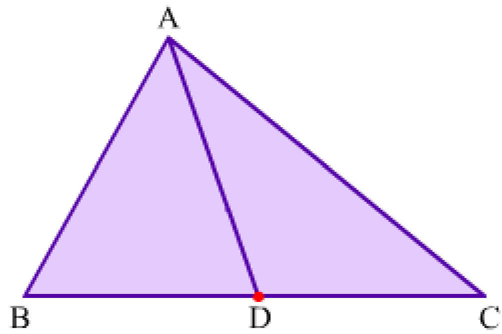
The Triangle and its Properties

Medians of Triangles

Let us consider the following triangle ABC.



In the given figure, A is the vertex of the triangle ABC and $\overline{BC}$ is the side opposite to vertex A. A line segment $\overline{AD}$ is drawn joining the point A and the point D, where D is the mid-point of $\overline{BC}$.

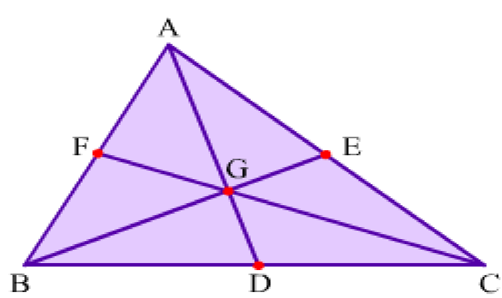
Then, we say that $\overline{\mathbf{AD}}$ **is the median of ΔABC.**

A median can be defined as follows.

> **"The line segment joining any vertex of a triangle to the mid-point of its opposite side is called the median of the triangle."**

Now we know what a median is, can we tell how many medians can be drawn inside a triangle?

In a triangle, there are three vertices. Therefore, ***a triangle can have three medians***, as shown in the following figure.



Here, $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are the three medians of ΔABC.

The medians of a triangle always lie inside the triangle.

From the figure, it can be observed that the medians $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ intersect each other at a common point G.

"The point of intersection of the medians is called the **centroid** of the triangle."

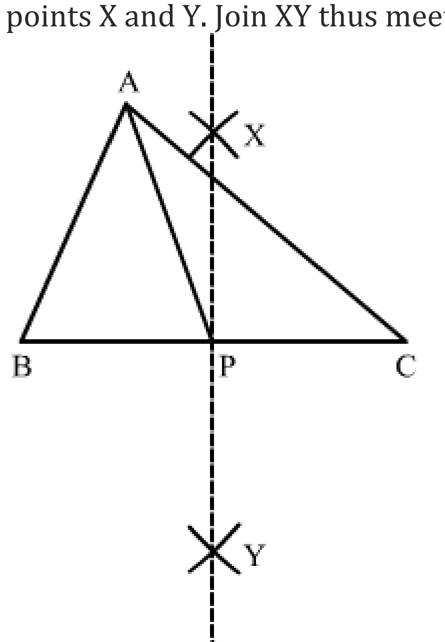
Thus, medians of a triangle are concurrent.

The point where medians intersect each other is known as the point of concurrence.

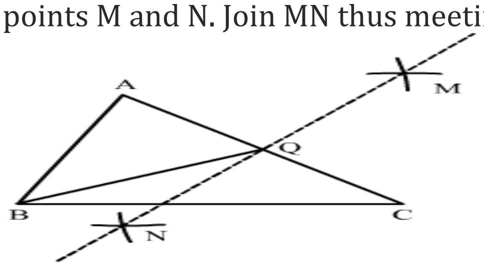
In the above given figure, G is the point of concurrence.

Construction of Median of Triangle

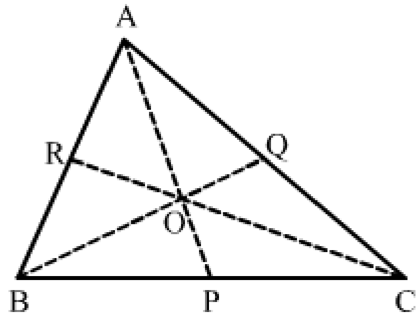
. Draw a △△ABC.
2. With B and C as centres and radius more than half of BC, draw two arcs intersecting at points X and Y. Join XY thus meeting the line BC at point P.



3. With A and C as centres and radius more than half of AC, draw two arcs intersecting at points M and N. Join MN thus meeting the line AC at point Q.



4. Similarly, draw the perpendicular bisector of line AB meeting AB at point R.
5. Join AP, BQ and CR. Let the meeting point be O.

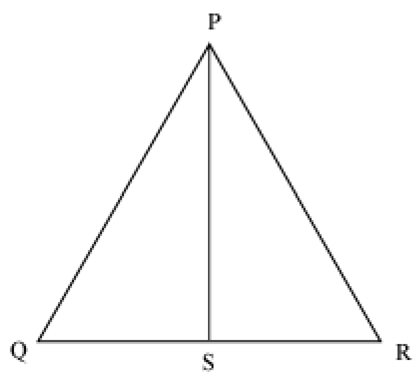


Point O is the centroid of △△ABC and AP, BQ and CR are the medians of sides BC, AC and AB respectively.

Now, let us look at an example.

Example 1:

In the triangle PQR, PS is a median and the length of $\overline{SR}$ = 6.5 cm. Find the length of $\overline{QR}$.



Solution:

Here, PS is the median to the side $\overline{QR}$ and we know that the median connects vertex to the midpoint of other side. Therefore, S is the mid-point of QR.

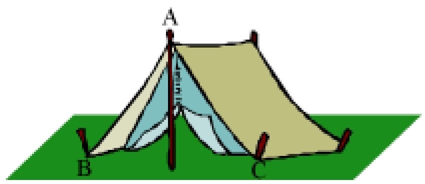
Therefore, $\overline{QR} = 2\overline{SR}$

$= 2 \times 6.5$ cm

$= 13$ cm

Altitudes of Triangles

The students of class VII were being taken to a tour to Corbett National Park. They stayed there in a tent. The entrance in the tent was of a triangular shape as shown in the following figure.

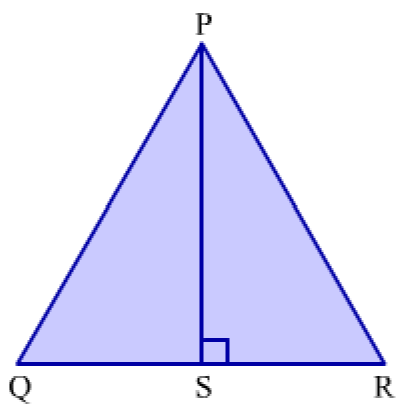


Now can you tell what the height of the tent is?

As we can see in the above figure, the height of the tent is the length of the vertical pole which is standing in the centre of the tent.

Similarly, in any triangle, we can draw a perpendicular which represents its height. The perpendicular representing the **height** of a triangle is called the **altitude** of the triangle.

Look at the triangle PQR below.

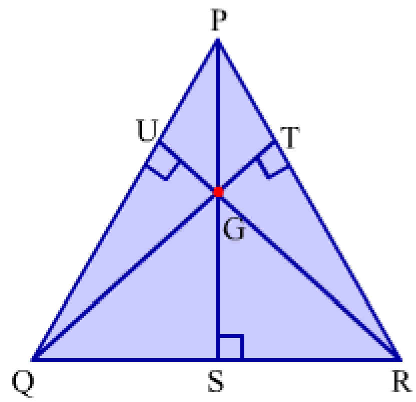


Here, P is a vertex of ΔPQR and $\overline{QR}$ is the opposite side of the vertex P. $\overline{PS}$ is a perpendicular drawn from P to $\overline{QR}$. Line segment $\overline{PS}$ is called the height or altitude of the triangle.

An altitude can be defined as follows.

"An altitude of a triangle is the perpendicular drawn from a vertex to the opposite side of the triangle."

Note: ***A triangle can have three altitudes.***



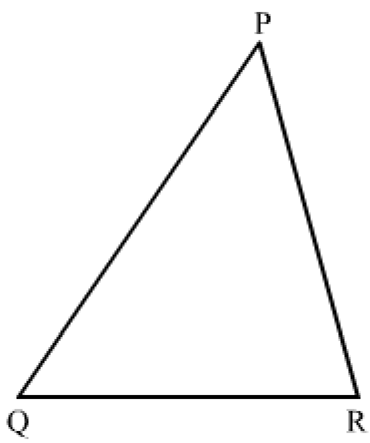
In the above figure, $\overline{PS}$, $\overline{QT}$, and $\overline{RU}$ are the three altitudes of ΔPQR.

"The point of intersection of the altitudes is called the **orthocentre** of the triangle."

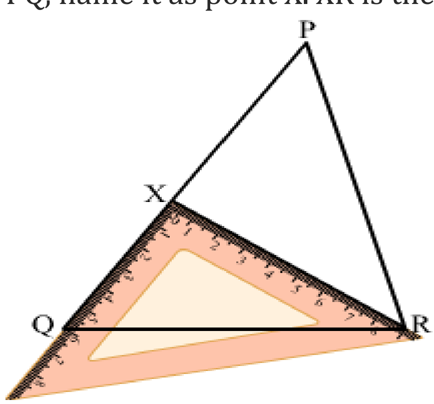
<u>**Construction of Altitudes of a triangle**</u>

I. Using set-square

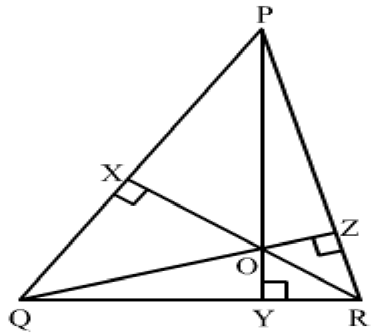
1. Draw a △△PQR.



2. From point R, draw a perpendicular on side PQ. Where the perpendicular meets the side PQ, name it as point X. XR is the altitude formed on side PQ.

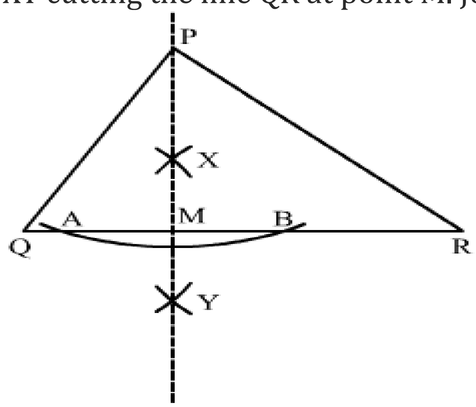


3. In a similar way construct the altitude from point P to side QR and from side Q to line PR. Thus, the altitudes obtained are XR, QZ and PY.

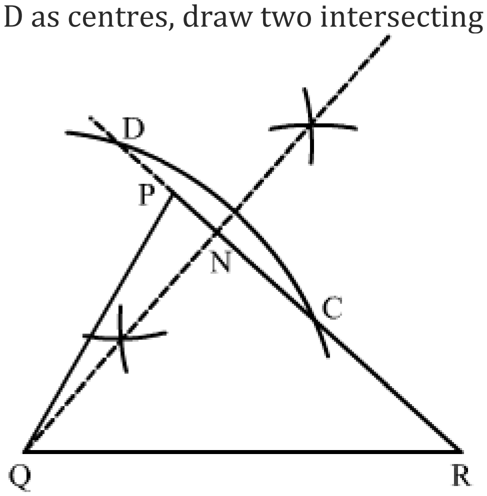


II. Using Compasses

1. Draw a △△PQR.
2. With P as centre, draw an arc on line QR cutting it at points A and B.
3. With A and B as centres, draw two intersecting arcs at points X and Y. Draw a line joining XY cutting the line QR at point M. Join PM.

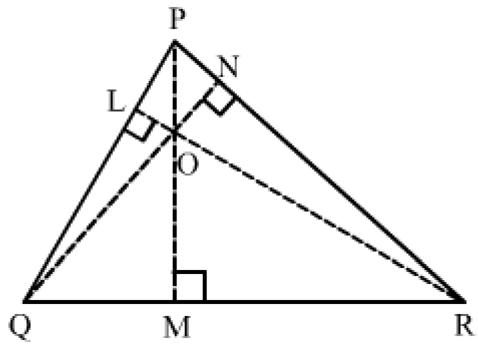


4. With Q as centre draw an arc on side RP extended to cut it at points C and D. With C and D as centres, draw two intersecting arcs. Let this line intersect PR at point N. Join QN.



5. Similarly, draw altitude from point R on PQ cutting the line PQ at point L.

6. Join PM, RL and QN and name the meeting point of these three altitudes as O.



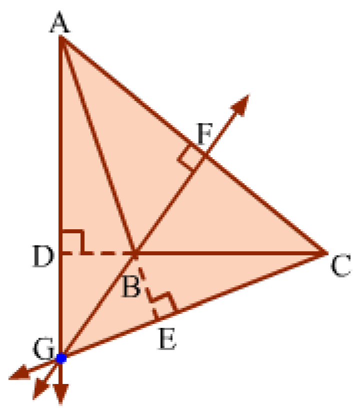
O is called the orthocentre of the △△PQR.

Remember

The altitudes of a triangle may not always lie inside it.

In an obtuse-angled triangle, the altitude drawn from the vertex of an acute angle lies outside the triangle. In this case, we have to extend the opposite side of the vertex from which the altitude is drawn.

For example,



In the above figure, ΔABC is an obtuse-angled triangle where ∠ABC is an obtuse angle. $\overline{AD}$ is the altitude of ΔABC drawn from the vertex A to extended side $\overline{BC}$. Similarly, $\overline{CE}$ is the altitude drawn from the vertex C to extended side $\overline{AB}$. And, $\overline{BF}$ is the altitude drawn from B to $\overline{CA}$.

Now, observe the altitudes drawn in the triangles ΔPQR and ΔABC.

It can be seen that altitudes in each triangle intersect each other at a common point.

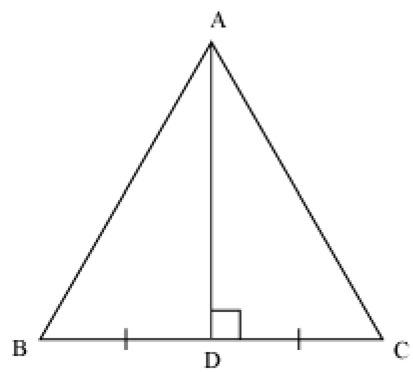
Thus, altitudes of a triangle are concurrent.

In ΔPQR and ΔABC, G is the point of concurrence.

Let us look at another example now.

Example:

In triangle ABC, $\overline{AD}$ is perpendicular to $\overline{BC}$ such that $\overline{BD} = \overline{CD}$. Are the median and the altitude drawn from A to $\overline{BC}$ same?



Solution:

Here,

$\overline{AD} \perp \overline{BC}$

Therefore, $\overline{AD}$ is an altitude of ΔABC drawn from the vertex A to $\overline{BC}$.

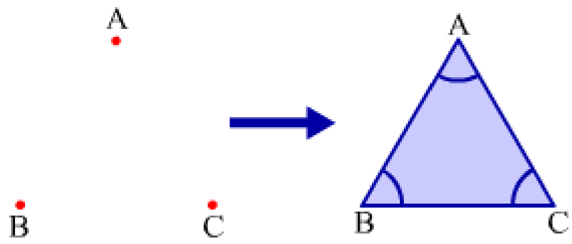
Also, $\overline{BD} = \overline{CD}$

Therefore, $\overline{AD}$ is a median of ΔABC drawn from the vertex A to $\overline{BC}$.

Thus, the altitude and the median drawn from A to $\overline{BC}$ are the same.

Angle Sum Property of Triangles

If we join any three non-collinear points in a plane, then we **get a triangle. There are three angles in a triangle.**



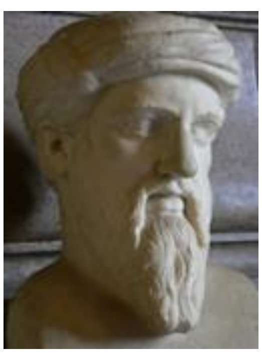
The sum of the three **interior angles** of a triangle is 180° and this property of a triangle is known as the angle sum property. This property holds true for all types of triangles, i.e., **acute-angled triangles**, **obtuse-angled triangles** and **right-angled triangles**. The angle sum property was identified by the Pythagorean school of Greek mathematicians (or the Pythagoreans) and proved by Euclid.

We will study the proof of the angle sum property of triangles and then solve some examples based on this property.

Proving the Angle Sum Property of Triangles

Know Your Scientist

Pythagoras

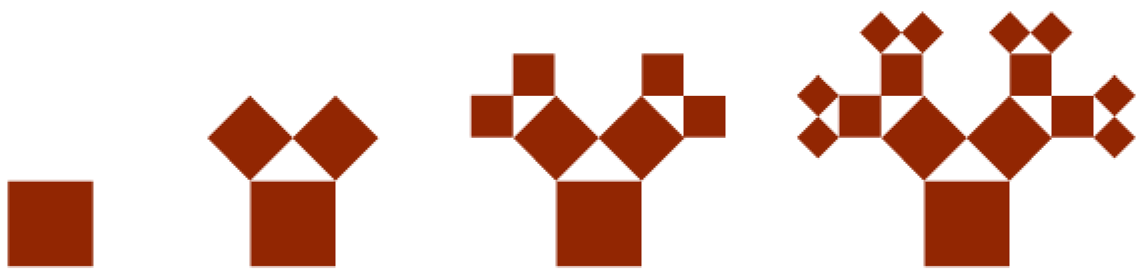
	Pythagoras (570 BC–495 BC) was a great Greek mathematician and philosopher, often described as the first pure mathematician. He was born on the island of Samos and is best known for the Pythagoras theorem about right-angled triangles. He also made influential contributions to philosophy and religious teaching. He led a society that was part religious and part scientific. This society followed a code of secrecy, which is the reason why a sense of mystery surrounds the figure of Pythagoras.

Euclid

Euclid of Alexandria(325 BC–265 BC) was a great Greek mathematician. He is referred to as ‘the father of geometry’. Euclid taught at Alexandria during the reign of Ptolemy I, who ruled Egypt from 323 BC to 285 BC. Euclid wrote a series of books which are collectively known as the *Elements*.It is considered one of the most influential works in the history of mathematics. The *Elements* served as the main textbook for teaching mathematics (especially geometry) from the time of its publication up until the early 20th century. In the *Elements*, Euclid defined most of the basic geometrical figures and deduced the principles of geometry through different sets of axioms.	

Did You Know?

In 1942, a Dutch mathematics teacher Albert E. Bosman invented a plane fractal constructed from a square. He named it the Pythagoras tree because of the presence of right-angled triangles in the figure.



Order 0 Order 1 Order 2 Order 3

Construction process of Pythagoras tree

Facts about the Angle Sum Property

An important fact deduced through the angle sum property of triangles is that ***there can be no triangle with two right angles or two obtuse angles*. This fact can be proved as is shown.**

Consider a ΔABC such that $\angle A = 90^\circ$ and $\angle B = 90^\circ$.

According to the angle sum property, we have:

$\angle A + \angle B + \angle C = 180^\circ$

$\Rightarrow 90^\circ + 90^\circ + \angle C = 180^\circ$

$\Rightarrow \angle C = 180^\circ - 180^\circ$

$\Rightarrow \angle C = 0^\circ$

However, the above is not possible. So, ΔABC (or any other triangle) cannot have two right angles.

Similarly, we can prove that a triangle cannot have two obtuse angles.

Whiz Kid

Relationship between the side lengths and the angle measurements of a triangle

- The largest interior angle is **opposite** the largest side.

- The smallest interior angle is **opposite** the smallest side.

- The middle-sized interior angle is **opposite** the middle-sized side.

Facts about the Angle Sum Property

By the angle sum property, we can deduce the fact that ***there can be no triangle with all angles less than or greater than* 60°.** This fact can be proved as is shown.

Consider a **ΔABC** with all angles equal to 59°.

According to the angle sum property, we should have $\angle A + \angle B + \angle C = 180^\circ$.

By adding the given angles, we obtain:

$59^\circ + 59^\circ + 59^\circ = 177^\circ \neq 180^\circ$

Since ΔABC does not satisfy the angle sum property, it cannot exist.

Now, consider a **ΔABC** with all angles equal to 61°.

According to the angle sum property, we should have $\angle A + \angle B + \angle C = 180^\circ$.

By adding the given angles, we obtain:

61° **+** 61° **+** 61° **= 183° ≠ 180°**

Since ΔABC does not satisfy the angle sum property, it cannot exist.

Thus, we have proved that a triangle cannot have all angles less than or greater than 60°.

Whiz Kid

Sum of the interior angles of an n-sided polygon = $(n - 2) \times 180°$

For example:

Sum of the interior angles of a 6-sided polygon = (6 − 2) × 180° = 720°

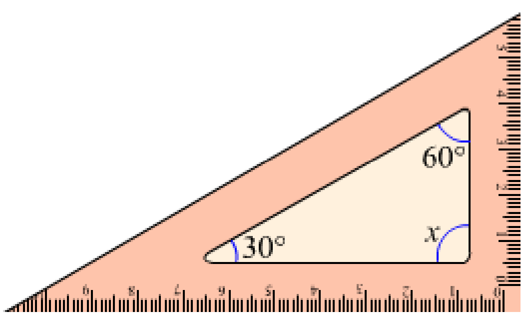
Relation between the Vertex Angle and the Angles Made by the Bisectors of the Remaining Angles

Solved Examples

Easy

Example 1:

Find the measurement of *x* in the following figure.



Solution:

We know that the sum of the three angles of a triangle is 180°.

So, we have:

30° + 60° + x = 180°

⇒ 90° + x = 180°

⇒ x = 90°

Example 2:

If the angles of a triangle are in the ratio 1 : 3 : 5, then what is the measure of each angle?

Solution:

It is given that the angles of the triangle are in the ratio 1 : 3 : 5.

Let the angles be x, $3x$ and $5x$.

Now, $x + 3x + 5x = 180°$ (By the angle sum property of triangles)

 $9x = 180°$

 $x = 20°$

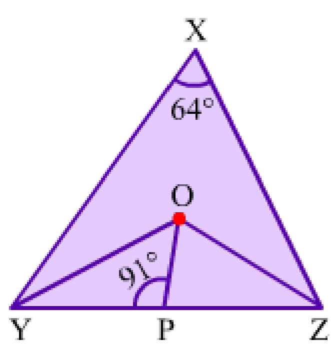
So, $3x = 3 \times 20° = 60°$ and $5x = 5 \times 20° = 100°$

Thus, the measures of the angles of the triangle are 20°, 60° and 100°.

Medium

Example 1:

In the given ΔXYZ, YO, ZO and PO are the respective bisectors of XYZ, XZY and YOZ. Find the measure of OYX.



Solution:

As per the relation between the vertex angle and the angles made by the bisectors of the remaining two angles, we have:

$$\angle YOZ = 90° + \frac{1}{2}\angle YXZ$$
$$\Rightarrow \angle YOZ = 90° + \frac{1}{2} \times 64°$$
$$\Rightarrow \angle YOZ = 90° + 32°$$
$$\Rightarrow \angle YOZ = 122°$$

In ΔOYP, we have:

$\angle YOP = \frac{1}{2} \angle YOZ$ (&because PO is the bisector of ∠YOZ)

$\Rightarrow \angle YOP = \frac{1}{2} \times 122°$

⇒ ∠YOP = 61°

Again in ΔOYP, we have:

∠YOP + ∠OPY + ∠PYO = 180° (By the angle sum property)

⇒ 61° + 91° + ∠PYO = 180°

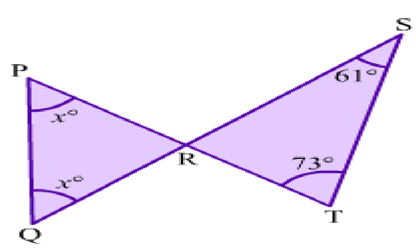
⇒ ∠PYO = 180° − 152°

⇒ ∠PYO = 28°

We know that YO is the bisector of ∠XYZ.

So, ∠OYX = ∠PYO = 28°

Example 2:
Find the value of *x* in the given figure.



Solution:

Using the angle sum property in ΔRST, we obtain:

∠RST + ∠RTS + ∠SRT = 180°

⇒ 61° + 73° + ∠SRT = 180°

⇒ 134° + ∠SRT = 180°

⇒ ∠SRT = 180° − 134°

⇒ ∠SRT = 46°

In ΔRST and ΔRPQ, we have:

∠PRQ = ∠SRT = 46° (Vertically opposite angles)

Using the angle sum property in ΔRPQ, we obtain:

∠RPQ + ∠RQP + ∠PRQ = 180°

⇒ x° + x° + 46° = 180°

⇒ 2x° + 46° = 180°

⇒ 2x° = 180° − 46°

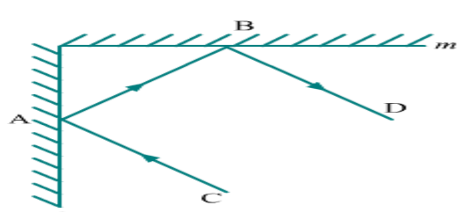
⇒ 2x° = 134°

⇒ x = 67

Hard

Example 1:

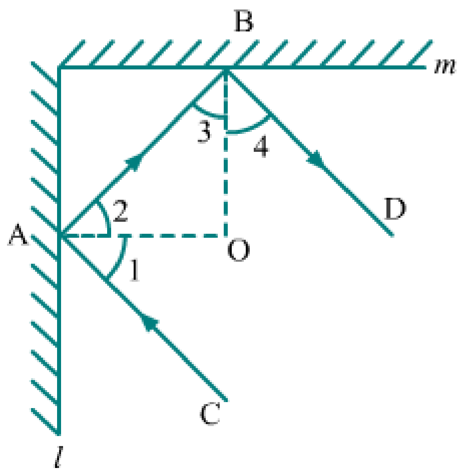
In the figure, *l* and *m* are two plane mirrors placed perpendicular to each other. Show that the incident ray CA is parallel to the reflected ray BD.



Solution:

It is given that mirrors *l* and *m* are perpendicular to each other.

Construction: Draw two perpendiculars OA and OB to *l* and *m* respectively. Mark the angles made by these perpendiculars with the incident and reflected rays as is shown.



OA ⊥ OB

∴ $\boldsymbol{\angle BOA = 90^\circ}$

In ΔBOA, we have:

$\angle 2 + \angle 3 + \angle BOA = 180^\circ$

$\Rightarrow \angle 2 + \angle 3 + 90^\circ = 180^\circ$

$\Rightarrow \angle 2 + \angle 3 = 90^\circ$

$\Rightarrow 2(\angle 2 + \angle 3) = 180^\circ$

$\Rightarrow 2\angle 2 + 2\angle 3 = 180^\circ$ **... (1)**

We know that:

Angle of incidence = Angle of reflection

For mirror *l*, $\angle 1$ is the angle of incidence and $\angle 2$ is the angle of reflection.

∴ $\angle 1 = \angle 2$

For mirror *m*, ∠3 is the angle of incidence and ∠4 is the angle of reflection.

∴ ∠3 = ∠4

So, by using equation 1, we get:

(∠2 + ∠2) + (∠3 + ∠3) = 180°

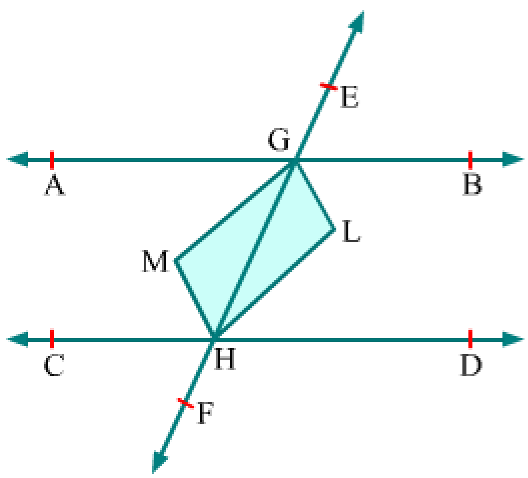
⇒ (∠1 + ∠2) + (∠3 + ∠4) = 180° **(**Since ∠1 = ∠2 and ∠3 = ∠4**)**

⇒ ∠CAB + ∠ABD = 180°

Since ∠CAB and ∠ABD are interior angles on the same side of transversal AB and their sum is 180°**, lines CA and BD must be parallel to each other. Therefore, the incident ray CA is parallel to the reflected ray BD.**

Example 2:

In the given figure, AB is parallel to CD; GM, HM, GL and HL are the bisectors of the two pairs of interior angles. Prove that ∠GLH = 90°.



Solution:

From the figure, we have:

∠AGH = ∠DHG (Alternate interior angles)

⇒ $\frac{1}{2}$ ∠AGH = $\frac{1}{2}$ ∠DHG

⇒ ∠HGM = ∠GHL (&because GM bisects ∠AGH and HL bisects ∠DHG)

It can be said that lines GM and HL are intersected by the transversal GH at G and H respectively such that the alternate interior angles are equal, i.e., $\angle HGM = \angle GHL$.

$\therefore$ GM||HL

Similarly, we can prove that GL||HM. So, GMHL is a parallelogram.

We know that AB||CD and EF is the transversal.

$\therefore \angle BGH + \angle DHG = 180°$ (Interior angles on the same side of a transversal)

$\Rightarrow \frac{1}{2} \angle BGH + \frac{1}{2} \angle DHG = 90°$

Since GL bisects $\angle BGH$ and HL bisects $\angle DHG$, we obtain:

$\angle LGH + \angle GHL = 90°$ … (1)

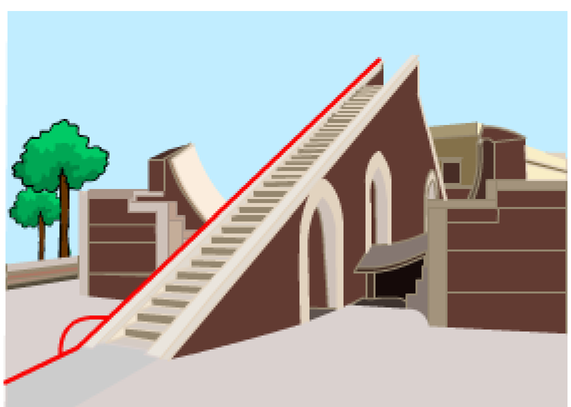
Also, $\angle LGH + \angle GHL + \angle GLH = 180°$ (By the angle sum property)

$\therefore 90° + \angle GLH = 180°$ (Using equation 1)

$\Rightarrow \angle GLH = 90°$

Exterior Angle Property of Triangles

Look at the triangular structure in the figure.

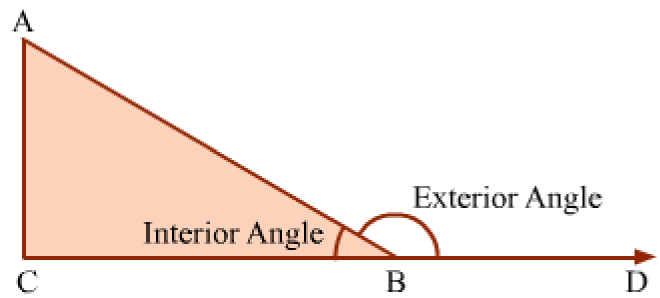


In the given figure, an open angle formed by the edge of the triangular structure with the horizontal plane is marked. This angle lies outside the triangle. Such angles are known as **exterior angles**.

In this lesson, we will study about exterior angles of triangles and the theorem based on them.

Exterior Angles of Triangles

Look at the triangle shown.



It can be seen that in **Δ**ABC, side CB is extended up to point D. This extended side forms an angle with side AB, i.e., ∠ABD. This angle lies exterior to the triangle. Hence, ∠**ABD is an exterior angle of ΔABC.**

An exterior angle of a triangle can be defined as follows:

The angle formed by a side of a triangle with an extended adjacent side

is called an exterior angle of the triangle.

It can be seen that exterior ∠ABD forms linear pair with interior ∠ABC of **Δ**ABC. The other two interior angles of the triangle such as ∠ACB and ∠CAB do not form linear pair with ∠ABD.

Such angles are known as the **remote interior angles** of an exterior angle.

So, ∠ACB and ∠CAB are remote interior angles of exterior ∠ABD.

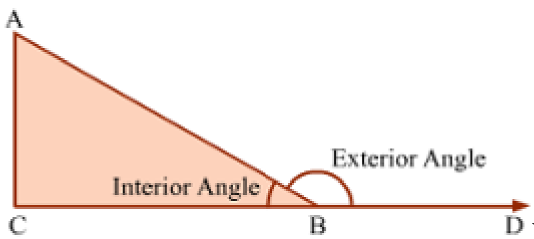
Exterior Angle Theorem or Remote Interior Angles Theorem and Its Proof

Corollary Related to Exterior Angle Theorem

There is a corollary related to exterior angle theorem which states that:

An exterior angle is always greater than each of its remote interior angles.

Let us prove this corollary with the help of **Δ**ABC shown in the figure.



Here, $\angle ABD$ is an exterior angle of the triangle and its interior opposite or remote interior angles are $\angle ACB$ and $\angle CAB$.

In a triangle, no interior angle can be zero angle or straight angle.

Thus, $0° < \angle ABC < 180°$, $0° < \angle ACB < 180°$ and $0° < \angle CAB < 180°$

Now, $\angle ABC < 180°$

$\therefore 180° - \angle ABC > 0°$

$\Rightarrow \angle ABD > 0°$

In triangle ΔABC, we have

$\angle ABD = \angle ACB + \angle CAB$ (By exterior angle theorem)

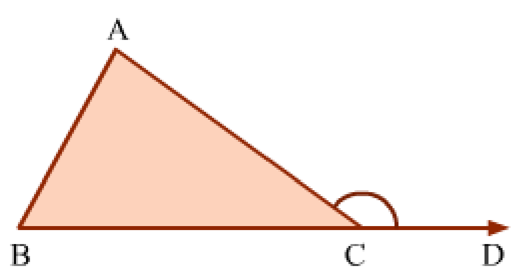
And, $\angle ABD > 0$, $\angle ACB > 0°$ and $\angle CAB > 0°$ (Property of triangle)

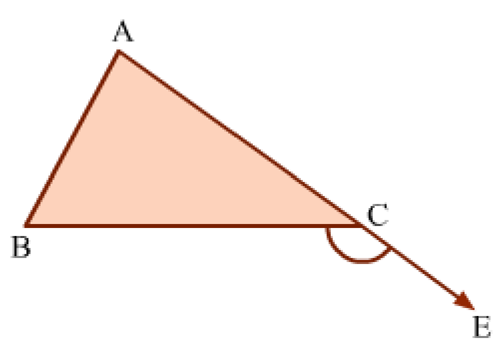
Therefore, $\angle ABD > \angle ACB$ and $\angle ABD > \angle CAB$

Thus, an exterior angle is always greater than each of its remote interior angles.

Two Exterior Angles at the Same Vertex are Equal

At any vertex, two exterior angles can be drawn by extending each of the two sides forming that vertex. These exterior angles are always of equal measure. Let us prove this using the ΔABC shown in the figure.





The figure clearly shows that two exterior angles can be drawn at vertex C—one by producing BC up to point D and the other by producing AC up to point E. The exterior angles thus obtained are ∠ACD and ∠BCE.

According to the exterior angle theorem, the measure of an exterior angle of a triangle is equal to the sum of the measures of the two opposite interior angles of the triangle.

∴ ∠ACD = ∠ABC + ∠BAC … (1)

And, ∠BCE = ∠ABC + ∠BAC … (2)

Using equations 1 and 2, we get:

∠ACD = ∠BCE

So, we can conclude that two exterior angles can be drawn at any vertex. The two angles thus drawn have an equal measure and are equal to the sum of the two opposite interior angles.

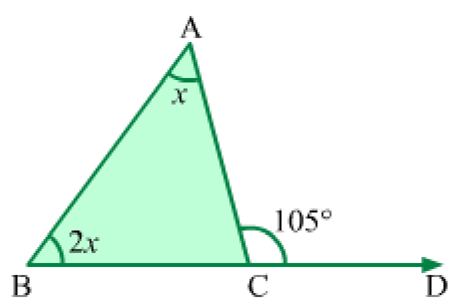
Practical Verification of Exterior Angle Property

Solved Examples

Easy

Example 1:

Find the value of *x* in the given figure.



Solution:

According to the exterior angle property of triangles, the measure of an exterior angle of a triangle is equal to the sum of the measures of the two opposite interior angles of the triangle.

So, we have:

$x + 2x = 105°$

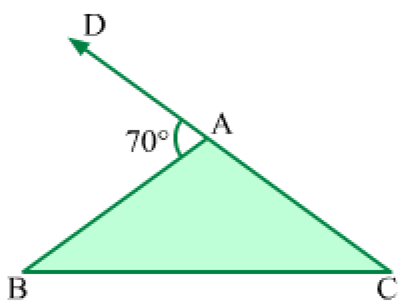
$\Rightarrow 3x = 105°$

On dividing both sides of the equation by 3, we obtain:

$$\frac{3x}{3} = \frac{105°}{3}$$
$$\Rightarrow x = 35°$$

Example 2:

If ∠ABC = ∠ACB in ΔABC, then find the measure of ∠ABC.



Solution:

∠ABC and ∠ACB are interior angles opposite to the exterior angle at vertex A, i.e., ∠BAD.

Therefore, by the exterior angle property of triangles, we obtain:

∠ABC + ∠ACB = ∠BAD

⇒ ∠ABC + ∠ACB = 70°

It is given that ∠ABC = ∠ACB

So, we obtain:

∠ABC + ∠ABC = 70°

⇒ 2∠ABC = 70°

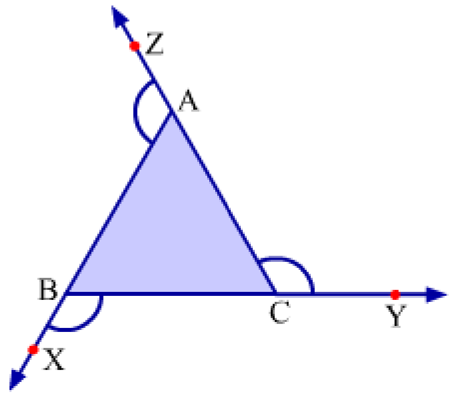
On dividing both sides of the equation by 2, we obtain:

$$\frac{2\angle ABC}{2} = \frac{70°}{2}$$
$$\Rightarrow \angle ABC = 35°$$

Medium

Example 1:

The sides AB, BC and CA of ΔABC are produced up to points X, Y and Z respectively. Find the sum of the three exterior angles so formed.



Solution:

Using the exterior angle property, we obtain:

$\angle BAZ = \angle ABC + \angle ACB$ … (1)

$\angle CBX = \angle BAC + \angle ACB$ … (2)

$\angle ACY = \angle BAC + \angle ABC$ … (3)

On adding equations 1, 2 and 3, we obtain:

$\angle BAZ + \angle CBX + \angle ACY = \angle ABC + \angle ACB + \angle BAC + \angle ACB + \angle BAC + \angle ABC$

$\Rightarrow \angle BAZ + \angle CBX + \angle ACY = 2(\angle ABC + \angle ACB + \angle BAC)$

According to the angle sum property of triangles, we have:

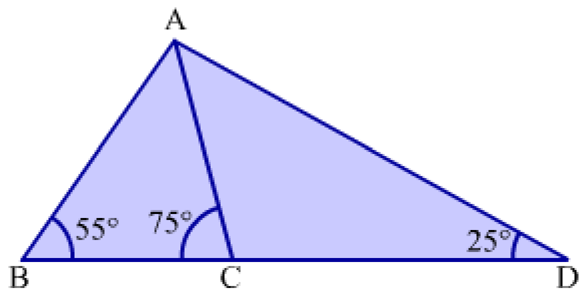
$\angle ABC + \angle ACB + \angle BAC = 180°$

$\therefore \angle BAZ + \angle CBX + \angle ACY = 2 \times 180° = 360°$

Thus, the sum of the three exterior angles is 360°.

Example 2:

Show that AC is the bisector of ∠BAD in the given figure.



Solution:

On applying the angle sum property in ΔABC, we get:

$\angle BAC + \angle ABC + \angle ACB = 180°$

$\Rightarrow \angle BAC + 55° + 75° = 180°$

$\Rightarrow \angle BAC = 180° - 130°$

$\Rightarrow \angle BAC = 50°$ ... (1)

Now, by using the exterior angle property, we get:

$\angle ACB = \angle ADC + \angle CAD$

$\Rightarrow 75° = 25° + \angle CAD$

$\Rightarrow \angle CAD = 75° - 25°$

$\Rightarrow \angle CAD = 50°$

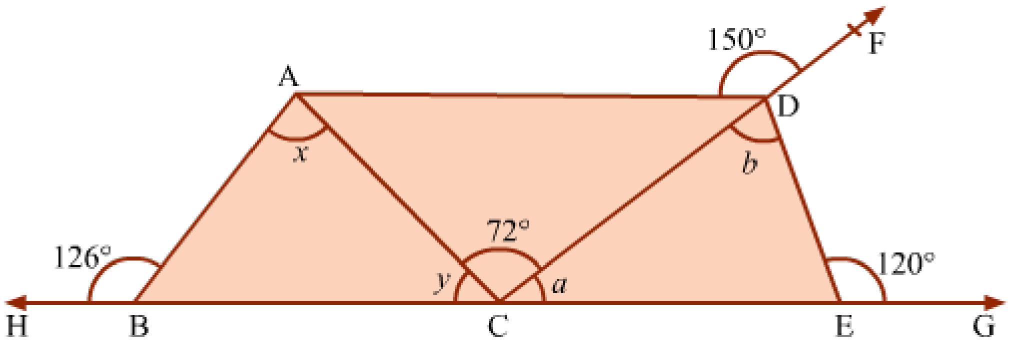
We know that $\angle BAC + \angle CAD = \angle BAD$. We have found $\angle BAC = \angle CAD = 50°$.

Thus, AC is the bisector of ∠BAD.

Hard

Example 1:

If AD||BE in the given figure, then find the values of *a*, *b*, *x* and *y*.



Solution:

From the figure, we have:

∠ADC + ∠ADF = 180° (Linear pair of angles)

⇒ ∠ADC + 150° = 180°

⇒ ∠ADC = 30°

Consider the parallel lines AD and BE and the transversal CF.

∠ADF = ∠DCB (Corresponding angles)

⇒ 150° = 72° + y

⇒ y = 78° ... (1)

Now, y + 72° + a = 180° (As they form line BCE)

⇒ 78° + 72° + a = 180° (Using equation 1)

⇒ a = 180° − 150°

⇒ a = 30° ... (2)

Consider ΔCDE.

∠DEG = a + b (Exterior angle property)

⇒ 120° = 30° + b (Using equation 2)

$\Rightarrow b = 90°$

Now, consider ΔABC.

$\angle ABH = x + y$ (Exterior angle property)

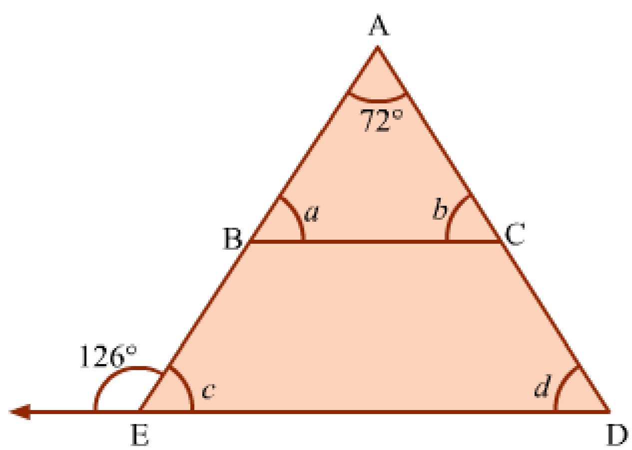
$\Rightarrow 126° = x + 78°$

$\Rightarrow x = 48°$

Hence, $a = 30°$, $b = 90°$, $x = 48°$ and $y = 78°$.

Example 2:

ΔABC is placed atop trapezium EBCD in the given figure. Find the values of *a*, *b*, *c* and *d*.



Solution:

The exterior angle at E forms a linear pair with c.

$\therefore 126° + c = 180°$

$\Rightarrow c = 180° - 126°$

$\Rightarrow c = 54°$

On using the exterior angle property in ΔAED,we get:

$126° = 72° + d$

$\Rightarrow d = 126° - 72°$

$\Rightarrow d = 54°$

Since EBCD is a trapezium, BC is parallel to ED. Also, BE and CD are transversals lying on the two parallel lines.

So, we have:

$a = c = 54°$ (Pair of corresponding angles)

$b = d = 54°$ (Pair of corresponding angles)

Thus, $a = b = c = d = 54°$.

Properties Of Equilateral Triangles

An equilateral triangle is a special case of a triangle. And why is it special?

The three most important properties of equilateral triangles are as follows.

All the sides are of equal length.

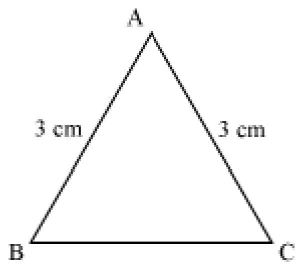
All the angles are equal and the measure of each angle is 60°.

All the exterior angles are equal and the measure of each exterior angle is 120°.

Isosceles Triangles

We can classify triangles into different categories according to its sides (or angles). Now, let us study about a special type of triangle known as isosceles triangle.

"A triangle in which two sides are of equal length is called isosceles triangle."



In ΔABC, two sides AB and AC are of equal length. Therefore, ΔABC is an isosceles triangle.

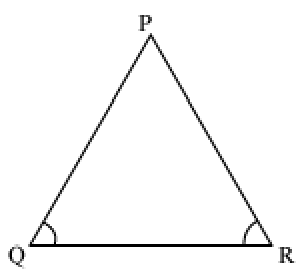
"In an isosceles triangle, angles opposite to equal sides are also equal."

In the above figure, ∠B and ∠C are the angles opposite to equal sides AC and AB respectively.

∴ ∠B = ∠C

Therefore, we can also define isosceles triangle in terms of angles.

"If two angles of a triangle are equal in measure, then the triangle is called isosceles triangle."



In the given figure,

∠Q = ∠R

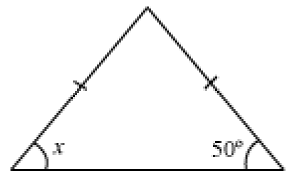
∴ ΔPQR is an isosceles triangle.

Now, let us solve some examples involving isosceles triangles.

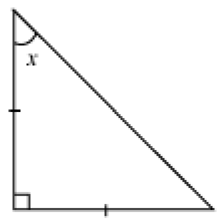
Example 1:

Find the value of *x* for the following figures.

(i)



(ii)



Solution:

(i) The given triangle is an isosceles triangle.

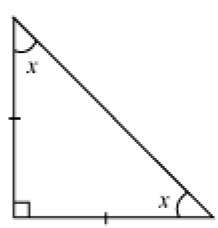
The angles opposite to equal sides are equal.

$\therefore x = 50°$

(ii) The given triangle is an isosceles triangle.

The angles opposite to equal sides are equal.

Thus, we have



Now, using angle sum property of triangles, we obtain

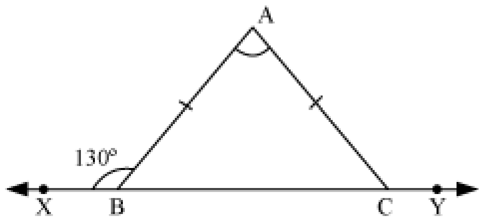
$x + x + 90° = 180°$

$2x = 90°$

$x = 45°$

Example 2:

In the given figure, find ∠BAC.



Solution:

It is given that ΔABC is an isosceles triangle. We know that in an isosceles triangle, angles opposite to equal sides are equal.

$\therefore \angle ABC = \angle ACB$ (1)

Now, $\angle ABC = 180° - \angle ABX$ (By linear pair axiom)

$= 180° - 130°$

$= 50°$

From equation (1), we obtain

$\angle ACB = 50°$

Using angle sum property in ΔABC, we obtain

$\angle ABC + \angle ACB + \angle BAC = 180°$

$50° + 50° + \angle BAC = 180°$

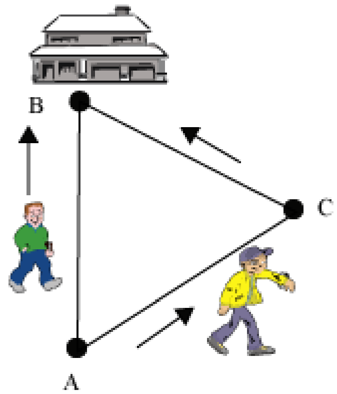
$\angle BAC = 180° - 100°$

$\angle BAC = 80°$

Thus, the measure of ∠BAC is 80°.

Triangle Inequalities

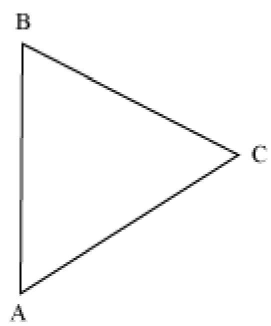
Rohit and Mohit are brothers. Their house is located at position B, as shown in the following figure.



They are standing at a position A and want to reach their house. Rohit chooses the path AB to reach his house while Mohit first goes to position C starting from A and then travels distance CB to reach his house. Both of them walk at the same speed. **Who will take less time to reach the house?**

Since their speeds are the same, more time will be taken by the person who has to cover the larger distance. Therefore, we can say that Rohit will take less time in reaching the house as the path followed by Rohit is shorter than the path followed by Mohit.

We can see that the paths followed by Rohit and Mohit form a triangle.



i.e. ABC is a triangle. From the above example, we can clearly see that length AB is shorter than length BC + CA. Or, we can say that length BC + CA is longer than length AB.

Therefore, we can conclude that

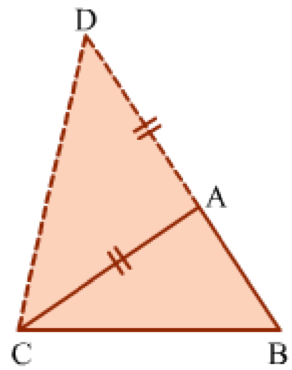
"The sum of the lengths of any two sides of a triangle is always greater than the length of the third side of the triangle".

Now, from the above discussion, we can say that in ΔABC, $AB + BC > CA$, $AB + CA > BC$ and $CA + BC > AB$.

Let us go through the proof of the above property.

Proof:

Let us take any ΔABC and extend the side AB to D such that AD = CA**.**



In ΔACD, we have

AD = CA

$\Rightarrow \angle ACD = \angle ADC$

$\Rightarrow \angle ACD + \angle ACB > \angle ADC$

$\Rightarrow \angle BCD > \angle BDC$ ($\angle BDC$ is same as $\angle ADC$)

$\Rightarrow$ BD > BC (Side opposite to greater angle is longer)

$\Rightarrow$ AB + AD > BC

$\Rightarrow$ AB + CA > BC (AD = CA)

Similarly, we can prove that AB + BC > CA and CA + BC > AB.

From the result CA + BC > AB, we can deduce that CA − AB < BC. Similarly, we can deduce results from the remaining two inequalities.

Thus, from this, we can conclude that

"The difference between the lengths of any two sides of a triangle is always smaller than the length of the third side of the triangle".

The inequalities AB + CA > BC, AB + BC > CA and CA + BC > AB are called triangle inequalities. They are necessary condition for the existence of a triangle with sides AB, BC and CA. This tells that the straight line is the shortest path between any two points. Given three numbers a, b, c, then the necessary condition for the existence of a triangle with sides a, b, c are that $b + c > a$, $c + a > b$ and $a + b > c$.

Let us now look at some examples to understand this property of triangles better.

Example 1:

Check whether it is possible to have a triangle with the following sides or not.

1. **11.5 cm, 7.8 cm, 14.7 cm**
2. **7.4 cm, 18.5 cm, 10.9 cm**

Solution:

1. 11.5 cm, 7.8 cm, 14.7 cm

For a triangle, the sum of the lengths of any two sides must be greater than the third side.

Now,

11.5 cm + 7.8 cm = 19.3 cm > 14.7 cm

11.5 cm + 14.7 cm = 26.2 cm > 7.8 cm

7.8 cm + 14.7 cm = 22.5 cm > 11.5 cm

We can see that the sum of the lengths of any two sides is greater than the length of the third side.

Therefore, a triangle with sides of given lengths is possible.

2. 7.4 cm, 18.5 cm, 10.9 cm

For a triangle, the sum of the lengths of any two sides must be greater than the third side.

But in this case,

7.4 cm + 10.9 cm = 18.3 cm < 18.5 cm

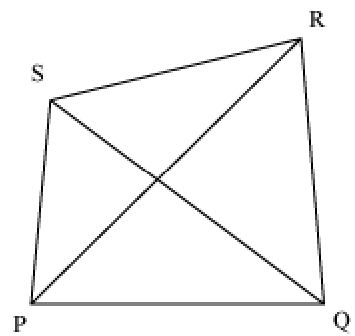
Therefore, a triangle with sides of given lengths is not possible.

Example 2:

Can the sum of the four sides of a quadrilateral be equal to the sum of the lengths of its diagonals?

Solution:

Consider a quadrilateral PQRS in which PR and QS are its two diagonals.



Now, we know that the sum of any two sides of a triangle is always greater than the third side. Therefore,

In ΔPQR,

PQ + QR > PR … (1)

In ΔPRS,

PS + RS > PR … (2)

In ΔPQS,

PQ + PS > QS … (3)

In ΔSRQ,

QR + RS > QS … (4)

On adding (1), (2), (3), and (4), we obtain

PQ + QR + PS + RS + PQ + PS + QR + RS > PR + PR + QS + QS

⇒ (PQ + PQ) + (QR + QR) + (RS + RS) + (PS + PS) > (PR + PR) + (QS + QS)

⇒ 2PQ + 2QR + 2RS + 2PS > 2PR + 2QS

⇒ 2(PQ + QR + RS + PS) > 2 (PR + QS)

On dividing both sides by 2, we obtain

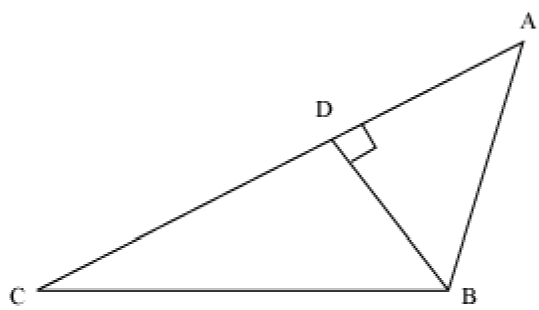
PQ + QR + RS + PS > PR + QS

Therefore, the sum of the lengths of the four sides of a quadrilateral is always greater than the sum of the lengths of its diagonals.

Hence, the sum of the lengths of the four sides cannot be equal to the sum of the lengths of its two diagonals.

Example 3:

In the given figure, BD $\perp$ AC.



Which of the following statements is correct?

(i) 2 BD > AB + BC + CA

(ii) 2 BD < AB + BC + CA

Solution:

We know that the sum of any two sides of a triangle is always greater than the third side.

Therefore, in ΔABD DA + AB > BD … (i)

In ΔBCD,

BC + CD > BD … (ii)

On adding (i) and (ii), we obtain

AB + DA + BC + CD > BD + BD

⇒ AB + BC + (CD + DA) > 2 BD

Now, from the figure,

CD + DA = CA

Therefore,

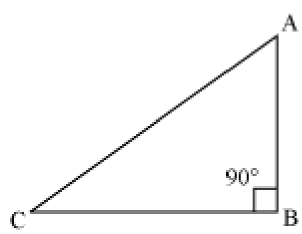
AB + BC + CA > 2 BD

i.e. 2 BD < AB + BC + CA

Therefore, statement **(ii)** is correct.

Pythagoras Theorem and Its Converse

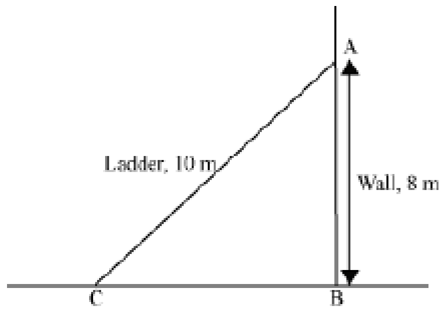
Look at the following right-angled triangle ABC. Here, m∠ABC = 90°.



In a right-angled triangle, special names are given to the sides. The side opposite to the right angle is termed as **hypotenuse** and the remaining two sides are termed as the **legs** of the right-angled triangle.

In real life, we come across many situations where a right angle is formed. Let us consider such a situation.

A 10m long ladder is placed on a wall such that the ladder touches the wall at 8m above the ground. This situation can be shown geometrically as follows.



In the above figure, AB is the wall of height 8 m and AC is the ladder of length 10 m. We know that a wall is perpendicular to the floor, i.e. AB is perpendicular to BC. Thus, ∠ABC is a right angle.

Now, can we calculate the distance of the foot of the ladder from the base of the wall?

We can calculate the distance of the foot of the ladder from the base of the wall by using Pythagoras theorem.

In this way, we can use Pythagoras theorem in many situations where right-angled triangle is formed.

In the reverse case, we can say that *for a triangle to be right-angled, the sum of the squares of two sides must be equal to the square of the third side.*

This is the converse of Pythagoras theorem. Using this converse, we can check whether a triangle is right-angled or not.

For example, let ABC be a triangle where $\overline{AB}$ = 10 cm, $\overline{BC}$ = 6 cm, and $\overline{AC}$ = 8 cm. Let us find whether ΔABC is a right-angled triangle or not.

Here, $\left(\overline{BC}\right)^2 + \left(\overline{AC}\right)^2 = 6^2 + 8^2$

$= (6 \times 6) + (8 \times 8)$

$= 36 + 64$

$= 100$

Again, $\left(\overline{AB}\right)^2 = 10^2$

$= 10 \times 10$

= 100

Therefore, $\left(\overline{AB}\right)^2 = \left(\overline{BC}\right)^2 + \left(\overline{AC}\right)^2$

Using converse of Pythagoras theorem, we can say that ΔABC is a right-angled triangle.

Here, AB = 10 cm is the **hypotenuse** of the triangle.

Let us now solve some more examples based on Pythagoras theorem and its converse.

Example 1:

Check whether the triangles having sides of given lengths are right-angled or not. Also identify the hypotenuse and the right angle in the right-angled triangles.

1. 5 cm, 6 cm, and 8 cm
2. 5 cm, 13 cm, and 12 cm

Solution:

1. $5^2 + 6^2 = 25 + 36 = 61$

   $8^2 = 8 \times 8 = 64$

   As, $5^2 + 6^2 \neq 8^2$

   Therefore, the triangle having the given sides is not a right-angled triangle.

2. $(5)^2 + (12)^2 = 25 + 144 = 169$

   $(13)^2 = 13 \times 13 = 169$

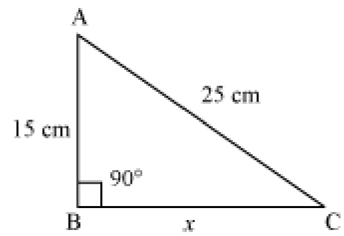
   As, $(5)^2 + (12)^2 = (13)^2$

   Therefore, the triangle having the given sides is a right-angled triangle.

   Here, the side of length 13 cm is the hypotenuse and the angle opposite to the hypotenuse is the right angle.

Example 2:

Find the value of *x* in the given figure.



Solution:

ΔABC is a right-angled triangle, right-angled at B.

Therefore, using Pythagoras theorem,

$15^2 + x^2 = 25^2$

On transposing 15^2 from L.H.S. to R.H.S., we obtain

$x^2 = 25^2 - 15^2$

$= 625 - 225$

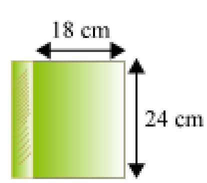
$= 400$

$= 20^2$

$\therefore x = 20$ cm

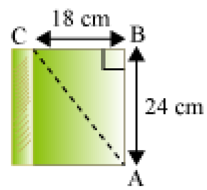
Example 3:

The lengths of two adjacent edges of a book are 18 cm and 24 cm. Find the length of the diagonal of the book.



Solution:

Let AC be the diagonal of the book as shown in the following figure.



Here, we can see that ∠ABC is a right angle. Therefore, using Pythagoras theorem,

$AC^2 = AB^2 + BC^2$

$\Rightarrow AC^2 = 24^2 + 18^2$

$\Rightarrow AC^2 = 576 + 324$

$\Rightarrow AC^2 = 900$

$\Rightarrow AC = 30$

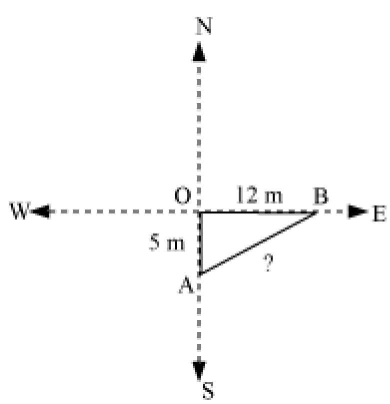
Therefore, the length of the diagonal of the book is 30 cm.

Example 4:

Gopal and Khushboo are friends. They are standing in front of a pole. Gopal is at a distance of 5 m south of the pole and Khushboo is standing 12 m east of the pole. What is the distance between Gopal and Khushboo?

Solution:

Let O, A, and B be the positions of the pole, Gopal, and Khushboo.



Using Pythagoras theorem, we obtain

$AB^2 = OA^2 + OB^2$

$AB^2 = 5^2 + 12^2$

$AB^2 = 169$

$AB^2 = 13^2$

$AB = 13$

Therefore, the distance between Gopal and Khushboo is 13 m.