Fourier Series

3

Introduction

It is an approximated process by which an non-standard signal is converted into a standard signal. The approximation of a given function by Fourier series gives a smooth function even when the function being approximated has discontinuities.

Advantage

- We can find spectral width very easily.
- We can find steady state response very easily due to Periodic input.

Convergence of Fourier Series (Dirichlet Condition)

Periodic signal x(t) has a Fourier series representation if it satisfies the following Dirichlet condition.

- x(t) is absolutely integrable over any period i.e.

$$\int_{T_0} |x(t)|dt < \infty$$

- x(t) has a finite number of maxima and minima within any finite interval of t.
- x(t) has a finite number of discontinuities within any finite interval of t, and each of these discontinuities is finite.

Fourier Series Representation of Continuous Time

By using Fourier series, a non-sinusoidal periodic function can be expressed as an infinite sum of sinusoidal function.

1. Trigonometric Fourier series

Any practical periodic function of frequency ω_0 can be expressed as an infinite sum of sine (or) cosine functions that are integral multiples of ω_0

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos \omega_0 t + b_n \sin \omega_0 t)$$

where, ω_0 = Fundamental frequency

a_0, a_n, b_n = Trigonometric Fourier series coefficient

$n\omega_0$ = n^{th} harmonic of ω_0

$$a_0 = \frac{1}{T}\int_0^T f(t)dt$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega_0 t\, dt$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega_0 t\, dt$$

Note:

- Here a_0 is the value of constant component of the signal f(t)
- The Fourier coefficient a_n and b_n are maximum amplitude of n^{th} harmonic component.

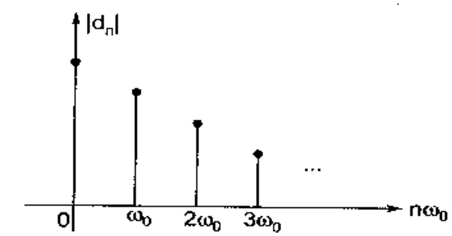
2. Polar form of trigonometric Fourier series

$$f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t - \phi_n)$$

where, $C_0 = a_0$

$|C_n| = \sqrt{a_n^2 + b_n^2}$ ... Magnitude spectrum

$\phi_n = \tan^{-1}\left(\frac{b_n}{a_n}\right)$ ... Phase spectrum



Magnitude spectrum

3. Exponential Fourier series

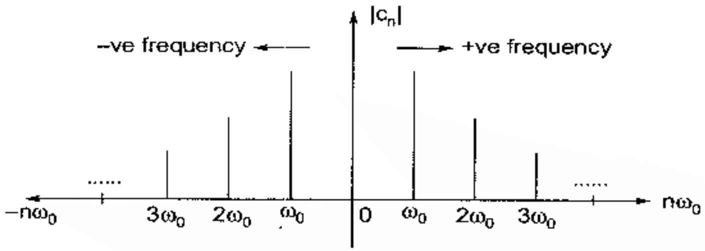
$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

where, $C_0 = a_0 = \frac{1}{T}\int_0^T f(t)dt$

$$C_n = \frac{1}{T}\int_0^T f(t)e^{-jn\omega_0 t}dt$$

Note:

Exponential Fourier series is compact form of Fourier series



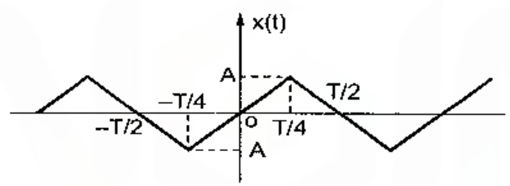
Positive and negative frequency indicate different phase of rotation, they maintain same magnitude but different phase.

Relation between exponential and trigonometric Fourier series

$$a_n = (C_n + C_{-n})$$
$$b_n = j(C_n - C_{-n})$$

Effect of Symmetry of Fourier Coefficients

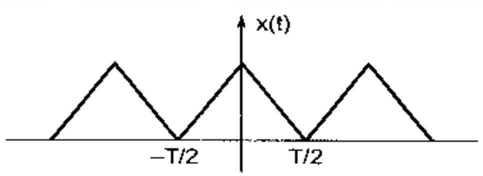
1. Odd symmetry



For signals with odd symmetry, the Fourier coefficient a_0 and a_n are zero.

$$x(t) = -x(-t)$$

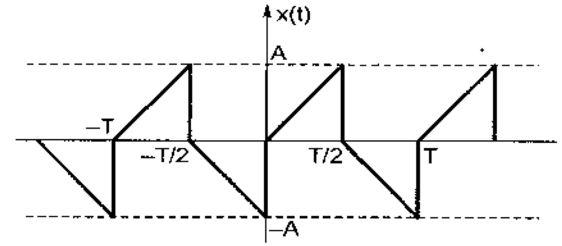
2. Even symmetry



$$x(t) = x(-t)$$

For signals with even symmetry, the Fourier coefficient b_n are zero.

3. Half-wave symmetry



$$x(t \pm T/2) = -x(t)$$

For signals with half-wave symmetry, the Fourier series will consist of odd harmonic terms of sine and cosine signals.

Summary

Function	C_n	Fourier Coefficient	Trigonometric Fourier Series
Real (Neither even nor odd)	Generally Complex $C_n = C_n^*$	$a_0 \neq 0$ $a_n = b_n \neq 0$	DC term, sine terms and cosine terms are present.
Even	Real (Even in nature)	$a_0 \neq 0$ $a_n \neq 0$ $b_n = 0$	DC term and cosine terms are present.
Odd	Imaginary (odd in nature)	$a_0 = 0$ $a_n = 0$ $b_n \neq 0$	Only sine terms are present.
Half wave symmetry	$C_n = 0$; For n = even	$\left.\begin{matrix} a_n = 0 \\ b_n = 0 \end{matrix}\right\}$; n = even	Odd sine and odd cosine terms are present.
Even and Half wave symmetry	C_n = Real & even $C_n = 0$; n = even	$\left.\begin{matrix} a_n = 0 \\ b_n = 0 \end{matrix}\right\}$; n = even	Only odd cosine terms are present.
Odd and Half wave symmetry	C_n = Imaginary & odd $C_n = 0$; n = even	$\left.\begin{matrix} a_n = 0 \\ b_n = 0 \end{matrix}\right\}$; n = even	Only odd sine terms are present.

Parseval's Power Theorem

The average power P, if x(t) has Fourier series coefficient C_n then Parseval's power theorem is given by

$$\frac{1}{T}\int_0^T |x(t)|^2\, dt = \sum_{n=-\infty}^{\infty} |C_n|^2$$

Properties of Exponential Form of Fourier Series Coefficients

Property	Continuous Time Periodic Signal	Fourier Series Coefficients
Linearity	$A\,x(t) + B\,y(t)$	$AC_n + Bd_n$
Time shifting	$x(t \pm t_0)$	$C_n e^{\pm jm\omega_0 t_0}$
Frequency shifting	$e^{\pm jm\omega_0 t}\,x(t)$	$C_{n \mp m}$
Conjugation	$x^*(t)$	C^*_{-n}
Time reversal	$x(-t)$	C_{-n}
Time scaling	$x(\alpha t)$; $\alpha > 0$ {x(t) is period with period T/α}	C_n (No change in Fourier coefficient
Multiplication	$x(t)\,y(t)$	$\sum_{m=-\infty}^{+\infty} C_m d_{n-m}$
Differentiation	$\frac{d}{dt}x(t)$	$jn\omega_0 t_0$
Integration	$\int_{-\infty}^{t} x(t)\,dt$ (Finite valued and periodic only if $a_0 = 0$)	$\frac{1}{jn\omega_0} C_n$
Periodic Convolution	$\int_T x(\tau)y(t-\tau)d\tau$	$TC_n d_n$
Symmetry of real signals	x(t) is real	$C_n = C^*_{-n}$ $\lvert C_n\rvert = \lvert C_{-n}\rvert$; $\angle C_n = -\angle C_{-n}$ $\mathrm{Re}\{C_n\} = \mathrm{Re}\{C_{-n}\}$ $\mathrm{Im}\{C_n\} = -\mathrm{Im}\{C_{-n}\}$
Real and even	x(t) real and even	C_n are real and even
Real and odd	x(t) real and odd	C_n are imaginary and odd