05

In class XI, we have studied about the concept of limit, left hand and right hand limits. We had also learnt how to differentiate polynomial function and trigonometric functions. In this chapter, we will introduce the very important concepts of continuity, differentiability and relations between them. We will also illustrate certain geometrical obvious conditions through differential calculus.

CONTINUITY AND DIFFERENTIABILITY

|TOPIC 1|

Continuity

CHAPTER CHECKLIST

- Continuity
- Properties of Continuous Functions
- Differentiability and Derivatives of Various Functions
- Derivatives of Implicit Functions
- Derivatives of Inverse Trigonometric Functions
- Differentiation of Exponential and Logarithmic Functions
- Differentiation of Parametric Functions
- Differentiation of a Function w.r.t. Another Function
- Differentiation of Infinite Series
- Second Order Derivative

DEFINITION OF LIMIT

If x approaches a, i.e. $x \to a$ implies $f(x)$ approaches l, i.e. $f(x) \to l$, where l is a real number, then l is called limit of the function $f(x)$. In symbolic form, it can be written as $\lim_{x \to a} f(x) = l$.

LEFT HAND LIMIT (LHL)

$\lim_{x \to a^-} f(x)$ is the expected value of f at $x = a$ given the value of f near x to the left of a. This value is called the left hand limit of $f(x)$ at $x = a$,

i.e. LHL $= \lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a - h)$, where h is very small and $h > 0$.

RIGHT HAND LIMIT (RHL)

$\lim_{x \to a^+} f(x)$ is the expected value of f at $x = a$ given the value of f near x to the right of a. This value is called the right hand limit of $f(x)$ at $x = a$,

i.e. RHL $= \lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a + h)$, where h is very small and $h > 0$.

EXISTENCE OF LIMIT

If the left hand limit and right hand limit both exist (i.e. finite and unique) and coincide (i.e. same), then we can say that limit exists and that common value l (say) is called the limit of $f(x)$ at $x = a$, i.e if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = l$, then we say that $\lim_{x \to a} f(x)$ exists and we write

(vii) $\lim_{x \to 0} \dfrac{a^x - 1}{x} = \log_e a$ (viii) $\lim_{x \to 0} \dfrac{\log(1 + x)}{x} = 1$

(ix) $\lim_{x \to 0} (1 + x)^{1/x} = e$ (x) $\lim_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^x = e$

(xi) $\lim_{x \to \infty} \dfrac{\sin x}{x} = \lim_{x \to \infty} \dfrac{\cos x}{x} = 0$

$\lim_{x \to a} f(x) = l.$

EXAMPLE |1| Evaluate the left hand and right hand limits of the following function at $x = 2$.

$$f(x) = \begin{cases} 2x+3, & \text{if } x \le 2 \\ x+5, & \text{if } x > 2 \end{cases}.$$ Does $\lim_{x \to 2} f(x)$ exist?

Sol. Given, $f(x) = \begin{cases} 2x+3, & \text{if } x \le 2 \\ x+5, & \text{if } x > 2 \end{cases}$

$$\text{LHL} = \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2x+3)$$

[if $x < 2$, then $f(x) = 2x + 3$]

$$= \lim_{h \to 0} [2(2-h)+3]$$

[put $x = 2 - h$; when $x \to 2^-$, then $h \to 0$]

$$= 2(2-0) + 3 = 4 + 3 = 7$$

$$\text{RHL} = \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x+5)$$

[if $x > 2$, then $f(x) = x + 5$]

$$= \lim_{h \to 2} (2 + h + 5)$$

[put $x = 2 + h$; when $x \to 2^+$, then $h \to 0$]

$$= 2 + 0 + 5 = 7$$

$\therefore$ LHL of f (at $x = 2$) = RHL of f (at $x = 2$)

Hence, $\lim_{x \to 2} f(x)$ exists and is equal to 7.

EXAMPLE |2| Find the value of $\lim_{x \to 3} \frac{x^2 + 2x - 15}{x - 3}$.

Factorise numerator in linear factors and then solve it.

Sol. We have, $\lim_{x \to 3} \frac{x^2 + 2x - 15}{x - 3}$

$$= \lim_{x \to 3} \frac{(x-3)(x+5)}{(x-3)}$$

$$= \lim_{x \to 3} (x+5) = 3 + 5 = 8$$

SOME STANDARD RESULTS BASED ON LIMIT

(i) $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$ (ii) $\lim_{x \to 0} \frac{\sin x}{x} = 1$

(iii) $\lim_{x \to 0} \frac{\tan x}{x} = 1$ (iv) $\lim_{x \to 0} \frac{\sin^{-1} x}{x} = 1$

(v) $\lim_{x \to 0} \frac{\tan^{-1} x}{x} = 1$ (vi) $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

Note

(i) A function $f(x)$ is said to be **continuous** at $x = c$ from left, if $\lim_{x \to c^-} f(x) = f(c)$.

(ii) A function $f(x)$ is said to be **continuous** at $x = c$ from right, if $\lim_{x \to c^+} f(x) = f(c)$.

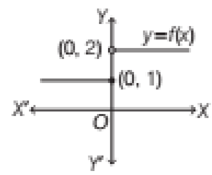
Working Rule to Check Continuity

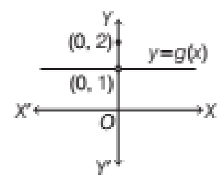
Some Illustrations to Understand Idea of Continuity (with Geometrical Approach)

Consider the following functions

$$f(x) = \begin{cases} 1, & \text{if } x \le 0 \\ 2, & \text{if } x > 0 \end{cases} \text{ and } g(x) = \begin{cases} 1, & \text{if } x \ne 0 \\ 2, & \text{if } x = 0 \end{cases}$$

and their graphs are as shown below





Here, both functions are defined at every point of the real line. The value of the function $f(x)$ at the points near and to the left of O (i.e. at points $-0.2, -0.1, -0.001, \ldots$) is 1 and at the points near and to the right of O (i.e. at points $0.2, 0.01, 0.001, \ldots$) is 2. Also, we can say that left hand limit of f at $x = 0$ is 1, right hand limit of f at $x = 0$ is 2 and value of f at $x = 0$ is 1.

Thus, LHL and RHL do not coincide and LHL = Value of function, so we cannot draw the graph of $f(x)$ without lifting pen from the plane of the paper. So, $f(x)$ is not continuous at $x = 0$. Similarly, we have left hand limit of g at $x = 0$ is 1, right hand limit of g at $x = 0$ is also 1 and value of g at $x = 0$ is 2. Thus, LHL and RHL coincide but the value of the function is different. So, we have to lift the pen at $x = 0$. Thus, we may say that if we can draw the graph of the function around the given fixed point without lifting the pen from the plane of a paper, then function is continuous at that fixed point.

CONTINUITY OF A FUNCTION AT A POINT

Suppose f is a real function and c is a point in the domain of f. Then, f is continuous at $x = c$, if $f(x)$ is defined at $x = c$, $\lim_{x \to c} f(x)$ exits and $\lim_{x \to c} f(x) = f(c)$, i.e. left hand limit, right hand limit and the value of the function at $x = c$ coincide. In other words, a function $f(x)$ is said to be continuous at $x = c$, if $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$.

EXAMPLE |3| Examine the continuity of the following functions at given points.

(i) $f(x) = x^2$ at $x = -25$ (ii) $f(x) = 2x^2 - 1$ at $x = 3$

(iii) $f(x) = |x - 5|$ at $x = 5$ **[NCERT]**

Sol. (i) Given, $f(x) = x^2$

At point $x = -2$, $f(-2) = (-2)^2 = 4$

of a Function at a Given Point

Suppose a function and a point is given to us and we have to check/examine/discuss the continuity of function at this point.

For this, we use the following steps

I. Write the given function say $f(x)$ and given point say $x = c$, at which we have to check continuity.

II. Find the value of $f(x)$ at $x = c$.

III. Find LHL (Left Hand Limit) of $f(x)$ at $x = c$ by using following steps

(i) First, consider $\lim_{x \to c^-} f(x)$.

(ii) Put $x = c - h$ and change the limit as $x \to c^-$ by $h \to 0$, then obtained limit from point (i) is $\lim_{h \to 0} f(c - h)$.

(iii) Simplify $\lim_{h \to 0} f(c - h)$ by using the appropriate formula.

IV. Find RHL (Right Hand Limit) of $f(x)$ at $x = c$ by using these steps

(i) First, consider $\lim_{x \to c^+} f(x)$.

(ii) Put $x = c + h$ and change the limit as $x \to c^+$ by $h \to 0$, then obtained limit from point (i) is $\lim_{h \to 0} f(c + h)$.

(iii) Simplify $\lim_{h \to 0} f(c + h)$ by using the appropriate formula.

V. Now, if LHL = RHL = $f(c)$. Then, f is continuous at $x = c$, otherwise not continuous.

Note

(i) If a function is not continuous at $x = c$, then it is said to be discontinuous at c and c is called the point of discontinuity of the function.

(ii) $f(x)$ is discontinuous at $x = c$, if any of the following cases arise.

(a) $f(c)$ is not defined.

(b) $\lim_{x \to c} f(x)$ does not exist.

(c) $\lim_{x \to c} f(x) \neq f(c)$.

Also, at $x = -2$, $f(x) = -5$

i.e. $f(-2) = -5$

Thus, $\lim_{x \to -2} f(x) = f(-2)$

Hence, $f(x)$ is continuous at $x = -2$.

EXAMPLE |5| Discuss the continuity of the function

Now, $\lim_{x \to -2} f(x) = \lim_{x \to -2} x^2 = (-2)^2 = 4$

Here, $f(-2) = \lim_{x \to -2} f(x)$

Hence, $f(x)$ is continuous at $x = -2$.

(ii) Given, $f(x) = 2x^2 - 1$

At point $x = 3$, $f(3) = 2(3)^2 - 1 = 2(9) - 1 = 17$

Now, $\lim_{x \to 3} f(x) = \lim_{x \to 3} (2x^2 - 1) = 2(3)^2 - 1 = 17$

Here, $f(3) = \lim_{x \to 3} f(x)$

Hence, $f(x)$ is continuous at $x = 3$.

(iii) Given, $f(x) = |x - 5| = \begin{cases} x - 5, & \text{if } x \geq 5 \\ 5 - x, & \text{if } x < 5 \end{cases}$

At point $x = 5$, $f(5) = |5 - 5| = 0$

Now, LHL $= \lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} (5 - x)$

$= \lim_{h \to 0} [5 - (5 - h)]$

[put $x = 5 - h$; when $x \to 5^-$, then $h \to 0$]

$= \lim_{h \to 0} h = 0$

and RHL $= \lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} (x - 5)$

$= \lim_{h \to 0} [(5 + h) - 5]$

[put $x = 5 + h$; when $x \to 5^+$, then $h \to 0$]

$= \lim_{h \to 0} h = 0$

Thus, LHL = RHL = $f(5) = 0$

Hence, $f(x)$ is continuous at $x = 5$.

EXAMPLE |4| Discuss the continuity of the function

$$f(x) = \begin{cases} \dfrac{x^2 - x - 6}{x + 2}, & \text{if } x \neq -2 \\ -5, & \text{if } x = -2 \end{cases} \text{ at } x = -2.$$

Sol. We have, $f(x) = \begin{cases} \dfrac{x^2 - x - 6}{x + 2}, & \text{if } x \neq -2 \\ -5, & \text{if } x = -2 \end{cases}$

At $x = -2$, $\lim_{x \to -2} f(x) = \lim_{x \to -2} \dfrac{x^2 - x - 6}{x + 2}$ ...(i)

Now, factorising the numerator, we get

$\lim_{x \to -2} f(x) = \lim_{x \to -2} \dfrac{(x - 3)(x + 2)}{(x + 2)}$

$= \lim_{x \to -2} (x - 3) = -2 - 3 = -5$

EXAMPLE |7| Discuss the continuity of the function $f(x)$ at $x = 1/2$, when $f(x)$ is defined as follows.

$$f(x) = \begin{cases} 1/2 + x, & \text{if } 0 \leq x < 1/2 \\ 1, & \text{if } x = 1/2 \\ 3/2 + x, & \text{if } 1/2 < x \leq 1 \end{cases}.$$

[Delhi 2011]

Sol. The given function is

$$f(x) = \begin{cases} \dfrac{\sin 5x}{x}, & \text{if } x \neq 0 \\ 5, & \text{if } x = 0 \end{cases} \text{ at } x = 0.$$

Sol. We have, $f(x) = \begin{cases} \dfrac{\sin 5x}{x}, & \text{if } x \neq 0 \\ 5, & \text{if } x = 0 \end{cases}$

At $x = 0$, $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} \dfrac{\sin 5x}{x}$

On multiplying numerator and denominator by 5 and then using result $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$, we get

$$= \lim_{x \to 0} \frac{5 \sin 5x}{5x} = 5 \times 1 = 5$$

Also, at $x = 0$, $f(0) = 5$

Thus, $\lim\limits_{x \to 0} f(x) = f(0)$

Hence, $f(x)$ is continuous at $x = 0$.

EXAMPLE |6| Discuss the continuity of the function

$$f(x) = \begin{cases} \dfrac{|x|}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} \text{ at } x = 0.$$

[NCERT]

Sol. We have, $f(x) = \begin{cases} \dfrac{|x|}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

At $x = 0$, LHL $= \lim\limits_{x \to 0^-} f(x)$

$= \lim\limits_{x \to 0^-} \dfrac{|x|}{x} = \lim\limits_{h \to 0} \dfrac{|0 - h|}{0 - h}$

[put $x = 0 - h$; when $x \to 0^-$, then $h \to 0$]

$= \lim\limits_{h \to 0} \dfrac{h}{-h}$ $\quad [\because |0 - h| = |-h| = h]$

$= \lim\limits_{h \to 0} (-1) = -1$

and RHL $= \lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} \dfrac{|x|}{x} = \lim\limits_{h \to 0} \dfrac{|0 + h|}{0 + h}$

[put $x = 0 + h$; when $x \to 0^+$, then $h \to 0$]

$= \lim\limits_{h \to 0} \dfrac{h}{h}$ $\quad [\because |0 + h| = |h| = h]$

$= \lim\limits_{h \to 0} (1) = 1$

Thus, LHL $\neq$ RHL.

Hence, $f(x)$ is not continuous at $x = 0$.

At $x = 0$, $f(0) = 0$

LHL $= \lim\limits_{x \to 0^-} f(x) = \lim\limits_{h \to 0} f(0 - h)$

$= \lim\limits_{h \to 0} f(-h)$

$= \lim\limits_{h \to 0} (-h) \sin\left(\dfrac{1}{-h}\right)$

$= \lim\limits_{h \to 0} h \sin \dfrac{1}{h}$ $\quad [\because \sin(-\theta) = -\sin\theta]$

$$f(x) = \begin{cases} \dfrac{1}{2} + x, & \text{if } 0 \le x < \dfrac{1}{2} \\ 1, & \text{if } x = \dfrac{1}{2} \\ \dfrac{3}{2} + x, & \text{if } \dfrac{1}{2} < x \le 1 \end{cases}$$

We have to check continuity of $f(x)$ at $x = \dfrac{1}{2}$.

Now, LHL $= \lim\limits_{x \to \frac{1}{2}^-} f(x) = \lim\limits_{x \to \frac{1}{2}^-} \left(\dfrac{1}{2} + x\right)$

$= \lim\limits_{h \to 0} \left(\dfrac{1}{2} + \dfrac{1}{2} - h\right)$

$\left[\text{put } x = \dfrac{1}{2} - h; \text{ when } x \to \dfrac{1^-}{2}, h \to 0\right]$

$= \lim\limits_{h \to 0} (1 - h) = 1$ $\quad$ [put $h = 0$]

RHL $= \lim\limits_{x \to \frac{1}{2}^+} f(x) = \lim\limits_{x \to \frac{1}{2}^+} \left(\dfrac{3}{2} + x\right)$

$= \lim\limits_{h \to 0} \left(\dfrac{3}{2} + \dfrac{1}{2} + h\right)$

$\left[\text{put } x = \dfrac{1}{2} + h; \text{ when } x \to \dfrac{1^+}{2}, h \to 0\right]$

$= \lim\limits_{h \to 0} (2 + h) = 2$ $\quad$ [put $h = 0$]

We know that a function $f(x)$ is said to be continuous at point $x = a$, if LHL = RHL = $f(a)$.

Here, LHL $\neq$ RHL

Hence, $f(x)$ is discontinuous at $x = \dfrac{1}{2}$.

EXAMPLE |8| Show that the function $f(x)$ given by

$$f(x) = \begin{cases} x \sin \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

is continuous at $x = 0$.

[NCERT Exemplar]

Sol. Given, $f(x) = \begin{cases} x \sin \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ and point is $x = 0$.

EXAMPLE |9| Discuss the continuity of the function f given by $f(x) = x^3 + x^2 - 1$.

Sol. Given, $f(x) = x^3 + x^2 - 1$.

Here, $f(x)$ is always defined in R.

Let c be any real number, i.e. $c \in R$.

Then, at $x = c$, $f(c) = c^3 + c^2 - 1$

and $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} (x^3 + x^2 - 1)$

$= c^3 + c^2 - 1$

$= 0 \times$ (an oscillating value lies between -1 and 1)

$= 0$

$$\text{RHL} = \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h)$$

$$= \lim_{h \to 0} f(h) = \lim_{h \to 0} h \sin\left(\frac{1}{h}\right)$$

$= 0 \times$ (an oscillating value lies between -1 and 1)

$= 0$

Here, LHL $= f(0) =$ RHL $= 0$

Hence, $f(x)$ is continuous at $x = 0$.

CONTINUITY OF A FUNCTION IN ITS DOMAIN

A function f is said to be continuous in its domain, if it is continuous at every point in its domain.

Continuity of a Function in an Interval

(i) A function $f(x)$ is said to be continuous in open interval (a, b), if it is continuous for every value of x in the interval (a, b).

(ii) A function $f(x)$ is said to be continuous in closed interval $[a, b]$, if

(a) it is continuous for every value of x in the open interval (a, b).

(b) $f(x)$ is continuous at $x = a$ from right,

i.e. $\lim_{x \to a^+} f(x) = f(a)$.

(c) $f(x)$ is continuous at $x = b$ from left,

i.e. $\lim_{x \to b^-} f(x) = f(b)$.

Note

(i) $\lim_{x \to a^-} f(x)$ and $\lim_{x \to b^+} f(x)$ do not make any sense as function is not defined.

(ii) If the domain of function f is a singleton i.e. it is defined only at one point, then it is a continuous function.

TOPIC PRACTICE 1

OBJECTIVE TYPE QUESTIONS

1 The function $f(x) = \begin{cases} 1, & \text{if } x \neq 0 \\ 2, & \text{if } x = 0 \end{cases}$ is not continuous at

(a) $x = 0$ (b) $x = 1$

(c) $x = -1$ (d) None of these

2 The point of discontinuity of the function $f(x) = \begin{cases} 2x + 3, & \text{if } x \le 2 \\ 2x - 3, & \text{if } x > 2 \end{cases}$ is

(a) $x = 0$ (b) $x = 1$

(c) $x = 2$ (d) None of these

Thus, $f(c) = \lim_{x \to c} f(x)$

So, $f(x)$ is continuous at every real number.

Hence, f is continuous for every real values of x.

EXAMPLE |10| Discuss the continuity of the function f, where f is defined by

$$f(x) = \begin{cases} -2, & \text{if } x \le -1 \\ 2x, & \text{if } -1 < x \le 1. \\ 2, & \text{if } x > 1 \end{cases}$$

[NCERT]

Sol. We have, $f(x) = \begin{cases} -2, & \text{if } x \le -1 \\ 2x, & \text{if } -1 < x \le 1. \\ 2, & \text{if } x > 1 \end{cases}$

At $x = -1$, $f(-1) = -2$

$$\text{LHL} = \lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (-2) = -2$$

$$\text{RHL} = \lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} 2x = \lim_{h \to 0} 2(-1 + h)$$

[put $x = -1 + h$; when $x \to -1^+$, then $h \to 0$]

$$= \lim_{h \to 0} (-2 + 2h) = -2$$

Thus, $f(-1) =$ LHL $=$ RHL

$\therefore$ $f(x)$ is continuous at $x = -1$.

At $x = 1$, $f(1) = 2$

$$\text{LHL} = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 2x = \lim_{h \to 0} 2(1 - h)$$

[put $x = 1 - h$; when $x \to 1^-$, then $h \to 0$]

$$= \lim_{h \to 0} (2 - 2h) = 2$$

$$\text{RHL} = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2 = 2$$

Thus, $f(1) =$ LHL $=$ RHL

So, $f(x)$ is continuous at $x = 1$.

Clearly $f(x)$ is continuous at all real numbers other than 1 and -1.

Hence, $f(x)$ is continuous for every value of x.

SHORT ANSWER Type I Questions

8 Is the function f defined by $f(x) = \begin{cases} x, & \text{if } x \le 1 \\ 5, & \text{if } x > 1 \end{cases}$ continuous at $x = 0$, at $x = 1$ and at $x = 2$? [NCERT]

9 Show that the function

$$f(x) = \begin{cases} \dfrac{\sin x}{x} + \cos x, & \text{if } x \neq 0 \\ 2, & \text{if } x = 0 \end{cases}$$

is continuous at $x = 0$.

SHORT ANSWER Type II Questions

10 Show that the function $f(x) = 2x - |x|$ is continuous at $x = 0$.

11 Let $f(x) = x - |x - x^2|$, $x \in [-1, 1]$. Find the point of

3 If $f(x) = \begin{cases} \lambda (x^2 - 2x), & \text{if } x \le 0 \\ 4x + 1, & \text{if } x > 0 \end{cases}$, then which one of the following is correct?

(a) $f(x)$ is continuous at $x = 0$ for any value of λ
(b) $f(x)$ is discontinuous at $x = 0$ for any value of λ
(c) $f(x)$ is discontinuous at $x = 1$ for any value of λ
(d) None of the above

4 The function $f(x) = \dfrac{4 - x^2}{4x - x^3}$ is **[NCERT Exemplar]**

(a) discontinuous at only one point
(b) discontinuous at exactly two points
(c) discontinuous at exactly three points
(d) None of the above

5 The function $f(x) = \cot x$ is discontinuous on the set

(a) $\{x = n\pi : n \in Z\}$ (b) $\{x = 2n\pi : n \in Z\}$
(c) $\left\{x = (2n+1)\dfrac{\pi}{2}; n \in Z\right\}$ (d) $\left\{x = \dfrac{n\pi}{2}; n \in Z\right\}$

VERY SHORT ANSWER Type Questions

6 Examine the continuity of the following functions at the given point. (Each part carries 1 Mark)

(i) $f(x) = 5x - 3$ at $x = -3$ **[NCERT]**
(ii) $f(x) = x^2 + 5$ at $x = -1$
(iii) $f(x) = x^2 + 3x + 4$ at $x = 1$

7 Is the function $f(x) = [x]$, where $[x]$ is the greatest integer function, continuous at $x = 1.5$? Support your answer. **[NCERT]**

18 Show that the function

$$f(x) = \begin{cases} \dfrac{e^{1/x} - 1}{e^{1/x} + 1}, & \text{when } x \ne 0 \\ 0, & \text{when } x = 0 \end{cases}$$

is discontinuous at $x = 0$. **[NCERT Exemplar]**

19 Show that the function

$$f(x) = \begin{cases} \dfrac{e^{1/x}}{1 + e^{1/x}}, & \text{if } x \ne 0 \\ 0, & \text{if } x = 0 \end{cases}$$

is discontinuous at $x = 0$. **[NCERT Exemplar]**

HINTS & SOLUTIONS

1. (a) **Hint** $\lim_{x \to 0^-} f(x) = 1 = \lim_{x \to 0^+} f(x) \ne f(0) = 2$

2. (c) **Hint** $\lim_{x \to 2^-} f(x) = 7$ and $\lim_{x \to 2^+} f(x) = 1$

discontinuity, (if any) of this function on $[-1, 1]$.

12 Discuss the continuity of the following function at $x = 0$

$$f(x) = \begin{cases} \dfrac{x^4 + 2x^3 + x^2}{\tan^{-1} x}, & \text{if } x \ne 0 \\ 0, & \text{if } x = 0 \end{cases}$$

13 Is the function $f(x) = \dfrac{3x + 4\tan x}{x}$ continuous at $x = 0$? If not, then how may the function be defined to make it continuous at this point?

14 If $f(x) = \dfrac{\sqrt{2}\cos x - 1}{\cot x - 1}, x \ne \dfrac{\pi}{4}$. Then, find the value of $f\left(\dfrac{\pi}{4}\right)$, so that $f(x)$ becomes continuous at $x = \dfrac{\pi}{4}$. **[NCERT Exemplar]**

15 Show that the function defined by $g(x) = x - [x]$ is discontinuous at all integral points. Here, $[x]$ denotes the greatest integer less than or equal to x. **[NCERT]**

16 Determine $f(0)$, so that the function $f(x)$ defined by $f(x) = \dfrac{(4^x - 1)^3}{\sin \dfrac{x}{4} \log\left(1 + \dfrac{x^2}{3}\right)}$ becomes continuous at $x = 0$.

17 Examine the continuity of a function

$$f(x) = \begin{cases} |x| \cos \dfrac{1}{x}, & \text{if } x \ne 0 \\ 0, & \text{if } x = 0 \end{cases} \text{ at } x = 0.$$

[NCERT Exemplar]

8. Hint For $x = 0$; $f(x) = x$, which is continuous and at $x = 1$, LHL $= 1$ and RHL $= 5$, hence discontinuous and at $x = 2$; $f(x) = 5$, which is continuous.

[**Ans.** At $x = 0$, $f(x)$ is continuous; at $x = 1$, $f(x)$ is not continuous and at $x = 2$, $f(x)$ is continuous]

9. We have, $f(x) = \begin{cases} \dfrac{\sin x}{x} + \cos x, & x \ne 0 \\ 2, & x = 0 \end{cases}$

Here, $\lim_{x \to 0} f(x) = \lim_{x \to 0}\left[\dfrac{\sin x}{x} + \cos x\right]$

$= \lim_{x \to 0} \dfrac{\sin x}{x} + \lim_{x \to 0} \cos x$

$= 1 + 1$ $\left[\because \lim_{x \to 0} \dfrac{\sin x}{x} = 1 \text{ and } \cos 0 = 1\right]$

$= 2$

Given, $f(0) = 2$

$\therefore \lim_{x \to 0} f(x) = f(0)$

Hence, $f(x)$ is continuous at $x = 0$.

3. (b) **Hint** $\lim_{x \to 0^-} f(x) = 0$ and $\lim_{x \to 0^+} f(x) = 1$

4. (c) We have, $f(x) = \dfrac{4 - x^2}{4x - x^3} = \dfrac{(4 - x^2)}{x(4 - x^2)}$

$$= \frac{(4 - x^2)}{x(2^2 - x^2)}$$

$$= \frac{4 - x^2}{x(2 + x)(2 - x)}$$

Clearly, $f(x)$ is discontinuous at exactly three points $x = 0$, $x = -2$ and $x = 2$.

5. (a) We know that $f(x) = \cot x$ is continuous in $R - \{n\pi : n \in Z\}$.

Since, $f(x) = \cot x = \dfrac{\cos x}{\sin x}$ [since, $\sin x = 0$ at $n\pi$, $n \in Z$]

Hence, $f(x) = \cot x$ is discontinuous on the set $\{x = n\pi : n \in Z\}$.

6. (i) Given, $f(x) = 5x - 3$

At $x = -3$,

$$\lim_{x \to -3} f(x) = \lim_{x \to -3} (5x - 3) = 5(-3) - 3$$

$$= -15 - 3 = -18$$

and $f(-3) = 5(-3) - 3 = -15 - 3 = -18$

$\therefore \lim_{x \to -3} f(x) = f(-3)$.

Hence, $f(x)$ is continuous at $x = -3$.

(ii) Solve as part (i). [**Ans.** Yes]

(iii) Solve as part (i). [**Ans.** Yes]

7. Yes, since greatest integer function is not continuous for all integer values.

$$\text{RHL} = \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h)$$

[put $x = 0 + h$; when $x \to 0^+$, then $h \to 0$]

$$= \lim_{h \to 0} f(h) = \lim_{h \to 0} h^2 = 0$$

Thus, $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$

$\Rightarrow f(x)$ is continuous at $x = 0$.

Hence, there is no point of discontinuity on $[-1, 1]$.

12. **Hint** $\lim_{x \to 0} f(x) = \lim_{x \to 0} \left(\dfrac{x^4 + 2x^3 + x^2}{\tan^{-1} x} \right) = \lim_{x \to 0} \dfrac{x^2 (x + 1)^2}{\tan^{-1} x}$

$$= \lim_{x \to 0} x(x + 1)^2 = 0 \quad \left[\because \lim_{x \to 0} \frac{\tan^{-1} x}{x} = 1 \right]$$

[**Ans.** $f(x)$ is continuous at $x = 0$]

13. **Hint** $\lim_{x \to 0} f(x) = \lim_{x \to 0} \left(\dfrac{3x + 4\tan x}{x} \right)$

$$\Rightarrow \lim_{x \to 0} \left(\frac{3x}{x} + \frac{4\tan x}{x} \right) = 3 + 4 = 7 \neq f(0)$$

10. The given function is $f(x) = 2x - |x|$.

$$\therefore \quad f(x) = \begin{cases} 2x - x, & \text{if } x \geq 0 \\ 2x - (-x), & \text{if } x < 0 \end{cases} \left[\because |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases} \right]$$

i.e. $f(x) = \begin{cases} x, & \text{if } x \geq 0 \\ 3x, & \text{if } x < 0 \end{cases}$

At $x = 0$, LHL $= \lim_{x \to 0^-} f(x)$

$$= \lim_{x \to 0^-} 3x = 3 \times 0 = 0$$

and RHL $= \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0$

Also, $f(0) = 0$

Thus, LHL = RHL $= f(0)$

Hence, $f(x)$ is continuous function at $x = 0$.

11. Given, $f(x) = x - |x - x^2|$

For simplifying the given function, put $x - x^2 = 0$

$\Rightarrow \quad x(1 - x) = 0$

$\Rightarrow \quad x = 0$ or $1 - x = 0$

$\Rightarrow \quad x = 0 \in [-1, 1]$

So, we have doubt for continuity of $f(x)$ at $x = 0$.

$$f(x) = \begin{cases} 2x - x^2, & \text{if } -1 \leq x < 0 \\ 0, & \text{if } x = 0 \\ x^2, & \text{if } 0 < x \leq 1 \end{cases}$$

At $x = 0$, $f(0) = 0$

LHL $= \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h)$

[put $x = 0 - h$; when $x \to 0^-$, then $h \to 0$]

$$= \lim_{h \to 0} f(-h) = \lim_{h \to 0} \{2(-h) - (-h)^2\}$$

$$= \lim_{h \to 0} (-2h - h^2) = 0$$

At $x = a$, LHL $= \lim_{x \to a^-} g(x) = \lim_{x \to a^-} (x - [x])$

Put $x = a - h$; when $x \to a^-$, then $h \to 0$

LHL $= \lim_{h \to 0} (a - h - [a - h])$

$$= \lim_{h \to 0} (a - h - (a - 1))$$

$$= \lim_{h \to 0} (-h + 1) = 1 \qquad [\because [a - h] = a - 1]$$

RHL $= \lim_{x \to a^+} g(x) = \lim_{x \to a^+} (x - [x])$

Put $x = a + h$; when $x \to a^+$, then $h \to 0$

RHL $= \lim_{h \to 0} (a + h - [a + h]) = \lim_{h \to 0} (a + h - a)$

$$= \lim_{h \to 0} h = 0 \qquad [\because [a + h] = a]$$

$\therefore$ LHL $\neq$ RHL

Thus, $g(x)$ is discontinuous at all integral points.

16. **Hint** Use the formula, $\lim_{x \to 0} \dfrac{a^x - 1}{x} = \log a$

Thus, for f to be continuous $f(0)$ should be 7.

$$\left[\text{Ans. } f(x) = \begin{cases} \dfrac{3x + 4\tan x}{x}, & \text{when } x \neq 0 \\ 7, & \text{when } x = 0 \end{cases}\right]$$

14. **Hint** For $f(x)$ to be continuous at $x = \frac{\pi}{4}$,

$$\lim_{x \to \pi/4} f(x) = f\left(\frac{\pi}{4}\right)$$

Consider, $\lim_{x \to \frac{\pi}{4}} \dfrac{\sqrt{2}\cos x - 1}{\cot x - 1}$

Put $x - \frac{\pi}{4} = t$, then $x = \frac{\pi}{4} + t$

$$\because \lim_{t \to 0} \frac{\sqrt{2}\cos\left(\frac{\pi}{4} + t\right) - 1}{\cot\left(\frac{\pi}{4} + t\right) - 1}$$

$$= \lim_{t \to 0} \frac{\sqrt{2}\left[\cos t \cdot \frac{1}{\sqrt{2}} - \sin t \cdot \frac{1}{\sqrt{2}}\right] - 1}{\frac{\cot t - 1}{\cot t + 1}}$$

$$\left[\because \cot(A + B) = \frac{\cot A \cot B - 1}{\cot B + \cot A}\right]$$

$$= \lim_{t \to 0} \frac{(\cos t - \sin t - 1)(\cot t + 1)}{\cot t - 1} \left[\text{Ans. } \frac{1}{2}\right]$$

15. Here, $g(x) = x - [x]$

Let a be an integer and h is very small, $h > 0$, then

$[a - h] = a - 1, [a + h] = a$

and $[a] = a$

18. We have, $\begin{cases} \dfrac{e^{1/x} - 1}{e^{1/x} + 1}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$

At $x = 0$, $f(0) = 0$

$$\text{LHL} = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{(e^{1/x} - 1)}{(e^{1/x} + 1)}$$

$$= \lim_{h \to 0} \frac{e^{\frac{1}{0-h}} - 1}{e^{\frac{1}{0-h}} + 1}$$

$$= \lim_{h \to 0} \frac{e^{-1/h} - 1}{e^{-1/h} + 1}$$

[put $x = 0 - h$; when $x \to 0^-$, then $h \to 0$]

$$= \lim_{h \to 0} \frac{\frac{1}{e^{1/h}} - 1}{\frac{1}{e^{1/h}} + 1} = \frac{0 - 1}{0 + 1} = -1 \qquad \left[\because \lim_{h \to 0} \frac{1}{e^{1/h}} = 0\right]$$

and $\lim_{x \to 0} \dfrac{\log(1 + x)}{x} = 1$, $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$

[**Ans.** $12(\log_e 4)^3$]

17. Given, $f(x) = \begin{cases} |x|\cos\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

At $x = 0$,

$$\text{LHL} = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} |x|\cos\frac{1}{x}$$

$$= \lim_{h \to 0} |0 - h|\cos\left(\frac{1}{0 - h}\right)$$

[put $x = 0 - h$; when $x \to 0^-$, then $h \to 0$]

$$= \lim_{h \to 0} h\cos\left(-\frac{1}{h}\right)$$

$= 0 \times$ (an oscillating value lie between -1 and 1)

[$\because \cos\theta$ lies between -1 to1]

$= 0$

$$\text{RHL} = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} |x|\cos\frac{1}{x}$$

$$= \lim_{h \to 0} |0 + h|\cos\left(\frac{1}{0 + h}\right)$$

[put $x = 0 + h$; when $x \to 0^+$, then $h \to 0$]

$$= \lim_{h \to 0} h\cos\left(\frac{1}{h}\right)$$

$= 0 \times$ (an oscillating value lie between -1 and 1) $= 0$

and $f(0) = 0$

$\therefore$ LHL = RHL = $f(0)$

Hence, $f(x)$ is continuous at $x = 0$.

and $$\text{RHL} = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{e^{1/x} - 1}{e^{1/x} + 1}$$

$$= \lim_{h \to 0} \frac{e^{\frac{1}{(0+h)}} - 1}{e^{\frac{1}{(0+h)}} + 1}$$

[put $x = 0 + h$; when $x \to 0^+$, then $h \to 0$]

$$= \lim_{h \to 0} \frac{e^{1/h} - 1}{e^{1/h} + 1}$$

On dividing numerator and denominator by $e^{1/h}$, we get

$$\text{RHL} = \lim_{h \to 0} \frac{\left(1 - \frac{1}{e^{1/h}}\right)}{\left(1 + \frac{1}{e^{1/h}}\right)} = \frac{1 - 0}{1 + 0} = 1 \qquad \left[\because \lim_{h \to 0} \frac{1}{e^{1/h}} = 0\right]$$

Here, LHL $\neq$ RHL

Hence, $f(x)$ is not continuous at $x = 0$.

19. Solve as Question 18.

|TOPIC 2|

Properties of Continuous Functions

There are some functions which are always continuous in their respective domain. **e.g.**

(i) Every constant function is continuous.

(ii) Every identity function is continuous.

(iii) Every rational function is continuous.

(iv) Every polynomial function is continuous.

(v) Modulus function $f(x) = |x|$ is continuous.

(vi) All trigonometric functions are continuous.

Algebra of Continuous Functions

We know that continuity of a function at a point is entirely dictated by the limit of the function at that point, so some algebra of continuous function similar as algebra of limits are given below

Theorem 1 Let f and g be two real functions continuous at a real number c, then

(i) $(f + g)$ is continuous at $x = c$.

(ii) $(f - g)$ is continuous at $x = c$.

(iii) fg is continuous at $x = c$.

(iv) $\dfrac{f}{g}$ is continuous at $x = c$, provided that $g(c) \neq 0$.

Proof We have, f and g are two real continuous functions.

So, $\lim_{x \to c} f(x) = f(c)$ and $\lim_{x \to c} g(x) = g(c)$

EXAMPLE |1| Discuss the continuity of sine function. **[NCERT]**

Sol. We know that $\lim_{x \to 0} \sin x = 0$ and $f(x) = \sin x$ is defined for every real number. To show $f(x)$ is continuous for every real number. Let c be a real number.

Then, $\lim_{x \to c} f(x) = \lim_{x \to c} \sin x = \lim_{h \to 0} \sin(c + h)$

[put $x = c + h$; when $x \to c$, then $h \to 0$]

$= \lim_{h \to 0} [\sin c \cos h + \cos c \sin h]$

$= \lim_{h \to 0} (\sin c \cos h) + \lim_{h \to 0} (\cos c \sin h)$

$= \sin c \lim_{h \to 0} \cos h + \cos c \lim_{h \to 0} \sin h$

$= \sin c \times 1 + \cos c \times 0$

$\left[\because \lim_{x \to 0} \cos x = 1 \text{ and } \lim_{x \to 0} \sin x = 0\right]$

$= \sin c = f(c)$

Hence, f is a continuous function.

EXAMPLE |2| A function $f(x)$ is defined below

Now, $\lim_{x \to c} (f + g)x = \lim_{x \to c} [f(x) + g(x)]$

$= \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$

$= f(c) + g(c)$

$= (f + g)(c)$

Hence, $(f + g)$ is continuous at $x = c$.

Similarly, we can prove other parts.

Note

(i) If we take f = constant function say λ, then $\lambda \cdot g$ is continuous at $x = c$. $[\because (\lambda \cdot g)(x) = \lambda \cdot g(x)]$

(ii) In particular, if $\lambda = -1$, then $-g$ is continuous at $x = c$. Thus, if g is continuous, then $-g$ is also continuous.

(iii) In part (iv) of above theorem, if we take f = constant function say λ, then $\dfrac{\lambda}{g}$ is continuous at $x = c$. $\left[\because \left(\dfrac{\lambda}{g}\right)x = \dfrac{\lambda}{g(x)}\right]$

In particular, if $\lambda = 1$, then $\dfrac{1}{g}$ is continuous at $x = c$.

Thus, if g is continuous, then $\dfrac{1}{g}$ is also continuous provided that $g \neq 0$.

Theorem 2 Suppose f and g are real valued functions such that (fog) is defined at c. If g is continuous at c and if f is continuous at $g(c)$, then (fog) is continuous at c.

Note If f and g are two real valued functions, then $(fog)(x) = f(g(x))$ is defined, whenever the range of g is a subset of domain of f.

$\Rightarrow \lim_{x \to \pi} g(x) = g(\pi)$

So, g is continuous at $x = \pi$.

Now, $g(\pi) = -1$

$\therefore k[g(\pi)] = k(-1) = |-1| = 1$

Now, $\lim_{y \to (-1)^+} k(y) = \lim_{y \to (-1)^+} |y| = \lim_{h \to 0} |-1 + h| = 1$

[put $y = -1 + h$; when $y \to -1^+$, then $h \to 0$]

and $\lim_{y \to (-1)^-} k(y) = \lim_{y \to (-1)^-} |y| = \lim_{h \to 0} |-1 - h| = 1$

[put $y = -1 - h$; when $y \to -1^-$, then $h \to 0$]

$\therefore k[g(\pi)] = \lim_{y \to g(\pi)} k(y)$

This shows that k is continuous at $g(\pi)$.

Consequently, (kog) is continuous at $x = \pi$.

But $kog(x) = k[g(x)] = k(\sin x + \cos x)$

$= |\sin x + \cos x|$

$= f(x)$

Hence, $f(x)$ is continuous at $x = \pi$.

$$f(x)=\begin{vmatrix} 1 & \cos x & 1-\cos x \\ 1+\sin x & \cos x & 1+\sin x-\cos x \\ \sin x & \sin x & 1 \end{vmatrix}$$

Determine $f(x)$ and check continuity of $f(x)$, $\forall x \in R$.

Sol. Given, $f(x)=\begin{vmatrix} 1 & \cos x & 1-\cos x \\ 1+\sin x & \cos x & 1+\sin x-\cos x \\ \sin x & \sin x & 1 \end{vmatrix}$

Applying $C_3 \to C_3 + C_2 - C_1$, we get

$$f(x)=\begin{vmatrix} 1 & \cos x & 0 \\ 1+\sin x & \cos x & 0 \\ \sin x & \sin x & 1 \end{vmatrix}$$

Expanding along C_3, we get

$$f(x) = 0 + 0 + 1[\cos x - \cos x(1+\sin x)]$$
$$= \cos x - \cos x - \cos x \sin x$$
$$= -\frac{1}{2}(2\sin x \cos x)$$
$$= -\frac{1}{2}\sin 2x, \text{ which is continuous everywhere.}$$

[$\because$ $\sin x$ is continuous everywhere, so $\lambda \sin x$ is also continuous for any constant value λ]

EXAMPLE |3| Discuss the continuity of $f(x) = |\sin x + \cos x|$ at $x = \pi$.

Sol. Let $g(x) = \sin x + \cos x$ and $k(y) = |y|$

We shall first show that g is continuous at $x = \pi$ and k is continuous at $y = g(\pi)$.

Now, $\lim\limits_{x \to \pi} g(x) = \lim\limits_{x \to \pi} (\sin x + \cos x)$

$$= \sin \pi + \cos \pi = -1$$

Also, $g(\pi) = \sin \pi + \cos \pi = -1$

V. Use the condition of continuity, i.e. $f(c) = \text{LHL} = \text{RHL}$ and simplify them by taking two terms at a time to get values of unknown.

EXAMPLE |5| Find the value of a, so that $f(x) = \begin{cases} ax+5, & \text{if } x \le 2 \\ x-1, & \text{if } x > 2 \end{cases}$ is continuous at $x = 2$.

Sol. Given, $f(x) = \begin{cases} ax+5, & \text{if } x \le 2 \\ x-1, & \text{if } x > 2 \end{cases}$

and at $x = 2$, $f(x)$ is continuous.

$\therefore$ $x = c = 2$

Here, a is an unknown constant.

At $x = 2$, $f(2) = a(2) + 5 = 2a + 5$

$$\text{LHL} = \lim_{x \to 2^-} f(x)$$
$$= \lim_{x \to 2^-} (ax + 5)$$

EXAMPLE |4| Show that the function defined by $f(x) = \sin(x^2)$ is a continuous function. **[NCERT]**

Sol. Given, $f(x) = \sin(x^2)$

It is defined for every real number.

Suppose $g(x) = \sin x$ and $h(x) = x^2$, then $f(x) = goh(x)$

Also, both $g(x)$ and $h(x)$ are continuous functions.

So, their composition is also continuous function.

Hence, $f(x) = \sin(x^2)$ is a continuous function.

Problems Based on Continuity

There are many problems which are based on continuity. Some of them with their method of solving are given below.

|TYPE I|

PROBLEMS BASED ON FINDING UNKNOWN CONSTANT IN CONTINUOUS FUNCTION

Suppose a function having one or two unknown constants is given to us. Also, suppose function is continuous at some point say $x = c$, then for finding the value of unknown or unknowns, we use the following steps

I. First, write the given function having unknown constants and the point say $x = c$ at which it is continuous.

II. Find the value of function at $x = c$, which may or may not be in terms of unknown.

III. Find LHL at $x = c$, which may or may not be in terms of unknown.

IV. Find RHL at $x = c$, which may or may not be in terms of unknown.

and $\text{RHL} = \lim\limits_{x \to 3^+} f(x) = \lim\limits_{x \to 3^+} \{bx + 3\}$

$$= \lim_{h \to 0} \{b(3+h) + 3\}$$

[put $x = 3 + h$; when $x \to 3^+$, then $h \to 0$]

$$= \lim_{h \to 0} \{3b + bh + 3\} = 3b + 3$$

Since, $f(x)$ is continuous at $x = 3$.

At $x = 3$, $f(3) = \text{LHL} = \text{RHL}$

i.e. $3a + 1 = 3a + 1 = 3b + 3$

$\Rightarrow$ $3a + 1 = 3b + 3$

$\Rightarrow$ $3a = 3b + 3 - 1$

$\Rightarrow$ $a = \dfrac{(3b+2)}{3} \Rightarrow a = b + \dfrac{2}{3}$

which is the required relation between a and b.

|TYPE II|

$$= \lim_{h \to 0} [a(2 - h) + 5]$$
$$= \lim_{h \to 0} (2a - ah + 5) = 2a + 5$$
$$\text{RHL} = \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x - 1)$$
$$= \lim_{h \to 0} [(2 + h) - 1]$$
$$= \lim_{h \to 0} (1 + h) = 1$$

Since, $f(x)$ is continuous at $x = 2$.

At $x = 2$,

$$\text{LHL} = \text{RHL} = f(2)$$

$\Rightarrow \quad 2a + 5 = 1 = 2a + 5$

$\therefore \quad 2a + 5 = 1$

$\Rightarrow \quad 2a = -4 \Rightarrow a = -2$

Hence, the required value of a is -2 for which $f(x)$ is continuous at $x = 2$.

EXAMPLE |6| Find the relationship between a and b so that the function f defined by $f(x) = \begin{cases} ax + 1, & \text{if } x \le 3 \\ bx + 3, & \text{if } x > 3 \end{cases}$ is continuous at $x = 3$. **[NCERT; All India 2011]**

Sol. Given, $f(x) = \begin{cases} ax + 1, & \text{if } x \le 3 \\ bx + 3, & \text{if } x > 3 \end{cases}$

and $f(x)$ is continuous at $x = 3$.

Here, a and b are two unknowns.

Now, at $x = 3$, $f(3) = a(3) + 1 = 3a + 1$

$$\text{LHL} = \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (ax + 1)$$
$$= \lim_{h \to 0} \{a(3 - h) + 1\}$$

[put $x = 3 - h$; when $x \to 3^-$ then $h \to 0$]

$$= \lim_{h \to 0} (3a - ah + 1) = 3a + 1$$

$\because$ LHL $\ne$ RHL at $x = 1$.

So, $f(x)$ is not continuous at $x = 1$.

Hence, $x = 1$ is the only point of discontinuity of f.

EXAMPLE |8| Find all points of discontinuity of f, where f is defined by $f(x) = \begin{cases} |x| + 3, & \text{if } x \le -3 \\ -2x, & \text{if } -3 < x < 3 \\ 6x + 2, & \text{if } x \ge 3 \end{cases}$. **[NCERT]**

Sol. Given, $f(x) = \begin{cases} |x| + 3, & \text{if } x \le -3 \\ -2x, & \text{if } -3 < x < 3 \\ 6x + 2, & \text{if } x \ge 3 \end{cases}$

Here, we have to check the continuity only at points $x = -3$ and $x = 3$.

[$\because$ functions defined in each interval is a polynomial, so it is continuous in each interval]

At $x = -3$,

$$\text{LHL} = \lim_{x \to -3^-} f(x) = \lim_{x \to -3^-} \{|x| + 3\}$$

PROBLEMS BASED ON FINDING POINTS OF DISCONTINUITY OF A FUNCTION

Suppose a function $f(x)$ is given to us and we have to find points of discontinuity, then we use following steps

I. First, find all possible points of given domain at which we have doubt for continuity of $f(x)$, say it is $x = a$. (here, more than one point is also possible)

II. Now, find LHL and RHL of $f(x)$ at $x = a$, if
   (i) $f(x)$ is continuous at $x = a$, so there is no point of discontinuity.
   (ii) $f(x)$ is not continuous at $x = a$, then $x = a$, is the required point of discontinuity.

EXAMPLE |7| Discuss the continuity of the function f defined by $f(x) = \begin{cases} x + 2, & \text{if } x \le 1 \\ x - 2, & \text{if } x > 1 \end{cases}$.

Sol. Given, $f(x) = \begin{cases} x + 2, & \text{if } x \le 1 \\ x - 2, & \text{if } x > 1 \end{cases}$

When $x < 1$, we have $f(x) = x + 2$, which is a polynomial function, so it is continuous at each point for $x < 1$.

Also, when $x > 1$, we have $f(x) = x - 2$, which is a polynomial function, so it is continuous at each point for $x > 1$.

Now, we have to check the continuity at $x = 1$.

$$\text{LHL} = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x + 2) = \lim_{h \to 0} (1 - h + 2) = 3$$

[put $x = 1 - h$; when $x \to 1^-$, then $h \to 0$]

$$\text{and RHL} = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x - 2)$$
$$= \lim_{h \to 0} (1 + h - 2) = -1$$

[put $x = 1 + h$; when $x \to 1^+$, then $h \to 0$]

| TOPIC PRACTICE 2 |

OBJECTIVE TYPE QUESTIONS

1 The function defined by $g(x) = x - [x]$ is discontinuous at
(a) all rational points
(b) all irrational points
(c) all integral points
(d) None of the above

2 The function $f(x) = \begin{cases} \dfrac{k \cos x}{\pi - 2x}, & \text{if } x \ne \dfrac{\pi}{2} \\ 3, & \text{if } x = \dfrac{\pi}{2} \end{cases}$ is continuous at $x = \dfrac{\pi}{2}$, when k equals

(a) -6 (b) 6
(c) 5 (d) -5

$= \lim_{h \to 0} \{|-3-h|+3\}$

[put $x=-3-h$; when $x \to -3^-$, then $h \to 0$]

$= \lim_{h \to 0} (3+h+3) = \lim_{h \to 0} (6+h) = 6$

$\text{RHL} = \lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} (-2x)$

$= \lim_{h \to 0} -2(-3+h)$

[put $x=-3+h$; when $x \to -3^+$, then $h \to 0$]

$= \lim_{h \to 0} (6-2h) = 6$ and $f(-3) = |-3| + 3 = 6$

Thus, $f(-3) = \text{LHL} = \text{RHL}$.

So, $f(x)$ is continuous at $x = -3$.

At $x = 3$,

$\text{LHL} = \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (-2x) = \lim_{h \to 0} -2(3-h)$

[put $x = 3-h$; when $x \to 3^-$, then $h \to 0$]

$= \lim_{h \to 0} (-6+2h) = -6$

$\text{RHL} = \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (6x+2) = \lim_{h \to 0} \{6(3+h)+2\}$

[put $x = 3+h$; when $x \to 3^+$, then $h \to 0$]

$= \lim_{h \to 0} (18+6h+2) = 20$

Thus, LHL $\neq$ RHL.

So, $f(x)$ is not continuous at $x = 3$.

Hence, point of discontinuity is $x = 3$.

(iii) $f(x) = \dfrac{x^2-25}{x+5}, x \neq -5$ **[NCERT]**

(iv) $f(x) = x - 5$ **[NCERT]**

7 Determine the value of 'k' for which the following function is continuous at $x = 3$.

$$f(x) = \begin{cases} \dfrac{(x+3)^2 - 36}{x-3}, & x \neq 3 \\ k, & x = 3 \end{cases}$$ **[All India 2017]**

SHORT ANSWER Type I Questions

8 Discuss the continuity of the function $f(x) = \sin x \cdot \cos x$. **[NCERT Exemplar]**

9 Discuss the continuity of the function $f(x) = \sin x - \cos x$. **[NCERT]**

10 Show that the function defined by $f(x) = \cos x^2$ is a continuous function. **[NCERT]**

11 Examine that $\sin|x|$ is a continuous function.

12 Show that the function $f(x) = \cos|x|$ is continuous.

13 Find the value of k for which the function

$$f(x) = \begin{cases} \dfrac{x^2+3x-10}{x-2}, & x \neq 2 \\ k, & x = 2 \end{cases}$$ is continuous at $x = 2$. **[Delhi 2017C]**

3 The number of points at which the function $f(x) = \dfrac{1}{x - [x]}$, $[\cdot]$ denotes the greatest integer function, is not continuous, is **[NCERT Exemplar]**

(a) 1 (b) 2

(c) 3 (d) None of these

4 If $f(x) = \begin{cases} \dfrac{\sqrt{1+kx} - \sqrt{1-kx}}{x}, & \text{for} -1 \leq x < 0 \\ 2x^2 + 3x - 2, & \text{for} 0 \leq x \leq 1 \end{cases}$

is continuous at $x = 0$, then k is equal to

(a) -4 (b) -3

(c) -2 (d) -1

5 If $f(x) = 2x$ and $g(x) = \dfrac{x^2}{2} + 1$, then which of the following can be a discontinuous function?

(a) $f(x) + g(x)$ (b) $f(x) - g(x)$

(c) $f(x) \cdot g(x)$ (d) $\dfrac{g(x)}{f(x)}$

VERY SHORT ANSWER Type Questions

6 Examine the following functions for continuity. (Each part carries 1 Mark)

(i) $f(x) = \dfrac{1}{x-5}, x \neq 5$ **[NCERT]**

(ii) $f(x) = \dfrac{1}{x+3}, x \in R$

SHORT ANSWER Type II Questions

18 If the function f defined as

$f(x) = \begin{cases} \dfrac{x^2-9}{x-3}, & x \neq 3 \\ k, & x = 3 \end{cases}$ is continuos at $x = 3$, find the value of k. **[All India 2020]**

19 Discuss the continuity of function f defined by

$$f(x) = \begin{cases} \dfrac{1}{2} - x, & \text{if } 0 \leq x < \dfrac{1}{2} \\ 1, & \text{if } x = \dfrac{1}{2} \\ \dfrac{3}{2} - x, & \text{if } \dfrac{1}{2} < x \leq 1 \end{cases}.$$

20 Given, $f(x) = \dfrac{1}{x-1}$. Find the point(s) of discontinuity of composite function $y = f\{f(x)\}$.

21 Show that the function f defined by $f(x) = |1 - x + |x||$, where x is any real number is a continuous function. **[NCERT]**

22 Find the value of k for which the function

$f(x) = \begin{cases} \dfrac{\sin x - \cos x}{4x - \pi}, & x \neq \dfrac{\pi}{4} \\ k, & x = \dfrac{\pi}{4} \end{cases}$ is continuous at $x = \dfrac{\pi}{4}$. **[Delhi 2017C]**

14 If $(x^2+y^2)^2 = xy$, find $\dfrac{dy}{dx}$. **[CBSE 2018]**

15 If the function $f(x) = \begin{cases} \dfrac{\sin x}{x} + \cos x, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$ is continuous at $x = 0$, then find the value of k. **[NCERT Exemplar]**

16 Find the value of a, so that the function $f(x)$ is defined by

$$f(x) = \begin{cases} \dfrac{\sin^2 ax}{x^2}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

may be continuous at $x = 0$.

17 Find the value of p for which the function $f(x) = \begin{cases} \dfrac{1-\cos 4x}{x^2}, & x \neq 0 \\ p, & x = 0 \end{cases}$ is continuous at $x = 0$. **[Delhi 2017C]**

23 Find the value of m, such that the function $f(x) = \begin{cases} m(x^2-2x), & \text{if } x < 0 \\ \cos x, & \text{if } x \geq 0 \end{cases}$ is continuous at $x = 0$.

24 For what value of λ is the function $f(x) = \begin{cases} \lambda(x^2-2x), & \text{if } x \leq 0 \\ 4x+1, & \text{if } x > 0 \end{cases}$ continuous at $x = 0$? What about continuity at $x = 1$? **[Foreign 2011]**

25 Find the value of k, so that the function defined by $f(x) = \begin{cases} kx+1, & \text{if } x \leq \pi \\ \cos x, & \text{if } x > \pi \end{cases}$ is continuous at $x = \pi$. **[Foreign 2011]**

26 For what value of k, the following function is continuous at $x = 0$?

$$f(x) = \begin{cases} \dfrac{1-\cos 4x}{8x^2}, & x \neq 0 \\ k, & x = 0 \end{cases}$$

[All India 2014C]

27 Prove that the function $f(x) = \begin{cases} \dfrac{x}{|x| + 2x^2}, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$ is continuous at $x = 0$ regardless of the value of k.

28 If $f(x) = \begin{cases} \dfrac{\cos^2 x - \sin^2 x - 1}{\sqrt{x^2+1} - 1}, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$ is continuous at $x = 0$, then find k.

29 Show that the function $f(x) = \begin{cases} |x-a| \sin \dfrac{1}{x-a}, & \text{if } x \neq a \\ 0, & \text{if } x = a \end{cases}$ is continuous at $x = a$. **[NCERT]**

30 Find the values of a and b such that the function defined by

$$f(x) = \begin{cases} 5, & \text{if } x \leq 2 \\ ax+b, & \text{if } 2 < x < 10 \\ 21, & \text{if } x \geq 10 \end{cases}$$

is a continuous function. **[NCERT]**

31 Find the value of k for which

$$f(x) = \begin{cases} \dfrac{\sqrt{1+kx} - \sqrt{1-kx}}{x}, & \text{if } -1 \leq x < 0 \\ \dfrac{2x+1}{x-1}, & \text{if } 0 \leq x < 1 \end{cases}$$

is continuous at $x = 0$. **[All India 2013]**

32 Let $f(x) = \begin{cases} \dfrac{1-\cos 4x}{x^2}, & \text{if } x < 0 \\ a, & \text{if } x = 0 \\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}} - 4}, & \text{if } x > 0 \end{cases}$ for what value

35 Find the values of a and b such that the function f defined by $f(x) = \begin{cases} \dfrac{x-4}{|x-4|} + a, & \text{if } x < 4 \\ a+b, & \text{if } x = 4 \\ \dfrac{x-4}{|x-4|} + b, & \text{if } x > 4 \end{cases}$ is a continuous function at $x = 4$. **[NCERT Exemplar]**

36 If $f(x) = \begin{cases} \dfrac{\sin(a+1)x + 2\sin x}{x}, & x < 0 \\ 2, & x = 0 \\ \dfrac{\sqrt{1+bx} - 1}{x}, & x > 0 \end{cases}$

is continuous at $x = 0$, then find the values of a and b. **[All India 2016]**

37 Find the values of p and q for which

$$f(x) = \begin{cases} \dfrac{1-\sin^3 x}{3\cos^2 x}, & \text{if } x < \dfrac{\pi}{2} \\ p, & \text{if } x = \dfrac{\pi}{2} \\ \dfrac{q(1-\sin x)}{(\pi-2x)^2}, & \text{if } x > \dfrac{\pi}{2} \end{cases}$$

is continuous at $x = \dfrac{\pi}{2}$. **[Delhi 2016]**

HINTS & SOLUTIONS

1. (c) **Hint** $f(x) = [x]$ is discontinuous at every integer.

2. (b) We have, $f(x) = \begin{cases} \dfrac{k\cos x}{\pi - 2x}, & \text{if } x \neq \dfrac{\pi}{2} \\ 3, & \text{if } x = \dfrac{\pi}{2} \end{cases}$

$f(x)$ is continuous at $x = \dfrac{\pi}{2}$

of a, f is continuous at $x = 0$?

[NCERT Exemplar; All India 2011C]

33 Find the value of a, if the function $f(x)$ defined by $f(x) = \begin{cases} 2x - 1, & \text{if } x < 2 \\ a, & \text{if } x = 2 \\ x + 1, & \text{if } x > 2 \end{cases}$ is continuous at $x = 2$.

[All India 2011C]

34 Find the value of k, so that the function

$$f(x) = \begin{cases} \dfrac{2^{x+2} - 16}{4^x - 16}, & \text{if } x \neq 2 \\ k, & \text{if } x = 2 \end{cases}$$ is continuous at $x = 2$.

4. (c) $\text{LHL} = \lim_{x \to 0^-} \dfrac{\sqrt{1 + kx} - \sqrt{1 - kx}}{x}$

$$= \lim_{x \to 0^-} \frac{2kx}{x(\sqrt{1 + kx} + \sqrt{1 - kx})} = k$$

$$\text{RHL} = \lim_{x \to 0^+} (2x^2 + 3x - 2) = -2$$

$f(0) = -2$

$\because$ It is given that $f(x)$ is continuous at $x = 0$.

$\therefore$ LHL = RHL = $f(0) \Rightarrow k = -2$

5. (d) We know that if f and g are continuous functions, then

(a) $f + g$ is continuous

(b) $f - g$ is continuous.

(c) fg is continuous

(d) $\dfrac{f}{g}$ is continuous at these points, where $g(x) \neq 0$.

Here, $\dfrac{g(x)}{f(x)} = \dfrac{\frac{x^2}{2} + 1}{2x} = \dfrac{x^2 + 2}{4x}$

which is discontinuous at $x = 0$.

6. (i) Here, $f(x) = \dfrac{1}{x - 5}$ is a rational function $\left[\text{i.e. } \dfrac{p(x)}{q(x)}\right]$.

So, $f(x)$ is continuous for all values of x provided $x \neq 5$.

(ii) Solve as part (i).

[**Ans.** $f(x)$ is not continuous at $x = -3$]

(iii) Here, $f(x) = \dfrac{x^2 - 25}{x + 5}, x \neq -5$

$$= \frac{(x - 5)(x + 5)}{x + 5}, x \neq -5 = x - 5$$

$\therefore f(x) = x - 5$ is a polynomial function, so $f(x)$ is continuous at all values of x, provided $x \neq -5$.

(iv) Here, $f(x) = x - 5$ is a polynomial function, so $f(x)$ is continuous for all values of x.

Note In a rational function $f(x) = \dfrac{p(x)}{q(x)}$, if $q(x)$ is zero at any value of x, then $f(x)$ is not continuous at that point.

$\therefore \lim_{x \to \frac{\pi}{2}^-} \dfrac{k \cos x}{\pi - 2x} = 3$

$\Rightarrow \lim_{h \to 0} \dfrac{k \cos\left(\frac{\pi}{2} - h\right)}{\pi - 2\left(\frac{\pi}{2} - h\right)} = 3$

$\Rightarrow \lim_{h \to 0} \dfrac{k \sin h}{2h} = 3$

$\therefore \dfrac{k}{2} = 3 \Rightarrow k = 6$

3. (d) $x - [x] = 0$ when x is an integer, so that $f(x)$ is discontinuous for all $x \in I$ i.e. $f(x)$ is discontinuous at infinite number of points.

$\Rightarrow \lim_{x \to 3} \dfrac{(x - 3)(x + 9)}{(x - 3)} = k$

$\Rightarrow \lim_{x \to 3} (x + 9) = k$

$\Rightarrow 3 + 9 = k \Rightarrow k = 12$

8. Since, $\sin x$ and $\cos x$ both are continuous function, therefore $\sin x \cdot \cos x$ is also continuous.

[$\because$ product of two continuous function is also continuous]

9. **Hint** Difference of two continuous function is also continuous. [**Ans.** Continuous]

10. Similar as Example 4.

11. Let $g(x) = |x|$ and $h(x) = \sin x$

Now, $hog(x) = h[g(x)] = h(|x|) = \sin|x| = f(x)$

Since, $g(x)$ and $h(x)$ are both continuous function for all $x \in R$, so composition of $g(x)$ and $h(x)$ is also a continuous function for all $x \in R$.

Hence, $f(x) = \sin|x|$ is a continuous function.

12. Solve as Question 11.

13. **Hint** $\lim_{x \to 2} f(x) = f(2)$

$\therefore \lim_{x \to 2} \dfrac{(x + 5)(x - 2)}{(x - 2)} = k$ [**Ans.** $k = 7$]

14. We have, $(x^2 + y^2)^2 = xy$

On differentiating both sides w.r.t. x, we get

$$2(x^2 + y^2)\left[2x + 2y\frac{dy}{dx}\right] = \left[x\frac{dy}{dx} + y\right]$$

$$\Rightarrow 4(x^2 + y^2)\left(x + y\frac{dy}{dx}\right) = \left(y + x\frac{dy}{dx}\right)$$

$$\Rightarrow 4(x^2 + y^2)x + 4(x^2 + y^2)y\frac{dy}{dx} = y + x\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}[4(x^2 + y^2)y - x] = y - 4x(x^2 + y^2)$$

$$\Rightarrow \frac{dy}{dx} = \frac{y - 4x(x^2 + y^2)}{4(x^2 + y^2)y - x}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{[y - 4x(x^2 + y^2)]}{[x - 4y(x^2 + y^2)]}$$

7. Given, $f(x) = \begin{cases} \dfrac{(x+3)^2 - 36}{x-3}, & x \neq 3 \\ k, & x = 3 \end{cases}$

Since, $f(x)$ is continuous at $x = 3$.

$$\therefore \quad \lim_{x \to 3} f(x) = f(3)$$

$$\Rightarrow \quad \lim_{x \to 3} \frac{(x+3)^2 - 36}{x-3} = k$$

$$\Rightarrow \quad \lim_{x \to 3} \frac{(x+3)^2 - 6^2}{x-3} = k$$

$$\Rightarrow \quad \lim_{x \to 3} \frac{(x+3-6)(x+3+6)}{x-3} = k$$

$$[\because a^2 - b^2 = (a-b)(a+b)]$$

18. Given, $f(x) = \begin{cases} \dfrac{x^2 - 9}{x-3}, & x \neq 3 \\ k, & x = 3 \end{cases}$

Since, $f(x)$ is continuous at $x = 3$.

$$\therefore \quad \lim_{x \to 3} f(x) = f(3)$$

$$\Rightarrow \quad \lim_{x \to 3} \frac{x^2 - 9}{x-3} = k$$

$$\Rightarrow \quad \lim_{x \to 3} \frac{(x-3)(x+3)}{(x-3)} = k$$

$$\Rightarrow \quad \lim_{x \to 3} (x+3) = k$$

$$\Rightarrow \quad 3 + 3 = k$$

$$\Rightarrow \quad k = 6$$

19. We have, $f(x) = \begin{cases} \dfrac{1}{2} - x, & 0 \leq x < \dfrac{1}{2} \\ 1, & x = \dfrac{1}{2} \\ \dfrac{3}{2} - x, & \dfrac{1}{2} < x \leq 1 \end{cases}$

Here, given function is a polynomial function, so it is continuous in the given intervals.

Now, we have to check the continuity at $x = \dfrac{1}{2}$.

$$\text{LHL} = \lim_{x \to \frac{1}{2}^-} f(x) = \lim_{x \to \frac{1}{2}^-} \left(\frac{1}{2} - x\right)$$

$$= \lim_{h \to 0} \left[\frac{1}{2} - \left(\frac{1}{2} - h\right)\right]$$

$$\left[\text{put } x = \frac{1}{2} - h; \text{ when } x \to \frac{1}{2}^-, \text{ then } h \to 0\right]$$

$$= \lim_{h \to 0} h = 0$$

$$\text{and} \quad \text{RHL} = \lim_{x \to \frac{1}{2}^+} f(x) = \lim_{x \to \frac{1}{2}^+} \left(\frac{3}{2} - x\right)$$

$$= \lim_{h \to 0} \left[\frac{3}{2} - \left(\frac{1}{2} + h\right)\right]$$

15. **Hint** $\lim_{x \to 0} f(x) = f(0)$

$$\Rightarrow \quad \lim_{x \to 0} \frac{\sin x}{x} + \lim_{x \to 0} \cos x = k$$

$$\Rightarrow \quad 1 + 1 = k \quad [\textbf{Ans. } 2]$$

16. **Hint** $\lim_{x \to 0} f(x) = f(0)$

$$\Rightarrow \quad a^2 \lim_{x \to 0} \left(\frac{\sin ax}{ax}\right)^2 = 1 \quad [\textbf{Ans. } a = \pm 1]$$

17. **Hint** $\lim_{x \to 0} f(x) = f(0)$

$$\Rightarrow \quad \lim_{x \to 0} \frac{2\sin^2 2x}{x^2} = p \Rightarrow 8 \lim_{x \to 0} \left(\frac{\sin 2x}{2x}\right)^2 = p$$

[**Ans.** $p = 8$]

20. **Hint** We have, $f(x) = \dfrac{1}{x-1}$

$$y = f\{f(x)\} = \frac{1}{f(x) - 1} = \frac{1}{\dfrac{1}{x-1} - 1} = \frac{x-1}{1-x+1} = \frac{x-1}{2-x}$$

At $x = 2$, $f\{f(x)\}$ is not defined.

[**Ans.** -2]

21. Let $g(x) = 1 - x + |x|$ and $h(x) = |x|$ be two functions defined on R.

Then, $(hog)(x) = h\{g(x)\} = h(1 - x + |x|)$

$$= |1 - x + |x|| = f(x), \forall x \in R.$$

Since, $(1 - x)$ is a polynomial function and $|x|$ is a modulus function, so both are continuous function on R.

Therefore, $g(x) = 1 - x + |x|$ is continuous function on R. and $h(x) = |x|$ is continuous function on R.

Hence, $f = hog$ is continuous for all real numbers.

22. **Hint** $\lim_{x \to \pi/4} f(x) = f(\pi/4)$

$$\Rightarrow \quad \lim_{x \to \pi/4} \frac{\sqrt{2}\left(\dfrac{1}{\sqrt{2}} \sin x - \cos x \cdot \dfrac{1}{\sqrt{2}}\right)}{4\left(x - \dfrac{\pi}{4}\right)} = k$$

$$\Rightarrow \quad \frac{1}{2\sqrt{2}} \lim_{x \to \pi/4} \frac{\sin\left(x - \dfrac{\pi}{4}\right)}{\left(x - \dfrac{\pi}{4}\right)} = k$$

$$\left[\textbf{Ans. } k = \frac{1}{2\sqrt{2}}\right]$$

23. We have, $f(x) = \begin{cases} m(x^2 - 2x), & \text{if } x < 0 \\ \cos x, & \text{if } x \geq 0 \end{cases}$

Given, $f(x)$ is continuous at $x = 0$.

$$\therefore \quad \text{LHL} = \text{RHL} = f(0) \quad \text{...(i)}$$

$$\Rightarrow \quad \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

$$\Rightarrow \quad \lim_{x \to 0^-} m(x^2 - 2x) = \lim_{x \to 0^+} (\cos x) = \cos 0$$

$$\Rightarrow \quad m(0 - 0) = \cos 0 = \cos 0 \Rightarrow 0 = 1 = 1$$

$\text{RHL} = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (4x + 1)$

$\therefore$ $\text{RHL} = \lim_{h \to 0} [4(0 + h) + 1] = \lim_{h \to 0} [4h + 1] = 0 + 1 = 1$

[put $x = 0 + h$, when $x \to 0^+$, then $h \to 0$] [1]

$\therefore$ LHL $\neq$ RHL

Thus, $f(x)$ is not continuous at $x = 0$ for any value of λ.

At $x = 1$,

$\text{LHL} = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (4x + 1)$

$\therefore \text{LHL} = \lim_{h \to 0} [4(1 - h) + 1] = \lim_{h \to 0} [5 - 4h] = 5 - 0 = 5$

[put $x = 1 - h$; when $x \to 1^-$, then $h \to 0$]

$\text{RHL} = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (4x + 1)$

$\therefore \text{RHL} = \lim_{h \to 0} [4(1 + h) + 1] = \lim_{h \to 0} (5 + 4h) = 5 + 0 = 5$

[put $x = 1 + h$; when $x \to 1$, then $h \to 0$]

Also, $f(1) = 4 \times 1 + 1 = 5$ $\quad [\because f(x) = 4x + 1]$

Thus, $f(x)$ is continuous at $x = 1$ for all values of λ.

25. Similar as Example 5. $\left[\textbf{Ans. } k = \frac{-2}{\pi}\right]$

26. Solve as Question 17. [**Ans.** $k = 1$]

27. **Hint** $f(x) = \begin{cases} \dfrac{x}{|x| + 2x^2}, & x \neq 0 \\ k, & x = 0 \end{cases}$

$$\Rightarrow \quad f(x) = \begin{cases} \dfrac{1}{2x - 1}, & x < 0 \\ k, & x = 0 \\ \dfrac{1}{2x + 1}, & x > 0 \end{cases}$$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{1}{2x - 1} = -1$

and $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{1}{2x + 1} = 1$

Hence, $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$

$\therefore$ $f(x)$ is discontinuous at $x = 0$.

[**Ans.** $f(x)$ is not continuous for any value of k]

28. **Hint** $\lim_{x \to 0} f(x) = f(0) \Rightarrow \lim_{x \to 0} \frac{\cos 2x - 1}{\sqrt{x^2 + 1} - 1} \times \frac{\sqrt{x^2 + 1} + 1}{\sqrt{x^2 + 1} + 1} = k$

$\Rightarrow \quad \lim_{x \to 0} \frac{\cos 2x - 1}{x^2} \cdot \lim_{x \to 0} \left(\sqrt{x^2 + 1} + 1\right) = k$ [**Ans.** $k = -4$]

29. **Hint** Find LHL and RHL at $x = a$. Both are 0.

Hence, $f(x)$ is continuous at $x = a$.

30. Here, $f(x) = \begin{cases} 5, & \text{if } x \leq 2 \\ ax + b, & \text{if } 2 < x < 10 \\ 21, & \text{if } x \geq 10 \end{cases}$

At $x = 2$, $\text{LHL} = \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (5) = 5$

and $\quad \text{RHL} = \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (ax + b)$

$\therefore \quad \text{RHL} = \lim_{h \to 0} [a(2 + h) + b]$

[put $x = 2 + h$; when $x \to 2^+$, then $h \to 0$]

$= \lim_{h \to 0} (2a + ah + b) = 2a + b$

Also, $f(2) = 5$

Since, $f(x)$ is continuous at $x = 2$.

$\therefore$ LHL = RHL = $f(2) \Rightarrow 2a + b = 5$ ...(i)

Now, at $x = 10$,

$\text{LHL} = \lim_{x \to 10^-} f(x) = \lim_{x \to 10^-} (ax + b)$

$\therefore \quad \text{LHL} = \lim_{h \to 0} [a(10 - h) + b] = \lim_{h \to 0} (10a - ah + b)$

[put $x = 10 - h$; when $x \to 10^-$, then $h \to 0$]

$= 10a + b$

and $\text{RHL} = \lim_{x \to 10^+} f(x) = \lim_{x \to 10^+} (21) = 21$

Also, $f(10) = 21$. Since, $f(x)$ is continuous at $x = 10$.

$\therefore$ LHL = RHL = $f(10)$

$\Rightarrow 10a + b = 21$ ...(ii)

On subtracting Eq. (i) from Eq. (ii), we get

$8a = 16 \Rightarrow a = 2$

On putting $a = 2$ in Eq. (i), we get $2 \times 2 + b = 5 \Rightarrow b = 1$

Hence, the values of a and b are 2 and 1, respectively.

31. Given, $f(x) = \begin{cases} \dfrac{\sqrt{1 + kx} - \sqrt{1 - kx}}{x}, & \text{if } -1 \leq x < 0 \\ \dfrac{2x + 1}{x - 1}, & \text{if } 0 \leq x < 1 \end{cases}$

is continuous at $x = 0$.

Now, $f(0) = \frac{2 \cdot 0 + 1}{0 - 1} = \frac{1}{-1} = -1$

$\text{LHL} = \lim_{h \to 0} f(0 - h) = \lim_{h \to 0} \frac{\sqrt{1 - kh} - \sqrt{1 + kh}}{-h}$

$= \lim_{h \to 0} \frac{\sqrt{1 - kh} - \sqrt{1 + kh} \times (\sqrt{1 - kh} + \sqrt{1 + kh})}{-h(\sqrt{1 - kh} + \sqrt{1 + kh})}$

$= \lim_{h \to 0} \frac{(1 - kh) - (1 + kh)}{-h(\sqrt{1 - kh} + \sqrt{1 + kh})}$

$[\because (a + b)(a - b) = a^2 - b^2]$

$= \lim_{h \to 0} \frac{-2kh}{-h(\sqrt{1 - kh} + \sqrt{1 + kh})}$

$= \lim_{h \to 0} \frac{2k}{\sqrt{1 - kh} + \sqrt{1 + kh}} = \frac{2k}{1 + 1} = \frac{2k}{2} = k$

$\because$ $f(x)$ is continuous at $x = 0$.

$\therefore f(0) = \text{LHL} \Rightarrow -1 = k$

So, $\quad k = -1$

32. **Hint** At $x = 0$, we have $f(0) = a$.

$$\text{LHL} = \lim_{x \to 0^-} f(x) = \lim_{h \to 0} \frac{1 - \cos 4h}{h^2} = \lim_{h \to 0} \frac{4 \cdot 2\sin^2 2h}{4 \cdot h^2}$$

$$= \lim_{h \to 0} 8\left(\frac{\sin^2 h}{2h}\right)^2 \quad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

$$\text{RHL} = \lim_{h \to 0^+} f(x) = \lim_{h \to 0} \frac{\sqrt{h}}{\sqrt{16 + \sqrt{h}} - 4} \times \frac{\sqrt{16 + \sqrt{h}} + 4}{\sqrt{16 + \sqrt{h}} + 4}$$

$$= \lim_{h \to 0} \frac{\sqrt{h}\,[\sqrt{16 + \sqrt{h}} + 4]}{16 + \sqrt{h} - 16} = 4 + 4 = 8$$

$\therefore$ LHL = RHL = $f(0)$ **[Ans. 8]**

33. Solve as Question 32. **[Ans.** $a = 3$**]**

34. **Hint** $\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{2^{x+2} - 16}{4^x - 16} = \lim_{x \to 2} \frac{4(2^x - 4)}{(2^x)^2 - 4^2}$

$$= \lim_{x \to 2} \frac{4(2^x - 4)}{(2^x + 4)(2^x - 4)} = \lim_{x \to 2} \frac{4}{2^x + 4} = \frac{1}{2} \left[\text{Ans. } k = \frac{1}{2}\right]$$

35. **Hint** $f(x) = \begin{cases} a - 1, & x < 4 \\ a + b, & x = 4 \\ b + 1, & x > 4 \end{cases}$

$\because f(x)$ is continuous at $x = 4$.

$\therefore \; a - 1 = a + b = b + 1$ **[Ans.** $a = 1, b = -1$**]**

36. Given function is

$$f(x) = \begin{cases} \dfrac{\sin (a+1)x + 2\sin x}{x}, & x < 0 \\ 2, & x = 0 \\ \dfrac{\sqrt{1 + bx} - 1}{x}, & x > 0 \end{cases}$$

Also, given that $f(x)$ is continuous at $x = 0$.

$\therefore \quad (\text{LHL})_{x=0} = (\text{RHL})_{x=0} = f(0)$ ...(i)

Now, $\text{LHL} = \lim_{x \to 0^-} f(x)$

$$= \lim_{h \to 0} f(0 - h)$$

[put $x = 0 - h$; when $x \to 0^-$, then $h \to 0$]

$$= \lim_{h \to 0} \frac{\sin(a+1)(0 - h) + 2\sin(0 - h)}{(0 - h)}$$

$$= \lim_{h \to 0} \frac{-\sin(a+1)h - 2\sin h}{-h} \quad [\because \sin(-\theta) = -\sin\theta]$$

$$= \lim_{h \to 0} \frac{\sin(a+1)h + 2\sin h}{h}$$

$$= \lim_{h \to 0} \frac{\sin(a+1)h}{h} + \lim_{h \to 0} 2\frac{\sin h}{h}$$

$$= \lim_{h \to 0} \frac{\sin(a+1)h}{(a+1)h} \times (a+1) + 2\lim_{h \to 0} \frac{\sin h}{h}$$

$$= 1 \times (a+1) + 2 \times 1 \quad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

$$= a + 1 + 2 = a + 3$$

and $\text{RHL} = \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h)$

[put $x = 0 + h$; when $x \to 0^+$, then $h \to 0$]

$$= \lim_{h \to 0} \frac{\sqrt{1 + b(0 + h)} - 1}{0 + h} = \lim_{h \to 0} \frac{\sqrt{1 + bh} - 1}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{1 + bh} - 1}{h} \times \frac{\sqrt{1 + bh} + 1}{\sqrt{1 + bh} + 1}$$

[multiplying numerator and denominator by $\sqrt{1 + bh} + 1$]

$$= \lim_{h \to 0} \frac{(1 + bh) - 1}{h(\sqrt{1 + bh} + 1)} = \lim_{h \to 0} \frac{bh}{h(\sqrt{1 + bh} + 1)}$$

$$= \lim_{h \to 0} \frac{b}{(\sqrt{1 + bh} + 1)} = \frac{b}{\sqrt{1 + 0} + 1} = \frac{b}{2}$$

From Eq. (i), we get

$$a + 3 = \frac{b}{2} = 2$$

$$\Rightarrow \quad a + 3 = 2 \text{ and } \frac{b}{2} = 2$$

$$\Rightarrow \quad a = -1 \text{ and } b = 4$$

37. Given, $f(x) = \begin{cases} \dfrac{1 - \sin^3 x}{3\cos^2 x}, & \text{if } x < \dfrac{\pi}{2} \\ p, & \text{if } x = \dfrac{\pi}{2} \\ \dfrac{q(1 - \sin x)}{(\pi - 2x)^2}, & \text{if } x > \dfrac{\pi}{2} \end{cases}$

is continuous at $x = \frac{\pi}{2}$.

$$\therefore \quad (\text{LHL})_{x=\pi/2} = (\text{RHL})_{x=\pi/2} = f\left(\frac{\pi}{2}\right) \quad \text{...(i)}$$

Now, $\text{LHL} = \lim_{x \to \frac{\pi}{2}^-} f(x) = \lim_{h \to 0} f\left(\frac{\pi}{2} - h\right)$

$\left[\text{put } x = \frac{\pi}{2} - h, \text{ when } x \to \frac{\pi}{2}^-, \text{ then } h \to 0\right]$

$$= \lim_{h \to 0} \frac{1 - \sin^3\left(\frac{\pi}{2} - h\right)}{3\cos^2\left(\frac{\pi}{2} - h\right)} = \lim_{h \to 0} \frac{1 - \cos^3 h}{3\sin^2 h}$$

$$\left[\because \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta, \sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta\right]$$

$$= \lim_{h \to 0} \frac{(1 - \cos h)(1^2 + \cos^2 h + 1 \times \cos h)}{3(1 - \cos^2 h)}$$

$$= \lim_{h \to 0} \frac{(1 - \cos h)(1 + \cos^2 h + \cos h)}{3(1 - \cos h)(1 + \cos h)}$$

$$= \lim_{h \to 0} \frac{(1 + \cos^2 h + \cos h)}{3(1 + \cos h)}$$

$$= \frac{1 + \cos^2 0 + \cos 0}{3(1 + \cos 0)} = \frac{1+1+1}{3(1+1)} = \frac{3}{3 \times 2} = \frac{1}{2}$$

and RHL $= \lim_{x \to \frac{\pi}{2}^+} f(x) = \lim_{h \to 0} f\left(\frac{\pi}{2} + h\right)$

$$\left[\text{put } x = \frac{\pi}{2} + h, \text{ when } x \to \frac{\pi^+}{2} \text{ then } h \to 0\right]$$

$$= \lim_{h \to 0} \frac{q\left[1 - \sin\left(\frac{\pi}{2} + h\right)\right]}{\left[\pi - 2\left(\frac{\pi}{2} + h\right)\right]^2}$$

$$= \lim_{h \to 0} \frac{q(1 - \cos h)}{(\pi - \pi - 2h)^2} = \lim_{h \to 0} \frac{q(1 - \cos h)}{4h^2}$$

$$= \lim_{h \to 0} \frac{q\left(2\sin^2 \frac{h}{2}\right)}{4h^2} \quad \left[\because \cos x = 1 - 2\sin^2 \frac{x}{2}\right]$$

$$= \lim_{h \to 0} \frac{q \sin^2 \frac{h}{2}}{2\left(\frac{h}{2}\right)^2 \times 4}$$

$$= \frac{q}{8} \lim_{h \to 0} \left[\frac{\sin\left(\frac{h}{2}\right)}{h/2}\right]^2 \quad \left[\because \lim_{h \to 0} \frac{\sin h}{h} = 1\right]$$

$$= \frac{q}{8} \times 1 = \frac{q}{8}$$

From Eq. (i), $\frac{1}{2} = \frac{q}{8} = p \Rightarrow \frac{1}{2} = \frac{q}{8}$ and $\frac{1}{2} = p$

$\therefore \quad q = 4$ and $p = \frac{1}{2}$

|TOPIC 3|

Differentiability and Derivatives of Various Functions

DIFFERENTIABILITY OR DERIVABILITY

A real function f is said to be **derivable** or **differentiable** at a point c in its domain, if its **left hand** and **right hand** derivatives at c exist (i.e. finite and unique) and are equal, i.e. $Lf'(c) = Rf'(c)$.

Here, at $x = c$ left hand derivative,

$$\text{LHD} = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = \lim_{h \to 0} \frac{f(c-h) - f(c)}{-h} = Lf'(c)$$

and right hand derivative,

$$\text{RHD} = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = Rf'(c).$$

The common value of $Lf'(c)$ and $Rf'(c)$ is known as the derivative of $f(x)$ at $x = c$ and denoted by $\frac{d}{dx}\{f(x)\}|_c$ or $f'(c)$.

A function is said to be differentiable in an interval (a, b), if it is differentiable at every point of (a, b). Similarly, a function is said to be differentiable in an interval $[a, b]$, if it is differentiable at every point of $[a, b]$. As in case of continuity we take right hand derivative and left hand derivative at points a and b, respectively.

Working Rule to Show Differentiability of a Function

Suppose a function $f(x)$ is given to us and we have to check its differentiability at point $x = c$ in its domain. Then, we use the following steps

I. First, write the given function say $f(x)$ and the point say $x = c$ at which we have to check differentiability.

II. Find left hand derivative (LHD) at $x = c$ by using the formula, LHD $= \lim_{h \to 0} \frac{f(c-h) - f(c)}{-h}$.

III. Find right hand derivative (RHD) at $x = c$ by using the formula, RHD $= \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$.

IV. If LHD = RHD at $x = c$, then $f(x)$ is differentiable at $x = c$, otherwise $f(x)$ is not differentiable at $x = c$.

EXAMPLE |1| Discuss the differentiability of $f(x) = x^2$ at $x = 1$.

Sol. We have, $f(x) = x^2$

Test for differentiability at $x = 1$,

$$\text{LHD} = \lim_{h\to 0} \frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0} \frac{(1-h)^2-1}{-h}$$

$$= \lim_{h\to 0} \frac{1+h^2-2h-1}{-h} = \lim_{h\to 0}(-h+2) = 2$$

$$\text{RHD} = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{(1+h)^2-1}{h}$$

$$= \lim_{h\to 0} \frac{h^2+2h+1-1}{h} = \lim_{h\to 0}(h+2) = 2$$

$\because$ LHD = RHD = 2

Hence, $f(x)$ is differentiable at $x = 1$.

EXAMPLE |2| Prove that $f(x) = |x|$ is not differentiable at $x = 0$.

Sol. We have, $f(x) = |x|$.

Test for differentiability at $x = 0$,

$$Lf'(0) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{f(-h)-f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{|-h|-|0|}{-h} \qquad [\because f(x) = |x|]$$

$$= \lim_{h\to 0} \frac{h}{-h} = \lim_{h\to 0}(-1) = -1$$

and $$Rf'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{f(h)-f(0)}{h}$$

$$= \lim_{h\to 0} \frac{|h|-|0|}{h}$$

$$= \lim_{h\to 0}\left(\frac{h}{h}\right) = \lim_{h\to 0}(1) = 1$$

$\because \quad Lf'(0) \neq Rf'(0)$

Hence, f is not differentiable at $x = 0$.

EXAMPLE |3| Examine the following function $f(x)$ for continuity at $x = 1$ and differentiability at $x = 2$.

$$f(x) = \begin{cases} 5x-4, & \text{if } 0 < x < 1 \\ 4x^2-3x, & \text{if } 1 \le x < 2 \\ 3x+4, & \text{if } x \ge 2 \end{cases}$$

Sol. We have, $$f(x) = \begin{cases} 5x-4, & \text{if } 0 < x < 1 \\ 4x^2-3x, & \text{if } 1 \le x < 2 \\ 3x+4, & \text{if } x \ge 2 \end{cases}$$

For continuity, at $x = 1$

$$\text{LHL} = \lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h) \qquad \begin{bmatrix} \text{put } x = 1-h; \\ \text{when } x \to 1^-, \\ \text{then } h \to 0 \end{bmatrix}$$

$$= \lim_{h\to 0}\{5(1-h)-4\} = \lim_{h\to 0}(5-5h-4)$$

$$= \lim_{h\to 0}(1-5h) = 1$$

$$\text{RHL} = \lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h)$$

$$= \lim_{h\to 0}\{4(1+h)^2-3(1+h)\}$$

$$= \lim_{h\to 0}(4+8h+4h^2-3-3h)$$

$$= \lim_{h\to 0}(1+5h+4h^2) = 1$$

Also, $f(1) = 4(1)^2 - 3(1) = 1$.

Here, LHL = RHL = $f(1)$.

Hence, $f(x)$ is continuous at $x = 1$.

For differentiability at $x = 2$,

$$\text{LHD} = \lim_{h\to 0} \frac{f(2-h)-f(2)}{-h}$$

$$= \lim_{h\to 0} \frac{\{4(2-h)^2-3(2-h)\}-\{3(2)+4\}}{-h}$$

$$= \lim_{h\to 0} \frac{\{4(4-4h+h^2)-6+3h\}-\{6+4\}}{-h}$$

$$= \lim_{h\to 0} \frac{16-16h+4h^2-6+3h-10}{-h}$$

$$= \lim_{h\to 0} \frac{4h^2-13h}{-h} = \lim_{h\to 0}(-4h+13) = 13$$

$$\text{RHD} = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h\to 0} \frac{3(2+h)+4-(3\times 2+4)}{h}$$

$$= \lim_{h\to 0} \frac{(6+3h+4-10)}{h}$$

$$= \lim_{h\to 0}\left(\frac{3h}{h}\right) = \lim_{h\to 0}(3) = 3$$

Since, LHD $\neq$ RHD

Hence, f is not differentiable at $x = 2$.

Some Basic Differentiable Functions

There are some functions which are differentiable in their respective domain.

(i) Every polynomial function is differentiable at each $x \in R$.

(ii) The exponential function a^x, $a > 0$ is differentiable at each $x \in R$.

(iii) Every constant function is differentiable at each $x \in R$.

(iv) The logarithmic function is differentiable at each point in its domain.

(v) The trigonometric and inverse trigonometric functions are differentiable in their respective domains.

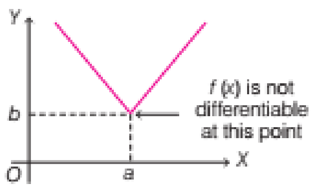
(vi) The sum, difference, product and quotient of two differentiable functions is differentiable.

(vii) The composition of differentiable functions is a differentiable function.

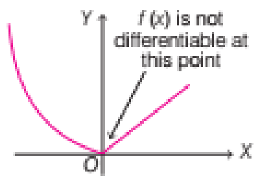
Graphical Meaning of Differentiability

Graphically, we interpret $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ as the slope of the tangent to the graph of $y = f(x)$ at the point $[x, f(x)]$. Thus, if the tangent to the given curve $y = f(x)$ at a certain point from left and right exist and are same straight line. Then, function is said to be differentiable at the certain point. In other words, function is always not differentiable at a corner point of a curve, i.e. a point, where the curve suddenly changes direction. See the following graphs:

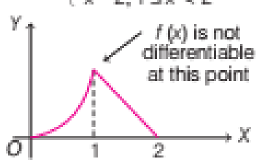
(i) $f(x) = |x - a| + b$



(ii) $f(x) = \begin{cases} x^2, & x \le 0 \\ x, & x > 0 \end{cases}$



(iii) $f(x) = \begin{cases} x^2, & 0 \le x < 1 \\ x - 2, & 1 \le x < 2 \end{cases}$



Relation between Continuity and Differentiability

The following theorem gives the relation between continuity and differentiability

> **Theorem** If a function is differentiable at a point c, then it is also continuous at that point.

Proof Given f is a differentiable function at c.

Then, $\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$

But for $x \neq c$, we have

$$f(x) - f(c) = \frac{f(x) - f(c)}{(x - c)} \cdot (x - c)$$

Taking $\lim_{x \to c}$ on both sides, we get

$$\lim_{x \to c} [f(x) - f(c)] = \lim_{x \to c} \left[\frac{f(x) - f(c)}{x - c} \cdot (x - c) \right]$$

$$\Rightarrow \lim_{x \to c} f(x) - \lim_{x \to c} f(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \cdot \lim_{x \to c} (x - c)$$

$$\Rightarrow \lim_{x \to c} f(x) - f(c) = f'(c) \cdot 0$$

$$\Rightarrow \lim_{x \to c} f(x) - f(c) = 0 \quad \Rightarrow \quad \lim_{x \to c} f(x) = f(c)$$

Hence, f is continuous at $x = c$.

Note Every differentiable function is continuous but converse may not be true e.g. $f(x) = |x|$.

Derivative or Differential Coefficient

Let f be a real valued function and c be a point in its domain, then derivative of f at c is given by

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

The derivative is defined, whenever the limit exists.

In general, the derivative of f at x is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \text{ provided limit exists.}$$

(i) The derivative of $f(x)$ is denoted by $f'(x)$ or $\frac{d}{dx} f(x)$.

(ii) If $y = f(x)$, then derivative or the differential coefficient of f w.r.t. x is denoted by $\frac{dy}{dx}$ or y' or $f'(x)$ or $Df(x)$.

DIFFERENTIATION

The process of finding derivative of a function is called differentiation.

Derivatives of Some Standard Functions

Derivatives of some standard functions which have been derived by first principle in class XI are given below

(i) $\frac{d}{dx}(\text{constant}) = 0$

(ii) $\frac{d}{dx}(x^n) = nx^{n-1}$

(iii) $\frac{d}{dx}(cx^n) = cnx^{n-1}$

(iv) $\frac{d}{dx}(\sin x) = \cos x$

(v) $\frac{d}{dx}(\cos x) = -\sin x$

(vi) $\frac{d}{dx}(\tan x) = \sec^2 x$

(vii) $\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cot x$

(viii) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(ix) $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

(x) $\frac{d}{dx}(e^x) = e^x$

(xi) $\frac{d}{dx}(a^x) = a^x \log_e a, a > 0$

(xii) $\frac{d}{dx}(\log_e x) = \frac{1}{x}, x > 0$

(xiii) $\frac{d}{dx}(\log_a x) = \frac{1}{x \log_e a}, a > 0, a \neq 1$

Algebra of Derivatives

The following theorems are important parts of algebra of derivatives.

SUM AND DIFFERENCE RULE

Let u and v be the two functions of x.

Then, $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$.

PRODUCT OR LEIBNITZ RULE

Derivative of the product of two functions is given as

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

i.e. Derivative of the product of two functions

= (First function × Derivative of second function)
+ (Second function × Derivative of first function)

QUOTIENT RULE

If u and v are differentiable functions, then

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\left[v\frac{du}{dx} - u\frac{dv}{dx}\right]}{v^2}$$

$$= \frac{\left[\text{Denominator} \times \frac{d}{dx}(\text{Numerator}) - \text{Numerator} \times \frac{d}{dx}(\text{Denominator})\right]}{(\text{Denominator})^2}$$

EXAMPLE |4| Find the derivative of $(x^2 + \cos x)$.

Sol. Let $y = x^2 + \cos x$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 + \cos x)$$

$$= \frac{d}{dx}(x^2) + \frac{d}{dx}(\cos x)$$

$$= 2x - \sin x \qquad \left[\because \frac{d}{dx}(x^n) = nx^{n-1}\right]$$

EXAMPLE |5| If $y = (2 + 3\sin x)(3 - 2\cos x)$, then find $\frac{dy}{dx}$.

Sol. We have, $y = (2 + 3\sin x)(3 - 2\cos x)$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[(2 + 3\sin x)(3 - 2\cos x)]$$

$$= (2+3\sin x)\frac{d}{dx}(3-2\cos x) + (3-2\cos x)\frac{d}{dx}(2+3\sin x)$$

$$\left[\because \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}\right]$$

$$= (2 + 3\sin x)(0 + 2\sin x) + (3 - 2\cos x)(0 + 3\cos x)$$

$$= (2 + 3\sin x)(2\sin x) + (3 - 2\cos x)(3\cos x)$$

$$= 4\sin x + 6\sin^2 x + 9\cos x - 6\cos^2 x$$

$$= 4\sin x + 9\cos x + 6\sin^2 x - 6\cos^2 x$$

$$= 4\sin x + 9\cos x - 6(\cos^2 x - \sin^2 x)$$

$$= 4\sin x + 9\cos x - 6\cos 2x$$

$$[\because \cos^2 x - \sin^2 x = \cos 2x]$$

EXAMPLE |6| Differentiate $\frac{\sec x - 1}{\sec x + 1}$ w.r.t. x.

Sol. Let $y = \frac{\sec x - 1}{\sec x + 1}$

$$\Rightarrow y = \frac{\left(\frac{1}{\cos x} - 1\right)}{\left(\frac{1}{\cos x} + 1\right)} \qquad \left[\because \sec x = \frac{1}{\cos x}\right]$$

$$\Rightarrow y = \frac{1 - \cos x}{1 + \cos x}$$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{\left[(1+\cos x)\frac{d}{dx}(1-\cos x) - (1-\cos x)\frac{d}{dx}(1+\cos x)\right]}{(1+\cos x)^2}$$

[using quotient rule]

$$= \frac{[(1 + \cos x)(0 + \sin x) - (1 - \cos x)(0 - \sin x)]}{(1 + \cos x)^2}$$

$$= \frac{[(1+\cos x)\sin x - (1-\cos x)(-\sin x)]}{(1+\cos x)^2}$$

$$= \frac{[\sin x + \cos x \cdot \sin x + \sin x - \cos x \cdot \sin x]}{(1+\cos x)^2}$$

$$= \frac{2\sin x}{(1+\cos x)^2}$$

DERIVATIVES OF COMPOSITE FUNCTIONS

Derivative of a composite function can be found out by chain rule.

Chain Rule

Chain rule is applied when the given function is the function of function, i.e. a function is in the form of $fog(x)$ or $f[g(x)]$.

Let y be a real valued function which is a composite of two functions, say $y = f(u)$ and $u = g(x)$.

Then, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = f'(u) \cdot g'(x)$

i.e. $\frac{d}{dx}[f\{g(x)\}] = f'\{g(x)\} \cdot g'(x)$

EXTENSION OF CHAIN RULE

Extension of chain rule is applied when the given function (say y) is a composite of three functions.

e.g.

If $y = f(u)$, $u = g(v)$ and $v = h(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx} = f'(u) \cdot g'(v) \cdot h'(x)$$

i.e. $\frac{d}{dx}[f\{g(h(x))\}] = f'[g\{h(x)\}].\ g'\{h(x)\} \cdot h'(x)$.

EXAMPLE |7| Differentiate $\sin(x^2 + 5)$ w.r.t. x.

[NCERT]

Sol. Let $y = \sin u$, where $u = x^2 + 5$.

Using chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos u \cdot \frac{du}{dx} = \cos(x^2+5)\frac{d}{dx}(x^2+5)$$

$$= \cos(x^2+5) \cdot (2x) = 2x \cdot \cos(x^2+5)$$

EXAMPLE |8| Differentiate $\sin^2(3x+1)$ w.r.t. x.

Sol. Let $y = \sin^2(3x+1)$

Here, y is a composite of three functions.

Let $y = t^2$, where $t = \sin u$ and $u = 3x + 1$.

Then, $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{du} \cdot \frac{du}{dx}$ [by chain rule]

$$= \frac{d}{dt}[t^2] \cdot \frac{d}{du}(\sin u) \cdot \frac{d}{dx}(3x+1)$$

$$= 2t(\cos u) \cdot 3 = 6t\cos u$$

$= 6\sin u \cos u$ [$\because t = \sin u$]

$= 3\sin 2u$ [$\because 2\sin\theta\cos\theta = \sin 2\theta$]

$= 3\sin 2(3x+1)$ [put $u = 3x+1$]

$= 3\sin(6x+2)$

EXAMPLE |9| Differentiate $\sin 5x \cdot \cos 7x$ w.r.t. x.

Sol. Let $y = \sin 5x \cdot \cos 7x = \frac{1}{2}[2\sin 5x \cdot \cos 7x]$

$$= \frac{1}{2}[\sin(5x+7x) + \sin(5x-7x)]$$

[$\because 2\sin A\cos B = \sin(A+B) + \sin(A-B)$]

$$= \frac{1}{2}[\sin 12x - \sin 2x]$$

$$\Rightarrow y = \frac{1}{2}\sin 12x - \frac{1}{2}\sin 2x$$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{2} \cdot \frac{d}{dx}(\sin 12x) - \frac{1}{2} \cdot \frac{d}{dx}(\sin 2x)$$

[by difference rule]

$$= \frac{1}{2}\cos 12x \frac{d}{dx}(12x) - \frac{1}{2}\cos 2x \frac{d}{dx}(2x)$$

$$= \frac{1}{2}\cos 12x \cdot 12 - \frac{1}{2}\cos 2x \cdot 2$$

$$= 6\cos 12x - \cos 2x$$

EXAMPLE |10| Differentiate $2\sqrt{\cot(x^2)}$ w.r.t. x.

[NCERT]

Sol. Let $y = 2\sqrt{\cot(x^2)}$.

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}2(\cot x^2)^{1/2} = 2 \cdot \frac{1}{2}\{\cot(x^2)\}^{\frac{1}{2}-1} \cdot \frac{d}{dx}\cot(x^2)$$

$\left[\text{using chain rule, } \frac{d}{dx}[fg(x)] = f'\{g(x)\}\frac{d}{dx}g(x)\right]$

$= \frac{1}{\sqrt{\cot(x^2)}}[-\operatorname{cosec}^2 x^2]\frac{d}{dx}(x^2)$ [using chain rule]

$$= -\frac{\operatorname{cosec}^2(x^2)\,2x}{\sqrt{\cot(x^2)}}$$

$$= \frac{-2x\operatorname{cosec}^2(x^2)}{\sqrt{\cot(x^2)}}$$

TOPIC PRACTICE 3

OBJECTIVE TYPE QUESTIONS

1 The set of points, where the function f given by $f(x) = |2x - 1|\sin x$ is differentiable, is

[NCERT Exemplar]

(a) R (b) $R - \left\{\frac{1}{2}\right\}$

(c) $(0, \infty)$ (d) None of these

2 If $f(x) = |\sin x|$, then **[NCERT Exemplar]**

(a) f is everywhere differentiable

(b) f is everywhere continuous but not differentiable at $x = n\pi$, $n \in Z$

(c) f is everywhere continuous but not differentiable at $x = (2n+1)\frac{\pi}{2}, n \in Z$

(d) None of the above

3 The differential coefficient of $\sin(\cos(x^2))$ with respect to x is.

(a) $-2x \sin x^2 \cos(\cos x^2)$

(b) $2x \sin(x^2) \cos(x^2)$

(c) $2x \sin(x^2) \cos(x^2) \cos x$

(d) None of the above

4 If $y = \sqrt{3x+2} + \frac{1}{\sqrt{2x^2+4}}$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{3}{2\sqrt{3x+2}} - \frac{2x}{(2x^2+4)^{3/2}}$

(b) $\frac{3}{2\sqrt{3x+2}} + \frac{2x}{(2x^2+4)^{3/2}}$

(c) $\frac{3}{2\sqrt{3x+2}} + \frac{2}{(2x^2+4)^{3/2}}$

(d) None of the above

5 Let $f(x) = \begin{cases} (x-1)\sin\frac{1}{(x-1)} & \text{,if } x \neq 1 \\ 0 & \text{,if } x = 1 \end{cases}$. Then, which of the following is true?

(a) f is differentiable at $x = 1$ but not at $x = 0$

(b) f is neither differentiable at $x = 0$ nor at $x = 1$

(c) f is differentiable at $x = 0$ and at $x = 1$

(d) f is differentiable at $x = 0$ but not at $x = 1$

VERY SHORT ANSWER Type Questions

6 Differentiate the following functions. (Each part carries 1 Mark)

(i) $\tan(x^2 + 5)$

(ii) $\tan(\sin x)$

(iii) $\operatorname{cosec}(\sqrt{x^2+2})$

(iv) $\sec\left(\frac{5}{x}\right)$

(v) $e^{\sqrt{3x}}$ **[All India 2019]**

7 If $x^{2/3} + y^{2/3} = a^{2/3}$, then find $\frac{dy}{dx}$.

SHORT ANSWER Type I Questions

8 Examine the differentiability of the function $f(x) = \begin{cases} x[x], & \text{if } 0 \le x < 2 \\ (x-1)x, & \text{if } 2 \le x < 3 \end{cases}$ at $x = 2$.

[NCERT Exemplar]

9 Prove that the greatest integer function defined by $f(x) = [x], 0 < x < 3$ is not differentiable at $x = 1$.

10 If $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$, then prove that $\frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$.

[NCERT]

11 If $f(x) = \sin 2x - \cos 2x$, then find $f'\left(\frac{\pi}{6}\right)$.

[All India 2017C]

12 If $f(x) = |\cos x|$, then find $f'\left(\frac{3\pi}{4}\right)$.

[NCERT Exemplar]

13 If $f(x) = |\cos x - \sin x|$, then find $f'\left(\frac{\pi}{6}\right)$.

14 Differentiate $\sqrt{\tan\sqrt{x}}$ w.r.t. x. **[NCERT Exemplar]**

15 Differentiate $\cos x^3 \cdot \sin^2(x^5)$ w.r.t. x. **[NCERT]**

16 If $y = \sin\left[\sqrt{\sin\sqrt{x}}\right]$, then find $\frac{dy}{dx}$.

17 If $y = x\tan x + \sec x$, then find the value of $\frac{dy}{dx}$ at $x = \frac{\pi}{4}$.

SHORT ANSWER Type II Questions

18 Prove that the function f given by $f(x) = |x - 1|$, $x \in R$ is not differentiable at $x = 1$. [NCERT]

19 Show that the function $f(x) = |x - 1| + |x + 1|$, $\forall x \in R$ is not differentiable at the points $x = -1$ and $x = 1$. [All India 2015]

20 A function $f(x)$ is defined as follows

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}.$$

Show that $f(x)$ is differentiable at $x = 0$.

21 Let $f(x) = x|x|, \forall x \in R$. Discuss the derivability of $f(x)$ at $x = 0$. [NCERT Exemplar]

22 Show that the function $f(x) = |x - 5|$ is continuous but not differentiable at $x = 5$. [Delhi 2013; NCERT Exemplar]

23 Examine the differentiability of the function f defined by $f(x) = \begin{cases} 2x + 3, & \text{if } -3 \leq x < -2 \\ x + 1, & \text{if } -2 \leq x < 0 \\ x + 2, & \text{if } 0 \leq x \leq 1 \end{cases}$.

24 Discuss the continuity and differentiability of function $f(x) = |x| + |x - 1|$ in the interval $(-1, 2)$.

25 Using mathematical induction, prove that $\frac{d}{dx}(x^n) = nx^{n-1}$, for all positive integers n. [NCERT]

26 If $y = \sin^3 \sqrt{ax^2 + bx + c}$, then find $\frac{dy}{dx}$.

27 If $y = \left[x + \sqrt{x^2 + a^2}\right]^n$, then prove that $\frac{dy}{dx} = \frac{ny}{\sqrt{x^2 + a^2}}$.

28 If $y = \sqrt{\frac{1 - \sin 2x}{1 + \sin 2x}}$, then show that

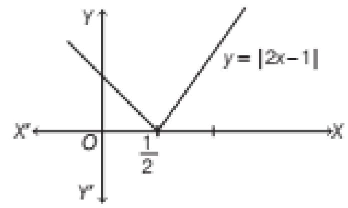
$$\frac{dy}{dx} + \sec^2\left(\frac{\pi}{4} - x\right) = 0, x \in \left[0, \frac{\pi}{4}\right]$$

29 Let $g(x) = ax^2 + bx + c$ be a polynomial function of the second degree. If $g(1) = g(-1)$ and $g'(a_1)$, $g'(a_2)$ and $g'(a_3)$ are in an AP, then prove that a_1, a_2 and a_3 are in an AP.

30 Draw the graph of $|\sin x|$ in the interval $[0, 2\pi]$ and find the derivative of $|\sin x|$ at $x = \frac{\pi}{4}$. Also, check whether the function $|\sin x|$ is many-one or one-one by using horizontal line test.

HINTS & SOLUTIONS

1. (b)



$\because$ At $x = \frac{1}{2}$, curve have two tangents at that point, therefore from the graph it is clear that $y = |2x - 1|$ is not differentiable at $x = \frac{1}{2}$.

$\therefore \quad f(x)$ is differentiable in $R - \left\{\frac{1}{2}\right\}$.

2. (b) Let $u(x) = \sin x$

$v(x) = |x|$

$\therefore \quad f(x) = vou(x) = v\{u(x)\}$

$= v(\sin x)$

$= |\sin x|$

$\because u(x) = \sin x$ is a continuous function and $v(x) = |x|$ is a continuous function

$\therefore$ $f(x) = vou(x)$ is also continuous everywhere but $v(x)$ is not differentiable at $x = 0$

$\Rightarrow f(x)$ is not differentiable where $\sin x = 0$

$\Rightarrow \quad x = n\pi, n \in Z$

Hence, $f(x)$ is continuous everywhere but not differentiable at $x = n\pi, n \in Z$.

3. (a) $\quad y = \sin(\cos x^2)$

Therefore, $\frac{dy}{dx} = \frac{d}{dx}\sin(\cos x^2)$

$= \cos(\cos x^2)\frac{d}{dx}(\cos x^2)$

$= \cos(\cos x^2)(-\sin x^2)\frac{d}{dx}(x^2)$

$= -\sin x^2 \cos(\cos x^2)(2x)$

$= -2x \sin x^2 \cos(\cos x^2)$

4. (a) Let $y = \sqrt{3x + 2} + \frac{1}{\sqrt{2x^2 + 4}}$

$= (3x + 2)^{\frac{1}{2}} + (2x^2 + 4)^{-\frac{1}{2}}$

Therefore, $\frac{dy}{dx} = \frac{1}{2}(3x+2)^{\frac{1}{2}-1} \cdot \frac{d}{dx}(3x+2)$

$$+\left(-\frac{1}{2}\right)(2x^2+4)^{-\frac{1}{2}-1} \cdot \frac{d}{dx}(2x^2+4)$$

$$= \frac{1}{2}(3x+2)^{-\frac{1}{2}} \cdot (3) - \left(\frac{1}{2}\right)(2x^2+4)^{-\frac{3}{2}} \cdot 4x$$

$$= \frac{3}{2\sqrt{3x+2}} - \frac{2x}{(2x^2+4)^{\frac{3}{2}}}$$

5. (d) We observe that

$$\lim_{x \to 1} \frac{f(x) - f(1)}{x-1}$$

$$= \lim_{x \to 1} \frac{(x-1)\sin\left(\frac{1}{x-1}\right)}{x-1} = \lim_{x \to 1} \sin\left(\frac{1}{x-1}\right)$$

= An oscillating number between -1 and 1

$\therefore \lim_{x \to 1} \frac{f(x) - f(1)}{x-1}$ does not exist.

$\Rightarrow f(x)$ is not differentiable at $x = 1$.

and $\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{(x-1)\sin\left(\frac{1}{x-1}\right) - \sin 1}{x}$

$$= \lim_{x \to 0} \frac{x \sin\left(\frac{1}{x-1}\right)}{x} - \lim_{x \to 0} \frac{\sin\left(\frac{1}{x-1}\right) + \sin 1}{x}$$

$$= -\sin 1 - \lim_{x \to 0} \frac{2\sin\frac{x}{2(x-1)} \cos\left\{\frac{2-x}{2(x-1)}\right\}}{\left\{\frac{x}{2(x-1)}\right\} 2(x-1)}$$

$$= -\sin 1 + \cos 1$$

$\therefore f(x)$ is differentiable at $x = 0$.

6. (i) Let $y = \tan u$, where $u = x^2 + 5$

By using chain rule of derivative, we get

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{d}{du}(\tan u) \times \frac{d}{dx}(x^2+5) = \sec^2 u \times 2x$$

$$= 2x \sec^2(x^2+5) \qquad [\text{put } u = x^2+5]$$

(ii) Solve as part (i).

[**Ans.** $\sec^2(\sin x) \cdot \cos x$]

(iii) Let $y = \operatorname{cosec} u$, where $u = \sqrt{v}$ and $v = x^2 + 2$

By using chain rule of derivative, we get

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{du}(\operatorname{cosec} u) \times \frac{d}{dv}(\sqrt{v}) \times \frac{d}{dx}(x^2+2)$$

$$= -\operatorname{cosec} u \cot u \times \frac{1}{2} v^{-1/2} \times 2x$$

$$= -\operatorname{cosec}\sqrt{v} \cot\sqrt{v} \times \frac{1}{2\sqrt{v}} \times 2x \qquad [\text{put } u = \sqrt{v}]$$

$$= -\operatorname{cosec}\sqrt{x^2+2} \cot\sqrt{x^2+2} \times \frac{x}{\sqrt{x^2+2}}$$

$[\text{put } v = x^2+2]$

(iv) Solve as part (i).

$\left[\textbf{Ans. } \sec\left(\frac{5}{x}\right)\tan\left(\frac{5}{x}\right) \times \frac{-5}{x^2}\right]$

(v) Let $y = e^{\sqrt{3x}}$

Then, $\frac{dy}{dx} = \frac{d(e^{\sqrt{3x}})}{dx} = \frac{3 \cdot e^{\sqrt{3x}}}{2 \cdot \sqrt{3x}} = \frac{3e^{\sqrt{3x}}}{2\sqrt{3x}}$

7. We have, $x^{2/3} + y^{2/3} = a^{2/3}$

On differentiating both sides w.r.t. x, we get

$$\frac{2}{3} x^{\frac{2}{3}-1} + \frac{2}{3} y^{\frac{2}{3}-1} \frac{dy}{dx} = 0$$

$$\Rightarrow \quad x^{-1/3} + y^{-1/3} \frac{dy}{dx} = 0$$

$$\Rightarrow \quad y^{-1/3} \frac{dy}{dx} = -x^{-1/3}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-x^{-1/3}}{y^{-1/3}} = -\left(\frac{y}{x}\right)^{1/3}$$

8. Given, $f(x) = \begin{cases} x[x], & \text{if } 0 \le x < 2 \\ (x-1)x & \text{if } 2 \le x < 3 \end{cases}$

At $x = 2$, LHD $= \lim_{h \to 0} \frac{f(2-h) - f(2)}{2-h-2}$

$$= \lim_{h \to 0} \frac{(2-h)[2-h] - (2-1)2}{-h}$$

$[\because [a-h] = [a-1]$, where a is any positive number$]$

$$= \lim_{h \to 0} \frac{(2-h)(1) - 2}{-h}$$

$$= \lim_{h \to 0} \frac{-h}{-h} = 1$$

$$\text{RHD} = \lim_{h \to 0} \frac{f(2+h) - f(2)}{2+h-2}$$

$$= \lim_{h \to 0} \frac{(2+h-1)(2+h) - (2-1) \cdot 2}{h}$$

$$= \lim_{h \to 0} \frac{(1+h)(2+h) - 2}{h}$$

$$= \lim_{h \to 0} \frac{h(h+3)}{h} = 3$$

$\because$ LHD $\neq$ RHD

Hence, $f(x)$ is not differentiable at $x = 2$.

9. Here, $f(x) = [x], 0 < x < 3$

For differentiability at $x = 1$,

$$\text{LHD} = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h \to 0} \frac{[1-h] - [1]}{-h} \qquad [\because [1-h] = 0 \text{ and } [1] = 1]$$

$$= \lim_{h \to 0} \frac{0-1}{-h} = \lim_{h \to 0} \frac{1}{h}, \text{ which does not exist.}$$

Hence, $f(x)$ is not differentiable at $x = 1$.

10. Given, $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

On expanding along R_1, we get

$$y = (mc - nb)f(x) - (lc - na)g(x) + (lb - ma)h(x)$$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = (mc - nb)f'(x) + (na - lc)g'(x) + (lb - ma)h'(x)$$

$$= \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$ **Hence proved.**

11. We have, $f(x) = \sin 2x - \cos 2x$

$\therefore \quad f'(x) = 2\cos 2x + 2\sin 2x$

Now, $f'\left(\frac{\pi}{6}\right) = 2\left[\cos\frac{\pi}{3} + \sin\frac{\pi}{3}\right] = 1 + \sqrt{3}$

12. Hint $f(x) = -\cos x$, if $\frac{\pi}{2} < x < \pi$

$\therefore \quad f'(x) = \sin x$

or $f'\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}$ $\left[\textbf{Ans. } \frac{1}{\sqrt{2}}\right]$

13. Hint For $x \in \left(0, \frac{\pi}{4}\right)$, $\cos x > \sin x$, take

$f(x) = \cos x - \sin x$. $\left[\text{Ans. } \frac{-1}{2}(1+\sqrt{3})\right]$

$f(x) = \cos x - \sin x$. $\left[\text{Ans. } \frac{-1}{2}(1+\sqrt{3})\right]$

14. Similar as Example 10. $\left[\text{Ans. } \frac{\sec^2(\sqrt{x})}{4\sqrt{x}\sqrt{\tan\sqrt{x}}}\right]$

15. Hint Apply product and chain rule.

[**Ans.** $10x^4 \sin x^5 \cos x^5 \cos x^3 - 3x^2 \sin x^3 \sin^2 x^5$]

16. Similar as Example 10.

$\left[\text{Ans. } \frac{\cos(\sqrt{\sin\sqrt{x}})}{4\sqrt{x}\sqrt{\sin\sqrt{x}}} \cdot \cos\sqrt{x}\right]$

17. We have, $y = x \tan x + \sec x$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x \tan x) + \frac{d}{dx}(\sec x)$$

$$= x\frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(x) + \frac{d}{dx}(\sec x)$$

$$\left[\because \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}\right]$$

$$= x\sec^2 x + \tan x + \sec x \tan x$$ [1]

At $x = \frac{\pi}{4}$, $\left(\frac{dy}{dx}\right)_{x = \pi/4} = \frac{\pi}{4}\sec^2\frac{\pi}{4} + \tan\frac{\pi}{4} + \sec\frac{\pi}{4}\tan\frac{\pi}{4}$

$$= \frac{\pi}{4} \times (\sqrt{2})^2 + 1 + \sqrt{2} \times 1$$

$$= \frac{\pi}{4} \times 2 + 1 + \sqrt{2}$$

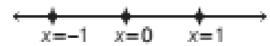
$$= \frac{\pi}{2} + 1 + \sqrt{2}$$

18. Similar as Example 2.

19. Hint Here, given $f(x) = |x-1| + |x+1|$

Put $x - 1 = 0$ and $x + 1 = 0$

$\Rightarrow x = 1$ and $x = -1$



***Case* I** *When* $x < -1$

In this case $x - 1 < 0$ and $x + 1 < 0$

$\therefore |x-1| = -(x-1)$ and $|x+1| = -(x+1)$

$\therefore \quad f(x) = -(x-1) + \{-(x+1)\}$

$= -x + 1 - x - 1$

$\Rightarrow \quad f(x) = -2x$

***Case* II** *When* $-1 \le x < 1$

In this case $x - 1 < 0$ and $x + 1 \ge 0$

$\therefore \quad |x-1| = -(x-1)$ and $|x+1| = x+1$

$\therefore \quad f(x) = -(x-1) + (x+1)$

$= -x + 1 + x + 1 = 2$

***Case* III** *When* $x \ge 1$

In this case $x - 1 \ge 0$ and $x + 1 \ge 0$

$\therefore |x-1| = x - 1$ and $|x+1| = x + 1$

In this case $x - 1 \ge 0$ and $x + 1 \ge 0$

$\therefore |x-1| = x - 1$ and $|x+1| = x + 1$

$\therefore \quad f(x) = (x-1) + (x+1) = 2x$

Thus, $f(x) = \begin{cases} -2x, & \text{if } x < -1 \\ 2, & \text{if } -1 \le x < 1 \\ 2x, & \text{if } x \ge 1 \end{cases}$

Now, examine the differentiability of the function $f(x)$, at $x = -1$ and $x = 1$.

20. Given, $f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & \text{if } x \ne 0 \\ 0, & \text{if } x = 0 \end{cases}$

At $x = 0$,

$$\text{LHD} = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{\left[(-h)^2 \sin\left(-\frac{1}{h}\right)\right] - 0}{-h}$$

$$= \lim_{h \to 0} \frac{h^2 \sin\left(-\frac{1}{h}\right)}{-h}$$

$$= \lim_{h \to 0} \frac{-h^2 \sin\left(\frac{1}{h}\right)}{-h} = \lim_{h \to 0} h \sin\left(\frac{1}{h}\right)$$

$= 0 \times$ (an oscillating value between -1 and 1) $= 0$

and $\text{RHD} = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$

$$= \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin\frac{1}{h} - 0}{h}$$

$$= \lim_{h \to 0} \frac{h^2 \sin\frac{1}{h}}{h} = \lim_{h \to 0} h \sin\left(\frac{1}{h}\right)$$

$= 0 \times$ (an oscillating value between -1 and 1)

$= 0$

$\because$ LHD = RHD

Hence, $f(x)$ is differentiable at $x = 0$.

21. **Hint** $f(x) = \begin{cases} -x^2, & x < 0 \\ 0, & x = 0 \\ x^2, & x > 0 \end{cases}$, then similar as Example 3.

[**Ans.** Derivable at $x = 0$].

22. Given, $f(x) = |x - 5|$

$\therefore$ $f(x) = \begin{cases} -(x-5), & \text{if } x < 5 \\ x - 5, & \text{if } x \geq 5 \end{cases}$

For continuity at $x = 0$,

$\text{LHL} = \lim_{x \to 5^-} |x - 5| = \lim_{x \to 5^-} [-(x-5)]$

$$= \lim_{h \to 0} [-(5-h) + 5] = \lim_{h \to 0} h = 0$$

and $\text{RHL} = \lim_{x \to 5^+} (x - 5)$

$$= \lim_{h \to 0} (5 + h - 5) = \lim_{h \to 0} h = 0$$

Also, $f(5) = 5 - 5 = 0$

$\because$ LHL = RHL = $f(5)$

Hence, $f(x)$ is continuous at $x = 5$.

Now, $\text{LHD} = \lim_{x \to 5^-} \frac{f(x) - f(5)}{x - 5}$

$$= \lim_{x \to 5^-} \frac{-x + 5 - 0}{x - 5} = \lim_{x \to 5^-} \frac{-(x-5)}{x-5} = -1$$

$$\text{RHD} = \lim_{x \to 5^+} \frac{f(x) - f(5)}{x - 5}$$

$$= \lim_{x \to 5^+} \frac{x - 5 - 0}{x - 5} = 1$$

$\therefore$ LHD $\neq$ RHD

Hence, $f(x) = |x - 5|$ is not differentiable at $x = 5$.

23. Similar as Example 3.

[**Ans.** Not differentiable at $x = -2$]

24. Similar as Example 3.

Hint $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

and $|x - 1| = \begin{cases} x - 1, & \text{if } x \geq 1 \\ -(x-1), & \text{if } x < 1 \end{cases}$

$$f(x) = \begin{cases} -x - (x-1), & \text{if } -1 < x \leq 0 \\ x - (x-1), & \text{if } 0 < x \leq 1 \\ x + (x-1), & \text{if } 1 < x < 2 \end{cases}$$

$$= \begin{cases} -2x + 1, & \text{if } -1 < x \leq 0 \\ 1, & \text{if } 0 < x \leq 1 \\ 2x - 1, & \text{if } 1 < x < 2 \end{cases}$$

25. Let $P(n) = \frac{d}{dx}(x^n) = nx^{n-1}$

Now, $P(1)$ is $\frac{d}{dx}(x^1) = 1 \cdot x^{1-1}$,

i.e. $\frac{d}{dx}(x) = 1$, which is true.

$\therefore$ $P(1)$ is true.

Let $P(k)$ be true, for some $k \in N$.

Then, $\frac{d}{dx}(x^k) = kx^{k-1}$ ...(i)

$\Rightarrow \frac{d}{dx}(x^{k+1}) = \frac{d}{dx}(x^k x)$

$$= x^k \frac{d}{dx}(x) + x \frac{d}{dx}(x^k)$$

[by product rule of derivative]

$= x^k \cdot 1 + x(kx^{k-1}) = (k+1)x^k$ [from Eq. (i)]

$\Rightarrow \frac{d}{dx}(x^{k+1}) = (k+1)x^{(k+1)-1}$

$\Rightarrow P(k+1)$ is true.

Thus, $P(1)$, $P(k)$ and $P(k+1)$, $n \in N$ is true. Hence, by mathematical induction, $P(n)$ is true for all $k \in N$.

26. **Hint** $\frac{dy}{dx} = 3\sin^2 \sqrt{ax^2 + bx + c} \cdot \frac{d}{dx}(\sin\sqrt{ax^2 + bx + c})$

$$= 3\sin^2 \sqrt{ax^2 + bx + c} \cdot \cos\sqrt{ax^2 + bx + c} \cdot \frac{d}{dx}(\sqrt{ax^2 + bx + c})$$

$$\left[\text{Ans. } \frac{3}{2} \sin^2 \sqrt{ax^2 + bx + c} \cdot \cos\sqrt{ax^2 + bx + c} \cdot \frac{(2ax + b)}{\sqrt{ax^2 + bx + c}}\right]$$

27. We have, $y = \left[x + \sqrt{x^2 + a^2}\right]^n$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = n\left[x + \sqrt{x^2 + a^2}\right]^{n-1} \cdot \frac{d}{dx}\left[x + \sqrt{x^2 + a^2}\right]$$

[by chain rule of derivative]

$$= n\left[x + \sqrt{x^2 + a^2}\right]^{n-1}\left[1 + \frac{1}{2\sqrt{x^2 + a^2}}\frac{d}{dx}(x^2 + a^2)\right]$$

[by chain rule of derivative]

$$= n\left[x + \sqrt{x^2 + a^2}\right]^{n-1}\left[1 + \frac{1}{2\sqrt{x^2 + a^2}}\cdot 2x\right]$$

$$= n\left[x + \sqrt{x^2 + a^2}\right]^{n-1}\left[\frac{\sqrt{x^2 + a^2} + x}{\sqrt{x^2 + a^2}}\right]$$

$$= \frac{n\left[x + \sqrt{x^2 + a^2}\right]^{n}}{\sqrt{x^2 + a^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{ny}{\sqrt{x^2 + a^2}} \qquad \left[\because y = (x + \sqrt{x^2 + a^2})^n\right]$$

Hence proved.

28. We have, $y = \sqrt{\frac{1 - \sin 2x}{1 + \sin 2x}}$

$$= \sqrt{\frac{\sin^2 x + \cos^2 x - 2\sin x \cos x}{\sin^2 x + \cos^2 x + 2\sin x \cos x}}$$

$[\because \sin^2 x + \cos^2 x = 1 \text{ and } \sin 2x = 2\sin x \cos x]$

$$= \sqrt{\frac{(\cos x - \sin x)^2}{(\cos x + \sin x)^2}}$$

$\left[\because \text{ for } x \in \left[0, \frac{\pi}{4}\right], \cos x \geq \sin x, \text{ then we take } (\cos x - \sin x)^2 \text{ instead of } (\sin x - \cos x)^2\right]$

$$= \frac{\cos x - \sin x}{\cos x + \sin x}$$

$$= \frac{1 - \tan x}{1 + \tan x}$$

[dividing numerator and denominator both by $\cos x$]

$$\Rightarrow y = \tan\left(\frac{\pi}{4} - x\right) \qquad \left[\because \tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}\right]$$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \sec^2\left(\frac{\pi}{4} - x\right)\frac{d}{dx}\left(\frac{\pi}{4} - x\right)$$

[by chain rule of derivative]

$$= \sec^2\left(\frac{\pi}{4} - x\right)(-1)$$

$$= -\sec^2\left(\frac{\pi}{4} - x\right)$$

$$\Rightarrow \frac{dy}{dx} + \sec^2\left(\frac{\pi}{4} - x\right) = 0$$

Hence proved.

29. Given, $g(x) = ax^2 + bx + c$ ... (i)

Also given $g(1) = g(-1)$

$\therefore \quad a(1)^2 + b(1) + c = a(-1)^2 + b(-1) + c$

$\Rightarrow \quad a + b + c = a - b + c$

$\Rightarrow \quad 2b = 0$

$\Rightarrow \quad b = 0$

On differentiating both sides of Eq. (i) w.r.t. x, we get

$$g'(x) = 2ax + b$$

Put $b = 0$, we get $g'(x) = 2ax$

Now, $g'(a_1) = 2aa_1$, $g'(a_2) = 2aa_2$, $g'(a_3) = 2aa_3$

Since, $g'(a_1)$, $g'(a_2)$ and $g'(a_3)$ are in AP.

$\therefore \quad 2g'(a_2) = g'(a_1) + g'(a_3)$

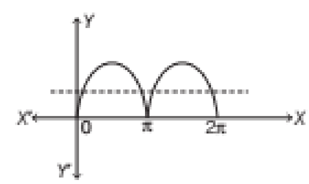
$\Rightarrow \quad 2\cdot 2aa_2 = 2aa_1 + 2aa_3$

$\Rightarrow \quad 2a_2 = a_1 + a_3$

Hence, a_1, a_2 and a_3 are in an AP.

30. Let $f(x) = |\sin x| = \begin{cases} \sin x, & 0 < x \leq \pi \\ -\sin x, & \pi < x \leq 2\pi \end{cases}$

The graph of $f(x)$ is shown below



For $x = \frac{\pi}{4}$, we take the function $f(x) = \sin x$ ...(i)

On differentiating Eq. (i), w.r.t. x, we get

$$f'(x) = \cos x$$

$$\Rightarrow \quad f'\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

When, we draw a horizontal line (dotted), it intersects the curve more than one point.
Hence, $f(x)$ is many-one function.

|TOPIC 4|

Derivatives of Implicit Functions

Let $f(x, y) = 0$ be a function in the form of x and y. If it is not possible to express y as a function of x in the form of $y = \phi(x)$, then y is said to be an implicit function of x.
e.g. $x^2 + y^2 = xy$, $\cos(x + y) = \log(x \cdot y)$

Sol. We have, $x^3 + y^3 = \sin(x + y)$

On differentiating both sides w.r.t. x, we get

$$3x^2 + 3y^2 \cdot \frac{dy}{dx} = \cos(x + y)\frac{d}{dx}(x + y)$$

Working Rule to Find *dy/dx* of Implicit Functions

Following steps are used to find dy/dx of implicit functions.

I. Write the given equation in x and y and then differentiate both sides of the given equation w.r.t. x.

II. Take all terms involving $\frac{dy}{dx}$ on LHS and transfer the remaining terms on RHS to get equation in the form of $\frac{dy}{dx} f(x, y) = g(x, y)$.

III. Now, find $\frac{dy}{dx}$, i.e. $\frac{dy}{dx} = \frac{g(x, y)}{f(x, y)}$.

EXAMPLE |1| If $x^3 + y^3 = 3axy$, then find $\frac{dy}{dx}$.

Sol. Given, $x^3 + y^3 = 3axy$

On differentiating both sides w.r.t. x, we get

$$3x^2 + 3y^2 \frac{dy}{dx} = 3a \frac{d}{dx}(xy)$$

$$\Rightarrow \quad 3x^2 + 3y^2 \frac{dy}{dx} = 3a\left(1 \cdot y + x\frac{dy}{dx}\right)$$

[by product rule of derivative]

$$\Rightarrow \quad 3x^2 + 3y^2 \frac{dy}{dx} = 3ay + 3ax\frac{dy}{dx}$$

$$\Rightarrow \quad 3y^2 \frac{dy}{dx} - 3ax\frac{dy}{dx} = 3ay - 3x^2$$

[taking terms involving dy/dx in LHS]

$$\Rightarrow \quad 3(y^2 - ax)\frac{dy}{dx} = 3(ay - x^2)$$

$$\Rightarrow \quad (y^2 - ax)\frac{dy}{dx} = ay - x^2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$$

EXAMPLE |2| Find $\frac{dy}{dx}$ for the equation

$$x^3 + y^3 = \sin(x + y).$$

Given, $\sin y = x\cos(a + y)$

When $x = 0$, then $\sin y = 0$

$$\Rightarrow \quad y = n\pi, n \in I.$$

$\therefore$ At $x = 0$,

$$\frac{dy}{dx} = \frac{\cos^2(a + n\pi)}{\cos a}$$

$$= \frac{\cos^2 a}{\cos a} = \cos a$$

$[\because \cos^2(n\pi + \theta) = \cos^2\theta]$

[by chain rule of derivative]

$$\Rightarrow \quad 3x^2 + 3y^2 \frac{dy}{dx} = \cos(x + y)\left(1 + \frac{dy}{dx}\right)$$

$$\Rightarrow \quad 3x^2 + 3y^2 \frac{dy}{dx} = \cos(x + y) + \cos(x + y) \cdot \frac{dy}{dx}$$

$$\Rightarrow \quad 3y^2 \frac{dy}{dx} - \cos(x + y)\frac{dy}{dx} = \cos(x + y) - 3x^2$$

$\left[\text{taking terms involving } \frac{dy}{dx} \text{ in LHS}\right]$

$$\Rightarrow \quad \frac{dy}{dx}[3y^2 - \cos(x + y)] = \cos(x + y) - 3x^2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\cos(x + y) - 3x^2}{3y^2 - \cos(x + y)}$$

EXAMPLE |3| If $\sin y = x\cos(a + y)$, then show that $\frac{dy}{dx} = \frac{\cos^2(a + y)}{\cos a}$. Also, show that $\frac{dy}{dx} = \cos a$, when $x = 0$.

Sol. We have, $\sin y = x\cos(a + y)$

$$\Rightarrow \quad x = \frac{\sin y}{\cos(a + y)}$$

On differentiating both sides w.r.t. y, we get

$$\frac{dx}{dy} = \frac{\left[\cos(a + y)\frac{d}{dy}\sin y - \sin y\frac{d}{dy}\cos(a + y)\right]}{[\cos(a + y)]^2}$$

[by quotient rule of derivative]

$$= \frac{\cos(a + y)\cos y - \sin y[-\sin(a + y) \cdot 1]}{\cos^2(a + y)}$$

$$= \frac{\cos(a + y)\cos y + \sin y \sin(a + y)}{\cos^2(a + y)}$$

$$= \frac{\cos(a + y - y)}{\cos^2(a + y)}$$

$[\because \cos A\cos B + \sin A\sin B = \cos(A - B)]$

$$= \frac{\cos a}{\cos^2(a + y)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos^2(a + y)}{\cos a}$$

VERY SHORT ANSWER Type Questions

6 If $y = \tan(x + y)$, then find $\frac{dy}{dx}$. **[NCERT Exemplar]**

7 Find the derivative dy/dx of $xy + y^2 = \tan x + y$. **[NCERT]**

8 Find the derivative of $\sec(x + y) = xy$ w.r.t. x. **[NCERT Exemplar]**

9 If $ax^2 + by^2 = \cos y$, then find dy/dx.

TOPIC PRACTICE 4

OBJECTIVE TYPE QUESTIONS

1 The derivative of $2x + 3y = \sin y$ is

(a) $\frac{2}{\cos y}$ (b) $\frac{2}{\cos y + 3}$

(c) $\frac{2}{\cos y - 3}$ (d) None of these

2 If $y + \sin y = \cos x$, then $\frac{dy}{dx}$ is equal to

(a) $-\frac{\sin x}{1+\cos y}, y = (2n+1)\pi$

(b) $\frac{\sin x}{1+\cos y}, y \neq (2n+1)\pi$

(c) $-\frac{\sin x}{1+\cos y}, y \neq (2n+1)\pi$

(d) None of the above

3 If $2x + 3y = \sin x$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{\cos x + 2}{3}$ (b) $\frac{\cos x - 2}{3}$

(c) $\cos x + 2$ (d) None of these

4 If $y = \sqrt{\sin x + y}$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{\cos x}{2y-1}$ (b) $\frac{\cos x}{1-2y}$

(c) $\frac{\sin x}{1-2y}$ (d) $\frac{\sin x}{2y-1}$

5 If $\cos y = x\cos(a + y)$ with $\cos a \neq 1$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{\sin^2(a+y)}{\sin a}$ (b) $\frac{\cos^2(a+y)}{\sin a}$

(c) $\sin^2(a+y)\sin a$ (d) None of these

SHORT ANSWER Type I Questions

10 Find dy/dx, if $x^3 + x^2y + xy^2 + y^3 = 81$. **[NCERT]**

11 Find the derivative of $(x^2 + y^2)^2 = xy$ w.r.t. x. **[NCERT Exemplar]**

12 Find dy/dx, when $\sin(x + y) = x^2 + y^2$.

13 Find dy/dx at $x = 1$, $y = \pi/4$, if $\sin^2 y + \cos xy = K$. **[Delhi 2017]**

SHORT ANSWER Type II Questions

14 If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, then show that $\frac{dy}{dx}\cdot\frac{dx}{dy} = 1$. **[NCERT Exemplar]**

15 If $x\sqrt{1+y} + y\sqrt{1+x} = 0$ for $-1 < x, y < 1$, then prove that $\frac{dy}{dx} = \frac{-1}{(x+1)^2}$. **[All India 2019]**

16 Find the derivative of $\sin(xy) + \frac{x}{y} = x^2 - y$ w.r.t. x. **[NCERT Exemplar]**

17 If $\cos y = x\cos(a + y)$, where $\cos a \neq \pm 1$, then prove that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$. **[NCERT; Foreign 2014]**

HINTS & SOLUTIONS

1. (c) Given, $2x + 3y = \sin y$

On differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}(2x+3y) = \frac{d}{dx}(\sin y)$$

$$\Rightarrow \quad 2 + 3\frac{dy}{dx} = \cos y\frac{dy}{dx}$$

$$\Rightarrow \quad 3\frac{dy}{dx} - \cos y\frac{dy}{dx} = -2$$

$$\Rightarrow \quad (3 - \cos y)\frac{dy}{dx} = -2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2}{\cos y - 3}$$

2. (c) We differentiate the relationship directly with respect to x, we get

$$\frac{dy}{dx} + \frac{d}{dx}(\sin y) = \frac{d}{dx}(\cos x)$$

[by chain rule of derivative]

$$\frac{dy}{dx} + \cos y\cdot\frac{dy}{dx} = -\sin x$$

This gives $\quad \frac{dy}{dx} = -\frac{\sin x}{1+\cos y}$

where, $\quad y \neq (2n+1)\pi$

6. Similar as Example 2.

[**Ans.** $-\operatorname{cosec}^2(x+y)$]

7. Similar as Example 2.

$\left[\text{Ans. } \frac{\sec^2 x - y}{(x+2y-1)}\right]$

8. Similar as Example 2.

$\left[\text{Ans. } \frac{y - \sec(x+y)\tan(x+y)}{\sec(x+y)\tan(x+y) - x}\right]$

9. We have, $ax^2 + by^2 = \cos y$

On differentiating both sides w.r.t. x, we get

3. (b) Given, $2x + 3y = \sin x$

On differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}(2x+3y) = \frac{d}{dx}(\sin x)$$

$$2 + 3\frac{dy}{dx} = \cos x$$

$$\Rightarrow \quad 3\frac{dy}{dx} = \cos x - 2 \Rightarrow \frac{dy}{dx} = \frac{\cos x - 2}{3}$$

4. (a) $\because \quad y = (\sin x + y)^{1/2}$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}(\sin x + y)^{-1/2} \cdot \frac{d}{dx}(\sin x + y)$$

[by chain rule of derivative]

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{(\sin x + y)^{1/2}} \cdot \left(\cos x + \frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2y}\left(\cos x + \frac{dy}{dx}\right) \qquad [\because (\sin x + y)^{1/2} = y]$$

$$\Rightarrow \quad \frac{dy}{dx}\left(1 - \frac{1}{2y}\right) = \frac{\cos x}{2y}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\cos x}{2y} \cdot \frac{2y}{2y-1} = \frac{\cos x}{2y-1}$$

5. (b) Given, $\cos y = x\cos(a + y)$

$$\Rightarrow \quad x = \frac{y}{\cos(a+y)}$$

$$\frac{dx}{dy} = \frac{d}{dy}\left\{\frac{\cos y}{\cos(a+y)}\right\}$$

$$= \frac{\cos(a+y)(-\sin y) - \cos y\{-\sin(a+y)1\}}{\cos^2(a+y)}$$

$$= \frac{\sin(a + y - y)}{\cos^2(a+y)}$$

$[\because \sin(A - B) = \sin A \cos B - \cos A \sin B]$

$$= \frac{\sin a}{\cos^2(a+y)}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{\cos^2(a+y)}{\sin a}$$

13. We have, $\sin^2 y + \cos xy = K$

On differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}(\sin^2 y + \cos xy) = \frac{d}{dx}(K)$$

$$\Rightarrow \quad \frac{d}{dx}(\sin^2 y) + \frac{d}{dx}(\cos xy) = 0$$

$$\Rightarrow \quad 2\sin y \cos y\frac{dy}{dx} + (-\sin xy)\frac{d}{dx}(xy) = 0$$

$$\Rightarrow \quad \sin 2y\frac{dy}{dx} - \sin xy\left(x\frac{dy}{dx} + y \cdot 1\right) = 0$$

$$\Rightarrow \quad \sin 2y\frac{dy}{dx} - x\sin xy\frac{dy}{dx} = y\sin xy$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y\sin(xy)}{\sin 2y - x\sin(xy)}$$

Now, at $x = 1$, $y = \frac{\pi}{4}$.

$$2ax + 2by\frac{dy}{dx} = -\sin y\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}(2by + \sin y) = -2ax$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2ax}{2by + \sin y}$$

10. We have, $x^3 + x^2y + xy^2 + y^3 = 81$

On differentiating both sides w.r.t. x, we get

$$3x^2 + x^2\frac{dy}{dx} + y \cdot 2x + x \cdot 2y\frac{dy}{dx} + y^2 \cdot 1 + 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow \quad x^2\frac{dy}{dx} + 2xy\frac{dy}{dx} + 3y^2\frac{dy}{dx} = -3x^2 - 2xy - y^2$$

$$\Rightarrow \quad \frac{dy}{dx}(x^2 + 2xy + 3y^2) = -3x^2 - 2xy - y^2$$

$$\frac{dy}{dx} = \frac{-(3x^2 + 2xy + y^2)}{x^2 + 2xy + 3y^2}$$

11. Similar as Example 1. $\left[\text{Ans. } \frac{y - 4x^3 - 4xy^2}{4yx^2 + 4y^3 - x}\right]$

12. We have, $\sin(x + y) = x^2 + y^2$

On differentiating both sides w.r.t. x, we get

$$\cos(x+y)\frac{d}{dx}(x+y) = 2x + 2y\frac{dy}{dx}$$

$$\Rightarrow \quad \cos(x+y)\left(1 + \frac{dy}{dx}\right) = 2x + 2y\frac{dy}{dx}$$

$$\Rightarrow \cos(x+y) + \cos(x+y)\frac{dy}{dx} = 2x + 2y\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}[\cos(x+y) - 2y] = 2x - \cos(x+y)$$

$$\therefore \quad \frac{dy}{dx} = \frac{2x - \cos(x+y)}{\cos(x+y) - 2y}$$

$$= \frac{-(ax + hy + g)}{(hx + by + f)} \qquad \text{...(ii)}$$

Now, differentiating both sides of Eq. (i) w.r.t. y, we get

$$\frac{d}{dy}(ax^2) + \frac{d}{dy}(2hxy) + \frac{d}{dy}(by^2) + \frac{d}{dy}(2gx) + \frac{d}{dy}(2fy) + \frac{d}{dy}(c) = 0$$

$$\Rightarrow \quad a \cdot 2x \cdot \frac{dx}{dy} + 2h \cdot \left(x \cdot 1 + y \cdot \frac{d}{dy}x\right) + b \cdot 2y + 2g \cdot \frac{dx}{dy} + 2f + 0 = 0$$

$$\Rightarrow \quad \frac{dx}{dy}[2ax + 2hy + 2g] = -2hx - 2by - 2f$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{-2(hx + by + f)}{2(ax + hy + g)} = \frac{-(hx + by + f)}{(ax + hy + g)} \qquad \text{...(iii)}$$

$$\therefore \left(\frac{dy}{dx}\right)_{(1,\pi/4)} = \frac{\frac{\pi}{4}\sin\left(1\cdot\frac{\pi}{4}\right)}{\sin\left(2\cdot\frac{\pi}{4}\right) - 1\sin\left(1\cdot\frac{\pi}{4}\right)}$$

$$= \frac{\frac{\pi}{4}\sin\left(\frac{\pi}{4}\right)}{\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{4}\right)} = \frac{\frac{\pi}{4}\left(\frac{1}{\sqrt{2}}\right)}{1 - \frac{1}{\sqrt{2}}}$$

$$= \frac{\frac{\pi}{4\sqrt{2}}}{(\sqrt{2}-1)/\sqrt{2}}$$

$$= \frac{\pi}{4\sqrt{2}} \times \frac{\sqrt{2}}{(\sqrt{2}-1)}$$

$$= \frac{\pi}{4(\sqrt{2}-1)}$$

14. We have, $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ...(i)

On differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}(ax^2) + \frac{d}{dx}(2hxy) + \frac{d}{dx}(by^2) + \frac{d}{dx}(2gx) + \frac{d}{dx}(2fy) + \frac{d}{dx}(c) = 0$$

$$\Rightarrow 2ax + 2h\left(x\cdot\frac{dy}{dx} + y\cdot 1\right) + b\cdot 2y\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} + 0 = 0$$

$$\Rightarrow \frac{dy}{dx}[2hx + 2by + 2f] = -2ax - 2hy - 2g$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2(ax + hy + g)}{2(hx + by + f)}$$

$$\therefore \frac{dy}{dx}\cdot\frac{dx}{dy} = \frac{-(ax + hy + g)}{(hx + by + f)}\cdot\frac{-(hx + by + f)}{(ax + hy + g)}$$

[using Eqs. (ii) and (iii)]

$= 1 = \text{RHS}$ **Hence proved.**

15. We have, $x\sqrt{1+y} + y\sqrt{1+x} = 0$

$$\Rightarrow x\sqrt{1+y} = -y\sqrt{1+x}$$

On squaring both sides, we get

$$x^2(1+y) = y^2(1+x)$$

$$\Rightarrow x^2 - y^2 + x^2y - y^2x = 0$$

$$\Rightarrow (x+y)(x-y) + xy(x-y) = 0$$

$$\Rightarrow (x-y)(x+y+xy) = 0$$

But $x \neq y$

$$\therefore x + y + xy = 0$$

$$\Rightarrow y(1+x) = -x$$

$$\Rightarrow y = \frac{-x-1+1}{1+x}$$

$$\Rightarrow y = -1 + \frac{1}{1+x}$$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{-1}{(1+x)^2}$$ **Hence proved.**

16. Similar as Example 2.

$$\left[\textbf{Ans. } \frac{2xy^2 - y^3\cos(xy) - y}{xy^2\cos(xy) - x + y^2}\right]$$

17. Similar as Example 3.

Hint $x = \frac{\cos y}{\cos(a+y)}$, find $\frac{dx}{dy}$.

|TOPIC 5|

Derivatives of Inverse Trigonometric Functions

Derivatives of inverse trigonometric functions are given below

(i) $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

(ii) $\frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$

(iii) $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

(iv) $\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}$

(v) $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$

EXAMPLE |2| Differentiate $\tan^{-1}(\sec x + \tan x)$ w.r.t. x.

Sol. Let $y = \tan^{-1}(\sec x + \tan x) = \tan^{-1}\left(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right)$

$$\Rightarrow y = \tan^{-1}\left(\frac{1+\sin x}{\cos x}\right) = \tan^{-1}\left(\frac{1-\cos\left(\frac{\pi}{2}+x\right)}{\sin\left(\frac{\pi}{2}+x\right)}\right)$$

$$\left[\because \cos\left(\frac{\pi}{2}+x\right) = -\sin x \text{ and } \sin\left(\frac{\pi}{2}+x\right) = \cos x\right]$$

$$\Rightarrow y = \tan^{-1}\left[\frac{2\sin^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}{2\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cdot\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)}\right]$$

(vi) $\frac{d}{dx}(\operatorname{cosec}^{-1}x) = \frac{-1}{x\sqrt{x^2-1}}$

Proof (i) $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

Let $y = \sin^{-1}x \Rightarrow x = \sin y$

On differentiating both sides w.r.t. y, we get

$$\frac{dx}{dy} = \cos y \geq 0 \qquad \left[\because y \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]\right]$$

$$\Rightarrow \frac{dx}{dy} = \sqrt{1-\sin^2 y}$$

$$\Rightarrow \frac{dx}{dy} = \sqrt{1-x^2} \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

Hence, $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

Similarly, we can prove other parts.

EXAMPLE |1| Differentiate $\cos^{-1}(5x^2+4)$ w.r.t. x.

Sol. Let $y = \cos^{-1}(5x^2+4)$. Put $z = 5x^2+4$. Then, $y = \cos^{-1}z$.

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(\cos^{-1}z) = \frac{d}{dz}(\cos^{-1}z)\frac{dz}{dx} \qquad \text{[by chain rule]}$$

$$= \frac{-1}{\sqrt{1-z^2}}\frac{d}{dx}(5x^2+4) \qquad [\text{put } z = 5x^2+4]$$

$$= -\frac{1}{\sqrt{1-(5x^2+4)^2}} \times [5 \times 2x + 0] \qquad [\text{put } z = 5x^2+4]$$

$$= -\frac{10x}{\sqrt{1-(5x^2+4)^2}}$$

On differentiating both sides of Eq. (ii) w.r.t. u, we get

$$\frac{dy}{du} = \frac{-1}{1+u^2} = \frac{-1}{1+\left(\frac{1-x}{1+x}\right)^2} \qquad \text{[from Eq. (i)]}$$

$$\Rightarrow \frac{dy}{du} = \frac{-(1+x)^2}{(1+x)^2+(1-x)^2} = \frac{-(1+x)^2}{2(1+x^2)}$$

On putting the values of $\frac{du}{dx}$ and $\frac{dy}{du}$ in Eq. (iii), we get

$$\frac{dy}{dx} = \frac{-(1+x)^2}{2(1+x^2)} \cdot \frac{-2}{(1+x)^2} \Rightarrow \frac{dy}{dx} = \frac{1}{1+x^2}$$

$$\left[\because \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}\right]$$

$$\Rightarrow y = \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right] \Rightarrow y = \frac{\pi}{4}+\frac{x}{2}$$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 0 + \frac{1}{2} \Rightarrow \frac{dy}{dx} = \frac{1}{2}$$

EXAMPLE |3| Differentiate $\cot^{-1}\left(\frac{1-x}{1+x}\right)$ w.r.t. x.

Sol. Let $y = \cot^{-1}\left(\frac{1-x}{1+x}\right)$.

Put $\frac{1-x}{1+x} = u$ ...(i)

Then, $y = \cot^{-1}u$ ...(ii)

Thus, y is a composite function, so we use chain rule of derivative.

i.e. $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ ...(iii)

Now, differentiating Eq. (i) w.r.t. x, we get

$$\frac{du}{dx} = \frac{d}{dx}\left(\frac{1-x}{1+x}\right)$$

$$= \frac{(1+x)(-1)-(1-x)(1)}{(1+x)^2}$$

[by quotient rule of derivative]

$$= \frac{-1-x-1+x}{(1+x)^2} = \frac{-2}{(1+x)^2}$$

For this, we can use the following trigonometric and inverse trigonometric formulae

(i) $\sin 2x = 2\sin x \cos x$

(ii) $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

(iii) $\sin 2x = \frac{2\tan x}{1+\tan^2 x}$

(iv) $\cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}$

(v) $\tan 2x = \frac{2\tan x}{1-\tan^2 x}$

EXAMPLE |4| Show that

$$\frac{d}{dx}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]=\sqrt{a^2-x^2}.$$

Sol. We have, LHS $=\frac{d}{dx}\left[\frac{x}{2}\cdot\sqrt{a^2-x^2}+\frac{a^2}{2}\cdot\sin^{-1}\frac{x}{a}\right]$

$$=\frac{d}{dx}\left[\frac{x}{2}\cdot\sqrt{a^2-x^2}\right]+\frac{a^2}{2}\cdot\frac{d}{dx}\left[\sin^{-1}\frac{x}{a}\right]$$

$$=\frac{x}{2}\cdot\frac{d}{dx}(\sqrt{a^2-x^2})+(\sqrt{a^2-x^2})\cdot\frac{d}{dx}\left(\frac{x}{2}\right)+\frac{a^2}{2}\cdot\frac{1}{\sqrt{1-\frac{x^2}{a^2}}}\cdot\frac{1}{a}$$

[by product rule of derivative]

$$=\frac{x}{2}\cdot\frac{1}{2}(a^2-x^2)^{-1/2}(-2x)+(\sqrt{a^2-x^2})\cdot\frac{1}{2}+\frac{a^2}{2\sqrt{a^2-x^2}}$$

$$=\frac{-x^2}{2\sqrt{a^2-x^2}}+\frac{\sqrt{a^2-x^2}}{2}+\frac{a^2}{2\sqrt{a^2-x^2}}$$

$$=\frac{-x^2+(a^2-x^2)+a^2}{2\sqrt{a^2-x^2}}$$

$$=\frac{2(a^2-x^2)}{2\sqrt{a^2-x^2}}=\sqrt{a^2-x^2}=\text{RHS}$$

Hence, $\frac{d}{dx}\left[\frac{x}{2}\cdot\sqrt{a^2-x^2}+\frac{a^2}{2}\cdot\sin^{-1}\frac{x}{a}\right]=\sqrt{a^2-x^2}.$

Trigonometrical Transformations Used for Differentiation

Sometimes differentiation of inverse trigonometric functions becomes difficult by chain rule, then we can differentiate them easily by using trigonometrical transformation.

Working Rule to Find Derivatives of Inverse Trigonometric Functions by Trigonometrical Transformations

To find the derivative of given inverse trigonometric function by trigonometrical transformation, we use the following steps

I. First, put the given inverse function equal to y, say $y=f^{-1}(x)$.

II. Convert $f^{-1}(x)$ to simplest form by using suitable trigonometric formulae or substitution such that $f^{-1}(x)$ can differentiate easily.

III. Now, differentiate y with respect to x to get required dy/dx.

EXAMPLE |5| If $y=\sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)$, then find $\frac{dy}{dx}$.

(vi) $\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}$

(vii) $\tan(x-y)=\frac{\tan x-\tan y}{1+\tan x\tan y}$

(viii) $\sin^{-1}(-x)=-\sin^{-1}(x)$,

$\cos^{-1}(-x)=\pi-\cos^{-1}x$, $\tan^{-1}(-x)=-\tan^{-1}x$

(ix) $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$, $\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}$,

$\sec^{-1}x+\text{cosec}^{-1}x=\frac{\pi}{2}$

(x) $\sin^{-1}(\sqrt{1-x^2})=\cos^{-1}x$, $\cos^{-1}(\sqrt{1-x^2})=\sin^{-1}x$

(xi) $2\tan^{-1}x=\sin^{-1}\left(\frac{2x}{1+x^2}\right)=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$

(xii) $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$, $xy<1$

(xiii) $\tan^{-1}x-\tan^{-1}y=\tan^{-1}\left(\frac{x-y}{1+xy}\right)$, $xy>-1$

Some Important Substitutions

S.No.	Expression	Substitution
(i)	a^2-x^2	$x=a\cos\theta$ or $x=a\sin\theta$
(ii)	a^2+x^2	$x=a\tan\theta$ or $x=a\cot\theta$
(iii)	x^2-a^2	$x=a\sec\theta$ or $x=a\,\text{cosec}\,\theta$
(iv)	$\sqrt{\frac{a+x}{a-x}}$ or $\sqrt{\frac{a-x}{a+x}}$	$x=a\cos\theta$
(v)	$\sqrt{\frac{a^2-x^2}{a^2+x^2}}$ or $\sqrt{\frac{a^2+x^2}{a^2-x^2}}$	$x^2=a^2\cos 2\theta$

Put $x=\tan\theta\Rightarrow\tan^{-1}x=\theta$

$\therefore\quad y=\sin^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)=\sin^{-1}(\cos 2\theta)$

$\left[\because\cos 2x=\frac{1-\tan^2 x}{1+\tan^2 x}\right]$

$\Rightarrow\quad y=\sin^{-1}\left[\sin\left(\frac{\pi}{2}-2\theta\right)\right]\quad\left[\because\cos x=\sin\left(\frac{\pi}{2}-x\right)\right]$

$\Rightarrow\quad y=\frac{\pi}{2}-2\theta\quad[\because\sin^{-1}(\sin\theta)=\theta]$

$\Rightarrow\quad y=\frac{\pi}{2}-2\tan^{-1}x\quad[\because\theta=\tan^{-1}x]$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{\pi}{2}\right)-2\frac{d}{dx}(\tan^{-1}x)$$

$$\frac{dy}{dx}=0-\frac{2}{1+x^2}\Rightarrow\frac{dy}{dx}=\frac{-2}{1+x^2}$$

$\left[\because\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}\right]$

Here $f^{-1}(x)$ expression is of the form $\frac{x^2+a^2}{x^2-a^2}$, so we substitute $x = \tan\theta$ and then use suitable trigonometrical formula to write it in simplest form and then differentiate.

Sol. We have, $y = \sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)$

Here, putting $x = \tan\theta$, we get

$$y = \sec^{-1}\left(\frac{\tan^2\theta+1}{\tan^2\theta-1}\right)$$

$$\Rightarrow \quad y = \cos^{-1}\left(\frac{\tan^2\theta-1}{\tan^2\theta+1}\right) \quad \left[\because \sec^{-1}\theta = \cos^{-1}\left(\frac{1}{\theta}\right)\right]$$

$$\Rightarrow \quad y = \cos^{-1}\left[-\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)\right]$$

$$\Rightarrow \quad y = \cos^{-1}[-\cos 2\theta] \quad \left[\because \cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}\right]$$

$$\Rightarrow \quad y = \pi - \cos^{-1}(\cos 2\theta) [\because \cos^{-1}(-x) = \pi - \cos^{-1} x]$$

$$\Rightarrow \quad y = \pi - 2\theta \quad [\because \cos^{-1}(\cos x) = x]$$

$$\Rightarrow \quad y = \pi - 2\tan^{-1} x \; [\because \tan\theta = x \Rightarrow \theta = \tan^{-1} x] \quad ...(i)$$

On differentiating both sides of Eq. (i) w.r.t. x, we get

$$\frac{dy}{dx} = 0 - \frac{2}{1+x^2} \Rightarrow \frac{dy}{dx} = \frac{-2}{1+x^2}$$

EXAMPLE |6| Find $\frac{dy}{dx}$, if

$$y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0 < x < 1.$$

[NCERT]

Sol. We have, $y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Let $\frac{12}{13} = \cos\alpha$, then $\frac{5}{13} = \sin\alpha$ and $\tan\alpha = \frac{5}{12}$

$$\therefore \quad y = \sin^{-1}(\cos\alpha \sin\theta + \sin\alpha \cos\theta)$$

$$= \sin^{-1}(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$$

$$= \sin^{-1}\{\sin(\theta+\alpha)\}$$

$$= \theta + \alpha$$

$$\Rightarrow \quad y = \cos^{-1} x + \tan^{-1}\left(\frac{5}{12}\right)$$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}} + 0 = \frac{-1}{\sqrt{1-x^2}}$$

EXAMPLE |9| Differentiate $\tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right)$ $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and $\frac{a}{b}\tan x > -1$. **[NCERT Exemplar]**

First, convert the given inverse trigonometric function into the simplest form and then differentiate it.

EXAMPLE |7| If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, then prove that $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$.

Sol. We have, $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$.

On putting $x = \sin\theta$ and $y = \sin\phi$, it becomes

$$\cos\theta + \cos\phi = a(\sin\theta - \sin\phi)$$

$$\Rightarrow 2\cos\left(\frac{\theta+\phi}{2}\right)\cdot\cos\left(\frac{\theta-\phi}{2}\right) = a2\cos\left(\frac{\theta+\phi}{2}\right)\cdot\sin\left(\frac{\theta-\phi}{2}\right)$$

$$\Rightarrow \cot\frac{\theta-\phi}{2} = a \Rightarrow \theta - \phi = 2\cot^{-1} a$$

$$\Rightarrow \quad \sin^{-1} x - \sin^{-1} y = 2\cot^{-1} a$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = 0$$

Hence, $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$

EXAMPLE |8| Find $\frac{dy}{dx}$, if $y = \sin^{-1}\left(\frac{5x + 12\sqrt{1-x^2}}{13}\right)$.

Sol. Given, $y = \sin^{-1}\left(\frac{5x+12\sqrt{1-x^2}}{13}\right)$.

Put $x = \cos\theta$, then $\sqrt{1-x^2} = \sin\theta$.

$$\therefore \quad y = \sin^{-1}\left(\frac{5\cos\theta + 12\sin\theta}{13}\right)$$

$$= \sin^{-1}\left(\frac{12}{13}\sin\theta + \frac{5}{13}\cos\theta\right) \quad ...(i)$$

| TOPIC PRACTICE 5 |

OBJECTIVE TYPE QUESTIONS

1 If $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{1}{1+x^2}$ (b) $\frac{2}{1+x^2}$

(c) $\frac{2}{1-x^2}$ (d) $\frac{-2}{1+x^2}$

2 If $y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$, $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$, then $\frac{dy}{dx}$ is

(a) $\frac{3}{1+x^2}$ (b) $\frac{1}{1+x^2}$

(c) $\frac{-3}{1+x^2}$ (d) $\frac{3}{1-x^2}$

3 If $y = \sin^{-1} x + \sin^{-1}\sqrt{1-x^2}$, $-1 \le x < 1$, then $\frac{dy}{dx}$ is

Sol. Let $y = \tan^{-1}\left(\dfrac{a\cos x - b\sin x}{b\cos x + a\sin x}\right)$

$$\Rightarrow\ y = \tan^{-1}\left[\frac{\left(\dfrac{a\cos x - b\sin x}{b\cos x}\right)}{\left(\dfrac{b\cos x + a\sin x}{b\cos x}\right)}\right]$$

[dividing the numerator and denominator by $b\cos x$]

$$\Rightarrow\quad y = \tan^{-1}\left[\frac{\left(\dfrac{a}{b} - \tan x\right)}{\left(1 + \dfrac{a}{b}\tan x\right)}\right]$$

Put $\dfrac{a}{b} = \tan\theta \Rightarrow \theta = \tan^{-1}\left(\dfrac{a}{b}\right)$

Then, $y = \tan^{-1}\left[\dfrac{\tan\theta - \tan x}{1 + \tan\theta \cdot \tan x}\right]$

$\Rightarrow\quad y = \tan^{-1}[\tan(\theta - x)]$

$$\left[\because \tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \cdot \tan y}\right]$$

$\Rightarrow\quad y = \theta - x \qquad [\because \tan^{-1}(\tan\theta) = \theta]$

$\Rightarrow\quad y = \tan^{-1}\left(\dfrac{a}{b}\right) - x \qquad \left[\because \theta = \tan^{-1}\left(\dfrac{a}{b}\right)\right]$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 0 - 1 = -1$$

10 Differentiate $\tan^{-1}\left(\dfrac{1+\cos x}{\sin x}\right)$ with respect to x.

[CBSE 2018]

11 If $y = \sin^{-1}(6x\sqrt{1-9x^2}),\ -\dfrac{1}{3\sqrt{2}} < x < \dfrac{1}{3\sqrt{2}}$, then find $\dfrac{dy}{dx}$.

[Delhi 2017]

12 If $y = \sin^{-1}(2x\sqrt{1-x^2}),\ -\dfrac{1}{\sqrt{2}} < x < \dfrac{1}{\sqrt{2}}$, then find $\dfrac{dy}{dx}$.

[NCERT]

13 If $y = \tan^{-1}\left[\dfrac{\sqrt{1+a^2x^2} - 1}{ax}\right]$, then find $\dfrac{dy}{dx}$.

14 If the derivative of $\tan^{-1}(a+bx)$ takes the value $\dfrac{dy}{dx} = 1$ at $x = 0$, then prove that $b = 1 + a^2$.

SHORT ANSWER Type II Questions

15 Find the derivative of $\tan^{-1}\left(\dfrac{\cos x + \sin x}{\cos x - \sin x}\right)$ w.r.t. x.

16 Find the derivative of $\tan^{-1}\left(\dfrac{\cos x}{1 + \sin x}\right)$ w.r.t. x.

equal to

(a) 0 (b) 1

(c) 2 (d) 3

4 Derivative of $\cot^{-1}\left[\dfrac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right]$, $0 < x < \dfrac{\pi}{2}$ is

(a) $\dfrac{1}{2}$

(b) 1

(c) 2

(d) None of the above

VERY SHORT ANSWER Type Questions

5 If $f(x) = \cos^{-1}(\sin x)$, then find $f'(x)$.

6 If $y = \sin^{-1}(3x+2)$, then find dy/dx.

7 Differentiate $\cos^{-1}\sqrt{\dfrac{1+\cos x}{2}}$ w.r.t. x.

8 Find the derivative of

$$\cos^{-1}\left(\frac{\sin x + \cos x}{\sqrt{2}}\right), \frac{-\pi}{4} < x < \frac{\pi}{4}.$$

SHORT ANSWER Type I Questions

9 Differentiate $\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ w. r. t. x.

[NCERT]

HINTS & SOLUTIONS

1. (b) **Hint** Put $x = \tan\theta$
2. (a) **Hint** Put $x = \tan\theta$
3. (a) **Hint** $\sin^{-1}\sqrt{1-x^2} = \cos^{-1} x$
4. (a) **Hint** $\sqrt{1+\sin x} = \cos\dfrac{x}{2} + \sin\dfrac{x}{2}$ and $\sqrt{1-\sin x} = \cos\dfrac{x}{2} - \sin\dfrac{x}{2}$
5. We have, $f(x) = \cos^{-1}(\sin x)$

$$\Rightarrow\quad f(x) = \cos^{-1}\left[\cos\left(\frac{\pi}{2} - x\right)\right]$$

$$= \frac{\pi}{2} - x \qquad [\because \cos^{-1}(\cos x) = x]$$

On differentiating both sides w.r.t. x, we get

$$f'(x) = 0 - 1 = -1$$

6. We have, $y = \sin^{-1}(3x+2)$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(3x+2)^2}}\frac{d}{dx}(3x+2)$$

17 Find the derivative of $\cos^{-1}\left(\dfrac{3\cos x - 4\sin x}{5}\right)$ w. r. t. x.

18 If $y = \sin^{-1}[x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}]$ and $0 < x < 1$, then find dy/dx. **[All India 2014C]**

19 Differentiate the function w.r.t. x,
$\tan^{-1}\left(\dfrac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}\right)$.

20 If $y = \tan^{-1}\left[\dfrac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}}\right]$, $-1 < x < 1$, then find the derivative of y. **[Delhi 2015; NCERT Exemplar]**

21 Find the derivative of
$\tan^{-1}\left(\dfrac{3a^2x - x^3}{a^3 - 3ax^2}\right), -\dfrac{1}{\sqrt{3}} < \dfrac{x}{a} < \dfrac{1}{\sqrt{3}}$.
[NCERT Exemplar]

22 Find $\dfrac{dy}{dx}$, if $y = \sin^{-1}\left[\dfrac{6x - 4\sqrt{1-4x^2}}{5}\right]$. **[All India 2016]**

23 $\sqrt{1-x^4} + \sqrt{1-y^4} = a(x^2 - y^2)$, then prove that
$\dfrac{dy}{dx} = \dfrac{x}{y}\sqrt{\dfrac{1-y^4}{1-x^4}}$.
[All India 2016C]

9. Let $y = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

Put $x = \tan\theta \Rightarrow \theta = \tan^{-1} x$

$\therefore \quad y = \cos^{-1}\left(\dfrac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$= \cos^{-1}(\cos 2\theta) = 2\theta \qquad \left[\because \cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}\right]$

$\Rightarrow \quad y = 2\tan^{-1} x \qquad [\because \theta = \tan^{-1} x]$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 2 \times \frac{1}{1+x^2} = \frac{2}{1+x^2}$$

10. Let $\quad y = \tan^{-1}\left(\dfrac{1+\cos x}{\sin x}\right)$

$$= \tan^{-1}\left(\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right)$$

$\left[\because 1 + \cos A = 2\cos^2\dfrac{A}{2} \text{ and } \sin A = 2\sin\dfrac{A}{2}\cos\dfrac{A}{2}\right]$

$$= \tan^{-1}\left(\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}\right)$$

$$= \tan^{-1}\left(\cot\frac{x}{2}\right)$$

$= \dfrac{\pi}{2} - \cot^{-1}\left(\cot\dfrac{x}{2}\right) \qquad \left[\because \tan^{-1} x + \cot^{-1} x = \dfrac{\pi}{2}\right]$

$\left[\because \dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}\right]$

$$= \frac{3}{\sqrt{1-(3x+2)^2}}$$

7. Let $\; y = \cos^{-1}\sqrt{\dfrac{1+\cos x}{2}}$

$= \cos^{-1}\sqrt{\dfrac{2\cos^2\frac{x}{2}}{2}} \qquad \left[\because 1 + \cos\theta = 2\cos^2\dfrac{\theta}{2}\right]$

$$= \cos^{-1}\sqrt{\cos^2\frac{x}{2}}$$

$$= \cos^{-1}\left[\cos\left(\frac{x}{2}\right)\right]$$

$\Rightarrow \quad y = \dfrac{x}{2} \qquad [\because \cos^{-1}(\cos\theta) = \theta]$

On differentiating both sides w.r.t. x, we get $\dfrac{dy}{dx} = \dfrac{1}{2}$

8. Hint $\cos^{-1}\dfrac{(\sin x + \cos x)}{\sqrt{2}}$

$$= \cos^{-1}\left[\cos\left(\frac{\pi}{4} - x\right)\right]$$

$$= \left(\frac{\pi}{4} - x\right)$$

[Ans. −1]

13. Hint Put $ax = \tan\theta$. $\left[\text{Ans. } \dfrac{a}{2(1+a^2x^2)}\right]$

14. Hint $y = \tan^{-1}(a + bx)$

$$\Rightarrow \frac{dy}{dx} = \frac{b}{1+(a+bx)^2}$$

Then, put $x = 0$ and $\dfrac{dy}{dx} = 1$, we get $b = 1 + a^2$

15. Let $y = \tan^{-1}\left(\dfrac{\cos x + \sin x}{\cos x - \sin x}\right)$

$$= \tan^{-1}\left[\frac{\frac{\cos x}{\cos x} + \frac{\sin x}{\cos x}}{\frac{\cos x}{\cos x} - \frac{\sin x}{\cos x}}\right]$$

$$= \tan^{-1}\left[\frac{1+\tan x}{1-\tan x}\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4} + x\right)\right]$$

$\left[\because \tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}\right]$

$\Rightarrow y = \pi/4 + x \qquad [\because \tan^{-1}(\tan\theta) = \theta]$

On differentiating both sides w.r.t. x, we get

$$dy/dx = 0 + 1 = 1$$

16. Let $\; y = \tan^{-1}\left(\dfrac{\cos x}{1+\sin x}\right) = \tan^{-1}\left[\dfrac{\sin\left(\frac{\pi}{2} - x\right)}{1+\cos\left(\frac{\pi}{2} - x\right)}\right]$

$\left[\because \cos x = \sin\left(\dfrac{\pi}{2} - x\right) \text{and } \sin x = \cos\left(\dfrac{\pi}{2} - x\right)\right]$

$$= \frac{\pi}{2} - \frac{x}{2} \qquad [\because \cot^{-1}(\cot x) = x]$$

Now, on differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 0 - \frac{1}{2} = -\frac{1}{2}$$

11. Given, $y = \sin^{-1}(6x\sqrt{1-9x^2})$

$\Rightarrow \quad y = \sin^{-1}(2 \cdot 3x\sqrt{1-(3x)^2})$

Put $3x = \sin\theta$.

Then, $\quad y = \sin^{-1}(2\sin\theta \cdot \cos\theta)$

$\Rightarrow \quad y = \sin^{-1}(\sin 2\theta)$

$\Rightarrow \quad y = 2\theta$

$\Rightarrow \quad y = 2\sin^{-1}(3x)$

$\therefore \quad \frac{dy}{dx} = \frac{2}{\sqrt{1-9x^2}}(3)$

$\Rightarrow \quad \frac{dy}{dx} = \frac{6}{\sqrt{1-9x^2}}$

12. Solve as Question 11. $\left[\text{Ans. } \frac{2}{\sqrt{1-x^2}}\right]$

On squaring and adding Eqs. (ii) and (iii), we get

$$r^2\cos^2\theta + r^2\sin^2\theta = \frac{9}{25} + \frac{16}{25} = \frac{25}{25} = 1$$

$\Rightarrow \quad r^2(\cos^2\theta + \sin^2\theta) = 1$

$\Rightarrow \quad r^2 = 1$

$\Rightarrow \quad r = 1$

$\therefore \quad \frac{3}{5} = \cos\theta$ and $\frac{4}{5} = \sin\theta$

Now, $\frac{r\sin\theta}{r\cos\theta} = \frac{4/5}{3/5} = \frac{4}{3}$

$\Rightarrow \quad \tan\theta = \frac{4}{3}$

$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{4}{3}\right)$

From Eq. (i), we get

$y = \cos^{-1}(\cos\theta\cos x - \sin\theta\sin x)$

$\Rightarrow \quad y = \cos^{-1}[\cos(\theta + x)]$

$[\because \cos(A+B) = \cos A\cos B - \sin A\sin B]$

$\Rightarrow \quad y = \theta + x$

$\Rightarrow \quad y = \tan^{-1}\frac{4}{3} + x$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 0 + 1 = 1$$

18. Let $y = \sin^{-1}[x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}]$

Put $x = \sin\alpha$.

Then, $\sqrt{1-x^2} = \sqrt{1-\sin^2\alpha} = \sqrt{\cos^2\alpha} = \cos\alpha$

Again, put $\sqrt{x} = \sin\beta$

$$= \tan^{-1}\left[\frac{2\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\cdot\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}\right]$$

$$\left[\because \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \text{ and } 1+\cos\theta = 2\cos^2\frac{\theta}{2}\right]$$

$\Rightarrow \quad y = \tan^{-1}\left[\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right]$

$\Rightarrow \quad y = \frac{\pi}{4} - \frac{x}{2} \qquad [\because \tan^{-1}(\tan\theta) = \theta]$

On differentiating both sides w.r.t. x, we get $\frac{dy}{dx} = -\frac{1}{2}$

17. Let $y = \cos^{-1}\left(\frac{3\cos x - 4\sin x}{5}\right)$

$\Rightarrow \quad y = \cos^{-1}\left(\frac{3}{5}\cos x - \frac{4}{5}\sin x\right) \qquad$ ...(i)

Let $\frac{3}{5} = r\cos\theta \qquad$ ...(ii)

and $\frac{4}{5} = r\sin\theta \qquad$ ...(iii)

19. **Hint** Put $x = \cos\theta$ and then solve as Question 15.

$\left[\text{Ans. } \frac{1}{2\sqrt{1-x^2}}\right]$

20. **Hint** Put $x^2 = \cos\theta$ and then solve as Question 15.

$\left[\text{Ans. } \frac{-x}{\sqrt{1-x^4}}\right]$

21. **Hint** Put $\frac{x}{a} = \tan\theta$. $\left[\text{Ans. } \frac{3a}{a^2+x^2}\right]$

22. Given, $\quad y = \sin^{-1}\left[\frac{6x - 4\sqrt{1-4x^2}}{5}\right]$

Put $x = \frac{1}{2}\sin\theta$

$$\therefore \quad y = \sin^{-1}\left[\frac{6 \times \frac{\sin\theta}{2} - 4\sqrt{1 - 4\times\left(\frac{\sin\theta}{2}\right)^2}}{5}\right]$$

$$= \sin^{-1}\left(\frac{3\sin\theta - 4\sqrt{1-\sin^2\theta}}{5}\right)$$

$$= \sin^{-1}\left(\frac{3\sin\theta - 4\cos\theta}{5}\right)$$

$$= \sin^{-1}\left(\frac{3}{5}\sin\theta - \frac{4}{5}\cos\theta\right) \qquad \text{...(i)}$$

Let $\cos\phi = 3/5$, then

$$\sin\phi = \sqrt{1-\cos^2\phi} = \sqrt{1-\left(\frac{3}{5}\right)^2}$$

Then, $\sqrt{1-x} = \sqrt{1-\sin^2\beta} = \sqrt{\cos^2\beta} = \cos\beta$

$\therefore \quad y = \sin^{-1}[\sin\alpha\cos\beta - \cos\alpha\sin\beta]$

$= \sin^{-1}[\sin(\alpha-\beta)] = \alpha - \beta \qquad [\because \sin^{-1}(\sin\theta) = \theta]$

$= \alpha - \beta$

$\Rightarrow \quad y = \sin^{-1}x - \sin^{-1}\sqrt{x} \qquad \text{...(i)}$

$[\because x = \sin\alpha \Rightarrow \alpha = \sin^{-1}x$

and $\sqrt{x} = \sin\beta \Rightarrow \beta = \sin^{-1}\sqrt{x}]$

On differentiating both sides of Eq. (i) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-(\sqrt{x})^2}}\frac{d}{dx}(\sqrt{x})$$

[by chain rule of derivative]

$$= \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x}}\cdot\frac{1}{2}x^{-1/2}$$

$$= \frac{1}{\sqrt{1-x^2}} - \frac{1}{2\sqrt{x}\sqrt{1-x}}$$

$$= \frac{1}{\sqrt{1-x^2}} - \frac{1}{2\sqrt{x-x^2}}$$

$$= \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

Now, Eq. (i) becomes

$y = \sin^{-1}(\cos\phi\sin\theta - \sin\phi\cos\theta)$

$= \sin^{-1}[\sin(\theta-\phi)] = \theta - \phi$

$\Rightarrow \quad y = \sin^{-1}(2x) - \cos^{-1}\left(\frac{3}{5}\right)$

$\left[\because x = \frac{1}{2}\sin\theta \Rightarrow \sin\theta = 2x \Rightarrow \theta = \sin^{-1}(2x)\right]$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(2x)^2}}\frac{d}{dx}(2x) - 0$$

$$= \frac{2}{\sqrt{1-4x^2}}$$

23. Similar as Example 7.

Hint Put $x^2 = \sin\theta$ and $y^2 = \sin\phi$.

|TOPIC 6|

Differentiation of Exponential and Logarithmic Functions

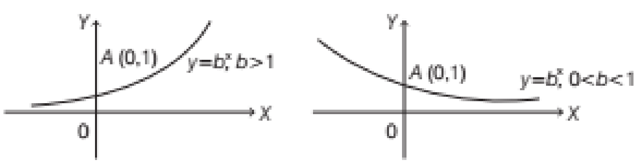
EXPONENTIAL FUNCTION

The function defined as $y = f(x) = b^x$ with positive base $b > 0$, $b \neq 1$ is called the exponential function.

(i) If the base of an exponential function is e, i.e. $b = e$, then it is called **natural exponential** function.

(ii) If the base of an exponential function is 10, i.e. $b = 10$, then it is called the **common exponential function.**

The graph of $y = b^x$ is as shown in the following figure



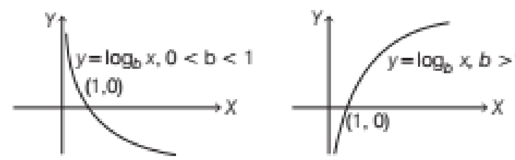
Main Features of Exponential Function

(i) The domain of exponential function is R i.e. the set of all real numbers.

(ii) The range of exponential function is R^+ i.e. the set of all positive real numbers.

(iii) The point (0,1) always lies on the graph of exponential function.

If base $b = 10$, then it is called **common logarithm** and if $b = e$, then it is called **natural logarithm**.



Main Features of Logarithmic Function

(i) The domain of logarithmic function is R^+ because logarithm of non-positive numbers cannot be defined meaningfully.

(ii) The range of logarithmic function is the set of all real numbers.

(iii) The logarithmic function having base greater than 1 is always increasing, i.e. as we move from left to right, the graph rises above.

(iv) For x, which is very near to zero and $b > 1$, the value of $\log_b x$ can be made lesser than any given real number.

(v) The point (1, 0) always lies on the graph of logarithmic function.

(vi) The graph of functions $f(x) = e^x$ and $f(x) = \ln(x)$

(iv) For every large negative values of x and $b > 1$, the exponential function is very close to 0, i.e. it tends to zero.

(v) Exponential function having base greater than 1 is always increasing, i.e. as we move from left to right, the graph rises above.

Note The important property of the natural exponential function in differential calculus is that it does not change during the process of differentiation. In this chapter, if base is not defined consider it as 'e'.

LOGARITHMIC FUNCTION

Let b be a positive real number such that $b > 0, b \neq 1$, then we say logarithm of a to base b is x, if $b^x = a$. Logarithm of a to base b is denoted by $\log_b a$. Thus, $\log_b a = x$, if $b^x = a$. In other words, the function from positive real numbers to real numbers is defined as $\log_b : R^+ \to R \Rightarrow \log_b(x) = y$, if $b^y = x$, is called logarithmic function.

IMPORTANT THEOREMS

There are following two theorems which are gives as

Theorem 1 The derivative of e^x w.r.t. x is e^x,

i.e. $\frac{d}{dx}(e^x) = e^x$.

Theorem 2 The derivative of $\log x$ w.r.t. x is $\frac{1}{x}$,

i.e. $\frac{d}{dx}(\log x) = \frac{1}{x}$.

EXAMPLE |1| Differentiate the following w.r.t. x.

(i) e^{x^3} (ii) $\log(\cos e^x)$ **[NCERT]**

Sol. (i) Let $y = e^{x^3}$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = e^{x^3}\frac{d}{dx}(x^3) \quad \text{[by chain rule of derivative]}$$

$$= e^{x^3} \cdot 3x^2 = 3x^2 e^{x^3}$$

(ii) Let $y = \log(\cos e^x)$, it is a composite function.

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\cos e^x} \cdot \frac{d}{dx}(\cos e^x)$$

$$\left[\text{by chain rule of derivative and } \frac{d}{dx}(\log t) = \frac{1}{t}\right]$$

$$= \frac{1}{\cos e^x}(-\sin e^x)\frac{d}{dx}(e^x)$$

[again by chain rule of derivative]

$$= -\tan e^x \times e^x = -e^x \tan e^x$$

EXAMPLE |2| Differentiate each of the following

are mirror images of each other reflected in the line $y = x$.

Basic Properties of Logarithms

(i) $\log_a(mn) = \log_a m + \log_a n$, $m > 0, n > 0, a > 0, a \neq 1$

(ii) $\log_a x = \log_a e \log_e x$, $a > 0, a \neq 1, x > 0$

(iii) $\log_a \frac{m}{n} = \log_a m - \log_a n, m > 0, n > 0, a > 0, a \neq 1$

(iv) $\log_a m^n = n \log_a m, m > 0, a > 0, a \neq 1$

(v) $e^{\log_e x} = x, x > 0$

(vi) $\log_a x = \frac{\log x}{\log a}, a > 0, a \neq 1, x > 0$

(vii) $\log_n m = \frac{1}{\log_m n}$, $m, n > 0$ and $m, n \neq 1$

(viii) $\log_{10} 10 = 1, \log_e e = 1$ and $\log_a 1 = 0, a > 0, a \neq 1$

$$\frac{dy}{dx} = \frac{1}{\tan\frac{x}{2}}\frac{d}{dx}\left(\tan\frac{x}{2}\right)$$

$$= \frac{1}{\tan\frac{x}{2}} \times \sec^2\frac{x}{2} \times \frac{d}{dx}\left(\frac{x}{2}\right)$$

$$= \frac{\cos\frac{x}{2}}{\sin\frac{x}{2}} \times \frac{1}{\cos^2\frac{x}{2}} \times \frac{1}{2} = \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sin x} = \operatorname{cosec} x$$

EXAMPLE |3| Differentiate $y = \log_{\cos x} \sin x$ w.r.t. x.

Sol. Given, $y = \log_{\cos x} \sin x = \frac{\log_e \sin x}{\log_e \cos x}$ $\quad \left[\because \log_a b = \frac{\log b}{\log a}\right]$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{\log_e \cos x \frac{d}{dx}(\log_e \sin x) - \log_e \sin x \frac{d}{dx}(\log_e \cos x)}{(\log_e \cos x)^2}$$

[by quotient rule of derivative]

$$= \frac{\log_e \cos x \frac{1}{\sin x}\frac{d}{dx}(\sin x) - \log_e \sin x \frac{1}{\cos x}\frac{d}{dx}(\cos x)}{(\log_e \cos x)^2}$$

$$= \frac{\log_e \cos x \times \frac{\cos x}{\sin x} - \log_e \sin x \times \frac{(-\sin x)}{\cos x}}{(\log_e \cos x)^2}$$

$$= \frac{\cot x \log_e \cos x + \tan x \log_e \sin x}{(\log_e \cos x)^2}$$

w.r.t. x.

(i) $e^{\sqrt{\cot x}}$ (ii) $\log \tan \frac{x}{2}$

Sol. (i) Let $y = e^{\sqrt{\cot x}}$

Taking log on both sides, we get

$$\log y = \sqrt{\cot x} \cdot \log e = \sqrt{\cot x} \quad [\because \log_e e = 1]$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2\sqrt{\cot x}}\frac{d}{dx}(\cot x)$$

$$\Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{2\sqrt{\cot x}} \times (-\operatorname{cosec}^2 x)$$

$$\Rightarrow \frac{dy}{dx} = y\left[-\frac{\operatorname{cosec}^2 x}{2\sqrt{\cot x}}\right]$$

$$\Rightarrow \frac{dy}{dx} = -e^{\sqrt{\cot x}}\frac{\operatorname{cosec}^2 x}{2\sqrt{\cot x}}$$

(ii) Let $y = \log \tan \frac{x}{2}$

On differentiating both sides w.r.t. x, we get

II. Take log on both sides and use suitable property of logarithm to convert it into simplest form for differentiation,

i.e. $\log y = v(x) \log u(x)$ ...(i)

III. Differentiate Eq. (i) obtained in step II with respect to x by using suitable formula,

i.e. $\frac{1}{y}\frac{dy}{dx} = v(x)\frac{1}{u(x)}\cdot u'(x) + v'(x)\log u(x)$ ...(ii)

IV. Now, transfer (or shift) the terms other than dy/dx from LHS to RHS and put the value of y. Then, we get the required answer.

Note Here, functions should always be positive otherwise their logarithms are not defined.

EXAMPLE |4| Differentiate a^x w.r.t. x, where a is a positive constant. **[NCERT]**

Sol. Let $y = a^x$.

Taking log on both sides, we get log

$$y = x \log a$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = \log a \Rightarrow \frac{dy}{dx} = y \log a$$

Thus, $\frac{d}{dx}(a^x) = a^x \log a$.

EXAMPLE |5| If $y = \log\sqrt{\frac{1+\sin^2 x}{1-\sin x}}$, then find $\frac{dy}{dx}$.

Sol. Given, $y = \log\sqrt{\frac{1+\sin^2 x}{1-\sin x}}$

LOGARITHMIC DIFFERENTIATION

Sometimes, a function of the following type or form is given

(i) A function having power which is a variable or function or numerical value and cannot be differentiated by another previous method.

(ii) A function having product or division of two or more than two functions, etc., is given to us and we have to differentiate it.

Then, we take logarithm and use properties of logarithm to simplify it and then differentiate it. This process is known as **logarithmic differentiation.** For this process, we use the following steps

I. Suppose given function is of the form $u(x)^{v(x)}$ or $u(x)^m$ or $u(x)\cdot v(x)$. Then, first assume the given function as y, say $y = u(x)^{v(x)}$.

$$\Rightarrow y = \left\{\frac{(x-3)(x^2+4)}{3x^2+4x+5}\right\}^{1/2}$$

Taking log on both sides, we get

$$\log y = \log\left\{\frac{(x-3)(x^2+4)}{3x^2+4x+5}\right\}^{1/2}$$

$$= \frac{1}{2}\log\left\{\frac{(x-3)(x^2+4)}{3x^2+4x+5}\right\} \quad [\because \log m^n = n \log m]$$

$$= \frac{1}{2}[\log(x-3)(x^2+4) - \log(3x^2+4x+5)]$$

$$[\because \log(m/n) = \log m - \log n]$$

$$= \frac{1}{2}[\log(x-3) + \log(x^2+4) - \log(3x^2+4x+5)] \quad ...(i)$$

$$[\because \log mn = \log m + \log n]$$

On differentiating both sides of Eq. (i) w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{(x-3)}\frac{d}{dx}(x-3) + \frac{1}{(x^2+4)}\frac{d}{dx}(x^2+4) - \frac{1}{(3x^2+4x+5)}\frac{d}{dx}(3x^2+4x+5)\right]$$

[by chain rule of derivative]

$$= \frac{1}{2}\left[\frac{1}{x-3} + \frac{2x}{x^2+4} - \frac{6x+4}{3x^2+4x+5}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{2}\left[\frac{1}{x-3} + \frac{2x}{x^2+4} - \frac{6x+4}{3x^2+4x+5}\right]$$

$$= \frac{1}{2}\sqrt{\frac{(x-3)(x^2+4)}{3x^2+4x+5}}\left[\frac{1}{x-3} + \frac{2x}{x^2+4} - \frac{6x+4}{3x^2+4x+5}\right]$$

$\Rightarrow \quad y = \frac{1}{2}\log\left(\frac{1+\sin^2 x}{1-\sin x}\right) \qquad [\because \log m^n = n\log m]$

$\Rightarrow \quad y = \frac{1}{2}[\log(1+\sin^2 x) - \log(1-\sin x)]$

$\left[\because \log\left(\frac{m}{n}\right) = \log m - \log n\right]$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{1+\sin^2 x}\times(0+2\sin x\cdot\cos x) - \frac{1}{1-\sin x}(-\cos x)\right]$$

$$= \frac{1}{2}\left[\frac{\sin 2x}{1+\sin^2 x} + \frac{\cos x}{1-\sin x}\right]$$

EXAMPLE |6| Differentiate $\sqrt{\frac{(x-3)(x^2+4)}{3x^2+4x+5}}$ with respect to x. **[NCERT]**

Sol. Let given function be $y = \sqrt{\frac{(x-3)(x^2+4)}{3x^2+4x+5}}$

$\Rightarrow \quad \frac{du}{dx} = u[-x\tan x + \log\cos x]$

$\Rightarrow \quad \frac{du}{dx} = (\cos x)^x[-x\tan x + \log\cos x]$

Again, $\quad v = \sin^{-1}\sqrt{3x}$

$$\frac{dv}{dx} = \frac{1}{\sqrt{1-(\sqrt{3x})^2}}\frac{d}{dx}[(3x)^{1/2}]$$ **[1]**

$\Rightarrow \quad \frac{dv}{dx} = \frac{1}{\sqrt{1-3x}}\frac{1}{2}(3x)^{-1/2}\times 3$

$\Rightarrow \quad \frac{dv}{dx} = \frac{1}{\sqrt{1-3x}}\times\frac{3}{2\sqrt{3x}}$

$\Rightarrow \quad \frac{dv}{dx} = \frac{3}{2\sqrt{3x}\sqrt{1-3x}} \Rightarrow \frac{dv}{dx} = \frac{\sqrt{3}}{2\sqrt{x-3x^2}}$

On putting the values of $\frac{du}{dx}$ and $\frac{dv}{dx}$ in Eq. (i), we get

$$\frac{dy}{dx} = (\cos x)^x[-x\tan x + \log\cos x] + \frac{\sqrt{3}}{2\sqrt{x-3x^2}}$$

EXAMPLE |8| If $y = x^{x^x}$, then find $\frac{dy}{dx}$.

Sol. Given, $y = x^{x^x}$ ...(i)

Taking log on both sides, we get

$\log y = \log x^{x^x}$ ...(ii)

$\Rightarrow \quad \log y = x^x \log x \qquad [\because \log m^n = n\log m]$

Again, taking log on both sides, we get

$\log\log y = \log(x^x \log x)$

$\Rightarrow \quad \log\log y = \log x^x + \log\log x$

$[\because \log(mn) = \log m + \log n]$

$\Rightarrow \quad \log\log y = x\log x + \log\log x$

EXAMPLE |7| If $y = (\cos x)^x + \sin^{-1}\sqrt{3x}$, find $\frac{dy}{dx}$.

[All India 2017C]

Sol. Let $y = (\cos x)^x + \sin^{-1}\sqrt{3x}$

Again, let $u = (\cos x)^x$, $v = \sin^{-1}\sqrt{3x}$

$\therefore \quad y = u + v$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \text{...(i)}$$

Now, $\quad u = (\cos x)^x$

Taking log on both sides, we get

$$\log u = \log(\cos x)^x = x\log(\cos x)$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{u}\frac{du}{dx} = x\frac{d}{dx}\log(\cos x) + \log(\cos x)\frac{d}{dx}(x)$$

$$= \frac{x}{\cos x}(-\sin x) + \log(\cos x)$$

TOPIC PRACTICE 6

OBJECTIVE TYPE QUESTIONS

1 If $y = \log x^x$, then the value of $\frac{dy}{dx}$ is

(a) $x^x(1+\log x)$ (b) $\log(ex)$

(c) $\log\frac{e}{x}$ (d) $\log\left(\frac{x}{e}\right)$

2 If $y = \log_a x + \log_x a + \log_x x + \log_a a$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{1}{x} + x\log a$ (b) $\frac{\log a}{x} + \frac{x}{\log a}$

(c) $\frac{1}{x\log a} + x\log a$ (d) None of these

3 If $y^x = e^{y-x}$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{1+\log y}{y\log y}$ (b) $\frac{(1+\log y)^2}{y\log y}$

(c) $\frac{1+\log y}{(\log y)^2}$ (d) $\frac{(1+\log y)^2}{\log y}$

4 If $x = e^{x/y}$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{x-y}{x\log x}$ (b) $\frac{y-x}{\log x}$

(c) $\frac{y-x}{x\log x}$ (d) $\frac{x-y}{\log x}$

5 If $x^y = y^x$, then $x(x - y\log x)\frac{dy}{dx}$ is equal to

On differentiating both sides w.r.t. x, we get

$$\frac{1}{\log y}\cdot\frac{d}{dx}(\log y)=\left[x\frac{d}{dx}(\log x)+\log x\frac{d}{dx}(x)\right]+\left[\frac{1}{\log x}\cdot\frac{d}{dx}(\log x)\right]$$

[by product rule and chain rule of derivative]

$$\Rightarrow \quad \frac{1}{\log y}\cdot\frac{1}{y}\cdot\frac{dy}{dx}=\left[x\cdot\frac{1}{x}+\log x\cdot 1\right]+\left[\frac{1}{\log x}\cdot\frac{1}{x}\right]$$

$$\Rightarrow \quad \frac{1}{y\log y}\cdot\frac{dy}{dx}=1+\log x+\frac{1}{x\log x}$$

$$\Rightarrow \quad \frac{dy}{dx}=y\log y\left[1+\log x+\frac{1}{x\log x}\right]$$

$$=x^{x^x}\log x^{x^x}\left[1+\log x+\frac{1}{x\log x}\right]$$

[from Eqs. (i) and (ii)]

SHORT ANSWER Type II Questions

13 If $\log\sqrt{x^2+y^2}=\tan^{-1}\left(\frac{x}{y}\right)$, then show that
$$\frac{dy}{dx}=\frac{y-x}{y+x}.$$
[Delhi 2019]

14 If $\frac{x}{x-y}=\log\left(\frac{a}{x-y}\right)$, then prove that $\frac{dy}{dx}=2-\frac{x}{y}$.

15 If $(x-y)\cdot e^{\frac{x}{x-y}}=a$, then prove that $y\frac{dy}{dx}+x=2y$. **[Delhi 2014C]**

16 If $y^x=x^y$, then find $\frac{dy}{dx}$. **[NCERT]**

17 If $(\cos x)^y=(\cos y)^x$, then find $\frac{dy}{dx}$. **[All India 2019]**

18 If $x^p y^q=(x+y)^{p+q}$, then prove that $\frac{dy}{dx}=\frac{y}{x}$. **[Foreign 2014]**

19 If u, v and w are functions of x, then show that
$$\frac{d}{dx}(u\cdot v\cdot w)=\frac{du}{dx}v\cdot w+u\cdot\frac{dv}{dx}\cdot w+u\cdot v\frac{dw}{dx}.$$
In two ways, first by repeated application of product rule, second by logarithmic differentiation. **[NCERT]**

20 Find the derivative of $y=(x+3)^2(x+4)^3(x+5)^4$. **[NCERT]**

21 Find the derivative of the function given by $f(x)=(1+x)(1+x^2)(1+x^4)(1+x^8)$ and hence find $f'(1)$. **[NCERT]**

22 Differentiate $\cos x\cos 2x\cos 3x$ w.r.t. x. **[NCERT]**

23 If $x^y+y^x=a^b$, then find $\frac{dy}{dx}$. **[All India 2017]**

24 Differentiate the following function w.r.t. x, $(\log x)^x+x^{\log x}$. **[All India 2020]**

(a) $y(y-x\log y)$ (b) $y(y+x\log y)$

(c) $x(x+y\log x)$ (d) $x(y-x\log y)$

VERY SHORT ANSWER Type Questions

6 Differentiate $e^{\sin^{-1}x}$ w.r.t. x. **[NCERT]**

7 If $y=\frac{\log x}{x}$, then find $\frac{dy}{dx}$.

8 Differentiate $\log(\log x)$, $x>1$ w.r.t. x. **[NCERT]**

SHORT ANSWER Type I Questions

9 If $f(x)=x\cos x+e^x$, then find $f'(0)$.

10 Differentiate $(\sin x)^{\log x}$ w.r.t. x.

11 Find the derivative of $xe^x\sin x$ w.r.t. x.

12 Find the derivative of $\log[\log(\log x^5)]$ w.r.t. x.

28 If $y=\frac{ax^2}{(x-a)(x-b)(x-c)}+\frac{bx}{(x-b)(x-c)}+\frac{c}{(x-c)}+1$, then prove that $\frac{y'}{y}=\frac{1}{x}\left[\frac{a}{a-x}+\frac{b}{b-x}+\frac{c}{c-x}\right]$.

29. Let y be an implicit function of x defined by $x^{2x}-2x^x\cot y-1=0$. Then, find the value of $y'(1)$.

HINTS & SOLUTIONS

1. (b) Given, $y=x\log x$

$$\frac{dy}{dx}=\frac{x}{x}+\log x$$

$$\Rightarrow \quad \frac{dy}{dx}=\log e+\log x$$

$$\Rightarrow \quad \frac{dy}{dx}=\log(ex)$$

2. (d) Given, $y=\frac{\log x}{\log a}+\frac{\log a}{\log x}+1+1$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1}{x\log a}-\frac{\log a}{x(\log x)^2}$$

3. (d) We have, $y^x=e^{y-x}$

Taking log both sides, we get

$$x\log y=y-x$$

$$\Rightarrow \quad x(\log y+1)=y$$

$$\Rightarrow \quad \frac{y}{1+\log y}=x$$

On differentiating w.r.t. x, we get

$$\left(\frac{(1+\log y)-\left(\frac{1}{y}\right)y}{(1+\log y)^2}\right)\frac{dy}{dx}=1$$

$$\Rightarrow \quad \frac{\log y}{(1+\log y)^2}\frac{dy}{dx}=1$$

25 If $y = x^{\sin x} + (\sin x)^{\cos x}$, then find $\frac{dy}{dx}$.

[NCERT; All India 2016]

26 If $e^x + e^y = e^{x+y}$, then prove that $\frac{dy}{dx} + \frac{e^x(e^y - 1)}{e^y(e^x - 1)} = 0$.

Or

If $e^x + e^y = e^{x+y}$, then prove that $\frac{dy}{dx} + e^{y-x} = 0$.

[Foreign 2014]

27 Find $\frac{dy}{dx}$, when $y = x^{\cot x} + \frac{2x^2 - 3}{x^2 + x + 2}$. **[All India 2012]**

5. (a) Since, $x^y = y^x \Rightarrow y \log x = x \log y$

On differentiating w.r.t. x, we get

$$\Rightarrow \quad y \cdot \frac{1}{x} + \log x \frac{dy}{dx} = \frac{x}{y}\frac{dy}{dx} + \log y$$

$$\Rightarrow \quad \frac{dy}{dx}\left(\frac{x}{y} - \log x\right) = \frac{y}{x} - \log y$$

$$\Rightarrow \quad x(x - y\log x)\frac{dy}{dx} = y(-x\log y + y)$$

6. Let $y = e^{\sin^{-1} x}$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = e^{\sin^{-1} x}\frac{d}{dx}(\sin^{-1} x)$$

$$= e^{\sin^{-1} x} \times \frac{1}{\sqrt{1 - x^2}} = \frac{e^{\sin^{-1} x}}{\sqrt{1 - x^2}}$$

7. We have, $y = \frac{\log x}{x}$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{\left(x\frac{d}{dx}(\log x) - \log x\frac{d}{dx}(x)\right)}{x^2}$$

[by quotient rule of derivative]

$$= \frac{\left(x \times \frac{1}{x} - \log x\right)}{x^2}$$

$$= \frac{1 - \log x}{x^2}$$

8. Similar as Example 2 (ii). $\left[\text{Ans. } \frac{1}{x\log x}\right]$

9. We have, $f(x) = x\cos x + e^x$

On differentiating both sides w.r.t. x, we get

$$f'(x) = \frac{d}{dx}(x\cos x) + \frac{d}{dx}(e^x)$$

$$\Rightarrow \quad f'(x) = x\frac{d}{dx}(\cos x) + \cos x\frac{d}{dx}(x) + \frac{d}{dx}(e^x)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(1 + \log y)^2}{\log y}$$

4. (a) Given that $x = e^{x/y}$

Taking log on both sides, we get

$$\log x = \frac{x}{y} \cdot \log e = \frac{x}{y}$$

$$\Rightarrow \quad x = y\log x$$

Now, differentiating w.r.t. x, we get

$$\Rightarrow \quad 1 = y \cdot \frac{1}{x} + \log x \cdot \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x - y}{x\log x}$$

13. Given, $\log\sqrt{x^2 + y^2} = \tan^{-1}\left(\frac{x}{y}\right)$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{\sqrt{x^2 + y^2}}\frac{d}{dx}(\sqrt{x^2 + y^2}) = \frac{1}{1 + \frac{x^2}{y^2}}\frac{d}{dx}\left(\frac{x}{y}\right)$$

$$\Rightarrow \quad \frac{1}{\sqrt{x^2 + y^2}} \cdot \frac{1}{2\sqrt{x^2 + y^2}}\left(2x + 2y\frac{dy}{dx}\right)$$

$$= \frac{y^2}{y^2 + x^2}\left(\frac{y - x\frac{dy}{dx}}{y^2}\right)$$

$$\Rightarrow \quad \frac{1}{(x^2 + y^2)}\left(x + y\frac{dy}{dx}\right) = \frac{y^2}{x^2 + y^2}\left(\frac{y - x\frac{dy}{dx}}{y^2}\right)$$

$$\Rightarrow \quad x + y\frac{dy}{dx} = y - x\frac{dy}{dx}$$

$$\Rightarrow \quad y\frac{dy}{dx} + x\frac{dy}{dx} = y - x$$

$$\Rightarrow \quad (y + x)\frac{dy}{dx} = y - x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y - x}{y + x}$$ **Hence proved.**

14. **Hint** $x = (x - y)[\log a - \log(x - y)]$

Then, differentiate w.r.t. y on both sides and simplify.

15. Given, $(x - y) \cdot e^{\frac{x}{x-y}} = a$

Taking log on both sides, we get

$$\log(x - y) + \log e^{\frac{x}{x-y}} = \log a$$

$$\Rightarrow \quad \log(x - y) + \frac{x}{x - y} = \log a \quad \text{...(i)}$$

$$\Rightarrow \quad (x - y)\log(x - y) + x = (x - y)\log a$$

$$= -x\sin x + \cos x + e^x$$

At $x = 0$,

$f'(0) = -(0)\sin 0 + \cos 0 + e^0 = 0 + 1 + 1 = 2$

10. Similar as Example 7.

$$\left[\text{Ans. } (\sin x)^{\log x}\left(\frac{\log \sin x}{x} + \cot x \cdot \log x\right)\right]$$

11. **Hint** Use product rule of derivative.

[Ans. $xe^x \cos x + (x+1)e^x \sin x$]

12. **Hint** Use chain rule of derivative.

$$\left[\text{Ans. } \frac{5}{x\log(x^5)\log\log(x^5)}\right]$$

$$\Rightarrow \quad \frac{dx}{dy}\left(2 - \frac{x}{x-y}\right) = -\frac{x}{x-y} + 1$$

$$\Rightarrow \quad \frac{dx}{dy}\left(\frac{x-2y}{x-y}\right) = -\frac{y}{x-y}$$

$$\Rightarrow \quad \frac{dx}{dy}(x-2y) = -y$$

$$\Rightarrow \quad \frac{ydy}{dx} + x = 2y$$

Hence proved.

16. We have, $y^x = x^y$

Taking log on both sides, we get

$$\log y^x = \log x^y$$

$$\Rightarrow \quad x\log y = y\log x$$

On differentiating both sides w.r.t. x, we get

$$x\frac{d}{dx}(\log y) + \log y\frac{d}{dx}(x) = y\frac{d}{dx}(\log x) + \log x\frac{d}{dx}(y)$$

[by product rule of derivative]

$$\Rightarrow \quad x\frac{1}{y}\cdot\frac{dy}{dx} + \log y = y \times \frac{1}{x} + \log x \times \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{x}{y}\frac{dy}{dx} - \log x\frac{dy}{dx} = \frac{y}{x} - \log y$$

$$\Rightarrow \quad \frac{dy}{dx}\left[\frac{x}{y} - \log x\right] = \frac{y}{x} - \log y$$

$$\Rightarrow \quad \frac{dy}{dx}\left[\frac{x - y\log x}{y}\right] = \left[\frac{y - x\log y}{x}\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x}\left[\frac{y - x\log y}{x - y\log x}\right]$$

17. Given, $(\cos x)^y = (\cos y)^x$

Taking log on both sides, we get

$$\log(\cos x)^y = \log(\cos y)^x$$

$$\therefore \quad y\log(\cos x) = x\log(\cos y)$$

On differentiating both sides w.r.t. x, we get

On differentiating both sides w.r.t. y, we get

$$\left(\frac{dx}{dy} - 1\right)\log(x-y) + (x-y) \times \frac{1}{x-y} \times \left(\frac{dx}{dy} - 1\right) + \frac{dx}{dy}$$
$$= \left(\frac{dx}{dy} - 1\right)\log a$$

$$\Rightarrow \left(\frac{dx}{dy} - 1\right)\log(x-y) + 2\frac{dx}{dy} - 1 = \left(\frac{dx}{dy} - 1\right)$$
$$\left[\log(x-y) + \frac{x}{x-y}\right] \text{[from Eq. (i)]}$$

$$\Rightarrow \quad 2\frac{dx}{dy} - 1 = \left(\frac{dx}{dy} - 1\right)\left(\frac{x}{x-y}\right)$$

$$\Rightarrow \quad [x\tan y + \log(\cos x)]\frac{dy}{dx} = \log(\cos y) + y\tan x$$

$$\therefore \quad \frac{dy}{dx} = \frac{\log(\cos y) + y\tan x}{x\tan y + \log(\cos x)}$$

18. We have, $x^p y^q = (x+y)^{p+q}$

Taking log on both sides, we get

$$\log x^p + \log y^q = \log(x+y)^{p+q}$$

$$\Rightarrow p\log x + q\log y = (p+q)\log(x+y)$$

On differentiating both sides w.r.t. x, we get

$$p\frac{1}{x} + q\frac{1}{y}\cdot\frac{dy}{dx} = (p+q)\frac{1}{(x+y)}\cdot\frac{d}{dx}(x+y)$$

$$\Rightarrow \quad \frac{p}{x} + \frac{q}{y}\cdot\frac{dy}{dx} = \left(\frac{p+q}{x+y}\right)\left[1 + \frac{dy}{dx}\right]$$

$$\Rightarrow \quad \frac{p}{x} + \frac{q}{y}\frac{dy}{dx} = \frac{p+q}{x+y} + \left(\frac{p+q}{x+y}\right)\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{q}{y}\frac{dy}{dx} - \left(\frac{p+q}{x+y}\right)\frac{dy}{dx} = \frac{p+q}{x+y} - \frac{p}{x}$$

$$\Rightarrow \quad \frac{dy}{dx}\left[\frac{qx + qy - py - qy}{y(x+y)}\right] = \frac{px + qx - px - py}{(x+y)x}$$

$$\Rightarrow \quad \frac{dy}{dx}\left[\frac{qx - py}{y(x+y)}\right] = \frac{(qx - py)}{x(x+y)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(qx - py)}{(x+y)x}\cdot\frac{y(x+y)}{(qx - py)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x}$$

Hence proved.

19. First, we use product rule and find that

$$\frac{d}{dx}(uvw) = \frac{d}{dx}[(uv)w] = (uv)\frac{dw}{dx} + w\frac{d}{dx}(uv)$$

[consider here uv as one function and w as another function]

$$y\cdot\frac{d}{dx}\log(\cos x)+\log(\cos x)\cdot\frac{d}{dx}(y)$$
$$=x\frac{d}{dx}\log(\cos y)+\log(\cos y)\frac{d}{dx}(x)$$
$$\left[\because \frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}\right]$$
$$\Rightarrow\quad y\cdot\frac{1}{\cos x}\frac{d}{dx}(\cos x)+\log(\cos x)\frac{dy}{dx}$$
$$=x\cdot\frac{1}{\cos y}\frac{d}{dx}(\cos y)+\log(\cos y)\cdot 1$$
$$\Rightarrow\quad y\cdot\frac{1}{\cos x}(-\sin x)+\log(\cos x)\cdot\frac{dy}{dx}$$
$$=x\frac{1}{\cos y}(-\sin y)\cdot\frac{dy}{dx}+\log(\cos y)\cdot 1$$
$$\Rightarrow\quad -y\tan x+\log(\cos x)\frac{dy}{dx}=-x\tan y\frac{dy}{dx}+\log(\cos y)$$

$$=(uvw)\left\{\frac{1}{u}\frac{du}{dx}+\frac{1}{v}\frac{dv}{dx}+\frac{1}{w}\frac{dw}{dx}\right\}$$
$$=(vw)\frac{du}{dx}+(uw)\frac{dv}{dx}+(uv)\frac{dw}{dx}$$
$$=(uv)\frac{dw}{dx}+wu\frac{dv}{dx}+wv\frac{du}{dx}\qquad\text{...(ii)}$$

From Eqs. (i) and (ii), we find that the result obtained is same in the two cases.

20. **Hint** Taking log on both sides and then differentiate it.

[**Ans.** $(x+3)(x+4)^2(x+5)^3[9x^2+70x+133]$]

21. Given, $f(x)=(1+x)(1+x^2)(1+x^4)(1+x^8)$

Taking log on both sides, we get

$$\log\{f(x)\}=\log\{(1+x)(1+x^2)(1+x^4)(1+x^8)\}$$
$$\Rightarrow\quad \log\{f(x)\}=\log(1+x)+\log(1+x^2)+\log(1+x^4)+\log(1+x^8)$$

On differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}\log\{f(x)\}=\frac{d}{dx}\log(1+x)+\frac{d}{dx}\log(1+x^2)+\frac{d}{dx}\log(1+x^4)+\frac{d}{dx}\log(1+x^8)$$
$$\frac{1}{f(x)}f'(x)=\frac{1}{1+x}(0+1)+\frac{1}{1+x^2}(0+2x)+\frac{1}{1+x^4}(0+4x^3)+\frac{1}{1+x^8}(0+8x^7)$$
$$\Rightarrow\quad f'(x)=f(x)\left\{\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\frac{8x^7}{1+x^8}\right\}$$
$$=(1+x)(1+x^2)(1+x^4)(1+x^8)\left\{\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\frac{8x^7}{1+x^8}\right\}$$
$$\therefore\quad f'(1)=(1+1)(1+1)(1+1)(1+1)\left\{\frac{1}{1+1}+\frac{2}{1+1}+\frac{4}{1+1}+\frac{8}{1+1}\right\}$$
$$=2^4\left\{\frac{1}{2}+1+2+4\right\}=16\left(\frac{15}{2}\right)=120$$

22. Let $y=\cos x\cos 2x\cos 3x$ ...(i)

$$=uv\frac{dw}{dx}+w\left(u\frac{dv}{dx}+v\frac{du}{dx}\right)$$
$$=uv\frac{dw}{dx}+wu\frac{dv}{dx}+wv\frac{du}{dx}\qquad\text{...(i)}$$

Next, we use logarithmic differentiation to obtain $d(uvw)/dx$.

Let $y=uvw$

Taking log on both sides, we get

$$\log y=\log(uvw)$$
$$\Rightarrow\quad \log y=\log u+\log v+\log w$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{u}\frac{du}{dx}+\frac{1}{v}\frac{dv}{dx}+\frac{1}{w}\frac{dw}{dx}$$
$$\Rightarrow\quad \frac{dy}{dx}=y\left\{\frac{1}{u}\frac{du}{dx}+\frac{1}{v}\frac{dv}{dx}+\frac{1}{w}\frac{dw}{dx}\right\}$$

$$\Rightarrow\quad \frac{dy}{dx}=y\left[\frac{1}{\cos x}(-\sin x)+\frac{1}{\cos 2x}(-\sin 2x)\frac{d}{dx}(2x)+\frac{1}{\cos 3x}(-\sin 3x)\frac{d}{dx}(3x)\right]$$

[by chain rule of derivative]

$$=y[-\tan x-2\tan 2x-3\tan 3x]$$
$$=-\cos x\cos 2x\cos 3x[\tan x+2\tan 2x+3\tan 3x]$$

[from Eq. (i)]

23. Given, $x^y+y^x=a^b$ ...(i)

Let $x^y=v$ and $y^x=u$ ...(ii)

On putting these values in Eq. (i), we get

$$v+u=a^b$$

On differentiating both sides w.r.t. x, we get

$$\frac{dv}{dx}+\frac{du}{dx}=0\qquad\text{...(iii)}$$

Now, $x^y=v$ [from Eq. (ii)]

Taking log on both sides, we get

$$\log x^y=\log v\ \Rightarrow\ y\log x=\log v$$

On differentiating both sides w.r.t. x, we get

$$y\cdot\frac{1}{x}+\log x\cdot\frac{dy}{dx}=\frac{1}{v}\frac{dv}{dx}$$
$$\Rightarrow\quad v\left(\frac{y}{x}+\log x\cdot\frac{dy}{dx}\right)=\frac{dv}{dx}$$
$$\Rightarrow\quad \frac{dv}{dx}=x^y\left(\frac{y}{x}+\log x\cdot\frac{dy}{dx}\right)\qquad\text{...(iv)}$$

[from Eq. (ii)]

and $y^x=u$ [from Eq. (ii)]

Taking log on both sides, we get

$$\log y^x=\log u$$
$$\Rightarrow\quad x\log y=\log u$$

On differentiating both sides w.r.t. x, we get

$$x\cdot\frac{1}{y}\frac{dy}{dx}+1\cdot\log y=\frac{1}{u}\frac{du}{dx}$$
$$\Rightarrow\quad \frac{x}{y}\frac{dy}{dx}+\log y=\frac{1}{u}\frac{du}{dx}$$

Taking log on both sides, we get

$$\log y = \log(\cos x \cdot \cos 2x \cdot \cos 3x)$$

$$\Rightarrow \quad \log y = \log \cos x + \log \cos 2x + \log \cos 3x$$

$$[\because \log(mnp) = \log m + \log n + \log p]$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{\cos x} \cdot \frac{d}{dx}(\cos x) + \frac{1}{\cos 2x} \cdot \frac{d}{dx}(\cos 2x) + \frac{1}{\cos 3x} \cdot \frac{d}{dx}(\cos 3x)$$

[by chain rule of derivative]

$$\Rightarrow \quad x^y \frac{y}{x} + x^y \log x \cdot \frac{dy}{dx} + y^x \cdot \frac{x}{y}\frac{dy}{dx} + y^x \log y = 0$$

$$\Rightarrow \quad x^y \log x \cdot \frac{dy}{dx} + y^x \frac{x}{y} \cdot \frac{dy}{dx} = -x^y \frac{y}{x} - y^x \log y$$

$$\Rightarrow \quad \frac{dy}{dx}\left[x^y \log x + y^x \cdot \frac{x}{y}\right] = -x^y \cdot \frac{y}{x} - y^x \log y$$

$$\therefore \quad \frac{dy}{dx} = \frac{-x^{y-1} \cdot y - y^x \log y}{x^y \log x + y^{x-1} \cdot x}$$

24. Let $y = (\log x)^x + x^{\log x}$ and $u = (\log x)^x$ and $v = x^{\log x}$.

Then, $y = u + v$.

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \text{...(i)}$$

Now, $u = (\log x)^x$

Taking log on both sides, we get

$$\log u = \log(\log x)^x \Rightarrow \log u = x \log(\log x)$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{u} \cdot \frac{du}{dx} = x \frac{d}{dx} \log(\log x) + \log(\log x) \frac{d}{dx}(x)$$

[by chain rule of derivative]

$$\Rightarrow \quad \frac{1}{u} \cdot \frac{du}{dx} = \frac{x}{\log x} \times \frac{1}{x} + \log(\log x)$$

$$\Rightarrow \quad \frac{du}{dx} = u\left[\frac{1}{\log x} + \log(\log x)\right]$$

$$\Rightarrow \quad \frac{du}{dx} = (\log x)^x \left[\frac{1}{\log x} + \log(\log x)\right]$$

Again, $v = x^{\log x}$

Taking log on both sides, we get

$$\log v = \log x^{\log x} \Rightarrow \log v = (\log x)(\log x)$$

$$\Rightarrow \quad \log v = (\log x)^2$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{v} \cdot \frac{dv}{dx} = 2 \log x \frac{d}{dx}(\log x) = 2 \log x \times \frac{1}{x}$$

$$\Rightarrow \quad \frac{dv}{dx} = v\left[\frac{2 \log x}{x}\right] \Rightarrow \frac{dv}{dx} = x^{\log x}\left[\frac{2 \log x}{x}\right]$$

Now, putting the values of $\frac{du}{dx}$ and $\frac{dv}{dx}$ in Eq. (i), we get

$$\Rightarrow \quad u\left[\frac{x}{y}\frac{dy}{dx} + \log y\right] = \frac{du}{dx}$$

$$\Rightarrow \quad y^x\left[\frac{x}{y}\frac{dy}{dx} + \log y\right] = \frac{du}{dx} \quad \text{[from Eq. (ii)] ...(v)}$$

On putting the values of $\frac{dv}{dx}$ and $\frac{du}{dx}$ from Eqs. (iv) and (v) respectively in Eq. (iii), we get

$$x^y\left(\frac{y}{x} + \log x \cdot \frac{dy}{dx}\right) + y^x\left(\frac{x}{y}\frac{dy}{dx} + \log y\right) = 0$$

Now, $u = x^{\sin x}$

Taking log on both sides, we get

$$\log u = \log x^{\sin x}$$

$$\Rightarrow \quad \log u = \sin x \log x$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{u} \cdot \frac{du}{dx} = \sin x \frac{d}{dx}(\log x) + \log x \frac{d}{dx}(\sin x)$$

[by product rule of derivative]

$$\Rightarrow \quad \frac{1}{u} \cdot \frac{du}{dx} = \frac{1}{x} \sin x + \log x \cdot \cos x$$

$$\Rightarrow \quad \frac{du}{dx} = u\left[\frac{\sin x}{x} + \log x \cdot \cos x\right]$$

$$= x^{\sin x}\left[\frac{\sin x}{x} + \cos x \log x\right] \quad \text{...(ii)}$$

Also, $v = (\sin x)^{\cos x}$.

Taking log on both sides, we get

$$\log v = \log(\sin x)^{\cos x}$$

$$\Rightarrow \quad \log v = \cos x \log(\sin x)$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{v}\frac{dv}{dx} = \cos x \frac{d}{dx}[\log(\sin x)] + \log \sin x \frac{d}{dx}(\cos x)$$

[by product rule of derivative]

$$\Rightarrow \quad \frac{1}{v}\frac{dv}{dx} = \cos x \times \frac{1}{\sin x} \cdot \cos x + \log \sin x(-\sin x)$$

$$\Rightarrow \quad \frac{dv}{dx} = v[\cos x \cdot \cot x - \sin x \log \sin x]$$

$$\Rightarrow \quad \frac{dv}{dx} = (\sin x)^{\cos x}[\cos x \cot x - \sin x \log \sin x] \quad \text{...(iii)}$$

On adding Eqs. (ii) and (iii), we get

$$\frac{du}{dx} + \frac{dv}{dx} = x^{\sin x}\left[\frac{\sin x}{x} + \cos x \log x\right] + (\sin x)^{\cos x}[\cos x \cot x - \sin x \log \sin x]$$

$$\therefore \quad \frac{dy}{dx} = x^{\sin x}\left[\frac{\sin x}{x} + \cos x \log x\right] + (\sin x)^{\cos x}[\cos x \cot x - \sin x \log \sin x]$$

[from Eq. (i)]

26. We have, $e^x + e^y = e^{x+y}$.

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = (\log x)^x\left[\frac{1}{\log x} + \log(\log x)\right] + x^{\log x}\left[\frac{2\log x}{x}\right]$$

$$\frac{dy}{dx} = (\log x)^{x-1}[1 + \log x \log(\log x)] + 2x^{\log x - 1} \cdot \log x$$

25. We have, $y = x^{\sin x} + (\sin x)^{\cos x}$

Let $u = x^{\sin x}$ and $v = (\sin x)^{\cos x}$

Then, $y = u + v$

$\therefore \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ [differentiate both sides]...(i)

$\Rightarrow \quad \frac{dy}{dx} = \frac{-e^x(e^y - 1)}{e^y(e^x - 1)}$

$\Rightarrow \quad \frac{dy}{dx} + \frac{e^x(e^y - 1)}{e^y(e^x - 1)} = 0$

Or $\quad \frac{dy}{dx} = \frac{-e^{x+y} + e^x}{e^{x+y} - e^y} = \frac{-e^x - e^y + e^x}{e^x + e^y - e^y}$ [given]

$\Rightarrow \quad \frac{dy}{dx} = \frac{-e^y}{e^x} = -e^{y-x}$

$\Rightarrow \quad \frac{dy}{dx} + e^{y-x} = 0$ **Hence proved.**

27. Similar as Example 7.

$$\left[\text{Ans. } x^{\cot x}\left(\frac{\cot x}{x} - \text{cosec}^2 x \cdot \log x\right) + \frac{2x^2 + 14x + 3}{(x^2 + x + 2)^2}\right]$$

28. We have,

$$y = \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx}{(x-b)(x-c)} + \frac{c}{(x-c)} + 1$$

$$= \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx}{(x-b)(x-c)} + \frac{c + x - c}{x - c}$$

$$= \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx}{(x-b)(x-c)} + \frac{x}{x-c}$$

$$= \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx + x(x-b)}{(x-b)(x-c)}$$

$$= \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx + x^2 - xb}{(x-b)(x-c)}$$

$$= \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{x^2}{(x-b)(x-c)}$$

$$= \frac{ax^2 + x^2(x-a)}{(x-a)(x-b)(x-c)} = \frac{ax^2 + x^3 - ax^2}{(x-a)(x-b)(x-c)}$$

$$\Rightarrow y = \frac{x^3}{(x-a)(x-b)(x-c)}$$

Taking log on both sides, we get

$$\log y = \log x^3 - [\log(x-a) + \log(x-b) + \log(x-c)]$$

$$\Rightarrow \log y = 3\log x - \log(x-a) - \log(x-b) - \log(x-c)$$

On differentiating both sides w.r.t. x, we get

$$e^x + e^y \frac{dy}{dx} = e^{x+y} \frac{d}{dx}(x+y)$$

[by chain rule of derivative]

$$\Rightarrow \quad e^x + e^y \frac{dy}{dx} = e^{x+y}\left[1 + \frac{dy}{dx}\right]$$

$$\Rightarrow \quad e^x + e^y \frac{dy}{dx} = e^{x+y} + e^{x+y}\frac{dy}{dx}$$

$$\Rightarrow e^y \frac{dy}{dx} - e^{x+y}\frac{dy}{dx} = e^{x+y} - e^x$$

$$\Rightarrow \quad -e^y \frac{dy}{dx}(e^x - 1) = e^x(e^y - 1)$$

$$= \left[\frac{x - a - x}{x(x-a)}\right] + \left[\frac{x - b - x}{x(x-b)}\right] + \left[\frac{x - c - x}{x(x-c)}\right]$$

$$= -\frac{a}{x(x-a)} - \frac{b}{x(x-b)} - \frac{c}{x(x-c)}$$

$$\Rightarrow \quad \frac{y'}{y} = \frac{1}{x}\left[\frac{a}{(a-x)} + \frac{b}{(b-x)} + \frac{c}{c-x}\right]$$

Hence proved.

29. Given implicit function is

$$x^{2x} - 2x^x \cot y - 1 = 0 \quad ...(i)$$

Let $z = x^x$. Then, Eq. (i) reduces to

$$z^2 - 2z\cot y - 1 = 0 \quad ...(ii)$$

Now, consider $z = x^x$

Taking log on both sides, we get

$$\log z = x\log(x)$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{z}\frac{dz}{dx} = x\frac{d}{dx}(\log(x)) + \log(x)\frac{d}{dx}(x)$$

$$\Rightarrow \quad \frac{dz}{dx} = z\left[\frac{x}{x} + \log(x) \times 1\right]$$

$$\Rightarrow \quad \frac{dz}{dx} = x^x[1 + \log(x)] \quad ...(iii)$$

Now, differentiating both sides of Eq. (ii) w.r.t. x, we get

$$2z\frac{dz}{dx} - 2\left[\frac{dz}{dx} \times \cot y + z \times (-\text{cosec}^2 y)\frac{dy}{dx}\right] - 0 = 0$$

$$\Rightarrow \quad 2x^{2x}[1 + \log(x)]$$

$$-2\left[x^x\{1 + \log(x)\}\cot y - x^x(\text{cosec}^2 y)\frac{dy}{dx}\right] = 0 \quad ...(iv)$$

Now, put $x = 1$ in Eq. (i), we get

$$(1)^2 - 2(1)^1 \cot y - 1 = 0$$

$$\Rightarrow \quad -2\cot y = 0$$

$$\Rightarrow \quad \cot y = 0$$

$$\Rightarrow \quad y = \pi/2$$

Now, put $x = 1$ and $y = \frac{\pi}{2}$ in Eq. (iv), we get

$$2(1)^{2(1)}(1 + \log 1) - 2\left[1^1(1 + \log 1)\cot\frac{\pi}{2}\right.$$

$$\frac{1}{y}\cdot\frac{dy}{dx}=3\times\frac{1}{x}-\frac{1}{x-a}-\frac{1}{x-b}-\frac{1}{x-c}$$

$$\Rightarrow \frac{y'}{y}=\frac{3}{x}-\frac{1}{x-a}-\frac{1}{x-b}-\frac{1}{x-c}$$

$$=\frac{1}{x}+\frac{1}{x}+\frac{1}{x}-\frac{1}{x-a}-\frac{1}{x-b}-\frac{1}{x-c}$$

$$=\left[\frac{1}{x}-\frac{1}{x-a}\right]+\left[\frac{1}{x}-\frac{1}{x-b}\right]+\left[\frac{1}{x}-\frac{1}{x-c}\right]$$

$$-1^1\left(\operatorname{cosec}^2\frac{\pi}{2}\right)\frac{dy}{dx}\Bigg]=0$$

$$\Rightarrow 2(1+0)-2\left[(1+0)0-(1)^2\frac{dy}{dx}\right]=0$$

$$\Rightarrow 2-2\left[0-\frac{dy}{dx}\right]=0$$

$$\Rightarrow \left[\frac{dy}{dx}\right]_{x=1}=-1 \text{ or } y'(1)=-1$$

|TOPIC 7|

Differentiation of Parametric Functions

Sometimes, x and y are given as functions of one another variable, say $x=\phi(t)$, $y=\psi(t)$ are two functions and t is a variable. In such a case, x and y are called **parametric functions** or **parametric equations** and t is called the **parameter**.

More precisely, a relation expressed between two variables x and y in the form $x=\phi(t)$, $y=\psi(t)$ is said to be parametric form with t as a parameter. To find the derivatives of parametric functions, we use following steps

I. First, write the given parametric function. Suppose $x=f(t)$ and $y=g(t)$, where t is a parameter.

II. Differentiate both functions separately with respect to parameter t by using suitable formula, i.e. find $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

III. Divide the derivative of one function w.r.t. parameter by the derivative of second function w.r.t. parameter, to get required value, i.e. $\frac{dy}{dx}$.

$$\text{Thus, } \frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{g'(t)}{f'(t)}, \text{ where } f'(t)\neq 0.$$

Note $\frac{dy}{dx}$ is expressed in terms of parameter only without directly involving the main variables x and y.

EXAMPLE |1| Find $\frac{dy}{dx}$, if $x=a\log t$ and $y=b\sin t$.

Sol. We have, $x=a\log t$ and $y=b\sin t$.

On differentiating both sides w.r.t. t, we get

$$\frac{dx}{dt}=\frac{a}{t} \text{ and } \frac{dy}{dt}=b\cos t$$

$$\text{Now,} \quad \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{b\cos t}{a/t}=\frac{bt\cos t}{a}$$

$$\therefore \quad \frac{dy}{dx}=\frac{b}{a}t\cos t$$

Sol. We have, $x=a\left[\cos t+\log\left(\tan\frac{t}{2}\right)\right]$

On differentiating both sides w.r.t. t, we get

$$\frac{dx}{dt}=a\left(-\sin t+\frac{1}{\tan t/2}\cdot\sec^2\frac{t}{2}\cdot\frac{1}{2}\right)\quad\left[\because \frac{d}{dx}\{\log(x)\}=\frac{1}{x}\right]$$

$$=a\left(-\sin t+\frac{1}{2\sin t/2\cos t/2}\right)$$

$$=a\left(-\sin t+\frac{1}{\sin t}\right)\quad\left[\because \sin t=2\sin\frac{t}{2}\cos\frac{t}{2}\right]$$

$$=a\left(\frac{1-\sin^2 t}{\sin t}\right)$$

$$\Rightarrow \frac{dx}{dt}=\frac{a\cos^2 t}{\sin t}\quad[\because 1-\sin^2 t=\cos^2 t]\ \ldots(i)$$

Also, $y=a\sin t$

On differentiating both sides w.r.t. t, we get

$$\frac{dy}{dt}=a\cos t \quad \ldots(ii)$$

$$\text{Now, } \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{a\cos t}{\left(\frac{a\cos^2 t}{\sin t}\right)}\quad\text{[from Eqs. (i) and (ii)]}$$

$$\therefore \quad \frac{dy}{dx}=\frac{\sin t}{\cos t}=\tan t$$

EXAMPLE |3| If $x=\cos\theta-\cos 2\theta$ and $y=\sin\theta-\sin 2\theta$, then find $\frac{dy}{dx}$ at $\theta=\frac{\pi}{3}$. **[Delhi 2020]**

Sol. We have, $x=\cos\theta-\cos 2\theta$

On differentiating both sides w.r.t. θ, we get

$$\frac{dx}{d\theta}=-\sin\theta+\sin 2\theta\cdot 2$$

$$\Rightarrow \frac{dx}{d\theta}=2\sin 2\theta-\sin\theta \text{ and } y=\sin\theta-\sin 2\theta$$

On differentiating both sides w.r.t. θ, we get

$$\frac{dy}{d\theta}=\cos\theta-2\cos 2\theta$$

$$\text{Now, } \frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{\cos\theta-2\cos 2\theta}{2\sin 2\theta-\sin\theta}$$

$$\left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{3}}=\frac{\cos\pi/3-2\cos(2\pi/3)}{-\sin\frac{\pi}{3}+2\sin(2\pi/3)}$$

EXAMPLE |2| If $x = a\left(\cos t + \log \tan \frac{t}{2}\right)$ and $y = a \sin t$, then find $\frac{dy}{dx}$. **[NCERT]**

EXAMPLE |4| If $x = ae^t (\sin t + \cos t)$ and $y = ae^t(\sin t - \cos t)$, then prove that $\frac{dy}{dx} = \frac{x+y}{x-y}$. **[All India 2019]**

Sol. Given, $x = ae^t (\sin t + \cos t)$

On differentiating x w.r.t. t, we get

$$\frac{dx}{dt} = ae^t \frac{d}{dt}(\sin t + \cos t) + \frac{d}{dt}(ae^t)\cdot(\sin t + \cos t)$$

$$\Rightarrow \frac{dx}{dt} = ae^t (\cos t - \sin t) + ae^t(\sin t + \cos t)$$

$$= -ae^t (\sin t - \cos t) + ae^t(\sin t + \cos t)$$

$$\Rightarrow \frac{dx}{dt} = -y + x$$

Also, $y = ae^t(\sin t - \cos t)$

On differentiating y both sides w.r.t. t, we get

$$\frac{dy}{dt} = ae^t \frac{d}{dt}(\sin t - \cos t) + \frac{d}{dt}(ae^t)\cdot(\sin t - \cos t)$$

$$\Rightarrow \frac{dy}{dt} = ae^t(\cos t + \sin t) + ae^t(\sin t - \cos t)$$

$$\Rightarrow \frac{dy}{dt} = ae^t(\sin t + \cos t) + ae^t(\sin t - \cos t) = x + y$$

Now, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$$= \frac{x+y}{-y+x} = \frac{x+y}{x-y}$$ **Hence proved.**

EXAMPLE |5| If $x = a\left(t + \frac{1}{t}\right)$ and $y = a\left(t - \frac{1}{t}\right)$, prove that $\frac{dy}{dx} = \frac{x}{y}$. **[Delhi 2016C]**

Sol. We have, $x = a\left(t + \frac{1}{t}\right)$ ...(i)

and $y = a\left(t - \frac{1}{t}\right)$ ...(ii)

Now, $\frac{dx}{dt} = a\left(1 - \frac{1}{t^2}\right) = a\left(\frac{t^2 - 1}{t^2}\right)$ ...(iii)

and $\frac{dy}{dt} = a\left(1 + \frac{1}{t^2}\right) = a\left(\frac{t^2+1}{t^2}\right)$ ...(iv)

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\left(\frac{t^2+1}{t^2}\right)}{a\left(\frac{t^2-1}{t^2}\right)}$$ [using Eqs. (iii) and (iv)]

$$= \frac{a(t^2+1)}{t^2}\cdot\frac{t^2}{a(t^2-1)} = \frac{t^2+1}{t^2-1}$$ ...(v)

$$= \frac{\frac{1}{2} + 2\left(\frac{1}{2}\right)}{-\frac{\sqrt{3}}{2} + 2\left(\frac{\sqrt{3}}{2}\right)} = \frac{\frac{3}{2}}{\frac{\sqrt{3}}{2}} = \frac{3}{\sqrt{3}} = \sqrt{3}$$

Now consider, $\frac{x}{y} = \frac{a\left(\frac{t^2+1}{t}\right)}{a\left(\frac{t^2-1}{t}\right)} = \frac{t^2+1}{t^2-1}$ ...(vi)

From Eqs. (v) and (vi), we get

$$\frac{dy}{dx} = \frac{x}{y}$$ **Hence proved.**

TOPIC PRACTICE 7

OBJECTIVE TYPE QUESTIONS

1 If $x = a\cos\theta$ and $y = a\sin\theta$, then $\frac{dy}{dx}$ is equal to

(a) $-\sin\theta$ (b) $-\cos\theta$ (c) $-\cot\theta$ (d) $\tan\theta$

2 If $x = at^2$ and $y = 2at$, then $\frac{dy}{dx}$ is equal to

(a) t (b) $\frac{1}{t}$

(c) $\frac{-1}{t^2}$ (d) None of these

3 If $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$, then $\frac{dy}{dx}$ is equal to

(a) $\tan\theta$ (b) $\cot\theta$

(c) $\sin\theta$ (d) $\cos\theta$

4 If $x = \sqrt{a^{\sin^{-1} t}}$ and $y = \sqrt{a^{\cos^{-1} t}}$, then

(a) $x\frac{dy}{dx} + y = 0$ (b) $x\frac{dy}{dx} = y$

(c) $y\frac{dy}{dx} = x$ (d) None of these

5 If $x = \log(1+t^2)$ and $y = t - \tan^{-1} t$, then $\frac{dy}{dx}$ is equal to

(a) $e^x - 1$ (b) $t^2 - 1$

(c) $\frac{\sqrt{e^x - 1}}{2}$ (d) $e^x - y$

SHORT ANSWER Type I Questions

6 If $x = a(2\theta - \sin 2\theta)$ and $y = a(1 - \cos 2\theta)$, find $\frac{dy}{dx}$ when $\theta = \frac{\pi}{3}$. **[CBSE 2018]**

7 Find dy/dx, if $x = a(\theta - \sin\theta)$ and $y = a(1 + \cos\theta)$. **[NCERT]**

8 If $x = a\sec^3\theta$ and $y = a\tan^3\theta$, then find $\frac{dy}{dx}$ at $\theta = \frac{\pi}{3}$. **[NCERT Exemplar]**

9 If $x = t + \frac{1}{t}$ and $y = t - \frac{1}{t}$, then find $\frac{dy}{dx}$.

SHORT ANSWER Type II Questions

10 If $x = 2\cos\theta - \cos 2\theta$ and $y = 2\sin\theta - \sin 2\theta$, then prove that $\frac{dy}{dx} = \tan\left(\frac{3\theta}{2}\right)$.

[All India 2016C, Delhi 2013C]

11 Find the value of $\frac{dy}{dx}$ at $\theta = \frac{\pi}{4}$, if $x = ae^{\theta}(\sin\theta - \cos\theta)$ and $y = ae^{\theta}(\sin\theta + \cos\theta)$. **[All India 2014]**

12 If $x = \cos t\,(3 - 2\cos^2 t)$ and $y = \sin t\,(3 - 2\sin^2 t)$, then find the value of $\frac{dy}{dx}$ at $x = \frac{\pi}{4}$. **[All India 2014]**

13 If $x = e^{\theta}\left(\theta + \frac{1}{\theta}\right)$, $y = e^{-\theta}\left(\theta - \frac{1}{\theta}\right)$, then find $\frac{dy}{dx}$.

[NCERT Exemplar]

14 If $x = \sqrt{a^{\sin^{-1} t}}$, $y = \sqrt{a^{\cos^{-1} t}}$, then show that $\frac{dy}{dx} = -\frac{y}{x}$. **[NCERT]**

15 If $x = a\sin 2t(1 + \cos 2t)$ and $y = b\cos 2t(1 - \cos 2t)$, find the values of $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$ and $t = \frac{\pi}{3}$.

[Delhi 2016]

Or

If $x = a\sin 2t\,(1+\cos 2t)$ and $y = b\cos 2t\,(1 - \cos 2t)$, then show that at $t = \frac{\pi}{4}$, $\frac{dy}{dx} = \frac{b}{a}$. **[All India 2014]**

16 If $x = e^{\cos 2t}$ and $y = e^{\sin 2t}$, then prove that

$$\frac{dy}{dx} = \frac{-y\log x}{x\log y}.$$

17 If $x = \frac{\sin^3 t}{\sqrt{\cos 2t}}$ and $y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$, then find $\frac{dy}{dx}$.

[NCERT]

HINTS & SOLUTIONS

1. (c) Given that $x = a\cos\theta$, $y = a\sin\theta$

Therefore, $\frac{dx}{d\theta} = -a\sin\theta$, $\frac{dy}{d\theta} = a\cos\theta$

Hence, $$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\cos\theta}{-a\sin\theta} = -\cot\theta$$

2. (b) Given that $x = at^2$, $y = 2at$

So, $$\frac{dx}{dt} = 2at$$

and $$\frac{dy}{dt} = 2a$$

Therefore, $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}$$

3. (a) Given, $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$

On differentiating w.r.t. θ, we get

$$\frac{dx}{d\theta} = a\frac{d}{d\theta}(\cos\theta + \theta\sin\theta)$$
$$= a\left\{\frac{d}{d\theta}(\cos\theta) + \frac{d}{d\theta}(\theta\sin\theta)\right\}$$
$$= a\{-\sin\theta + (\theta\cos\theta + \sin\theta\cdot 1)\} = a\theta\cos\theta$$

[using product rule in $\frac{d}{d\theta}(\theta\sin\theta)$]

and $$\frac{dy}{d\theta} = a\frac{d}{d\theta}(\sin\theta - \theta\cos\theta)$$
$$= a\left\{\frac{d}{d\theta}(\sin\theta) - \frac{d}{d\theta}(\theta\cos\theta)\right\}$$
$$= a[\cos\theta - \{\theta(-\sin\theta) + \cos\theta.1\}] = a\theta\sin\theta$$

[using product rule in $\frac{d}{d\theta}(\theta\cos\theta)$]

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$$

4. (a) $x = \sqrt{a^{\sin^{-1} t}}$ ...(i)

and $y = \sqrt{a^{\cos^{-1} t}}$ ...(ii)

On multiplying Eqs. (i) and (ii), we get

$$xy = \sqrt{a^{\sin^{-1} t}} \times \sqrt{a^{\cos^{-1} t}} \Rightarrow xy = \sqrt{a^{\sin^{-1} t}.a^{\cos^{-1} t}}$$

$$\Rightarrow xy = \sqrt{a^{\sin^{-1} t + \cos^{-1} t}} \qquad \left[\because \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}\right]$$

$$\Rightarrow xy = \sqrt{a^{\frac{\pi}{2}}}$$

On differentiating w.r.t. x, we get

$$x\frac{dy}{dx} + y = 0 \Rightarrow x\frac{dy}{dx} = -y \Rightarrow x\frac{dy}{dx} + y = 0$$

5. (c) Given, $x = \log(1 + t^2)$ and $y = t - \tan^{-1} t$

$$\frac{dx}{dt} = \frac{1}{1+t^2}(2t)$$

$$\frac{dy}{dt} = 1 - \frac{1}{1+t^2} = \frac{t^2}{1+t^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{t^2/1+t^2}{2t/1+t^2} = t/2 \quad \text{...(i)}$$

Also, $x = \log(1+t^2) \Rightarrow t^2 = e^x - 1$ ...(ii)

From Eqs. (i) and (ii), we get

$$\frac{dy}{dx} = \frac{\sqrt{e^x - 1}}{2}$$

6. We have, $x = a(2\theta - \sin 2\theta)$ and $y = a(1 - \cos 2\theta)$

Clearly, $\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}$ ...(i)

Here, $\frac{dx}{d\theta} = a(2 - \cos 2\theta \cdot 2) = 2a(1 - \cos 2\theta)$

and $\frac{dy}{d\theta} = a(0 + 2\sin 2\theta) = 2a\sin 2\theta$

From Eq. (i), we get

$$\frac{dy}{dx} = \frac{2a\sin 2\theta}{2a(1-\cos 2\theta)} = \frac{\sin 2\theta}{1-\cos 2\theta} = \frac{2\sin\theta\cos\theta}{2\sin^2\theta}$$

$[\because \cos 2\theta = 1 - 2\sin^2\theta]$

$$= \cot\theta$$

Now, $\left(\frac{dy}{dx}\right)_{\theta=\pi/3} = \cot\frac{\pi}{3} = \frac{1}{\sqrt{3}}$

7. We have, $x = a(\theta - \sin\theta)$ and $y = a(1 + \cos\theta)$

On differentiating x and y w.r.t. θ, we get

$$\frac{dx}{d\theta} = a(1 - \cos\theta)$$

and $\frac{dy}{d\theta} = a(0 - \sin\theta) = -a\sin\theta$

Now, $\frac{dy}{dx} = \left(\frac{dy}{d\theta}\right) \times \left(\frac{d\theta}{dx}\right) = \frac{-a\sin\theta}{a(1-\cos\theta)}$

$$= \frac{\left(-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)}{2\sin^2\frac{\theta}{2}}$$

$\left[\because \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \text{ and } 1 - \cos\theta = 2\sin^2\frac{\theta}{2}\right]$

$$= -\cot\left(\frac{\theta}{2}\right)$$

8. Solve as Question 6. $\left[\text{Ans. } \frac{\sqrt{3}}{2}\right]$

9. **Hint** $\frac{dx}{dt} = 1 - \frac{1}{t^2}$ and $\frac{dy}{dt} = 1 + \frac{1}{t^2}$ $\left[\text{Ans. } \frac{t^2+1}{t^2-1}\right]$

10. Similar as Example 3.

11. We have, $x = ae^\theta(\sin\theta - \cos\theta) = a(e^\theta\sin\theta - e^\theta\cos\theta)$

On differentiating both sides w.r.t. θ, we get

$$\frac{dx}{d\theta} = a\frac{d}{d\theta}[e^\theta\sin\theta - e^\theta\cos\theta]$$

$$= a\left[\frac{d}{d\theta}(e^\theta\sin\theta) - \frac{d}{d\theta}(e^\theta\cos\theta)\right]$$

$$= a\left[e^\theta\frac{d}{d\theta}(\sin\theta) + \sin\theta\frac{d}{d\theta}(e^\theta) - e^\theta\frac{d}{d\theta}(\cos\theta) - \cos\theta\frac{d}{d\theta}(e^\theta)\right]$$

$$= a[e^\theta\cos\theta + e^\theta\sin\theta - e^\theta(-\sin\theta) - e^\theta\cos\theta]$$

$$= a[e^\theta\cos\theta + e^\theta\sin\theta + e^\theta\sin\theta - e^\theta\cos\theta]$$

$$= a[2e^\theta\sin\theta] = 2ae^\theta\sin\theta \quad \text{...(i)}$$

and $y = ae^\theta(\sin\theta + \cos\theta) = a(e^\theta\sin\theta + e^\theta\cos\theta)$

On differentiating both sides w.r.t. θ, we get

$$\frac{dy}{d\theta} = a\left[\frac{d}{d\theta}(e^\theta\sin\theta) + \frac{d}{d\theta}(e^\theta\cos\theta)\right]$$

$$= a\left[e^\theta\frac{d}{d\theta}(\sin\theta) + \sin\theta\frac{d}{d\theta}(e^\theta) + e^\theta\frac{d}{d\theta}(\cos\theta) + \cos\theta\frac{d}{d\theta}(e^\theta)\right]$$

$$= a[e^\theta\cos\theta + e^\theta\sin\theta + e^\theta(-\sin\theta) + e^\theta\cos\theta]$$

$$= a[e^\theta\cos\theta + e^\theta\sin\theta - e^\theta\sin\theta + e^\theta\cos\theta]$$

$$\Rightarrow \frac{dy}{d\theta} = 2ae^\theta\cos\theta \quad \text{...(ii)}$$

Now, $\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx} = \frac{2ae^\theta\cos\theta}{2ae^\theta\sin\theta}$ [from Eqs. (i) and (ii)]

$$= \cot\theta$$

At $\theta = \frac{\pi}{4}$, $\frac{dy}{dx} = \cot\frac{\pi}{4} \Rightarrow \frac{dy}{dx} = 1$ $\left[\because \cot\frac{\pi}{4} = 1\right]$

12. Solve as Question 11. $\left[\text{Ans. } \frac{dy}{dx} = 0\right]$

13. Solve as Question 11. $\left[\text{Ans. } e^{-2\theta}\left(\frac{-\theta^3+\theta^2+\theta+1}{\theta^3+\theta^2+\theta-1}\right)\right]$

14. Given, $x = \sqrt{a^{\sin^{-1}t}}$

Taking log on both sides, we get

$$\log x = \log(a^{\sin^{-1}t})^{1/2} \Rightarrow \log x = \frac{1}{2}\log a^{\sin^{-1}t}$$

$$\Rightarrow \log x = \frac{1}{2}\sin^{-1}t\log a \Rightarrow \log x = \frac{1}{2}\log a\sin^{-1}t$$

On differentiating both sides w.r.t. t, we get

$$\frac{1}{x}\cdot\frac{dx}{dt} = \frac{1}{2}\log a\frac{d}{dt}(\sin^{-1}t) = \frac{1}{2}\log a\frac{1}{\sqrt{1-t^2}}$$

$$\Rightarrow \frac{dx}{dt} = \frac{x\log a}{2}\cdot\frac{1}{\sqrt{1-t^2}}$$

Again, $y = \sqrt{a^{\cos^{-1}t}}$

Taking log on both sides, we get

$$\log y = \log(a^{\cos^{-1}t})^{1/2} \Rightarrow \log y = \frac{1}{2}\log a^{\cos^{-1}t}$$

$$\Rightarrow \log y = \frac{1}{2}\cos^{-1}t\log a$$

On differentiating both sides w.r.t. t, we get

$$\frac{1}{y}\cdot\frac{dy}{dt}=\frac{1}{2}\log a\cdot\left[\frac{-1}{\sqrt{1-t^2}}\right]\Rightarrow\frac{dy}{dt}=\frac{y\log a}{2}\cdot\frac{-1}{\sqrt{1-t^2}}$$

Now, $\frac{dy}{dx}=\left(\frac{dy}{dt}\right)\cdot\left(\frac{dt}{dx}\right)$

$$\Rightarrow\quad\frac{dy}{dx}=\frac{-y\log a}{2}\cdot\frac{1}{\sqrt{1-t^2}}\cdot\frac{2\cdot\sqrt{1-t^2}}{x\log a}\Rightarrow\frac{dy}{dx}=\frac{-y}{x}$$

Hence proved.

15. Given, $x=a\sin 2t\,(1+\cos 2t)$
and $y=b\cos 2t\,(1-\cos 2t)$

On differentiating both sides w.r.t. t, we get

$$\frac{dx}{dt}=a\left[\sin 2t\frac{d}{dt}(1+\cos 2t)+(1+\cos 2t)\frac{d}{dt}(\sin 2t)\right]$$

[by using product rule of derivative]

$$=a[\sin 2t\cdot(0-2\sin 2t)+(1+\cos 2t)(2\cos 2t)]$$
$$=a[-2\sin^2 2t+2\cos 2t+2\cos^2 2t]$$
$$=a[2(\cos^2 2t-\sin^2 2t)+2\cos 2t]$$
$$=a(2\cos 4t+2\cos 2t)$$
$$=2a(\cos 4t+\cos 2t)\quad[\because \cos^2 2\theta-\sin^2 2\theta=\cos 4\theta]$$

and $\frac{dy}{dt}=b\left[\cos 2t\frac{d}{dt}(1-\cos 2t)+(1-\cos 2t)\frac{d}{dt}(\cos 2t)\right]$

[by using product rule of derivative]

$$=b[\cos 2t\times(0+2\sin 2t)+(1-\cos 2t)(-2\sin 2t)]$$
$$=b(2\sin 2t\cos 2t-2\sin 2t+2\sin 2t\cos 2t)$$
$$=b(4\sin 2t\cos 2t-2\sin 2t)$$

$[\because 2\sin 2\theta\cos 2\theta=\sin 4\theta]$

$$=b(2\sin 4t-2\sin 2t)$$
$$=2b[\sin 4t-\sin 2t]$$

Now, $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{2b[\sin 4t-\sin 2t]}{2a[\cos 4t+\cos 2t]}$

$$=\frac{b(\sin 4t-\sin 2t)}{a(\cos 4t+\cos 2t)}$$

At $t=\frac{\pi}{4}$, $\frac{dy}{dx}=\frac{b\left(\sin 4\times\frac{\pi}{4}-\sin 2\times\frac{\pi}{4}\right)}{a\left(\cos 4\times\frac{\pi}{4}+\cos 2\times\frac{\pi}{4}\right)}$

$$=\frac{b\left(\sin\pi-\sin\frac{\pi}{2}\right)}{a\left(\cos\pi+\cos\frac{\pi}{2}\right)}=\frac{b(0-1)}{a(-1+0)}=\frac{b}{a}$$

At $t=\frac{\pi}{3}$, $\frac{dy}{dx}=\frac{b\left(\sin 4\times\frac{\pi}{3}-\sin 2\times\frac{\pi}{3}\right)}{a\left(\cos 4\times\frac{\pi}{3}+\cos 2\times\frac{\pi}{3}\right)}$

$$=\frac{b\left[\sin\left(\pi+\frac{\pi}{3}\right)-\sin\left(\pi-\frac{\pi}{3}\right)\right]}{a\left[\cos\left(\pi+\frac{\pi}{3}\right)+\cos\left(\pi-\frac{\pi}{3}\right)\right]}$$

$$=\frac{b\left(-\sin\frac{\pi}{3}-\sin\frac{\pi}{3}\right)}{a\left(-\cos\frac{\pi}{3}-\cos\frac{\pi}{3}\right)}$$

$$=\frac{b\left(-2\sin\frac{\pi}{3}\right)}{-a\times 2\cos\frac{\pi}{3}}=\frac{b}{a}\frac{\left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{1}{2}\right)}=\frac{b\sqrt{3}}{a}$$

16. Hint $\frac{dx}{dt}=e^{\cos 2t}(-\sin 2t)\cdot 2$ and $\frac{dy}{dt}=e^{\sin 2t}(\cos 2t)\cdot 2$

17. Given, $x=\frac{\sin^3 t}{\sqrt{\cos 2t}}$, $y=\frac{\cos^3 t}{\sqrt{\cos 2t}}$.

On differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt}=\frac{d}{dt}\left(\frac{\sin^3 t}{\sqrt{\cos 2t}}\right)$$

$$=\frac{\sqrt{\cos 2t}(3\sin^2 t\cos t)-\sin^3 t\left(\frac{-2\sin 2t}{2\sqrt{\cos 2t}}\right)}{(\sqrt{\cos 2t})^2}$$

[using quotient rule of derivative]

$$=\frac{3(\cos 2t)\sin^2 t\cos t+\sin 2t\sin^3 t}{\cos 2t\sqrt{\cos 2t}}$$

$$=\frac{3(1-2\sin^2 t)\sin^2 t\cos t+(2\sin t\cos t)\sin^3 t}{\cos 2t\sqrt{\cos 2t}}$$

$[\because \cos 2t=1-2\sin^2 t$ and $\sin 2t=2\sin t\cos t]$

$$=\frac{3\sin^2 t\cos t-4\sin^4 t\cos t}{\cos 2t\sqrt{\cos 2t}}$$

and $\frac{dy}{dt}=\frac{d}{dt}\left(\frac{\cos^3 t}{\sqrt{\cos 2t}}\right)$

$$=\frac{\sqrt{\cos 2t}(-3\cos^2 t\sin t)-\cos^3 t\left(\frac{-2\sin 2t}{2\sqrt{\cos 2t}}\right)}{(\sqrt{\cos 2t})^2}$$

[using quotient rule of derivative]

$$=\frac{-3(\cos 2t)\cos^2 t\sin t+\sin 2t\cos^3 t}{\cos 2t\sqrt{\cos 2t}}$$

$$=\frac{-3(2\cos^2 t-1)\cos^2 t\sin t+\cos^3 t(2\sin t\cos t)}{\cos 2t\sqrt{\cos 2t}}$$

$$=\frac{3\cos^2 t\sin t-4\cos^4 t\sin t}{\cos 2t\sqrt{\cos 2t}}$$

Now, $\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=\frac{3\cos^2 t\sin t-4\cos^4 t\sin t}{3\sin^2 t\cos t-4\sin^4 t\cos t}$

$$=\frac{\cos^2 t\sin t(3-4\cos^2 t)}{\sin^2 t\cos t(3-4\sin^2 t)}=\frac{\cos t(3-4\cos^2 t)}{\sin t(3-4\sin^2 t)}$$

$$=\frac{3\cos t-4\cos^3 t}{3\sin t-4\sin^3 t}=\frac{-\cos 3t}{\sin 3t}=-\cot 3t$$

$[\because \cos 3t=4\cos^3 t-3\cos t$ and $\sin 3t=\sin t-4\sin^3 t]$

|TOPIC 8|

Differentiation of a Function w.r.t. Another Function

Suppose it is required to differentiate a function $f(x)$ with respect to another function $g(x)$.
Let $y = f(x)$ and $z = g(x)$.
Here, both functions are different but both are in same variable x. Then, to find the derivative of $f(x)$ with respect to $g(x)$ or derivative of $g(x)$ with respect to $f(x)$, i.e. to find $\frac{dy}{dz}$ or $\frac{dz}{dy}$. We first differentiate both functions $f(x)$ and $g(x)$ with respect to x separately and then put these values in the following formulae

$$\frac{dy}{dz} = \frac{dy/dx}{dz/dx} \text{ or } \frac{dz}{dy} = \frac{dz/dx}{dy/dx}$$

EXAMPLE |1| Write the derivative of $\sin x$ w.r.t. $\cos x$. **[Delhi 2014C]**

Sol. Let $y = \sin x$ and $z = \cos x$

On differentiating y and z w.r.t. x, we get

$$\frac{dy}{dx} = \cos x \text{ and } \frac{dz}{dx} = -\sin x$$

Now, $$\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{\cos x}{-\sin x} = -\tan x$$

EXAMPLE |2| Differentiate x^6 w.r.t. $\frac{1}{\sqrt{x}}$.

Sol. Let $f = x^6$ and $g = \frac{1}{\sqrt{x}} = x^{-1/2}$

$$\Rightarrow \frac{df}{dx} = 6x^5 \text{ and } \frac{dg}{dx} = -\frac{1}{2}x^{-\frac{1}{2}-1} = -\frac{1}{2}x^{-\frac{3}{2}}$$

$$\because \frac{df}{dg} = \frac{6x^5}{-\frac{1}{2}x^{-\frac{3}{2}}} = -12x^{5+\frac{3}{2}} = -12x^{\frac{13}{2}}$$

EXAMPLE |3| Differentiate $\sin^2 x$ w.r.t. $e^{\cos x}$. **[NCERT]**

Sol. Let $y = \sin^2 x$ and $z = e^{\cos x}$.

On differentiating y and z w.r.t. x, we get

$$\frac{dy}{dx} = 2\sin x \cos x$$

and $$\frac{dz}{dx} = e^{\cos x}(-\sin x) = -(\sin x)e^{\cos x}$$

Now, $$\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{2\sin x\cos x}{-(\sin x)e^{\cos x}} = \frac{-2\cos x}{e^{\cos x}}$$

EXAMPLE |4| Differentiate $\log(1+x^2)$ w.r.t. $\tan^{-1} x$.

Sol. Let $y = \log(1+x^2)$.

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{1+x^2}\frac{d}{dx}(1+x^2)$$

$$\left[\text{by chain rule of derivative and } \frac{d}{dx}(\log x) = \frac{1}{x}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x}{(1+x^2)}$$

Again, let $z = \tan^{-1} x$

On differentiating both sides w.r.t. x, we get

$$\frac{dz}{dx} = \frac{1}{1+x^2} \quad \left[\because \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}\right]$$

Now, $$\frac{dy}{dz} = \frac{(dy/dx)}{(dz/dx)} = \frac{\left(\frac{2x}{1+x^2}\right)}{\left(\frac{1}{1+x^2}\right)} = 2x$$

EXAMPLE |5| Find derivative of $\sin^{-1} x$ w.r.t. $\tan^{-1} x$.

Sol. Let $f(x) = \sin^{-1} x$ and $g(x) = \tan^{-1} x$

On differentiating $f(x)$ and $g(x)$ w.r.t. x, we get

$$\frac{df}{dx} = \frac{1}{\sqrt{1-x^2}} \text{ and } \frac{dg}{dx} = \frac{1}{1+x^2}$$

Now, $$\frac{df}{dg} = \frac{df}{dx} \times \frac{dx}{dg} = \frac{1}{\sqrt{1-x^2}} \times (1+x^2) = \frac{1+x^2}{\sqrt{1-x^2}}$$

|TOPIC PRACTICE 8|

OBJECTIVE TYPE QUESTIONS

1 The derivative of $\cos^{-1}(2x^2-1)$ w.r.t. $\cos^{-1} x$ is

(a) 2 (b) $\frac{-1}{2\sqrt{1-x^2}}$ (b) $\frac{2}{x}$ (d) $1-x^2$

2 The derivative of $\sin^2 x$ with respect to $e^{\cos x}$ is

(a) $\frac{2\cos x}{e^{\cos x}}$ (b) $-\frac{2\cos x}{e^{\cos x}}$

(c) $\frac{2}{e^{\cos x}}$ (d) None of these

3 The derivative of $b\tan\theta$ with respect to $a\sec\theta$ is

(a) $\frac{b}{a}\operatorname{cosec}\theta$ (b) $\frac{a}{b}\operatorname{cosec}\theta$

(c) $\frac{b}{a}\cot\theta$ (d) $\frac{a}{b}\cot\theta$

SHORT ANSWER Type I Questions

4 Differentiate $\sin x^2$ w.r.t. x^2.

5 Differentiate $\tan e^x$ w.r.t. e^x.

6 Differentiate $(\sin x)^{\log x}$ w.r.t. $\sin x$.

SHORT ANSWER Type II Questions

7 Differentiate $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ w.r.t. $\cos^{-1}(2x\sqrt{1-x^2})$, when $x \neq 0$. **[Delhi 2014]**

8 Find the derivative of $\tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$ w.r.t. $\tan^{-1}\frac{2x}{1-x^2}$.

9 Find the derivative of $\sec^{-1}\left(\frac{1}{2x^2-1}\right)$ w.r.t. $\sqrt{1-x^2}$ at $x=\frac{1}{2}$.

10 Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ w.r.t. $\tan^{-1}x$, when $x \neq 0$. **[NCERT Exemplar]**

11 Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ w.r.t. $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$, when $x \neq 0$. **[Delhi 2016, 2014]**

12 Differentiate $\cos^{-1}(4x^3-3x)$ w.r.t. $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$, if $\frac{1}{2}<x<1$.

13 Differentiate $\sin^{-1}(2ax\sqrt{1-a^2x^2})$ w.r.t. $\sqrt{1-a^2x^2}$, if $-\frac{1}{\sqrt{2}}<ax<\frac{1}{\sqrt{2}}$.

HINTS & SOLUTIONS

1. (a) Let $u=\cos^{-1}(2x^2-1)$ and $v=\cos^{-1}x$

$$\therefore \quad \frac{du}{dx}=-\frac{1}{\sqrt{1-(2x^2-1)^2}}\cdot 4x$$

$$=\frac{-4x}{\sqrt{1-(4x^4+1-4x^2)}}$$

$$=\frac{-4x}{\sqrt{-4x^4+4x^2}}=\frac{-4x}{\sqrt{4x^2(1-x^2)}}=\frac{-2}{\sqrt{1-x^2}}$$

and $\quad \frac{dv}{dx}=\frac{-1}{\sqrt{1-x^2}}$

$$\therefore \quad \frac{du}{dv}=\frac{du/dx}{dv/dx}=\frac{-2/\sqrt{1-x^2}}{-1/\sqrt{1-x^2}}=2$$

2. (b) Let $u(x)=\sin^2 x$ and $v(x)=e^{\cos x}$.

We want to find $\frac{du}{dv}=\frac{du/dx}{dv/dx}$.

Clearly, $\frac{du}{dx}=2\sin x\cos x$

and $\quad \frac{dv}{dx}=e^{\cos x}(-\sin x)=-(\sin x)\,e^{\cos x}$

$$\frac{du}{dv}=\frac{2\sin x\cos x}{-\sin x e^{\cos x}}=-\frac{2\cos x}{e^{\cos x}}$$

3. (a) Let $x=b\tan\theta$ and $y=a\sec\theta$

On differentiating w.r.t. θ, we get

$$\frac{dx}{d\theta}=b\sec^2\theta$$

and $\quad \frac{dy}{d\theta}=a\sec\theta\tan\theta$

$$\therefore \quad \frac{dx}{dy}=\frac{dx/d\theta}{dy/d\theta}=\frac{b\sec^2\theta}{a\sec\theta\tan\theta}$$

$$=\frac{b}{a}\left(\frac{\sec\theta}{\tan\theta}\right)=\frac{b}{a}\operatorname{cosec}\theta$$

4. Let $u=\sin x^2$ and $v=x^2$

On differentiating u and v w.r.t. x, we get

$$\frac{du}{dx}=\cos x^2\cdot\frac{d}{dx}(x^2)=2x\cdot\cos x^2$$

and $\quad \frac{dv}{dx}=2x$

$$\text{Now, } \frac{du}{dv}=\frac{\left(\frac{du}{dx}\right)}{\left(\frac{dv}{dx}\right)}=\left(\frac{du}{dx}\right)\times\left(\frac{dx}{dv}\right)=\frac{\cos x^2\times 2x}{2x}$$

$$\therefore \quad \frac{du}{dv}=\cos x^2$$

5. Solve as Question 4. **[Ans.** $\sec^2 e^x$**]**

6. **Hint** $u=(\sin x)^{\log x}$

$\Rightarrow \log u=\log x\log\sin x$

$$\Rightarrow \frac{1}{u}\frac{du}{dx}=\frac{\log x}{\sin x}\times\cos x+\frac{\log\sin x}{x}$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^{\log x}\left[\cot x \log x + \frac{\log \sin x}{x}\right]$$

$$\left[\textbf{Ans. } \frac{(\sin x)^{\log x}\left[\cot x \log x + \dfrac{\log \sin x}{x}\right]}{\cos x}\right]$$

7. Let $u = \tan^{-1}\left[\dfrac{\sqrt{1-x^2}}{x}\right]$.

Put $x = \cos\theta \Rightarrow \theta = \cos^{-1} x$

$$\therefore \quad u = \tan^{-1}\left[\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}\right] = \tan^{-1}\left[\frac{\sqrt{\sin^2\theta}}{\cos\theta}\right]$$

$$= \tan^{-1}\left[\frac{\sin\theta}{\cos\theta}\right] = \tan^{-1}[\tan\theta] = \theta$$

$\Rightarrow u = \cos^{-1} x$

On differentiating both sides w.r.t. x, we get

$$\frac{du}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

Again, let $v = \cos^{-1}(2x\sqrt{1-x^2})$

Put $x = \cos\theta \Rightarrow \theta = \cos^{-1} x$

$$\therefore \quad v = \cos^{-1}[2\cos\theta\sqrt{1-\cos^2\theta}]$$

$$= \cos^{-1}[2\cos\theta\sin\theta]$$

$$[\because \cos^2\theta + \sin^2\theta = 1 \Rightarrow \sin\theta = \sqrt{1-\cos^2\theta}]$$

$$= \cos^{-1}[\sin 2\theta]$$

$$= \cos^{-1}\left[\cos\left(\frac{\pi}{2} - 2\theta\right)\right] \qquad \left[\because \sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)\right]$$

$$= \frac{\pi}{2} - 2\theta = \frac{\pi}{2} - 2\cos^{-1} x$$

On differentiating both sides, w.r.t. x, we get

$$\frac{dv}{dx} = \frac{2}{\sqrt{1-x^2}}$$

Now, $\dfrac{du}{dv} = \left(\dfrac{du}{dx}\right) \times \left(\dfrac{dx}{dv}\right) = -\dfrac{1}{\sqrt{1-x^2}} \times \dfrac{\sqrt{1-x^2}}{2} = -\dfrac{1}{2}$

8. Let $u = \tan^{-1}\left[\dfrac{\sqrt{1+x^2}-1}{x}\right]$.

Put $x = \tan\theta \Rightarrow \theta = \tan^{-1} x$

$$\therefore \quad u = \tan^{-1}\left[\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right]$$

$$= \tan^{-1}\left[\frac{\sec\theta - 1}{\tan\theta}\right] \qquad [\because \sec^2\theta = 1 + \tan^2\theta]$$

$$= \tan^{-1}\left[\frac{\dfrac{1}{\cos\theta} - 1}{\dfrac{\sin\theta}{\cos\theta}}\right] = \tan^{-1}\left[\frac{1-\cos\theta}{\cos\theta} \times \frac{\cos\theta}{\sin\theta}\right]$$

$$= \tan^{-1}\left[\frac{1-\cos\theta}{\sin\theta}\right]$$

$$= \tan^{-1}\left[\frac{2\sin^2\dfrac{\theta}{2}}{2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}}\right]$$

$$\left[\because \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \text{ and } 1 - \cos\theta = 2\sin^2\frac{\theta}{2}\right]$$

$$= \tan^{-1}\left[\tan\frac{\theta}{2}\right] = \frac{\theta}{2} \qquad \left[\because \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2}\right]$$

$$\Rightarrow u = \frac{1}{2}\tan^{-1} x \qquad [\because \theta = \tan^{-1} x]$$

On differentiating both sides w.r.t. x, we get

$$\frac{du}{dx} = \frac{1}{2} \times \frac{1}{1+x^2} = \frac{1}{2(1+x^2)}$$

Again, let $v = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right) = 2\tan^{-1} x$

On differentiating both sides w.r.t. x, we get

$$\frac{dv}{dx} = 2 \times \frac{1}{1+x^2}$$

Now, $\dfrac{du}{dv} = \dfrac{du}{dx} \times \dfrac{dx}{dv} = \dfrac{1}{2(1+x^2)} \times \dfrac{(1+x^2)}{2} = \dfrac{1}{4}$

9. **Hint** Put $x = \cos\theta$ in first function and then solve as Question 8. [**Ans.** 4]

10. Solve as Question 8. $\left[\textbf{Ans. } \dfrac{1}{2}\right]$

11. Solve as Question 8. $\left[\textbf{Ans. } \dfrac{1}{4}\right]$

12. **Hint** Put $x = \cos\theta$ in both functions and then solve as Question 8. [**Ans.** 3]

13. **Hint** Put $ax = \sin\theta$ in first function and then solve as Question 8. $\left[\textbf{Ans. } -\dfrac{2}{ax}\right]$

|TOPIC 9|

Differentiation of Infinite Series

When the value of y is given as an infinite series, then the process to find the derivative of such infinite series is called differentiation of infinite series. In such cases, we use the fact that, if one term is deleted from an infinite series, it remains unaffected, so we can replace all terms except first term by y. Thus, we convert it into a finite series or function. Then, we differentiate it to find the required value.

EXAMPLE |1| If $y=\sqrt{\log x+\sqrt{\log x+\sqrt{\log x+\ldots+\infty}}}$, then find $\frac{dy}{dx}$.

Sol. We have, $y=\sqrt{\log x+\sqrt{\log x+\sqrt{\log x+\ldots+\infty}}}$

Then, $y=\sqrt{\log x+y}$

$\Rightarrow \quad y^2=\log x+y$ [squaring on both sides]

On differentiating both sides w.r.t. x, we get

$$2y\cdot\frac{dy}{dx}=\frac{1}{x}+\frac{dy}{dx}$$

$$\Rightarrow 2y\frac{dy}{dx}-\frac{dy}{dx}=\frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx}(2y-1)=\frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{x(2y-1)}$$

EXAMPLE |2| If $y=(\sin x)^{(\sin x)^{(\sin x)\ldots\infty}}$, then prove that $\frac{dy}{dx}=\frac{y^2\cos x}{1-y\log\sin x}$.

Sol. We have, $y=(\sin x)^{(\sin x)^{(\sin x)\ldots\infty}}$. Then, $y=(\sin x)^y$

Taking log on both sides, we get

$$\log y=y\log\sin x$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx}=y\frac{d}{dx}(\log\sin x)+\log\sin x\frac{d}{dx}(y)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx}=y\cdot\frac{1}{\sin x}\cdot\cos x+\log\sin x\cdot\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{1}{y}-\log\sin x\right)=y\cot x$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{1-y\log\sin x}{y}\right)=y\cot x$$

$$\therefore \frac{dy}{dx}=\frac{y^2\cot x}{1-y\log\sin x}$$

Hence proved.

|TOPIC PRACTICE 9|

OBJECTIVE TYPE QUESTIONS

1 If $y=x^{x^{x^{x\ldots}}}$, then $\frac{dy}{dx}$ is equal to

(a) yx^{y-1} (b) $\frac{y^2}{x(1-y\log x)}$

(c) $\frac{y}{x(1+y\log x)}$ (d) None of these

2 If $y=(\cos x)^{(\cos x)^{(\cos x)\ldots\infty}}$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{y\tan x}{y\log\cos x-1}$

(b) $\frac{y^2\tan x}{y\log\cos x-1}$

(c) $\frac{y\tan x}{1+y\log\cos x}$

(d) None of the above

SHORT ANSWER Type I Questions

3 If $y=\sqrt{\sin x+\sqrt{\sin x+\sqrt{\sin x+\ldots+\infty}}}$, then find $\frac{dy}{dx}$.

4 If $y=ax-\frac{(ax)^2}{2}+\frac{(ax)^3}{3}-\ldots$, then find $\frac{dy}{dx}$.

SHORT ANSWER Type II Questions

5 If $y=e^{x+e^{x+e^{x+e^{x+\ldots+\infty}}}}$, then find $\frac{dy}{dx}$.

6 If $y=x+\cfrac{1}{x+\cfrac{1}{x+\cfrac{1}{x+\ldots+\infty}}}$, then find $\frac{dy}{dx}$.

7 If $y=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}$, then show that $\frac{dy}{dx}-y+\frac{x^n}{n!}=0$.

8 If $y=a^{x^{a^{x\ldots\infty}}}$, then prove that

$$\frac{dy}{dx}=\frac{y^2(\log y)}{x[1-y(\log x)(\log y)]}.$$

HINTS & SOLUTIONS

1. (b) We have, $y = x^{x^{x^{x^{x\ldots}}}}$

Then, $y = x^y \Rightarrow y = e^{y \log x}$

$$\Rightarrow \frac{dy}{dx} = e^{y \log x} \frac{d}{dx}(y \log x)$$

$$\Rightarrow \frac{dy}{dx} = x^y \left(\frac{dy}{dx} \log x + \frac{y}{x}\right)$$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{dy}{dx} \log x + \frac{y}{x}\right) \quad [\because y = x^y]$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2}{x(1 - y \log x)}$$

2. (b) Given that $y = (\cos x)^{(\cos x)^{(\cos x)^{\ldots\infty}}}$

$$\Rightarrow y = (\cos x)^y$$

Taking log on both sides, we get

$$\log y = y \log(\cos x)$$

Now, differentiating w.r.t. x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = y \cdot \frac{1}{\cos x}(-\sin x) + \log(\cos x) \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = y\left\{-y \tan x + \log \cos x \cdot \frac{dy}{dx}\right\}$$

$$\Rightarrow (1 - y \log \cos x)\frac{dy}{dx} = -y^2 \tan x$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2 \tan x}{(y \log \cos x - 1)}$$

3. We have, $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \ldots + \infty}}}$

$$\therefore \quad y = \sqrt{\sin x + y}$$

$$\Rightarrow y^2 = \sin x + y \quad \text{[squaring both sides]}$$

On differentiating both sides w.r.t. x, we get

$$2y \frac{dy}{dx} = \cos x + \frac{dy}{dx} \Rightarrow 2y \frac{dy}{dx} - \frac{dy}{dx} = \cos x$$

$$\Rightarrow \frac{dy}{dx}(2y - 1) = \cos x$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{2y - 1}$$

4. Given, $y = ax - \frac{(ax)^2}{2} + \frac{(ax)^3}{3} - \ldots$

$$\Rightarrow y = \log(1 + ax)$$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{(1 + ax)} \frac{d}{dx}(1 + ax) = \frac{a}{1 + ax}$$

Note Series expansion of $\log(1 + x)$ is $x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$

5. We have, $y = e^{x + e^{x + e^{x + \ldots\infty}}}$.

Then, $y = e^{x + y}$

Taking log on both sides, we get

$$\log y = \log e^{(x + y)}$$

$$\Rightarrow \log y = (x + y) \log e$$

$$\Rightarrow \log y = (x + y) \quad [\because \log e = 1]$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{y} \frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} - \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{1}{y} - 1\right) = 1$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{1 - y}{y}\right) = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{1 - y}$$

6. We have, $y = x + \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \ldots + \infty}}}$

$$\therefore \quad y = x + \frac{1}{y} \Rightarrow y = \frac{xy + 1}{y}$$

$$\Rightarrow y^2 = xy + 1$$

On differentiating both sides w.r.t. x, we get

$$2y \frac{dy}{dx} = x \frac{dy}{dx} + y \cdot 1 + 0$$

[by product rule of derivative]

$$\Rightarrow 2y \frac{dy}{dx} - x \frac{dy}{dx} = y$$

$$\Rightarrow (2y - x)\frac{dy}{dx} = y$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{2y - x}$$

7. **Hint** Find $\frac{dy}{dx}$, then substitute this value in the LHS of shown result.

8. **Hint** $y = a^{x^y} \Rightarrow \log y = x^y \log a$

$$\Rightarrow \log \log y = \log x^y + \log \log a$$

Now, find $\frac{dy}{dx}$ by differentiating the above expression.

|TOPIC 10|
Second Order Derivative

Second order derivative of a function is the derivative of the first order derivative of the function.

Let $\quad y = f(x) \quad \ldots(i)$

Then, $\dfrac{dy}{dx} = f'(x)$ is called the first derivative of y or $f(x)$.

It can also be written as y' or y_1.

If $f'(x)$ is differentiable, then derivative of dy/dx exists.

Again, differentiating Eq. (i) both sides w.r.t. x, we get

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}[f'(x)] \Rightarrow \frac{d^2y}{dx^2} = f''(x)$$

where, $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$ is called the **second order derivative** of y w.r.t. x and is denoted by $\dfrac{d^2y}{dx^2}$ or $f''(x)$ or y'' or D^2y or y_2.

Similarly, $\dfrac{d}{dx}\left(\dfrac{d^2y}{dx^2}\right) = \dfrac{d}{dx}\{f''(x)\}$

$\Rightarrow \dfrac{d^3y}{dx^3} = f'''(x)$ is called the **third order derivative** of y w.r.t. x and is denoted by $\dfrac{d^3y}{dx^3}$ or $f'''(x)$ or y''' or D^3y or y_3.

In general, $\dfrac{d^ny}{dx^n} = f^n(x)$ is called the nth order **derivative** of y w.r.t. x and is denoted by $\dfrac{d^ny}{dx^n}$ or $f^n(x)$ or $y'''^{\ldots n}$ or D^ny or y_n.

Problems Based on Second Order Derivative

There are many problems in which we have to find second order derivative. Some of them are given below

|TYPE I|

PROBLEMS BASED ON FINDING SECOND ORDER DERIVATIVE

In this type of problems, a function say $y = f(x)$ is given to us and we have to find its second order derivative. For this, we differentiate $y = f(x)$ two times w.r.t. x by using suitable formula.

EXAMPLE |1| Find the second order derivative of

(i) $x^4 + \cot x$ (ii) $\sin(\log x)$

Sol. (i) Let $y = x^4 + \cot x$.

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 4x^3 - \operatorname{cosec}^2 x$$

Again, differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = 12x^2 - 2\operatorname{cosec} x\,(-\operatorname{cosec} x \cdot \cot x)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 12x^2 + 2\operatorname{cosec}^2 x \cdot \cot x$$

(ii) Let $y = \sin(\log x)$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \cos(\log x)\frac{d}{dx}(\log x)$$

$$\Rightarrow \quad \frac{dy}{dx} = \cos(\log x) \cdot \frac{1}{x}$$

$$\Rightarrow \quad x.\frac{dy}{dx} = \cos(\log x) \quad \ldots(i)$$

Again, differentiating both sides w.r.t. x, we get

$$x \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx} = -\frac{\sin(\log x)}{x}$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -\sin(\log x)$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} + \cos(\log x) = -\sin(\log x) \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2} = -\sin(\log x) - \cos(\log x)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -\frac{1}{x^2}[\sin(\log x) + \cos(\log x)]$$

EXAMPLE |2| If $y = e^x \sin 5x$, then find $\dfrac{d^2y}{dx^2}$. **[NCERT]**

Sol. We have, $y = e^x \sin 5x$.

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = e^x \cdot \frac{d}{dx}\sin 5x + \sin 5x \cdot \frac{d}{dx}e^x$$

[by product rule of derivative]

$$= e^x \cdot \cos 5x \cdot \frac{d}{dx}(5x) + \sin 5x \cdot e^x$$

[by chain rule of derivative]

$$= e^x \cdot \cos 5x \cdot 5 + e^x \sin 5x$$

$$\Rightarrow \quad \frac{dy}{dx} = e^x(5\cos 5x + \sin 5x)$$

Again, differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = e^x \cdot \frac{d}{dx}[5\cos 5x + \sin 5x] + [5\cos 5x + \sin 5x]\frac{d}{dx}\cdot(e^x)$$

[by product rule of derivative]

$$= e^x\left[5(-\sin 5x)\cdot\frac{d}{dx}(5x) + \cos 5x \cdot \frac{d}{dx}(5x)\right] + [5\cos 5x + \sin 5x]\cdot e^x$$ [by chain rule of derivative]

$$= e^x[5(-\sin 5x)5 + \cos 5x(5)] + [5\cos 5x + \sin 5x]e^x$$

$$= e^x[-25\sin 5x + 5\cos 5x + 5\cos 5x + \sin 5x]$$

$$= e^x[10\cos 5x - 24\sin 5x]$$

$$\therefore \frac{d^2y}{dx^2} = 2e^x[5\cos 5x - 12\sin 5x]$$

EXAMPLE |3| A curve has the equation $y = \dfrac{x+6}{2x-3}$, when $x \neq \dfrac{3}{2}$.

(i) Show that the derivative of the curve can be expressed as $\dfrac{k}{(2x-3)^2}$, where k is a constant.

(ii) Find the value of x for which $\dfrac{d^2y}{dx^2} = \dfrac{12}{25}$.

Sol. (i) Given equation of curve is $y = \dfrac{x+6}{2x-3}$.

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(2x-3)\frac{d}{dx}(x+6) - (x+6)\frac{d}{dx}(2x-3)}{(2x-3)^2}$$

$$= \frac{(2x-3)(1) - (x+6)(2)}{(2x-3)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-15}{(2x-3)^2} \text{ is of the form of } \frac{k}{(2x-3)^2} \quad \text{...(i)}$$

(ii) Again, differentiating both sides of Eq. (i) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -15\frac{d}{dx}(2x-3)^{-2}$$

$$= -15 \times (-2)(2x-3)^{-2-1} \times \frac{d}{dx}(2x-3)$$

$$= 30(2x-3)^{-3} \times 2 = \frac{60}{(2x-3)^3}$$

Also, given $\dfrac{d^2y}{dx^2} = \dfrac{12}{25}$

$$\Rightarrow \frac{12}{25} = \frac{60}{(2x-3)^3}$$

$$\Rightarrow (2x-3)^3 = (5)^3$$

$$\Rightarrow 2x - 3 = 5 \quad \text{[taking cube root]}$$

$$\Rightarrow 2x = 8 \Rightarrow x = 4$$

| TYPE II |

PROBLEMS BASED ON FINDING SECOND ORDER DERIVATIVES OF PARAMETRIC FUNCTIONS

Suppose two parametric functions, say $y = f(t)$ and $x = g(t)$, are given to us and we have to find their second order derivative, i.e. $\dfrac{d^2y}{dx^2}$, then we use the following steps

I. First, write the given functions and differentiate them with respect to parameters.
i.e. let $y = f(t)$ and $x = f(t)$, then find $\dfrac{dy}{dt}$ and $\dfrac{dx}{dt}$.

II. Divide derivative of first function by derivative of second function, i.e. put the values of $\dfrac{dy}{dt}$ and $\dfrac{dx}{dt}$ in the formula

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad \text{...(i)}$$

III. Now, differentiate Eq. (i) w.r.t. x to get required second order derivative,

i.e. $$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy/dt}{dx/dt}\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy/dt}{dx/dt}\right)\frac{dt}{dx}$$ and simplify it.

EXAMPLE |4| If $x = a(\theta + \sin\theta)$ and $y = a(1 - \cos\theta)$, then find $\dfrac{d^2y}{dx^2}$ at $\theta = \dfrac{\pi}{2}$. **[All India 2011]**

Sol. We have, $x = a(\theta + \sin\theta)$ ...(i)

and $y = a(1 - \cos\theta)$ ...(ii)

Here, θ is the parameter.

On differentiating Eqs. (i) and (ii), respectively w.r.t. θ, we get

$$\frac{dx}{d\theta} = \frac{d}{d\theta}[a(\theta + \sin\theta)] = a(1 + \cos\theta) \quad \text{...(iii)}$$

and $$\frac{dy}{d\theta} = \frac{d}{d\theta}[a(1 - \cos\theta)] = a(0 + \sin\theta) = a\sin\theta \quad \text{...(iv)}$$

On putting the values from Eqs. (iii) and (iv) in the formula $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta}$, we get

$$\frac{dy}{dx} = \frac{a\sin\theta}{a(1+\cos\theta)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin\theta}{1+\cos\theta} = \frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\theta}{2}\right)} = \tan\frac{\theta}{2}$$

Now, differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\tan\frac{\theta}{2}\right)=\frac{d}{d\theta}\left(\tan\frac{\theta}{2}\right)\cdot\frac{d\theta}{dx}$$

$$=\frac{1}{2}\sec^2\frac{\theta}{2}\cdot\frac{1}{a(1+\cos\theta)} \quad \text{[from Eq. (iii)]}$$

$$=\frac{\sec^2\frac{\theta}{2}}{2a(1+\cos\theta)}$$

Now, at $\theta=\frac{\pi}{2}$,

$$\frac{d^2y}{dx^2}=\frac{\sec^2\frac{\pi}{4}}{2a\left(1+\cos\frac{\pi}{2}\right)}=\frac{2}{2a(1+0)}=\frac{1}{a}$$

|TYPE III|

PROBLEMS BASED ON PROVING THE RESULT/EQUATION

Sometimes a function, say $y=f(x)$, is given to us and we have to show/prove that given equation having first and second order derivatives of y is satisfied or not. For this, we use following steps

I. First, write the given function say $y=f(x)$.

II. Differentiate it w.r.t. x, i.e. find first derivative of function, i.e. $\frac{dy}{dx}$.

III. Find second derivative of function, i.e. $\frac{d^2y}{dx^2}$.

IV. Put the values of y, first derivative and second derivative in LHS of given equation and check whether LHS = RHS or not, for proving the given equation.

Note Sometimes, we differentiate given function two times to get required equation.

Sol. We have, $y=x^x$

Taking log on both sides, we get

$$\log y=\log x^x \Rightarrow \log y = x\log x$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx}=x\frac{d}{dx}(\log x)+\log x\frac{d}{dx}(x)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx}=x\times\frac{1}{x}+\log x \Rightarrow \frac{dy}{dx}=y(1+\log x) \quad \text{...(i)}$$

Again, differentiating both sides of Eq. (i) w.r.t. x, we get

$$\frac{d^2y}{dx^2}=y\frac{d}{dx}(1+\log x)+(1+\log x)\frac{dy}{dx}$$

$$=y\times\frac{1}{x}+(1+\log x)\frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2}=\frac{y}{x}+(1+\log x)\frac{dy}{dx} \quad \text{...(ii)}$$

Now, LHS $=\frac{d^2y}{dx^2}-\frac{1}{y}\left(\frac{dy}{dx}\right)^2-\frac{y}{x}$

On putting the values of $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ from Eqs. (i) and (ii), we get

$$\text{LHS}=\frac{y}{x}+(1+\log x)\,y(1+\log x)-\frac{1}{y}[y^2(1+\log x)^2]-\frac{y}{x}$$

$$=y(1+\log x)^2-y(1+\log x)^2=0=\text{RHS}$$

Hence proved.

EXAMPLE |6| If $y=\sin^{-1}x$, then show that $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=0$.

Sol. We have, $y=\sin^{-1}x$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \sqrt{1-x^2}\frac{dy}{dx}=1$$

On squaring both sides, we get

$$(1-x^2)\left(\frac{dy}{dx}\right)^2=1$$

Again, differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}\left[(1-x^2)\left(\frac{dy}{dx}\right)^2\right]=\frac{d}{dx}(1) \quad \text{...(i)}$$

$$\Rightarrow \frac{2dy}{dx}\left[(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}\right]=0$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=0$$

EXAMPLE |7| If $y=Pe^{ax}+Qe^{bx}$, then show that $\frac{d^2y}{dx^2}-(a+b)\frac{dy}{dx}+aby=0$. **[All India 2015C]**

Sol. We have, $y=Pe^{ax}+Qe^{bx}$ ...(i)

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx}=P\frac{d}{dx}(e^{ax})+Q\frac{d}{dx}(e^{bx})$$

$$\Rightarrow \frac{dy}{dx}=Pae^{ax}+Qbe^{bx} \quad \text{...(ii)}$$

Again, differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2}=Pa\frac{d}{dx}(e^{ax})+bQ\frac{d}{dx}(e^{bx})$$

|TOPIC PRACTICE 10|

OBJECTIVE TYPE QUESTIONS

1 If $y=\cos^{-1}x$, then the value of $\frac{d^2y}{dx^2}$ in terms of y alone is

(a) $-\cot y\,\text{cosec}^2 y$

$= Pa(ae^{ax}) + bQ(be^{bx})$

$= a^2Pe^{ax} + b^2Qe^{bx}$ ...(iii)

Now, LHS $= \frac{d^2y}{dx^2} - (a+b)\frac{dy}{dx} + aby$

On putting the values from Eqs. (i), (ii) and (iii), we get

LHS $= a^2Pe^{ax} + b^2Qe^{bx} - (a+b)(aPe^{ax} + bQe^{bx}) + ab(Pe^{ax} + Qe^{bx})$

$= a^2Pe^{ax} + b^2Qe^{bx} - a^2Pe^{ax} - abQe^{bx} - abPe^{ax} - b^2Qe^{bx} + abPe^{ax} + abQe^{bx}$

$= 0 =$ RHS

EXAMPLE [8] If $x = a\cos\theta + b\sin\theta$ and $y = a\sin\theta - b\cos\theta$, then show that $y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$. **[Delhi 2015]**

Sol. Given, $x = a\cos\theta + b\sin\theta$ ...(i)

and $y = a\sin\theta - b\cos\theta$ ...(ii)

Here, θ is the parameter.

On differentiating Eqs. (i) and (ii), respectively w.r.t. θ, we get

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(a\cos\theta + b\sin\theta)$$

$$\Rightarrow \frac{dx}{d\theta} = -a\sin\theta + b\cos\theta$$

and $\frac{dy}{d\theta} = \frac{d}{d\theta}(a\sin\theta - b\cos\theta)$

$$\Rightarrow \frac{dy}{d\theta} = a\cos\theta + b\sin\theta$$

Now, $\frac{dy}{dx} = \frac{(dy/d\theta)}{(dx/d\theta)} = \frac{a\cos\theta + b\sin\theta}{-(a\sin\theta - b\cos\theta)} = -\frac{x}{y}$ [from Eqs. (i) and (ii)]

Again, differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -\left[\frac{y\frac{d}{dx}(x) - x\frac{dy}{dx}}{y^2}\right]$$

$$\Rightarrow y^2\frac{d^2y}{dx^2} = -y + x\frac{dy}{dx} \Rightarrow y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$$

9 If $y = x^x$, then find $\frac{d^2y}{dx^2}$.

10 If $f(x) = \tan^{-1}x$, then find $f''(x)$. **[NCERT]**

11 If $y = \tan^{-1}x$, then find $\frac{d^2y}{dx^2}$ in terms of y.

12 If $y = A\sin x + B\cos x$, then prove that $\frac{d^2y}{dx^2} + y = 0$. **[NCERT]**

13 If $y = 5\cos x - 3\sin x$, then prove that $\frac{d^2y}{dx^2} + y = 0$. **[NCERT]**

(b) $\text{cosec}\, y \cot^2 y$

(c) $-\cot y\, \text{cosec}\, y$

(d) None of the above

2 If $y = 3\cos(\log x) + 4\sin(\log x)$, then

(a) $xy_2 + y_1 + y = 0$ (b) $xy_2 + y_1 - y = 0$

(c) $x^2y_2 + xy_1 + y = 0$ (d) None of these

3 If $y = Ae^{mx} + Be^{nx}$, then $\frac{d^2y}{dx^2}$ is equal to

(a) $(m+n)\frac{dy}{dx} + mny$ (b) $(m+n)\frac{dy}{dx} - mny$

(c) $-(m+n)\frac{dy}{dx} + mny$ (d) None of these

4 If $y = (\tan^{-1}x)^2$, then the value of $(x^2+1)^2y_2 + 2x(x^2+1)y_1$ is

(a) 2 (b) 3

(c) 4 (d) None of these

5 If $x = t^2$ and $y = t^3$, then $\frac{d^2y}{dx^2}$ is equal to

(a) $\frac{3}{2}$ (b) $\frac{3}{4t}$

(c) $\frac{3}{2t}$ (d) $\frac{3}{2t}$

VERY SHORT ANSWER Type Questions

6 Find the second order derivative of the following. (Each part carries 1 Mark)

(i) $x^3 - x^2 + 2$

(ii) $x^{10} + x^9$

7 Find $\frac{d^2y}{dx^2}$, when $y = e^{ax}$.

SHORT ANSWER Type I Questions

8 Find the second derivative of the following functions. (Each part carries 2 Marks)

(i) $y = \log(\log x)$ (ii) $x^3\log x$

(iii) $e^{6x}\cos 3x$ **[NCERT]**

22 If $y = e^{ax}\cdot\cos bx$, then prove that

$$\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2+b^2)\,y = 0.$$

23 If $y = x\log\left(\frac{x}{a+bx}\right)$, then prove that

$$x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right).$$ **[Delhi 2013]**

24 If $y = \sqrt{x+1} - \sqrt{x-1}$, then prove that

$$(x^2-1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - \frac{1}{4}y = 0.$$

25 If $y = 500e^{7x} + 600e^{-7x}$, then show that

SHORT ANSWER Type II Questions

14 If $y = 3e^{2x} + 2e^{3x}$, then prove that

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0.$$ **[NCERT]**

15 If $y = \tan x + \sec x$, then prove that

$$\frac{d^2y}{dx^2} = \frac{\cos x}{(1-\sin x)^2}.$$ **[NCERT Exemplar]**

16 If $x\cos(a+y) = \cos y$, then prove that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$. Hence show that

$$\sin a\frac{d^2y}{dx^2} + \sin 2(a+y)\frac{dy}{dx} = 0.$$ **[All India 2016]**

17 If $e^y(x+1) = 1$, then show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$. **[All India 2017]**

18 If $y = \sin(\sin x)$, prove that

$$\frac{d^2y}{dx^2} + \tan x\frac{dy}{dx} + y\cos^2 x = 0.$$ **[CBSE 2018]**

19 If $y = e^{a\cos^{-1}3x}$, prove that

$$(1-9x^2)\frac{d^2y}{dx^2} - 9x\frac{dy}{dx} - 9a^2y = 0.$$ **[All India 2016]**

20 $x = \tan\left(\frac{1}{a}\log y\right)$, then show that

$$(1+x^2)\frac{d^2y}{dx^2} + (2x-a)\frac{dy}{dx} = 0.$$ **[All India 2011]**

21 If $y = x^3\log\left(\frac{1}{x}\right)$, then prove that

$$x\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 3x^2 = 0.$$

$$\frac{d^2y}{dx^2} = 49y.$$ **[NCERT]**

26 If $y = \log(1+\cos x)$, then prove that

$$\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2}\cdot\frac{dy}{dx} = 0.$$

27 If $y = Ae^{-kt}\cos(pt+c)$, prove that

$$\frac{d^2y}{dt^2} + 2k\frac{dy}{dt} + n^2y = 0, \text{ where } n^2 = p^2 + k^2.$$

28 If $y = (\sec^{-1}x)^2$, then show that

$$x^2(x^2-1)\frac{d^2y}{dx^2} + (2x^3-x)\frac{dy}{dx} = 2$$ **[All India 2017C]**

29 If $y = \log[x + \sqrt{x^2+1}]$, then prove that

$$(x^2+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0.$$ **[Foreign 2011]**

30 If $y = \left(x + \sqrt{1+x^2}\right)^n$, then show that

$$(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = n^2y.$$ **[Foreign 2015]**

31 If $x^m y^n = (x+y)^{m+n}$, then prove that $\frac{d^2y}{dx^2} = 0$. **[Delhi 2017]**

32 If $y = 3\cos(\log x) + 4\sin(\log x)$, then show that $x^2y_2 + xy_1 + y = 0.$ **[NCERT]**

33 If $(x-a)^2 + (y-b)^2 = c^2$, then prove that

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$ is a constant and independent of a and b. **[All India 2019, NCERT]**

34 If $f(x) = |x|^3$, then show that $f''(x)$ exists for all real x and find it. **[NCERT]**

35 If $x = \sin t$ and $y = \sin pt$, then prove that

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0.$$ **[NCERT Exemplar]**

36 If $x = 3\cos t - 2\cos^3 t$ and $y = 3\sin t - 2\sin^3 t$, then find $\frac{d^2y}{dx^2}$. **[Delhi 2017C]**

37 If $x = a\sec^3\theta$, $y = a\tan^3\theta$, then find $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{4}$. **[Delhi 2015C]**

Again, differentiating w.r.t. x, we get

$$xy_2 + y_1 \cdot 1 = -3\cos(\log x)\frac{d}{dx}(\log x) - 4\sin(\log x)\frac{d}{dx}\log x$$

$$= -3\cos(\log x)\frac{1}{x} - 4\sin(\log x)\frac{1}{x}$$

Multiplying throughout by x, we have

$$x^2y_2 + xy_1 = -\{3\cos(\log x) + 4\sin(\log x)\}$$

$\Rightarrow$ $x^2y_2 + xy_1 = -y$ [from Eq. (i)]

$\Rightarrow$ $x^2y_2 + xy_1 + y = 0$

3. (b) Given, $y = Ae^{mx} + Be^{nx}$ ...(i)

On differentiating twice w.r.t. x, we get

38 If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, then find $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$. **[Delhi 2019, 2017C, 2014C]**

39 If $y = a\left(\cos t + \log\tan\frac{t}{2}\right)$, $x = a\sin t$, then evaluate $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{3}$. **[Delhi 2013]**

HINTS & SOLUTIONS

1. (a) Given, $y = \cos^{-1} x$

$\Rightarrow \quad x = \cos y$

On differentiating w.r.t. y, we get

$$\frac{dx}{dy} = -\sin y$$

$$\Rightarrow \quad \frac{dy}{dx} = -\operatorname{cosec} y \quad \text{...(i)}$$

Again, differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(-\operatorname{cosec} y) = -(-\operatorname{cosec} y \cot y)\frac{dy}{dx}$$

$$= \operatorname{cosec} y \cot y(-\operatorname{cosec} y) \quad \text{[from Eq. (i)]}$$

$$= -\cot y \cdot \operatorname{cosec}^2 y$$

2. (c) Given, $y = 3\cos(\log x) + 4\sin(\log x)$ ...(i)

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = -3\sin(\log x)\frac{d}{dx}(\log x) + 4\cos(\log x)\frac{d}{dx}\log x$$

$$y_1 = -3\sin(\log x)\frac{1}{x} + 4\cos(\log x)\frac{1}{x} \quad \left[\because \frac{dy}{dx} = y_1\right]$$

Multiplying by x, we get

$$xy_1 = -3\sin(\log x) + 4\cos(\log x) \quad \text{...(ii)}$$

On further differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{3}{2}\cdot\frac{d}{dt}t\cdot\frac{dt}{dx} = \frac{3}{2}\cdot\frac{1}{2t} \quad \left[\because \frac{dt}{dx} = \frac{1}{2t}\right]$$

$$= \frac{3}{4t}$$

6. (i) Let $y = x^3 - x^2 + 2$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 3x^2 - 2x + 0$$

Again, differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = 6x - 2$$

(ii) Solve as part (i). [**Ans.** $18x^7(5x+4)$]

7. We have, $y = e^{ax}$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = e^{ax}\frac{d}{dx}(ax) \quad \text{[by chain rule of derivative]}$$

$$\frac{dy}{dx} = Ae^{mx}\frac{d}{dx}(mx) + Be^{nx}\frac{d}{dx}(nx)$$

$$= A\,e^{mx}\,m + B\,e^{nx}\,n \quad \text{...(ii)}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = m^2 Ae^{mx} + n^2Be^{nx} \quad \text{...(iii)}$$

Using Eqs. (i), (ii) and (iii), we get

$$\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny$$

$$= m^2 Ae^{mx} + n^2 Be^{nx} - (m+n)\{Ame^{mx} + n\,Be^{nx}\} + mn\{Ae^{mx} + Be^{nx}\}$$

i.e. $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0.$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = (m+n)\frac{dy}{dx} - mny$$

4. (a) Given, $y = (\tan^{-1} x)^2$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 2\tan^{-1} x\frac{d}{dx}(\tan^{-1} x)$$

$$= 2(\tan^{-1} x)\frac{1}{1+x^2}$$

$$\Rightarrow \quad (1+x^2)\,y_1 = 2\tan^{-1} x \quad \left[\because \frac{dy}{dx} = y_1\right]$$

Again, differentiating w.r.t. x, we get

$$(1+x^2)\frac{dy_1}{dx} + y_1\frac{d}{dx}(1+x^2) = \frac{2}{1+x^2}$$

$$\Rightarrow \quad (1+x^2)y_2 + y_1(0+2x) = \frac{2}{1+x^2} \quad \left[\because \frac{d}{dx}y_1 = y_2\right]$$

$$\Rightarrow \quad (1+x^2)^2 y_2 + 2x(1+x^2)y_1 = 2$$ **Hence proved.**

5. (b) We have, $x = t^2$ and $y = t^3$

$\therefore \quad \frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 3t^2$

$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2}{2t} = \frac{3}{2}t$

10. We have, $f(x) = \tan^{-1} x$

On differentiating both sides w.r.t. x, we get

$$f'(x) = \frac{1}{1+x^2} = (1+x^2)^{-1}$$

Again, differentiating both sides w.r.t. x, we get

$$f''(x) = -1(1+x^2)^{-2}\frac{d}{dx}(1+x^2)$$

[by chain rule of derivative]

$$= \frac{-1}{(1+x^2)^2}\times(0+2x) = \frac{-2x}{(1+x^2)^2}$$

11. **Hint** $y = \tan^{-1} x \Rightarrow x = \tan y$

$\Rightarrow \frac{dx}{dy} = \sec^2 y \Rightarrow \frac{dy}{dx} = \cos^2 y$

$\Rightarrow \frac{d^2y}{dx^2} = 2\cos y(-\sin y)\cdot\frac{dy}{dx}$ $\left[\text{Ans. } \frac{d^2y}{dx^2} = -2\sin y\cos^3 y\right]$

$$\Rightarrow \quad \frac{dy}{dx} = ae^{ax}$$

Again, differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = ae^{ax} \cdot \frac{d}{dx}(ax) \quad \text{[by chain rule of derivative]}$$

$$= a^2 e^{ax}$$

8. (i) **Hint** $\frac{dy}{dx} = \frac{1}{\log x} \cdot \frac{1}{x} = (x \cdot \log x)^{-1}$

and $\frac{d^2y}{dx^2} = (-1)(x \log x)^{-2} \cdot \left(x \cdot \frac{1}{x} + \log x\right)$

$\left[\textbf{Ans. } \frac{1 + \log x}{(x \log x)^2}\right]$

(ii) Let $y = x^3 \log x$.

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x^3 \log x) = x^3 \frac{d}{dx}(\log x) + \log x \frac{d}{dx}(x^3)$$

[by product rule of derivative]

$$= x^3 \left(\frac{1}{x}\right) + (\log x)(3x^2) = x^2 (1 + 3 \log x)$$

Again, differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = x^2 \frac{d}{dx}(1 + 3 \log x) + \frac{d}{dx}(x^2) \cdot (1 + 3 \log x)$$

$$= x^2 \left(0 + \frac{3}{x}\right) + (1 + 3 \log x)(2x)$$

$$= 3x + 2x(1 + 3 \log x) = x(5 + 6 \log x)$$

(iii) Similar as Example 2.

[**Ans.** $9e^{6x}(3\cos 3x - 4\sin 3x)$]

9. Similar as Example 5. $\left[\textbf{Ans. } x^x\left\{(1 + \log x)^2 + \frac{1}{x}\right\}\right]$

12. **Hint** $\frac{dy}{dx} = A\cos x + B(-\sin x) = A\cos x - B\sin x$

$$\frac{d^2y}{dx^2} = A(-\sin x) - B\cos x = -A\sin x - B\cos x$$

13. **Hint** Find $\frac{d^2y}{dx^2}$ and then put the value of y and $\frac{d^2y}{dx^2}$ in LHS of given equation.

14. We have, $y = 3e^{2x} + 2e^{3x}$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 3e^{2x} \frac{d}{dx}(2x) + 2e^{3x} \frac{d}{dx}(3x)$$

[by chain rule of derivative]

$$\Rightarrow \quad \frac{dy}{dx} = 3e^{2x} \cdot 2 + 2e^{3x} \cdot 3 = 6e^{2x} + 6e^{3x}$$

Again, differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = 6e^{2x} \frac{d}{dx}(2x) + 6e^{3x} \frac{d}{dx}(3x)$$

[by chain rule of derivative]

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 6e^{2x} \cdot 2 + 6e^{3x} \cdot 3 = 12e^{2x} + 18e^{3x}$$

Now, LHS $= \frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y$

On putting the values of $\frac{d^2y}{dx^2}, \frac{dy}{dx}$ and y, we get

$$\text{LHS} = (12e^{2x} + 18e^{3x}) - 5(6e^{2x} + 6e^{3x}) + 6(3e^{2x} + 2e^{3x})$$

$$= 12e^{2x} - 30e^{2x} + 18e^{2x} + 18e^{3x} - 30e^{3x} + 12e^{3x}$$

$$= 30e^{2x} - 30e^{2x} + 30e^{3x} - 30e^{3x} = 0$$

$$\therefore \quad \frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0 = \text{RHS}$$

Hence proved.

15. Given, $y = \tan x + \sec x$

$$\Rightarrow \quad y = \frac{1 + \sin x}{\cos x}$$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{\cos x \cdot \cos x - (1 + \sin x)(-\sin x)}{\cos^2 x}$$

[by quotient rule of derivative]

$$\Rightarrow \frac{dy}{dx} = \frac{\cos^2 x + \sin x + \sin^2 x}{\cos^2 x} = \frac{1 + \sin x}{1 - \sin^2 x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1 + \sin x}{(1 + \sin x)(1 - \sin x)} = \frac{1}{1 - \sin x}$$

Again, differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -(1 - \sin x)^2 \frac{d}{dx}(1 - \sin x) = \frac{\cos x}{(1 - \sin x)^2}$$

Hence proved.

16. Given, $x\cos(a + y) = \cos y$

$$\Rightarrow \quad x = \frac{\cos y}{\cos(a + y)}$$

On differentiating both sides w.r.t. y, we get

$$\frac{dx}{dy} = \frac{\cos(a + y)\frac{d}{dy}(\cos y) - \cos y \frac{d}{dy}\{\cos(a + y)\}}{\cos^2(a + y)}$$

17. Given, $e^y(x + 1) = 1$

Taking log on both sides, we get

$$\log[e^y(x + 1)] = \log 1 \Rightarrow \log e^y + \log(x + 1) = \log 1$$

$$\Rightarrow \quad y + \log(x + 1) = \log 1$$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} + \frac{1}{x + 1} = 0 \quad \text{...(i)}$$

Again, differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} - \frac{1}{(x + 1)^2} = 0$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} - \left(-\frac{dy}{dx}\right)^2 = 0 \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = 0 \Rightarrow \frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$$

18. Given, $y = \sin(\sin x)$ ...(i)

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \cos(\sin x) \cdot \cos x \quad \text{...(ii)}$$

Again, on differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \cos(\sin x) \cdot (-\sin x) + \cos x\{-\sin(\sin x)\} \cdot \cos x$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{\cos x} \cdot \left(\frac{dy}{dx}\right)(-\sin x) - y\cos^2 x$$

[by using quotient rule of derivative]

$$= \frac{\cos(a+y)\cdot(-\sin y)+\cos y\cdot\sin(a+y)}{\cos^2(a+y)}$$

$$= \frac{\sin(a+y)\cos y-\cos(a+y)\sin y}{\cos^2(a+y)}$$

$$\Rightarrow \frac{dx}{dy} = \frac{\sin(a+y-y)}{\cos^2(a+y)} = \frac{\sin a}{\cos^2(a+y)}$$

$[\because \sin A\cos B-\cos A\sin B = \sin(A-B)]$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a} \quad ...(i)$$

On differentiating both sides of Eq. (i) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{1}{\sin a}\frac{d}{dx}[\cos^2(a+y)]$$

$$= \frac{1}{\sin a}\times\frac{d}{dy}[\cos^2(a+y)]\times\frac{dy}{dx}$$

$$= \frac{1}{\sin a}\times 2\cos(a+y)[-\sin(a+y)]\times\frac{dy}{dx}$$

$$= -\frac{2\sin(a+y)\cos(a+y)}{\sin a}\times\frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{\sin 2(a+y)}{\sin a}\frac{dy}{dx} \quad [\because 2\sin\theta\cos\theta = \sin 2\theta]$$

$$\Rightarrow \sin a\frac{d^2y}{dx^2}+\sin 2(a+y)\frac{dy}{dx} = 0$$

Hence proved.

20. Given, $x = \tan\left(\frac{1}{a}\log y\right) \Rightarrow a\tan^{-1}x = \log y$

On differentiating both sides w.r.t. x, we get

$$a\times\frac{1}{1+x^2} = \frac{1}{y}\cdot\frac{dy}{dx}$$

$$\Rightarrow (1+x^2)\frac{dy}{dx} = ay$$

Again, differentiating both sides w.r.t. x, we get

$$(1+x^2)\frac{d}{dx}\left(\frac{dy}{dx}\right)+\frac{dy}{dx}\cdot\frac{d}{dx}(1+x^2) = \frac{d}{dx}(ay)$$

$$\left[\because \frac{d}{dx}(u\cdot v) = u\frac{dv}{dx}+v\frac{du}{dx}\right]$$

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2}+\frac{dy}{dx}\cdot(2x) = a\cdot\frac{dy}{dx}$$

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-a\frac{dy}{dx} = 0$$

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2}+(2x-a)\frac{dy}{dx} = 0$$

21. Hint $y = x^3(\log 1-\log x) = -x^3\log x$

$$\Rightarrow \frac{dy}{dx} = -x^3\cdot\frac{1}{x}+\log x(-3x^2) = -x^2(1+3\log x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = -x^2\frac{3}{x}+(1+3\log x)(-2x)$$

$$= -3x-2x(1+3\log x)$$

$$= -5x-6x\log x$$

On substituting these values in LHS, we get RHS.

[using Eqs. (i) and (ii)]

$$\Rightarrow \frac{d^2y}{dx^2} = -\tan x\frac{dy}{dx}-y\cos^2 x$$

$$\Rightarrow \frac{d^2y}{dx^2}+\tan x\frac{dy}{dx}+y\cos^2 x = 0$$

Hence proved.

19. We have, $y = e^{a\cos^{-1}3x}$ ...(i)

$$\Rightarrow \frac{dy}{dx} = e^{a\cos^{-1}3x}\cdot\left(a\cdot\frac{-1}{\sqrt{1-(3x)^2}}\cdot 3\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-3ay}{\sqrt{1-9x^2}} \quad \text{[using Eq. (i)]}$$

$$\Rightarrow \sqrt{1-9x^2}\frac{dy}{dx} = -3ay$$

$$\Rightarrow (1-9x^2)\left(\frac{dy}{dx}\right)^2 = 9a^2y^2 \quad \text{[squaring both sides]}$$

On differentiating both sides, we get

$$(1-9x^2)2\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2(-18x) = 18a^2y\frac{dy}{dx}$$

On dividing both sides by $2\frac{dy}{dx}$, we get

$$(1-9x^2)\frac{d^2y}{dx^2}-9x\frac{dy}{dx} = 9a^2y$$

$$\Rightarrow (1-9x^2)\frac{d^2y}{dx^2}-9x\frac{dy}{dx}-9a^2y = 0$$

Hence proved.

25. Solve as Question 14.

26. Hint Firstly, find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$, then substitute these value in the given equation, we get LHS = RHS.

27. Similar as Example 2.

28. Given, $y = (\sec^{-1}x)^2$

$$\Rightarrow \frac{dy}{dx} = 2\sec^{-1}x\cdot\frac{1}{x\sqrt{x^2-1}}$$

$$\therefore x\sqrt{x^2-1}\cdot\frac{dy}{dx} = 2\sec^{-1}x$$

$$\Rightarrow x\sqrt{x^2-1}\cdot\frac{d^2y}{dx^2}+\frac{dy}{dx}\left(\frac{x^2}{\sqrt{x^2-1}}+\sqrt{x^2-1}\right) = \frac{2}{x\sqrt{x^2-1}}$$

$$\Rightarrow x^2(x^2-1)\frac{d^2y}{dx^2}+\frac{dy}{dx}\cdot x(2x^2-1) = 2$$

i.e. $x^2(x^2-1)\frac{d^2y}{dx^2}+(2x^3-x)\frac{dy}{dx} = 2$

29. Hint $\frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}} \Rightarrow \sqrt{x^2+1}\frac{dy}{dx} = 1$ and then again differentiating both sides w.r.t. x. Then, put values of $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ in LHS to get RHS.

30. Given, $y = \left(x+\sqrt{1+x^2}\right)^n$. ...(i)

22. Similar as Example 2.

23. **Hint** Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ and then putting these values in the given equation, we get LHS = RHS.

24. Given, $y = \sqrt{x+1} - \sqrt{x-1}$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x+1}} - \frac{1}{2\sqrt{x-1}} = \frac{1}{2}\left[\frac{\sqrt{x-1}-\sqrt{x+1}}{\sqrt{x^2-1}}\right]$$

$$\Rightarrow \quad 2\sqrt{x^2-1}\cdot\frac{dy}{dx} = -y$$

Now, squaring both sides, we get $4(x^2-1)\left(\frac{dy}{dx}\right)^2 = y^2$

Again, differentiating both sides w.r.t. x, we get

$$4(x^2-1)\, 2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\cdot 8x = 2y\left(\frac{dy}{dx}\right)$$

On dividing both sides by $2\frac{dy}{dx}$, we get

$$4(x^2-1)\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} - y = 0$$

$$\therefore \quad (x^2-1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - \frac{y}{4} = 0$$ **Hence proved.**

$$\Rightarrow \quad (1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = n\sqrt{1+x^2}\cdot\frac{ny}{\sqrt{1+x^2}}$$

$$\Rightarrow \quad (1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = n^2 y$$

31. We have, $x^m y^n = (x+y)^{m+n}$

On taking log both sides, we get

$$\log(x^m y^n) = \log(x+y)^{m+n}$$

$$\Rightarrow \quad \log x^m + \log y^n = \log(x+y)^{m+n}$$

$$\Rightarrow \quad m\log x + n\log y = (m+n)\log(x+y)$$

On differentiating both sides w.r.t. x, we get

$$\frac{m}{x} + \frac{n}{y}\cdot\frac{dy}{dx} = \frac{(m+n)}{x+y}\cdot\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{dy}{dx}\left(\frac{n}{y} - \frac{m+n}{x+y}\right) = \frac{m+n}{x+y} - \frac{m}{x}$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{nx+ny-my-ny}{y(x+y)}\right) = \left(\frac{mx+nx-mx-my}{x(x+y)}\right)$$

$$\Rightarrow \quad \frac{dy}{dx}\left(\frac{nx-my}{y(x+y)}\right) = \left(\frac{nx-my}{x(x+y)}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x} \qquad \ldots\text{(i)}$$

Again, differentiating both sides w.r.t. x, we get

On differentiating both sides w.r.t. x, we get

$$\Rightarrow \quad \frac{dy}{dx} = n(x+\sqrt{1+x^2})^{n-1}\left(1+\frac{2x}{2\sqrt{1+x^2}}\right)$$

[by chain rule of derivative]

$$\Rightarrow \quad \frac{dy}{dx} = n(x+\sqrt{1+x^2})^{n-1}\left(\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{n(x+\sqrt{1+x^2})^n}{\sqrt{1+x^2}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{ny}{\sqrt{1+x^2}}$$ [from Eq. (i)]

$$\Rightarrow \sqrt{1+x^2}\,\frac{dy}{dx} = ny$$

Again, differentiating both sides w.r.t. x, we get

$$\sqrt{1+x^2}\,\frac{d^2y}{dx^2} + \frac{2x}{2\sqrt{1+x^2}}\cdot\frac{dy}{dx} = n\frac{dy}{dx}$$

$$\Rightarrow \quad (1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = n\cdot\sqrt{1+x^2}\,\frac{dy}{dx}$$

[multiplying both sides by $\sqrt{1+x^2}$]

33. Given, $(x-a)^2 + (y-b)^2 = c^2$ ...(i)

On differentiating both sides w.r.t. x, we get

$$2(x-a) + 2(y-b)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad (x-a) + (y-b)\frac{dy}{dx} = 0$$ [dividing both sides by 2]

$$\Rightarrow \quad \frac{dy}{dx} = -\left(\frac{x-a}{y-b}\right) \qquad \ldots\text{(ii)}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -\left[\frac{(y-b)\cdot 1 - (x-a)\frac{dy}{dx}}{(y-b)^2}\right]$$

[by quotient rule of derivative]

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -\left[\frac{(y-b) + \frac{(x-a)^2}{(y-b)}}{(y-b)^2}\right]$$ [using Eq. (ii)]

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -\frac{c^2}{(y-b)^3}$$ [using Eq. (i)] ...(iii)

$$\frac{d^2y}{dx^2}=\frac{x\cdot\frac{dy}{dx}-y\cdot 1}{x^2}=\frac{x\left(\frac{y}{x}\right)-y}{x^2} \quad \text{[using Eq. (i)]}$$

$$=\frac{y-y}{x^2}=0 \quad \textbf{Hence proved.}$$

32. Given, $y=3\cos(\log x)+4\sin(\log x)$ ...(i)

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx}=y_1=-3\sin(\log x)\frac{d}{dx}(\log x)+4\cos(\log x)\frac{d}{dx}(\log x)$$

[by chain rule of derivative]

$$\Rightarrow \quad y_1=-3\sin(\log x)\times\frac{1}{x}+4\cos(\log x)\times\frac{1}{x}$$

$$\Rightarrow x y_1=-3\sin(\log x)+4\cos(\log x)$$

[multiplying both sides by x]

Again, differentiating both sides w.r.t. x, we get

$$x\frac{d}{dx}(y_1)+y_1\times\frac{d}{dx}(x)=-3\cos(\log x)\frac{d}{dx}(\log x)$$

$$-4\sin(\log x)\frac{d}{dx}(\log x) \text{ [by chain rule of derivative]}$$

$$\Rightarrow \quad x y_2+y_1=-3\cos(\log x)\times\frac{1}{x}-4\sin(\log x)\times\frac{1}{x}$$

$$\Rightarrow \quad x^2y_2+x y_1=-3\cos(\log x)-4\sin(\log x)$$

[multiplying both sides by x]

$$\Rightarrow \quad x^2y_2+x y_1=-[3\cos(\log x)+4\sin(\log x)]$$

$$\Rightarrow \quad x^2y_2+x y_1=-y \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad x^2y_2+x y_1+y=0$$

and $\quad Rf'(0)=\lim\limits_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim\limits_{h\to 0}\frac{f(h)-f(0)}{h}$

$$=\lim_{h\to 0}\frac{h^3-0}{h}=\lim_{h\to 0}h^2=0$$

$\because \quad Lf'(0)=Rf'(0)$

$\therefore f(x)$ is differentiable at $x=0$, i.e. $f'(x)$ exist at $x=0$ also.

Thus, $f'(x)=\begin{cases}3x^2, & \text{if } x\ge 0\\ -3x^2, & \text{if } x<0\end{cases}$

Now, $Lf''(0)=\lim\limits_{h\to 0}\frac{f'(0-h)-f'(0)}{-h}$

$$=\lim_{h\to 0}\frac{f'(-h)-f(0)}{-h}$$

$$=\lim_{h\to 0}\frac{-3(-h)^2-0}{-h}$$

$$=\lim_{h\to 0}3h=0$$

and $\quad Rf''(0)=\lim\limits_{h\to 0}\frac{f'(0+h)-f'(0)}{h}$

$$=\lim_{h\to 0}\frac{f'(h)-f'(0)}{h}$$

$$=\lim_{h\to 0}\frac{3h^2-0}{h}=\lim_{h\to 0}3h=0$$

$\because \quad Lf''=Rf''(0)$

$\therefore f'(x)$ is differentiable at $x=0$, i.e. $f''(x)$ exist at $x=0$ also.

Hence, $f''(x)=\begin{cases}6x, & \text{if } x\ge 0\\ -6x, & \text{if } x<0\end{cases}$

Now consider, $\dfrac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}=-\dfrac{\left[1+\frac{(x-a)^2}{(y-b)^2}\right]^{3/2}}{\frac{c^2}{(y-b)^3}}$

[using Eqs. (ii) and (iii)]

$$=-\frac{\left(\frac{c^2}{(y-b)^2}\right)^{3/2}}{\frac{c^2}{(y-b)^3}} \quad \text{[using Eq. (i)]}$$

$$=-\frac{c^3}{(y-b)^3}\times\frac{(y-b)^3}{c^2}$$

$=-c$, which is constant and independent of a and b.

Hence proved.

34. We have, $f(x)=|x|^3=\begin{cases}x^3, & \text{if } x\ge 0\\ -x^3, & \text{if } x<0\end{cases}$

Clearly $f''(x)$ exist for all $x\in R$ except possibly at $x=0$.

[$\because f(x)$ is an polynomial function for $(-\infty,0)\cup(0,\infty)$]

Now, $\quad Lf'(0)=\lim\limits_{h\to 0}\frac{f(0-h)-f(0)}{-h}$

$$=\lim_{h\to 0}\frac{f(-h)-f(0)}{-h}$$

$$=\lim_{h\to 0}\frac{-(-h)^3-0}{-h}=\lim_{h\to 0}\frac{h^3}{-h}=\lim_{h\to 0}-h^2=0$$

$$\Rightarrow \quad \frac{d^2y}{dx^2}=-\operatorname{cosec}^2 t\cdot\frac{dt}{dx}$$

$$=-\operatorname{cosec}^2 t\cdot\frac{1}{3\sin t\cos 2t} \quad \text{[using Eq. (i)]}$$

$$=\frac{-1}{3\sin^3 t\cos 2t}$$

37. Hint $\frac{dx}{d\theta}=3a\sec^2\theta\cdot\sec\theta\tan\theta$

and $\quad \frac{dy}{d\theta}=3a\tan^2\theta\cdot\sec^2\theta\left[\text{Ans. }\frac{1}{12a}\right]$

38. Given, $x=a(\cos t+t\sin t)$ and $y=a(\sin t-t\cos t)$

On differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt}=a\left[\frac{d}{dt}(\cos t)+\left(t\frac{d}{dt}(\sin t)+\sin t\frac{d}{dt}(t)\right)\right]$$

$$\left[\because \frac{d}{dx}(u\cdot v)=u\frac{dv}{dx}+v\frac{du}{dx}\right]$$

$$\therefore \quad \frac{dx}{dt}=a\{-\sin t+(t\cos t+\sin t\cdot 1)\}=at\cos t$$

and $\quad \frac{dy}{dt}=a\left[\frac{d}{dt}(\sin t)-\left(t\frac{d}{dt}(\cos t)+\cos t\frac{d}{dt}(t)\right)\right]$

$$\Rightarrow \quad \frac{dy}{dt}=a\{\cos t-t(-\sin t)-\cos t\cdot 1\}=at\sin t$$

Now, $\quad \frac{dy}{dx}=\dfrac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{at\sin t}{at\cos t}=\tan t$

35. Similar as Example 8.

36. We have, $x = 3\cos t - 2\cos^3 t$ and $y = 3\sin t - 2\sin^3 t$

Now, $\frac{dx}{dt} = -3\sin t + 6\cos^2 t \sin t = 3\sin t(2\cos^2 t - 1)$

$= 3\sin t \cdot \cos 2t$ ...(i)

and $\frac{dy}{dt} = 3\cos t - 6\sin^2 t \cos t = 3\cos t(1 - 2\sin^2 t)$

$= 3\cos t \cdot \cos 2t$ ...(ii)

$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3\cos t \cos 2t}{3\sin t \cos 2t}$ [using Eqs. (i) and (ii)]

$= \cot t$

Again, differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\tan t) = \frac{d}{dt}(\tan t)\frac{dt}{dx} = \frac{\sec^2 t}{at\cos t}$$

$$= \frac{1}{at}\sec^3 t \qquad \left[\because \frac{dx}{dt} = at\cos t\right]$$

At $t = \frac{\pi}{4}$, $\frac{d^2y}{dx^2} = \frac{1}{a \times \frac{\pi}{4}}\sec^3\left(\frac{\pi}{4}\right) = \frac{4}{a\pi}(\sqrt{2})^3 = \frac{8\sqrt{2}}{a\pi}$

39. Solve as Question 38. $\left[\text{Ans. } \frac{-8}{3a}\right]$

SUMMARY

- A real valued function $f(x) = f(c)$ is said to be **continuous** at a point 'c' in its domain iff $\lim_{x \to c} f(x) = f(c)$, i.e. left hand limit, right hand limit and the value of the function at $x = c$ exist and equal to each other.
- A function is said to be continuous in its domain if it is continuous at every point of its domain.
- If f and g are continuous functions, then
  - (i) $(f \pm g)(x) = f(x) \pm g(x)$ is continuous.
  - (ii) $(f \cdot g)(x) = f(x) \cdot g(x)$ is continuous.
  - (iii) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ is continuous, provided $g(x) \neq 0$.
- If g is continuous at $x = a$ and f is continuous at $g(a)$, then fog is continuous at $x = a$.
- A real valued function $f(x)$ is said to be differentiable at $x = c$ iff $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists finitely.

  or

  $\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}$ and $\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$ are finite and equal.

  or

  $\lim_{h \to 0} \frac{f(c - h) - f(c)}{-h}$ and $\lim_{h \to 0} \frac{f(c + h) - f(c)}{h}$ are finite and equal.
- A function is said to differentiable, if it is differentiable at every point in its domain.
- Every differentiable function is continuous but the converse is not true.
- The derivative of f at any point x is given by $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$, provided limit exist.

  The process of finding derivative of a function is called differentiation.
- **Algebra of Derivatives** Let u and v be the two functions of x. Then,
  - (i) $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$
  - (ii) $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$ [product rule]
  - (iii) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\left[v\frac{du}{dx} - u\frac{dv}{dx}\right]}{v^2}$ [quotient rule]
- **Derivatives of Composite Functions** Let y be a real valued function which is a composite of two functions say $y = f(u)$ and $u = g(x)$.

  Then, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = f'(u) \cdot g'(x)$.

  i.e. $\frac{d}{dx}[f\{g(x)\}] = f'(g(x)) \cdot g'(x)$.
- **Derivative of Implicit Functions** Let $f(x, y) = 0$ be an implicit function of x. Then, to find $\frac{dy}{dx}$ we first differentiate both sides of equation w.r.t. x and then take all terms involving $\frac{dy}{dx}$ on LHS and remaining terms on RHS to get required value.
- To differentiate the function of the form $f(x) = (u(x))^{v(x)}$, we first take log on both sides and use properties of logarithm to simplify it and then differentiate it.
- If $x = \phi(t)$ and $y = \psi(t)$, then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$.
- If $y = f(x)$ and $z = g(x)$, then $\frac{dy}{dz} = \frac{dy/dx}{dz/dx}$.
- Let $y = f(x)$, then $\frac{dy}{dx} = f'(x)$ is called the **first derivative** of y or $f(x)$ and $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is called the **second derivative** of w.r.t. x and is denoted by $\frac{d^2y}{dx^2}$ or $f''(x)$ or y'' or D^2y or y_2.

CHAPTER PRACTICE

OBJECTIVE TYPE QUESTIONS

1 If $f(x) = \begin{cases} mx+1, \text{ if } x \le \frac{\pi}{2} \\ \sin x + n, \text{ if } x > \frac{\pi}{2} \end{cases}$ is continuous at $x = \frac{\pi}{2}$, then

(a) $m = 1, n = 0$ (b) $m = \frac{n\pi}{2} + 1$

(c) $n = \frac{m\pi}{2}$ (d) $m = n = \frac{\pi}{2}$

2 If $f(x) = |\cos x|$, then **[NCERT Exemplar]**

(a) f is everywhere differentiable

(b) f is everywhere continuous but not differentiable at $x = n\pi, n \in Z$

(c) f is everywhere continuous but not differentiable at $x = (2n+1)\frac{\pi}{2}, n \in Z$

(d) None of the above

3 If $y = \log\left(\frac{1-x^2}{1+x^2}\right)$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{4x^3}{1-x^4}$ (b) $\frac{-4x}{1-x^4}$ (c) $\frac{1}{4-x^4}$ (d) $\frac{-4x^3}{1-x^4}$

4 If $y = \log_7(\log x)$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{1}{x\log x \log 7}$ (b) $\frac{-1}{x\log x \log 7}$

(c) $\frac{1}{x\log x}$ (d) None of these

5 If $y = a^{t+\frac{1}{t}}$ and $x = \left(t+\frac{1}{t}\right)^a$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{a^{\left(1+\frac{1}{t}\right)}\log a}{\left(t+\frac{1}{t}\right)^{a-1}}$ (b) $\frac{a^{\left(1+\frac{1}{t}\right)}}{\left(t+\frac{1}{t}\right)^{a-1}}$

(c) $\frac{a^{\left(1+\frac{1}{t}\right)}\log a}{a\left(t+\frac{1}{t}\right)^{a-1}}$ (d) None of these

6 If $(\cos x)^y = (\cos y)^x$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{\log(\cos y) + y(\tan x)}{\log(\cos x) + x\tan y}$ (b) $\frac{\log(\cos y) - y(\tan x)}{\log(\cos x) - x(\tan y)}$

(c) $\frac{\log(\tan x) + y(\cos x)}{\log(\cos x) + x(\tan y)}$ (d) None of these

7 If $y = x\cos x$, then $\frac{d^2y}{dx^2}$ is

(a) $-x\cos x - 2\sin x$ (b) $x\cos x + 2\sin x$

(c) $x\sin x + \cos x$ (d) None of these

8 If $y = (x + \sqrt{1+x^2})^n$, then $(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx}$ is equal to

(a) n^2y (b) $-n^2y$

(c) $-y$ (d) $2x^2y$

9 If $x = \sin t$ and $y = \sin pt$, then $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx}$ is equal to

(a) $-y$ (b) y (c) py (d) $-p^2y$

10 If $y = Ae^{5x} + Be^{-5x}$, then $\frac{d^2y}{dx^2}$ is equal to

(a) $25y$ (b) $5y$

(c) $-25y$ (d) $15y$

VERY SHORT ANSWER Type Questions

11 If the following function $f(x)$ is continuous at $x = 0$, then write the value of k.

$$f(x) = \begin{cases} \dfrac{\sin\frac{3x}{2}}{x}, & x \ne 0 \\ k, & x = 0 \end{cases}$$

[All India 2017C]

12 Determine the value of the constant 'k', so that function $f(x) = \begin{cases} \frac{kx}{|x|}, & \text{if } x < 0 \\ 3, & \text{if } x \ge 0 \end{cases}$ is continuous at $x = 0$. **[Delhi 2017]**

13 Find the derivative of $\cos(\sqrt{x})$. [NCERT]

14 Differentiate $\sin[\cos(x^2)]$ w.r.t. x. [NCERT]

15 Find the derivative of $\log \sin x$ w.r.t. x.

16 Differentiate $e^{\sqrt{x}}$ w.r.t. x.

17 Differentiate $e^{3x} \cdot \cos 2x$ w.r.t. x.

18 Find $\frac{dy}{dx}$, when $2x + 3y = \sin y$.

19 Find the second derivative of $\log x$. [NCERT]

20 Find the derivative of $\sin(\tan^{-1} x)$ w.r.t. x.

SHORT ANSWER Type I Questions

21 Show that the function $f(x) = \begin{cases} \frac{|x-a|}{x-a}, & x \neq a \\ 1, & x = a \end{cases}$ is discontinuous at $x = a$. [NCERT]

22 Discuss the continuity of secant. [NCERT]

23 If a function f is differentiable at a point c, then prove that it is also continuous at that point.

24 If $x = a\cos\theta$ and $y = b\cos\theta$, then find $\frac{dy}{dx}$. [NCERT]

25 Differentiate $x^{\sin x}$ w.r.t. x. [NCERT]

SHORT ANSWER Type II Questions

26 Examine the continuity of the function

$$f(x) = \begin{cases} \frac{|\sin x|}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases} \text{ at } x = 0.$$

27 Discuss the continuity of the function

$$f(x) = \begin{cases} \frac{x}{|x| + 2x^2}, & x \neq 0 \\ 2, & x = 0 \end{cases} \text{ at } x = 0.$$

28 If the function $f(x) = \begin{cases} 3ax + b, & \text{if } x > 1 \\ 11, & \text{if } x = 1 \\ 5ax - 2b, & \text{if } x < 1 \end{cases}$

is continuous at $x = 1$, then find the values of a and b. [Delhi 2011; All India 2010]

29 Find the value of k, if the function,

$$f(x) = \begin{cases} \frac{1 - \sin x}{(\pi - 2x)^2}, & x \neq \frac{\pi}{2} \\ k, & x = \frac{\pi}{2} \end{cases} \text{ is continuous at } x = \pi/2.$$

30 Find the value of k, so that the function f is continuous at $x = \pi/2$, $f(x) = \begin{cases} \frac{k\cos x}{\pi - 2x}, & \text{if } x \neq \frac{\pi}{2} \\ 3, & \text{if } x = \frac{\pi}{2} \end{cases}$. [Delhi 2012]

31 The function $f(x) = \begin{cases} x^2 a, & \text{if } 0 \le x < 1 \\ a, & \text{if } 1 \le x < \sqrt{2} \\ \frac{2b^2 - 4b}{x^2}, & \text{if } \sqrt{2} \le x < \infty \end{cases}$ is continuous on $[0, \infty)$. Find the most suitable values of a and b.

32 If the function $f(x)$ defined by

$$f(x) = \begin{cases} \frac{\log(1 + ax) - \log(1 - bx)}{x}, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$$ is continuous at $x = 0$, then find the value of k.

33 Examine the continuity of the following function $f(x) = \begin{cases} \frac{x}{2|x|}, & x \neq 0 \\ \frac{1}{2}, & x = 0 \end{cases}$ at $x = 0$.

34 Find the point of discontinuity, if any of the function $f(x) = \begin{cases} x^{10} - 1, & \text{if } x \le 1 \\ x^2, & \text{if } x > 1 \end{cases}$. [NCERT]

35 Find the point of discontinuity of the function

$$f(x) = \begin{cases} \frac{\sin 2x}{x}, & \text{if } x < 0 \\ x + 2, & \text{if } x \ge 0 \end{cases}.$$

36 Show that the function $f(x) = |x - 3|$, $x \in R$ is continuous but not differentiable at $x = 3$.

37 Test the continuity and differentiability at $x = 1$ of the function f defined by

$$f(x) = \begin{cases} 3^x, & -1 \le x < 1 \\ 4 - x, & 1 \le x < 4 \end{cases}.$$

38 For what value of λ the function defined by $f(x) = \begin{cases} \lambda(x^2 + 2), & \text{if } x \le 0 \\ 4x + 6, & \text{if } x > 0 \end{cases}$ is continuous $x = 0$?

Hence, check the differentiability of $f(x)$ at $x = 0$. [All India 2015C]

39 If function $f(x) = |x-3| + |x-4|$, then show that $f(x)$ is not differentiable at $x = 3$ and $x = 4$.

40 Find the set of points, where the function f given by $f(x) = |2x-1| \sin x$ is non-differentiable.

41 If $y = \dfrac{\sin(ax+b)}{\cos(cx+d)}$, then find $\dfrac{dy}{dx}$. **[NCERT]**

42 Find the derivative of $\dfrac{x+\sin x}{x+\cos x}$ w.r.t. x.

43 Differentiate w.r.t. x, $\sin^{-1}\left[\dfrac{2^{x+1}\cdot 3^x}{1+(36)^x}\right]$.

44 If $y = \tan^{-1}\left(\dfrac{a}{x}\right) + \log\sqrt{\dfrac{x-a}{x+a}}$, then prove that $\dfrac{dy}{dx} = \dfrac{2a^3}{x^4 - a^4}$. **[NCERT; All India 2012]**

45 If $\log(\sqrt{1+x^2} - x) = y\sqrt{1+x^2}$, then show that

$$(1+x^2)\frac{dy}{dx} + xy + 1 = 0.$$ **[All India 2011C]**

46 If $y = (\sin x - \cos x)^{(\sin x - \cos x)}$, $\dfrac{\pi}{4} < x < \dfrac{3\pi}{4}$, then find $\dfrac{dy}{dx}$.

47 Find the derivative of the following function w.r.t. x at $x = 1$.

$$\cos^{-1}\left[\sin\sqrt{\frac{1+x}{2}}\right] + x^x$$

48 Find $\dfrac{dy}{dx}$, if $y = (\cos x)^x + (\sin x)^{1/x}$. **[Delhi 2010]**

49 If $y = (\sin x)^x + \sin^{-1}\sqrt{x}$, then find $\dfrac{dy}{dx}$.

Or

Differentiate the function $(\sin x)^x + \sin^{-1}\sqrt{x}$ w.r.t. x. **[Delhi 2017, 2015C]**

50 If $y = x^{\sin x - \cos x} + \dfrac{x^2-1}{x^2+1}$, then find $\dfrac{dy}{dx}$. **[Delhi 2016C, 2012]**

51 Find $\dfrac{dy}{dx}$, when $y = x^{\cot x} + \dfrac{2x^2-3}{x^2+x+2}$. **[All India 2012C]**

52 Differentiate $x^{x\cos x} + \dfrac{x^2+1}{x^2-1}$ w.r.t. x. **[Delhi 2011]**

53 Find $\dfrac{dy}{dx}$, if $y^x + x^y + x^x = a^b$. **[NCERT]**

54 If $x = \alpha \sin 2t\,(1+\cos 2t)$ and $y = \beta\cos 2t\,(1-\cos 2t)$, then show that

$$\frac{dy}{dx} = \frac{\beta}{\alpha}\tan t.$$

55 If $x = a\left(\dfrac{1+t^2}{1-t^2}\right)$ and $y = \dfrac{2t}{1-t^2}$, then find $\dfrac{dy}{dx}$.

56 If $x = a\left(\cos\theta + \log\tan\dfrac{\theta}{2}\right)$ and $y = a\sin\theta$, then find the value of $\dfrac{dy}{dx}$ at $\theta = \dfrac{\pi}{4}$. **[Delhi 2011C]**

57 Differentiate $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1+x^2}}\right)$ w.r.t. $\cos^{-1}x^2$.

58 Differentiate $\tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$ w.r.t. $\sin^{-1}(2x\sqrt{1-x^2})$. **[Delhi 2014]**

59 Differentiate $\tan^{-1}\left(\dfrac{x}{1+\sqrt{1-x^2}}\right)$ w.r.t. $\sin\left(2\cot^{-1}\sqrt{\dfrac{1+x}{1-x}}\right)$.

[Hint put $x = \sin\theta$ in first function and $x = \cos\theta$ in second function]

60 If $y = ae^{2x} + be^{-x}$, then show that $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 2y = 0$. **[Delhi 2016C]**

61 If $y = e^{m\sin^{-1}x}$, then show that

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2 y = 0.$$ **[All India 2015]**

62 If $x = a\cos^3\theta$ and $y = a\sin^3\theta$, then find the value of $\dfrac{d^2y}{dx^2}$ at $\theta = \dfrac{\pi}{6}$. **[All India 2013]**

63 If $y = e^x \sin x$, then prove that $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = 0$.

64 If $y = \dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$, then prove that

$$(1-x^2)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - y = 0.$$

65 If $y = (\sin^{-1}x)^2$, then prove that

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 2 = 0.$$ **[Delhi 2019]**

CASE BASED Questions

|5 Marks |

66. Consider the following values

$x = a\cos^3\theta$ and $y = a\sin^3\theta$ **[CBSE Question Bank]**

On the basis of above information, answer the following questions.

(i) $\left.\frac{dx}{d\theta}\right|_{\theta=\frac{\pi}{4}}$ is equal to

(a) $\frac{3a}{2\sqrt{2}}$ (b) $\frac{-3a}{2\sqrt{2}}$

(c) $\frac{4\sqrt{2}}{3a}$ (d) $\frac{-1}{\sqrt{3}}$

(ii) $\left.\frac{dy}{d\theta}\right|_{\theta=\frac{\pi}{4}}$ is equal to

(a) $\frac{3a}{2\sqrt{2}}$ (b) $\frac{-3a}{2\sqrt{2}}$

(c) $\frac{4\sqrt{2}}{3a}$ (d) $\frac{-1}{\sqrt{3}}$

(iii) $\frac{dy}{dx}$ is equal to

(a) $\tan\theta$ (b) $-\tan\theta$ (c) $\cot\theta$ (d) $-\cot\theta$

(iv) $\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}}$ is equal to

(a) $\frac{3a}{2\sqrt{2}}$ (b) $\frac{-3a}{2\sqrt{2}}$

(c) $\frac{4\sqrt{2}}{3a}$ (d) $\frac{-1}{\sqrt{3}}$

(v) $\frac{d^2y}{dx^2}$ is equal to

(a) $\frac{3a}{2\sqrt{2}}$ (b) $\frac{-3a}{2\sqrt{2}}$

(c) $\frac{4\sqrt{2}}{3a}$ (d) $\frac{-1}{\sqrt{3}}$

67. A potter made a mud vessel, where the shape of the pot is based on $f(x) = |x-3| + |x-2|$, where $f(x)$ represents the height of the pot.

[CBSE Question Bank]

Answer the following questions using the above information.

(i) When $x > 4$ What will be the height in terms of x?

(a) $x-2$ (b) $x-3$

(c) $2x-5$ (d) $5-2x$

(ii) Will the slope vary with x value?

(a) Yes

(b) No

(c) Slope is not defined for any value of x.

(d) Insufficient data for the slope.

(iii) What is $\frac{dy}{dx}$ at $x = 3$?

(a) 2

(b) -2

(c) Function is not differentiable

(d) 1

(iv) When the x value lies between (2, 3), then the function $f(x)$ is

(a) $2x-5$

(b) $5-2x$

(c) 1

(d) 5

(v) If the potter is trying to make a pot using the function $f(x) = [x]$, will he get a pot or not? Why?

(a) Yes, because it is a continuous function

(b) Yes, because it is not continuous

(c) No , because it is a continuous function

(d) No , because it is not continuous

ANSWERS

1. (c) **2.** (c) **3.** (b) **4.** (a) **5.** (c) **6.** (a) **7.** (a) **8.** (a)

9. (d) **10.** (a) **11.** $k=\frac{3}{2}$ **12.** $k=-3$ **13.** $\frac{-\sin\sqrt{x}}{2\sqrt{x}}$ **14.** $-2x\cos(\cos x^2)\sin x^2$ **15.** $\cot x$

16. $\frac{e^{\sqrt{x}}}{2\sqrt{x}}$ **17.** $e^{3x}(-2\sin 2x+3\cos 2x)$ **18.** $\frac{2}{\cos y-3}$ **19.** $-\frac{1}{x^2}$ **20.** $\frac{\cos(\tan^{-1}x)}{1+x^2}$

22. Continuous everywhere except $x=(2n+1)\frac{\pi}{2}, n\in I$. **24.** $\frac{b}{a}$ **25.** $x^{\sin x}\left[\frac{\sin x}{x}+(\log x)\cos x\right]$

26. Discontinuous at $x=0$ **27.** Discontinuous at $x=0$ **28.** $a=3, b=2$ **29.** $k=\frac{1}{8}$

30. $k=6$ **31.** $a=-1, b=1$ or $a=1, b=1\pm\sqrt{2}$ **32.** $f(x)$ is continuous at $x=0$, if $k=a+b$.

33. $f(x)$ is discontinuous at $x=0$. **34.** $f(x)$ is discontinuous at $x=1$

35. $f(x)$ is everywhere continuous **37.** Continuous at $x=1$ but not differentiable at $x=1$

38. $\lambda=3$, not differentiable at $x=0$. **40.** $x=\frac{1}{2}$

41. $\frac{[a\cos(cx+d)\cos(ax+b)+c\sin(ax+b)\sin(cx+d)]}{\cos^2(cx+d)}$ **42.** $\frac{x(\cos x+\sin x)+\cos x-\sin x+1}{(x+\cos x)^2}$

43. $\left[\frac{2^{x+1}\cdot 3^x}{1+(36)^x}\right]\log 6$ **46.** $(\sin x-\cos x)^{(\sin x-\cos x)}\times(\cos x+\sin x)[1+\log(\sin x-\cos x)]$ **47.** $\frac{3}{4}$

48. $(\cos x)^x[-x\tan x+\log\cos x]+(\sin x)^{1/x}\left[\frac{\cot x}{x}-\frac{\log(\sin x)}{x^2}\right]$ **49.** $(\sin x)^x[x\cot x+\log\sin x]+\frac{1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}}$

50. $x^{\sin x-\cos x}\left[\frac{\sin x-\cos x}{x}+(\cos x+\sin x)\log x\right]+\frac{4x}{(x^2+1)^2}$ **51.** $x^{\cot x}\left(\frac{\cot x}{x}-\operatorname{cosec}^2 x\cdot\log x\right)+\frac{2x^2+14x+3}{(x^2+x+2)^2}$

52. $x^{x\cos x}[\cos x-x\sin x\log x+\cos x\log x]-\frac{4x}{(x^2-1)^2}$ **53.** $\frac{-[y^x\log y+y\cdot x^{y-1}+x^x(1+\log x)]}{x\cdot y^{x-1}+x^y\log x}$

55. $\frac{1+t^2}{2at}$ **56.** 1 **57.** $\frac{-1}{2}$ **58.** $\frac{1}{2}$ **59.** $-\frac{1}{2x}$ **62.** $\frac{32}{27a}$

66. (i) → (b), (ii) → (a), (iii) → (b), (iv) → (d), (v) → (c) **67.** (i) → (c), (ii) → (a), (iii) → (c), (iv) → (c), (v) → (d)