) < f'(2-x)$

Thus, $\forall x > \dfrac{4}{3}, g'(x) < 0$

$\Rightarrow g(x)$ decreasing in $\left(\dfrac{4}{3}, 2\right)$

and increasing in $\left(0, \dfrac{4}{3}\right)$.

5. d. $f(x) = x^{100} + \sin x - 1$

$\Rightarrow f'(x) = 100x^{99} + \cos x$

If $0 < x < \dfrac{\pi}{2}$, then $f'(x) > 0$, therefore $f(x)$ is increasing on $(0, \pi/2)$.

If $0 < x < 1$, then

$100x^{99} > 0$ and $\cos x > 0$ [$\because x$ lies between 0 and 1 radian]

$\Rightarrow f'(x) = 100x^{99} + \cos x > 0$

$\Rightarrow f(x)$ is increasing on $(0, 1)$.

If $\dfrac{\pi}{2} < x < \pi$, then

$100x^{99} > 100$ $\quad [\because x > 1 \Rightarrow x^{99} > 1]$

$\Rightarrow 100x^{99} + \cos x > 0$ [$\because \cos x \ge -1 \Rightarrow 100x^{99} + \cos x > 99$]

$\Rightarrow f'(x) > 0 \Rightarrow f(x)$ is increasing in $(\pi/2, \pi)$.

6. d. When $f(x) = 3x^2 - 2x + 1$

$\therefore f'(x) = 6x - 2$

f is increasing $\Rightarrow f'(x) \ge 0$

$\Rightarrow 6x - 2 \ge 0 \Rightarrow x \ge \dfrac{1}{3}$

7. a. Here $f(x) = \tan^{-1}(\sin x + \cos x)$

$\therefore f'(x) = \dfrac{1}{1+(\sin x + \cos x)^2}(\cos x - \sin x)$

$= \dfrac{\cos x - \sin x}{2 + \sin 2x}$

For $-\dfrac{\pi}{2} < x < \dfrac{\pi}{4}$, $\cos x > \sin x$

Hence, $y = f(x)$ is increasing in $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{4}\right)$.

8. c. $\because f'(x) = x^x[1 + \log x] = x^x \log(ex)$

$f'(x) < 0$

$\Rightarrow \log(ex) < 0$

$\Rightarrow 0 < ex < 1$

$\Rightarrow 0 < x < 1/e$

9. c. $f(x) = (x-1)^2 + (x-2)^2 + (x-3)^2 + (x-4)^2 + (x-5)^2$

$f'(x) = 2[x-1+x-2+x-3+x-4+x-5] = 2[5x-15]$

$f'(x) = 0$ gives $x = 3$ and $f''(x) > 0$ for all x

$\therefore f(x)$ is minimum for $x = 3$.

10. d. If $f(x)$ increases then $f^{-1}(x)$ increases. Refer Fig. 6.65.

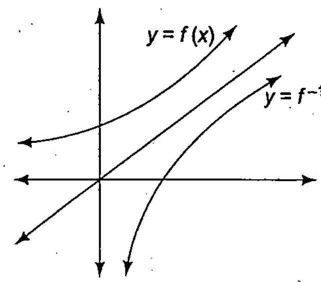


Fig. 6.65

If $f(x)$ increases, then $f'(x) > 0$

$\Rightarrow \dfrac{d}{dx}\left(\dfrac{1}{f(x)}\right) = -\dfrac{f'(x)}{f^2(x)} < 0 \Rightarrow \dfrac{1}{f(x)}$ decreases

$\dfrac{d}{dx}\left(\dfrac{f}{g}\right) = \dfrac{f'g - fg'}{g^2}$

If f and g are + ve functions and $f' < 0$ and $g' > 0$, then $\dfrac{d}{dx}\left(\dfrac{f}{g}\right) < 0$.

11. c. $f(x) f'(x) < 0\ \forall x \in R$

$\Rightarrow \dfrac{1}{2}\dfrac{d}{dx}(f^2(x)) < 0$

$\Rightarrow \dfrac{d}{dx}(f^2(x)) < 0$

$\Rightarrow f^2(x)$ is a decreasing function.

12. a. $f(x) = x\sqrt{4ax - x^2}$ $\quad$ (domain is $[0, 4a]$

$\Rightarrow f'(x) = \sqrt{4ax - x^2} + \dfrac{x(4a-2x)}{2\sqrt{4ax - x^2}}$

$= \dfrac{2x(3a-x)}{\sqrt{4ax - x^2}}$

Now if $f'(x) > 0$

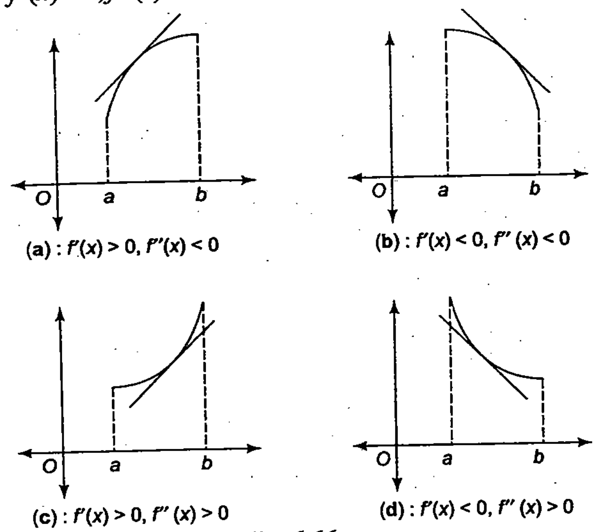
$\Rightarrow 2x(3a - x) > 0$

$\Rightarrow 2x(x - 3a) < 0$

$\Rightarrow x \in (0, 3a)$

Thus, $f(x)$ increases in $(0, 3a)$ and decreases in $(3a, 4a)$.

13. c. $f'(x)<0, f''(x)<0$



(a) : $f'(x) > 0$, $f''(x) < 0$ (b) : $f'(x) < 0$, $f''(x) < 0$

(c) : $f'(x) > 0$, $f''(x) > 0$ (d) : $f'(x) < 0$, $f''(x) > 0$

Fig. 6.66

Clearly for $f'(x) > 0$, $f''(x) > 0$ [in Fig. (6.66(c)] tangent always lies below the graph.

Or $f'(x)<0, f''(x)>0$ [in Fig. 6.66 (d)) tangent always lies below the graph.

14.a. $f'(x)=|x|-\{x\}=|x|-(x-[x])=|x|-x+[x]$

For $x \in (-1/2, 0)$,

$f'(x)=-x-x-1=-2x-1$

Also, for $-\dfrac{1}{2}<x<0 \Rightarrow 0<-2x<1 \Rightarrow -1<-2x-1<0$

$\Rightarrow$ $f'(x)<0 \Rightarrow f(x)$ decreases in $(-1/2, 0)$.

Similarly, we can check for other given options say for $x \in (-1/2, 2)$,

$$f'(x)=\begin{cases}(-x)-x-1, & -\dfrac{1}{2}<x<0\\ x-x+0, & 0\le x<1\\ x-x+1, & 1\le x<2\\ \vdots & \end{cases}$$

Here $f(x)$ decreases only in $(-1/2, 0)$, otherwise $f(x)$ in other intervals is constant.

15. d. $f(x)=|x|-|x-1|$

$$=\begin{cases}-x-(1-x), & x<0\\ x-(1-x), & 0\le x<1\\ x-(x-1), & x\ge 0\end{cases}$$

$$=\begin{cases}-1, & x<0\\ 2x-1, & 0\le x<1\\ 1, & x\ge 1\end{cases}$$

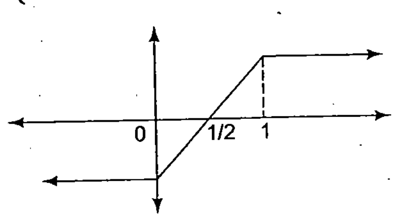


Fig. 6.67

Graph of the function is that $f(x)$ clearly increases in $(0, 1)$.

16. d. $|f(x)|=\begin{cases}f(x), & f(x)\ge 0\\ -f(x), & f(x)<0\end{cases}$

$\Rightarrow$ $\dfrac{d}{dx}|f(x)|=\begin{cases}f'(x), & f(x)>0\\ -f'(x), & f(x)<0\end{cases}$

Now as $f(x)$ and $f'(x)$ keep opposite sign, then

$\dfrac{d}{dx}|f(x)|<0.$

Hence $|f|$ is decreasing.

17. c. $\phi'(x)=2f(x)f'(x)$

We do not know the sign of $f(x)$ in (a, b), so we cannot say about the sign of $\phi'(x)$.

18. a. Given $\phi'(x)-\phi(x)>0$ $\forall x\ge 1$

$\Rightarrow e^{-x}\{\phi'(x)-\phi(x)\}>0$ $\forall x\ge 1$

$\Rightarrow \dfrac{d}{dx}e^{-x}\phi(x)>0$ $\forall x\ge 1$

$\therefore$ $e^{-x}\phi(x)$ is an increasing function $\forall x\ge 1$

Since $\phi(x)$ is a polynomial

$\Rightarrow$ $e^{-x}\phi(x)>e^{-1}\phi(1)\Rightarrow e^{-x}\phi(x)>0$ $[\because \phi(1)=0]$

$\Rightarrow$ $\phi(x)>0$.

19. c. Function is increasing in $(-\infty, -2)\cup(0,\infty)$, function is decreasing in $(-2, 0)$.

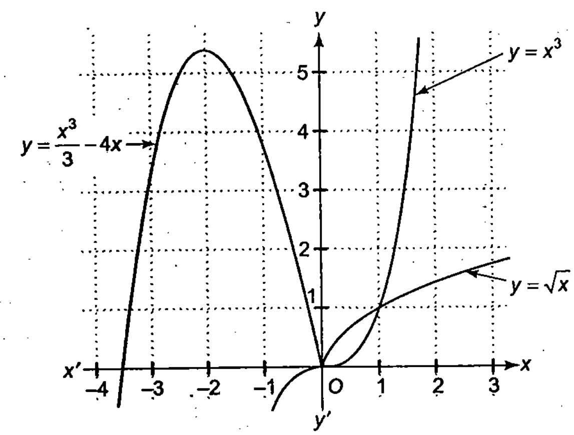


Fig. 6.68

$x=-2$ is local maxima, $x=0\rightarrow$ local minima

Derivable $\forall x\in R-\{0, 1\}$

Continuous $\forall$ $x\in R$.

20. d. $g'(x)=(f'((\tan x-1)^2+3))\,2(\tan x-1)\sec^2 x$

Since $f''(x)>0\Rightarrow f'(x)$ is increasing

So, $f'((\tan x-1)^2+3)>f'(3)=0$ $\forall x\in\left(0,\dfrac{\pi}{4}\right)\cup\left(\dfrac{\pi}{4},\dfrac{\pi}{2}\right)$

Also, $(\tan x-1)>0$ $x\in\left(\dfrac{\pi}{4},\dfrac{\pi}{2}\right)$.

So, $g(x)$ is increasing in $\left(\dfrac{\pi}{4},\dfrac{\pi}{2}\right)$.

21. b. We must have $\log_{1/3}(\log_3(\sin x+a))<0$ $\forall x\in R$

$\Rightarrow$ $\log_3(\sin x+a)>1$ $\forall x\in R$

$\Rightarrow$ $\sin x+a>3$ $\forall x\in R$

$\Rightarrow$ $a>3-\sin x$ $\forall x\in R$

$\Rightarrow$ $a>4$

22. c. $u = \sqrt{c+1} - \sqrt{c}$

$$u = \frac{1}{\sqrt{c+1}+\sqrt{c}} \text{ and } v = \frac{1}{\sqrt{c-1}+\sqrt{c}}$$

Clearly $u < v$

Also, f is increasing whereas g is decreasing.

Thus $u < v$

$\Rightarrow f(u) < f(v)$

$\Rightarrow gof(u) > gof(v)$

23. b. $f'(x) = 4 - 2\sec^2 2x = 2(1 - \tan^2 2x)$

For the continuous domain $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$, $f'(x) \geq 0$ in $\left[-\frac{\pi}{8}, \frac{\pi}{8}\right]$

and $f'(x) \leq 0$ in $\left(-\frac{\pi}{4}, -\frac{\pi}{8}\right] \cup \left[\frac{\pi}{8}, \frac{\pi}{4}\right)$

So the required largest continuous interval is $\left[-\frac{\pi}{8}, \frac{\pi}{8}\right]$,

length $= \frac{\pi}{4}$.

24. a.

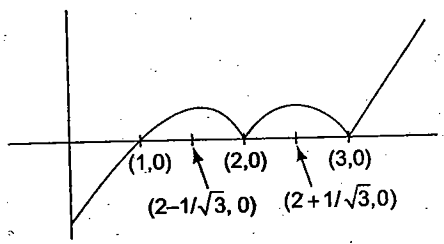


Fig. 6.69

$f(x) = (x-1)\,|(x-2)(x-3)|$

let $g(x) = (x-1)(x-2)(x-3)$

$= x^3 - 6x^2 + 11x - 6$

$\Rightarrow g'(x) = 3x^2 - 12x + 11$

$$g'(x) = 0 \Rightarrow x = \frac{12 \pm \sqrt{144-132}}{6} = \frac{12 \pm \sqrt{12}}{6} = 2 \pm \frac{1}{\sqrt{3}}$$

Hence, $f(x)$ decreases in $\left(2 - \frac{1}{\sqrt{3}}, 2\right) \cup \left(2 + \frac{1}{\sqrt{3}}, 3\right)$.

25. d. Let $f(x) = x^3 + 2x^2 + 5x + 2\cos x$

$\Rightarrow f'(x) = 3x^2 + 4x + 5 - 2\sin x$

Now the least value of $3x^2 + 4x + 5$ is

$$-\frac{D}{4a} = -\frac{(4)^2 - 4(3)(5)}{4(3)} = \frac{11}{3}$$

and the greatest value of $2\sin x = 2$

$\Rightarrow 3x^2 + 4x + 5 > 2\sin x \ \forall x \in R$

$\Rightarrow f'(x) = 3x^2 + 4x + 5 - 2\sin x > 0 \ \forall x \in R$

$\Rightarrow f(x)$ is strictly an increasing function

also $f(0) = 2$ and $f(2\pi) > 0$.

Thus, for the given interval, $f(x)$ never becomes zero.

Hence, the number of roots is zero.

26. a. $f(x) = (x-2)\,|x-3|$

For, $f(x) = (x-2)(x-3) = x^2 - 5x + 6$

$f'(x) = 2x - 5 = 0 \Rightarrow x = 5/2$

Now, the graph of $f(x) = (x-2)\,|x-3|$ is

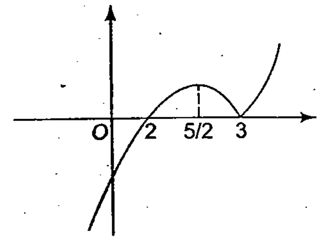


Fig. 6.70

Clearly from the graph, $f(x)$ increases in $(-\infty, 5/2) \cup (3, \infty)$.

27. b. $f(x) = (x-8)^4 (x-9)^5, 0 \leq x \leq 10$

$\Rightarrow f'(x) = 4(x-8)^3 (x-9)^5 + 5(x-9)^4 (x-8)^4$

$= (x-8)^3 (x-9)^4 [4(x-9) + 5(x-8)]$

$= 9(x-8)^3 (x-9)^4 \left(x - \frac{76}{9}\right)$.

Sign scheme of $f'(x)$

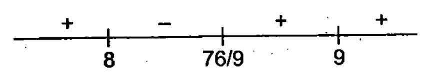


Fig. 6.71

$f'(x) < 0$, if $x \in \left(8, \frac{76}{9}\right) \Rightarrow f(x)$ decreases if $x \in \left(8, \frac{76}{9}\right)$.

28. a. Here $f'(x) \leq 0$

$\Rightarrow 3x^2 + 8x + \lambda \leq 0 \ \forall x \in \left(-2, -\frac{2}{3}\right)$

Then situations for $f'(x)$ is as follow:

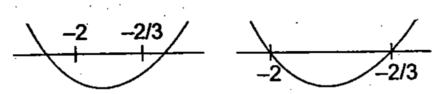


Fig. 6.72

Given that $f(x)$ decreases in the largest possible interval $\left(-2, -\frac{2}{3}\right)$, then $f'(x) = 0$ must have roots –2 and –2/3.

$\Rightarrow$ Product of roots is $(-2)\left(-\frac{2}{3}\right) = \frac{\lambda}{3} \Rightarrow \lambda = 4$.

29. b. $f(x) = |x \log_e x|$

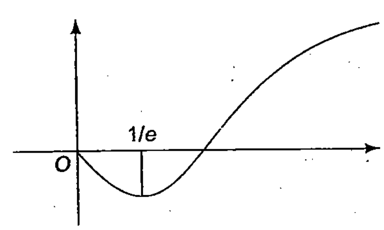
For $g(x) = x \log_e x$,

$$g'(x) = x\,\frac{1}{x} + \log_e x = 1 + \log_e x$$

$\Rightarrow g(x)$ increases for $\left(\frac{1}{e}, \infty\right)$ and decreases for $\left(0, \frac{1}{e}\right)$.

Graph of $y = g(x) = x \log_e x$

(a)



Graph of $y = f(x) = |x \log_e x|$

(b)

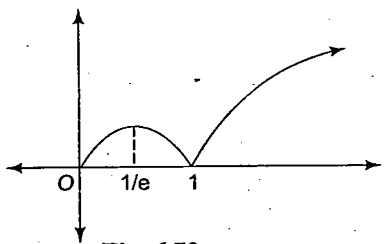


Fig. 6.73

From the graph, $f(x) = |x \log_e x|$ decreases in $\left(\frac{1}{e}, 1\right)$.

30. b. Let $h(x) = f(x) - g(x)$

$h'(x) = f'(x) - g'(x) > 0 \ \forall x \in R$

$\Rightarrow h(x)$ is an increasing function and

$h(0) = f(0) - g(0) = 0$

Therefore, $h(x) > 0 \ \forall x \in (0, \infty)$ and $h(x) < 0 \ \forall x \in (-\infty, 0)$.

31. b. $g'(x) = xf'(2x^2 - 1) - xf'(1 - x^2) = x(f'(2x^2 - 1) - f'(1 - x^2))$

$g'(x) > 0$

if $x > 0$, $2x^2 - 1 > 1 - x^2$ (as f' is an increasing function)

$\Rightarrow \quad 3x^2 > 2 \Rightarrow x \in \left(-\infty, -\sqrt{\frac{2}{3}}\right) \cup \left(\sqrt{\frac{2}{3}}, \infty\right)$

$\Rightarrow \quad x \in \left(\sqrt{\frac{2}{3}}, \infty\right)$

If $x < 0$, $2x^2 - 1 < 1 - x^2$

$\Rightarrow \quad 3x^2 < 2 \Rightarrow x \in \left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right) \Rightarrow x \in \left(-\sqrt{\frac{2}{3}}, 0\right)$.

32. b. $f(0) = \sin 0 = 0$

$f(0^+) \to 0^+$

$f(0^-) = \lim_{x \to 0^-} \sin(x^2 - 3x) = \lim_{h \to 0} \sin(h^2 + 3h) \to 0^+$

Thus, $f(0^+) > f(0)$ and $f(0^-) > f(0)$

Hence, $x = 0$ is a point of minima.

33. b. Since $\cos \theta \le 1$ for all θ. Therefore, $f(x) \le 1$ for all x.

34. b. $\frac{dy}{dx} = 5x^2 (x - 1)(x - 3) = 0$

$\therefore \quad x = 0, 1, 3$

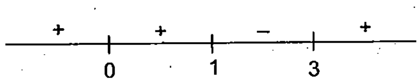


Fig. 6.74

Clearly $x = 0$ is neither a point of maxima nor a point of minima as derivative does not change sign at $x = 0$.

$x = 1$ is a point of maxima and $x = 3$ is a point of minima.

35. d. $y = \frac{\log x}{x}$

$\Rightarrow \quad \frac{dy}{dx} = -\frac{1}{x^2} \log x + \frac{1}{x} \frac{1}{x}$

$= \frac{1}{x^2} (1 - \log x) = 0$

$\frac{dy}{dx} = 0 \Rightarrow \log x = 1$ or $x = e$

For $x < e \Rightarrow \log x < 1$

and $x > e \Rightarrow \log x > 1$

At $x = e$, $\frac{dy}{dx}$ changes sign from + ive to –ive and hence y is maximum at $x = e$ and its value is

$\frac{\log e}{e} = e^{-1}$.

36. c. When $f''(a) = 0$, then $f'''(a)$ must also be zero and sign of $f''''(a)$ will decide about maximum or minimum.

37. c. Then given expression is minimum when

$y = (x^2 - 3)^3 + 27$ is minimum, which is so when $x = 0$

Hence $y_{\text{min.}} = 0$.

$\Rightarrow \quad$ Min. value of $2^{(x^2-3)^3+27}$ is $2^0 = 1$.

38. a. Let $f(x) = e^{x-1} + x - 2 \Rightarrow f'(x) = e^{x-1} + 1 > 0 \ \forall x \in R$

Also when $x \to \infty, f(x) \to \infty$ and when $x \to -\infty, f(x) \to -\infty$

Further $f(x)$ is continuous, hence its graph cuts x-axis only at one point.

Hence, equation $f(x) = 0$ has only one root.

Alternative method:

Also $e^{x-1} = 2 - x$

As shown in the figure, graphs of $y = e^{x-1}$ and $y = 2 - x$ cuts at only one point. Hence, there is only one root.

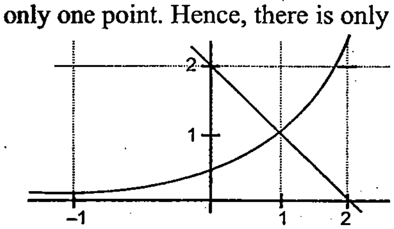


Fig. 6.75

39. c. $f(x) = \frac{(\sin x + \cos x)^2 - 1}{\frac{1}{\sqrt{2}}(\sin x + \cos x)} = \sqrt{2} \, \frac{t^2 - 1}{t}$

or $f(x) = \phi(t) = \sqrt{2}\left(t - \frac{1}{t}\right)$

where $t = g(x) = \sin x + \cos x, x \in [0, \pi/2]$

$g'(x) = \cos x - \sin x = 0 \Rightarrow \tan x = 1$

$\Rightarrow \quad x = \pi/4$ and $g''(x) = -$ ive

At $x = 0, t = 1 \therefore t \in [1, \sqrt{2}]$

Now $\phi(t) = \sqrt{2}\left(t - \frac{1}{t}\right)$ where $t \in [1, \sqrt{2}]$

$\phi'(t) = \sqrt{2}\left(1 + \frac{1}{t^2}\right) = +$ ive

Therefore, $\quad \phi(t)$ is increasing.

Hence $\phi(t)$ is greatest at the endpoint of interval $[1, \sqrt{2}]$ i.e. $t = \sqrt{2}$.

$\therefore \quad f(x) = \phi(t) = \sqrt{2}\left[\sqrt{2} - \frac{1}{\sqrt{2}}\right] = 1$

Alternative method:

$f(x) \frac{\sin 2x}{\sin\left(x + \frac{\pi}{4}\right)} = \frac{2 \sin x \cos x}{\frac{1}{\sqrt{2}}(\sin x + \cos x)} = 2\sqrt{2} \frac{1}{\sec x + \operatorname{cosec} x}$

For $x \in (0, \pi/2)$, maximum value of $\sec x + \operatorname{cosec} x$ occurs when $\sec x = \operatorname{cosec} x$ or $x = \pi/4$

Hence, $f_{max} = \dfrac{2\sqrt{2}}{\sec\frac{\pi}{4} + \text{cosec}\frac{\pi}{4}} = \dfrac{2\sqrt{2}}{2\sqrt{2}} = 1$

40. a. $f(x) = (4\sin^2 x - 1)^n (x^2 - x + 1)$

$x^2 - x + 1 > 0\ \forall x$

$f\left(\frac{\pi}{6}\right) = 0$

$f\left(\frac{\pi^+}{6}\right) = \lim_{x\to\frac{\pi^+}{6}} (4\sin^2 x - 1)^n (x^2 - x + 1)$

$= \to 0^+$

$f\left(\frac{\pi^-}{6}\right) = \lim_{x\to\frac{\pi^-}{6}} (4\sin^2 x - 1)^n (x^2 - x + 1)$

$= (\to 0^-)^n$ (a positive value)

$f\left(\frac{\pi^-}{6}\right) > 0$ if n is an even number.

41. d. $f(x) = x\ln x - x + 1$

$\therefore\ f(1) = 0$

$f'(x) = 1 + \ln x - 1 = \ln x$

$\therefore\ f'(x) < 0$ if $0 < x < 1$

and $f'(x) > 0$ if $x > 1$

42. b. We have $f(x) = (x+1)^{1/3} - (x-1)^{1/3}$

$\therefore\ f'(x) = \frac{1}{3}(x+1)^{\frac{-2}{3}} - \frac{1}{3}(x-1)^{\frac{-2}{3}}$

$= \dfrac{(x-1)^{2/3} - (x+1)^{2/3}}{3(x^2-1)^{2/3}}$

Clearly $f'(x)$ does not exist at $x = \pm 1$

Now, $f'(x) = 0$

$\Rightarrow\ (x-1)^{2/3} = (x+1)^{2/3}$

$\Rightarrow\ (x-1)^2 = (x+1)^2$

$\Rightarrow\ -2x = 2x \Rightarrow 4x = 0 \Rightarrow x = 0.$

Clearly, $f'(x) \neq 0$ for any other values of $x \in [0, 1]$.

The value of $f(x)$ at $x = 0$ is 2.

Hence, the greatest value of $f(x) = 2$.

43. b. $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$

$\therefore\ f'(x) = 6x^2 - 18ax + 12a^2$ and $f''(x) = 12x - 18a$

For maximum/minimum, $6x^2 - 18ax + 12a^2 = 0$

$\Rightarrow\ x^2 - 3ax + 2a^2 = 0$

$\Rightarrow\ (x-a)(x-2a) = 0$

$\Rightarrow\ x = a$ or $x = 2a$

Now, $f''(a) = 12a - 18a = -6a < 0$

and $f''(2a) = 24a - 18a = 6a > 0$

$\therefore\ f(x)$ is maximum at $x = a$ and minimum at $x = 2a$

$\Rightarrow\ p = a$ and $q = 2a$

Given that $p^2 = q \Rightarrow a^2 = 2a \Rightarrow a(a-2) = 0 \Rightarrow a = 2.$

44. a. Let $f(x) = x + \frac{1}{x}$

$\therefore\ f'(x) = 1 - \frac{1}{x^2}$ and $f''(x) = \frac{2}{x^3}$

For maximum/minimum, $f'(x) = 0 \Rightarrow 1 - \frac{1}{x^2} = 0$

$\Rightarrow\ x^2 = 1 \Rightarrow x \pm 1$

$f(x)$ is minimum at $x = 1$ $\quad [\because f''(x) = \frac{2}{1} = 2 > 0]$

45. a. We have $f(x) = \frac{x}{2} + \frac{2}{x}$

$\therefore\ f'(x) = \frac{1}{2} - \frac{2}{x^2}$ and $f''(x) = \frac{4}{x^3}$

Now $f'(x) = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

$\therefore\ f''(x) > 0$ for $x = 2$

Therefore, f has local minima at $x = 2$.

46. a. $f(x) = \sin\left(x + \frac{\pi}{6}\right) + \cos\left(x + \frac{\pi}{6}\right)$

$= \sqrt{2}\sin\left(x + \frac{\pi}{6} + \frac{\pi}{4}\right)$

$= \sqrt{2}\sin\left(x + \frac{5\pi}{12}\right)$

Its maximum value $= \sqrt{2}$ when $x + \frac{5\pi}{12} = \frac{\pi}{2}$

i.e., when $x = \frac{\pi}{2} - \frac{5\pi}{12} = \frac{6\pi - 5\pi}{12} = \frac{\pi}{12}$.

47. d. $f(0) > f(0^+)$ and $f(0) < f(0^-)$, hence $x = 0$ is neither a maximum nor a minimum.

48. c. $f'(x) = \dfrac{(1+4x+x^2)1 - x(4+2x)}{(1+4x+x^2)^2} = \dfrac{1-x^2}{(1+4x+x^2)^2}$

For maximum or minimum $f'(x) = 0 \Rightarrow x = \pm 1$

For $x = 1$, $f'(x)$ changes sign from positive to negative as x passes through 1.

Therefore, $f(x)$ is maximum for $x = 1$, and maximum value $= \dfrac{1}{1+4+1} = \dfrac{1}{6}$.

49. b. $y = -x^3 + 3x^2 + 9x - 27$

$\therefore\ \frac{dy}{dx} = -3x^2 + 6x + 9$

Let the slope of tangent to the curve at any point be m (say)

$\Rightarrow\ m = -3x^2 + 6x + 9 \Rightarrow \frac{dm}{dx} = -6x + 6$

$\frac{d^2m}{dx^2} = -6 < 0$ for all x

Therefore, m is maximum when $\frac{dm}{dx} = 0$, i.e., when $x = 1$

Therefore, maximum slope $= -3 + 6 + 9 = 12$.

50. b. $f'(x) = -\pi\sin\pi x + 10 + 6x + 3x^2$

$= 3(x+1)^2 + 7 - \pi\sin\pi x > 0$ for all x.

$\therefore\ f(x)$ is increasing in $-2 \le x \le 3$.

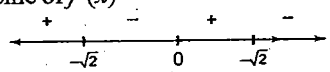
So, the absolute minimum $= f(-2) = 1 - 20 + 12 - 8$.

51. c. Given $y = e^{(2x^2-2x+1)\sin^2 x} = e^{2\left[\left(x-\frac{1}{2}\right)^2 + \frac{1}{4}\right]\sin^2 x}$

Clearly, the minimum value occurs when $\sin^2 x = 0$ as $\left[\left(x - \frac{1}{2}\right)^2 + \frac{1}{4}\right] \ge 1/4$.

52. d. $f(x)=x^4 e^{-x^2} \Rightarrow f'(x)=4x^3e^{-x^2}+x^4e^{-x^2}(-2x)$
$=2x^3 e^{-x^2}(z-x^2)$

Sign scheme of $f'(x)$



Hence, $f(x)$ is maximum at $x=\sqrt{2} \Rightarrow$ Maximum value $=4e^{-2}$

53. c. $a^2x^4+b^2y^4=c^6$

$$\Rightarrow y=\left(\frac{c^6-a^2x^4}{b^2}\right)^{1/4}$$

$$\Rightarrow f(x)=xy=x\left(\frac{c^6-a^2x^4}{b^2}\right)^{1/4}$$

$$\Rightarrow f(x)=\left(\frac{c^6x^4-a^2x^8}{b^2}\right)^{1/4}$$

Differentiate $f(x)$ w.r.t x,

$$\Rightarrow f'(x)=\frac{1}{4}\left(\frac{c^6x^4-a^2x^8}{b^2}\right)^{-3/4}\left(\frac{4x^3c^6}{b^2}-\frac{8x^7a^2}{b^2}\right)$$

Put $f'(x)=0$, $\frac{4x^3c^6}{b^2}-\frac{8x^7a^2}{b^2}=0$

$$\Rightarrow x^4=\frac{c^6}{2a^2} \Rightarrow x=\pm\frac{c^{3/2}}{2^{1/4}\sqrt{a}}$$

At $x=\frac{c^{3/2}}{2^{1/4}\sqrt{a}}$, $f(x)$ will be maximum,

$$\text{so } f\left(\frac{c^{3/2}}{2^{1/4}\sqrt{a}}\right)=\left(\frac{c^{12}}{2a^2b^2}-\frac{c^{12}}{4a^2b^2}\right)^{1/4}=\left(\frac{c^{12}}{4a^2b^2}\right)^{1/4}$$

$$=\frac{c^3}{\sqrt{2ab}}$$

lternative method:

Since A.M. $\geq$ G.M.

$$\frac{a^2x^4+b^2y^4}{2}\geq\sqrt{a^2x^4b^2y^4}$$

$$\Rightarrow abx^2y^2\leq\frac{c^6}{2}$$

$$\Rightarrow xy\leq\frac{c^3}{\sqrt{2ab}}$$

Hence, maximum value of xy is $\frac{c^3}{\sqrt{2ab}}$.

54. b. Let $g(x)=4x^3-12x^2+11x-3$

$\Rightarrow g'(x)=12x^2-24x+11$
$=12(x-1)^2-1$

$\Rightarrow g'(x)>0$ for $x\in[2,3]$

$\Rightarrow g(x)$ is increasing in $[2,3]$

$f(x)_{max}=f(3)=\log_{10}(4.27-12.9+11.3-3)$
$=\log_{10}(30)$
$=1+\log_{10}3.$

55. b. Let $f(x)=x+ax^{-2}-2$

$\Rightarrow f'(x)=1-2ax^{-3}=0\Rightarrow x=(2a)^{1/3}$

Also $f''(x)=6ax^{-4}\Rightarrow f''\left((2a)^{1/3}\right)>0$

$\Rightarrow x=(2a)^{1/3}$ is the point of minima.

For $x+ax^{-2}-2>0\ \forall x$ we must have $f\left((2a)^{1/3}\right)>0$

$\Rightarrow (2a)^{1/3}+a(2a)^{-2/3}-2>0$

$\Rightarrow 2a+a-2(2a)^{2/3}>0$

$\Rightarrow 3a>2(2a)^{2/3}$

$\Rightarrow 27a^3>32a^2$

$\Rightarrow a>32/27$

Hence, the least value of a is 2.

56. c. We have
$$f(x)=\begin{cases}(-1)^{m+n}x^m(x-1)^n, & \text{if } x<0\\ (-1)^n x^m(x-1)^n, & \text{if } 0\leq x<1\\ x^m(x-1)^n, & \text{if } x\geq 1\end{cases}$$

Let $g(x)=x^m(x-1)^n$, then

$g'(x)=mx^{m-1}(x-1)^n+nx^m(x-1)^{n-1}$
$=x^{m-1}(x-1)^{n-1}\{mx-m+nx\}$

Now $f'(x)=0\Rightarrow g'(x)=0\Rightarrow x=0,1$ or $\frac{m}{m+n}$

$f(0)=0, f(1)=0$ and

$$f\left(\frac{m}{m+n}\right)=(-1)^n\frac{m^mn^n(-1)^n}{(m+n)^{m+n}}$$

$$=\frac{m^mn^n}{(m+n)^{m+n}}>0$$

$\therefore$ the maximum value $=\frac{m^mn^n}{(m+n)^{m+n}}$.

57. b. Clearly, $f(x)$ in decreasing just before $x=3$ and increasing after $x=3$. For $x=3$ to be the point of local minima, $f(3)\leq f(3^-)$.

$\Rightarrow -15\leq 12-27+\ln(a^2-3a+3)$

$\Rightarrow a^2-3a+3\geq 1$

$\Rightarrow a\in(-\infty,1)\cup(2,\infty)$.

58. c.

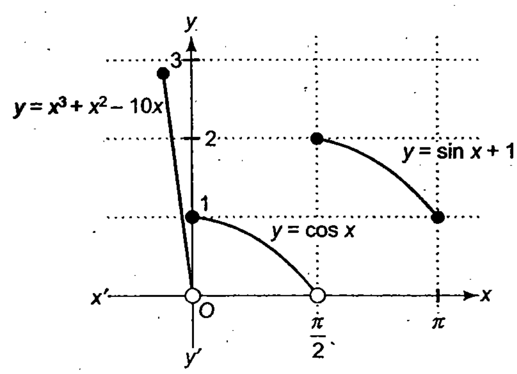


Fig. 6.76

59. b. Since $f(x)$ has a relative minimum at $x=0$, therefore $f'(0)=0$ and $f''(0)>0$.

If the function $y = f(x) + ax + b$ has a relative minimum at $x = 0$, then

$\frac{dy}{dx} = 0$ at $x = 0 \Rightarrow f'(x) + a = 0$ for $x = 0$

$\Rightarrow f'(0) + a = 0 \Rightarrow 0 + a = 0 \quad [\because f'(0) = 0] \Rightarrow a = 0$

Now, $\frac{d^2y}{dx^2} = f''(x) \Rightarrow \left(\frac{d^2y}{dx^2}\right)_{x=0} = f''(0) > 0$

$[\because f''(0) > 0]$

Hence, y has a relative minimum at $x = 0$ if $a = 0$ and b can attain any real value.

60. d. $f'(x) = 12x^2 - 2x - 2 = 2[6x^2 - x - 1] = 2(3x + 1)(2x - 1)$

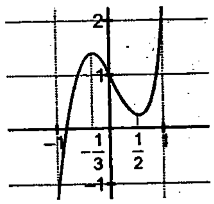


Fig. 6.77

$$\text{Hence } g(x) = \begin{cases} f(x), & \text{if } 0 \le x < \frac{1}{2} \\ f\left(\frac{1}{2}\right), & \text{if } \frac{1}{2} \le x \le 1 \\ 3 - x, & \text{if } 1 < x \le 2 \end{cases}$$

$$\Rightarrow g\left(\frac{1}{4}\right) + g\left(\frac{3}{4}\right) + g\left(\frac{5}{4}\right) = f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right) + g\left(\frac{5}{4}\right)$$

$$= \frac{5}{2}$$

61. a. $f'(x) = ax^2 + 2(a + 2)x + (a - 1)$,

$f''(x) = 2ax + 2(a + 2) = 0$

$\Rightarrow x = -\frac{a + 2}{a}$ which is the point of inflection

Given that, we must have $-\frac{a + 2}{a} < 0$

$\Rightarrow (-\infty, -2) \cup (0, \infty)$.

62. d. The derivative of a degree 3 polynomial is quadratic. This must have either 0, 1 or 2 roots. If this has precisely one root, then this must be repeated. Hence, we have $f'(x) = m(x - \alpha)^2$, where α is the repeated root and $m \in R$. So, our original function f has a critical point at $x = \alpha$.

Also, $f''(x) = 2m(x - \alpha)$, in which case $f''(\alpha) = 0$. But we are told that the 2nd derivative is non-zero at critical point. Hence, there must be either 0 or 2 critical points.

63. c. $f'(x) = \frac{0.6(1+x)^{-0.4}(1+x^{0.6}) - 0.6x^{-0.4}(1+x)^{0.6}}{(1+x^{0.6})^2}$

$= 0.6\frac{(1+x^{0.6}) - x^{-0.4}(1+x)^1}{(1+x^{0.6})^2(1+x)^{0.4}} = 0.6\frac{(1+x^{0.6})x^{0.4} - (1+x)}{(1+x^{0.6})^2(1+x)^{0.4}x^{0.4}}$

$= 0.6\frac{x^{0.4} - 1}{(1+x^{0.6})^2(1+x)^{0.4}x^{0.4}} < 0 \quad \forall x \in (0, 1)$

Hence, $f(x)$ is decreasing.

$\Rightarrow f(x)_{\max} = f(0) = 1$

64. c. $f(f(x)) = k(x^5 + x) \Rightarrow f'(f(x))f'(x) = k(5x^4 + 1)$

$\Rightarrow f(x)$ is always increasing or decreasing as $k(5x^4 + 1)$ is either always negative or positive.

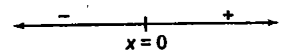
65. d. We have $f(x) = \frac{x^2 - a}{x^2 + a} = 1 - \frac{2a}{x^2 + a}$

Clearly range of f is $[-1, 1)$

Now, $f'(x) = \frac{4ax}{(x^2 + a)^2}$

and $f''(x) = \frac{4a}{(x^2 + a)^3}(a - 3x^2)$

Sign scheme of $f'(x)$



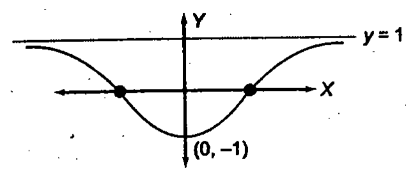


Fig. 6.78

$\Rightarrow f(x)$ is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$

Therefore, $f(x)$ has a local minimum at $x = 0$.

66. a. $f(x) + f''(x) = -x\,g(x)f'(x)$

Let $h(x) = f^2(x) + (f'(x))^2$

$\Rightarrow h'(x) = 2f(x)f'(x) + 2f'(x)f''(x)$

$= 2f'(x)[-x]g(x)f'(x)$

$= -2x(f'(x))^2 g(x)$

$\Rightarrow x = 0$ is a point of maxima for $h(x)$.

67. a. $h'(x) = \frac{m}{n}x^{\frac{m-n}{n}} = \frac{m}{n}x^{\left(\frac{\text{even}}{\text{odd}}\right)}$

As $h'(x)$ is undefined at $x = 0$ and $h'(x)$ does not change its sign in the neighbourhood. So, no extremums.

68. a. Here, $f(x) = 4\tan x - \tan^2 x + \tan^3 x$

$\Rightarrow f'(x) = 4\sec^2 x - 2\tan x\sec^2 x + 3\tan^2 x\sec^2 x$

$= \sec^2 x(4 - 2\tan x + 3\tan^2 x)$

$= 3\sec^2 x\left\{\tan^2 x - \frac{2}{3}\tan x + \frac{4}{3}\right\}$

$= 3\sec^2 x\left\{\left(\tan x - \frac{1}{3}\right)^2 + \left(\frac{4}{3} - \frac{1}{9}\right)\right\}$

$= 3\sec^2 x\left\{\left(\tan x - \frac{1}{3}\right)^2 + \frac{11}{9}\right\} > 0, \forall x$

Therefore, $f(x)$ is increasing for all $x \in$ domain.

69. c. It is a fundamental property.

70. a. $f'(x) = -\frac{1}{2}e^{-\frac{x}{2}}(x^2 - 8)$.

Clearly, $x = 2\sqrt{2}$ is the point of local maxima.

71. a. $f(0)=\pi/2, f(0^+)=0, f(0^-)=0.$
Hence $x=0$ is the point of maxima.

72. a. $f(x)$ will have maxima at $x=-2$ only if $a^2+1\geq 2 \Rightarrow |a|\geq 1$

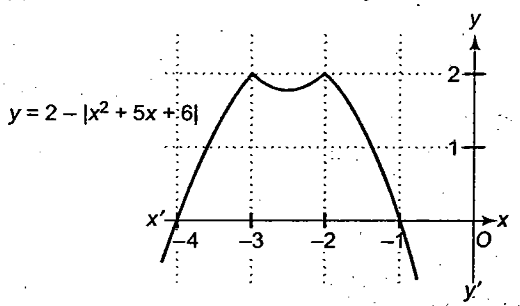


Fig. 6.79

73. b. Given $A+B=60° \Rightarrow B=60°-A$

$\Rightarrow \quad \tan B=\tan(60°-A)=\dfrac{\sqrt{3}-\tan A}{1+\sqrt{3}\tan A}$

Now $z=\tan A\tan B$

or $z=\dfrac{t(\sqrt{3}-t)}{1+\sqrt{3}t}=\dfrac{\sqrt{3}t-t^2}{1+\sqrt{3}t}$

where $t=\tan A$

$\dfrac{dz}{dt}=-\dfrac{(t+\sqrt{3})(\sqrt{3}t-1)}{(1+\sqrt{3}t)^2}=0$

$\Rightarrow \quad t=1/\sqrt{3}$

$\Rightarrow \quad t=\tan A=\tan 30°$

The other value is rejected as both A and B are + ive acute angles.

If $t<\dfrac{1}{\sqrt{3}}, \dfrac{dz}{dt}=$ positive and if $t>\dfrac{1}{\sqrt{3}}, \dfrac{dz}{dt}=-$ive.

Hence maximum when $t=\dfrac{1}{\sqrt{3}}$ and maximum value $=\dfrac{1}{3}$.

74. c. $f(x)=\dfrac{t+3x-x^2}{x-4}; f'(x)=\dfrac{(x-4)(3-2x)-(t+3x-x^2)}{(x-4)^2}$

for maximum or minimum, $f'(x)=0$

$-2x^2+11x-12-t-3x+x^2=0$

$-x^2+8x-(12+t)=0$

for one maxima and minima,

$D>0$

$\Rightarrow \quad 64-4(12+t)>0$

$\Rightarrow \quad 16-12-t>0 \Rightarrow 4>t$ or $t<4$.

75. d. If $f(x)$ has an extremum at $x=\pi/3$, then $f'(x)=0$ at $x=\pi/3$

Now, $f(x)=a\sin x+\dfrac{1}{3}\sin 3x$

$\Rightarrow \quad f'(x)=a\cos x+\cos 3x$

$f'(\pi/3)=0$

$\Rightarrow \quad a\cos(\pi/3)+\cos\pi=0$

$\Rightarrow \quad a=2$

76. a. Since $a=\left(\dfrac{4}{\sin x}+\dfrac{1}{1-\sin x}\right)$, a is least.

$\Rightarrow \dfrac{da}{dx}=\left[-\dfrac{4}{\sin^2 x}+\dfrac{1}{(1-\sin x)^2}\right]\cos x=0$

We have to find the values of x in the interval $(0, \pi/2)$.

$\Rightarrow \quad \cos x\neq 0$ and the other factor when equated to zero gives $\sin x=2/3$.

Now, $\dfrac{d^2a}{dx^2}=\left[-\dfrac{4}{\sin^2 x}+\dfrac{1}{(1-\sin x)^2}\right](-\sin x)$

$+\left[\dfrac{8}{\sin^3 x}+\dfrac{2}{(1-\sin x)^3}\right]\cos^2 x$

Put $\sin x=\dfrac{2}{3}$ and $\cos^2 x=1-\dfrac{4}{9}=\dfrac{5}{9}$

$\therefore \dfrac{d^2a}{dx^2}=0+\left[\dfrac{8}{8/27}+2\times 27\right]\dfrac{5}{9}=81\times\dfrac{5}{9}=45>0$

$\Rightarrow \quad a$ is minimum and its value is

$\dfrac{4}{2/3}+\dfrac{1}{1-(2/3)}=6+3=9.$

77. c. Consider the function $f(x)=\dfrac{x^2}{(x^3+200)}$ (1)

$f'(x)=x\dfrac{(400-x^3)}{(x^3+200)^2}=0$

When $x=(400)^{1/3}$, $(\because x\neq 0)$

$x=(400)^{1/3}-h \Rightarrow f'(x)>0$

$x=(400)^{1/3}+h \Rightarrow f'(x)<0$

$\therefore \quad f(x)$ has maxima at $x=(400)^{1/3}$

Since $7<(400)^{1/3}<8$, either a_7 or a_8 is the greatest term of the sequence.

$\because a_7=\dfrac{49}{543}$ and $a_8=\dfrac{8}{89}$ and $\dfrac{49}{543}>\dfrac{8}{89}$

$\Rightarrow \quad a_7=\dfrac{49}{543}$ is the greatest term.

78. d. Let there be a value of k for which $x^3-3x+k=0$ has two distinct roots between 0 and 1.
Let a, b be two distinct roots of $x^3-3x+k=0$ lying between 0 and 1 such that $a<b$. Let $f(x)=x^3-3x+k$. Then $f(a)=f(b)=0$. Since between any two roots of a polynomial $f(x)$, there exists at least one root of its derivative $f'(x)$. Therefore, $f'(x)=3x^2-3$ has at least one root between a and b. But $f'(x)=0$ has two roots equal to ± 1 which do not lie between a and b. Hence $f(x)=0$ has no real roots lying between 0 and 1 for any value of k.

79. b. $f'(x)=-x\sin x=0$ when $x=0$ or π

$\left.\begin{array}{l} f'(0^-)=(-)(-)(-)<0 \\ f'(0^+)=(-)(+)(+)<0 \end{array}\right\}$ no sign change

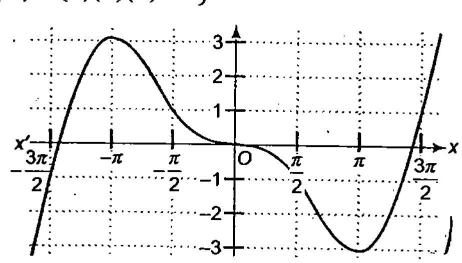


Fig. 6.80

This also implies that f is decreasing at $x = 0$
$\Rightarrow (b)$ is correct.
$f''(x) = -(x \cos x + \sin x)$
$f''(\pi) = -(-\pi) > 0$ minima at $x = \pi$
$f''(-\pi) = -(\pi) < 0$ maxima at $x = -\pi$

80. d. From the given data, graph of $f(x)$ can be shown as

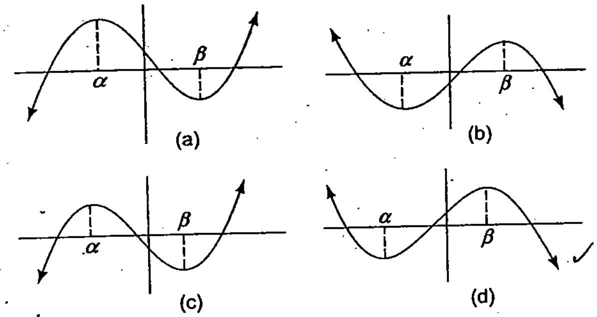


Fig. 6.81

Thus from graph, nothing can be said about roots when the sign of $f(\alpha)$ and $f(\beta)$ is given.

81. a.

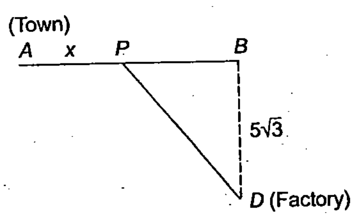


Fig. 6.82

Let the charges for railway line be k ₹/km.

Now the total freight charges, $T = kx + 2k\sqrt{(20-x)^2 + 75}$

Let $\frac{dT}{dx} = 0 \Rightarrow k + 2k\, \frac{2(x-20)}{2\sqrt{(x-20)^2+75}} = 0$

$\Rightarrow \quad 4(x-20)^2 = 75 + (x-20)^2$

$\Rightarrow \quad (x-20)^2 = 25 \Rightarrow x = 25, 15 \Rightarrow x = 15$ (as $AP < AB$).

$\Rightarrow \quad PB = AB - AP = 20 - 15 = 5$ km

82. a.

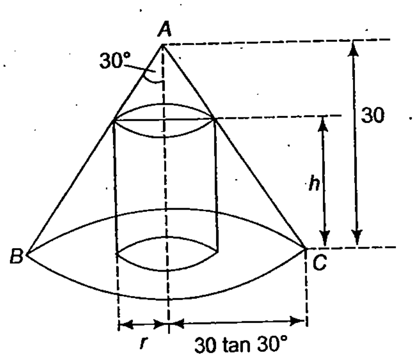


Fig. 6.83

From geometry, we have $\frac{r}{30 \tan 30°} = \frac{30-h}{30}$

$\Rightarrow \quad h = 30 - \sqrt{3}r$

Now, the volume of cylinder, $V = \pi r^2 h = \pi r^2 (30 - \sqrt{3}r)$

Now, let $\frac{dV}{dr} = 0 \Rightarrow \pi(60r - 3\sqrt{3}r^2) = 0 \Rightarrow r = \frac{20}{\sqrt{3}}$

Hence, $V_{max} = \pi\left(\frac{20}{\sqrt{3}}\right)^2\left(30 - \sqrt{3}\frac{20}{\sqrt{3}}\right) = \pi\, \frac{400}{3} \times 10$

$= \frac{4000\pi}{3}$

83. b.

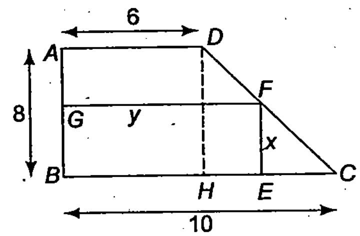


Fig. 6.84

Let rectangle $BEFG$ is inscribed.
Its area, $A = xy$
Now ΔFEC and ΔDHC are similar, i.e.,

$\Rightarrow \quad \frac{x}{8} = \frac{10-y}{4} \Rightarrow y = 10 - \frac{x}{2} \Rightarrow A = x\left(10 - \frac{x}{2}\right)$ where $x \in (0, 8]$

Now $\frac{dA}{dx} = 10 - x$. Now for $x \in (0, 8)$

$\frac{dA}{dx} > 0 \Rightarrow A$ increases. Hence A_{max} occurs when $x = 8$.

Hence, max area $= A_{max} = 8(10 - \frac{8}{2}) = 48$ cm^2.

84. a.

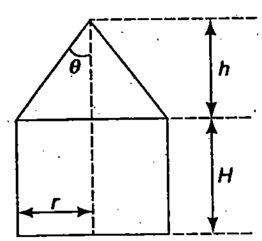


Fig. 6.85

Given volume and r
Now, V = volume of cone + volume of cylinder

$$= \frac{\pi}{3}r^2 h + \pi r^2 H$$

$$V = \frac{\pi}{3}r^2(h + 3H) \Rightarrow H = \frac{\frac{3V}{\pi r^2} - h}{3}$$

Now, surface area, $S = \pi r l + 2\pi r H = \pi r\sqrt{h^2 + r^2} + 2\pi r \times \left(\frac{\frac{3V}{\pi r^2} - h}{3}\right)$

Now, let $\frac{dS}{dh} = 0 \Rightarrow \pi r \frac{h}{\sqrt{h^2+r^2}} - \frac{2\pi r}{3} = 0$

$\Rightarrow \dfrac{h}{\sqrt{h^2+r^2}}=\dfrac{2}{3} \Rightarrow 5h^2=4r^2 \Rightarrow \dfrac{r}{h}=\dfrac{\sqrt{5}}{2}=\tan\theta$

$\Rightarrow \cos\theta=\dfrac{2}{3} \Rightarrow \theta=\cos^{-1}\dfrac{2}{3}$

85. d.

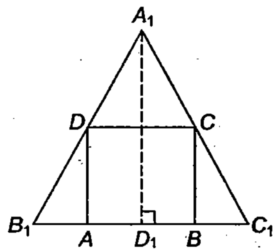


Fig. 6.86

Let $BD_1=x \Rightarrow BC_1=(a-x)$

$\Rightarrow BC=(a-x)\tan\dfrac{\pi}{3}=\sqrt{3}(a-x).$

Now, area of rectangle $ABCD$,

$\Delta=(AB)(BC)=2\sqrt{3}x(a-x)$.

$\Rightarrow \Delta \le 2\sqrt{3}\left(\dfrac{x+a-x}{2}\right)^2=\dfrac{\sqrt{3}a^2}{2}$ (using A.M. ≥ G.M.)

86. d.

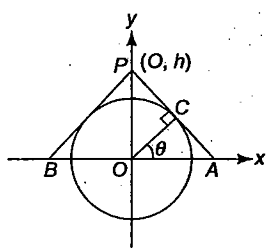


Fig. 6.87

Let $\angle COA=\theta \Rightarrow OA=OC\sec\theta=4\sec\theta$

Also $\angle OPC=\theta \Rightarrow OP=OC\,\mathrm{cosec}\,\theta=4\,\mathrm{cosec}\,\theta$

Now, $\Delta_{PAB}=OA\ OP=\dfrac{32}{\sin 2\theta}$

For Δ_{PAB} to be minimum $\sin 2\theta=1 \Rightarrow \theta=\dfrac{\pi}{4}$

$\Rightarrow P=(0, 4\sqrt{2})$

87. a.

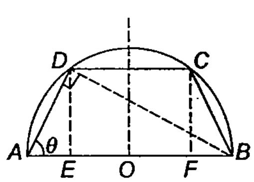


Fig. 6.88

$AD=AB\cos\theta=2R\cos\theta,\ AE=AD\cos\theta=2R\cos^2\theta$

$\Rightarrow EF=AB-2AE=2R-4R\cos^2\theta$

$DE=AD\sin\theta=2R\sin\theta\cos\theta$

$\Rightarrow$ Area of trapezium,

$S=\dfrac{1}{2}(AB+CD)\times DE$

$=\dfrac{1}{2}(2R+2R-4R\cos^2\theta)\times 2R\sin\theta\cos\theta$

$=4R^2\sin^3\theta\cos\theta$

$\dfrac{dS}{d\theta}=12R^2\sin^2\theta\cos^2\theta-4R^2\sin^4\theta$

$=4R^2\sin^2\theta(3\cos^2\theta-\sin^2\theta)$

For maximum area, $\dfrac{dS}{d\theta}=0 \Rightarrow \tan^2\theta=3 \Rightarrow \tan\theta=\sqrt{3}$

(θ is acute) $\Rightarrow S_{\max}=\dfrac{3\sqrt{3}}{4}R^2$.

88. c.

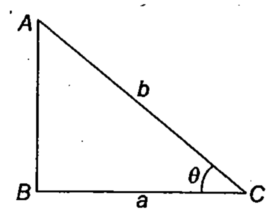


Fig. 6.89

$b\cos\theta=a \Rightarrow b\cos\theta+b=4$ or $b=\dfrac{4}{1+\cos\theta}$

$\Rightarrow a=\dfrac{4\cos\theta}{1+\cos\theta}$

$\Rightarrow$ area $=\Delta=\dfrac{1}{2}ba\sin\theta$

$=\dfrac{1}{2}\dfrac{4}{1+\cos\theta}\dfrac{4\cos\theta}{1+\cos\theta}\times\sin\theta=\dfrac{4\sin 2\theta}{(1+\cos\theta)^2}$

$\Rightarrow \dfrac{d\Delta}{d\theta}=4\dfrac{2\cos 2\theta(1+\cos\theta)^2+2\sin 2\theta(1+\cos\theta)\sin\theta}{(1+\cos\theta)^4}$

$\Rightarrow \dfrac{d\Delta}{d\theta}=0 \Rightarrow \cos 2\theta\,(1+\cos\theta)+\sin 2\theta\sin\theta=0$

or $\cos 2\theta+\cos\theta=0$ or $\cos 2\theta=-\cos\theta=\cos(\pi-\theta)$

or $\theta=\dfrac{\pi}{3}$

Therefore, Δ is maximum when $\theta=\dfrac{\pi}{3}$.

89. b.

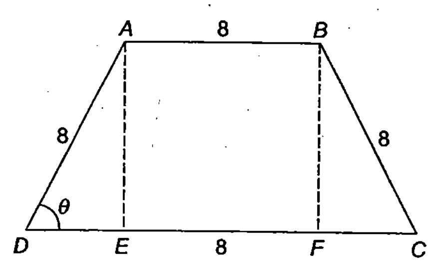


Fig. 6.90

$\Delta=(AB\times AE)+2\left(\dfrac{1}{2}DE\times AE\right)$

$=(8\times 8\sin\theta)+8\sin\theta\times 8\cos\theta=64\sin\theta+32\sin 2\theta$

Let $\dfrac{d\Delta}{d\theta}=0 \Rightarrow 64\cos\theta+64\cos 2\theta=0$

$\Rightarrow\ 2\cos^2\theta+\cos\theta-1=0 \Rightarrow (2\cos\theta-1)(\cos\theta+1)=0$

$\Rightarrow\ \cos\theta=1/2 \Rightarrow \theta=\pi/3$

$\Rightarrow A_{max} = 64\dfrac{\sqrt{3}}{2} + 32\dfrac{\sqrt{3}}{2} = 32\sqrt{3} + 16\sqrt{3} = 48\sqrt{3}$.

90. b. $\because$ Fuel charges $\propto v^2$. Let F represents fuel charges

$\Rightarrow F \propto v^n \Rightarrow F = kv^2$ (1)

Given that $F =$ ₹48 per hour, $v = 16$ km per hour

$\Rightarrow 48 = k(16)^2 \Rightarrow k = \dfrac{3}{16}$

From (1), $F = \dfrac{3v^2}{16}$

Let the train covers λ km in t hours

$\Rightarrow \lambda = vt$ or $t = \dfrac{\lambda}{v}$

$\Rightarrow$ Fuel charges in time $t = \dfrac{3}{16}v^2 \times \dfrac{\lambda}{v} = \dfrac{3v\lambda}{16}$

$\Rightarrow$ Total cost for running the train,

$$C = \frac{3v\lambda}{16} + 300 \times \frac{\lambda}{v}$$

$$\Rightarrow \frac{dC}{dv} = \frac{3\lambda}{16} - \frac{300\lambda}{v^2} \text{ and } \frac{d^2C}{dv^2} = \frac{600\lambda}{v^3}$$

For the maximum or minimum value of C, $\dfrac{dC}{dv} = 0$

$\Rightarrow v = 40$ km/hr. Also, $\left.\dfrac{d^2C}{dv^2}\right|_{v=40} = \dfrac{60\lambda}{(40)^3} > 0$ $(\because \lambda > 0)$

$\Rightarrow$ C is minimum when $v = 40$ km/hr.

91. c.

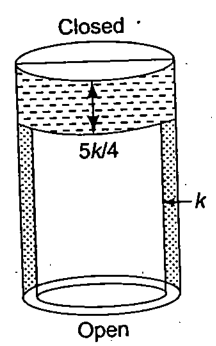


Fig. 6.91

Let x be the radius and y the height of the cylindrical gas container. Also let k be the thickness of the plates forming the cylindrical sides. Therefore, the thickness of the plate forming the top will be $5k/4$.

Capacity of the vessel = vol. of cylinder

$= \pi x^2 y = V$ (Given) $\Rightarrow y = V/(\pi x^2)$ (1)

Now, the volume V_1 of the iron plate used for construction of the container is given by

$V_1 = \pi(x+k)^2(y + 5k/4) - \pi x^2 y$

$$\Rightarrow \frac{dV_1}{dx} = 2Vk(x+k) \times \left(\frac{5\pi}{4V} - \frac{1}{x^3}\right)$$

For maximum or minimum of V_1, $dV_1/dx = 0$

$\Rightarrow x = [4V/(5\pi)]^{1/3}$

For this value of x, d^2V_1/dx^2 is +ve

Hence, V_1 is minimum when $x = [4V/(5\pi)]^{1/3}$.

Now $x = [4V/(5\pi)]^{1/3}$

$\Rightarrow 5\pi x^3 = 4V = 4\pi x^2 y \Rightarrow x/y = 4/5$

Hence, the required ratio is 4 : 5.

92. c.

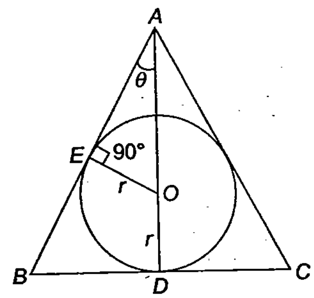


Fig. 6.92

Let ABC be an isosceles triangle in which a circle of radius r is inscribed.

Let $\angle BAD = \theta$ (semi-vertical angle).

In ΔOAE, $OA = OE$ cosec $\theta = r$ cosec θ, $AE = r\cot\theta$.

$\Rightarrow AD = OA + OD = r(\text{cosec}\,\theta + 1)$

In ΔABD, $BD = AD\tan\theta = r(\text{cosec}\,\theta + 1)\tan\theta$

$AB = AD\sec\theta = r(\text{cosec}\,\theta + 1)\sec\theta$

Now, the perimeter of the ΔABC is $S = AB + AC + BC$

$= 2AB + 2BD$

$(\because AC = AB)$

$S = 2r(\text{cosec}\,\theta + 1)(\sec\theta + \tan\theta)$ or $S = \dfrac{4r(1+\sin\theta)^2}{\sin 2\theta}$

$$\Rightarrow \frac{dS}{d\theta} = 4r\,[2(1+\sin\theta)\cos\theta\sin 2\theta - (1+\sin\theta)^2\, 2\cos 2\theta]/(\sin 2\theta)^2$$

$= 8r(1+\sin\theta)[\sin 2\theta\cos\theta - \cos 2\theta\sin\theta - \cos 2\theta]/(\sin 2\theta)^2$

$= 8r(1+\sin\theta)(\sin\theta - 1 + 2\sin^2\theta)/(\sin 2\theta)^2$

$= 16r(1+\sin\theta)^2(\sin\theta - 1/2)/(\sin 2\theta)^2$

For maximum or minimum of S, $dS/d\theta = 0 \Rightarrow \sin\theta = 1/2$

$\therefore \theta = \pi/6$ $(\because \sin\theta \neq -1$ as θ is an acute angle)

Now if θ is little less and little greater than $\pi/6$, then sign of $dS/d\theta$ changes from -ve to +ve. Hence S is minimum when $\theta = \pi/6$, which is the point of minima.

Hence, the least perimeter of the

$\Delta = 4r[1 + \sin(\pi/6)]^2/\sin(\pi/3) = 6\sqrt{3}r$.

93. a.

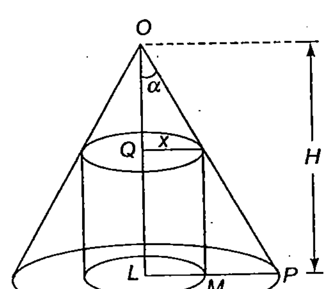


Fig. 6.93

Let H be the height of the cone and α be its semi-vertical angle. Suppose that x is the radius of the inscribed cylinder and h be its height $h = QL = OL - OQ = H - x\cot\alpha$,

V = volume of the cylinder $= \pi x^2(H - x\cot\alpha)$.

Also, $p=\frac{1}{3}\pi(H\tan\alpha)^2 H$ (1)

$\frac{dV}{dx}=\pi(2Hx-3x^2\cot\alpha)$

So, $\frac{dV}{dx}=0 \Rightarrow x=0$ or $x=\frac{2}{3}H\tan\alpha$;

$\left.\frac{d^2V}{dx^2}\right|_{x=\frac{2}{3}H\tan\alpha}=-2\pi H<0$

So, V is maximum when $x=\frac{2}{3}H\tan\alpha$

$q=V_{\max}=\pi\frac{4}{9}H^2\tan^2\alpha\frac{1}{3}H$

$=\frac{4}{27}\frac{\pi^3 p\tan^2\alpha}{\pi\tan^2\alpha}=\frac{4}{9}p.$ [from (1)]

Hence, $p:q=9:4$.

94. d. Given $4x+2\pi r=a$

where x is side length of the square and r is radius of the circle

$A=x^2+\pi r^2=\frac{1}{16}(a-2\pi r)^2+\pi r^2$

$\frac{dA}{dr}=0$ gives $r=\frac{a}{2(\pi+4)}$ for which $\frac{d^2A}{dr}$ is + ve and hence minimum.

$\Rightarrow\ 4x=a-2\pi r=a-\frac{a\pi}{\pi+4}=\frac{4a}{\pi+4}$

$\therefore\ x=\frac{a}{\pi+4}$

$\therefore\ A=x^2+r^2\pi=\frac{a^2}{4(\pi+4)}$.

95. c. The dimensions of the box after cutting equal squares of side x on the corner will be $21-2x$, $16-2x$ and height x.

$V=x(21-2x)(16-2x)$

$=x(336-74x+4x^2)$

or $V=4x^3+336x-74x^2 \Rightarrow \frac{dV}{dx}=12x^2+336-148x$

$\Rightarrow \frac{dV}{dx}=0$ gives $x=3$ for which $\frac{d^2V}{dx^2}$ is –ve and hence maximum.

96. a. $f(x)=\frac{\sin^3 x\cos x}{2}$

$\Rightarrow f'(x)=\frac{3\sin^2 x\cos^2 x-\sin^4 x}{2}$

$f'(x)=0 \Rightarrow 3\sin^2 x\cos^2 x-\sin^4 x=0$

$\Rightarrow 3\cos^2 x-\sin^2 x=0$

$\Rightarrow 4\cos^2 x-1=0$

$\Rightarrow \cos x=\frac{1}{2}$

$\Rightarrow x=\frac{\pi}{3}$, which is the point of maxima.

$\Rightarrow f_{\max}=\frac{(\sqrt{3}/2)^3(1/2)}{2}=\frac{3\sqrt{3}}{32}$

Multiple Correct Answers Type

1. a, b, c, d.

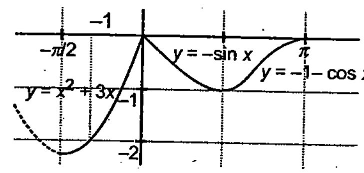


Fig. 6.94

From the graph global minimum value is $f(-1)=-2$ and global maximum value is $f(0)=f(\pi)=0$.

2. a, c.

$f'(x)=4(x^3-3x^2+3x-1)=4(x-1)^3>0$ for $x>1$.

Hence, f increases in $[1,\infty)$. Moreover, $f'(x)<0$ for $x<1$.

Hence, f has a minimum at $x=1$.

3. a, b, c, d.

$f(x)=2x-\sin x \Rightarrow f'(x)=2-\cos x>0\ \forall x$. Hence, $f(x)$ is strictly increasing, hence one-one and onto $g(x)=x^{1/3}$.

$\Rightarrow\ g'(x)=\frac{1}{3}x^{-2/3}>0\ \forall x$, hence $g(x)$ is strictly increasing and hence one-one and onto.

Also, gof is one-one.

$gof(x)=(2x-\sin x)^{1/3}$ has range R as the range of $2x-\sin x$ is R.

4. a, d.

We have

$f(x)=2x+\cot^{-1}x+\log(\sqrt{1+x^2}-x)$

$\therefore\ f'(x)=2-\frac{1}{1+x^2}+\frac{1}{\sqrt{1+x^2}-x}\left(\frac{x}{\sqrt{1+x^2}}-1\right)$

$=\frac{1+2x^2}{1+x^2}-\frac{1}{\sqrt{1+x^2}}=\frac{1+2x^2}{1+x^2}-\frac{\sqrt{(1+x^2)}}{1+x^2}$

$=\frac{x^2+\sqrt{1+x^2}(\sqrt{1+x^2}-1)}{1+x^2}>0$ for all x

Hence, $f(x)$ is an increasing function in $(-\infty,\infty)$ and in particular in $(0,\infty)$.

5. a, b, c, d.

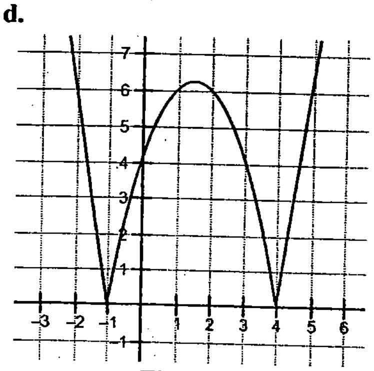


Fig. 6.95

Refer the graph for the answers.

6. b, d.

$f'(x)=\dfrac{\sin x}{x}$

For $f'(x)=0$, $\dfrac{\sin x}{x}=0 \Rightarrow x=n\pi\,(n\in I, n\neq 0)$

$f''(x)=\dfrac{x\cos x-\sin x}{x^2}$

$f''(n\pi)=\dfrac{\cos n\pi}{n\pi}<0$ if $n=2k-1$ and >0 if $n=2k$, $k\in I^+$.

Hence, $f(x)$ has local maxima at $x=n\pi$, where $n=2k-1$ and local minima at $x=n\pi$, $n=2k$, where $k\in I^+$.

7. a, b.

Given that $\dfrac{x^2+x+2}{x^2+5x+6}<0 \Rightarrow x\in(-3,-2)$

We have to find the extrema for the function
$f(x)=1+a^2x-x^3$

For maximum or minimum, $f'(x)=0$

$\Rightarrow\ a^2-3x^2=0$ or $x=\pm\dfrac{a}{\sqrt{3}}$ and $f''(x)=-6x$ is +ve when x is negative.

If a is positive, then the point of minima is $-\dfrac{a}{\sqrt{3}}$

i.e., $-3<-\dfrac{a}{\sqrt{3}}<-2$ or $2\sqrt{3}<a<3\sqrt{3}$.

And if a is negative, then the point of minima is $\dfrac{a}{\sqrt{3}}$

i.e., $-3<\dfrac{a}{\sqrt{3}}<-2$ or $-3\sqrt{3}<a<-2\sqrt{3}$

Then, $a\in(-3\sqrt{3},-2\sqrt{3})\cup(2\sqrt{3},3\sqrt{3})$.

8. a, c.

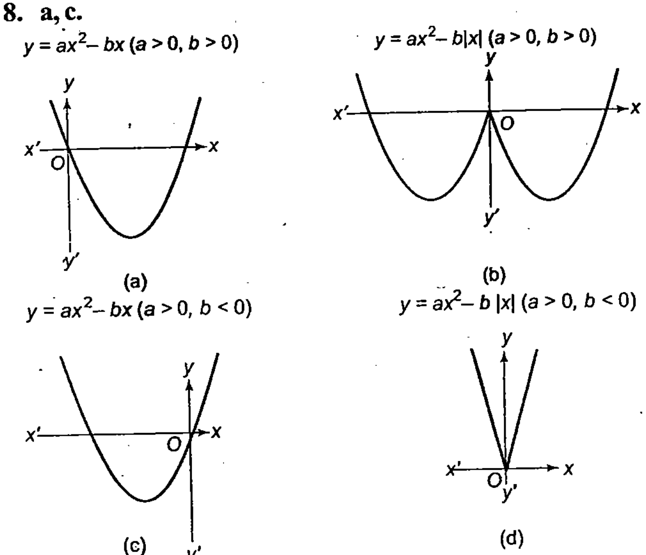


Fig. 6.96

9. a, b, d.

$y=\dfrac{2x-1}{x-2}$

$\dfrac{dy}{dx}=\dfrac{2(x-2)-(2x-1)}{(x-2)^2}=\dfrac{-3}{(x-2)^2}<0\ \forall x\neq 2$

Therefore, y is decreasing in $(-\infty, 2)$ as well as in $(2,\infty)$

$y=\dfrac{2x-1}{x-2}\Rightarrow x=\dfrac{2y-1}{y-2}$

$\therefore\ f^{-1}(x)=\dfrac{2x-1}{x-2}$ $\therefore f(x)$ is its own inverse.

10. b, c.

$\because g(x)$ is increasing and $f(x)$ is decreasing
$\Rightarrow\ g(x+1)>g(x-1)$ and $f(x+1)<f(x-1)$
$\Rightarrow\ f\{g(x+1)\}<f\{g(x-1)\}$ and
$g\{f(x+1)\}<g\{f(x-1)\}$.

11. b, c.

$f(x)=x^3-x^2+100x+2002$
$f'(x)=3x^2-2x+100>0\ \forall x\in R$
$\therefore\ f(x)$ is increasing (strictly)

$\therefore\ f\left(\dfrac{1}{2000}\right)>f\left(\dfrac{1}{2001}\right)$

Also, $f(x-1)>f(x-2)$ as $x-1>x-2$ for $\forall x$.

12. a, d.

Since $g(a)\neq 0$, therefore either $g(a)>0$ or $g(a)<0$.
Let $g(a)>0$. Since $g(x)$ is continuous at $x=a$, therefore there exists a neighbourhood of a in which $g(x)>0$.
$\Rightarrow\ f'(x)>0 \Rightarrow f(x)$ is increasing in the neighbourhood of a.
Let $g(a)<0$. Since $g(x)$ is continuous at $x=a$, therefore there exists a neighbourhood of a in which $g(x)<0$.
$\Rightarrow\ f'(x)<0 \Rightarrow f(x)$ is decreasing in the neighbourhood of a.

13. a, d.

We have $f(x)=(4a-3)(x+\log 5)+2(a-7)\cot\dfrac{x}{2}\sin^2\dfrac{x}{2}$

$=(4a-3)(x+\log 5)+(a-7)\sin x$
$\Rightarrow\ f'(x)=(4a-3)+(a-7)\cos x$
If $f(x)$ does not have critical points, then $f'(x)=0$ does not have any solution in R.

Now, $f'(x)=0\Rightarrow \cos x=\dfrac{4a-3}{7-a}$

$\Rightarrow\ \left|\dfrac{4a-3}{7-a}\right|\leq 1$ $\quad[\because |\cos x|\leq 1]$

$\Rightarrow\ -1\leq\dfrac{4a-3}{7-a}\leq 1 \Rightarrow a-7\leq 4a-3\leq 7-a$

$\Rightarrow\ a\geq -4/3$ and $a\leq 2$
Thus, $f'(x)=0$ has solutions in R if $-4/3\leq a\leq 2$
So, $f'(x)=0$ is not solvable in R if $a<-4/3$ or $a>2$, i.e., $a\in(-\infty,-4/3)\cup(2,\infty)$.

14. a, c, d.

Graph of $f(x)$

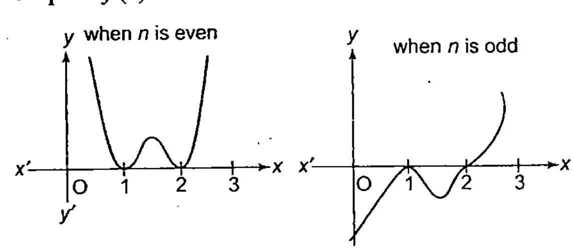


Fig. 6.97

15. a, b, c

$f'(x) = \cos x + a$

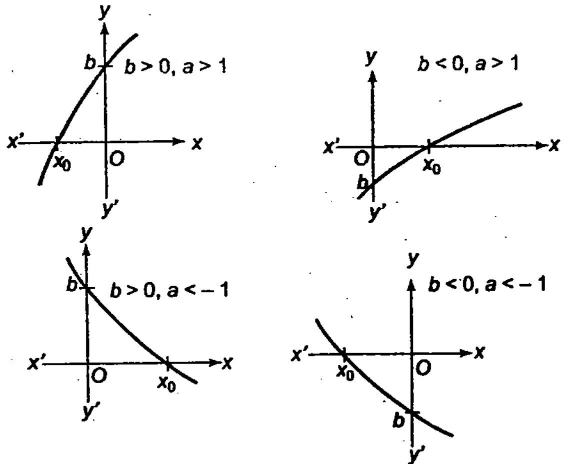


Fig. 6.98

If $a > 1$, then $f'(x) > 0$ or $f(x)$ is an increasing function, then $f(x) = 0$ has +ve root if $b < 0$ and –ve root if $b > 0$

$f'(x) = \cos x + a$

If $a < -1$, then $f'(x) < 0$ or $f(x)$ is a decreasing function, then $f(x) = 0$ has negative root if $b < 0$.

16. a, b, c.

$$f(x) = \frac{\sin(x+a)}{\sin(x+b)}$$

$$f'(x) = \frac{\sin(x+b)\cos(x+a) - \sin(x+a)\cos(x+b)}{\sin^2(x+b)}$$

$$= \frac{\sin(b-a)}{\sin^2(x+b)}$$

If $\sin(b-a) = 0$, then $f'(x) = 0 \Rightarrow f(x)$ will be a constant, i.e., $b - a = n\pi$ or $b - a = (2n+1)\pi$ or $b - a = 2n\pi$, then $f(x)$ has no minima.

17. c, d.

r must be an even integer because two decreasing functions are required to make it increasing function

Let $y = r(n - r)$

When n is odd, $r = \frac{n-1}{2}$ or $\frac{n+1}{2}$ for maximum values of y

when n is even, $r = \frac{n}{2}$ for maximum value of y.

Therefore, maximum $(y) = \frac{n^2-1}{4}$ when n is odd and $\frac{n^2}{4}$ when n is even.

18. a, b, d.

$$f'(x) = \frac{12x^2 - 12x + 5}{(2x-1)^2} > 0 \ \forall x \in R$$

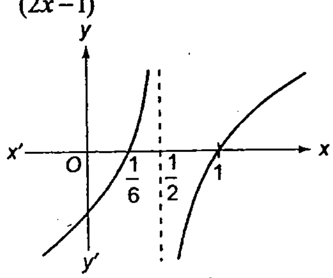


Fig. 6.99

Hence, f is increasing $\forall x \in R$.

$x = 1/2$ is the point of inflection as concavity changes at $x = 1/2$.

19. a, b, d.

At the point of inflection, concavity of the curve changes irrespective of any other factor.

20. b, c, d.

Since f is defined on $(0, \infty)$.

Therefore, $2a^2 + a + 1 > 0$ which is true as $D < 0$

also $3a^2 - 4a + 1 > 0$

$(3a-1)(a-1) > 0 \Rightarrow a < 1/3$ or $a > 1$ (1)

as f is increasing hence

$f(2a^2 + a + 1) > f(3a^2 - 4a + 1)$

$\Rightarrow \ 2a^2 + a + 1 > 3a^2 - 4a + 1$

$\Rightarrow \ 0 > a^2 - 5a$

$\Rightarrow \ a(a-5) < 0 \Rightarrow (0, 5)$ (2)

From (1) and (2), we get

hence, $a \in (0, 1/3) \cup (1, 5)$.

Therefore, possible integers are $\{2, 3, 4\}$.

21. a, d.

$f(x) = (\sin^2 x - 1)^n$

$$f\left(\frac{\pi}{2}\right) = 0$$

$$f\left(\frac{\pi^+}{2}\right) = (\to 0^-)^n \text{ and } f\left(\frac{\pi^-}{2}\right) = (\to 0^-)^n$$

If n is even $f\left(\frac{\pi^+}{2}\right)$ and $f\left(\frac{\pi^-}{2}\right) > 0$, then $x = \frac{\pi}{2}$ is the point of minima.

If n is odd $f\left(\frac{\pi^+}{2}\right)$ and $f\left(\frac{\pi^-}{2}\right) < 0$, then $x = \frac{\pi}{2}$ is the point of maxima.

22. a, b, d.

$f(x) = 2x^3 + 9x^2 + 12x + 1$

$\Rightarrow \ f'(x) = 6[x^2 + 3x + 2]$

$= 6(x+2)(x+1)$

$f'(x) < 0$ for $x \in (-2, -1)$, where $f(x)$ decreases.

$f'(x) > 0$ for $x \in (-\infty, -2) \cup (-1, \infty)$, where $f(x)$ increases.

$f''(x) = 2x + 3 = 0$

$\Rightarrow \ x = -3/2$ is the point of inflection.

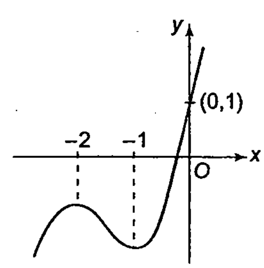


Fig. 6.100

From the graph, f is many-one, hence it is not bijective.

23. a, b, c.

$f(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x$

$\Rightarrow \ f(x) = 0$ has one root $x = 0$

Also, given that $f(x) = 0$ has positive root a_0.

Thus, the equation must have at least three real roots (as complex root occurs in conjugate pair). Thus $f'(x) = 0$ has at least two real roots as between two roots of $f(x) = 0$, there lies at least one root of $f'(x) = 0$.
Similarly, we can say that $f''(x) = 0$ has at least one real root.
Further, $f'(x) = 0$ has one root between roots $x = 0$ and $x = a_0$ of $f(x) = 0$.

24. a, b, c.
Let $y = f(x)^{g(x)}$

$$\Rightarrow \frac{dy}{dx} = f(x)^{g(x)}\left[g(x)\frac{f'(x)}{f(x)} + g'(x)\log f(x)\right]$$

$f(x)^{g(x)}$, $g(x)$, $f(x)$, $f'(x)$ and $g'(x)$ are positive, but $\log f(x)$ can be negative, which can cause $\frac{dy}{dx} < 0$, hence statement (a) is false.
If $f(x) < 1 \Rightarrow \log f(x) < 0$, which does not necessarily make $\frac{dy}{dx} < 0$, hence statement (b) is false.
$f(x) < 0$ can also cause $\frac{dy}{dx} > 0$, hence statement (c) is false.
But reverse of (c) is true.

25. a, c.

$$f(x) = \frac{2-x}{\pi}\cos\pi(x+3) + \frac{1}{\pi^2}\sin\pi(x+3)$$

$$f'(x) = -\frac{1}{\pi}\cos\pi(x+3) - (2-x)\sin\pi(x+3)$$

$$+\frac{1}{\pi}\cos\pi(x+3) = (x-2)\sin\pi(x+3) = 0$$

$x = 2, 1, 3$
$f''(x) = \sin\pi(x+3) + \pi(x-2)\cos\pi(x+3)$
$f''(1) = -\pi < 0, f''(2) = 0, f''(3) = \pi > 0$
Therefore, $x = 1$ is a maximum and $x = 3$ is a minimum, hence $x = 2$ is the point of inflection.

26. a, b, c, d.
$f(x) = x^4(12\log_e x - 7); x > 0$

$$\Rightarrow \frac{dy}{dx} = 16x^3(3\log_e x - 1) \text{ and } \frac{d^2y}{dx^2} = x^2(9\log_e x)$$

$$\frac{dy}{dx} = 0 \Rightarrow x = e^{1/3}$$

at $x = e^{1/3}$, $\frac{d^2y}{dx^2} > 0$, hence $x = e^{1/3}$ is point of minima.

Also, for $0 < x < 1$, $\frac{d^2y}{dx^2} < 0$ and for $x > 1$, $\frac{d^2y}{dx^2} > 0$
Hence $x = 1$ point of inflection and for $0 < x < 1$, graph is concave downward and for $x > 1$, graph is concave upward.

27. a, b, c.

$$f(x) = \log(2x - x^2) + \sin\frac{\pi x}{2}$$

$$= \log(1 - (x-1)^2) + \sin\frac{\pi x}{2}$$

$$f(1-x) = \log(1 - (1-(x-1)^2)) + \sin\frac{\pi(1-x)}{2}$$

$$= \log(1 - x^2) + \cos\frac{\pi x}{2}$$

$$\text{Also, } f(1+x) = \log(1 - (1+(x-1)^2)) + \sin\frac{\pi(1+x)}{2}$$

$$= \log(1 - x^2) + \cos x\frac{\pi x}{2}$$

Hence, function is symmetrical about line $x = 1$
Also, $f(1) = 1$
Also, for domain of the function is $2x - x^2 > 0$ or $x \in (0, 2)$
For $x > 1$, $f(x)$ decreases hence $x = 1$ is point of maxima.
Also, maximum value of the function is 1.
Also, $f(x) \to \infty$, when $x \to 2$, hence absolute minimum value of f does not exists.

28. a, b, c, d.

$$f'(x) = 2 - 2x^{-1/3} = 2\left(1 - \frac{1}{x^{1/3}}\right) = 2\left(\frac{x^{1/3} - 1}{x^{1/3}}\right)$$

Sign scheme of derivative is

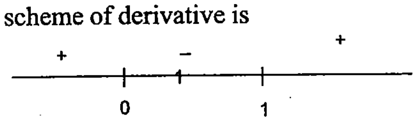


Fig. 6.101

$f(x)$ has point of maxima at $x = 0$ and point of minima at $x = 1$.
Also $f(x)$ is non-differentiable at $x = 0$.

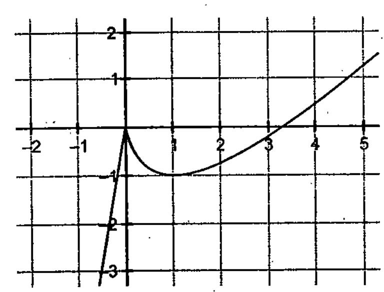


Fig. 6.102

29. a, b, c.

$$f(x) = \frac{e^x}{1+e^x}$$

$$\Rightarrow f'(x) = \frac{e^x(1+e^x) - e^x e^x}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2} > 0 \ \forall x \in R$$

$\Rightarrow f(x)$ is an increasing function.

$$\text{Also, } \lim_{x\to-\infty}\frac{e^x}{1+e^x} = 0 \text{ and } \lim_{x\to\infty}\frac{e^x}{1+e^x} = \lim_{x\to\infty}\frac{1}{1+\frac{1}{e^x}} = 1$$

Hence, the graph of $f(x) = \frac{e^x}{1+e^x}$ is as shown

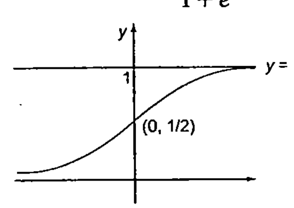


Fig. 6.103

Also, $f''(x)=\dfrac{e^x(1+e^x)^2-2(1+e^x)e^xe^x}{(1+e^x)^4}=0$

$\Rightarrow (1+e^x)-2e^x=0$

$\Rightarrow e^x=1$

$\Rightarrow x=0$ which is point of inflection

$x=0$ is the inflection point and f is bounded in (0, 1).

No maxima and f has two asymptotes.

30. a, b, c.

The following function is discontinuous at $x=2$, but has point of maxima.

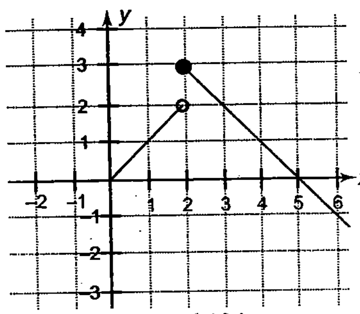


Fig. 6.104

$f(x)=|x|$ has point of minima at $x=0$, though it is non-differentiable at $x=0$

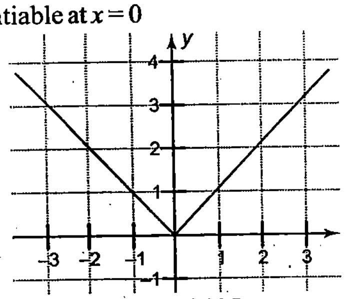


Fig. 6.105

$f(x)=x^{2/3}$ has point of inflection at $x=0$, as curve changes its concavity at $x=0$, however $x=0$ is point of minima for the function.

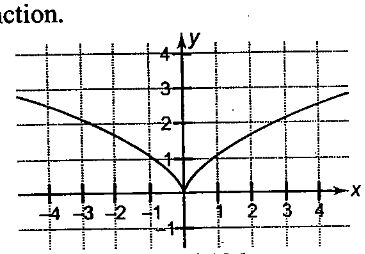


Fig. 6.106

31. a. b, d.

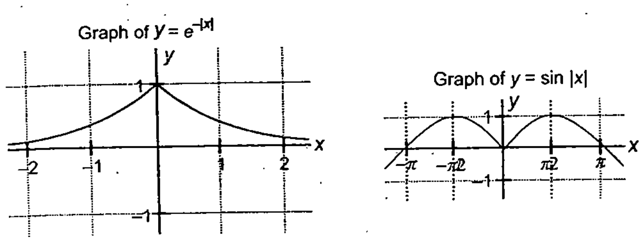


Fig. 6.107

Graph of $f(x)=\begin{cases}x^2+4x+3, & x<0\\ -x, & x\ge 0\end{cases}$

Graph of $f(x)=\begin{cases}|x|, & x<0\\ \{x\}, & x\ge 0\end{cases}$

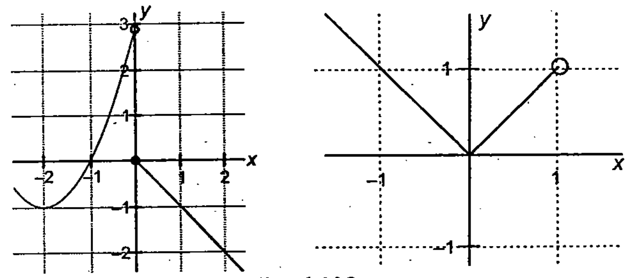


Fig. 6.108

32. c, d.

$f(x)=x^{6/7}$

$\Rightarrow f''(x)=-\dfrac{6}{7}\dfrac{1}{7}x^{\frac{-8}{7}}$, here $f''(x)$ does not change sign, hence has no point of inflection.

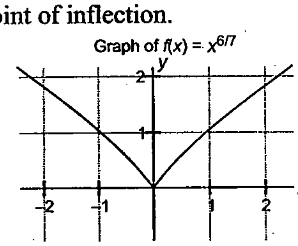


Fig. 6.109

For $f(x)=x^6, f''(x)=30x^4$, but $f''(x)$ does not change sign in the neighbourhood of $x=0$.

$f(x)=\cos x+2x$

$\Rightarrow f''(x)=-\cos x,$

$\Rightarrow f''(0)=0$ for $x=(2n+1)\pi/2, n\in Z.$

Also, sign of $f''(x)$ changes sign in the neighbourhood of $(2n+1)\pi/2$, hence function has infinite points of inflection.

$f(x)=x|x|=\begin{cases}-x^2, & x<0\\ x^2, & x\ge 0\end{cases}$

$\Rightarrow f''(x)=\begin{cases}-2, & x<0\\ 2, & x>0\end{cases}$, here $f''(x)$ changes sign in the neighbourhood of $x=0$, hence has point of inflection.

33. a. c, d.

$f'(x)=2x-\dfrac{\lambda}{x^2}\quad \therefore f'(x)=0\ \Rightarrow\ x=\left(\dfrac{\lambda}{2}\right)^{1/3}$

If $\lambda=16, x=2$

Now, $f''(x)=2+\dfrac{2\lambda}{x^3}$

$\therefore$ if $\lambda=16, f''(x)>0$, i.e. $f(x)$ has a minimum at $x=2$

Also, $f''\left\{\left(\dfrac{\lambda}{2}\right)^{1/3}\right\}=2+\dfrac{2\lambda}{\lambda/2}=2+4>0$

Hence, $f(x)$ has maximum for no real value of λ.

When $\lambda=-1, f''(x)=0$ if $x=1$. So, $f(x)$ has a point of inflection at $x=1$.

34. a. b. c. d.

$f(x)=x^{1/3}(x-1)$

$\Rightarrow \quad \dfrac{df(x)}{dx} = \dfrac{4}{3}x^{1/3} - \dfrac{1}{3}\cdot\dfrac{1}{x^{2/3}} = \dfrac{1}{3x^{2/3}}[4x-1]$

$\Rightarrow$ $f'(x)$ changes sign from –ve to +ve, at $x = \frac{1}{4}$, which is point of minima.

Also, $f'(x)$ does not exist at $x = 0$ as $f(x)$ has vertical tangent at $x = 0$.

$$f''(x) = \frac{4}{9}\cdot\frac{1}{x^{2/3}} + \frac{1}{3}\cdot\frac{2}{3}\cdot\frac{1}{x^{5/3}} = \frac{2}{9x^{2/3}}\left[2 + \frac{1}{x}\right]$$
$$= \frac{2}{9x^{2/3}}\left[\frac{2x+1}{x}\right]$$

$\therefore$ $f''(x) = 0$ at $x = -\dfrac{1}{2}$ which is the point of inflection

at $x = 0$, $f''(x)$ does not exists but $f''(x)$ changes sign, hence $x = 0$ is also the point of inflection.

From the above information the graph of $y = f(x)$ is as shown.

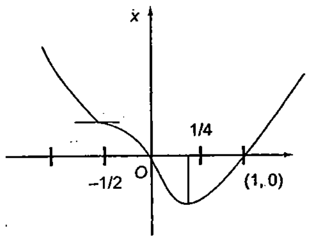


Fig. 6.110

Also, minimum value of $f(x)$ is at $x = 1/4$ which is $-3 \times 2^{-8/3}$

Hence, range is $[-3 \times 2^{-8/3}, \infty)$.

Reasoning Type

1. c. Statement 1 is true, but statement 2 is false as consider the functions in statement 1 in $\left(0, \dfrac{\pi}{2}\right)$.

2. a. $f(x) = \dfrac{\log_e x}{x} \Rightarrow f'(x) = \dfrac{1 - \log_e x}{x^2}$

$f'(x) > 0$ for $1 - \log_e x > 0$ or $x < e \Rightarrow f(x)$ is increasing.

$f(x)$ is decreasing for $x > e$.

$e < 2.91 < \alpha < \beta$

$\Rightarrow \quad f(\alpha) > f(\beta)$

$\Rightarrow \quad \dfrac{\log_e \alpha}{\alpha} > \dfrac{\log_e \beta}{\beta}$

$\Rightarrow \quad \beta \log_e \alpha > \alpha \log_e \beta$

$\Rightarrow \quad \alpha^{\beta} > \beta^{\alpha}$.

3. d. Statement 2 is true as $f(x)$ is non-differentiable at $x = 1, 2, 3$. But $f(x)$ has a point of minima at $x = 1$ and not at $x = 3$.

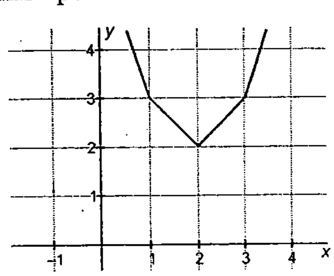


Fig. 6.111

4. c. Both $f(x) = x$ and $g(x) = x^3$ are increasing in $(-1, 0)$ but $h(x) = x.\,x^3$ is decreasing.

5. a. Suppose $f(x) = 0$ has real root say $x = a$, then $f(x) < 0$ for all $x < a$.

Thus $|f(x)|$ becomes strictly decreasing in $(-\infty, a)$ which is a contradiction.

6. d. Statement 1 is false as $f(x) = 5 - 4(x-2)^{2/3}$ attains the greatest value at $x = 2$, though it is not differentiable at $x = 2$, and for extreme value it is not necessary that $f'(x)$ exists at that point.

Statement 2 is obviously true.

7. b. $f(x) = x + \cos x$

$\therefore \quad f'(x) = 1 - \sin x > 0\ \forall x \in R$,

$\therefore \quad f(x)$ is increasing.

Statement 2 is true but does not explain statement 1.

Therefore, according to statement 2, $f'(x)$ may vanish at finite number of points but in statement 1 $f'(x)$ vanishes at infinite number of points.

8. a. Statement 2 is obviously true.

Also, for $f(x) = 2\cos x + 3\sin x = \sqrt{13}\sin\left(x + \tan^{-1}\dfrac{2}{3}\right)$

$\Rightarrow g(x) = \sin^{-1}\dfrac{x}{\sqrt{13}} - \tan^{-1}\dfrac{2}{3}$. Hence, statement 1 is true.

9. a. $f(x) = \dfrac{x^3}{3} + \dfrac{ax^2}{2} + x + 5$

$\Rightarrow \quad f'(x) = x^2 + ax + 1$

If $f(x)$ has positive point of maxima, then point of minima is also positive. Hence, both the roots of equation $x^2 + ax + 1 = 0$ must be positive.

$\Rightarrow$ sum or roots $-a > 0$, product of roots $1 > 0$ and discriminant $D = a^2 - 4 > 0$

$\Rightarrow \quad a < -2$.

10. a. $\dfrac{dy}{dx} = 12x(x^2 - x + 1) + a$ and $\dfrac{d^2y}{dx^2} = 12(3x^2 - 2x + 1) > 0$

$\Rightarrow \quad \dfrac{dy}{dx}$ is an increasing function.

But $\dfrac{dy}{dx}$ is a polynomial of degree 3 $\Rightarrow$ it has exactly one real root.

11. b. Let $f(x) = \sin x \tan x - x^2 \Rightarrow f'(x) = \sin x \sec^2 x + \sin x - 2x$

$\Rightarrow \quad f''(x) = 2\sin x \sec^2 x \tan x + \sec x + \cos x - 2$

$= 2\sin x \tan x \sec^2 x + (\cos x + \sec x - 2)$

$> 0\ \forall x \in \left(0, \dfrac{\pi}{2}\right)$

$\Rightarrow \quad f'(x)$ is an increasing function.

$\Rightarrow \quad f'(x) > f'(0) \Rightarrow \sin x \sec^2 x + \sin x - 2x > 0$

$\Rightarrow \quad f(x)$ is an increasing function

$\Rightarrow \quad f(x) > f(0)$

$\Rightarrow \quad \sin x \tan x - x^2 > 0$

$\Rightarrow \quad \sin x \tan x > x^2$

Thus, statement 1 is true, also statement 2 is true but it does not explain statement 1.

12. b. $f(x) = \sin(\cos x)$

$\Rightarrow f'(x) = -\sin x \cos(\cos x) < 0$ for $\forall x \in \left[0, \frac{\pi}{2}\right]$.

Statement 2 is also true, but it is not the only reason for statement 1 to be correct.

13. c. $f(x) = (x^3 - 6x^2 + 12x - 8)\, e^x$

$\Rightarrow f'(x) = e^x(x^3 - 6x^2 + 12x - 8) + e^x(3x^2 - 12x + 12)$
$= e^x(x^3 - 3x^2 + 4)$

$\Rightarrow f''(x) = e^x(x^3 - 3x^2 + 4) + e^x(3x^2 - 6x)$
$= e^x(x^3 - 6x + 4)$

$\Rightarrow f'''(x) = e^x(x^3 - 6x + 4) + e^x(3x^2 - 6)$
$= e^x(x^3 + 3x^2 - 6x - 2)$

Clearly, $f'(2) = f''(2) = 0$ and $f''' \neq 0$, hence $x = 2$ is the point of inflection and hence not a point of extrema. Thus, statement 1 is true.

But statement 2 is false, as it is not necessary that at point of inflection, extrema does not occur. Consider the following graph (Fig. 6.112).

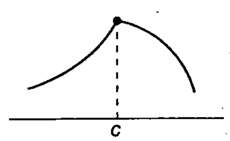


Fig. 6.112

14. a. $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$

$f'(x) = 4x^3 - 24x^2 + 44x - 24$
$= 4(x^3 - 6x^2 + 11x - 6)$
$= 4(x-1)(x-2)(x-3)$

Sign scheme of $f'(x)$

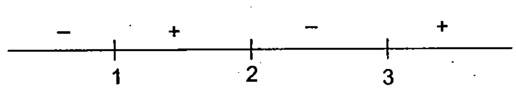


Fig. 6.113

From the sign scheme of $f'(x)$, $f(x)$ increases for $x \in (1, 2) \cup (3, \infty)$.

Since $f(x)$ is a polynomial function, which is continuous, and has no point of inflection, intervals of increase and decrease occur alternatively.

15. a. $f'(x) = \ln(x + \sqrt{1+x^2}) = -\ln(\sqrt{1+x^2} - x)$

$\Rightarrow f'(x) > 0$ for $x > 0$ and $f'(x) < 0$ for $x < 0$

$\Rightarrow f(x)$ is increasing when $x > 0$ and decreasing for $x < 0$

Hence, for $x > 0, f(x) > f(0) \Rightarrow f(x) > 0$.

Again $f(x)$ is decreasing in $(-\infty, 0)$

Then for $x < 0, f(x) > f(0) \Rightarrow f(x) > 0$.

$\Rightarrow f(x)$ is positive for all $x \in R_0$

Thus, Statement 1 is true and follows from Statement 2.

Linked Comprehension Type

For Problems 1–2

1. b, 2. a.

Sol. Let $f(x) = \sin^{-1} x + x^2 - 3x + \dfrac{x^3}{3}$

$\Rightarrow f'(x) = \dfrac{1}{\sqrt{1-x^2}} + 2x - 3 + x^2$

$\Rightarrow f'(x) = 0$ for some $x = x_1 \in (0, 1)$

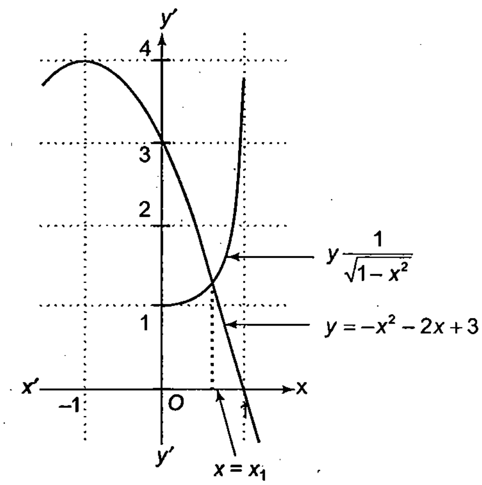


Fig. 6.114

and $f''(x) = \dfrac{x}{(1-x^2)^{3/2}} + 2 + 2x > 0, \forall x \in (0, 1)$

$\Rightarrow x = x_1$ is the point of minimum

$f(x)$ is continuous $\forall x \in [0, 1]$.

Hence, the global maxima exist at $x = 0$ or $x = 1$

$f(0) = 0, f(1) = \pi/2 - 5/3 < 0$

$f(0)$ is global maxima $\forall x \in [0, 1]$

$\Rightarrow f(x) \leq f(0), x \in [0, 1] \Rightarrow \sin^{-1} x + x^2 - 3x + x^3/3 \leq 0$

$\Rightarrow \sin^{-1} x + x^2 \leq \dfrac{x(9 - x^2)}{3} \quad \forall x \in [0, 1]$

For Problems 3–4

3. a, 4. d.

Sol. $g'(x) = f'(\sin x)\cos x - f'(\cos x)\sin x$

$\Rightarrow g''(x) = -f'(\sin x)\sin x + \cos^2 x\, f''(\sin x) + f''(\cos x)\sin^2 x - f'(\cos x)\cos x > 0\ \forall x \in (0, \pi/2)$

(as it is given $f'(\sin x) = f'(\cos x(\pi/2 - x)) < 0$ and $f''(\sin x) = f''(\cos x(\pi/2 - x)) > 0$)

$\Rightarrow g'(x)$ is increasing in $(0, \pi/2)$. Also $g'(\pi/4) = 0$

$\Rightarrow g'(x) > 0\ \forall x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ and $g'(x) < 0\ \forall x \in (0, \pi/4)$.

Thus $g(x)$ is decreasing in $(0, \pi/4)$.

For Problems 5–8

5. d, 6. a, 7. d, 8. c

Sol. If $f(x)$ is continuous then $f(3^-) = f(3^+) \Rightarrow -9 + 12 + a = 3a + b \Rightarrow 2a + b = 3$ (1)

Also $f(4^-) = f(4^+) \Rightarrow 4a + b = -b + 6 \Rightarrow 2a + b = 3$ (2)

$\Rightarrow$ $f(x)$ is continuous for infinite values of a and b.

Also, $f'(x) = \begin{cases} -2x+4, & x<3 \\ a, & 3<x<4 \\ -\dfrac{b}{4}, & x>4 \end{cases}$. For $f(x)$ to be differentiable, $f'(3^-) = f(3^+) \Rightarrow a = -2$ and $-\dfrac{b}{4} = a = -2$

$\Rightarrow$ $b = 8$.

Hence, $f(x)$ can be differentiable.

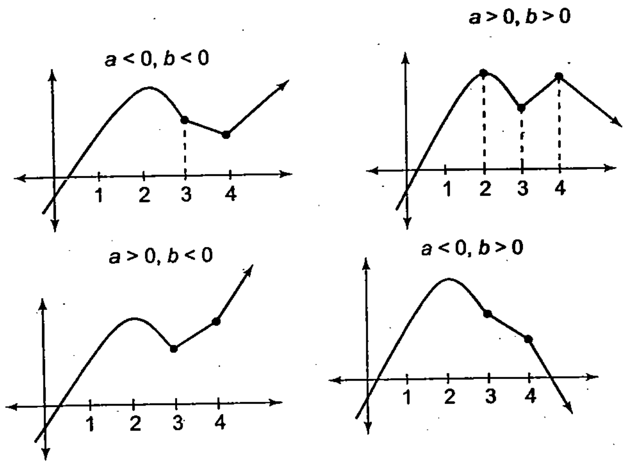


Fig. 6.115

For Problems 9–10

9.a, 10.b.

Sol.

9. a. $\dfrac{dP(x)}{dx} > P(x)$

$\Rightarrow$ $e^{-x}\dfrac{dP(x)}{dx} - e^{-x}P(x) > 0$

$\Rightarrow$ $\dfrac{d}{dx}\left(P(x)e^{-x}\right) > 0$

$\Rightarrow$ $P(x)e^{-x}$ is an increasing function.

$\Rightarrow$ $P(x)e^{-x} > P(1)e^{-1}\ \forall x \geq 1$

$\Rightarrow$ $P(x)e^{-x} > 0\ \forall x > 1 \Rightarrow P(x) > 0\ \forall x > 1$.

10. b. Given that $\dfrac{d}{dx}H(x) > 2cxH(x)$

$\Rightarrow$ $e^{-cx^2}\dfrac{d}{dx}H(x) - e^{-cx^2}2cxH(x) > 0$

$\Rightarrow$ $\dfrac{d}{dx}(H(x)e^{-cx^2}) > 0$

$\Rightarrow$ $H(x)\,e^{-cx^2}$ is an increasing function.

But $H(x_0) = 0$ and e^{-cx^2} is always positive.

$\Rightarrow$ $H(x) > 0$ for all $x > x_0$

$\Rightarrow$ $H(x)$ cannot be zero for any $x > x_0$.

For Problems 11–13

11. a. $h(x) = f(x) - a(f(x))^2 + a(f(x))^3$

$\Rightarrow$ $h'(x) = f'(x) - 2af(x)f'(x) + 3a(f(x))^2 f'(x)$

$\Rightarrow$ $h'(x) = f'(x)[3a(f(x))^2 - 2af(x) + 1]$

Now $h(x)$ increases if $f(x)$ increases and $3a(f(x))^2 - 2af(x) + 1 > 0$ for all $x \in R$

$\Rightarrow$ $3a > 0$ and $4a^2 - 12a \leq 0$

$\Rightarrow$ $a > 0$ and $a \in [0, 3]$

$\Rightarrow$ $a \in [0, 3]$.

12. d. $h(x)$ increases as $f(x)$ decreases for all real values of x if $3a(f(x))^2 - 2af(x) + 1 \leq 0$ for all $x \in R$

$\Rightarrow$ $3a < 0$ and $4a^2 - 12a \leq 0$

$\Rightarrow$ $a < 0$ and $a \in [0, 3]$

$\Rightarrow$ no such a is possible.

13. d. $h(x)$ is non-monotonic functions if $3a(f(x))^2 - 2af(x) + 1$ changes sign for which $D > 0$ or $4a^2 - 12a > 0$

$\Rightarrow$ $a \in (-\infty, 0) \cup (3, -\infty)$.

For Problems 14–16

14.a, 15.c, 16.b.

Sol. Let $g(x) = x^3 - 9x^2 + 24x = x(x^2 - 9x + 24)$

$\Rightarrow g'(x) = 3(x-2)(x-4)$

Sign scheme of $g(x)$

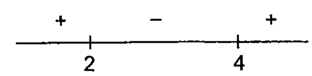


Fig. 6.116

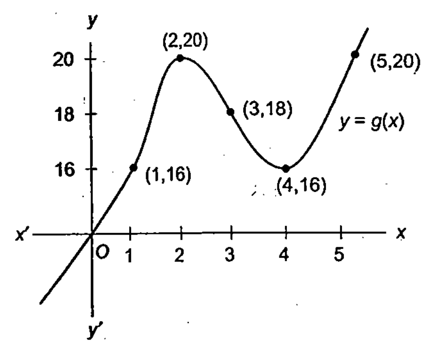


Fig. 6.117

For three real roots of

$f(x) = x^3 - 9x^2 + 24x + c = 0$, c must lie in the interval $(-20, -16)$

$f(0) = c < 0$

$f(1) = 1 - 9 + 24 + c = c + 16 < 0$ for $\forall c \in (-20, -16)$

$f(2) = 8 - 36 + 48 + c = c + 20 > 0$

$\alpha \in (1, 2) \Rightarrow [\alpha] = 1$

$f(3) = 27 - 81 + 72 + c = 18 + c$

$\Rightarrow f(3) < 0$ if $c \in (-20, -18)$ or $f(3) > 0$ if $c \in (-18, -16)$

or $\beta \in (2, 3)$ if $c \in (-20, -18)$

and $\beta \in (3, 4)$ if $c \in (-18, -16)$

Now $f(4) = 64 - 144 + 96 + c = 16 + c < 0\ \forall c \in (-20, -16)$

$f(5) = 125 - 225 + 120 + c = c + 20 > 0\ \forall c \in (-20, -16)$

$\Rightarrow$ $\gamma \in (4, 5) \Rightarrow [\gamma] = 4$

Thus, $[\alpha] + [\beta] + [\gamma] = \begin{cases} 1+2+4, & -20 < c < -18 \\ 1+3+4, & -18 < c < -16 \end{cases}$

Now if $c \in (-20, -18)$,

$\alpha \in (1, 2), \beta \in (2, 3), \gamma \in (4, 5)$

$\Rightarrow$ $[\alpha] + [\beta] + [\gamma] = 7$

if $c \in (-18, -16)$, $\alpha \in (1, 2), \beta \in (3, 4), \gamma \in (4, 5)$

$\Rightarrow$ $[\alpha] + [\beta] + [\gamma] = 8$.

For Problems 17–21

17.b, 18.d, 19.b, 20.d., 21.b.

Sol.

17. b. $f'(x) \le 0\ \forall x \in [a, b]$, so $f(x)$ is a decreasing function and $f(c) = 0 \Rightarrow f(x)$ cuts x-axis once when $x = c$.

18. d. We note that $f(c) = 0, f'(c) = 0$. Also tangent to $f'(x)$ at $x = c$ is $y = 0$. So $f''(c) = 0$.

$\therefore\ x = c$ is the repeated root of third order. That is, the equation $f(x) = 0$ has at least three repeated roots.

19. b. We have $f''(c) = 0$. So the graph of $y = f(x)$ has one point of inflection at $x = c$.

20. d. As $f(x)$ is a decreasing functions for all $x \in (a, b)$, so $f(x)$ has no local maxima or minima.

21. b. $f''(c) = 0 \Rightarrow x = c$ is a root of $f''(x) = 0$.

For Problems 22–24

22.b, 23.d, 24.a

Sol. $f(x) = 4x^2 - 4ax + a^2 - 2a + 2$. Vertex of this parabola is $\left(\frac{a}{2}, 2-2a\right)$.

Case 1: $0 < \frac{a}{2} < 2$

In this case, $f(x)$ will attain the minimum value at $x = \frac{a}{2}$.

Thus, $f\left(\frac{a}{2}\right) = 3$.

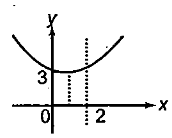


Fig. 6.118

$\Rightarrow\ 3 = -2a + 2 \Rightarrow a = -\frac{1}{2}$ (Rejected).

Case 2: $\frac{a}{2} \ge 2$

In this, $f(x)$ attains the global minimum value at $x = 2$.

Thus $f(2) = 3$

$\Rightarrow\ 3 = 16 - 8a + a^2 - 2a + 2 \Rightarrow a = 5 \pm \sqrt{10}$.

Thus $a = 5 + \sqrt{10}$.

Case 3: $\frac{a}{2} \le 0$

In this case, $f(x)$ attains the global minimum value at $x = 0$. Thus $f(0) = 3$

Convert the following graph

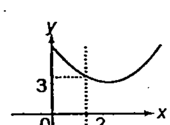


Fig. 6.119

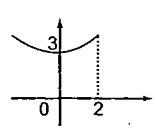


Fig. 6.120

$\Rightarrow\ 3 = a^2 - 2a + 2 \Rightarrow a = 1 \pm \sqrt{2}$. Thus, $a = 1 - \sqrt{2}$.

Hence, the permissible values of a are $1 - \sqrt{2}$ and $5 + \sqrt{10}$.

$f(x) = 4x^2 - 49x + a^2 - 2a + 2$ is monotonic in $[0, 2]$.

Hence, the point of minima of function should not lie in $[0, 2]$.

Now $f'(x) = 0 \Rightarrow 8x - 4a = 0 \Rightarrow x = a/2$. If $\frac{a}{2} \in [0, 2]$

$\Rightarrow\ a \in [0, 4]$.

For $f(x)$ to be monotonic in $[0, 2]$, $a \notin [0, 4] \Rightarrow a \le 0$ or $a \ge 4$.

For Problems 25–27

25.a, 26. b, 27. b

Sol. $f(x) = x^3 - 3(7-a)x^2 - 3(9-a^2)x + 2$

$\Rightarrow f'(x) = 3x^2 - 6(7-a)x - 3(9-a^2)$

For real root $D \ge 0$,

$\Rightarrow\ 49 + a^2 - 14a + 9 - a^2 \ge 0 \ \Rightarrow\ a \le \frac{58}{14}$ (1)

When point of minima is negative, point of maxima is also negative.

Hence, equation $f'(x) = 3x^2 - 6(7-a)x - 3(9-a^2) = 0$ has both roots negative.

For which sum of roots $= 2(7-a) < 0$ or $a > 0$, which is not possible as from (1), $a \le \frac{58}{14}$.

When point of maxima is positive, point of minima is also positive.

Hence, equation $f'(x) = 3x^2 - 6(7-a)x - 3(9-a^2) = 0$ has both roots positive.

For which sum of roots $= 2(7-a) > 0 \ \Rightarrow\ a < 7$ (2)

Also product of roots is positive $\Rightarrow -(9-a^2) > 0$ or $a^2 > 9$ or $a \in (-\infty, -3) \cup (3, \infty)$. (3)

From (1), (2) and (3); $a \in (-\infty, -3) \cup (3, 58/14)$

For points of extrema of opposite sign, equation (1) has roots of opposite sign.

$\Rightarrow\ a \in (-3, 3)$

For Problems 28–30

28. c, 29. d, 30. d

Sol. $f(x) = \left(1 + \frac{1}{x}\right)^x$, $f(x)$ is defined if $1 + \frac{1}{x} > 0$

$\Rightarrow\ \frac{x+1}{x} > 0 \Rightarrow (-\infty, -1) \cup (0, \infty)$

Now $f'(x) = \left(1 + \frac{1}{x}\right)^x \left[\ln\left(1 + \frac{1}{x}\right) + \frac{x}{1 + \frac{1}{x}} \frac{-1}{x^2}\right]$

$= \left(1 + \frac{1}{x}\right)^x \left[\ln\left(1 + \frac{1}{x}\right) - \frac{1}{x+1}\right]$

Now $\left(1 + \frac{1}{x}\right)^x$ is always positive, hence the sign of $f'(x)$ depends on sign of $\ln\left(1 + \frac{1}{x}\right) - \frac{1}{1+x}$

Let $g(x) = \ln\left(1 + \frac{1}{x}\right) - \frac{1}{x+1}$

$g'(x) = \frac{1}{1 + \frac{1}{x}} \frac{-1}{x^2} + \frac{1}{(x+1)^2} = \frac{-1}{x(x+1)^2}$

(1) for $x \in (0, \infty), g'(x) < 0$

$\Rightarrow$ $g(x)$ is monotonically decreasing for $x \in (0, \infty)$

$\Rightarrow$ $g(x) > \lim\limits_{x \to \infty} g(x)$

$\Rightarrow$ $g(x) > 0$

and since $g(x) > 0 \Rightarrow f'(x) > 0$

(2) for $x \in (-\infty, -1), g'(x) > 0$

$\Rightarrow$ $g(x)$ is monotonically increasing for $x \in (-\infty, -1)$

$\Rightarrow$ $g(x) > \lim\limits_{x \to -\infty} g(x)$

$\Rightarrow$ $g(x) > 0 \Rightarrow f'(x) > 0$

Hence from (1) and (2) we get $f'(x) > 0$ for all $x \in (-\infty, -1) \cup (0, \infty)$

$\Rightarrow$ $f(x)$ is monotonically increasing in its domain

Also $\lim\limits_{x \to \pm\infty} \left(1 + \frac{1}{x}\right)^x = e$

$\lim\limits_{x \to 0^+} \left(1 + \frac{1}{x}\right)^x = 1$ and $\lim\limits_{x \to -1} \left(1 + \frac{1}{x}\right)^x = \infty$

The graph of $f(x)$ is shown in Fig. 6.121.

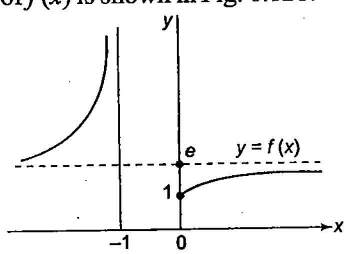


Fig. 6.121

Range is $y \in (1, \infty) - \{e\}$.

For Problems 31–33

31. d, 32. d, 33. b

Sol. $f(x) = x + \cos x - a \Rightarrow f'(x) = 1 - \sin x \geq 0 \ \forall x \in R$.

Thus $f(x)$ is increasing in $(-\infty, \infty)$, as for $f'(x) = 0$, x is not forming an interval.

Also $f''(x) = -\cos x = 0$

$\Rightarrow x = (2n + 1)\frac{\pi}{2}, n \in Z$

Hence infinite points of infletion

Now $f(0) = 1 - a$.

For positive root $1 - a < 0 \Rightarrow a > 1$. For negative root $1 - a > 0 \Rightarrow a < 1$.

For Problems 34–36

34. c, 35. b, 36. d.

Sol. $f(x) = 3x^4 + 4x^3 - 12x^2$

$\Rightarrow f'(x) = 12(x^3 + x^2 - 2x) = 12x(x - 1)(x + 2)$

The sign scheme of $f'(x)$ is as follows.

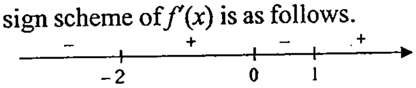
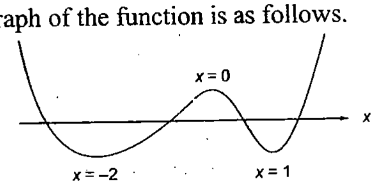


Fig. 6.122

The graph of the function is as follows.



Thus, we have,

$f(-2) = -32$ and $f(1) = -5$

Hence, range of the function is $[-32, \infty)$.

Also, $f(x) = a$ has no real roots if $a < -32$.

For Problems 37–39

Sol.

37. d, 38. d, 39. b

$$f(x) = \frac{x^2 - 6x + 4}{x^2 + 2x + 4}$$

$$= 1 - \frac{8x}{x^2 + 2x + 4}$$

$$f'(x) = -8\left[\frac{(x^2 + 2x + 4) - x(2x + 2)}{(x^2 + 2x + 4)^2}\right]$$

$$= -8\left[\frac{-x^2 + 4}{(x^2 + 2x + 4)^2}\right] = \frac{8(x^2 - 4)}{(x^2 + 2x + 4)^2}$$

$$f'(x) = 0 \Rightarrow x = 2 \text{ or } -2$$

$$f(2) = \frac{4 - 12 + 4}{4 + 4 + 4} = \frac{-4}{12} = \frac{-1}{3}$$

$$f(-2) = \frac{4 + 12 + 4}{4 - 4 + 4} = 5$$

the graph of $y = f(x)$ is as shown

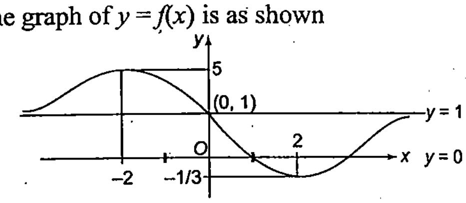


Fig. 6.124

hence $-\frac{1}{3} \leq f(x) \leq 5$

For Problems 40–42

40. c, 41. d, 42. c

Sol. Since two points of inflection occur at $x = 1$ and $x = 0$

$\Rightarrow P''(1) = P''(0) = 0$

$\therefore P''(x) = a(x^2 - x)$

$\Rightarrow P'(x) = a\left(\frac{x^3}{3} - \frac{x^2}{2}\right) + b$

Also, Given $\left(\frac{dy}{dx}\right)_{x=0} = \sec^{-1}\sqrt{2} = \tan^{-1} 1$

Hence, $P'(0) = 1$, so $b = 1$

$\Rightarrow P'(x) = a\left(\frac{x^3}{3} - \frac{x^2}{2}\right) + 1$

$\therefore P(x) = a\left(\frac{x^4}{12} - \frac{x^3}{6}\right) + x + c$

As $P(-1) = 1$

$\Rightarrow a\left(\frac{1}{12} + \frac{1}{6}\right) - 1 + c = 1 \Rightarrow \frac{a}{4} + c = 2$ (1)

$P(1) = 2$

$\Rightarrow a\left(\frac{1}{12} - \frac{1}{6}\right) + 1 + c = 1$

$\Rightarrow -\dfrac{a}{12}+c=0$ (2)

Solving (1) and (2),

We have $a = 6$ and $c = \dfrac{1}{2}$

$\Rightarrow P(x) = 6\left(\dfrac{x^4}{12}-\dfrac{x^3}{6}\right)+x+\dfrac{1}{2}$

$\Rightarrow P(-1) = 6\left(\dfrac{1}{12}+\dfrac{1}{6}\right)-1+\dfrac{1}{2}=1$ and $P(0)=\dfrac{1}{2}$

$\Rightarrow P'(x) = 6\left(\dfrac{x^3}{3}-\dfrac{x^2}{2}\right)+1=(x-1)^2(2x+1)$

For Problems 43–45

43. c, 44. d, 45. d.

Sol.

We have $f(x)=x^2e^{-|x|}=\begin{cases} x^2e^{-x}, & x\ge 0 \\ x^2e^{x}, & x<0 \end{cases}$

$\Rightarrow f'(x)=\begin{cases} e^{-x}(2x-x^2), & x\ge 0 \\ e^{x}(x^2+2x), & x<0 \end{cases}$

$f(x)$ increases in $(-\infty,-2)\cup(0,2)$

and $f(x)$ decreases in $(-2,0)\cup(2,\infty)$

$\Rightarrow f''(x)=\begin{cases} e^{-x}(x^2-4x+2), & x\ge 0 \\ e^{x}(x^2+4x+2), & x<0 \end{cases}$

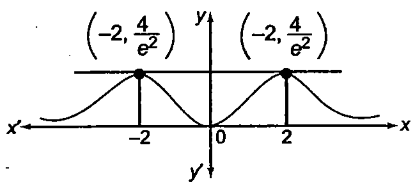


Fig. 6.125

$f''(x)=0$ has four roots. Hence, four points of inflection.

Matrix-Match Type

1. **a → r; b → s; c → q; d → p**

$f'(x)=4x^3-28x+24$

$=4(x^3-7x+6)$

$=4(x^3-x^2+x^2-x-6x+6)$

$=4(x-1)(x^2+x-6)$

$=4(x-1)(x+3)(x-2)$

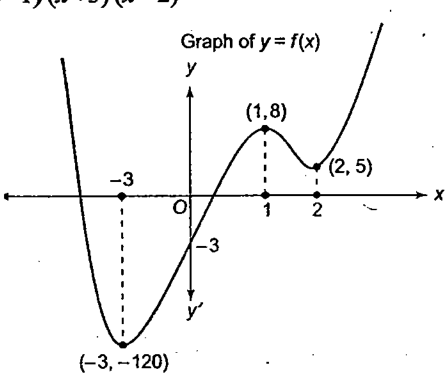


Fig. 6.126

Now, nature of roots of $f(x)+p=0$ can be obtained by shifting the graph of $y=f(x)$ by p units upward or downward depending on whether p is positive or negative.

2. **a → p, s; b → p, s; c → q, r; d → q**

a. $f(x)=x^2\log x$

for $f'(x)=x(2\log x+1)=0, \Rightarrow x=\dfrac{1}{\sqrt{e}}$ which is the point of minima as derivative changes sign from negative to positive.

Also, the function decreases in $\left(0,\dfrac{1}{\sqrt{e}}\right)$.

b. $y=x\log x$

$\Rightarrow \dfrac{dy}{dx}=x\times\dfrac{1}{x}+\log x\times 1=1+\log x$ and $\dfrac{d^2y}{dx^2}=\dfrac{1}{x}$

For $\dfrac{dy}{dx}=0\Rightarrow \log x=-1\Rightarrow x=\dfrac{1}{e}$

$\dfrac{d^2y}{dx^2}=\dfrac{1}{1/e}=e>0$ at $x=\dfrac{1}{e}$

$\Rightarrow$ y is min for $x=\dfrac{1}{e}$

c. $f(x)=\dfrac{\log x}{x}$

For $f'(x)=\dfrac{1-\log x}{x^2}=0, x=e$. Also, derivative changes sign from positive to negative at $x=e$, hence it is the point of maxima.

d. $f(x)=x^{-x}$

$f'(x)=-x^{-x}(1+\log x)=0\Rightarrow x=1/e$, which is clearly point of maxima.

3. **a → p, r; b → p, s; c → q, r; d → q, s**

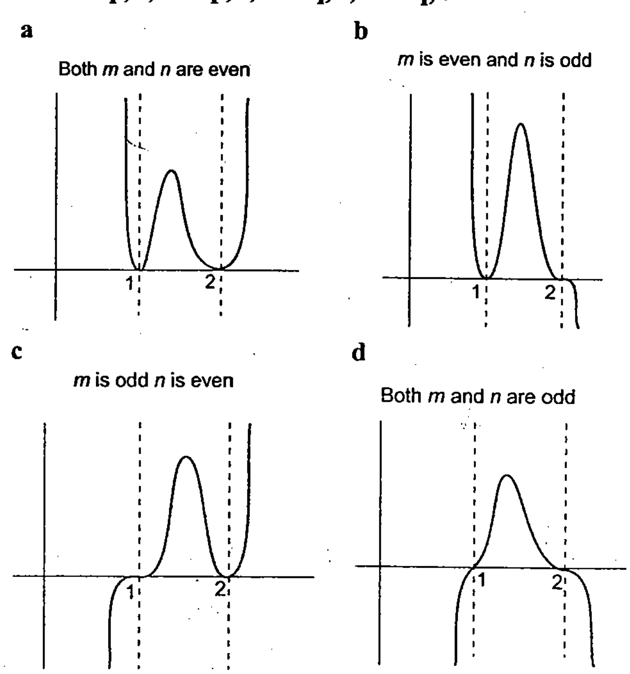


Fig. 6.127

4. a → s; b → p; c → q; d → r

Since $f(x)$ is minimum at $x=-2$ and maximum at $x=2$, let

$g(x)=ax^3+bx^2+cx+d$

$\therefore$ $g(x)$ is also minimum at $x=-2$ and maximum at $x=2$

$\therefore$ $a<0$

$\because$ a is a root of $x^2-x-6=0$, i.e., $x=3,-2$

$\therefore$ $a=-2$

Then, $g(x)=-2x^3+bx^2+cx+d$.

$\therefore$ $g'(x)=-6x^2+2bx+c=-6(x+2)(x-2)$

($\because g(x)$ is minimum at $x=-2$ and maximum at $x=2$)

On comparing, we get

$b=0$ and $c=24$

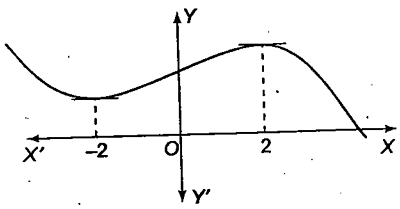


Fig. 6.128

Since minimum and maximum values are positive

$\therefore$ $g(-2)>0$ $\Rightarrow 16-48+d>0$ $\Rightarrow d>32$

and $g(2)>0$ $\Rightarrow -16+48+d>0$ $\Rightarrow d>-32$

It is clear $d>32$

Hence, $a=-2, b=0, c=24, d>32$.

5. a → q, b → p, c → s, d → r.

$f(x)=\sin x-x^2+1$

$f'(x)=\cos x-2x$

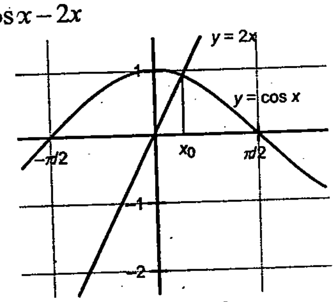


Fig. 6.129

$\Rightarrow f'(x)<0$ for $x>x_0$

$f'(x)>0$ for $x<x_0$

Hence $x=x_0$ is point of maxima.

b. p. $f(x)=x\log_e x-x+e^{-x}$

$f'(x)=\log_e x+1-1-e^{-x}=\log_e x-e^{-x}$

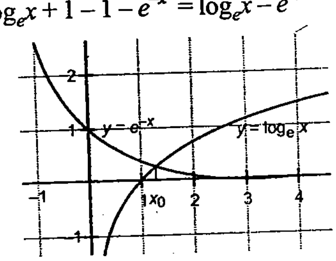


Fig. 6.130

From the graph for $x<x_0$, $e^{-x}>\log_e x$, $\Rightarrow f'(x)<0$

For $x>x_0$, $e^{-x}<\log_e x$, $\Rightarrow f'(x)>0$

Hence, $x=x_0$ is point of minima.

c. s $f(x)=-x^3+2x^2-3x+1$

$f'(x)=-3x^2+4x-3$

Now $D=16-4(-3)(-3)=-20<0$

Hence $f'(x)<0$, for all real x

$\Rightarrow f(x)$ is always decreasing.

d. r. $f(x)=\cos \pi x+10x+3x^2+x^3$

$\Rightarrow$ $f'(x)=-\pi\sin \pi x+10+6x+3x^2$

$=3(x^2+2x+10/3)-\pi\sin\pi x$

$=3((x+1)^2+7/3)-\pi\sin\pi x$

Now min. value of $3((x+1)^2+7/3)$ is 7 but maximum value of $\pi\sin\pi x$ is π.

Hence, $f'(x)>0$ for all real x.

Hence, $f(x)$ is always increasing.

6. a → s; b → s; c → r; d → q

a. s. Graph of $f(x)=|2x-1|+|2x-3|$

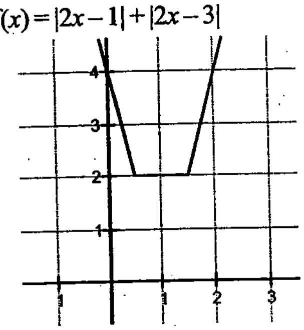


Fig. 6.131

From the graph $f(x)$ has infinite points of minima.

b. s. $f(x)=2\sin x-x \Rightarrow$ for $f'(x)=2\cos x-1=0$ we have $\cos x=½$ which has infinite points of extrema.

c. r. Graph of $f(x)=|x-1|+|2x-3|$

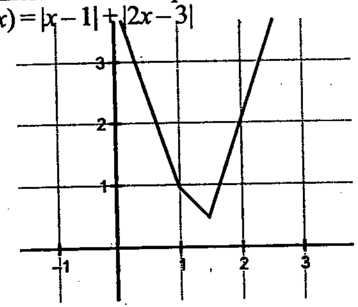


Fig. 6.132

From the graph $f(x)$ has one point of minima.

d. q. Graph of $f(x)=|x|-|2x-3|$

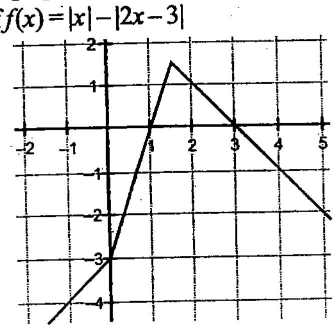


Fig. 6.133

From the graph $f(x)$ has one point of maxima.

7. a → q, r; b → r, s; c → p, r; d → r, s

a. q, r. $f(x)=(x-1)^3(x+2)^5$

$\Rightarrow$ $f'(x)=3(x-1)^2(x+2)^5+5(x-1)^3(x+2)^4$

$\Rightarrow$ $f'(x)=(x-1)^2(x+2)^4[3(x+2)+5(x-1)]$

$=(x-1)^2(x+2)^4[8x+1]$

Sign of derivative does not change at $x = 1$ and $x = -2$.
Sign of derivative changes sign at $x = -1/8$ from –ve to +ve.
Hence, function has point of minima.
Also, $f''(x) = 0$ for $x = 1$ and $x = -2$
Hence, function has two points of inflection.

b. r, s.

$f(x) = 3\sin x + 4\cos x - 5x$

$\Rightarrow f'(x) = 3\cos x - 4\sin x - 5 \le 0$, hence $f(x)$ is decreasing function

Also, $f''(x) = -3\sin x - 4\cos x = 0$ for infinite values of x, hence function has infinite points of inflection.

c. p, r

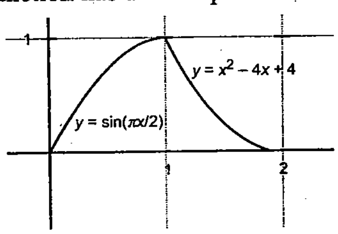


Fig. 6.134

From the graph $x = 1$ point of maxima as well as point of inflection.

d. r, s.

$$f(x) = (x-1)^{3/5} \Rightarrow f'(x) = \frac{3}{5}(x-1)^{-\frac{2}{5}} \ge 0 \text{ for all real } x$$

Also, $f''(x) = -\frac{3}{5}\frac{2}{5}(x-1)^{-\frac{7}{5}}$ which changes sign at $x = 1$

Hence, $x = 1$ is point of inflection.

8. a → r; b → s; c → p; d → q

a. r. From the graph $x = 1$ is point of maxima

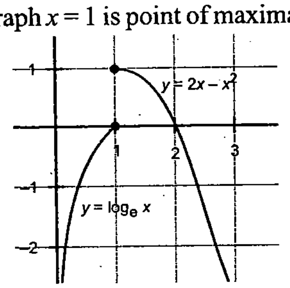


Fig. 6.135

b. s. $f(x) = \begin{cases} x-1, & x<2 \\ 0, & x=2 \\ \sin x, & x>2 \end{cases}$

$f(2) = 0, f(2^+) = \sin(2^+) > 0$ and $f(2^-) > 0$, hence $x = 2$ is point of minima.

c. p. $f(x) = \begin{cases} 2x+3 & x<0 \\ 5, & x=0 \\ x^2+7, & x>0 \end{cases}$

$f(0^-) = 3, f(0) = 5, f(0^+) = 7$, hence $f(0^-) < f(0) < f(0^+)$

Thus, $f(x)$ is increasing at $x = 0$

d. q. $f(x) = \begin{cases} e^{-x} & x<0 \\ 0, & x=0 \\ -\cos x, & x>0 \end{cases}$

$f(0) = 0, f(0^+) = -1, f(0^-) = 1$

Thus, $f(0^-) > f(0) > f(0^+)$

Hence, $f(x)$ is decreases at $x = 0$

9. (a) → s, (b) → r, (c) → q, (d) → p.

Integer Type

1. (9) Let $y = 2x\tan^{-1}x - \ln(1+x^2)$

$$y' = 2\tan^{-1}x + \frac{2x}{1+x^2} - \frac{2x}{1+x^2}$$

$\Rightarrow y' > 0\ \forall x \in R^+, y' < 0\ \forall x \in R^-$

$\Rightarrow y \ge 0, \forall x \in R$

$\therefore$ $4 - |[x]|$ takes the values 0, 1, 2, 3, 4 $\{\because |\alpha| \le 4 - |[x]|\}$

$|\alpha| \le 4 - |[x]|$ is satisfied by $\alpha = 0, \pm1, \pm2, \pm3, \pm4$,

Therefore, number of values of α is 9.

2. (3)

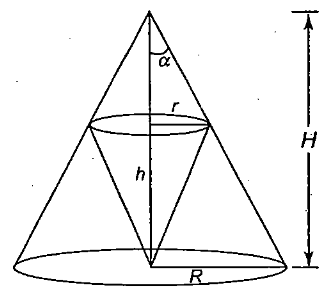


Fig. 6.136

$$\frac{r}{R} = \frac{H-h}{H}$$

$$r = \frac{R(H-h)}{H}$$

$$\text{Volume } V = \frac{1}{3}\pi\frac{R^2(H-h)^2}{H^2}\cdot h$$

$$\therefore \quad V = \frac{\pi R^2}{3H^2}(H-h)^2 h$$

$$\therefore \quad \frac{dV}{dh} = \frac{\pi R^2}{3H^2}\left[(H-h)^2 - 2h(H-h)\right]$$

$$= \frac{\pi R^2}{3H^2}(H-h)(H-h-2h)$$

$$\therefore \quad \frac{dV}{dh} = 0 \text{ if } h = \frac{H}{3}$$

and $h = \frac{H}{3}$ is a point of maximum $\Rightarrow \frac{H}{h} = 3$

3. (1) $f(x) = \begin{cases} |x^3 + x^2 + 3x + \sin x|\left(3 + \sin\left(\frac{1}{x}\right)\right), & x \ne 0 \\ 0, & x = 0 \end{cases}$

Let $g(x) = x^3 + x^2 + 3x + \sin x$

$\therefore f'(x) = 3x^2 + 2x + 3 + \cos x$

$=3\left(x^2+\frac{2x}{3}+1\right)+\cos x$

$=3\left\{\left(x+\frac{1}{3}\right)^2+\frac{8}{9}\right\}+\cos x>0$

and $2<3+\sin\left(\frac{1}{x}\right)<4$

Hence, minimum value of $f(x)$ is 0 at $x=0$.

Hence, number of points = 1.

4. (8) Let $f''(x)=6a(x-1)\ (a>0)$

$\Rightarrow f'(x)=6a\left(\frac{x^2}{2}-x\right)+b=3a(x^2-2x)+b.$

Given $f'(-1)=0$

$\Rightarrow 9a+b=0 \Rightarrow b=-9a.$

$\Rightarrow f'(x)=3a(x^2-2x-3)=0$

$\Rightarrow x=-1$ and 3.

So, $y=f(-1)$ and $y=f(3)$ are two horizontal tangents.

$\Rightarrow$ Distance between these tangents = $|f(3)-f(-1)|=|-22-10|$.

5. (2) Given $\lim_{x\to 0}\left(\frac{P(x)}{x^3}-2\right)=4$

$\therefore \lim_{x\to 0}\frac{P(x)}{x^3}=6$

Consider $P(x)=ax^5+bx^4+6x^3$

$\Rightarrow P'(x)=5ax^4+4bx^3+18x^2$

Now, $P'(-1)=0 \Rightarrow 5a-4b=-18$

and $P'(1)=0 \Rightarrow 5a+4b=-18$

$\therefore$ On solving, we get $a=\frac{-18}{5}$, $b=0$

Hence, $P(x)=\frac{-18}{5}x^5+6x^3$.

$\Rightarrow P(1)=\frac{12}{5}$

6. (3) We have $f(x,y)=x^2+y^2-4x+6y$

Let $(x,y)=(\cos\theta,\sin\theta)$, then $\theta\in[0,\pi/2]$ and

$f(x,y)=f(\theta)=\cos^2\theta+\sin^2\theta-4\cos\theta+6\sin\theta$

$f'(\theta)=6\cos\theta+4\sin\theta>0\ \forall\ \theta\in[0,\pi/2]$

$\therefore f(\theta)$ is strictly increasing in $[0,\pi/2]$

$\therefore f(\theta)_{\min}=f(0)=1-4+0=-3$

7. (3) $f''(x)=4x$

$f'(x)=2x^2+C$

given $f'(-2)=1 \Rightarrow C=-7$

$\therefore f'(x)=2x^2-7$

$f(x)=\frac{2}{3}x^3-7x+C$, $f(-2)=0$

$0=-\frac{16}{3}+14+C \Rightarrow C=-\frac{26}{3}$

$\therefore f(x)=\frac{2}{3}x^3-7x-\frac{26}{3}=\frac{1}{3}(2x^3-21x-26)$

$\therefore f(1)=-15$

8. (1) $f(x)=\frac{x^3}{3}-x-b;$

$\therefore f'(x)=x^2-1=0$

$\therefore x=1$ or -1

for three distinct roots $f(x_1)\cdot f(x_2)<0$ where x_1 and x_2 are the roots of $f'(x)=0$

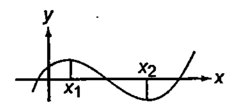


Fig. 6.137

$\Rightarrow \left(\frac{1}{3}-1-b\right)\left(-\frac{1}{3}+1-b\right)<0$

$\Rightarrow \left(b+\frac{2}{3}\right)\left(b-\frac{2}{3}\right)<0$

$\Rightarrow b\in\left(-\frac{2}{3},\frac{2}{3}\right)$

9. (4) $f'(x)=\begin{cases}1, & x<-1\\ 2x, & -1<x<1\\ 2(x-2), & x>1\end{cases}$ $f'(x)$ changes sign at $x=-1, 0, 1, 2$

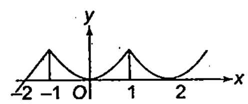


Fig. 6.138

10. (4) $f''(x)=12x^2+6ax+3\geq 0\ \forall x\in R$

$\Rightarrow 36a^2-144\leq 0$

$\Rightarrow a\in[-2,2]$

$\Rightarrow$ Number of non-zero integral values of 'a' is 4.

11. (3) A, B, C are the 3 critical points of $y=f(x)$

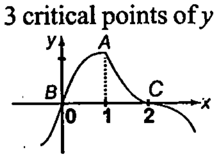


Fig. 6.139

At B, it has vertical tangent, hence non-differentiable

At A, it is non-differentiable

At C, $\frac{dy}{dx}=0$

12. (9)

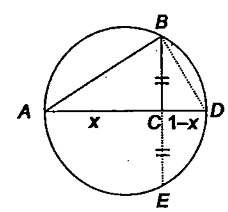


Fig. 6.140

$BC\times CE=AC\times CD$

$\Rightarrow (BC)(CE)=x(1-x)$

but $BC=CE$

$\therefore BC=\sqrt{x(1-x)}$

$\Rightarrow$ Area $\Delta=\frac{x\sqrt{x-x^2}}{2}$

$\Rightarrow \Delta^2 = \frac{x^3 - x^4}{2}$

$\Rightarrow \frac{d\Delta^2}{dx} = \frac{3x^2 - 4x^3}{2}$

If $\frac{d\Delta^2}{dx} = 0$

$\Rightarrow x = 3/4$ which is the point of maxima.

Hence, maximum area is $\frac{3\sqrt{3}}{32}$.

13.(5)

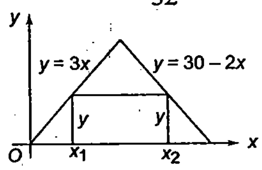


Fig. 6.141

$A = (x_2 - x_1)y$

$y = 3x_1$ and $y = 30 - 2x_2$

$A(y) = \left(\frac{30-y}{2} - \frac{y}{3}\right)y$

$6A(y) = (90 - 3y - 2y)y = 90y - 5y^2$

$6A'(y) = 90 - 10y = 0$

$\Rightarrow y = 9;\quad A''(y) = -10 < 0$

$x_1 = 3; x_2 = \frac{21}{2}$

$\Rightarrow A_{max} = \left(\frac{21}{2} - 3\right)9 = \frac{15 \cdot 9}{2} = \frac{135}{2}$

14. (4) $x^2 - 2x - 3 > 0$

$\Rightarrow (x-3)(x+1) > 0$

$\Rightarrow x < -1$ or $x > 3$ (1)

Now, $f(x) = \log_{1/2}(x^2 - 2x - 3)$

$= \frac{\log_e(x^2 - 2x - 3)}{\log_e(1/2)}$

$f'(x) = \frac{2x-2}{(\log_e(1/2))(x^2 - 2x - 3)}$

For $f(x)$ to be decreasing $f'(x) < 0$

$\Rightarrow \frac{x-1}{(\log_e(1/2))(x-3)(x+1)} < 0$

$\Rightarrow x > 1$ (2)

From (1) and (2); $x > 3$

15. (9)

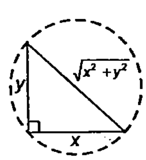


Fig. 6.142

$\frac{9}{\pi} = S = \frac{xy}{2} = \text{constant}$

Area of the circles $A(x) = \pi r^2 = \frac{\pi(x^2 + y^2)}{4}; (x^2 + y^2 = 4r^2)$

$A(x) = \frac{\pi}{4}\left[x^2 + \left(\frac{2S}{x}\right)^2\right]$

$A'(x) = \frac{\pi x}{2} - \frac{2\pi S^2}{x^3} = 0$

$\Rightarrow x^4 = 4S^2$

$\Rightarrow x^2 = 2S$

$\Rightarrow S^2 = \frac{x^2 y^2}{4} = \frac{2Sy^2}{4}$

$\Rightarrow y^2 = 2S$

Therefore, least area of circle $= \pi r^2 = \frac{\pi}{4}(x^2 + y^2) = \pi S = 9$ sq. units.

16. (9) $f\left(\frac{3}{2}\right) = 0 \Rightarrow \lim_{x \to \frac{3}{2}} |x^2 - 3x| + a \le 0 \Rightarrow a \le -\frac{9}{4}$

Hence, greatest value of $|4a|$ is 9.

Archives

Subjective

1. $y = \frac{(a+x)(b+x)}{(c+x)}$

Let $x + c = t$

$\Rightarrow \quad y = \frac{(a-c+t)(b-c+t)}{t}$

$= \frac{t^2 + [(a-c)+(b-c)]t + (a-c)(b-c)]}{t}$

$= t + \frac{(a-c)(b-c)}{t} + (a-c) + (b-c)$

$= \left(\sqrt{t} - \sqrt{\frac{(a-c)(b-c)}{t}}\right)^2 + \left(\sqrt{(a-c)} + \sqrt{(b-c)}\right)^2$

Hence, the minimum value of y is $\left(\sqrt{a-c} + \sqrt{b-c}\right)^2$

when $\sqrt{t} = \sqrt{\frac{(a-c)(b-c)}{t}}$.

2. We know that $A.M. \ge G.M.$

$\Rightarrow \frac{x+y}{2} \ge (xy)^{1/2}$

$\Rightarrow x + y \ge 2$

Hence, the minimum value $x + y$ is 2.

3. Let $f(x) = x^{1/x}$

$\Rightarrow \log f(x) = (1/x)\log x$

Differentiating w.r.t. x, we get

$\frac{f'(x)}{f(x)} = \frac{1}{x}\frac{1}{x} - \frac{1}{x^2}\log x = \frac{1}{x^2}(1 - \log x)$

$\Rightarrow f'(x) = \frac{x^{1/x}}{x^2}(1 - \log x)$

Obviously, for $x > e$, $\log x > 1$ so $f'(x) < 0$

$\therefore$ $f(x)$ is a monotonically decreasing function of x for $x \geq e$

Also, $\pi > e \Rightarrow f(\pi) < f(e)$

$\Rightarrow \pi^{1/\pi} < e^{1/e} \Rightarrow (\pi^{1/\pi})^{\pi} < (e^{1/e})^{\pi} \Rightarrow \pi < (e^{\pi})^{1/e} \Rightarrow \pi^{e} < e^{\pi}$.

4. $(0, c)\ y = x^2, 0 \leq c \leq 5$

Any point on the parabola is (x, x^2).

Distance between (x, x^2) and $(0, c)$ is

$$D = \sqrt{x^2 + (x^2 - c)^2}$$

$$\Rightarrow D^2 = x^4 - (2c-1)x^2 + c^2$$

$$= \left(x^2 - \frac{2c-1}{2}\right)^2 + c - \frac{1}{4}$$

which is minimum when $x^2 - \frac{2c-1}{2} = 0$

$$\Rightarrow D_{\min} = \sqrt{c - \frac{1}{4}}$$

5. Given $ax^2 + \frac{b}{x} \geq c$ (1)

$\forall x > 0, a > 0, b > 0$

To show that $27ab^2 \geq 4c^3$.

Let us consider the function $f(x) = ax^2 + \frac{b}{x} - c$,

then $f'(x) = 2ax - \frac{b}{x^2} = 0 \Rightarrow x^3 = b/2a \Rightarrow x = (b/2a)^{1/3}$

Also, $f''(x) = 2a + \frac{2b}{x^3} \Rightarrow f''\left(\left(\frac{b}{2a}\right)^{1/3}\right) = 6a > 0$

Therefore, f is minimum at $x = \left(\frac{b}{2a}\right)^{1/3}$

As (1) is true $\forall x$ $\therefore$ is so for $x = \left(\frac{b}{2a}\right)^{1/3}$

$$\Rightarrow a\left(\frac{b}{2a}\right)^{2/3} + \frac{b}{(b/2a)^{1/3}} \geq c$$

$$\Rightarrow \frac{a\left(\frac{b}{2a}\right) + b}{(b/2a)^{1/3}} \geq c \Rightarrow \frac{3b}{2}\left(\frac{2a}{b}\right)^{1/3} \geq c$$

As a, b are + ve, cubing both sides, we get $\frac{27b^3}{8}\frac{2a}{b} \geq c^3$

$$\Rightarrow 27ab^2 \geq 4c^3.$$

6. To show $1 + x\ln\left(x + \sqrt{x^2+1}\right) \geq \sqrt{1+x^2}$ for $x \geq 0$

consider $f(x) = 1 + x\ln\left(x + \sqrt{x^2+1}\right) - \sqrt{1+x^2}$

Here, $f'(x) = \ln\left(x + \sqrt{x^2+1}\right) + \frac{x}{x + \sqrt{x^2+1}}$

$$\times\left[1 + \frac{x}{\sqrt{x^2+1}}\right] - \frac{x}{\sqrt{1+x^2}}$$

$$= \ln\left(x + \sqrt{x^2+1}\right)$$

As $x + \sqrt{x^2+1} \geq 1$ for $x \geq 1$

$\therefore$ $\ln\left(x + \sqrt{x^2+1}\right) \geq 0$

$\therefore$ $f'(x) \geq 0, \forall x \geq 0$

Hence, $f(x)$ is an increasing function.

Now for $x \geq 0 \Rightarrow f(x) \geq f(0)$

$$\Rightarrow 1 + x\ln\left(x + \sqrt{x^2+1}\right) - \sqrt{1+x^2} \geq 0$$

$$\Rightarrow 1 + x\ln\left(x + \sqrt{x^2+1}\right) \geq \sqrt{1+x^2}.$$

7. Let the swimmer lands at the point P, x km. from A and then walks from P to the point B to be reached.

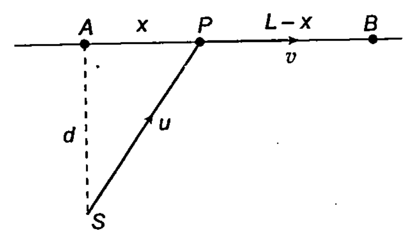


Fig. 6.143

Given that $AB = L$ km. $\therefore PB = (L - x)$ km.

t = total time from S to B

= (time taken from S to P) + (time taken from P to B)

$= SP/u + PB/v$

$$= \sqrt{(d^2 + x^2)}/u + (L - x)/v$$

$$\Rightarrow \frac{dt}{dx} = \frac{x}{u\sqrt{(d^2+x^2)}} - \frac{1}{v}$$

and $\frac{d^2t}{dx^2} = \frac{1}{u\sqrt{(d^2+x^2)}} - \frac{x^2}{u(d^2+x^2)^{3/2}}$

$= \frac{d^2}{u(d^2+x^2)^{3/2}}$ which is +ve.

For maximum or minimum of t, $dt/dx = 0$

$$\Rightarrow v^2x^2 = u^2(d^2 + x^2)$$

$$\Rightarrow x = \frac{ud}{\sqrt{v^2 - u^2}}$$

Therefore, t is minimum for this value of $x\left(\because \frac{d^2t}{dx^2} \text{ is +ve}\right)$

Hence, the swimmer will reach his house in the shortest possible time if he lands at a distance $L - x = L - \frac{ud}{\sqrt{v^2-u^2}}$ from his house to be reached.

8. $y = \frac{x}{1+x^2}$ is an odd function.

Also, $x > 0 \Rightarrow y > 0$ and $x < 0 \Rightarrow y < 0$

When $x \to \pm\infty, y \to 0$

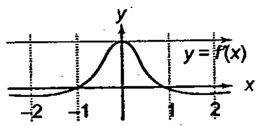


Fig. 6.144

$\frac{dy}{dx} = \frac{1+x^2-x(2x)}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$ which has greatest value at $x = 0$.

9. $f(x) = \sin^3 x + \lambda \sin^2 x$

$\therefore \quad f'(x) = 3\sin^2 x(\cos x) + \lambda\, 2\sin x(\cos x)$
$= \sin x \cos x(3\sin x + 2\lambda)$

For extremum, let $f'(x) = 0$

$\therefore \quad \sin x = 0, \cos x = 0, \sin x = -\frac{2\lambda}{3}$

Since $-\pi/2 < x < \pi/2$

$\therefore \quad \cos x \neq 0$

$\Rightarrow \quad \sin x = 0 \Rightarrow x = 0$

and $\sin x = \frac{-2\lambda}{3} \Rightarrow x = \sin^{-1}\left(\frac{-2\lambda}{3}\right)$ (1)

From (1), one of these will give maximum and one minimum, provided $-1 < \sin x = \frac{-2\lambda}{3} < 1$.

$\Rightarrow \quad -1 < \frac{-2\lambda}{3} < 1$

$\Rightarrow \quad -3 < -2\lambda < 3$

$\Rightarrow \quad -3 < 2\lambda < 3$

i.e., $\quad -3/2 < \lambda < 3/2$

But if $\lambda = 0$, then $\sin x = 0$ has only one solution.

$\therefore \quad \lambda \in (-3/2, 3/2) - \{0\}$

$\Rightarrow \quad \lambda \in (-3/2, 0) \cup (0, 3/2)$.

For this value of λ, there are two distinct solutions. Since, $f(x)$ is continuous, these solutions give one maximum and one minimum because for a continuous function, between two maxima there must lie one minima and vice versa.

10.

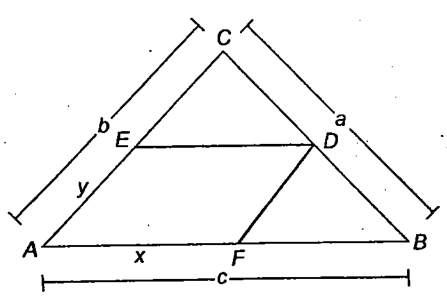


Fig. 6.145

From similar Δs FBD and ABC, $\frac{c-x}{c} = \frac{y}{b}$

or $y = (b/c)(c-x)$

$\Rightarrow Z = \text{Area of } AFDE = xy\sin A = \frac{b\sin A}{c}(cx - x^2)$ (1)

$0 < x < c$

$\Rightarrow \quad \frac{dZ}{dx} = \frac{b\sin A}{c}(c-2x) = 0 \Rightarrow x = c/2$

$\left(\frac{d^2Z}{dx^2}\right)_{x=c/2} = \frac{-2}{c} b\sin A < 0$

$\Rightarrow$ Z has maxima at $x = \frac{c}{2}$, so the greatest area of parallelogram $AFDE$

$= (b/c)\sin A\,(c^2/4) = \frac{1}{2}\left(\frac{1}{2}bc\sin A\right)$

$= \frac{1}{2}\Delta_{ABC}$

$= \frac{1}{2} \times \frac{1}{2}\begin{vmatrix} p^2 & -p & 1 \\ q^2 & q & 1 \\ r^2 & -r & 1 \end{vmatrix}$

$= \frac{1}{4}\begin{vmatrix} p^2 & -p & 1 \\ q^2 - p^2 & q+p & 0 \\ r^2 - p^2 & -r+p & 0 \end{vmatrix}$ $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$= \frac{1}{4}(q+p)(q+r)(p-r)$

11. Given curve is $4x^2 + a^2y^2 = 4a^2$

$\Rightarrow \quad \frac{x^2}{a^2} + \frac{y^2}{4} = 1$ (1)

Let point $P(a\cos\phi, 2\sin\phi)$ be on (1), also given a point $Q(0, -2)$.

Let $u = (PQ)^2 = (a\cos\phi)^2 + (2\sin\phi + 2)^2$

Differentiating both sides w.r.t. ϕ, we have

$\frac{du}{d\phi} = \cos\phi\,[(8-2a^2)\sin\phi + 8]$

For the extremum value of u, $\frac{du}{d\phi} = 0$

$\Rightarrow \quad \phi = \frac{\pi}{2}$ and $\sin\phi = \frac{4}{a^2-4}$

$\because \quad 4 < a^2 < 8 \Rightarrow 0 < a^2 - 4 < 4$

$\Rightarrow \quad \frac{a^2-4}{4} < 1 \Rightarrow \frac{4}{a^2-4} > 1$

$\Rightarrow \quad \sin\phi > 1$ (not possible)

$\therefore \quad \phi = \pi/2$

Again, $\frac{d^2u}{d\phi^2} = (8-2a^2)\cos^2\phi + (2a^2-8)\sin^2\phi - 8\sin\phi$

$\therefore \quad \left.\frac{d^2u}{d\phi^2}\right|_{\phi=\pi/2} = 0 + (2a^2-8) - 8 = 2(a^2-8) < 0$

$(\because 4 < a^2 < 8)$

$\therefore \quad u$ is maximum at $\phi = \pi/2$.

So, $\sqrt{PQ}$ is also maximum at $\phi = \pi/2$.

Hence, co-ordinates of required point P are $(0, 2)$.

12. We have

$$f(x)=\int_1^x \left[2(t-1)(t-2)^3+(t-1)^2 3(t-2)^2\right] dt$$

$\Rightarrow$ $f'(x)=2(x-1)(x-2)^3+3(x-1)^2(x-2)^2$

$=(x-1)(x-2)^2(2x-4+3x-3)$

$=(x-1)(x-2)^2(5x-7)$

Critical points are $x=1, 2, 7/5$

Sign scheme of $f'(x)$ is

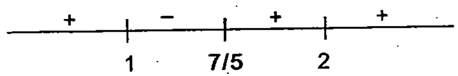


Fig. 6.146

Clearly $x=1$ is the point of maxima

$x=7/5$ is the point of minima

$x=2$ is the point of inflection (derivative does not change sign at $x=2$).

13. We have $y=x(x-1)^2, 0\le x\le 2$

$$\frac{dy}{dx}=(x-1)^2+2x(x-1)=(x-1)(3x-1)$$

Sign scheme of $f'(x)$ is

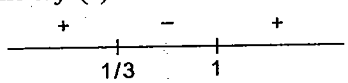


Fig. 6.147

Clearly $x=1$ is the point of minima(local) and $x=1/3$ is the point of maxima (local).

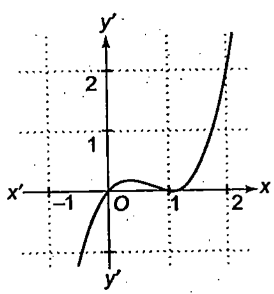


Fig. 6.148

14. Let $f(x)=2\sin x+\tan x-3x$

$\Rightarrow$ $f'(x)=2\cos x+\sec^2 x-3$

$=\sec^2 x(2\cos^3 x-3\cos^2 x+1)$

$=\sec^2 x(1-\cos x)^2(1+2\cos x)$

$\Rightarrow$ $f'(x)\ge 0, \forall\, 0\le x<\pi/2$

$\Rightarrow$ $f(x)$ is an increasing function of x, $\forall\, 0\le x<\pi/2$

$\therefore$ $f(x)\ge f(0)$ $\qquad (\because x>0)$

$\Rightarrow$ $f(x)\ge 0$

$\Rightarrow$ $2\sin x+\tan x\ge 3x, \forall 0\le x<\pi/2$.

15.

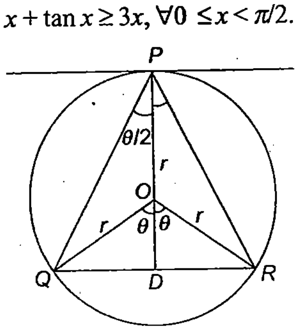


Fig. 6.149

Let O be the centre and r the radius of the circle.

Let QR be the chord parallel to the tangent at the point P on the circle.

Let $\angle QPR=\theta$, $\therefore$ $\angle QOD=\angle ROD=\theta$

Area of ΔPQR

$$=A=\frac{1}{2}(QR)(PD)=QD(OP+OD)$$

$$=r\sin\theta(r+r\cos\theta)$$

$$=\frac{1}{2}r^2(2\sin\theta+\sin 2\theta), 0<\theta\le\pi/2$$

$\Rightarrow$ $\dfrac{dA}{d\theta}=r^2(\cos\theta+\cos 2\theta)$,

and $\dfrac{d^2A}{d\theta^2}=-r^2(\sin\theta+2\sin 2\theta)$

For maximum or minimum of A, $dA/d\theta=0$

$\Rightarrow$ $\cos 2\theta+\cos\theta=0$

$\Rightarrow$ $2\cos^2\theta+\cos\theta-1=0$

$\Rightarrow$ $(2\cos\theta-1)(\cos\theta+1)=0$

$\Rightarrow$ $\cos\theta=1/2$ or $\cos\theta=-1$

$\Rightarrow$ $\theta=\pi/3$ $\qquad (\because 0<\theta<\pi/2)$

When $\theta=\dfrac{\pi}{3}$, $\dfrac{d^2A}{d\theta^2}=-\dfrac{3\sqrt{3}}{2}$ (−ve)

$\Rightarrow$ A is max., when $\theta=\pi/3$, the only critical point.

Thus, max. (greatest) area $A=\dfrac{1}{2}r^2[2\sin(\pi/3)+\sin(2\pi/3)]$

$$=\frac{1}{4}(3\sqrt{3})r^2.$$

16. Let x and y metres be the lengths of the sides of rectangle $ABCD$ and let there be a semi-circle on side CD of length x.

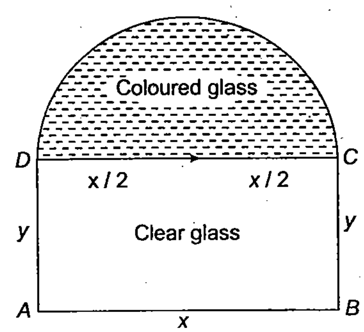


Fig. 6.150

Therefore, perimeter of the window (including the base of the arch) = perimeter of the rectangle + perimeter of the semi-circle

$$=2x+2y+\frac{1}{2}(2\pi x/2)$$

$$=2x+2y+\frac{1}{2}\pi x=c \text{ (constant)}$$

$$\therefore\quad y=\frac{1}{2}\left(c-2x-\frac{1}{2}\pi x\right) \qquad (1)$$

Let k be the light per square metre transmitted by coloured glass so that transmitted by clear glass will be $3k$ per square metre.

Hence, the total light transmitted by the window is given by

A = (Area of coloured glass) k + (Area of clear glass) $3k$

$$= \frac{1}{2}\pi(x/2)^2 k + xy\ (3k)$$

$$= \frac{1}{8}\pi kx^2 + 3kx\,\frac{1}{2}(c - 2x - \frac{1}{2}\pi x) \quad \text{[substituting for } y \text{ from (1)]}$$

$$= \frac{1}{8}k(-5\pi x^2 - 24x^2 + 12cx)$$

$$\therefore \quad \frac{dA}{dx} = \frac{1}{8}k(-10\pi x - 48x + 12c)$$

and $\frac{d^2A}{dx^2} = -\frac{1}{4}(5\pi + 24)k = -\text{ve}$

For maximum or minimum of A, $dA/dx = 0$

$\Rightarrow\ x = 6c/(5\pi + 24)$.

$\therefore$ From (1), $y = (\pi + 6)\,c/(5\pi + 24)$

Since $\frac{d^2A}{dx^2}$ is –ve, therefore A has maxima

Hence, ratio $x/y = 6(\pi + 6)$.

17. Let $f(x) = ax^3 + bx^2 + cx + d$

$f(x)$ vanishes at $x = -2$

$\Rightarrow\ -8a + 4b - 2c + d = 0$ (1)

and $f'(x) = 3a - 2b + c = 0$

Also, $f(x)$ has relative max./min at $x = -1$ and $x = \frac{1}{3}$

$\Rightarrow\ f'(-1) = 0 = f'(1/3)$

$\Rightarrow\ a + 2b + 3c = 0$ (2)

and $3a - 2b + c = 0$ (3)

Also, $\int_{-1}^{1} f(x)dx = \frac{14}{3}$

$$\Rightarrow \left(\frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx\right)_{-1}^{1} = \frac{14}{3}$$

$$\Rightarrow \left[\frac{a}{4} + \frac{b}{3} + \frac{c}{2} + d\right] - \left[\frac{a}{4} - \frac{b}{3} + \frac{c}{2} - d\right] = \frac{14}{3}$$

$$\Rightarrow \frac{b}{3} + d = \frac{7}{3}$$

$\Rightarrow\ b + 3d = 7$ (4)

From (1), (2), (3), (4) on solving, we get

$a = 1, b = 1, c = -1, d = 2$

$\therefore$ The required cubic is $x^3 + x^2 - x + 2$.

18. We have $f(x) = \begin{cases} -x^3 + \dfrac{(b^3 - b^2 + b - 1)}{(b^2 + 3b + 2)}, & 0 \le x < 1 \\ 2x - 3, & 1 \le x \le 3 \end{cases}$

Let $\dfrac{(b^3 - b^2 + b - 1)}{(b^2 + 3b + 2)} = a$ (constant)

Let $a = 0$, then $f(x) = \begin{cases} -x^3, & 0 \le x < 1 \\ 2x - 3, & 1 \le x \le 3 \end{cases}$

The graph is as follows

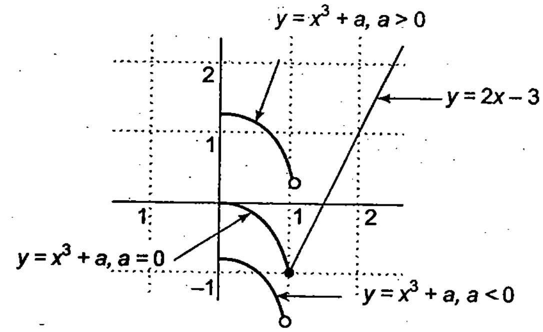


Fig. 6.151

If $a > 0$, then the graph of $-x^3 + a$ shifts upward.

If $a < 0$, then the graph of $-x^3 + a$ shifts downward.

For point of minima at $x = 1, a > 0$

$$\Rightarrow \frac{b^3 - b^2 + b - 1}{b^2 + 3b + 2} \ge 0$$

$$\Rightarrow \frac{(b^2 + 1)(b - 1)}{(b + 2)(b + 1)} \ge 0$$

$\Rightarrow\ (b - 1)(b + 1)(b + 2) \ge 0$

Sign Scheme is

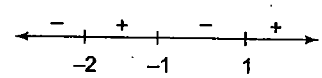
$\Rightarrow$ 

Fig. 6.144

$\Rightarrow\ b \in (-2, -1) \cup (1, \infty)$.

19.

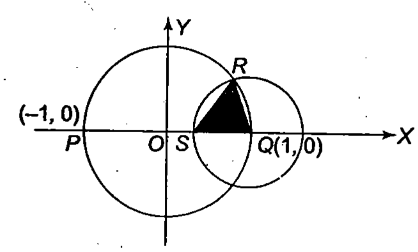


Fig. 6.152

$\because$ Circle $x^2 + y^2 = 1$ cuts x-axis at P and Q, then $P \equiv (-1, 0)$ and $Q \equiv (1, 0)$.

Equation of circle with centre at $Q\,(1, 0)$ and having variable radius r is $(x - 1)^2 + (y - 0)^2 = r^2$ or $(x - 1)^2 + y^2 = r^2$

Solving two curves

$\Rightarrow\ (x - 1)^2 + 1 - x^2 = r^2$ $[\because x^2 + y^2 = 1]$

$\Rightarrow\ (2x - 1) = 1 - r^2$

$\Rightarrow\ x = 1 - \frac{r^2}{2}$ and $y = \pm\sqrt{(1 - x^2)}$

$$\Rightarrow\ y = \sqrt{1 - \left(1 - \frac{r^2}{2}\right)^2} = \sqrt{\left(r^2 - \frac{r^4}{4}\right)}$$

($\because$ R is above the x-axis)

$\Rightarrow \quad A = \text{Area of triangle } QSR = \frac{1}{2}\, r\sqrt{\left(r^2 - \frac{r^4}{4}\right)}$

$\Rightarrow \quad A^2 = \frac{r^4}{4} - \frac{r^6}{16} = z \text{ (say)} \Rightarrow \frac{dz}{dr} = r^3 - \frac{6r^5}{16}$

For maximum or minimum of z, $\frac{dz}{dr} = 0$

$\Rightarrow \quad r = \sqrt{\left(\frac{8}{3}\right)}$ and $\frac{d^2z}{dr^2} = 3r^2 - \frac{30r^4}{16}$

$\Rightarrow \quad \left.\frac{d^2z}{dr^2}\right|_{r=\sqrt{8/3}} = 3\left(\frac{8}{3}\right) - \frac{15}{8}\,\frac{64}{9} = -\frac{16}{3} < 0$

$\Rightarrow \quad z$ is maximum $\Rightarrow A$ is also maximum when $r = \sqrt{\left(\frac{8}{3}\right)}$.

$\Rightarrow \quad \text{Maximum area of } \Delta QSR = \frac{r}{2}\sqrt{\left(r^2 - \frac{r^4}{4}\right)}$

$= \frac{1}{2}\sqrt{\frac{8}{3}}\sqrt{\left(\frac{8}{3} - \frac{64}{36}\right)}$

$= \frac{\sqrt{2}}{\sqrt{3}}\,\frac{2\sqrt{2}}{3} = \left(\frac{4}{3\sqrt{3}}\right)$ sq.units.

20.

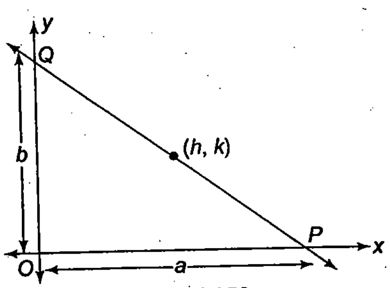


Fig. 6.153

Let the line in intercepts form be $\frac{x}{a} + \frac{y}{b} = 1$

It passes through $(h, k) \Rightarrow \frac{h}{a} + \frac{k}{b} = 1$

$\Rightarrow \quad \frac{k}{b} = 1 - \frac{h}{a} = \frac{a-h}{a} \Rightarrow b = \frac{ak}{a-h}$

$\Delta = \frac{1}{2}\,ab = \frac{1}{2}\,a\frac{ak}{a-h} = \frac{1}{2}\,\frac{k}{\frac{a-h}{a^2}}$ (1)

Δ is minimum when $y = \frac{a-h}{a^2} = \frac{1}{a} - \frac{h}{a^2}$ is max.

$\Rightarrow \quad \frac{dy}{da} = -\frac{1}{a^2} + \frac{2h}{a^3} = 0 \Rightarrow a = 2h$ (2)

$\frac{d^2y}{da^2} = \frac{2}{a^3} - \frac{6h}{a^4} = \frac{2}{a^3} - \frac{3}{a^3}$ (by (2))

$\Rightarrow \quad \frac{d^2y}{da^2} = -\frac{1}{a^3} = -\text{ive} \therefore \text{max.}$

Now, put $a = 2h$ in (1), $\Delta = \frac{1}{2}\,4h^2\,\frac{k}{h} = 2hk.$

21. Here $f(x) = \frac{1}{8}\log x - bx + x^2$ is defined and continuous for all $x > 0$.

Then $f'(x) = \frac{1}{8x} - b + 2x$

or $f'(x) = \frac{16x^2 - 8bx + 1}{8x}$

for extrema, let $f'(x) = 0$

$\Rightarrow \quad 16x^2 - 8bx + 1 = 0$

so, $x = \frac{8b \pm \sqrt{64(b^2-1)}}{2\times 16}$ or $x = \frac{b \pm \sqrt{b^2-1}}{4}$.

Obviously, the roots are real if $b^2 - 1 \geq 0$

$\Rightarrow \quad b > 1$ [as $x > 0$]

Sign scheme of $f'(x)$ as shown in Fig. 6.154.

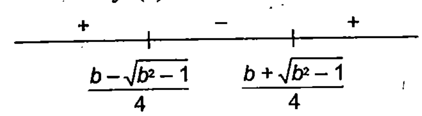


Fig. 6.154

$f'(x)$ changes sign from + ve to − ve at $x = \frac{b - \sqrt{b^2-1}}{4}$

$\therefore \quad f(x)_{\max}$ at $x = \frac{b - \sqrt{b^2-1}}{4}$

and $f'(x)$ changes sign from − ve to + ve at $x = \frac{b + \sqrt{b^2-1}}{4}$

$\therefore \quad f(x)_{\min}$ at $x = \frac{b + \sqrt{b^2-1}}{4}$

also if $b = 1$

$f'(x) = \frac{16x^2 - 8x + 1}{x} = \frac{(4x-1)^2}{x}$ no changes in sign.

$\therefore$ neither maximum or minimum if $b = 1$

Thus, $f(x)$

$$= \begin{cases} f(x)_{\max.}, \text{ when } x = \frac{b - \sqrt{b^2-1}}{4} \text{ and } b > 1 \\ f(x)_{\min.}, \text{ when } x = \frac{b + \sqrt{b^2-1}}{4} \text{ and } b > 1 \\ f(x) \text{ neither maximum nor minimum, when } b = 1 \end{cases}$$

22. Given that $f(x) = \begin{cases} xe^{ax}, & x \leq 0 \\ x + ax^2 - x^3, & x > 0 \end{cases}$

Differentiating both sides, we have

$f'(x) = \begin{cases} axe^{ax} + e^{ax}, & x < 0 \\ 1 + 2ax - 3x^2, & x > 0 \end{cases}$

Again differentiating both sides, we have

$f''(x) = \begin{cases} 2ae^{ax} + a^2 x e^{ax}; x < 0 \\ 2a - 6x\,;\, x > 0 \end{cases}$

For critical points, we put $f''(x) = 0$

$\Rightarrow x = -\dfrac{2}{a}$ if $x < 0$ and $x = \dfrac{a}{3}$ if $x > 0$

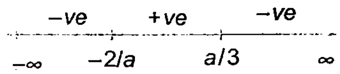


Fig. 6.155

It is clear from number line that $f''(x)$ is + ive in $\left(-\dfrac{2}{a}, \dfrac{a}{3}\right)$.

$\Rightarrow$ $f'(x)$ increases in $\left(-\dfrac{2}{a}, \dfrac{a}{3}\right)$.

23. Applying $R_3 \to R_3 - R_1 - 2R_2$

we get $f'(x) = \begin{vmatrix} 2ax & 2ax - a & 2ax + b + 1 \\ b & b+1 & -1 \\ 0 & 0 & 1 \end{vmatrix}$

$= \begin{vmatrix} 2ax & 2ax - 1 \\ b & b+1 \end{vmatrix} = \begin{vmatrix} 2ax & -1 \\ b & 1 \end{vmatrix}$ [Using $C_2 \to C_2 - C_1$]

$\Rightarrow$ $f'(x) = 2ax + b$

Integrating, we get $f(x) = ax^2 + bx + c$ where c is an arbitrary constant.

Since, f has a maximum at $x = 5/2$

$f'(5/2) = 0 \Rightarrow 5a + b = 0$ (1)

Also $f(0) = 2 \Rightarrow c = 2$

And $f(1) = 1 \Rightarrow a + b + c = 1$

$\therefore$ $a + b = -1$ (2)

Solving (1) and (2) for a, b we get, $a = 1/4, b = -5/4$

Thus, $f(x) = \dfrac{1}{4}x^2 - \dfrac{5}{4}x + 2.$

24. Given $-1 \le p \le 1$ and equation $4x^3 - 3x - p = 0$

Also, $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

Then let $x = \cos\theta$

Then $4x^3 - 3x - p = 0 \Rightarrow 4\cos^3\theta - 3\cos\theta - p = 0$

$\Rightarrow$ $\cos 3\theta = p$

Since, $x = \cos\theta \in [1/2, 1]$

$\Rightarrow$ $\theta \in [0, \pi/3]$

$\Rightarrow$ $3\theta \in [0, \pi]$ for which $\cos 3\theta = p \in [-1, 1]$

Hence proved.

Also $\cos 3\theta = p$

$\Rightarrow$ $3\theta = \cos^{-1} p$

$\Rightarrow$ $\theta = \dfrac{1}{3}\cos^{-1}(p)$

$\Rightarrow$ $x = \cos\theta = \cos\left(\dfrac{1}{3}\cos^{-1}(p)\right).$

25. Given that $2(1 - \cos x) < x^2, x \ne 0$

To prove $\sin(\tan x) \ge x, x \in [0, \pi/4]$

Let us consider $f(x) = \sin(\tan x) - x$

$\Rightarrow$ $f'(x) = \cos(\tan x)\sec^2 x - 1$

$= \dfrac{\cos(\tan x) - \cos^2 x}{\cos^2 x}$

As given $2(1 - \cos x) < x^2, x \ne 0$

$\Rightarrow$ $\cos x > 1 - \dfrac{x^2}{2}$

Similarly, $\cos(\tan x) > 1 - \dfrac{\tan^2 x}{2}$

$\therefore$ $f'(x) > \dfrac{1 - \frac{1}{2}\tan^2 x - \cos^2 x}{\cos^2 x}$

$= \dfrac{\sin^2 x\left[1 - \dfrac{1}{2\cos^2 x}\right]}{\cos^2 x}$

$= \dfrac{\sin^2 x(\cos 2x)}{2\cos^4 x} > 0, \forall x \in [0, \pi/4]$

$\therefore$ $f'(x) > 0 \Rightarrow f(x)$ is an increasing function

$\therefore$ For $x \in [0, \pi/4]$

we have $x \ge 0 \Rightarrow f(x) \ge f(0)$

$\Rightarrow$ $\sin(\tan x) - x \ge \sin(\tan 0) - 0$

$\Rightarrow$ $\sin(\tan x) - x \ge 0$

$\Rightarrow$ $\sin(\tan x) \ge x.$

26. Given that $\dfrac{dP(x)}{dx} > P(x), \forall x \ge 1$ and $P(1) = 0$

$\Rightarrow$ $\dfrac{dP(x)}{dx} - P(x) > 0$

Multiplying by e^{-x}, we get

$e^{-x}\dfrac{dP(x)}{dx} - e^{-x}P(x) > 0$

$\Rightarrow$ $\dfrac{d}{dx}\left[e^{-x}P(x)\right] > 0$

$\Rightarrow$ $e^{-x}P(x)$ is an increasing function.

$\therefore$ $\forall x > 1, e^{-x}P(x) > e^{-1}P(1) = 0$ [Using $P(1) = 0$]

$\Rightarrow$ $e^{-x}P(x) > 0, \forall x > 1$

$\Rightarrow$ $P(x) > 0, \forall x > 1.$ [$\because e^{-x} > 0$]

27. Let $f(x) = \sin x + 2x - \dfrac{3x(x+1)}{\pi}$

$\Rightarrow$ $f'(x) = \cos x + 2 - \dfrac{3}{\pi}(2x + 1)$

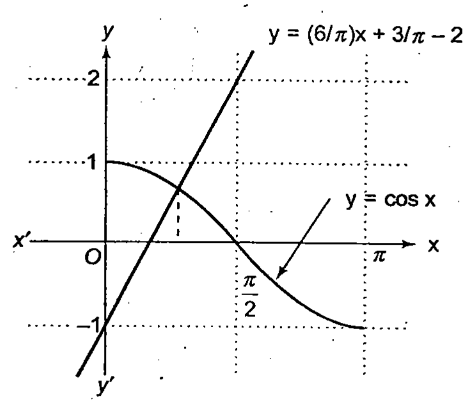


Fig. 6.156

$\Rightarrow \quad f''(x) = -\sin x - \dfrac{3}{\pi}(2) < 0$

$\Rightarrow \quad f'(x)$ is a decreasing function.

Also, for $x \in \left[0, \dfrac{\pi}{2}\right]$ if $f'(x) = 0$, then $\cos x = \dfrac{6}{\pi}x + \dfrac{3}{\pi} - 2$

As graph of $y = \cos x$ and $y = \dfrac{6}{\pi}x + \dfrac{3}{\pi} - 2$ intersect only once.

$f'(x) = 0$ has one root in $\left(0, \dfrac{\pi}{2}\right)$

Also $f'(x)$ changes its sign from + ve to – ve.

Hence, graph of $f(x)$ is as follows.

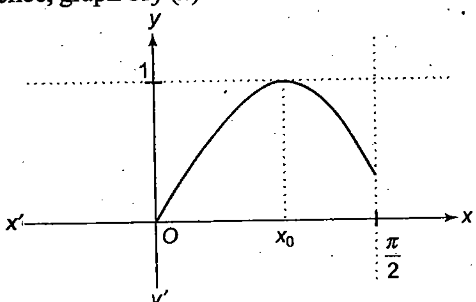


Fig. 6.157

$\Rightarrow \quad f(x) \geq 0 \Rightarrow \sin x + 2x - \dfrac{3x}{\pi}(x+1) \geq 0$

Also $f(x)$ has a point of maxima.

28. Let $p(x) = ax^3 + bx^2 + cx + d$

$p(-1) = 10 \Rightarrow -a + b - c + d = 10$ (1)

$p(1) = -6 \Rightarrow a + b + c + d = -6$ (2)

$p(x)$ has max. at $x = -1 \Rightarrow p'(-1) = 0$

$\Rightarrow \quad 3a - 2b + c = 0$ (3)

$p'(x)$ has min. at $x = 1 \Rightarrow p''(1) = 0$

$\Rightarrow \quad 6a + 2b = 0$ (4)

Solving (1), (2), (3) and (4), we get

From (4), $b = -3a$

from (3), $3a + 6a + c = 0 \Rightarrow c = -9a$

from (2) $a - 3a - 9a + d = -6 \Rightarrow d = 11a - 6$

from (1), $-a - 3a + 9a + 11a - 6 = 10$

$\Rightarrow \quad 16a = 16 \Rightarrow a = 1$

$\Rightarrow \quad b = -3, c = -9, d = 5$

$\Rightarrow \quad p(x) = x^3 - 3x^2 - 9x + 5$

$p'(x) = = 0 \Rightarrow 3x^2 - 6x - 9 = 0$

$\Rightarrow \quad 3(x+1)(x-3) = 0$

$\Rightarrow \quad x = -1$ is a point of maxima (given)

and $x = 3$ is a point of minima

[$\because$ maxima and minima occur alternatively]

$\therefore$ point of local maxima is $(-1, 10)$ and local minima is $(3, -22)$.

And the distance between them is

$= \sqrt{[3-(-1)]^2 + (-22-10)^2}$

$= \sqrt{16 + 1024}$

$= \sqrt{1040} = 4\sqrt{65}$.

Objective

Fill in the blanks

1. We have $e^{-\pi/2} < \theta < \pi/2$

$\Rightarrow \quad -\dfrac{\pi}{2} < \ln\theta < \ln \pi/2$

$\Rightarrow \quad \cos(-\pi/2) < \cos(\ln\theta) < \cos(\ln \pi/2)$

[$\because$ cos x is increasing in 4th quadrant]

$\Rightarrow \quad \cos(\ln\theta) > 0$ (1)

Also, $-1 \leq \cos\theta \leq 1 \ \forall \ \theta \in R$

$\therefore \quad -\infty < \ln(\cos\theta) \leq 0, \forall \ 0 < \cos\theta \leq 1$

$\Rightarrow \quad \ln(\cos\theta) \leq 0$ (2)

From (1) and (2), we get $\cos(\ln\theta) > \ln(\cos\theta)$

$\therefore \quad \cos(\ln\theta)$ is larger.

2. $y = 2x^2 - \ln|x|$

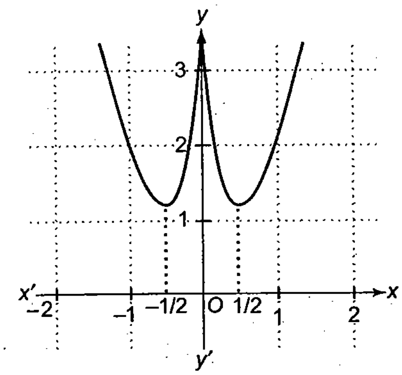


Fig. 6.158

$\dfrac{dy}{dx} = 4x - \dfrac{1}{x} = \dfrac{(2x+1)(2x-1)}{x}$

Critical pts are 0, 1/2, –1/2

Sign scheme of $\dfrac{dy}{dx}$ is

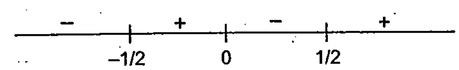


Fig. 6.159

Clearly $f(x)$ increases in $(-1/2, 0) \cup (1/2, \infty)$ and $f(x)$ decreases in $(-\infty, -1/2) \cup (0, 1/2)$.

3. Let $f(x) = \log(1+x) - x$ for $x > -1$

$\Rightarrow \quad f'(x) = \dfrac{1}{1+x} - 1 = \dfrac{-x}{1+x}$

$\Rightarrow \quad f'(x) > 0$ for $-1 < x < 0$ and $f'(x) < 0$ for $x > 0$

Therefore, f increases in $(-1, 0)$ and decreases in $(0, \infty)$

Also, $f(0) = \log 1 - 0 = 0$

$\therefore \quad x \geq 0 \Rightarrow f(x) \leq f(0)$

$\Rightarrow \quad \log(1+x) - x \leq 0$

$\Rightarrow \quad \log(1+x) \leq x$

Thus, we get $\log_e(1+x) \leq x, \forall \ x \geq 0$.

4. $f'(x) = 6(x-2)(x-3)$

So, $f(x)$ is increasing in $(-\infty, 2) \cup (3, \infty)$

Also, $A = \{4 \leq x \leq 5\}$

$\therefore f_{max} = f(5) = 7$

5. $f(x) = \frac{3}{2}(x)^{1/2}(3x-10) + (x)^{3/2} \times 3$

$= \frac{15}{2}(x)^{1/2}(x-2)$

$\therefore f(x)$ is increasing, when $x \geq 2$.

True or false

1. If $(x-r)$ is a factor of $f(x)$ repeated m times then $f'(x)$ is a polynomial with $(x-r)$ as factor repeated $(m-1)$ times. Therefore, statement is false.
2. Given that $0 < a < x$

   Let $f(x) = \log_a x + \log_x a = \log_a x + \frac{1}{\log_a x}$

   Consider $g(y) = y + \frac{1}{y}$ where $\log_a x = y$

   $\therefore y + \frac{1}{y} = \left(\sqrt{y} - \frac{1}{\sqrt{y}}\right)^2 + 2 \geq 2$

   But equality holds when $y = 1$

   $\Rightarrow x = a$ which is not possible

   Therefore, $y + \frac{1}{y} > 2 \Rightarrow g_{min}$ cannot be 2.

   Therefore, f_{min} cannot be 2.

   Therefore, statement is false.

Multiple choice questions with one correct answer

1. a

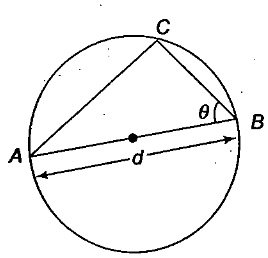


Fig. 6.160

Area of ΔABC, $A = \frac{1}{2} AC \times BC$

$= \frac{1}{2}(d \sin\theta)(d \cos\theta)$, where $\theta \in (0, \pi/2)$

$= \frac{d^2}{4}\sin 2\theta$

which is maximum when $\sin 2\theta = 1$ or $\theta = \pi/4$

Hence, $AC = BC$, then the triangle is isosceles.

2. b We have $f(x) = a \log|x| + bx^2 + x$

$\Rightarrow f'(x) = \frac{a}{x} + 2bx + 1$

Since, $f(x)$ attains its extremum values at $x = -1, 2$

$\Rightarrow f'(-1) = 0$ and $f'(2) = 0$

$\Rightarrow -a - 2b + 1 = 0$ and $\frac{a}{2} + 4b + 1 = 0 \Rightarrow a = 2$ and $b = -1/2$.

3. b $f'(x) = \dfrac{x \ln\left(\frac{e+x}{\pi+x}\right) + (e\ln(e+x) - \pi\ln(\pi+x))}{(\ln(e+x))^2(\pi+x)(e+x)}$

Now $\pi + x > e + x \Rightarrow \ln\left(\frac{e+x}{\pi+x}\right) < 0 \Rightarrow f'(x) < 0$

and also $e\ln(e+x) < \pi\ln(\pi+x) \Rightarrow f'(x) < 0$

Thus, $f(x)$ is decreasing.

4. b. Let $y = x^{25}(1-x)^{75}$

$\Rightarrow \frac{dy}{dx} = 25x^{24}(1-x)^{75} - 75x^{25}(1-x)^{74}$

$= 25x^{24}(1-x)^{74}(1-x-3x)$

$= 25x^{24}(1-x)^{74}(1-4x)$

Clearly, critical points are 0, 1/4 and 1.

Sign scheme of $\frac{dy}{dx}$

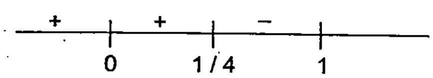


Fig. 6.161

Thus, $x = 1/4$ is the point of maxima.

5. c. We have $f(x) = \frac{x}{\sin x}$, $0 < x \leq 1$

$\Rightarrow f'(x) = \frac{\sin x - x\cos x}{\sin^2 x}$

$= \frac{\cos x(\tan x - x)}{\sin^2 x}$

We know that $\tan x > x$ for $0 < x < \pi/2$

$\Rightarrow f'(x) > 0$ for $0 < x \leq 1$

Hence, $f(x)$ is an increasing function.

$g(x) = \frac{x}{\tan x}$

$\Rightarrow g'(x) = \frac{\tan x - x\sec^2 x}{\tan^2 x} = \frac{\sin x\cos x - x}{\sin^2 x}$

$= \frac{\sin 2x - 2x}{2\sin^2 x}$

$= \frac{\sin\theta - \theta}{2\sin^2(\theta/2)}$, where $\theta \in (0, 2)$

We know that $\sin\theta < \theta$ for $\forall\ \theta > 0$

$\Rightarrow g'(x) < 0 \Rightarrow g(x)$ is a decreasing function.

6. b. We have $f(x) = \sin^4 x + \cos^4 x = \frac{3}{4} + \frac{1}{4}\cos 4x$

$\Rightarrow f'(x) = -\sin 4x$

Now, $f'(x) > 0 \Rightarrow -\sin 4x > 0 \Rightarrow \sin 4x < 0 \Rightarrow \pi < 4x < 2\pi$

$\Rightarrow \pi/4 < x < \pi/2$.

7. d. From the graph, it is clear that both $\sin x$ and $\cos x$ in the internal $(\pi/2, \pi)$ are the decreasing functions.

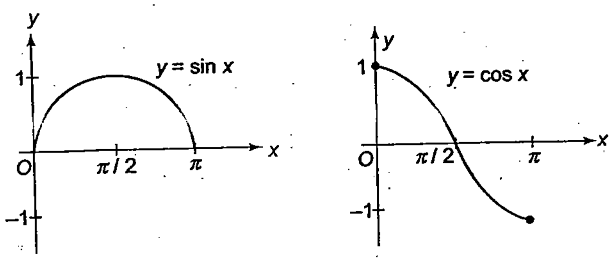


Fig. 6.162

Therefore, S is correct.
To disprove R let us consider the counter example,
$f(x) = \sin x$ in $(0, \pi/2)$
so that $f'(x) = \cos x$
again from the graph, it is clear that $f(x)$ is increasing in $(0, \pi/2)$, but $f'(x)$ is decreasing in $(0, \pi/2)$
Therefore, R is wrong. Therefore, **d.** is the correct option.

8. c. $f(x) = \int e^x (x-1)(x-2)\,dx$

For decreasing function, $f'(x) < 0$

$\Rightarrow\ e^x (x-1)(x-2) < 0$

$\Rightarrow\ (x-1)(x-2) < 0 \Rightarrow 1 < x < 2. \qquad (\because e^x > 0\ \forall x \in R)$

9. a.

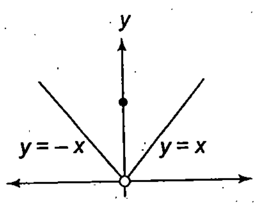


Fig. 6.163

From the graph $f(0^+) < f(0)$ and $f(0^-) < 0 \Rightarrow x = 0$ is the point of maxima.

10. b. $y = e^x \Rightarrow \dfrac{dy}{dx} = e^x$

Then, equation of the tangent at $x = 0$ is $y - 1 = 1(x - 0)$ or $y = x + 1$

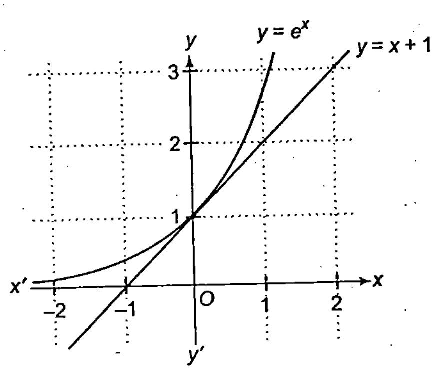


Fig. 6.164

Graph of $y = e^x$ always lies above the graph of $y = 1 + x$,
Hence, $e^x > 1 + x \Rightarrow x > \log_e(1 + x)$. Hence, **b.** is true.
c. is wrong as $\sin x < x$ for $x \in (0, 1)$
and **d.** is wrong as $x > \log_e x$ for $\forall\ x > 0$.

11. a. $f(x) = xe^{x(1-x)}$

$\Rightarrow\ f'(x) = e^{x(1-x)} + (1-2x)\,xe^{x(1-x)}$

$= -e^{x(1-x)}(2x^2 - x - 1)$

$= -e^{x(1-x)}(2x + 1)(x - 1)$

Sign scheme of $f'(x)$

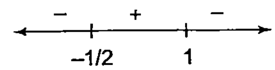


Fig. 6.165

$\therefore\ g(x)$ is increasing in $[-1/2, 1]$.

12. d. $f(x) = (1 + b^2)x^2 + 2bx + 1$

The graph of $f(x)$ is upward parabola as coefficient of x^2 is $1 + b^2 > 0$.

$\Rightarrow$ The range of $f(x)$ is $\left[\dfrac{-D}{4a}, \infty\right)$, where D is discriminant of $f(x)$.

$$\Rightarrow\ m(b) = -\frac{4b^2 - 4(1 + b^2)}{4(1 + b^2)}$$

$$\Rightarrow\ m(b) = \frac{1}{1 + b^2} \in (0, 1].$$

13. a. $3\sin x - 4\sin^3 x = \sin 3x$ which increases for $3x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) \Rightarrow x \in \left(-\dfrac{\pi}{6}, \dfrac{\pi}{6}\right)$ whose length is $\dfrac{\pi}{3}$.

14. b. Equation of the tangent to the ellipse $\dfrac{x^2}{27} + y^2 = 1$ at $\left(3\sqrt{3}\cos\theta, \sin\theta\right)$, $\theta \in (0, \pi/2)$

is $\dfrac{\sqrt{3}\,x\cos\theta}{9} + y\sin\theta = 1$

$\therefore$ Sum of the intercepts $= S = 3\sqrt{3}\sec\theta + \operatorname{cosec}\theta$

For minimum values of S, $\dfrac{dS}{d\theta} = 0$

$\Rightarrow\ 3\sqrt{3}\sec\theta\tan\theta - \operatorname{cosec}\theta\cot\theta = 0$

$$\Rightarrow\ \frac{3\sqrt{3}\sin\theta}{\cos^2\theta} - \frac{\cos\theta}{\sin^2\theta} = 0$$

$\Rightarrow\ 3\sqrt{3}\sin^3\theta - \cos^3\theta = 0$

$$\Rightarrow\ \tan^3\theta = \frac{1}{3\sqrt{3}} = \left(\frac{1}{\sqrt{3}}\right)^3$$

$$\Rightarrow\ \tan\theta = \frac{1}{\sqrt{3}} = \tan\pi/6 \Rightarrow \theta = \pi/6.$$

15. a. $f(x) = x^3 + bx^2 + cx + d,\ 0 < b^2 < c$

$f'(x) = 3x^2 + 2bx + c$

Discriminant $= 4b^2 - 12c = 4(b^2 - 3c) < 0$

$\therefore\ f'(x) > 0\ \forall x \in R$

$\Rightarrow\ f(x)$ is strictly increasing $\forall\ x \in R$.

16. d. $f'(x) = -(x + 2)e^{-x} + e^{-x} = -(x + 1)e^{-x} = 0$

$\Rightarrow\ x = -1$

For $x \in (-\infty, -1)$, $f'(x) > 0$ and for $x \in (-1, \infty)$, $f'(x) < 0$

$\therefore$ $f(x)$ is increasing in $(-\infty, -1)$ and decreasing in $(-1, \infty)$.

Multiple choice question with one or more than one correct answer

1. c. The given polynomial is $p(x) = a_0 + a_1x^2 + a_2x^4 + \cdots + a_nx^{2n}, x \in R$ and $0 < a_0 < a_1 < a_2 < \cdots < a_n$

Here we observe that all coefficients of different powers of x, i.e., $a_0, a_1, a_2, \ldots, a_n$ are positive.

Also, only even powers of x are involved.

Therefore, $P(x)$ cannot have any maximum value.

Moreover, $P(x)$ is minimum, when $x = 0$, i.e., a_0

Therefore, $P(x)$ has only one minimum.

Alternative method

We have

$P'(x) = 2a_1x + 4a_2x^3 + \cdots + 2na_nx^{2n-1}$

$= x(2a_1 + 4a_2x^2 + \cdots + 2na_nx^{2n-2})$

Clearly $P'(x) > 0$ for $x > 0$ and $P'(x) < 0$ for $x < 0$

$\Rightarrow$ $P(x)$ increases for all $x > 0$ and decreases for all $x < 0$

Therefore, $P'(x)$ has $x = 0$ as the point of maxima.

2. c.

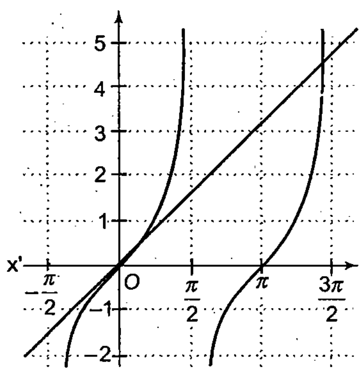


Fig. 6.166

It is clear from the graph that the curves $y = \tan x$ and $y = x$ intersect at P in $(\pi, 3\pi/2)$.

Thus, the smallest + ve root of $\tan x - x = 0$ is $(\pi, 3\pi/2)$.

3. a. Since g is decreasing in $[0, \infty)$

$\therefore$ For $x \geq y \geq 0, g(x) \leq g(y)$ (1)

Also $g(x), g(y) \in [0, \infty)$ and f is increasing from $[0, \infty)$ to $[0, \infty)$

$\therefore$ For $g(x), g(y) \in [0, \infty)$

such that $g(x) \leq g(y)$

$\Rightarrow$ $f(g(x)) \leq f(g(y))$ where $x \geq y$

$\Rightarrow$ $h(x) \leq h(y)$

$\Rightarrow$ h is a decreasing function from $[0, \infty)$ to $[0, \infty)$

$\therefore h(x) \leq h(0), \forall x \geq 0$

But $h(0) = 0$ (given)

$\therefore$ $h(x) \leq 0, \forall x \geq 0$ (2)

Also $h(x) \geq 0, \forall x \geq 0$ (3)

[as $h(x) \in [0, \infty)$]

From (2) and (3) we get $h(x) = 0, \forall x \geq 0$

Hence, $h(x) - h(1) = 0 - 0 = 0, \forall x \geq 0$.

4. a, b, c, d.

We are given that $f(x) = \begin{cases} 3x^2 + 12x - 1, -1 \leq x \leq 2 \\ 37 - x, 2 < x \leq 3 \end{cases}$

Then in $[-1, 2]$, $f'(x) = 6x + 12$

$f'(x) = 0 \Rightarrow x = -2$

$\Rightarrow$ $f(x)$ decreases in $(-\infty, -2)$ and increases in $(-2, \infty)$

Also $f(2^-) = 3(2)^2 + 12(2) - 1 = 35$

and $f(2^+) = 37 - 2 = 35$

Hence $f(x)$ is continuous.

$f'(x) = \begin{cases} 6x + 12, -1 < x < 2 \\ -1, 2 < x < 3 \end{cases}$

$\Rightarrow$ $f'(2^-) = 24$ and $f'(2^+) = -1$

Hence, $f(x)$ is non-differentiable at $x = 2$.

Also, $f(2^+) < f(2)$ and $f(2^-) < f(2)$

Hence, $x = 2$ is the point of maxima.

5. a, c. We have $h'(x) = f'(x)[1 - 2f(x) + 3f(x)^2]$

$= 3f'(x)\left[(f(x))^2 - \frac{2}{3}f(x) + \frac{1}{3}\right]$

$= 3f'(x)[(f(x) - 1/3)^2 + 2/9]$

Note that $h'(x) < 0$ whenever $f'(x) < 0$ and $h'(x) > 0$ whenever $f'(x) > 0$,

thus $h(x)$ increases (decreases) whenever $f(x)$ increases (decreases)

$\therefore$ **(a)** and **(c)** are the correct options.

6. d. $f(x) = \dfrac{x^2 - 1}{x^2 + 1} = \dfrac{(x^2 + 1) - 2}{x^2 + 1} = 1 - \dfrac{2}{x^2 + 1}$

For $f(x)$ to be min $\dfrac{2}{x^2 + 1}$ should be max, which is so if $x^2 + 1$ is min and $x^2 + 1$ is min at $x = 0$.

$\therefore$ $f_{\min} = \dfrac{0 - 1}{0 + 1} = -1$.

7. b. The maximum value of $f(x) = \cos x + \cos(\sqrt{2}x)$ occurs when $\cos x = 1$ and $\cos(\sqrt{2}x) = 1$

$\Rightarrow$ $x = 2n\pi, n \in Z$ and $\sqrt{2}x = 2m\pi, m \in Z$

Comparing the value of x, $2n\pi = \dfrac{2m\pi}{\sqrt{2}} \Rightarrow m = n = 0 \Rightarrow x = 0$ only.

8. b, d. $f(x) = \int_{-1}^{x} t(e^t - 1)(t - 1)(t - 2)^3(t - 3)^5 dt$

$\Rightarrow$ $f'(x) = x(e^x - 1)(x - 1)(x - 2)^3(x - 3)^5$

The critical points are 0, 1, 2, 3.

Sign scheme of $f'(x)$

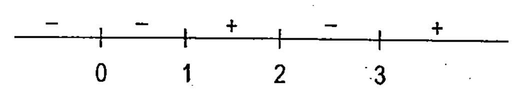


Fig. 6.167

Clearly $x = 1$ and $x = 3$ are the points of minima.

9. b, c. Let $f(x) = ax^3 + bx^2 + cx + d$

$f(2) = 18 \Rightarrow 8a + 4b + 2c + d = 18$ (1)

$f(1) = -1 \Rightarrow a + b + c + d = -1$ (2)

$f(x)$ has local max at $x = -1$

$\Rightarrow$ $f'(-1) = 0 \Rightarrow 3a - 2b + c = 0$ (3)

$f'(x)$ has local min. at $x = 0$

$\Rightarrow$ $f''(0) = 0 \Rightarrow b = 0$ (4)

$\therefore$ $f(x)$ is increasing in $(-\infty, -1)$ and decreasing in $(-1, \infty)$.

Multiple choice question with one or more than one correct answer

1. c. The given polynomial is $p(x) = a_0 + a_1x^2 + a_2x^4 + \cdots + a_n x^{2n}, x \in R$ and $0 < a_0 < a_1 < a_2 < \cdots < a_n$
Here we observe that all coefficients of different powers of x, i.e., $a_0, a_1, a_2, \ldots, a_n$ are positive.
Also, only even powers of x are involved.
Therefore, $P(x)$ cannot have any maximum value.
Moreover, $P(x)$ is minimum, when $x = 0$, i.e., a_0
Therefore, $P(x)$ has only one minimum.

Alternative method

We have
$P'(x) = 2a_1x + 4a_2x^3 + \cdots + 2na_nx^{2n-1}$
$= x(2a_1 + 4a_2x^2 + \cdots + 2na_nx^{2n-2})$
Clearly $P'(x) > 0$ for $x > 0$ and $P'(x) < 0$ for $x < 0$
$\Rightarrow$ $P(x)$ increases for all $x > 0$ and decreases for all $x < 0$
Therefore, $P'(x)$ has $x = 0$ as the point of maxima.

2. c.

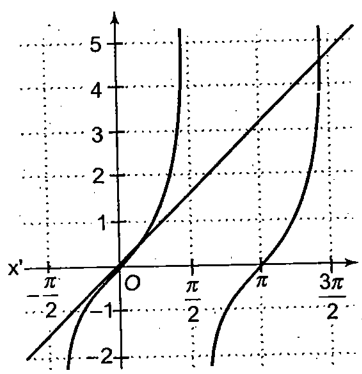


Fig. 6.166

It is clear from the graph that the curves $y = \tan x$ and $y = x$ intersect at P in $(\pi, 3\pi/2)$.
Thus, the smallest + ve root of $\tan x - x = 0$ is $(\pi, 3\pi/2)$.

3. a. Since g is decreasing in $[0, \infty)$
$\therefore$ For $x \geq y \geq 0, g(x) \leq g(y)$ (1)
Also $g(x), g(y) \in [0, \infty)$ and f is increasing from $[0, \infty)$ to $[0, \infty)$
$\therefore$ For $g(x), g(y) \in [0, \infty)$
such that $g(x) \leq g(y)$
$\Rightarrow$ $f(g(x)) \leq f(g(y))$ where $x \geq y$
$\Rightarrow$ $h(x) \leq h(y)$
$\Rightarrow$ h is a decreasing function from $[0, \infty)$ to $[0, \infty)$
$\therefore h(x) \leq h(0), \forall x \geq 0$
But $h(0) = 0$ (given)
$\therefore$ $h(x) \leq 0, \forall x \geq 0$ (2)
Also $h(x) \geq 0, \forall x \geq 0$ (3)
[as $h(x) \in [0, \infty)$]
From (2) and (3) we get $h(x) = 0, \forall x \geq 0$
Hence, $h(x) - h(1) = 0 - 0 = 0, \forall x \geq 0$.

4. a, b, c, d.

We are given that $f(x) = \begin{cases} 3x^2 + 12x - 1, -1 \leq x \leq 2 \\ 37 - x, 2 < x \leq 3 \end{cases}$

Then in $[-1, 2], f'(x) = 6x + 12$
$f'(x) = 0 \Rightarrow x = -2$
$\Rightarrow$ $f(x)$ decreases in $(-\infty, -2)$ and increases in $(-2, \infty)$
Also $f(2^-) = 3(2)^2 + 12(2) - 1 = 35$
and $f(2^+) = 37 - 2 = 35$
Hence $f(x)$ is continuous.

$f'(x) = \begin{cases} 6x + 12, -1 < x < 2 \\ -1, 2 < x < 3 \end{cases}$

$\Rightarrow$ $f'(2^-) = 24$ and $f'(2^+) = -1$
Hence, $f(x)$ is non-differentiable at $x = 2$.
Also, $f(2^+) < f(2)$ and $f(2^-) < f(2)$
Hence, $x = 2$ is the point of maxima.

5. a, c. We have $h'(x) = f'(x)[1 - 2f(x) + 3f(x)^2]$

$$= 3f'(x)\left[(f(x))^2 - \frac{2}{3}f(x) + \frac{1}{3}\right]$$

$$= 3f'(x)[(f(x) - 1/3)^2 + 2/9]$$

Note that $h'(x) < 0$ whenever $f'(x) < 0$ and $h'(x) > 0$ whenever $f'(x) > 0$,
thus $h(x)$ increases (decreases) whenever $f(x)$ increases (decreases)
$\therefore$ **(a)** and **(c)** are the correct options.

6. d. $f(x) = \dfrac{x^2 - 1}{x^2 + 1} = \dfrac{(x^2 + 1) - 2}{x^2 + 1} = 1 - \dfrac{2}{x^2 + 1}$

For $f(x)$ to be min $\dfrac{2}{x^2 + 1}$ should be max, which is so if $x^2 + 1$ is min and $x^2 + 1$ is min at $x = 0$.

$\therefore$ $f_{\min} = \dfrac{0 - 1}{0 + 1} = -1$.

7. b. The maximum value of $f(x) = \cos x + \cos(\sqrt{2}x)$ occurs when $\cos x = 1$ and $\cos(\sqrt{2}x) = 1$

$\Rightarrow$ $x = 2n\pi, n \in Z$ and $\sqrt{2}x = 2m\pi, m \in Z$

Comparing the value of x, $2n\pi = \dfrac{2m\pi}{\sqrt{2}} \Rightarrow m = n = 0 \Rightarrow x = 0$ only.

8. b, d. $f(x) = \int_{-1}^{x} t(e^t - 1)(t - 1)(t - 2)^3(t - 3)^5 dt$
$\Rightarrow$ $f'(x) = x(e^x - 1)(x - 1)(x - 2)^3(x - 3)^5$
The critical points are 0, 1, 2, 3.
Sign scheme of $f'(x)$

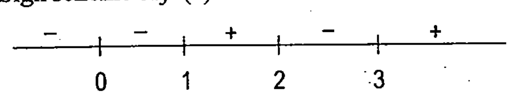


Fig. 6.167

Clearly $x = 1$ and $x = 3$ are the points of minima.

9. b, c. Let $f(x) = ax^3 + bx^2 + cx + d$
$f(2) = 18 \Rightarrow 8a + 4b + 2c + d = 18$ (1)
$f(1) = -1 \Rightarrow a + b + c + d = -1$ (2)
$f(x)$ has local max at $x = -1$
$\Rightarrow$ $f'(-1) = 0 \Rightarrow 3a - 2b + c = 0$ (3)
$f'(x)$ has local min. at $x = 0$
$\Rightarrow$ $f''(0) = 0 \Rightarrow b = 0$ (4)

Solving (1), (2), (3) and (4), we get

$$f(x)=\frac{1}{4}\left(19x^3-57x+34\right)\Rightarrow f(0)=\frac{17}{2}$$

Also, $f'(x)=\frac{57}{4}\left(x^2-1\right)>0,\ \forall x>1$

Also, $f'(x)=0\Rightarrow x=1,-1$

$f''(-1)<0, f''(1)>0$

$\Rightarrow$ $x=-1$ is a point of local max. and $x=1$ is a point of local minimum distance between $(-1, 2)$ and $(1, f(1))$, i.e., $(1,-1)$ is $=\sqrt{13}\neq 2\sqrt{5}$.

10. a, b.

$$g'(x)=f(x)=\begin{cases} c^x, & 0\le x\le 1 \\ 2-c^{x-1}, & 1<x\le 2 \\ x-c, & 2<x\le 3 \end{cases}$$

$g'(x)=0$, when $x=1+\ln 2$ and $x=c$

$$g''(x)=\begin{Bmatrix} -c^{x-1} & 1<x<2 \\ 1 & 2<x<3 \end{Bmatrix}$$

$g''(1+\ln 2)=-c^{\ln 2}<0$ hence at $x=1+\ln 2$, $g(x)$ has a local maximum. $g''(c)=1>0$ hence at $x=c$, $g(x)$ has local minimum.

Therefore, $f(x)$ is discontinuous at $x=1$, then we get local maxima at $x=1$ and local minima at $x=2$.

Match the column type

1. a → p, q, s; b → p, t; c → p, q, r, t; d → s

a. $(x-3)^2\frac{dy}{dx}+y=0$

$$\int\frac{dx}{(x-3)^2}=-\int\frac{dy}{y}$$

$$\Rightarrow \frac{1}{x-3}=\ln|y|+c$$

So, domain is $R-\{3\}$.

b. Put $x=t+3$

$$\int_{-2}^{2}(t+2)(t+1)\,t(t-1)(t-2)\,dt$$

$$=\int_{-2}^{2}t\left(t^2-1\right)\left(t^2-4\right)dt=0 \quad \text{(being odd function)}$$

c. $f(x)=\frac{5}{4}-\left(\sin x-\frac{1}{2}\right)^2$

Maximum value occurs when $\sin x=\frac{1}{2}$.

d. $f'(x)>0$ if $\cos x>\sin x$

Integer type

1. (1)

$f(x)=\ln\{g(x)\}$

$\therefore\ g(x)=e^{f(x)}$

$\therefore\ g'(x)=e^{f(x)}\cdot f(x)$

$g'(x)=0 \Rightarrow f'(x)=0$ as $e^{f(x)}\neq 0$

$\Rightarrow 2010(x-2009)(x-2010)^2(x-2011)^3(x-2012)^4=0$

so there is only one point of local maxima.