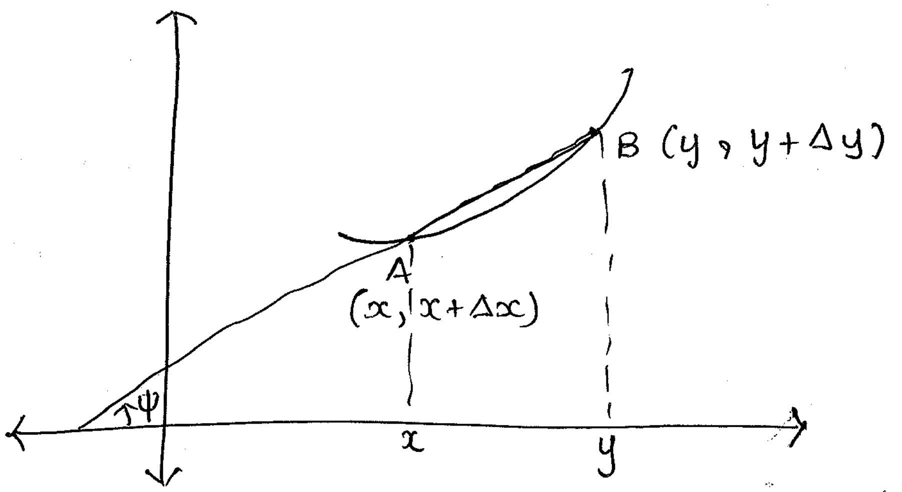
Differentiability

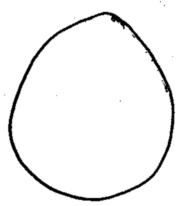
$\frac{dy}{dx}$ is used in minima & maxima.

increasing & decreasing function

⇒ The function $f(x)$ is said to be differentiable at $x = a$ if L.H.D. exists & R.H.D. exists and if both are equal.

$$\text{L.H.D.} = \underbrace{\lim_{h \to 0} \frac{f(a+h) - f(a)}{-h} \qquad \text{R.H.D.} = \lim_{h \to 0}}_{f'(a)} \frac{f(a+h) - f(a)}{h}$$





Volume change then radius changes.

① $\frac{d}{dx} \cdot x^n = n x^{n-1}$

② $\frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}}$

③ $\frac{d}{dx} \log x = \frac{1}{x}$

④ $\frac{d}{dx} a^x = a^x \log a$

⑤ $\frac{d}{dx} \sin x = \cos x$

⑥ $\frac{d}{dx} \cos x = -\sin x$

⑦ $\frac{d}{dx} \tan x = \sec^2 x$

⑧ $\frac{d}{dx} \sec x = \sec x \tan x$

⑨ $\frac{d}{dx} \cot x = -\operatorname{cosec}^2 x$

⑩ $\frac{d}{dx} \operatorname{cosec} x = -\operatorname{cosec} x \cot x$

⑪ $\frac{d}{dx} \sinh x = \cosh x$

⑫ $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$

⑬ $\frac{d}{dx} \cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}$

⑭ $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$

⑮ $\frac{d}{dx} \sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$

⑯ $\frac{d}{dx} x^x = x^x(1+\log x)$

⑰ $\frac{d}{dx} x^{1/x}$ $\cancel{= \frac{1}{x} \cdot x^{1/x-1}}\left(\frac{-1}{x^2}\right)$

$y = x^{1/x}$

$\log y = \frac{1}{x} \log x$

$\frac{1}{y}\frac{dy}{dx} = \log x \cdot \left(\frac{-1}{x^2}\right) + \frac{1}{x^2}$

$\frac{dy}{dx} = x^{1/x}\left[\frac{-\log x + 1}{x^2}\right]$

$\frac{dy}{dx} = x^{1/x}\left(\frac{1-\log x}{x^2}\right)$

Q:- $y = \tan^{-1}\left(\frac{2x}{1+x^2}\right)$. Find $\frac{dy}{dx}$

$x = \tan\theta$

$$y = \tan^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$= \tan^{-1}(\tan 2\theta)$$

$$y = 2\theta = 2\tan^{-1}x$$

$$\frac{dy}{dx} = 2$$

$$\boxed{\frac{dy}{dx} = \frac{2}{1+x^2}}$$

Q:- If $y = x^{x^{x^{x\cdots\infty}}}$ $\frac{dy}{dx} = ?$

$$y = x^{(x+y)}$$

$$\log y = (x+y)\log x$$

$$\frac{1}{y}\frac{dy}{dx} = \log x + \cancel{x}\cdot\frac{1}{\cancel{x}} + \frac{dy}{dx}\log x + \frac{y}{x}$$

$$\frac{dy}{dx}\left(\frac{1}{y} - \log x\right) = \frac{x\log x + y}{x} + 1$$

$$\frac{dy}{dx} = \frac{(x\log x + y + x)\,y}{x\,(1 - y\log x)}$$

$$y = x^y$$

$$\log y = y\log x$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{dy}{dx}\log x + \frac{y}{x}$$

$$\frac{dy}{dx}\left(\frac{1}{y} - \log x\right) = \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{y^2}{x(1 - y\log x)}$$

$$\frac{dy}{dx} = \frac{x^{2y}}{x(1 - x^y\log x)}$$

Q:- $x^m \cdot y^n = (x+y)^{m+n}$. Find $\frac{dy}{dx}$

Sol^n:-

$\cancel{n \cdot y^{n-1} \frac{dy}{dx} \cdot x^m +}$

$$\log(x^m \cdot y^n) = (m+n)\log(x+y)$$

$$m\log x + n\log y = m\log(x+y) + n\log(x+y)$$

$$\frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = \frac{m}{x+y}\left(1+\frac{dy}{dx}\right) + \frac{n \cdot 1}{(x+y)}\left(1+\frac{dy}{dx}\right)$$

$$\frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = \left(1+\frac{dy}{dx}\right)\left[\frac{m+n}{x+y}\right]$$

$$\frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = \frac{m+n}{x+y} + \frac{m+n}{x+y}\left(\frac{dy}{dx}\right)$$

$$\frac{m}{x} - \frac{m+n}{x+y} = \left(\frac{m+n}{x+y} - \frac{n}{y}\right)\frac{dy}{dx}$$

$$\frac{\cancel{mx}+my-\cancel{mx}-nx}{x(\cancel{x+y})} = \left(\frac{my+\cancel{ny}-mx-\cancel{ny}}{y(\cancel{x+y})}\right)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{\cancel{my-nx}}{my-nx}\,\frac{y}{x}$$

$$\boxed{\frac{dy}{dx} = \frac{y}{x}}$$

Q:- $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \dots}}}$ Find $\frac{dy}{dx}$

Sol^n:- $y = \sqrt{\sin x + y}$

$$y^2 = \sin x + y$$

$$2y\frac{dy}{dx} = \cos x + \frac{dy}{dx}$$

$$\boxed{\frac{dy}{dx} = \frac{\cos x}{2y-1}}$$

Q:- If $y = \frac{\sin^{-1}x}{\sqrt{1-x^2}} \Rightarrow (1-x^2)y_1 - xy$

$$y\sqrt{1-x^2} = \sin^{-1}x$$

$$\frac{dy}{dx}\sqrt{(1-x^2)} + y\cdot\frac{1}{\cancel{2}\sqrt{1-x^2}}(-\cancel{2}x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{\frac{dy}{dx}(1-x^2) + -xy}{\sqrt{\cancel{1-x^2}}} = \frac{1}{\sqrt{\cancel{1-x^2}}}$$

$$(1-x^2)y_1 - xy = \underline{1}$$

Q:- $y = a\cos(\log x) + b\sin(\log x) \Rightarrow x^2y_2 + xy_1 = \_\_\_$.

$$y_1 = \frac{a(-\sin(\log x))}{x} + \frac{b\cos(\log x)}{x}$$

$$y_1 x = -a\sin(\log x) + b\cos(\log x)$$

$$y_2 x + y_1 = \frac{-a\cos(\log x)}{x} - \frac{b\sin(\log x)}{x}$$

$$y_2 x^2 + y_1 x = -a\cos(\log x) - b\sin(\log x) = -y$$

Q:- $x = a(\theta + \sin\theta)$ and $y = a(1-\cos\theta)$

then $\frac{dy}{dx} = ?$ [ME-2004]

$$\frac{dx}{d\theta} = a(1+\cos\theta) \qquad \frac{dx}{d\theta} = a[1+\cos\theta]$$

$$\frac{dy}{d\theta} = a\cancel{(6}\cdot \cancel{1+}a\sin\theta$$

$$\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cancel{a(1+\cos\theta)}\ a\sin\theta}{a(1+\cos\theta)}$$

$$= \frac{\cancel{2}\sin\theta/2\ \cancel{\cos\theta/2}}{\cancel{2}\cos^2\theta/2}$$

$$= \tan\theta/2$$

$$\frac{dy}{dx} = \frac{\cancel{a(1+\cos\theta)}}{\cancel{a\sin\theta}}$$

$$\frac{dy}{dx} = \frac{\cancel{a}\ 2\cos^{\cancel{2}}\cancel{\theta/2}}{\cancel{a}\ \cancel{2}\sin\theta/2\ \cancel{\cos\theta/2}} = \cancel{a}\cot\theta/2$$

Q:- If y is the sol^n of $\frac{d^2y}{dx^2} = 0$, with boundary condition $y = 5$ at $x = 0$ and $\frac{dy}{dx} = 2$ at $x = 10$, $f(15) = ?$

Sol^n:-

$$\frac{d^2y}{dx^2} = 0$$

$$\frac{dy}{dx} = c = 2$$

$$y = cx + d$$

$$\boxed{5 = d}$$

$$\boxed{f(y) = 2(15) + 5 = 35}$$

Q :- Let $f(x)$ be polynomial and $g(x) = f'(x)$ be its derivative. If the degree of $[f(x)+f(-x)]$ is 10, then the degree of $(g(x)-g(-x))$ is ____.

Sol^n:- $f(x) = a_{10}x^{10} + a_9x^9 + a_8x^8 + \ldots$

$f(-x) = a_{10}x^{10} - a_9x^9 + a_8x^8 - \ldots$

$f(x) + f(-x) = 2[a_{10}x^{10} + a_8x^8 + a_6x^6 + \ldots] \rightarrow$ degree $= 10$

$g(x) = f'(x) = 10a_{10}x^9 + 9a_9x^8 + 8a_8x^7 + \ldots$

$g(-x) = -10a_{10}x^9 + 9a_9x^8 - 8a_8x^7 + \ldots$

$g(x) - g(-x) = 2[10a_{10}x^9 + 8a_8x^7 + \ldots] \rightarrow$ degree $= 9$.

Q :- $y = \tan^{-1}\left[\frac{(3-x)\sqrt{x}}{1-3x}\right]$ Find $\left(\frac{dy}{dx}\right)_{x=1}$

$x = \tan^2\theta$

$y = \tan^{-1}\left[\frac{(3-\tan^2\theta)\tan\theta}{1-3\tan^2\theta}\right]$

$= \tan^{-1}\left[\frac{(3-\tan^2\theta)\tan\theta}{1-3\tan^2\theta}\right]$

$= \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\right)$

$= \tan^{-1}(\tan 3\theta)$

$y = 3\theta = 3\tan^{-1}\sqrt{x}$

$\frac{dy}{dx} = \frac{3 \cdot 1}{1+x}$ $\qquad \frac{dy}{dx}\Big|_{x=1} = \frac{3}{4}$