APPLICATIONS OF DERIVATIVE (XII, R.S. AGGARWAL)

EXERCISE 11A (Pg. No.: 461)

1. The side of a square in increasing at the rate of 0.2 cm/s. Find the rate of increases of the perimeter of the square.

Sol. Let P be the perimeter of square, a be its side

Given $\frac{da}{dt}=0.2$ cm/s, Now $P=4a$ ... (i)

$\frac{dP}{dt}=4.\frac{da}{dt} \Rightarrow \frac{dP}{dt}=4\times 0.2$ cm/s $\Rightarrow \frac{dP}{dt}=0.8$ cm/s

Hence, the rate of increase of the perimeter of the square is 0.8 cm/s.

2. The radius of a circle is increasing at the rate of 0.7 cm/s. What is the rate of increases of its circumference?

Sol. Let C be the circumference of circle, r be its radius

Given $\frac{dr}{dt}=0.7$ cm/s Now $C=2\pi r$ ... (i)

$\Rightarrow \frac{dC}{dt}=2\pi.\frac{dr}{dt} \Rightarrow \frac{dC}{dt}=2\times 3.14\times 0.7 \Rightarrow \frac{dC}{dt}=4.4$ cm/s

Hence, the rate of increases of its circumference is 4.4 cm/s.

3. The radius of a circle is increasing uniformly at the rate of 0.3 centimeters per second. At what rate is the area increasing when the radius is 10 cm? (Take $\pi=3.14$.)

Sol. Let A be the area of circle, r be its radius

Given $\frac{dr}{dt}=0.3$ cm/s, Now $r=10$ cm, $A=\pi r^2$ ... (1)

$\Rightarrow \frac{dA}{dt}=\pi.\frac{d\left(r^2\right)}{dt} \Rightarrow \frac{dA}{dt}=\pi.2r\frac{dr}{dt} \Rightarrow \frac{dA}{dt}=3.14\times 2\times 10\times 0.3 \Rightarrow \frac{dA}{dt}=18.84 \text{ cm}^2/\text{s}$

Hence, the rate of change area of circle is 18.84 cm^2/s.

4. The side of a square sheet of metal is increasing at 3 centimeters per second. At what rate is the area increasing when the side is 10 cm long?

Sol. A be the area of square, a be its sides of square

Given $\frac{da}{dt}=3$ cm/s, $a=10$ cm Now $A=a^2$ ... (1)

$\Rightarrow \frac{dA}{dt}=2a.\frac{da}{dt} \Rightarrow \frac{dA}{dt}=2\times 10\times 3 \Rightarrow \frac{dA}{dt}=60 \text{ cm}^2/\text{s}$

Hence, the rate of change of area of square is 60 cm^2/s.

5. The radius of circular soap bubble is increasing at the rate of 0.2 cm/s. Find the rate of increase of its surface area when the radius is 7 cm.

Sol. A soap bubble is in the form of a sphere. At an instant t, let its radius be r and surface area S, then

Given $\frac{dr}{dt}=0.2$ cm/sec Now, $S=4\pi r^2$

$\Rightarrow \frac{dS}{dt}=8\pi r.\frac{dr}{dt}=8\pi\times 7\times 0.2 \Rightarrow \left(\frac{dS}{dt}\right)_{r=7}=\left(1.6\pi\times 7\right) \text{ cm}^2/\text{s} \Rightarrow \left.\frac{dS}{dt}\right]_{r=7}=35.2 \text{ cm}^2/\text{s}$

6. The radius of an air bubble is increasing at the rate of 0.5 centimetre per second. At what rate is the volume of the bubble increasing when the radius is 1 centimetre?

Sol. A bubble is in the form of a sphere.

At an instant t, let its radius be r and the volume of sphere V, given information $\frac{dr}{dt} = 0.5$ cm/sec given.

Now, $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = \frac{4}{3}\pi . 3r^2 \frac{dr}{dt} \Rightarrow \frac{dV}{dt} = 4\pi r^2 .(0.5)$

$\Rightarrow \left(\frac{dV}{dt}\right)_{r=1} = 4 \times 3.14 \times (1)^2 .0.5 \quad \therefore \frac{dV}{dt} = 6.28 \text{ cm}^3/\text{s}$

7. The volume of a spherical balloon is increasing at the rate of 25 cubic centimeters per second. Find the rate of change of its surface at the instant when its radius is 5 cm.

Sol. A spherical balloon is in the form of a sphere at an instant t.

Let its radius be r and the volume of sphere V, and S be the surface area given information

$\Rightarrow \frac{dV}{dt} = 25 \text{ cm}^3/\text{s}$

Now, $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = \frac{4}{3}\pi . 3r^2 \frac{dr}{dt} \Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \Rightarrow 25 = 4\pi r^2 \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{25}{4\pi r^2}$

$S = 4\pi r^2 \Rightarrow \frac{dS}{dt} = 4\pi \,.\, 2r \frac{dr}{dt} \Rightarrow \frac{dS}{dt} = 4\pi \,.\, 2r \,.\, \frac{25}{4\pi r^2} \Rightarrow \frac{dS}{dt} = \frac{50}{r}$

$\therefore \left(\frac{dS}{dt}\right)_{r=5} = \frac{50}{5} = 10 \text{ cm}^2/\text{sec}.$

Hence, the rate of change of surface area at the instant when $r = 5$ cm/s is $10 \text{ cm}^2/\text{sec}$

8. A balloon which always remains spherical is being inflated by pumping in 900 cubic centimeters of gas per second. Find the rate at which the radius of the balloon is increasing when the radius is 15 cm.

Sol. A spherical balloon is in the form of a sphere at an instant t.

Let its radius be r and the volume of the sphere V. $\frac{dV}{dt} = 900 \text{ cm}^3/\text{sec}$

Now, $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = \frac{4}{3}\pi . 3r^2 \frac{dr}{dt} \Rightarrow 900 = 4\pi r^2 \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{900}{4\pi r^2}$

$\Rightarrow \frac{dr}{dt} = \frac{225}{\pi r^2} \Rightarrow \left(\frac{dr}{dt}\right)_{r=15} = \frac{225}{\pi (15)^2} \Rightarrow \frac{dr}{dt} = \frac{1}{\pi} \quad \therefore \frac{dr}{dt} = 0.32 \text{ cm/s}$

Hence, the rate of change instant when $r = 15$ cm is 0.32 cm/s.

9. The bottom of a rectangular swimming tank is 25 m by 40 m. Water is pumped into the tank at the rate of 500 cubic metres per minute. Find the rate at which the level of water in the tank is rising.

Sol. The balloon of a rectangle swimming tank be l, b and h. At an instant t, let its radius be r and the volume of sphere V, $\frac{dV}{dt} = 500$ cubic metre per minute.

Now, $V = l \times b \times h \Rightarrow \frac{dV}{dt} = 25 \times 40 \times \frac{dh}{dt} \Rightarrow 500 = 25 \times 40 \times \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{500}{25 \times 40}$

$\Rightarrow \frac{dh}{dt} = \frac{1}{2} \quad \therefore \frac{dh}{dt} = 0.5 \text{ m/min}$

Hence, the rate at which the level of water in the tank is rising 0.5 m/min

10. A stone is dropped into a quiet lake and waves move in circles at a speed of 3.5 cm per second. At the instant when the radius of the circular wave is 7.5 cm, how fast is the enclosed area increasing? (Take $\pi = 22/7$.)

Sol. At any instant t, let the radius of the circle be r cm and its area be A then, $\frac{dr}{dt} = 3.5$ cm/s

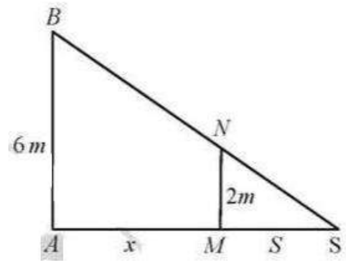
Now, $A = \pi r^2 \Rightarrow \frac{dA}{dt} = \pi \cdot 2r \cdot \frac{dr}{dt} \Rightarrow \frac{dA}{dt} = \pi . 2r . 3.5 \Rightarrow \frac{dA}{dt} = 7\pi r$

$\Rightarrow \left[\frac{dA}{dt}\right]_{(r=7.5)} = (7\pi \times 7.5) cm^2/s \quad \therefore \frac{dA}{dt} = 165 \text{ cm}^2/\text{s}$

Hence, the area of the circle is increasing at the rat of 165 cm^2/s at the instant when $r = 7.5$ cm

11. A 2-m-tall man walks at a uniform speed of 5 km per hour away from a 6-metre-high lamp post. Find the rate at which the length of his shadow increases.

Sol. Let AB be the lamp post the lamp being at B.



Then $AB = 6$ m

At any instant t, let MN be the position of the man and MS be his shadow, then MN = 2m let $AM = x$ meter and $MS = S$ meter It is given that $\frac{dx}{dt} = 5$ km/hr

Clearly, $\Delta SAB \sim \Delta SMN$

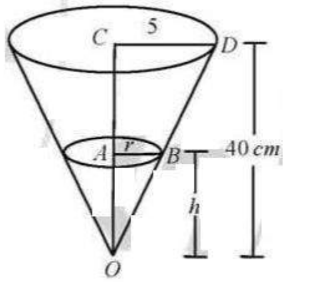
$\therefore \frac{AS}{MS} = \frac{AB}{MN} \Rightarrow \frac{x+S}{S} = \frac{6}{2} \Rightarrow 2x + 2S = 6S \Rightarrow 2x = 4S$

$\Rightarrow x = 2S \Rightarrow \frac{dx}{dt} = 2.\frac{dS}{dt} \Rightarrow 5 = 2.\frac{dS}{dt} \Rightarrow 2.5 = \frac{dS}{dt} \quad \therefore \frac{dS}{dt} = 2.5$ km/h

Hence, the length of the shadow is increasing at the rate of 2.5 km/h

12. An inverted cone has a depth of 40 cm and a base of radius 5 cm. Water is poured into it rate a of 1.5 cubic centimeters per minute. Find the rate which the level of water in the cone is rising when the depth is 4 cm.

Sol. At any instant t, let r be the radius of the water level, h be the height of the water level and V be the volume of the water in the conical funnel then.



$\frac{dV}{dt} = 1.5$ (given) ..(1)

From similar ΔOAB and ΔOCD, we have

$\frac{AB}{OA} = \frac{CD}{OC} \Rightarrow \frac{r}{h} = \frac{5}{40} \Rightarrow r = \frac{1}{8}h.$

Now, $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{1}{8}h\right)^2 h = \frac{1}{192}h^3\pi$

$\therefore \frac{dV}{dt} = \frac{1}{192}\pi . 3h^2 \frac{dh}{dt} \Rightarrow 1.5 = \frac{\pi h^2}{64}\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{96}{\pi h^2} \quad [\because h = 4\text{cm}]$

$\Rightarrow \left[\frac{dh}{dt}\right]_{(h=4\text{cm})} = \frac{96}{\pi(4)^2} = \frac{96}{\pi 16} = \frac{6}{\pi}$ cm/min.

13. Sand is pouring from a pipe at the rate of 18 cm^3/s. The falling sand forms a cone on the ground in such a way that the height of the cone is one-sixth of the radius of the base. How fast is the height of the sand cone increasing when its height is 3 cm?

Sol. At any time t, let r be the radius, h the height and V volume of the cone then given information

$\Rightarrow h = \frac{r}{6}$

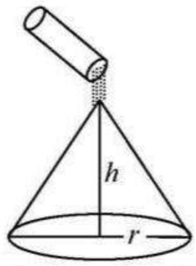
$\Rightarrow r = 6h \quad$ Now $V = \frac{1}{3}\pi r^2 h \Rightarrow V = \frac{1}{3}\pi(6h)^2 h = 12\pi h^3$

Now, $V = 12\pi h^3 \Rightarrow \frac{dV}{dt} = 12\pi . 3h^2 \frac{dh}{dt}$

$$\Rightarrow \frac{dV}{dt} = 36\pi h^2 \frac{dh}{dt} \quad \Rightarrow 18 = 36\pi h^2 \frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{18}{36\pi h^2} \quad \Rightarrow \frac{dh}{dt} = \frac{1}{2\pi h^2}$$

$$\Rightarrow \left[\frac{dh}{dt}\right]_{(h=3)} = \frac{1}{2\pi(3)^2} \quad \therefore \frac{dh}{dt} = \frac{1}{18\pi} \text{ cm/s}$$



Hence, the rate of increases of the height of the sand cone at the instant when $h = 3$ is $\frac{1}{18\pi}$ cm/sec.

14. Water is dripping through a tiny hole at the vertex in the bottom of a conical funnel at a uniform rate of 4 cm^3/s. When the slant height of the water is 3 cm, find the rate of decreases of the slant height of the water, given that the vertical angle of the funnel is 120°.

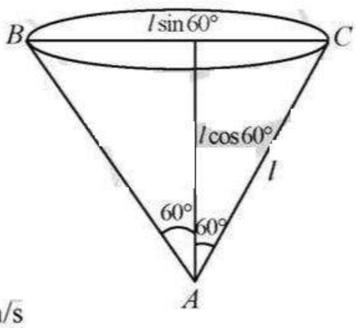
Sol. Let V be volume, of conical funnel and l be slant height. At an instant t, let its radius be r, $\frac{dV}{dt} = -4$ cm^3/s. - ve sign is taken $\therefore$ volume is decres with time

Now, $V = \frac{1}{3}\pi r^2 h \quad \Rightarrow V = \frac{1}{3}\pi(l\sin 60°)^2(l\cos 60)$

$$\Rightarrow V = \frac{\pi}{3}\left(l.\frac{\sqrt{3}}{2}\right)^2\left(l.\frac{1}{2}\right) \quad \Rightarrow V = \frac{\pi}{8}l^3$$

$$\Rightarrow \frac{dV}{dt} = \frac{\pi}{8}3l^2\frac{dl}{dt} \quad \Rightarrow -4 = \frac{3\pi l^2}{8}.\frac{dl}{dt} \quad \Rightarrow \frac{dl}{dt} = \frac{-32}{3\pi l^2}$$

$$\therefore \left[\frac{d\ell}{dt}\right]_{l=3} = \frac{-32}{3\pi(3)^2} = \frac{-32}{27\pi} \text{ cm/s.}$$

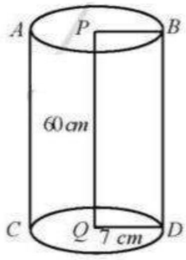


Hence the rate of decrease of the slant height of the water is $\frac{32}{27\pi}$ cm/s

15. Oil is leaking at the rate of 16mL/s from a vertically kept cylindrical drum containing oil. If the radius of the drum is 7 cm and its height is 60 cm, find the rate at which the level of the oil is changing when the oil level is 18 cm.

Sol. Let $ABCD$ be the vertically kept cylindrical drum connecting oil and QD the radius of the drum is 7 cm and its PQ is height of 60 cm and V be its volume.

$\Rightarrow r = 7$ cm, $h = 60$ cm, $\frac{dV}{dt} = -16$ cm^3/sec -ve sign is taken because volume is decreasing with time



Now, $V = \pi r^2 h \quad \Rightarrow V = \pi(7)^2 h \quad \Rightarrow V = 49\pi h$

$$\Rightarrow \frac{dV}{dt} = 49\pi.\frac{dh}{dt} \quad \Rightarrow -16 = 49\pi\frac{dh}{dt} \quad \therefore \frac{dh}{dt} = \frac{-16}{49\pi} \text{ cm/s}$$

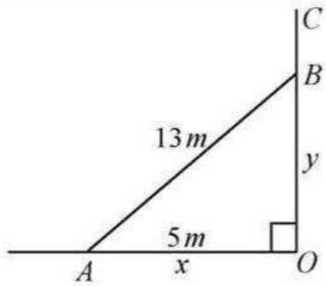
Hence, the rate at which the level of the oil is decreasing $\frac{16}{49\pi}$ cm/s.

16. A 13-m-long ladder is leaning against a wall. The bottom of the ladder is pulled along the ground, away form the wall, at the rate of 2m/s. How fast is its height on the wall decreasing when the foot of the ladder is 5m away from the wall?

Sol. Let OC be the wall. At a certain instant t, let AB be the position of the ladder such that $OA = x$ and $OB = y$.

$\Rightarrow$ Length of the ladder AB = 13 m. Given that $\frac{dx}{dt} = 2$ m/sec

From right $\Delta\ AOB$, we have, $(AO)^2+(OB)^2=(AB)^2$



$\Rightarrow x^2+y^2=(13)^2 \quad \Rightarrow x^2+y^2=169$

$\Rightarrow 2x.\dfrac{dx}{dt}+2y.\dfrac{dy}{dt}=0 \ \Rightarrow 2x.(2)+2y\dfrac{dy}{dt}=0 \ \left[\because \dfrac{dx}{dt}=2\right]$

$\Rightarrow \dfrac{dy}{dt}=-\dfrac{2x}{y} \qquad \ldots(1)$

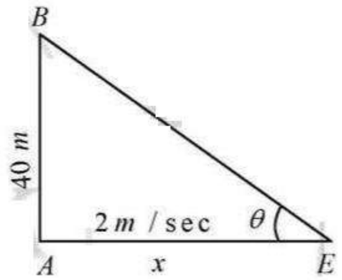
Now, $x=5 \Rightarrow \ y=\sqrt{(13)^2-(5)^2}=\sqrt{144}=12$

Now Putting $x=5,\ y=12$ in (1), we get $\dfrac{dy}{dt}=\dfrac{-2(5)}{12} \quad \Rightarrow \dfrac{dy}{dt}=\dfrac{-5}{6}$

Hence, the rate of decrease in the height of the ladder on the wall is $\dfrac{5}{6}$ m/s.

17. A man is moving away from 40-m-high tower at a speed of 2m/s. Find the rate at which the angle of elevation of the top of the tower is changing when he is at a distance of 30 metres from the foot of the tower. Assume that the eye level of the man is 1.6 m from the ground.

Sol. Let any time t, the man be at a distance of x metres from tower AB and let θ be the angle of elevation at that time. Then, $x=40\ \cot\theta$



$\Rightarrow \dfrac{dx}{dt}=-40\ \text{cosec}^2\theta\dfrac{d\theta}{dt} \quad \Rightarrow 2=-40\,\text{cosec}^2\theta\dfrac{d\theta}{dt}$

$\Rightarrow \dfrac{d\theta}{dt}=-\dfrac{1}{20}\sin^2\theta \quad \Rightarrow \dfrac{d\theta}{dt}=-\dfrac{1}{20}\left(\dfrac{AB}{BE}\right)^2$

$\Rightarrow \dfrac{d\theta}{dt}=-\dfrac{1}{20}\left\{\dfrac{(40)^2}{(40)^2+x^2}\right\}$ at $x=30 \quad \Rightarrow \dfrac{d\theta}{dt}=-\dfrac{1}{20}\left\{\dfrac{1600}{2500}\right\}$

$\Rightarrow \dfrac{d\theta}{dt}=-0.032$ radian/second

Hence, the rate at which the angle of elevation of the top of the tower is changing when he is at a distance of 30 meter from the foot of the tower is 0.032 radian/second

18. Find the angle x which increases twice as fast as its sine.

Sol. Let the angle be x which increase twice as fast at its sine $\Rightarrow \dfrac{d}{dt}(x)=\dfrac{d}{dt}(2\sin x)$

$\Rightarrow \dfrac{dx}{dt}=2\cos x.\dfrac{dx}{dt} \qquad \Rightarrow \cos x=\dfrac{1}{2} \qquad \Rightarrow \cos x=\cos\dfrac{\pi}{3} \qquad \therefore\ x=\dfrac{\pi}{3}$

19. The radius of a balloon is increasing at the rate of 10 cm/s. At what rate is the surface area of the balloon increasing when the radius is 15 cm?

Sol. At any instant t, let r be the radius, S the surface area of the balloon then $S=4\pi r^2$ and given $\dfrac{dr}{dt}=10$ cm/s

$\Rightarrow \dfrac{dS}{dt}=4\pi.2r.\dfrac{dr}{dt} \quad \Rightarrow \dfrac{dS}{dt}=(8\pi\times15\times10)\text{cm}^2/\text{s} \quad \therefore\ \dfrac{dS}{dt}=1200\pi\ \text{cm}^2/\text{s}$

Hence the rate at which the surface area of the balloon increasing is 1200 π cm^2/s.

20. An edge of a variable cube is increasing at the rate of 5 cm/s. How fast is the volume of the cube increasing when the edge is 10 cm long?

Sol. At any instant t, let the length of each edge of the cube be x, v be its volume

Now, $v=x^3 \quad \Rightarrow \dfrac{dv}{dt}=3x^2\dfrac{dx}{dt} \quad \Rightarrow \dfrac{dv}{dt}=3x^2.5 \quad \Rightarrow \dfrac{dv}{dt}=15x^2$

$\Rightarrow \left[\frac{dv}{dt}\right]_{t=10} = 15(10)^2 \qquad \therefore \frac{dv}{dt} = 1500 \text{ cm}^3/\text{s}$

Hence, the volume of the cube is increasing at the rate of $1500 \text{ cm}^3/\text{s}$.

21. the sides of an equilateral triangle are increasing at the rate of 2 cm/sec. Find the rate at which the area is increasing when the side is 10 cm

Sol. Let the side of an equilateral triangle be a and its area be A

$\therefore \frac{da}{dt} = 2 \text{ cm/sec}$

Area of equilateral triangle $= \frac{\sqrt{3}}{4}a^2$

Sides = 10 cm

$\because \frac{dA}{dt} = ?$

Differentiate both sides with respect to t $\frac{dA}{dt} = \frac{\sqrt{3}}{4}\frac{d}{dt}(a^2)$

$\Rightarrow \frac{dA}{dt} = \frac{\sqrt{3}}{4} \times 2a \times \frac{da}{dt} \Rightarrow \frac{dA}{dt} = \frac{\sqrt{3}a}{2} \times \frac{da}{dt} \Rightarrow \frac{dA}{dt} \times \frac{dt}{da} = \frac{\sqrt{3}}{a} \Rightarrow \frac{dA}{dt} \times \frac{1}{2} = \frac{\sqrt{3}}{2}a \Rightarrow \frac{dA}{dt} = \frac{\sqrt{3}}{2} \times 10 \times 2$

$\Rightarrow \frac{dA}{dt} = 10\sqrt{3} \ cm^2 s$

Hence Required area is increasing at the rate of $10\sqrt{3} \ cm^2/s$

EXERCISE 11B (Pg.No.: 469)

Using differentials, find the approximate values of:

1. $\sqrt{37}$

Sol. Let $f(x)=x^{1/2}$, then $f'(x)=\frac{1}{2x^{1/2}}$. Now $\{f(x+\delta x)-f(x)\}=f'(x).\delta x$

$\Rightarrow \{f(x+\delta x)-f(x)\}=\frac{1}{2x^{1/2}}.\delta x \qquad ...(1)$

We may write $37=36+1$, Hence putting $x=36$ and $\delta x=1$ in (1) we get

$f(36+1)-f(36)=\frac{1}{2(36)^{1/2}}.1 \Rightarrow f(37)-f(36)=\frac{1}{2.6}$

$\Rightarrow f(37)=\left\{f(36)+\frac{1}{12}\right\} \Rightarrow f(37)=\left\{(36)^{\frac{1}{2}}+\frac{1}{12}\right\} \Rightarrow f(37)=\left(6+\frac{1}{12}\right)$

$\Rightarrow f(37)=6+0.0833 \quad \therefore f(37)=6.083$

2. $\sqrt[3]{29}$

Sol. Let $f(x)=x^{1/3}$ then $f'(x)=\frac{1}{3x^{2/3}}$. Now $\{f(x+\delta x)-f(x)\}=f'(x).\delta x$

$\Rightarrow \{f(x+\delta x)-f(x)\}=\frac{1}{3x^{2/3}}.\delta x \qquad ...(1)$

We many write $29=27+2$, Hence putting $x=27$ and $\delta x=2$ in equation (1), we get

$f(27+2)-f(27)=\frac{2}{3(27)^{2/3}} \Rightarrow f(29)-f(27)=\frac{2}{27} \Rightarrow f(29)=(27)^{1/3}+\frac{2}{27}$

$\Rightarrow f(29)=3+\frac{2}{27} \Rightarrow f(29)=3+0.074 \quad \therefore f(29)=3.074$

3. $\sqrt[3]{127}$

Sol. Let $f(x)=x^{1/3}$ then $f'(x)=\frac{1}{3x^{2/3}}$. Now $\{f(x+\delta x)-f(x)\}=f'(x).\delta x$

$\Rightarrow \{f(x+\delta x)-f(x)\}=\frac{1}{3x^{2/3}}.\delta x \qquad ...(1)$

We many write, $127=125+2$, Hence putting $x=125$ and $\delta x=2$ in equation (1), we get

$f(125+2)-f(125)=\frac{1}{3\times(125)^{2/3}}\times 2 \Rightarrow f(127)-f(125)=\frac{2}{75}$

$\Rightarrow f(127)=\left\{(125)^{1/3}+\frac{2}{75}\right\} \Rightarrow f(127)=\left(5^3\right)^{\frac{1}{3}}+\frac{2}{75} \Rightarrow f(127)=5+\frac{2}{75}$

$\Rightarrow f(127)=\frac{377}{75} \quad \therefore f(127)=5.026$

4. $\sqrt{0.24}$

Sol. Let $f(x)=x^{1/2}$ then $f'(x)=\frac{1}{2x^{1/2}}$. Now $\{f(x+\delta x)-f(x)\}=f'(x).\delta x$

$$\Rightarrow \{f(x+\delta x)-f(x)\}=\frac{1}{2x^{1/2}}.\delta x \qquad \ldots(1)$$

We may write $0.24=0.25+(-0.01)$, Hence putting $x=0.25$ and $\delta x=-0.01$ in equation (1), we get

$$f(0.25-0.01)-f(0.25)=\frac{1}{2(0.25)^{1/2}}(-0.01)$$

$$\Rightarrow f(0.24)-f(0.25)=\frac{1}{2\times 0.5}\times(-0.01) \quad \Rightarrow f(0.24)=f(0.25)+\left(\frac{-0.01}{1}\right)$$

$$\Rightarrow f(0.24)=(0.25)^{1/2}-0.01 \quad \Rightarrow f(0.24)=0.5-0.01 \quad \therefore f(0.24)=0.49$$

5. $\sqrt{49.5}$

Sol. Let $y=\sqrt{x}$

Let $x=49$ and $\Delta x=0.5$

Since $y=\sqrt{x}$

$$\frac{dy}{dx}=\frac{d(\sqrt{x})}{dx}=\frac{1}{2\sqrt{x}}$$

Now, $\Delta y=\frac{dx}{dy}\Delta x=\frac{1}{2\sqrt{x}}(0.5) =\frac{1}{2\sqrt{49}}(0.5)=\frac{1}{2\times 7}\times 0.5=\frac{0.5}{14}=0.036$

Also, $\Delta y=f(x+\Delta x)-f(x)$

Putting values

$\Delta y=\sqrt{x+\Delta x}-\sqrt{x}$

$0.036=\sqrt{49+0.5}-\sqrt{49}$

$0.036=\sqrt{49.5}-7$

$0.36+7=\sqrt{49.5}$

$\sqrt{49.5}=7.036$

Hence approximate value $\sqrt{49.5}$ is 7.036

6. $\sqrt[4]{15}$

Sol. Let $f(x)=x^{1/4}$ Then, $f'(x)=\frac{1}{4x^{3/4}}$. Now, $\{f(x+\delta x)-f(x)\}=f'(x).\delta x$

$$\Rightarrow \{f(x+\delta x)-f(x)\}=\frac{1}{4x^{3/4}}.\delta x \qquad \ldots(1)$$

We may write $15=16-1$, Hence putting $x=16$ and $\delta x=-1$ in equation (1), we get

$$f(16-1)-f(16)=\frac{1}{4(16)^{3/4}}.-1 \Rightarrow f(15)-f(16)=\frac{-1}{4\times 8} \quad \Rightarrow f(15)=f(16)-\frac{1}{32}$$

$$\Rightarrow f(15)=(16)^{1/4}+-\frac{1}{32} \quad \Rightarrow f(15)=2-\frac{1}{32} \quad \Rightarrow f(15)=2-0.03125 \quad \therefore f(15)=1.96875$$

7. $\dfrac{1}{(2.002)^2}$

Sol. Let $f(x)=\dfrac{1}{x^2}$ then $f'(x)=\dfrac{-2}{x^3}$. Now $\{f(x+\delta x)-f(x)\}=f'(x).\delta x$

$$\{f(x+\delta x)-f(x)\}=\frac{1}{-2/x^3}.\delta x \quad \Rightarrow \{f(x+\delta x)-f(x)\}=\frac{2}{-x^3}.\delta x \qquad \ldots(1)$$

We may write 2.002 = 2 + 0.002, Hence putting $x=2$ and $\delta x=0.002$ in equation (1), we get

$$f(2+0.002)-f(2)=\frac{2}{-(2)^3}.(0.002) \quad \Rightarrow f(2.002)=f(2)-\frac{(0.002)}{4}$$

$$\Rightarrow f(2.002)=\frac{1}{(2)^2}-0.0005 \quad \Rightarrow f(2.002)=\frac{1}{4}-0.0005 \quad \therefore\ f(2.002)=0.2495$$

8. $\log_e 10.02$, given that $\log_e 10=2.3026$

Sol. Let $f(x)=\log_e x$ then $f'(x)=\dfrac{1}{x}$.

Now $f\{(x+\delta x)-f(x)\}=f'(x)\delta x \quad \Rightarrow f(x+\delta x)-f(x)=\dfrac{1}{x}.\delta x$

We may write 10.02 = 10 + 0.02, Hence putting $x=10$ and $\delta x=0.02$ in (1) we get

$$f(10+0.02)-f(10)=\frac{1}{10}(0.02) \quad \Rightarrow f(10.02)=\frac{0.02}{10}+\log_e 10$$

$$\Rightarrow f(10.02)=0.002+2.3026 \qquad \therefore\ f(10.02)=2.3046$$

9. $\log_{10}(4.04)$, it being given that $\log_{10} 4=0.6021$ and $\log_{10} e=0.4343$

Sol. Let $f(x)=\log x \Rightarrow f(x+\delta x)-f(x)=f'(x)\cdot\delta x \Rightarrow \log_e(4.04)-\log 4=0.01$

$$\Rightarrow \frac{\log_{10} 4.04}{\log_{10}^e}-\frac{\log_{10} 4}{\log_{10}^e}=0.01 \Rightarrow \log_{10} 4.04-0.6021=0.01\times .4343 \Rightarrow \log_{10} 4.04=0.606443.$$

10. cos 61°, it is being given that sin 60° = 0.8663 and 1° = 0.01745 radian

Sol. $y=\cos x,\ x=60^\circ=\left(\dfrac{\pi}{3}\right)^\circ,\ x+\delta x=61^\circ$ and $\delta x=1^\circ=0.01745$

$$\frac{dy}{dx}=-\sin x \quad \Rightarrow \delta y=-\sin 60^\circ\times 0.01745=\frac{-0.01745}{2}\times\sqrt{3}=-0.0872\times 1.732$$

$$y+\delta y=\frac{1}{2}+(-0.151)=0.5-0.151=0.4849$$

11. If $y=\sin x$ and x changes from $\dfrac{\pi}{2}$ to $\dfrac{22}{14}$, what is the approximate changes in y?

Sol. Take $x=\dfrac{\pi}{2},\ x+\delta x=\dfrac{22}{14} \quad \therefore\ \delta x=\dfrac{22}{14}-\dfrac{\pi}{2}$

Now, $y=\sin x \Rightarrow \dfrac{dy}{dx}=\cos x \quad \Rightarrow \left[\dfrac{dy}{dx}\right]_{\left(x=\frac{\pi}{2}\right)}=0$

$\therefore\ \delta y=\dfrac{dy}{dx}.\delta x=0 \quad \Rightarrow \delta y=0\left(\dfrac{22}{14}-\dfrac{\pi}{2}\right)=0 \quad \Rightarrow \delta y=0$. Hence there is no change.

12. A circular metal plate expands under heating so that its radius increases by 2%. Find the approximate increase in the area of the plate, if the radius of the plate before heating is 10 cm.

Sol. Let at any time t, r be the radius and A be the area of the plate, then $A=\pi r^2$

Let δr be the change in the radius and let δA be the corresponding in the area of the plate then

$$\delta A=\frac{dA}{dr}.\delta r \quad \Rightarrow \delta A=2\pi r.\delta r \quad \Rightarrow \delta A=2\pi\times 10\times\frac{2}{100}=\frac{4\pi}{10}=\frac{2\pi}{5}\text{ cm}^2 .$$

13. If the length of a simple pendulum is decreased by 2%, find the percentage decreases in its period T, where $T=2\pi\sqrt{\frac{l}{g}}$.

Sol. $T=2\pi\sqrt{\frac{l}{g}}$ ...(1) Taking log both side we get, $\log T=\log\left\{2\pi.\left(\frac{l}{g}\right)^{1/2}\right\}$

$$\Rightarrow \log T=\log(2\pi)+\log\left\{\left(\frac{l}{g}\right)^{1/2}\right\} \quad \Rightarrow \log T=\log(2\pi)+\frac{1}{2}\log\left(\frac{l}{g}\right)$$

$$\Rightarrow \log T=\log 2\pi+\frac{1}{2}\log(l)-\frac{1}{2}\log g \quad \Rightarrow \frac{1}{T}.\frac{dT}{dl}=0+\frac{1}{2}.\frac{1}{l}-0 \quad \Rightarrow \frac{1}{T}\frac{dT}{dl}=\frac{1}{2l}$$

$$\therefore\ \delta T=\frac{T}{2l}.\delta l \quad \Rightarrow \left(\frac{\delta T}{T}\times 100\right)=\left(\frac{1}{2}.\frac{\delta l}{l}\times 100\right)=\left\{\frac{1}{2}\times\frac{(-2)}{100}\times 100\right\}=-1$$

$\therefore$ Percentage decrease in $T=1\%$

14. The pressure p and the volume V of a gas are connected by the relation, $pV^{1/4}=k$, where k is a constant. Find the percentage increase in the pressure, corresponding to a diminution of 0.5% in the volume.

Sol. Given that % decrease in volume $=\frac{1}{2} \quad \Rightarrow \frac{dV}{V}(100)=\frac{-1}{2}$

Given $pV^{\frac{1}{4}}=a \quad \Rightarrow \log pV^{\frac{1}{4}}=\log a \quad \Rightarrow \log p+\frac{1}{4}\log V=\log a \quad \Rightarrow \frac{dp}{p}+\frac{1}{4}.\frac{dV}{V}=0$

$\Rightarrow \frac{dp}{p}.100=-\frac{1}{4}.\frac{dV}{V}.100=\left(-\frac{1}{4}\right)\left(-\frac{1}{2}\right)=\frac{1}{8}$. Hence % increase in pressure $=\frac{1}{8}\%=0.125\%$

15. The radius of a sphere shrinks from 10 cm to 9.8 cm. Find approximately the decrease in (i) volume, and (ii) surface area.

Sol. **(i)** Let r be radius and V be the volume than $y=\frac{4}{3}\pi r^3$

Let $r=10,\ r+\Delta r=9.8 \Rightarrow dr=\Delta r=-0.2$

Now $V=\frac{4}{3}\pi r^3 \quad \Rightarrow \frac{dV}{dr}=4\pi r^2 \quad \Rightarrow \left[\frac{dV}{dr}\right]_{x=10}=400\pi$

$\therefore\ \delta V=\frac{dV}{dr}.\delta r \quad \Rightarrow \delta V=400\pi(-0.2)=-80\pi\text{ cm}^3$

Decrease in its volume $=80\pi\ cm^3$.

(ii) Let r be the radius and S be the surface area $=4\pi r^2$

Let $r=10$ and $r+\Delta r=9.8 \Rightarrow \Delta r=-0.2$

Now, $S=4\pi r^2 \quad \Rightarrow \frac{dS}{dr}=8\pi r \quad \Rightarrow \left[\frac{dS}{dr}\right]_{r=10}=80\pi$

$\therefore\ \delta S = \frac{dS}{dr}.\delta r = \delta S = 80\pi(-0.2) = -16\pi \quad \therefore\ \delta S = -16\pi$

Decrease in its surface area $= 16\pi\ cm^2$.

16. If there is an error of 0.1% in the measurement of the radius of a sphere, find approximately the percentage error in the calculation of the volume of the sphere.

Sol. Let r be the radius of sphere Hence $V = \frac{4}{3}\pi r^3$

Let Δr be the error in r and Δy be the corresponding error in y.

Then $\frac{\Delta r}{r}\times 100 = 0.1$ (given) $\Rightarrow \frac{dr}{r}\times 100 = 0.1$

We have to find $\frac{\Delta V}{V}\times 100$. Now $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dy}{dr} = 4\pi r^2 \Rightarrow \delta V = \frac{dy}{dr}.\delta r \Rightarrow \frac{\delta V}{V} = \frac{4\pi r^2}{V}\delta r$

$\Rightarrow \frac{\delta V}{V} = \frac{4\pi r^2}{\frac{4}{3}\pi r^3}\delta r \Rightarrow \frac{\delta V}{V} = \frac{3\delta r}{r} \Rightarrow \frac{\delta V}{V}\times 100 = 3\left(\frac{\delta r}{r}\times 100\right) \Rightarrow \frac{\delta V}{V}\times 100 = 0.30\%$

$\therefore$ Percentage error in volume of sphere $= 0.3\%$

17. Show that the relative error in the volume of a sphere, due to an error in measuring the diameter, is three times the relative error in the diameter.

Sol. Let r be the radius of sphere and v be the volume of the sphere $V = \frac{4}{3}\pi(2r)^3$

$V = \frac{32}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = \frac{32}{3}.\pi.3r^2 = 32\pi r^2 \Rightarrow \delta V = \frac{dV}{dr}.\delta r \Rightarrow \delta V = 32\pi r^2\delta r$

$\Rightarrow \frac{\delta V}{V} = \frac{32\pi r^2}{V}\delta r \Rightarrow \frac{\delta V}{V} = \frac{32\pi r^2}{\frac{32}{3}\pi r^3}\delta r \Rightarrow \frac{\delta V}{V} = 3.\frac{\delta r}{r} = 3\frac{\delta d}{d}$

Relative error in volume of sphere $= 3$ (relative error in diameter of the sphere)

ROLLE'S AND LAGRANGE'S THEOREMS (XII, R.S. AGGARWAL)

EXERCISE 11 C [pg. no. 483]

Verify Rolle's theorem for each of the following functions:

1. $f(x)=x^2$ on $[-1, 1]$

Sol. Here, we observe that

(i) $f(x)$ being a polynomial function of x, is continuous on the interval $[-1, 1]$

(ii) $f'(x)=2x$, which clearly exits for all value of $x\in(-1, 1)$

So, $f(x)$ is differentiable on the open interval $(-1, 1)$

(iii) $f(-1)=(-1)^2=1$ and $f(1)=(1)^2=1 \quad \therefore\ f(-1)=f(1)$

Thus all the condition of Rolle's theorem are satisfied.

So, these must exists same $c\in(-1, 1)$ such that $f'(c)=0$

Now, $f'(c)=0 \quad \Rightarrow 2c=0 \quad \Rightarrow c=0$

Clearly, value of c lie in the interval $(-1, 1)$ hence, Rolle's theorem is verified.

2. $f(x)=x^2-x-12$ in $[-3, 4]$

Sol. Here, we observe that

(i) $f(x)$ being polynomial function of x, is continuous on the interval $[-3, 4]$

(ii) $f'(x)=2x-1$, which clearly exist for all value of $x\in(-3, 4)$

So, $f(x)$ is differentiable on the open interval $(-3, 4)$

(iii) $f(-3)=(-3)^2-(-3)-12=0$; $\quad f(4)=(4)^2-4-12=0 \quad \therefore\ f(-3)=f(4)$

Thus, all the conditions of Rolle's theorem are satisfied

So, there must exist some $c\in(-3, 4)$ such that $f'(c)=0$;

Now, $f'(c)=0 \quad \Rightarrow 2c-1=0 \qquad \Rightarrow 2c=1 \quad \Rightarrow c=\dfrac{1}{2}$

Clearly, the value of c lies in the interval $(-3,4)$. Hence, Rolle's theorem is verified.

3. $f(x)=x^2-5x+6$ in $[2, 3]$

Sol. Here, we observe that

(i) $f(x)$ being a polynomial function of x, is continuous on the interval $[2, 3]$.

(ii) $f'(x)=2x-5$, which clearly exists for all value of $x\in(2, 3)$.

So, $f(x)$ is differentiable on the open interval $(2, 3)$.

(iii) $f(2)=(2)^2-5\times2+6=0$; $\quad f(3)=(3)^2-5\times3+6=0 \quad \therefore\ f(2)=f(3)$

Thus all the condition of Rolle's theorem are satisfied.

So these must exist some $c\in(2, 3)$ such that $f'(c)=0$.

Now, $f'(c)=0 \quad \Rightarrow 2c-5=0 \quad \Rightarrow c=\frac{5}{2}$

Clearly, the value of c lies in the interval $(2, 3)$. Hence, Rolle's theorem is verified.

4. $f(x)=x^2-3x-18$ in $[-3, 6]$.

Sol. Here, we observe that

(i) $f(x)$ is being a polynomial function of x, is continuous on the interval $[-3, 6]$

(ii) $f'(x)=2x-3$, which clearly exist for all value of $x \in (-3, 6)$.

So, $f(x)$ is differentiable on the open interval $(-3, 6)$

(iii) $f(-3)=(-3)^2-3(-3)-18=0$; $f(6)=(6)^2-3(6)-18=0 \quad \therefore f(-3)=f(6)$

Thus all the condition of Rolle's theorem are satisfied.

So, there must exist some $c \in (-3, 6)$ such that $f'(c)=0$

Now $f'(c)=0 \quad \Rightarrow 2c-3=0 \quad \Rightarrow c=\frac{3}{2}$

Clearly, the value of c lie in the interval $(-3, 6)$. Hence, Rolle's theorem is verified.

5. $f(x)=x^2-4x+3$ in $[1, 3]$

Sol. Here, we observe that

(i) $f(x)$ being a polynomial function of x, is continuous on the interval $[1, 3]$.

(ii) $f'(x)=2x-4$ which clearly exists for all value of $x \in (1, 3)$.

So, $f(x)$ is differentiable on the open interval $(1, 3)$.

(iii) $f(1)=(1)^2-4(1)+3=0$; $f(3)=(3)^2-4(3)+3=0 \quad \therefore f(1)=f(3)$

Thus, all the condition of Rolle's theorems are satisfied.

So, there must exist some $c \in (1, 3)$ such that $f'(c)=0$

Now, $f'(c)=0 \quad \Rightarrow 2c-4=0 \quad \Rightarrow 2c=4 \quad \Rightarrow c=2$

Clearly, the value of c lie in the interval $(1,3)$. Hence Rolle's theorem is verified

6. $f(x)=x(x-4)^2$ in $[0, 4]$

Sol. $f(x)=x(x-4)^2 \quad \Rightarrow f(x)=x(x^2-8x+16) \quad \Rightarrow f(x)=x^3-8x^2+16x$

Here, we observe that

(i) $f(x)$ being a polynomial function of x, is continuous on the interval $[0, 4]$.

(ii) $f'(x)=3x^2-16x+16$ which clearly exists for all value of $x \in (0, 4)$.

So, $f(x)$ is differentiable on the open interval $(1, 3)$.

(iii) $f(0)=(0)-8.0+16.0=0$; $f(4)=(4)^3-8(4)^2+16(4)=0 \quad \therefore f(0)=f(4)$

Thus, all the condition of Rolle's theorems are satisfied.

So, there must exist some $c \in (0, 4)$ such that $f'(0)=0$.

Now, $f'(c)=0 \quad \Rightarrow 3c^2-16c+16=0 \quad \Rightarrow 3c^2-12c-4c+16=0$

$\Rightarrow 3c(c-4)-4(c-4)=0 \quad \Rightarrow (c-4)(3c-4)=0 \quad \Rightarrow c=\frac{4}{3}, 4.$

But $c = 4 \notin (0, 4)$. $\therefore c = \frac{4}{3} \in (0, 4)$

Clearly, the value of c lie in the interval $(0, 4)$. Hence Rolle's theorem is verified.

7. $f(x) = x^3 - 7x^2 + 16x - 12$ in $[2, 3]$

Sol. Here, we observe that

(i) $f(x)$ being a polynomial function of x, is continuous on the interval $[2, 3]$.

(ii) $f'(x) = 3x^2 - 14x + 16$, which clearly exists for all value of $x \in (2, 3)$.

So, $f(x)$ is differentiable on the open interval $(2, 3)$.

(iii) $f(2) = (2)^3 - 7(2)^2 + 16(2) - 12 = 0$; $f(3) = (3)^3 - 7(3)^2 + 16(3) - 12 = 0$ $\therefore f(2) = f(3)$

Thus, all the condition of Rolle's theorem are satisfied.

So, these must exist some $c \in (2, 3)$ such that $f'(c) = 0$

Now, $f'(c) = 0 \Rightarrow 3c^2 - 14c + 16 = 0 \Rightarrow 3c^2 - 6c - 8c + 16 = 0$

$\Rightarrow 3c(c-2) - 8(c-2) = 0 \Rightarrow (3c-8)(c-2) = 0 \Rightarrow 3c = 8 \therefore c = \frac{8}{3} \in (2, 3), c = 2 \notin (2,3)$

Clearly, the value of c lies in the interval $(2, 3)$. Hence, Rolle's theorem is verified.

8. $f(x) = x^3 + 3x^2 - 24x - 80$ in $[-4, 5]$

Sol. Here, we observe that

(i) $f(x)$ being a polynomial function of x, is continuous interval $[-4, 5]$.

(ii) $f'(x) = 3x^2 + 6x - 24$, which clearly exists for all value of $x \in (-4, 5)$

So, $f(x)$ is differentiable on the open interval $(-4, 5)$.

(iii) $f(-4) = (-4)^3 + 3(-4)^2 - 24(-4) - 80 = 0$

$\Rightarrow f(5) = (5)^3 + 3(5)^2 - 24(5) - 80 = 0$ $\therefore f(-4) = f(5)$

Thus, all the condition of Rolle's theorems are satisfied.

So, there must exist some $c \in (-4, 5)$ such that $f'(c) = 0$.

Now, $f'(c) = 0 \Rightarrow 3c^2 + 6c - 24 = 0 \Rightarrow 3(c^2 + 2c - 8) = 0$

$\Rightarrow (c-2)(c+4) = 0 \therefore c = 2, -4 \therefore c = -4 \notin (-4, 5), c = 2 \in (-4, 5)$

Clearly, the value of $c = 2$ lie in the interval $(-4,5)$. Hence Rolle's theorem is verified.

9. $f(x) = (x-1)(x-2)(x-3)$ in $[1, 3]$

Sol. $f(x) = (x-1)(x-2)(x-3) \Rightarrow f(x) = x^3 - 6x^2 + 11x - 6$

Here, we observe that

(i) $f(x)$ being a polynomial function of x, is continuous on the interval $[1, 3]$

(ii) $f'(x) = 3x^2 - 12x + 11$, which clearly exists for all value of $x \in (1, 3)$

So, $f(x)$ is differentiable on the open interval $(1, 3)$

(iii) $f(1) = (1)^3 - 6(1)^2 + 11(1) - 6 = 0$; $f(3) = (3)^3 - 6(3)^2 + 11(3) - 6 = 0$ $\therefore f(1) = f(3)$

Thus, all the condition of Rolle's theorems are satisfied.

So, these must exist some $c \in (1, 3)$ such that $f'(c) = 0$

Now, $f'(c) = 0 \Rightarrow 3c^2 - 12c + 11 = 0 \Rightarrow c = \dfrac{12 + \sqrt{144 - 132}}{6}$

$\Rightarrow c = \dfrac{12 + \sqrt{12}}{6} \Rightarrow c = 2 \pm \dfrac{1}{\sqrt{3}}$

Clearly, both the value of c lie in the interval $(1, 3)$. Hence, Rolle's theorems is verified.

10. $f(x) = (x-1)(x-2)^2$ in $[1, 2]$

Sol. $f(x) = (x-1)(x-2)^2 \Rightarrow f(x) = (x-1)(x^2 - 4x + 4) \Rightarrow f(x) = x^3 - 5x^2 + 8x - 4$

Here, we observe that

(i) $f(x)$ being a polynomial function of x, is continuous on the interval $[1, 2]$.

(ii) $f'(x) = 3x^2 - 10x + 8$, which clearly exists for all value of $x \in (1, 2)$.

So, $f(x)$ is differentiable on the open interval $(1, 2)$.

(iii) $f(1) = (1)^3 - 5(1)^2 + 8(1) - 4 = 0$; $f(2) = (2)^3 - 5(2)^2 + 8(2) - 4 = 0 \quad \therefore f(1) = f(2)$

Thus all the condition of Rolle's theorems are satisfied

So, these must exist some $c \in (1, 2)$ such that $f'(c) = 0$

Now, $f'(c) = 0 \Rightarrow 3c^2 - 10c + 8 = 0 \Rightarrow 3c^2 - 6c - 4c + 8 = 0 \Rightarrow 3c(c-2) - 4(c-2) = 0$

$\Rightarrow (3c-4)(c-2) = 0 \Rightarrow (3c-4) = 0$ or $(c-2) = 0 \Rightarrow c = \dfrac{4}{3}, 2$.

But $c = 2 \notin (1, 2)$ and $c = \dfrac{4}{3} \in (1, 2)$. Hence, Rolle's theorem is verified.

11. $f(x) = (x-2)^4 (x-3)^3$ in $[2, 3]$

Sol. Here, we observe that

(i) $f(x)$ being a polynomial function of x, is continuous on the interval $[2, 3]$

(ii) $f'(x) = (x-2)^4 \{3(x-3)^2\} + (x-3)^3 \{4(x-2)^3\}$

$= (x-2)^3 (x-3)^2 \{3(x-2) + 4(x-3)\} = (x-2)^3 (x-3)^2 (3x - 6 + 4x - 12)$

$\Rightarrow f'(x) = (x-2)^3 (x-3)^2 (7x - 18)$, which clearly exits for all value of $x \in (2, 3)$.

So, $f(x)$ is differentiable on the open interval $(2, 3)$.

(iii) $f(2) = (2-2)^4 (2-3)^3 = 0$; $f(3) = (3-2)^4 (3-3)^3 = 0 \quad \therefore f(2) = f(3)$

Thus all the condition of Rolle's theorem are satisfied.

So these must exists some $c \in (2, 3)$ such that $f'(c) = 0$.

Now, $f'(c) = 0 \Rightarrow (c-2)^3 (c-3)^2 (7c - 18) = 0 \Rightarrow 7c - 18 = 0$ rejecting 2 and 3 $\therefore c = \dfrac{18}{7}$

Clearly, the value of c lie in the interval $(2, 3)$. Here, Rolle's theorems is verified.

12. $f(x) = \sqrt{1 - x^2}$ in $[-1, 1]$

Sol. Here, we observe that

(i) $f(x)$ being a polynomial function of x, is continuous on the interval $[-1, 1]$

(ii) $f'(x)=\dfrac{1}{2\sqrt{1-x^2}}(-2x)=\dfrac{-x}{\sqrt{1-x^2}}$, which clearly exists for all value of $x \in (-1, 1)$.

So, $f(x)$ is differentiable on the open interval $(-1, 1)$

(iii) $f(-1)=\sqrt{1-(-1)^2}=0;\quad f(1)=\sqrt{1-1}=0 \quad \therefore f(-1)=f(1)$

Thus all the condition of Rolle's theorem are satisfied.

So, these must exists some $c \in (-1, 1)$ such that $f'(c)=0$.

Now, $f'(c)=0 \;\Rightarrow\; \dfrac{-c}{\sqrt{1-c^2}}=0 \;\therefore\; c=0$

Clearly, the value of c lies in the interval $(-1, 1)$. Hence, Rolle's theorem is verified.

13. $f(x)=\cos x$ in $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

Sol. Consider $f(x)=\cos x$ in $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.

Since the cosine function is continuous at each $x \in R$, it follows that $f(x)=\cos x$ is continuous on $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$. Also, $f'(x)=-\sin x$, which clearly exists for all $x \in \left(\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

So, $f(x)$ is differentiable on $\left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$. Also $f\left(-\dfrac{\pi}{2}\right)=f\left(\dfrac{\pi}{2}\right)=0$.

Thus, all the condition of Rolle's theorem are satisfied.

So, there must exist a real number $c \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ such that $f'(c)=0$.

Now, $f'(c)=0 \;\Rightarrow\; -\sin c=0 \;\Rightarrow\; c=0$.

Thus $c=0 \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ such that $f'(c)=0$. Hence, Rolle's theorem is verified.

14. $f(x)=\cos 2x$ in $[0, \pi]$

Sol. Consider $f(x)=\cos 2x$ in $[0, \pi]$

Since the cosine function is continuous at each $x \in R$, it follows that $f(x)=\cos 2x$ is continuous on $[0, \pi]$. Also $f'(x)=-2\sin 2x$, which clearly exists for all $x \in (0, \pi)$.

So, $f(x)$ is differentiable on $(0, \pi)$. Also $f(0)=f(\pi)=0$.

Thus, all the condition of Rolle's theorem are satisfied.

So, there must exist a real number $c \in (0, \pi)$ Such that $f'(c)=0$

Now, $f'(c)=0 \;\Rightarrow\; -2\sin 2c=0 \;\Rightarrow\; \sin 2c=0 \Rightarrow 2c=\pi=c=\dfrac{\pi}{2}\in(0, \pi)$

Here, Rolle's theorem is verified.

15. $f(x)=\sin 3x$ in $[0, \pi]$

Sol. Consider $f(x)=\sin 3x$ in $[0, \pi]$

Since the sine function is continuous at each $x \in R$, if follows that $f(x) = \sin 3x$ is continuous on $[0, \pi]$. Also, $f'(x) = 3\cos 3x$, which clearly exists for all $x \in (0, \pi)$.

So, $f(x)$ is differentiable in $(0, \pi)$. Also, $f(0) = f(\pi) = 0$.

Thus, all the condition of Rolle's theorem are satisfied.

So, these must exist a real number $c \in (0, \pi)$ such that $f'(c) = 0$

Now, $f'(c) = 0 \Rightarrow 3\cos 3c = 0 \Rightarrow \cos 3c = \cos\frac{\pi}{2} \Rightarrow 3c = \frac{\pi}{2} \quad \therefore\ c = \frac{\pi}{6}$

Thus, $c = \frac{\pi}{6} \in (0, \pi)$ such that $f'(c) = 0$. Hence, Rolle's theorem is verified.

16. $f(x) = \sin x + \cos x$ in $\left[0, \frac{\pi}{2}\right]$

Sol. Consider $f(x) = \sin x + \cos x$ in $\left[0, \frac{\pi}{2}\right]$

The sine function, the cosine function and its sum is continuous function, so that $f(x)$ is continuous on $\left[0, \frac{\pi}{2}\right]$.

Also, $f'(x) = \cos x - \sin x$, which clearly exists for all value of $x \in \left(0, \frac{\pi}{2}\right)$.

So, $f(x)$ is differentiable on $\left(0, \frac{\pi}{2}\right)$. Also $f(0) = f\left(\frac{\pi}{2}\right) = 1$.

Thus, all the condition of Rolle's theorem are satisfied.

So, these must exits some $c \in \left(0, \frac{\pi}{2}\right)$ such that $f'(c) = 0$.

Now, $f'(c) = 0 \Rightarrow \cos c - \sin c = 0 \Rightarrow \cos c = \sin c \Rightarrow \tan c = 1 = \tan\frac{\pi}{4} \ \therefore c = \frac{\pi}{4}$

Thus, $c = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$ such that $f'(c) = 0$. Hence, Rolle's theorem is verified.

17. $f(x) = e^{-x} \sin x$ in $[0, \pi]$

Sol. Consider $f(x) = \frac{\sin x}{e^x}$ in $[0, \pi]$.

Since $e^x \neq 0$, for any $x \in [0, \pi]$ and $f(x)$ is the quotient of two continuous function, if follows that $f(x)$ is continuous on $[0, \pi]$.

Also, $f'(x) = \frac{e^x \cos x - e^x \sin x}{e^{2x}} = \frac{\cos x - \sin x}{e^x}$, which clearly exists for all $x \in (0, \pi)$.

So $f(x)$ is differentiable on $(0, \pi)$ such that $f'(c) = 0$.

$\therefore \ \frac{\cos c - \sin c}{e^c} = 0 \Rightarrow \cos c - \sin c = 0$

$\Rightarrow \tan c = 1 = \tan\frac{\pi}{4} \therefore\ c = \frac{\pi}{4} \in (0, \pi)$ Hence, Rolle's theorem is verified.

18. $f(x)=e^{-x}(\sin x-\cos x)$ in $\left[\frac{\pi}{4},\frac{5\pi}{4}\right]$

Sol. Consider $f(x)=e^{-x}(\sin x-\cos x) \Rightarrow f(x)=\frac{\sin x-\cos x}{e^x}$

Since $e^x \neq 0$ for any $x \in \left[\frac{\pi}{4},\frac{5\pi}{4}\right]$ and $f(x)$ is the quotient of two continuous function, if follows that $f(x)$ is continuous on $\left[\frac{\pi}{4},\frac{5\pi}{4}\right]$.

Also $f'(x)=\frac{e^x(\cos x+\sin x)-(\sin x-\cos x)e^x}{(e^x)^2}=\frac{e^x(\cos x+\sin x-\sin x+\cos x)}{(e^x)^2}$

$f'(x)=\frac{2\cos x}{e^x}$, which clearly exists for all $x \in \left(\frac{\pi}{4},\frac{5\pi}{4}\right)$

So, $f(x)$ is differentiable on $(0,\pi)$. Also $f\left(\frac{\pi}{4}\right)=f\left(\frac{5\pi}{4}\right)=0$.

So, all the condition of Rolle's theorem are satisfied.

$\therefore$ These must exists $c \in \left(\frac{\pi}{4},\frac{5\pi}{4}\right)$, such that $f'(c)=0$.

Now $f'(c)=0 \quad \Rightarrow \frac{2\cos c}{e^x}=0 \quad \Rightarrow \cos c=0 \quad \therefore c=\frac{\pi}{2}$

Thus, $c=\frac{\pi}{2} \in \left(\frac{\pi}{4},\frac{5\pi}{4}\right)$ such that $f'(c)=0$. Hence, Rolle's theorem is verified.

19. $f(x)=\sin x-\sin 2x$ in $[0,2\pi]$

Sol. Consider $f(x)=\sin x-\sin 2x$ in $[0,2\pi]$

Since the sine function is continuous, it follows that $g(x)=\sin x$ and $h(x)=\sin 2x$ are both continuous and so their difference is also continuous.

Consequently, $f(x)=g(x)-h(x)$ is differentiable on $(0,2\pi)$.

Also, $f'(x)=(\cos x-2\cos 2x)$, which clearly exists for all $x \in [0,2\pi]$

$\therefore$ $f(x)$ is differentiable on $(0,2\pi)$

Thus, all the condition of Rolle's theorem are satisfied.

So, these must exist a real number $c \in (0,2\pi)$ such that $f'(c)=0$

Now, $f'(c)=0 \quad \Rightarrow \cos c-2\cos 2c=0 \quad \Rightarrow \cos c-2(2\cos^2 c-1)=0$

$\Rightarrow \cos c-4\cos^2 c+2=0 \quad \Rightarrow 4\cos^2 c-\cos c-2=0$

$\Rightarrow \cos c=\frac{1\pm\sqrt{33}}{8}=0.8431$ or $-0.5931 \quad \Rightarrow \cos c=0.8431$ or $\cos(180°-c)=0.5931$

$\Rightarrow c=32°32'$ or $c=126°23'$

Thus $c \in (0,2\pi)$ such that $f'(c)=0$. Hence, Rolle's theorem is satisfied.

20. $f(x)=x(x+2)e^x$ in $[-2,0]$

Sol. Since a polynomial function as well as an exponential function is continuous and the product of two continuous function is continuous.

If follows that $f(x)$ is continuous on the given interval $[-2, 0]$

Now $f'(x)=e^x(2x+2)+(x^2+2x)e^x \Rightarrow f'(x)=e^x(2x+2+x^2+2x)$

$\Rightarrow f'(x)=e^x(x^2+4x+2)$, which is clearly finite for all value of x in $(-2, 0)$.

So, $f(x)$ is differentiable on $(-2, 0)$. Also $f(-2)=f(0)=0$.

Thus, all the condition of Rolle's theorem are satisfied.

So, these must exist $c \in (-2, 0)$ such that $f'(c)=0$.

$\Rightarrow e^c(c^2+4c+4)=0 \quad \Rightarrow e^c=0$ or $c^2+4c+2=0$

$$\Rightarrow c=\frac{(-4)\pm\sqrt{(-4)^2-4.1.2}}{2\times1} \quad \Rightarrow c=\frac{-4\pm\sqrt{16-8}}{2}$$

$$\therefore c=\frac{-4\pm2\sqrt{2}}{2}=-2\pm\sqrt{2} \quad \Rightarrow c=-2+\sqrt{2},\ -2-\sqrt{2}$$

But $c=-2-\sqrt{2}\notin(-2,0) \quad \therefore c=-2+\sqrt{2}\in(-2,0)$. Hence Rolle's Theorem is verified.

21. Show that $f(x)=x(x-5)^2$ satisfies Rolle's theorem on $[0, 5]$ and that the value of c is $\left(\frac{5}{3}\right)$.

Sol. Since a polynomial function is every where continuous and differentiable, the given function is continuous as well as differentiable on every interval.

Also $f'(x)=(x-5)^2.1+x.2(x-5) \Rightarrow f'(x)=(x-5)\{x-5+2x\}$

$\Rightarrow f'(x)=(x-5)(3x-5)$, which clearly exits for each $x\in(0, 5)$.

Also $f(0)=f(5)=0$. Thus, all the condition of Rolle's theorem are satisfied.

So, there must exist $c\in(0, 5)$ such that $f'(c)=0$.

Now, $f'(c)=0 \quad \Rightarrow (c-5)(3c-5)=0 \quad \therefore c=\frac{5}{3}, 5$, rejecting 5.

Thus $c=\frac{5}{3}=(0, 5)$. Hence, Rolle's theorem is verified.

Discuss the applicability of Rolle's theorem, when:

22. $f(x)=(x-1)(2x-3)$, where $1\le x\le 3$

Sol. Consider $f(x)=(x-1)(2x-3)$ in $[1, 3] \quad \Rightarrow f(x)=2x^2-5x+3$

Since a polynomial function is continuous and differentiable every where, the first two conditions of Rolle's theorem are satisfied.

But $f(1)=2(1)^2-5(1)+3=0$ and $f(3)=2(3)^2-5(3)+3=6$

$\therefore f(1)\ne f(3)$. Hence, Rolle's theorem is not applicable for $f(x)=(x-1)(2^x-3)$ in $[1, 3]$.

23. $f(x)=x^{1/2}$ on $[-1, 1]$

Sol. Consider $f(x)=x^{1/2}$ in $[-1, 1]$

$f(-1)=(-1)^{\frac{1}{2}},\ f(1)=(1)^{\frac{1}{2}}$. $\qquad$ So, $f(-1)\ne f(1)$

Hence, Rolle's theorem is not applicable to $f(x)=x^{1/2}$ in $[-1, 1]$.

24. $f(x)=2+(x-1)^{2/3}$ on $[0, 2]$

Sol. Consider $f(x)=2+(x-1)^{2/3}$ in $[0, 2]$. $f(0)=1, f(2)=2+1^{2/3}=3$ Since $f(0)\neq f(2)$.

Hence, Rolle's theorem is not applicable to $f(x)=2+(x-1)^{2/3}$ in $[0, 2]$.

25. $f(x)=\cos\frac{1}{x}$ on $[-1, 1]$

Sol. Consider $f(x)=\cos\frac{1}{x}$ in $[-1, 1]$

It is undefined at $x=0$. Hence, discontinuous at $x=0$.

$\Rightarrow$ The conditions of Rolle's Theorem are not satisfied.

26. $f(x)=[x]$ on $[-1, 1]$, where $[x]$ denotes the greatest integer not exceeding x

Sol. Consider $f(x)=[x]$ in $[-1, 1]$

$f(-1)=[-1]=-1, f(1)=[1]=1$. So, $f(-1)\neq f(1)$

Hence, Rolle's theorem is not applicable to $f(x)=[x]$ in $[-1, 1]$.

27. Using Rolle's theorem, find the point on the curve $y=x(x-4), x\in[0, 4]$, where the tangent is parallel to the x-axis.

Sol. Consider $f(x)=x^2-4x$ in $[0, 4]$

Now, the polynomial function being continuous and the constant function being continuous, it follows that their difference x^2-4x is continuous.

So, $f(x)$ is continuous on $[0, 4]$. $f'(x)=2x-4$, which exists for all $x\in(0, 4)$.

$\therefore$ $f(x)$ is differentiable on $(0, 4)$. Also $f(0)=f(4)=0$.

So, all the condition of Rolle's theorem are satisfied.

So, these must exist some $c\in(0, 4)$ such that $f'(c)=0$.

Now, $f'(c)=0 \Rightarrow 2c-4=0 \Rightarrow 2c=4 \quad \therefore \ c=2$

Hence the point is $x=2$

$y=2^2-4\times 2=4-8=-4$.

EXERCISE 11 D (Pg. No.: 490)

Verify Lagrange's mean-value theorem for each of the following function:

1. $f(x)=x^2+2x+3$ on $[4, 6]$

Sol. Consider $f(x)=x^2+2x+3$ in $[4, 6]$. The given function is $f(x)=x^2+2x+3$

It being a polynomial function, is continuous on $[4, 6]$.

Also, $f'(x)=2x+2$, which exists for all x in $(4, 6)$. So, $f(x)$ is differentiable on $(4, 6)$.

Thus, both the condition of Lagrange's mean-value theorem are satisfied.

So, there must exist some $c \in (4, 6)$ such that

$$f(6)=(6)^2+2\times 6+3=51; \quad f(4)=(4)^2+2\times 4+3=27$$

$$f'(c)=\frac{f(6)-f(4)}{6-4} \Rightarrow 2c+2=\frac{51-27}{2} \Rightarrow 2c+2=12 \Rightarrow 2c=10 \quad \therefore c=5$$

$$c=5\in(4, 6) \text{ such that } f'(c)=\frac{f(6)-f(4)}{6-4}$$

Hence, Lagrange's mean- value theorem is verified.

2. $f(x)=x^2+x-1$ on $[0, 4]$

Sol. Consider $f(x)=x^2+x-1$ in $[0, 4]$. It being a polynomial function, is continuous on $[0, 4]$

Also $f'(x)=2x+1$, which exists for all x in $(0, 4)$. So, $f(x)$ is differentiable on $(0, 4)$.

Thus, both the condition of Lagrange's mean-value theorem are satisfied.

So, there must exist some $c \in (0, 4)$ such that

$$f(4)=(4)^2+4-1=19; \quad f(0)=-1$$

$$f'(c)=\frac{f(4)-f(0)}{4-0} \Rightarrow 2c+1=\frac{19-(-1)}{4} \Rightarrow 2c+1=\frac{20}{4}=5$$

Thus $2c=4 \Rightarrow c=2\in(0,4)$ such that $f'(c)=\dfrac{f(4)-f(0)}{4-0}$

Hence, Lagrange's mean-value theorem is verified.

3. $f(x)=2x^2-3x+1$ on $[1, 3]$

Sol. Consider, the given function is $f(x)=2x^2-3x+1$.

It being a polynomial function, is continuous on $[1, 3]$.

Also, $f'(x)=4x-3$, which exists for all x in $(1, 3)$. So, $f(x)$ is differentiable on $(1, 3)$.

Thus, both the condition of Lagrange's mean-value theorem are satisfied

So, there must exist some $c \in (1, 3)$ such that

$$f(1)=2(1)^2-3(1)+1=0; \quad f(3)=2(3)^2-3(3)+1=10$$

$$f'(c)=\frac{f(3)-f(1)}{3-1} \quad \Rightarrow 4c-3=\frac{10-0}{2} \quad \Rightarrow 4c-3=5 \quad \Rightarrow 4c=8 \quad \Rightarrow c=2$$

Thus $c=2\in(1,3)$ such that $f'(c)=\frac{f(3)-f(1)}{3-1}$

Here, Lagrange's mean-value theorem is verified.

4. $f(x)=x^3+x^2-6x$ on $[-1,4]$

Sol. Consider the given function is $f(x)=x^3+x^2-6x$

It being a polynomial function is continuous on $[-1,4]$.

Also, $f'(x)=3x^2+2x-6$, which exists for all x in $(-1,4)$. So, $f(x)$ is differentiable on $(-1,4)$.

Thus, both the condition of Lagrange's mean value theorem are satisfied.

So, these must exist some $c\in(-1,4)$ such that

$$f(-1)=(-1)^3+(-1)^2-6(-1)=6; \quad f(4)=(4)^3+(4)^2-6\times4=56$$

$$f'(c)=\frac{f(4)-f(-1)}{4-(-1)} \quad \Rightarrow 3c^2+2c-6=\frac{56-6}{5} \quad \Rightarrow 3c^2+2c-6=10$$

$$\Rightarrow 3c^2+2c-16=0 \quad \Rightarrow 3c^2+8c-6c-16=0 \quad \Rightarrow c(3c+8)-2(3c+8)=0 \quad \therefore (c-2)(3c+8)=0$$

$\therefore c=2,-8/3$, but $c\neq-8/3$. Hence, Lagrange's mean-value theorem is verified.

5. $f(x)=(x-4)(x-6)(x-8)$ on $[4,10]$

Sol. Consider $f(x)=(x-4)(x-6)(x-8)$ in $[4,10]$.

The given function is $f(x)=x^3-18x^2+104x-192$.

It being a polynomial function, is continuous on $[4,10]$.

Also $f'(x)=3x^2-36x+104$, which exists for all x in $(4,10)$. So, $f(x)$ is differentiable on $(4,10)$.

Thus, both the condition of Lagrange's mean-value theorem are satisfied.

So, there must exist some $c\in(4,10)$ such that

$$f(4)=(4-4)(4-6)(4-8)=0;$$

$$f(10)=(10-4)(10-6)(10-8)=6\times4\times2=48$$

$$f'(c)=\frac{f(10)-f(4)}{10-4} \quad \Rightarrow 3c^2-36c+104=\frac{48-0}{6} \quad \Rightarrow 3c^2-36c+104-8=0$$

$$\Rightarrow 3c^2-36c+96=0 \quad \Rightarrow 3(c^2-12c+32)=0 \quad \Rightarrow c^2-12c+32=0$$

$$\Rightarrow c^2-8c-4c+32=0 \quad \Rightarrow c(c-8)-4(c-8)=0 \quad \Rightarrow (c-4)(c-8)=0 \quad \therefore c=4,8$$

$\therefore \; c=8\in(4,10)$ and $c=4\notin(4,10)$ such that $f'(c)=\frac{f(10)-f(4)}{10-4}$.

Hence, Lagrange's mean-value theorem is verified

6. $f(x)=e^x$ on $[0,1]$

Sol. Consider the given function is $f(x)=e^x$.

It being a exponential function, is continuous on $[0,1]$.

Also, $f'(x)=e^x$, which exists for all x in $(0,1)$. So, $f(x)$ is differentiable on $(0,1)$.

Thus, both the condition of Lagrange's mean-value theorem are satisfied.

So, these must exist some $c \in (0, 1)$ such that

$$f(0) = (0)^{2/3} = 0 \quad \Rightarrow f(1) = (1)^{2/3} = 1$$

$$f'(c) = \frac{f(1) - f(0)}{1-0} \quad \Rightarrow \frac{2}{3}c^{-\frac{1}{3}} = \frac{1-0}{1-0} \quad \Rightarrow \frac{2}{3c^{1/3}} = 1 \quad \Rightarrow c^{1/3} = \frac{2}{3} \quad \therefore\ c = \left(\frac{2}{3}\right)^3$$

$c = \left(\frac{2}{3}\right)^3 \in (0, 1)$. Hence, Lagrange's mean-value theorem is verified.

7. $f(x) = x^{2/3}$ on $[0, 1]$

Sol. $f(x)$ is a polynomial function. So it is continuous on $[0, 1]$

$$f(x) = x^{2/3} \quad \Rightarrow f'(x) = \frac{2}{3}x^{-1/3}$$, it is differentiable at $(0, 1)$.

So, the conditions of Lagrange's Mean Value theorem are verified.

$$f(0) = 0^{2/3} = 0$$

$$f(1) = (1)^{2/3} = 1$$

$$f'(x) = \frac{f(b) - f(a)}{b-a} \quad \Rightarrow \frac{2}{3}x^{-1/3} = \frac{1-0}{1-0} \quad \Rightarrow \frac{2}{3}.\frac{1}{x^{1/3}} = 1$$

$\Rightarrow x^{1/3} = \frac{2}{3} \quad \Rightarrow x = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$. So, $x = \frac{8}{27} \in (0, 1)$.

Hence, Lagrange's Mean Value theorem is verified.

8. $f(x) = \log x$ on $[1, e]$

Sol. Consider $f(x) = \log x$ in $[1, e]$. The given function is $f(x) = \log x$.

It being a logarithmic function, is continuous on $[1, e]$.

Also, $f'(x) = \frac{1}{x}$, which exists for all x in $(1, e)$. So, $f(x)$ is differentiable on $(1, e)$.

Thus, both the condition of Lagrange's mean-value theorem are satisfied.

So, there must exists some $c \in (1, e)$ such that

$$f(e) = \log e = 1 \quad \Rightarrow f(1) = \log 1 = 0$$

$$f'(c) = \frac{f(e) - f(1)}{e-1} \quad \Rightarrow \frac{1}{c} = \frac{1-0}{e-1} \quad \Rightarrow \frac{1}{c} = \frac{1}{e-1} \quad \Rightarrow c = e-1$$

$c = e - 1 \in (1, e)$ such that $f'(c) = \frac{f(e) - f(1)}{e-1}$.

Hence, Lagrange's mean-value theorem is verified.

9. $f(x) = \tan^{-1} x$ on $[0, 1]$

Sol. Consider the given $f(x) = \tan^{-1}(x)$ in $[0, 1]$

It being a inverse trigonometric function, is continuous on $[0, 1]$.

Also, $f'(x) = \frac{1}{1+x^2}$, which exist for all x in $(0, 1)$. So, $f(x)$ is differentiable on $(0, 1)$.

Thus, both the condition of Lagrange's mean-value theorem are satisfied.

So, there must exist some $c \in (0, 1)$ such that

$$f(1) = \tan^{-1}(1) = \frac{\pi}{4}; \quad f(0) = \tan^{-1}(0) = 0$$

$$f'(c) = \frac{f(1) - f(0)}{1 - 0} \quad \Rightarrow \frac{1}{1+c^2} = \frac{\frac{\pi}{4} - 0}{1} \quad \Rightarrow \frac{1}{1+c^2} = \frac{\pi}{4} \quad \Rightarrow \frac{4}{\pi} = 1 + c^2$$

$\Rightarrow c^2 = \frac{4}{\pi} - 1 \quad \Rightarrow c = \pm\sqrt{\frac{4}{\pi} - 1}. \quad \therefore\ c = \pm\sqrt{\frac{4}{\pi} - 1} \in (1, e)$ such that $f'(c) = \frac{f(1) - f(0)}{1 - 0}$.

Hence, Lagrange's mean-value theorem is verified.

10. $f(x) = \sin x$ on $\left[\frac{\pi}{2}, \frac{5\pi}{2}\right]$

Sol. Consider the given function is $f(x) = \sin x$ in $\left[\frac{\pi}{2}, \frac{5\pi}{2}\right]$.

It being a trigonometric function, is continuous on $\left[\frac{\pi}{2}, \frac{5\pi}{2}\right]$.

Also, $f'(x) = \cos x$, which exist for all x in $\left(\frac{\pi}{2}, \frac{5\pi}{2}\right)$. So, $f(x)$ is differentiable on $\left(\frac{\pi}{2}, \frac{5\pi}{2}\right)$.

Thus, both the condition of Lagrange's mean-value theorem are satisfied

So, these must exist some $c \in \left(\frac{\pi}{2}, \frac{5\pi}{2}\right)$ such that

$$f\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} = 1; \quad f\left(\frac{5\pi}{2}\right) = \sin\left(\frac{5\pi}{2}\right) = \sin\frac{\pi}{2} = 1$$

$$f'(c) = \frac{f\left(\frac{\pi}{2}\right) - f(0)}{\frac{\pi}{2} - 0} \quad \Rightarrow \cos c - \sin c = 0 \quad \Rightarrow \cos c = \sin c \quad \therefore\ c = \frac{\pi}{4}, \frac{5\pi}{4}$$

But $c = \frac{\pi}{4} \notin \left(\frac{\pi}{2}, \frac{5\pi}{2}\right)$, so $c = \frac{5\pi}{4} \in \left(\frac{\pi}{2}, \frac{5\pi}{2}\right)$.

Hence the Lagrange's theorems is satisfied.

11. $f(x) = \sin x + \cos x$ on $\left[0, \frac{\pi}{2}\right]$

Sol. Consider the given function is $f(x) = \sin x + \cos x$ in $\left[0, \frac{\pi}{2}\right]$

$\because$ $\sin x$ and $\cos x$ are always continues on $\left[0, \frac{\pi}{2}\right]$.

Also $f'(x) = \cos x - \sin x$, which exist for all x in $\left(0, \frac{\pi}{2}\right)$. So, $f(x)$ is differentiable on $\left(0, \frac{\pi}{2}\right)$.

$$f(0) = \sin 0 + \cos 0 = 0 + 1 = 1 \Rightarrow f\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) + \cos\frac{\pi}{2} = 1 + 0 = 1$$

Thus both the condition of Lagrange's theorem is satisfied.

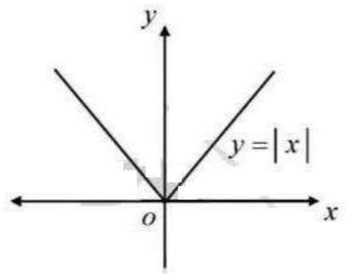
So, there must exist $c \in \left(0, \frac{\pi}{2}\right)$ such that $f'(c) = \dfrac{f\left(\frac{\pi}{2}\right) - f(0)}{\frac{\pi}{2} - 0}$ $\Rightarrow \cos c - \sin c = \dfrac{1-1}{\frac{\pi}{2} - 0}$

$\Rightarrow \cos c - \sin c = 0 \quad \Rightarrow \cos c = \sin c \Rightarrow \tan c = 1 \quad \therefore c = \frac{\pi}{4} \in (0, \pi/2)$

Hence Lagrange's mean value theorem is satisfied

12. Show that Lagrange's mean-value theorem is not applicable to $f(x) = |x|$ on $[-1, 1]$

Sol. As the graph of $y = |x|$ is shown. which is clearly not differentiable at $x = 0$. (As it has corner at $x = 0$) Hence, the Lagrange's mean-value theorem is not applicable to $f(x) = |x|$.



13. Show that Lagrange's mean-value theorem is not applicable to $f(x) = \frac{1}{x}$ on $[-1,1]$.

Sol. Clearly $y = \frac{1}{x}$ is undefined at $x = 0$. Hence discontinuous at $x = 0$.

Hence, the Lagrange's mean-value theorem is not applicable to $f(x) = \frac{1}{x}$.

14. Find c of Lagrange's mean-value theorem for

(i) $f(x) = (x^3 - 3x^2 + 2x)$ on $\left[0, \frac{1}{2}\right]$ (ii) $f(x) = \sqrt{25 - x^2}$ on $[1, 5]$

(iii) $f(x) = \sqrt{x+2}$ on $[4, 6]$

Sol. **(i)** $f(x) = x^3 - 3x^2 + 2x \Rightarrow f'(x) = 3x^2 - 6x + 2 \Rightarrow f(0) = 0$

$\Rightarrow f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3 - 3\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right) = \frac{3}{8}$

Now, $f'(c) = \dfrac{f\left(\frac{1}{2}\right) - f(0)}{\frac{1}{2} - 0} \Rightarrow 3c^2 - 6c + 2 = \dfrac{\frac{3}{8} - 0}{\frac{1}{2}} \Rightarrow 3c^2 - 6c + 2 = \frac{3}{8} \times \frac{2}{1}$

$\Rightarrow 3c^2 - 6c + 2 = \frac{3}{4} \Rightarrow 3c^2 - 6c + 2 - \frac{3}{4} = 0 \Rightarrow 3c^2 - 6c + \frac{5}{4} = 0 \Rightarrow 12c^2 - 24c + 5 = 0$

$\Rightarrow c = \dfrac{24 \pm \sqrt{(24)^2 - 4(12) \times 5}}{2 \times 12} \Rightarrow c = \dfrac{24 \pm \sqrt{336}}{24}$

$\therefore c = \dfrac{24 \pm 4\sqrt{21}}{24} \Rightarrow c = 1 \pm \frac{1}{6}\sqrt{21}$. So $c = 1 - \frac{1}{6}\sqrt{21} \in \left(0, \frac{1}{2}\right)$

(ii) The given function is $f(x) = \sqrt{25 - x^2} \Rightarrow f'(x) = \dfrac{1}{2\sqrt{25 - x^2}}(-2x) = \dfrac{-x}{\sqrt{25 - x^2}}$

$f(1) = \sqrt{25 - (1)^2} = \sqrt{24} = 2\sqrt{6}$; $f(5) = \sqrt{25 - 25} = 0$

$$f'(c)=\frac{f(5)-f(1)}{5-1} \Rightarrow \frac{-c}{\sqrt{25-c^2}}=\frac{0-2\sqrt{6}}{4} \Rightarrow \frac{c}{\sqrt{25-c^2}}=\frac{\sqrt{6}}{2} \Rightarrow 2c=\sqrt{6}\sqrt{25-c^2}$$

$$\Rightarrow 4c^2=6\left(25-c^2\right) \Rightarrow 4c^2=150-6c^2 \Rightarrow 10c^2=150 \Rightarrow c^2=15 \quad \therefore\ c=\pm\sqrt{15}$$

$\therefore\ c=+\sqrt{15}\in(1,5)$ and $c=-\sqrt{15}\notin(1,5)$

(iii) The given function is $f(x)=\sqrt{x+2} \Rightarrow f'(x)=\dfrac{1}{2\sqrt{x+2}}$

$$f(4)=\sqrt{4+2}=\sqrt{6};\quad f(6)=\sqrt{6+2}=\sqrt{8}$$

$$f'(c)=\frac{f(6)-f(4)}{6-4} \Rightarrow \frac{1}{2\sqrt{c+2}}=\frac{\sqrt{8}-\sqrt{6}}{2} \Rightarrow \frac{1}{\sqrt{8}-\sqrt{6}}=\sqrt{c+2}$$

$$\Rightarrow \sqrt{c+2}=\frac{1}{\sqrt{8}-\sqrt{6}}\times\frac{\sqrt{8}+\sqrt{6}}{\sqrt{8}+\sqrt{6}} \Rightarrow \sqrt{c+2}=\frac{\sqrt{8}+\sqrt{6}}{8-6} \Rightarrow \sqrt{c+2}=\frac{\sqrt{8}+\sqrt{6}}{2}$$

$$\Rightarrow c+2=\frac{8+6+2\sqrt{8}.\sqrt{6}}{4} \Rightarrow 2(c+2)=7+\sqrt{8}\sqrt{6} \Rightarrow 2c=3+\sqrt{8}\sqrt{6}$$

$$\Rightarrow 2c=3+2\sqrt{2}.\sqrt{2}.\sqrt{3} \Rightarrow 2c=3+4.(1.731) \Rightarrow 2c=3+6.924 \quad \therefore\ c=\frac{9.924}{2}\in(4,6)$$

15. Using Lagrange's mean-value theorem, find a point on the curve $y=x^2$, where the tangent is parallel to the line joining the points $(1, 1)$ and $(2, 4)$.

Sol. Let us apply Lagrange's mean value theorem for the function $f(x)=x^2$ in the interval $[1, 2]$.

Now, $f(x)$ being a polynomial function, it is continuous on $[1, 2]$.

Also, $f'(x)=2x$, which exists for all $x\in]1, 2[$.

Thus, both the condition of Lagrange's mean-value theorem are satisfied.

So, there must exist a point $c\in]1, 2[$ such that $f'(c)=\dfrac{f(2)-f(1)}{(2-1)}=\dfrac{4-1}{1}-3$

Now, $f'(c)=3 \Rightarrow 2c=3 \Rightarrow c=\dfrac{3}{2}\in]1, 2[$

Now $x=\dfrac{3}{2}$ and $y=x^2 \Rightarrow y=\left(\dfrac{3}{2}\right)^2-\dfrac{9}{4}$.

Thus, at the point $\left(\dfrac{3}{2},\dfrac{9}{4}\right)$ on the given curve is parallel to the chord joining $(1, 1)$ and $(2, 4)$.

16. Find a point on the curve $y=x^3$, when the tangent to the curve is parallel to the chord point $(1, 1)$ and $(3, 27)$.

Sol. Let us apply Lagrange's mean-value theorem for the function now $f(x)=x^3$ in the interval $[1, 3]$

Now, $f(x)$ being a polynomial function, so it is continuous on $[1, 3]$.

Also $f'(x)=3x^2$, which exists for all $x\in(1, 3)$. So, $f(x)$ is differentiable on $(1, 3)$.

Thus, both the condition of Lagrange's mean-value theorem are satisfied

So, these must exist a point $c\in(1, 3)$ such that

$$f(1)=(1)^3=1 \Rightarrow f(3)=(3)^3=27$$

$$f'(c)=\frac{f(3)-f(1)}{3-1} \Rightarrow 3c^2=\frac{27-1}{2} \Rightarrow 3c^2=\frac{26}{2} \Rightarrow c^2=\frac{26}{6}$$

$$\Rightarrow c=\sqrt{\frac{26}{6}} \Rightarrow c=\frac{\sqrt{39}}{3} \Rightarrow c=\frac{\sqrt{39}}{3}\in(1,3)$$

Now, $x=\frac{\sqrt{39}}{3}$, and $y=x^3 \Rightarrow y=\left(\frac{\sqrt{39}}{3}\right)^3=\frac{39\times\sqrt{39}}{3\times3\times3}=\frac{13\sqrt{39}}{9}$

Thus at the point $\left(\frac{\sqrt{39}}{3},\frac{13\sqrt{39}}{9}\right)$ on the given curve the tangent is parallel to the chord joining $(1, 1)$ and $(3, 27)$.

17. Find the point on the curve $y=x^3-3x$, where the tangent to the curve is parallel to the chord joining $(1,-2)$ and $(2, 2)$.

Sol. Let us apply lagrange's mean value-theorem for the function $f(x)=x^3-3x$ in the interval $[1, 2]$

Now, $f(x)$ being a polynomial function, it is continuous on $[1, 2]$. $\quad f'(x)=3x^2-3$

Also, $f'(x)$ is differentiable on $(1, 2)$.

Thus both the condition of lagrange's mean-value theorem are satisfied.

So, these must exist a point $c\in(1, 2)$ such that

$$f(1)=(1)^3-3(1)=-2;\quad f(2)=(2)^3-3(2)=2$$

$$f'(c)=\frac{f(2)-f(1)}{2-1} \Rightarrow 3c^2-3=\frac{2-(-2)}{2-1} \Rightarrow 3c^2-3=4$$

$$\Rightarrow 3c^2=7 \Rightarrow c=\pm\sqrt{\frac{7}{3}} \quad \therefore\ c=\sqrt{\frac{7}{3}}\in(1,2)$$

Now, $x=\sqrt{\frac{7}{3}}$ and $y=x^3-3x=\left(\sqrt{\frac{7}{3}}\right)^3-3\left(\sqrt{\frac{7}{3}}\right)=\sqrt{\frac{7}{3}}\left(\frac{7}{3}-3\right) \Rightarrow y=\frac{-2}{3}\sqrt{\frac{7}{3}}$

Thus at the point $\left(\sqrt{\frac{7}{3}},-\frac{2}{3}\sqrt{\frac{7}{3}}\right)$

On the given curve the tangent is parallel to the chord joining $(1,-2)$ and $(2, 2)$.

18. If $f(x)=x(1-\log x)$, where $x>0$ show that $(a-b)\log c=b(1-\log b)-a(1-\log a)$, where $0<a<c<b$.

Sol. Let us apply Lagrange's mean-value theorem for the function $f(x)=x(1-\log x)$.

Now, $f(x)$ being a polynomial and logarithms function, it is continuous for all $x>0$

and $f'(x)=x\left(-\frac{1}{x}\right)+1-\log x=-\log x$, which exist for all $x>0$.

$$\therefore\ f'(c)=\frac{f(b)-f(a)}{b-a} \Rightarrow -\log c=\frac{b(1-\log b)-a(1-\log a)}{b-a}$$

$\therefore\ (a-b)\log c=b(1-\log b)-a(1-\log a)$, where $0<a<c<b$.

MAXIMA AND MINIMA (XII, R.S. AGGARWAL)

EXERCISE 11 E (Pg.No.: 506)

Find the maximum or minimum values, if any, without using derivative of function :

1. $(5x-1)^2+4$

Sol. Let $f(x)=(5x-1)^2+4$ Its minimum value is 4, when $5x-1=0$.

But its maximum value can not be found because when x increases, then $f(x)$ increases, so $f(x)$ does not have maximum value.

2. $-(x-3)^2+9$

Sol. Let $f(x)=-(x-3)^2+9$, we know that $(x-3)^2\geq 0 \Rightarrow -(x-3)^2\leq 0$

$\Rightarrow -(x-3)^2+9\leq 0+9 \Rightarrow f(x)\leq 9$

$\Rightarrow$ maximum value of $f(x)=9$, which occurs as $x=3$ called point of absolute maxima and minimum value of $f(x)$ does not exist.

3. $-|x+4|+6$

Sol. We know that $|x+4|\geq 0 \Rightarrow -|x+4|\leq 0 \Rightarrow -(x+4)+6\leq 0+6 \Rightarrow f(x)\leq 6$

$\Rightarrow$ maximum value of $f(x)=6$, which occurs at $x=-4$ called point of absolute maxima and minimum value of $f(x)$ does not exits.

4. $\sin 2x+5$

Sol. We know that, for all $x \Rightarrow -1\leq \sin 2x\leq 1 \Rightarrow 5-1\leq \sin 2x+5\leq 5+1 \Rightarrow 4\leq f(x)\leq 6$

$\Rightarrow$ minimum value of $f(x)=4$ and maximum value of $f(x)=6$.

5. $|\sin 4x+3|$

Sol. We know that for all $x \Rightarrow -1\leq \sin 4x\leq 1 \Rightarrow 3-1\leq \sin 4x+3\leq 1+3$

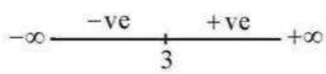
$\Rightarrow 2\leq \sin 4x+3\leq 4 \Rightarrow |2|\leq|\sin 4x+3|\leq|4| \Rightarrow 2\leq f(x)\leq 4$

Minimum value of $f(x)=2$ and maximum value of $f(x)=4$.

Find the point of local maxima or local minima and the corresponding local maximum and minimum value of each of the following functions.

6. $f(x)=(x-3)^4$

Sol. Given, $f(x)=(x-3)^4 \quad \therefore f'(x)=4(x-3)^3$



Now, $f'(x)=0 \Rightarrow 4(x-3)^3=0 \Rightarrow x-3=0 \Rightarrow x=3$

$f'(3^-)=-ve$ and $f'(3^+)=+ve$

$\therefore x=3$ is a point of local minima, and local minimum value $f(3)=0$

7. $f(x)=x^2$

Sol. Given, $f(x)=x^2 \quad \therefore f'(x)=2x$ and $f''(x)=2$

Now, $f'(x)=0 \Rightarrow 2x=0 \Rightarrow x=0$ and $f''(0)=2>0$

$\therefore$ $x=0$ is a point of local minimum, local minimum value is $f(0)=0$.

8. $f(x)=2x^3-21x^2+36x-20$

Sol. Given, $f(x)=2x^3-21x^2+36x-20 \quad \therefore f'(x)=6x^2-42x+36$

Now, $f'(x)=0 \Rightarrow 6x^2-42x+36=0$

$\Rightarrow 6(x^2-7x+6)=0 \Rightarrow 6(x^2-6x-x+6)=0$

$\Rightarrow 6\{x(x-6)-1(x-6)\}=0 \Rightarrow 6(x-6)(x-1)=0 \Rightarrow x=1, 6$

also $f''(x)=(2x-42) \Rightarrow f''(1)=-ve$.

$\therefore$ $x=1$ is a point of local maximum, local maximum value $=f(1)=2(1)^3-21(1)^2+36(1)-20=-3$

$\therefore$ $x=6$ is a point of local minimum, $\because f''(6)=30 \Rightarrow f''(6)$ is $+ve$ local minimum value

$f(6)=2(6)^3-21(6)^2+36(6)-20=-128$

9. $f(x)=x^3-6x^2+9x+15$

Sol. Given, $f(x)=x^3-6x^2+9x+15 \quad \therefore f'(x)=3x^2-12x+9$ and $f''(x)=6x-12$

Now, $f'(x)=0 \Rightarrow 3x^2-12x+9=0 \Rightarrow 3(x^2-4x+3)=0$

$\Rightarrow 3(x-1)(x-3)=0 \Rightarrow x=1, 3$

$\therefore$ $x=1$ is a point of local maximum. $\because f''(1)=-6 \therefore$ local maximum at $x=1$

Local maximum value $f(1)=(1)^3-6(1)^2+9(1)+15=1-6+9+15=19$

$\therefore$ $x=3$ is a point of local minimum. $\because f''(3)=+ve$. Hence, local minimum at $x=3$

Local minimum value $f(3)=(3)^2-6(3)^2+9\times3+15=27-54+27+15=15$

10. $f(x)=x^4-62x^2+120x+9$

Sol. Given, $f(x)=x^4-62x^2+120x+9$

$\therefore$ $f'(x)=4x^3-124x+120$ and $f''(x)=12x^2-124=4(3x^2-31)$

Now, $f'(x)=0 \Rightarrow 4x^3-124x+120=0 \Rightarrow 4(x^3-31x+30)=0 \Rightarrow 4(x+1)(x^2+x-30)=0$

$\Rightarrow 4(x+1)(x+6)(x-5)=0 \quad \therefore x=-6, -1, 5$

$-\infty$ —ve— -6 —+ve— -1 —−ve— 5 —+ve— $+\infty$

$\therefore$ $x=-6$ is a point of local minimum,

local minimum value $f(-6)=(-6)^4-62(-6)^2+120(-6)+9=1296-2232-720+9=-1647$

$\therefore$ $x=1$ is a point of local maximum,

local maximum value $f(-1)=(-1)^4-6^2(-1)^2+120(-1)+9=1-62-120+9=-173$

$\therefore$ $x=5$ is a point of local minimum,

local minimum $f(5)=(5)^4-62(5)^2+120(5)+9=625-1550+600+9=-316$

11. $f(x)=-x^3+12x^2-5$

Sol. $f(x)=-x^3+12x^2-5 \quad \therefore\ f'(x)=-3x^2+24x$ and $f''(x)=-6x+24=-6(x-4)$

Now, $f'(x)=0 \quad \Rightarrow -3x^2+24x=0 \quad \Rightarrow -3x(x-8)=0$

$\Rightarrow x=0, 8$ and $f''(0)=-6(0-4)=24>0,\ f''(8)=-6(8-4)=-6(4)=-24<0$

$\therefore\ x=0$ is a point of local minimum,

local minimum value $f(0)=-(0)^3+12(0)-5=-5$

$\therefore\ x=8$ is a point of local maximum,

local maximum value $f(8)=-(8)^3+12(8)^2-5=-512+768-5=251$

12. $f(x)=(x-1)(x+2)^2$

Sol. Given, $f(x)=(x-1)(x+2)^2$

$\therefore\ f'(x)=\{(x-1)2(x+2)+(x+2)^2.1\}=(x+2)\{2(x-1)+(x+2)\}$

$=(x+2)(2x-2+x+2)=(x+2)(3x)=3x^2+6x$ and $f''(x)=6x+6$

Now, $f'(x)=0 \quad \Rightarrow 3x(x+2)=0 \quad \Rightarrow x=0,-2$ and $f''(0)=6(0)+6=6>0$

$\therefore\ x=0$ is a point of local minimum,

local minimum value $f(0)=(0-1)(0+2)^2=-4$.

$\therefore\ x=-2$ is a point of local maximum,

local maximum value $f(-2)=f(-2-1)(-2+2)^2=0$

13. $f(x)=-(x-1)^3(x+1)^2$

Sol. Given, $f(x)=-(x-1)^3(x+1)^2$

$-\infty$ —ve— -1 —+ve— $-1/5$ —−ve— 1 —+ve— $+\infty$

$\therefore\ f'(x)=-\left[(x-1)^3 2(x+1)+(x+1)^2.3(x-1)^2\right]$

$=-(x-1)^2(x+1)[2(x-1)+3(x+1)]=-(x-1)^2(x+1)(2x-2+3x+3)$

$=-(x-1)^2(x+1)(5x+1)=-(x-1)^2(5x^2+6x+1)=-(x-1)^2(x+1)(5x+1)$

$f'(x)=0 \quad \therefore x=1,-1,-\frac{1}{5}$

$\therefore\ x=-1$ is a point of local minima, local minimum value $f(-1)=-(-1-1)^3(-1+1)^2=0$

$\therefore\ x=1$ is neither maximum nor minimum.

$\therefore\ x=\frac{-1}{5}$ is a point of local minimum,

local minimum value $f\left(\frac{-1}{3}\right)=-\left(\frac{-1}{5}-1\right)^3\left(-\frac{1}{5}+1\right)^2=-\left(\frac{-6}{5}\right)^3\left(\frac{4}{5}\right)^2=\frac{3456}{3125}$

14. $f(x)=\frac{x}{2}+\frac{2}{x},\ x>0$

Sol. $f(x)=\frac{x}{2}+\frac{2}{x} \quad \therefore\ f'(x)=\frac{1}{2}+2\left(\frac{-1}{x^2}\right)=\frac{1}{2}-\frac{2}{x^2}$ and $f''(x)=-2(-2)x^{-3}=\frac{4}{x^3}$

Now $f'(x)=0 \Rightarrow \frac{1}{2}-\frac{2}{x^2}=0 \Rightarrow \frac{1}{2}=\frac{2}{x^2} \Rightarrow x^2=4 \Rightarrow x=\pm 2$

$$f''(-2)=\frac{4}{(-2)^3}=\frac{4}{-8}=\frac{-1}{2}<0,\quad f''(2)=\frac{4}{8}=\frac{1}{2}>0$$

$\therefore$ $x=-2$ is a point of local maximum, local maximum value of $f(-2)=\frac{-2}{2}+\frac{2}{-2}=-1-1=-2$

$\therefore$ $x=2$ is a point of local minimum, local minimum value of $f(2)=\frac{2}{2}+\frac{2}{2}=1+1=2$

15. Find the maximum and minimum value of $2x^3-24x+107$ on the interval $[-3, 3]$

Sol. We have, $f(x)=2x^3-24x+107$

$\therefore$ $f'(x)=6x^2-24$ i.e., $f'(x)=6(x^2-4)$, $f'(x)=0 \Rightarrow 6(x^2-4)=0 \Rightarrow x=-2, 2$

Now, the critical point be $x=-3,-2, 2, 3$

$$f(-2)=2(-2)^3-24(-2)+107=-16+48+107=139$$

$$f(-3)=2(-3)^3-24(-3)+107=-54+72+107=125$$

$$f(2)=2(2)^3-24(2)+107=16-48+107=75$$

$$f(3)=2(3)^3-24(3)+107=54-72+107=89$$

Thus f has the maximum at $x=-2$ and the minimum at $x=2$.

The maximum of f is 139 and the minimum of f is 75.

16. Find the maximum and minimum value of $3x^4-8x^3+12x^2-48x+1$ on the interval $[1, 4]$

Sol. We have, $f(x)=3x^4-8x^3+12x^2-48x+1$

$\therefore$ $f'(x)=12x^3-24x^2+24x-48$ i.e., $f'(x)=12(x^3-2x^2+2x-4)$

$f'(x)=12\{x^2(x-2)+2(x-2)\}$, $f'(x)=12(x-2)(x^2+2)$

$f'(x)=0 \Rightarrow 12(x-2)(x^2+2)=0 \Rightarrow x=2$

Since real roots be $x=2$, so critical point be $x=1, 2, 4$

$$f(1)=3(1)^4-8(1)^3+12(1)^2-48(1)+1=3-8+12-48+1=-40$$

$$f(2)=3(2)^4-8(2)^3+12(2)^2-48(2)+1=48-64+48-96+1=-63$$

$$f(4)=3(4)^4-8(4)^3+12(4)^2-48(4)+1=768-512+768-192+1=257$$

Thus f has the maximum at $x=4$ and the minimum at $x=2$.

The maximum of f is 257 and the minimum of f is -63

17. Find the maximum and minimum value of $f(x)=\sin x+\frac{1}{2}\cos 2x$ in $0\le x\le\frac{\pi}{2}$

Sol. $f(x)=\sin x+\frac{1}{2}\cos 2x \Rightarrow f'(x)=\cos x+\frac{1}{2}(-\sin 2x).2$

$\Rightarrow f'(x)=\cos x-\sin 2x \Rightarrow f'(x)=\cos x-2\sin x\cos x \Rightarrow f'(x)=\cos x(1-2\sin x)$

$f'(x)=0 \Rightarrow \cos x(1-2\sin x)=0 \Rightarrow \cos x=0$ or $1-2\sin x=0$

$\Rightarrow x=\frac{\pi}{2}$ or $\sin x=\frac{1}{2} \Rightarrow x=\frac{\pi}{6}$

Now critical point be $x=0, \frac{\pi}{6}, \frac{\pi}{2}$

$$f(0)=\sin 0+\frac{1}{2}\cos 2.0=\frac{1}{2},\ f\left(\frac{\pi}{6}\right)=\sin\frac{\pi}{6}+\frac{1}{2}\cos\left(2.\frac{\pi}{6}\right)=\frac{1}{2}+\frac{1}{2}.\left(\frac{1}{2}\right)=\frac{3}{4}$$

$$f\left(\frac{\pi}{2}\right)=\sin\frac{\pi}{2}+\frac{1}{2}\cos\left(2.\frac{\pi}{2}\right)=1+\frac{1}{2}(-1)=\frac{1}{2}$$

Thus f has the maximum at $x=\frac{\pi}{6}$ and the minimum at $x=\frac{\pi}{2}$ the maximum of f is $\frac{3}{4}$ and the minimum of f is $\frac{1}{2}$.

18. Show that the maximum value of $x^{\frac{1}{x}}$ is $e^{\frac{1}{e}}$

Sol. Let $y=\left(\frac{1}{x}\right)^{x}=x^{-x}$ ...(1)

Taking log both side, we get, $\log y=\log\left(x^{-x}\right)$

$$\Rightarrow \log y=-x\log x \quad \Rightarrow \frac{1}{y}\frac{dy}{dx}=-\left(x.\frac{1}{x}+\log x.1\right) \quad \Rightarrow \frac{1}{y}\frac{dy}{dx}=-(1+\log x)$$

$$\Rightarrow \frac{dy}{dx}=-y(1+\log x) \quad \Rightarrow \frac{d^2y}{dx^2}=\frac{-dy}{dx}(1+\log x)-\frac{y}{x}$$

$$\Rightarrow \frac{d^2y}{dx^2}=y(1+\log x)^2-\frac{y}{x} \quad \Rightarrow \frac{d^2y}{dx^2}=x^{-x}(1+\log x)^2-\frac{x^{-x}}{x}$$

$$\Rightarrow \frac{d^2y}{dx^2}=x^{-x}(1+\log x)^2-x^{-x-1}$$

For maximum and minimum, we must have $\frac{dy}{dx}=0$.

$$\Rightarrow -y(1+\log x)=0 \quad \Rightarrow 1+\log x=0 \quad \Rightarrow \log x=-1 \quad \Rightarrow x=e^{-1}=\frac{1}{e}$$

Also, $\left(\frac{d^2y}{dx^2}\right)_{x=\frac{1}{e}}-\left(\frac{1}{e}\right)^{-\frac{1}{e}}\left(1+\log\frac{1}{e}\right)^2-\left(\frac{1}{e}\right)^{-\frac{1}{e}-1}$

$$=\left(e^{-1}\right)^{\frac{-1}{e}}(1-\log e)^2-\left(e^{-1}\right)^{\frac{-1}{e}-1}=e^{\frac{1}{e}}(1-1)^2-\left(e^{-1}\right)^{\frac{-1}{e}-1}<0$$

So, $x=\frac{1}{e}$ is a point of local maximum. The local maximum value of y is given by $y=(e)^{\frac{1}{e}}$

19. Show that $x+\frac{1}{x}$ has a maximum and minimum, but the maximum value is less than the minimum value.

Sol. Let $f(x)=x+\frac{1}{x}$. Now, $f'(x)=1-\frac{1}{x^2}$ and $f''(x)=\frac{2}{x^3}$

$$f'(x)=0 \quad \Rightarrow 1-\frac{1}{x^2}=0 \quad \Rightarrow \frac{x^2-1}{x^2}=0 \quad \Rightarrow x^2-1=0 \Rightarrow (x-1)(x+1)=0$$

$\Rightarrow (x-1)(x+1)=0 \quad \Rightarrow x=-1, 1$

and $f''(-1)=\frac{2}{(-1)^3}=-2<0$, $f''(1)=\frac{2}{(1)^3}=2>0$

$\therefore$ $x=-1$ is a point of local maximum, local maximum value $f(-1)=-1+\frac{1}{(-1)}=-2$

$\therefore$ $x=1$ is a point of local minimum, local minimum value $f(1)=1+\frac{1}{1}=2$

Hence, maximum value is less than the minimum value.

20. Find the maximum profit that a company can make, if the profit function is given by $P(x)=41+24x-18x^2$.

Sol. $P(x)=41+24x-18x^2 \quad \ldots(1)$

$\Rightarrow P'(x)=24-36x \quad \ldots(2)$

$\Rightarrow P''(x)=-36$ for all $x \quad \ldots(3)$

$P'(x)$ exists for all x. $P'(x)=0 \Rightarrow 24-36x=0 \Rightarrow x=\frac{2}{3}$

$P''\left(\frac{2}{3}\right)=-36=-ve \quad \Rightarrow P(x)$ has local maximum at $x=\frac{2}{3}$

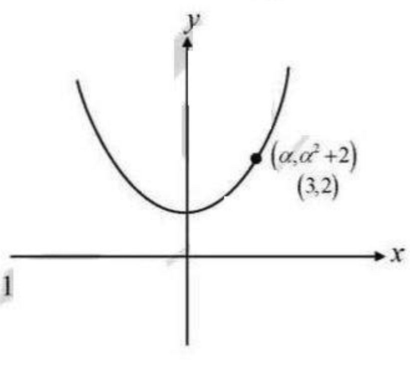
Also $P(x)$ is continuous for all x and has only one extreme point.

$\Rightarrow P(x)$ is absolute maximum at $x=\frac{2}{3}$ and maximum value of

$$P(x)=P\left(\frac{2}{3}\right)=41+24\left(\frac{2}{3}\right)-18\left(\frac{4}{9}\right)=41+16-8=49$$

21. An enemy set is flying along the curve $y=(x^2+2)$. A soldier is placed at the point $(3, 2)$. Find the nearest point between the soldier and jet.

Sol. If (α, α^2+2) is the point on the curve nearest to the soldiers then the normal at that point will pass through



$\therefore$ Equation of normal is $y-(\alpha^2+2)=\frac{-1}{2\alpha}(x-\alpha)$

As it passes through $(3, 2)$

$\therefore \quad 2-(\alpha^2+2)=-\frac{1}{2\alpha}(3-\alpha)$

$=2\alpha^3+\alpha-3=0 \quad \Rightarrow (\alpha-1)(2\alpha^2+2\alpha+3)=0 \quad \Rightarrow \alpha=1$

Hence the point is $(1, 3)$.

22. Find the maximum and minimum value of $f(x)=(-x+2\sin x)$ on $[0, 2\pi]$

Sol. We have, $f(x)=-x+2\sin x$, $f'(x)=-1+2\cos x$

$f'(x)=0 \Rightarrow -1+2\cos x=0 \Rightarrow 2\cos x=1 \Rightarrow \cos x=\frac{1}{2} \Rightarrow x=\frac{\pi}{3}, \frac{5\pi}{3}$

So critical point be $x=0, \frac{\pi}{3}, \frac{5\pi}{3}, 2\pi$

$f(0)=-0+2\sin 0=0$, $f\left(\frac{\pi}{3}\right)=\frac{-\pi}{3}+2\sin\frac{\pi}{3}=\frac{-\pi}{3}+2.\frac{\sqrt{3}}{2}=\left(\frac{-\pi}{3}+\sqrt{3}\right)$

$f\left(\frac{5\pi}{3}\right)=\frac{-5\pi}{3}+2\sin\frac{2\pi}{3}=\frac{-5\pi}{3}+2.\left(\frac{-\sqrt{3}}{2}\right)=-\left(\frac{5\pi}{3}+\sqrt{3}\right)$, $f(2\pi)=-2\pi+2\sin 2\pi=-2\pi$

Thus, f has the maximum at $x=\frac{\pi}{3}$ and the minimum at $x=\frac{5\pi}{3}$.

The maximum of f is $\left(\frac{-\pi}{3}+\sqrt{3}\right)$ and the minimum if f is $-\left(\frac{5\pi}{3}+\sqrt{3}\right)$.

EXERCISE 11 F (Pg. No.: 521)

1. Find two positive numbers whose product is 49 and the sum is minimum.

Sol. Let the number be x and $\frac{49}{x}$. Let $S=x+\frac{49}{x}$.

Then, $\frac{dS}{dx}=1+49\left(\frac{-1}{x^2}\right)$, $\frac{dS}{dx}=1-\frac{49}{x^2}$ and $\frac{d^2S}{dx^2}=\frac{98}{x^3}$

For a maximum or minimum, we have $\frac{dS}{dx}=0$

Now, $\frac{dS}{dx}=0 \Rightarrow 1-\frac{49}{x^2}=0 \Rightarrow x^2=49 \Rightarrow x=7$

Also $\left[\frac{d^2S}{dx^2}\right]_{x=7}=\frac{98}{7\times7\times7}=\frac{98}{343}>0$

So, $x=7$ is a point of minimum. Hence, the required number are 7 and 7

2. Find two positive numbers whose sum is 16 and the sum of whose squares is minimum.

Sol. Let the number be x and $16-x$.

Let $S=(x)^2+(16-x)^2=x^2+256-32x+x^2 \Rightarrow S=2x^2-32x+256$

Then, $\frac{dS}{dx}=4x-32$ and $\frac{d^2S}{dx^2}=4$

For a minimum or minimum, we have $\frac{dS}{dx}=0$, $4x-32=0 \Rightarrow x=8$

Also $\left[\frac{d^2S}{dx^2}\right]_{x=8}=4>0$. So $x=8$ is a point of minimum.

Hence, the required numbers are 8 and 8.

3. Divide 15 into two parts such that the square of one number multiplied with the cube of the other number is maximum.

Sol. Let be the number x and $15-x$

$P=(15-x)^2x^3$

$\frac{dP}{dx}=(15-x)^2 3x^2+x^3.2(15-x)(-1)=(15-x)x^2\left[3(15-x)-2x\right]$

$\Rightarrow \frac{dP}{dx}=(15-x)x^2(45-5x)$

For maximum or minimum value, $\frac{dP}{dx}=0 \Rightarrow x^2(15-x)(45-5x)=0$

So, $x = 0, 9, 15$

$$\frac{d^2P}{dx^2} = (15-x)x^2(-5) + (15-x)(45-5x)2x + x^2(45-5x)(-1)$$

At $x = 0, 15$, product is zero. So, it is rejected.

For $x = 9$, $\frac{d^2P}{dx^2} = -\text{ve}$. Hence, the product is maximum, when $x = 9$.

So, the required numbers are 9 and 6.

4. Divide 8 into two positive parts such that the sum of the square of one and the cube of the other is minimum.

Sol. Let be number x and $8-x$, $S = (x)^2 + (8-x)^3$

$$\Rightarrow \frac{dS}{dx} = 2x + 3(8-x)^2.(-1) \Rightarrow \frac{dS}{dx} = 2x - 3(64 - 16x + x^2) \Rightarrow \frac{dS}{dx} = 2x - 192 + 48x - 3x^2$$

$$\Rightarrow \frac{dS}{dx} = -(3x^2 - 50x + 192) \Rightarrow \frac{dS}{dx} = -(x-6)(3x-32) \Rightarrow \frac{d^2S}{dx^2} = -(6x-50)$$

For maximum or minimum than $\frac{dS}{dx} = 0$, $-(x-6)(3x-32) = 0$

$$x = 6 \text{ or } \frac{32}{3}, \left[\frac{d^2S}{dx^2}\right]_{x=6} = -(6\times 6 - 50) = -(36-50) = 14 > 0$$

So $x = 6$ is a point of minimum. Hence, the required point is 6 and 2.

5. Divide a into two parts such that the product of the p th power of one part and the q th power of the second part may be maximum.

Sol. Let be the number x and $a-x$, $y = x^p(a-x)^q$

$$\Rightarrow \frac{dy}{dx} = x^p.q(a-x)^{q-1}(-1) + (a-x)^q px^{p-1}$$

$$= x^{p-1}(a-x)^{q-1}(-qx + p(a-x)) = x^{p-1}(a-x)^{q-1}(ap - (p+q)x) = 0$$

But for $x = 0$ or a, product is zero and $\left.\frac{dy}{dx}\right]_{x<\frac{ap}{p+q}} = +ve$ and $\left.\frac{dy}{dx}\right]_{x>\frac{ap}{p+q}} = -ve$

Hence, y is maximum for $x = \frac{ap}{p+q}$.

6. The rate of working of an engine is given by $R = 15v + \frac{6000}{v}$, where $0 < v < 30$ and v is the speed of the engine. Show that R is the least when $v = 20$.

Sol. $R = 15v + \frac{6000}{v}$ ...(1)

$$\Rightarrow \frac{dR}{dv} = 15 + \left(-\frac{6000}{v^2}\right) \Rightarrow \frac{dR}{dv} = 15 - \frac{6000}{v^2} \Rightarrow \frac{d^2R}{dv^2} = \frac{12000}{v^3}$$

For maximum or minimum than $\frac{dR}{dv} = 0$

$$15 - \frac{6000}{v^2} = 0 \Rightarrow 15 = \frac{6000}{v^2} \Rightarrow v^2 = 400 \Rightarrow v = 20$$

$$\left[\frac{d^2R}{dv^2}\right]_{v=20}=\frac{12000}{(20)^3}=\frac{12000}{8000}=\frac{3}{2}>0.\ \ v=20 \text{ is a point of minimum.}$$

7. Find the dimensions of the rectangle of area 96 cm^2 whose perimeter is the least. Also, find the perimeter of the rectangle.

Sol. Let P be the fixed perimeter and x, y be the sides. Then area $= xy=96 \Rightarrow y=\frac{96}{x}$ ...(1)

$$P=2(x+y) \Rightarrow P=2\left(x+\frac{96}{x}\right) \Rightarrow \frac{dp}{dx}=2\left(1-\frac{96}{x^2}\right)$$

and $\frac{d^2P}{dx^2}=\frac{2(192)}{x^3}=\frac{384}{x^3}$

For maximum or minimum $\frac{dP}{dx}=0$, $2\left(1-\frac{96}{x^2}\right)=0 \Rightarrow 1-\frac{96}{x^2}=0 \Rightarrow x^2=96 \Rightarrow x=4\sqrt{6}$

At $x=4\sqrt{6}$, $\frac{d^2P}{dx^2}>0$. So, the perimeter is least.

Hence equation (1), $y=\frac{96}{4\sqrt{6}}=4\sqrt{6}$. Length be $x=4\sqrt{6}\ cm$, breadth be $y=4\sqrt{6}$ cm

Perimeter $=2(x+4)=2\left(4\sqrt{6}+4\sqrt{6}\right)=2\left(8\sqrt{6}\right)=16\sqrt{6}$ cm.

8. Prove that the largest rectangle with a given perimeter is a square.

Sol. Let a be the fixed perimeter. Consider a rectangle with sides x and y and perimeter a, let A be the area of the rectangle.

Then, $2x+2y=a$ ...(1)

And $A=xy=x\left(\frac{a-2x}{2}\right)$ [Using (1)]

$\therefore \frac{dA}{dx}=\frac{1}{2}(a-4x)$ and $\frac{d^2A}{dx^2}=-2$

Now, $\frac{dA}{dx}=0 \Rightarrow \frac{1}{2}(a-4x)=0 \Rightarrow x=\frac{a}{4}$. Also $\left[\frac{d^2A}{dx^2}\right]_{x=\left(\frac{a}{4}\right)}=-2<0$

Also, from (1), we have $y=\frac{1}{2}(a-2x)=\frac{1}{2}\left(a-\frac{a}{2}\right)=\frac{a}{4}$

Then, $x=y$ [Each $=\frac{a}{4}$]. Hence, the rectangle is a square.

9. Given the perimeter of a rectangle, show that its diagonal is minimum when it is a square.

Sol. Let P be the fixed perimeter. Consider a rectangle with sides x and y and perimeter P. D be the diagonal. $P=2(x+y) \Rightarrow y=\frac{1}{2}(P-2x)$ ...(1)

$$\therefore D^2=Z=x^2+y^2 \Rightarrow Z=x^2+\frac{1}{4}(P-2x)^2 \Rightarrow \frac{dZ}{dx}=2x+\frac{1}{4}(P-2x)(-2)(2)$$

$$\Rightarrow \frac{dZ}{dx}=2x-(P-2x) \Rightarrow \frac{dZ}{dx}=4x-P \text{ and } \frac{d^2Z}{dx^2}=4>0$$

Now, $\frac{dZ}{dx}=0 \Rightarrow 4x-P=0 \Rightarrow x=\frac{p}{4}$

Also $\left[\frac{d^2Z}{dx^2}\right]_{x=\left(\frac{P}{4}\right)} = 4 > 0$. So, the diagonal is minimum at $x = \frac{P}{4}$.

Also from (1), we have $y = \frac{1}{2}(P-2x) = \frac{1}{2}\left(P - \frac{P}{2}\right) = \frac{P}{4}$

Then $x = y$ $\left[\text{Each} = \frac{P}{4}\right]$. Hence, the rectangle is a square.

10. Show that a rectangle of maximum perimeters which can be inscribed in a circle of radius a is a square of side $\sqrt{2}a$.

Sol. Let $ABCD$ be the rectangle inscribed in a circle of radius a and with centre O.

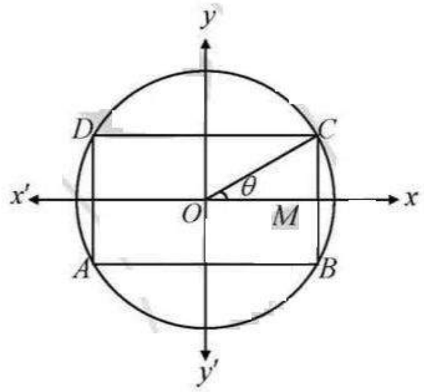
Join OC, let $\angle COX = \theta$.

Then, the co-ordinate of C are $(a\cos\theta, a\sin\theta)$

$\therefore$ $OM = a\cos\theta$ and $MC = a\sin\theta$, $BC = 2MC = 2a$

and $CD = 2OM = 2a\cos\theta$

$\therefore$ $P = 2a(\cos\theta + \sin\theta)$.



So, $\frac{dP}{d\theta} = 2a(-\sin\theta + \cos\theta)$ and $\frac{d^2P}{d\theta^2} = -2a(\cos\theta + \sin\theta)$

Now, $\frac{dP}{d\theta} = 0$

$\Rightarrow 2a(-\sin\theta + \cos\theta) = 0 \Rightarrow (\cos\theta - \sin\theta) = 0$

$\Rightarrow \cos\theta = \sin\theta \Rightarrow 1 = \tan\theta \Rightarrow \theta = \frac{\pi}{4}$

and at this value of θ, $\frac{d^2P}{d\theta^2} = -2a\left(\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\right) < 0$. So, P is maximum at $\theta = \frac{\pi}{4}$.

Then, $BC = \sqrt{2}a = CD$.

11. The sum of the perimeters of a square and a circle is given. show that the sum of their areas is least when the side of the square is equal to the diameter of the circle.

Sol. Let x be side of the square and r be the radius of circle. Given, $4x + 2\pi r = k$

$\Rightarrow x = \frac{k - 2\pi r}{4}$ ...(1)

$A = x^2 + \pi r^2 = \left[\frac{k - 2\pi r}{4}\right]^2 + \pi r^2 \Rightarrow A = \frac{1}{16}\left(k^2 - 4k\pi r + 4\pi^2 r^2\right) + \pi r^2$

$\Rightarrow \frac{dA}{dr} = \frac{1}{16}\left(-4k\pi + 8\pi^2 r\right) + 2\pi r$ and $\frac{d^2A}{dr^2} = \frac{\pi^2}{2} + 2\pi$

Now, $\frac{dA}{dr} = 0 \Rightarrow 2\pi r - \frac{4k\pi}{16} + \frac{8\pi^2 r}{16} = 0$

$\Rightarrow r\left(2\pi + \frac{\pi^2}{2}\right) = \frac{k\pi}{4} \Rightarrow r = \frac{\left(\frac{k\pi}{4}\right)}{2\pi + \frac{\pi^2}{2}} = \frac{k}{8 + 2\pi}$ ...(2)

And $\left(\frac{d^2A}{dr^2}\right)_{r=\frac{k}{8+2\pi}}$ = positive. $\therefore$ A is least when $r = \frac{k}{8 + 2\pi}$

And from (1), $x=\frac{k-2\pi r}{4}=\frac{1}{4}\left[k-2\pi.\frac{k}{8+2\pi}\right]$

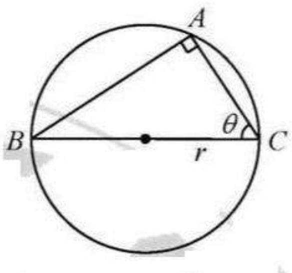
$$=\frac{1}{4}\left[\frac{8k+2\pi k-2\pi k}{8+2\pi}\right]=\frac{2k}{8+2\pi}=2\left(\frac{k}{8+2\pi}\right)=2r$$

$\therefore$ A is least when x (side of square) = diameter of the circle.

12. Show that the right triangle of maximum area that can be inscribed in a circle is an isosceles triangle.

Sol. Here, $AC=2r\cos\theta$

$AB=2r\sin\theta$



$\therefore$ $\Delta ABC=\frac{1}{2}.2r\cos\theta.2r\sin\theta=r^2\sin 2\theta$

Clearly, area will be maximum.

When $\sin 2\theta=1$ i.e., $2\theta=90°$ $\Rightarrow\theta=45°$

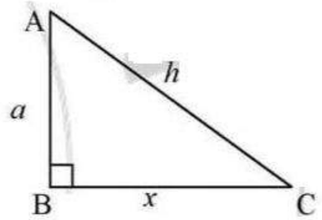
Hence, ΔABC is isosceles triangle.

13. Prove that the perimeter of a right-angled triangle of given hypotenuse is maximum when the triangle is isosceles.

Sol. Let ABC be the right triangle with given hypotenuse h.

Let base $BC=x$ and altitude $BA=a$. Then, $h^2=x^2+a^2$ $\Rightarrow a=\sqrt{h^2-x^2}$

$\therefore$ $P=a+x+h$ $\Rightarrow P=\sqrt{h^2-x^2}+x+h$



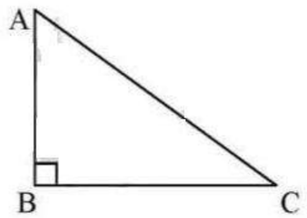
So, $\frac{dP}{dx}=\frac{-x}{\sqrt{h^2-x^2}}+1$ and $\frac{d^2P}{dx^2}=\frac{-h^2}{\left(h^2-x^2\right)^{3/2}}$

Now, $\frac{dP}{dx}=0$ $\Rightarrow x=\frac{h}{\sqrt{2}}$ and for $x=\frac{h}{\sqrt{2}}$, $\frac{d^2P}{dx^2}=\frac{-2^{3/2}}{h}<0$

$\therefore$ P is maximum when $x=\frac{h}{\sqrt{2}}$ and $a=\frac{h}{\sqrt{2}}$ i.e., when $x=a$

14. The perimeter of a triangle is 8 cm. If one of the sides of the triangle be 3 cm, what will be the other two sides for maximum area of the triangle?

Sol. Let the sides are x and $5-x$, $S=\frac{x+5-x+3}{2}=\frac{8}{2}=4$



Area $\Delta ABC=\sqrt{S(s-a)(s-b)(s-c)}$

$A=\sqrt{4(4-3)(4-x)\{4-(5-x)\}}$, $A=\sqrt{4\times1\times(4-x)(x-1)}$

$A=2\sqrt{(4-x)(x-1)}$, $A=2\sqrt{4x-4-x^2+x}$, $A=2\sqrt{5x-4-x^2}$

We will find maximum value of $5x-4-x^2$

$\therefore$ Let $y=-x^2+5x-4$ $\Rightarrow\frac{dy}{dx}=-2x+5$

$\frac{dy}{dx}=0$ $\Rightarrow x=\frac{5}{2}$ and $\frac{d^2y}{dx^2}=-2$

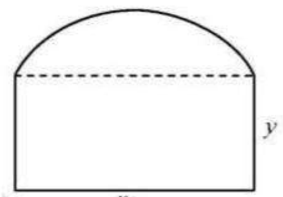
$\Rightarrow$ y will be maximum for $x=\frac{5}{2}$. Hence area will be maximum for $x=\frac{5}{2}$, so, the sides are $\frac{5}{2},\frac{5}{2}$.

15. A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 meters. Find the dimensions of the window to admit maximum light through it.

Sol. Let the length and breadth of the rectangle be x and y metres respectively. Then, radius of the semicircle $=(x/2)$ metres.

So, the perimeter of the window is given by

$$10 = x + 2y + \frac{\pi x}{2} \quad \Rightarrow y = 5 - \left(\frac{2+\pi}{4}\right)x$$



$$\therefore \quad A = xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2 \quad \Rightarrow A = x\left[5 - \frac{(2+\pi)}{4}.x\right] + \frac{\pi x^2}{8}$$

$$\therefore \quad \frac{dA}{dx} = \left(5 - x - \frac{\pi x}{4}\right) \text{ and } \frac{d^2A}{dx^2} = -\left(1 + \frac{\pi}{4}\right) < 0$$

Now, $\frac{dA}{dx} = 0 \quad \Rightarrow x = \frac{20}{(\pi+4)}$

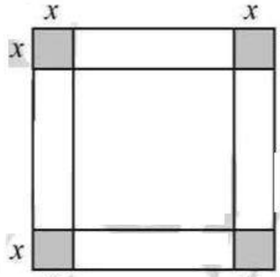
$\therefore \quad A$ is maximum when $x = \frac{20}{(\pi+4)}$ and $y = \frac{10}{(\pi+4)}$

16. A square piece of tin of side 12 cm is to be made into a box without a lid by cutting a square from each corner and folding up the flaps to form the sides. What should be the side of the square to be cut off so that the volume of the box is maximum? Also, find this maximum volume.

Sol. The length cut off from each corner be x cm.

Then, length of the box $=(12-2x)$ ($m =$ breadth and height $= x$ cm.

$$\therefore \quad V = (12-2x)^2 \times x \quad \Rightarrow V = 4x^3 - 48x^2 + 144x$$



$$\therefore \quad \frac{dV}{dx} = 12(x-6)(x-2) \text{ and } \frac{d^2V}{dx^2} = 24(x-4)$$

Then, $\frac{dV}{dx} = 0 \quad \Rightarrow x = 2 \quad [\because x = 6 \text{ is not possible}]$

Also at $x = 2$, $\frac{d^2V}{dx^2} < 0$

$\therefore \quad V$ is maximum when $x = 2$

and maximum $V = \left[(12-4)^2 \times 2\right]\text{cm}^3 \quad \Rightarrow V = [64 \times 2]\text{cm}^3, \; V = 128\,\text{cm}^3$

17. An open box with a square base is to be made out of a given cardboard of area c^2 square units. Show that the maximum volume of the box is $\frac{c^3}{6\sqrt{3}}$ cubic units.

Sol. Let each side, of the base a and height be h, then, $c^2 = (a^2 + 4ah)$

$$\Rightarrow h = \frac{c^2 - a^2}{4a}, \text{ so, } V = a^2 \times h \quad \Rightarrow V = \frac{(c^2a - a^3)}{4}$$

$$\therefore \quad \frac{dV}{da} = \frac{(c^2 - 3a^2)}{4} \text{ and } \frac{d^2V}{da^2} = \frac{-3}{2}a < 0$$

Thus, V is maximum when $a^2 = \frac{c^2}{3}$ and $h = \frac{c}{2\sqrt{3}}$

18. A cylindrical can is to be made to hold 1 litre of oil. Find the dimensions which will minimize the cost of the metal to make the can.

Sol. Volume of the cylindrical $= \pi r^2 h$, $1000 = \pi r^2 h$

$$\Rightarrow h = \frac{1000}{\pi r^2} \quad \left[\because v = 1 \text{ litre} = 1000\,cm^3\right]$$

$$\therefore \; S = \left(2\pi r^2 + 2\pi r h\right) \;\Rightarrow 2\left(\pi r^2 + \frac{1000}{r}\right) \;\Rightarrow \frac{dS}{dr} = 2\left(\pi . 2r - \frac{1000}{r^2}\right)$$

$$\Rightarrow \frac{dS}{dr} = 4\left(\pi r - \frac{500}{r^2}\right) \;\Rightarrow \frac{d^2S}{dr^2} = 4\left(1 + \frac{1000}{r^3}\right)$$

For maximum or minimum than $\frac{dS}{dr} = 0$, $4\left(\pi r - \frac{500}{r^2}\right) = 0$

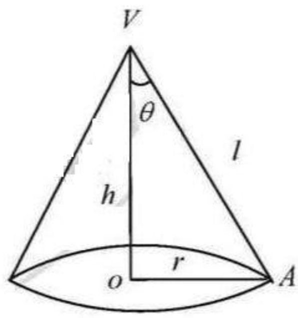
$$\Rightarrow \pi r = \frac{500}{r^3} \;\Rightarrow r^3 = \frac{500}{\pi} \;\Rightarrow r = \left(\frac{500}{\pi}\right)^{1/3} \text{ cm}$$

$$\text{Height} = \frac{1000}{\pi r^2} = \frac{1000}{\pi\left\{\left(\frac{500}{\pi}\right)^{1/3}\right\}^2} = \frac{1000}{\pi\left(\frac{500}{\pi}\right)^{2/3}} = \frac{1000}{\pi\frac{1}{3}(500)^{2/3}} \text{ cm}$$

19. Show that the right circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ times the radius of the base.

Sol. Least amount of convex will be required, if curved surface say S of conical tent is least.

Let r be radius of base, h is height and V is given volume of cone.



Now, $V = \frac{\pi r^2 h}{3}$ ...(1)

$\Rightarrow h = \frac{3V}{\pi r^2}$ ...(2)

Also $S = \pi r l = \pi r\sqrt{r^2 + h^2}$

$\Rightarrow S^2 = \pi^2 r^2\left(r^2 + h^2\right) = \pi^2 r^2\left(r^2 + \frac{9V^2}{\pi^2 r^4}\right)$ [using (2)]

$\Rightarrow Z = S^2 = \pi^2\left(r^4 + \frac{9V^2}{\pi^2 r^2}\right)$ ...(3)

$\Rightarrow \frac{dZ}{dr} = \pi^2\left(4r^3 - \frac{18V^2}{\pi^2 r^3}\right)$ ...(4)

$\Rightarrow \frac{d^2z}{dr^2} = \pi^2\left(12r^2 + \frac{54V^2}{\pi^2 r^4}\right)$ ...(5)

$$\frac{dZ}{dr} = 0 \;\Rightarrow 4r^3 - \frac{18V^2}{\pi^2 r^3} = 0 \;\Rightarrow 4r^3 = \frac{18V^2}{\pi^2 r^3} \;\Rightarrow 2r^6 = \frac{9V^2}{\pi^2} \;\Rightarrow 9V^2 = 2\pi^2 r^6$$

For $9V^2 = 2\pi^2 r^6$, $\frac{d^2Z}{dr^2} = +ve$

Hence, Z, i.e. S^2 i.e., S has minimum value is $9V^2 = 2\pi^2 r^6$

i.e., if $9.\left(\frac{\pi r^2 h}{3}\right)^2 = 2\pi^2 r^6$, i.e., if $h^2 = 2r^2$ i.e., if $h = \sqrt{2}r$.

Hence, S is least if $h=\sqrt{2}r$, which prove the required result.

20. Find the radius of a closed right circular cylinder of volume $100cm^3$ which has the minimum total surface area.

Sol. Let r be the radius and h the height of cylindrical, given volume $=100=\pi r^2 h$

$$\Rightarrow h=\frac{100}{\pi r^2} \quad \ldots(1)$$

$$S=2\pi rh+2\pi r^2=2\pi r\left(\frac{100}{\pi r^2}\right)+2\pi r^2=\frac{200}{r}+2\pi r^2$$

$$\frac{dS}{dr}=\frac{-200}{r^2}+4\pi r=\frac{-200+4\pi r^3}{r^2} \quad \ldots(2)$$

$$\frac{dS}{dr}=0 \quad \Rightarrow \frac{-200+4\pi r^3}{r^2}=0 \quad \Rightarrow 4\pi r^3=200$$

$$\Rightarrow \pi r^3=50 \quad \Rightarrow r=\left(\frac{50}{\pi}\right)^{1/3} \quad \ldots(3)$$

Now form (2), $\frac{dS}{dr}=-\frac{200}{r^2}+4\pi r \quad \Rightarrow \frac{d^2S}{dr^2}=\frac{400}{r^3}+4\pi$

At $r=\left(\frac{50}{\pi}\right)^{1/3}$, $\frac{d^2S}{dr^2}=+ve \quad \Rightarrow S$ is minimum when $r=\left(\frac{50}{\pi}\right)^{1/3}$

21. Show that the height of a closed cylinder of given volume and the least surface area is equal to its diameter.

Sol. Given volume of cylinder $=\pi r^2 h \quad \Rightarrow h=\frac{V}{\pi r^2} \quad \ldots(1)$

Now, $S=2\pi r^2+2\pi rh=2\pi r(r+h)=2\pi r\left(r+\frac{V}{\pi r^2}\right)=2\pi\left(r^2+\frac{V}{\pi r}\right)$

$$\therefore \frac{dS}{dr}=2\pi\left(2r-\frac{V}{\pi r^2}\right), \frac{dS}{dr}=0 \quad \Rightarrow 2\pi\left(2r-\frac{V}{r^2}\right)=0 \quad \Rightarrow V=2\pi r^3$$

Now, $\frac{d^2S}{dr^2}=2\pi\left(2+\frac{2V}{\pi r^3}\right)$. At $V=2\pi r^3$, $\frac{d^2S}{dr^2}=+ve$

$\Rightarrow S$ is minimum when $V=2\pi r^3$ and from (1), $h=\frac{2\pi r^3}{\pi r^2}=2r$

$\Rightarrow h$ is (height of the cylinder) $=2r$, i.e., diameter of the cylinder.

22. Prove that the volume of the largest cone that can be inscribed in a sphere is $\frac{8}{27}$ of the volume of the sphere.

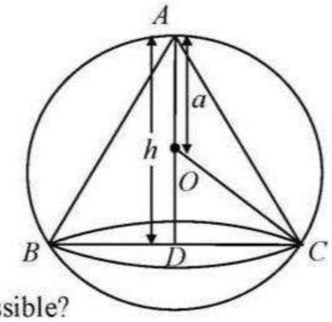
Sol. Let the radius of the given sphere be a. Let v be the volume of the inscribed cone, h be its height and r be its radius. Then, $v=\frac{1}{3}\pi r^2 h$. Now, $OD=(AD-OA)=(h-a)$

$$OC^2=OD^2+DC^2 \quad \Rightarrow a^2=(h-a)^2+r^2 \quad \Rightarrow r^2=h(2a-h)$$

$$\therefore v=\frac{1}{3}\pi h^3(2a-h)=\left[\frac{2\pi ah^2}{3}-\frac{1}{3}\pi h^3\right]$$

$$\therefore \frac{dv}{dh}=\left(\frac{4\pi ah}{3}-\pi h^2\right) \text{ and } \frac{d^2v}{dh^2}=\left(\frac{4\pi a}{3}-2\pi h\right)$$

So, $\frac{dv}{dh}=0 \Rightarrow h=\frac{4a}{3}$ and $\left[\frac{d^2v}{dh^2}\right]_{\left(h=\frac{4a}{3}\right)}=\frac{-4\pi a}{3}<0$



$\therefore$ v is maximum when $h=\frac{4a}{3}$ and $r^2=\frac{8a^2}{9}$

Maximum volume $=\frac{1}{3}\pi\times\frac{8a^2}{9}\times\frac{4a}{3}=\frac{8}{27}\times\left(\frac{4}{3}\pi a^3\right)$

23. Which fraction exceeds its p th power by the greatest number possible?

Sol. Let the required fraction be x. Let $y=x-x^p$.

Then, $\frac{dy}{dx}=1-px^{p-1} \Rightarrow \frac{dy}{dx}=0 \Rightarrow 1=px^{p-1} \Rightarrow \frac{1}{p}=x^{p-1} \Rightarrow x=\left(\frac{1}{p}\right)^{\frac{1}{p-1}}$

At $x=\left(\frac{1}{p}\right)^{\frac{1}{p+1}}$, $\frac{d^2y}{dx^2}<0$. $\therefore$ y is maximum when $x=\left(\frac{1}{P}\right)^{\frac{1}{P-1}}$

24. Find the point on the curve $y^2=4x$ which is nearest to the point $(2,-8)$.

Sol. Let the required point be (x, y). Then, $D=(x-2)^2+(y+8)^2$

$\Rightarrow D=x^2+y^2-4x+16y+68 \Rightarrow D=\frac{y^2}{16}+16y+68 \quad \left[\because x=\frac{y^2}{4}\right]$

$\Rightarrow \frac{dD}{dy}=\left(\frac{2y}{16}+16\right)$ and $\frac{d^2D}{dy^2}=\frac{1}{8}$

Now, $\frac{dD}{dy}=0 \Rightarrow \frac{y}{8}+16=0 \Rightarrow y=-16\times 8$

$\left[\frac{d^2D}{dy^2}\right]_{y=-4}=\frac{3}{4}\times 16=12>0$. $\therefore$ D is minimum when $y=-4$.

At this value, we have $x=4$. Hence, the required point is $(4,-4)$

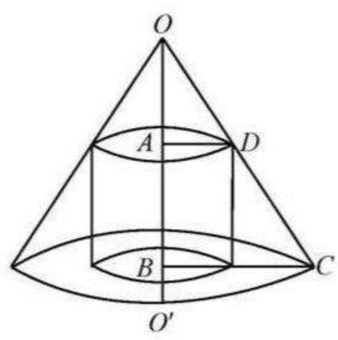
25. A right circular cylinder is inscribed in a cone. Show that the curved surface area of the cylinder is maximum when the diameter of the cylinder is equal to the radius of the base of the cone.

Sol. Let r_1 be the radius of the cone and h_1 be its height and let r be the radius and h the height of the inscribed cylinder. Clearly, ΔOAD and $\Delta O'BC$ are similar

$\therefore \frac{r}{r_1}=\frac{h_1-h}{h_1} \Rightarrow r=\frac{r_1}{h_1}(h_1-h) \qquad \ldots(1)$

Let S be the curved surface area of the cylinder.

Then $S=2\pi rh \Rightarrow S=\frac{2\pi r_1}{h_1}(h_1-h)h$



$\frac{dS}{dh}=\frac{2\pi r_1}{h_1}(h_1-2h)$ and $\frac{d^2S}{dh^2}=\frac{-4\pi r_1}{h_1}<0$

Now, $\frac{dS}{dh}=0$, $\frac{2\pi r_1}{h_1}(h_1-2h)=0 \Rightarrow h_1=2h$. Also $\frac{d^2S}{dh^2}<0$.

So, S is maximum when $h_1=2h$.

$\therefore \ \dfrac{r}{r_1}=\dfrac{2h-h}{2h}=\dfrac{1}{2} \quad \Rightarrow r_1=2r=$ diameter of the cylinder.

26. Show that the surface area of a closed cuboids with square base and given volume is minimum when it is a cube.

Sol. Let V be the volume, x the side of the square base and h the height of the cuboid.

Then, $V=x^2h \quad \Rightarrow h=\dfrac{V}{x^2}$ ...(1)

$\therefore \ S=2\left(x^2+xh+xh\right) \ \Rightarrow S=2x^2+4xh \quad \therefore \ S=2x^2+\dfrac{4V}{x}$ [using (1)]

$\Rightarrow \dfrac{dS}{dx}=4x-\dfrac{4V}{x^2}$ and $\dfrac{d^2S}{dx^2}=4+\dfrac{8V}{x^3}$

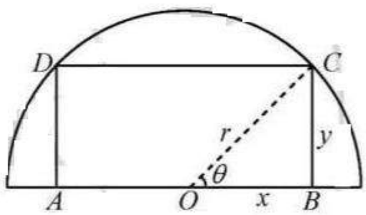
Now, $\dfrac{dS}{dx}=0 \quad \Rightarrow x^3=V$. For this value of x, $\dfrac{d^2S}{dx^2}=4+\dfrac{8V}{V}=4+8=12>0$

$\therefore$ Surface area is minimum when $x^3=V$. Also $h=\dfrac{V}{x^2}=\dfrac{x^3}{x^2}=x$

$\therefore$ S is minimum when the cuboid is a cube.

27. A rectangle is inscribed in a semicircle of radius r with one of its sides on the diameter of the semicircle. Find the dimensions of the rectangle so that its area is maximum. Find also this area.

Sol. Let $ABCD$ be the rectangle of length $2x$ and breadth y, inscribed in a semicircle of radius r and centre O. Let $\angle BOC=\theta$



Then $\dfrac{y}{r}=\sin\theta$ and $\dfrac{x}{r}=\cos\theta$

$\therefore$ area of the rectangle is given by $A=2xy=r^2\sin 2\theta$

$\Rightarrow \dfrac{dA}{d\theta}=2r^2\cos 2\theta$ and $\dfrac{d^2A}{d\theta^2}=-4r^2\sin 2\theta$

Now, $\dfrac{dA}{d\theta}=0 \quad \Rightarrow 2r^2\cos 2\theta=0 \quad \Rightarrow \cos 2\theta=0 \quad \Rightarrow 2\theta=\dfrac{\pi}{2} \quad \Rightarrow \theta=\dfrac{\pi}{4}$

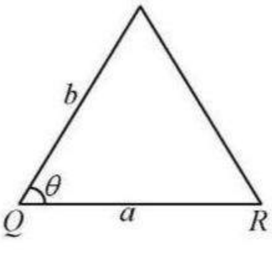
For this value of θ, we have $\dfrac{d^2A}{d\theta^2}=-4r^2<0$. So, area is maximum.

$\therefore \ x=r\cos\dfrac{\pi}{4}=\dfrac{r}{\sqrt{2}}, \ y=r\sin\dfrac{\pi}{4}=\dfrac{r}{\sqrt{2}}$

Maximum area $=2.\dfrac{r}{\sqrt{2}}.\dfrac{r}{\sqrt{2}}=r^2$ sq. units.

28. Two sides of a triangle have lengths a and b and the angle between them is θ. What value of θ will maximize the area of the triangle.

Sol. Let A be the area of ΔPQR. Then, $A=\dfrac{1}{2}ab\sin\theta$



$\Rightarrow \dfrac{dA}{d\theta}=\dfrac{1}{2}ab\cos\theta \quad \Rightarrow \dfrac{d^2A}{d\theta^2}=-\dfrac{1}{2}ab\sin\theta$

For maximum or minima, we have, $\dfrac{dA}{d\theta}=0$

$\Rightarrow \dfrac{1}{2}ab\cos\theta=0 \quad \Rightarrow \cos\theta=0 \quad \Rightarrow \theta=\dfrac{\pi}{2}$

and $\frac{d^2A}{d\theta^2} = -\frac{1}{2}ab\sin\frac{\pi}{2} < 0$ $\therefore$ A is maximum, when $\theta = \frac{\pi}{2}$

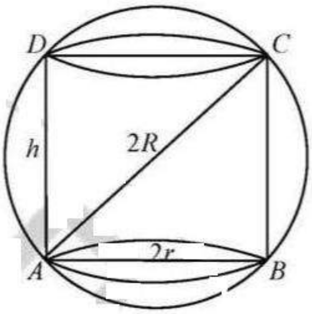
29. Show that the maximum volume of the cylinder which can be inscribed in a sphere of radius $5\sqrt{3}$ cm is $(500\pi)\text{cm}^3$.

Sol. Let r be the radius and h the height of the inscribed cylinder $ABCD$.

Let V be its volume. Then, $V = \pi r^2 h$ ...(1)

Clearly, $AC = 2R$, also $(AC)^2 = (AB)^2 + (BC)^2$

$\Rightarrow (2R)^2 = (2r)^2 + h^2 \Rightarrow r^2 = \frac{1}{4}(4R^2 - h^2)$ ...(2)



Using (2) in (1), we get, $V = \frac{\pi h}{4}(4R^2 - h^2)$

$\Rightarrow \frac{dV}{dh} = \left(\pi R^2 - \frac{3}{4}\pi h^2\right)$ and $\frac{d^2V}{dh^2} = \frac{-3}{2}\pi h$

For a maximum or minima, we have, $\left(\frac{dV}{dh} = 0\right)$

Now, $\frac{dV}{dh} = 0 \Rightarrow \pi R^2 - \frac{3}{4}\pi h^2 = 0 \Rightarrow \pi R^2 = \frac{3}{4}\pi h^2$

$\Rightarrow \frac{4R^2}{3} = h^2 \Rightarrow h = \frac{2R}{\sqrt{3}}$, if R is $5\sqrt{3}$ $\Rightarrow h = \frac{2 \cdot 5\sqrt{3}}{\sqrt{3}}$ $\Rightarrow h = 10$

$\therefore \left[\frac{d^2V}{dh^2}\right]_{h=10} = \frac{-3}{2}\pi.10 = -15\pi < 0$. So, V is maximum when $h = 10$.

If from equation (2), $r^2 = \frac{1}{4}(4R^2 - h^2)$, if $R = 5\sqrt{3}, h = 10$

$r^2 = \frac{1}{4}\left\{4(5.\sqrt{3})^2 - (10)^2\right\}$, $r^2 = \frac{1}{4}(300 - 100)$, $r^2 = \frac{1}{4} \times 200$, $r^2 = 50$

Largest volume of the cylinder $= \pi r^2 h = \pi.50 \times 10 = 500\pi\ cm^3$.

30. A square tank of capacity 250 cubic meters has to be dug out. The cost of the land is Rs. 50 per square metre. The cost of digging increases with the depth and for the whole tank, it is Rs $(400 \times h^2)$, where h metres is the depth of the tank? What should be the dimensions of the tank so that the cost is minimum?

Sol. Let the side of the square tank be x meters. Then, total cost is given by

$C = 50x^2 + 400h^2$ ...(1)

Also, $x^2 h = 250 \Rightarrow h = \frac{250}{x^2}$ ...(2)

$\therefore C = 50x^2 + \frac{400 \times 62500}{x^4} \Rightarrow \frac{dC}{dx} = (100x - 10^8.x^{-5})$ and $\frac{d^2C}{dx^2} = \left(100 + \frac{10^8 \times 5}{x^6}\right)$

Now, $\frac{dC}{dx} = 0 \Rightarrow 100x - 10^8.x^{-5} = 0$, $100x = 10^8.x^{-5}$

$x^6 = 10^6 \therefore x = 10$ and $\left[\frac{d^2C}{dx^2}\right]_{x=10} = \left(100 + \frac{10^8 \times 5}{10^6}\right) = (100 + 500) = 600 > 0$

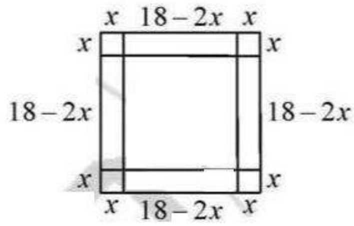
$\therefore$ Cost is minimum when $x = 10$ and $h = \left(\frac{250}{100}\right)m = 2.5\,m$

Hence, for minimum cost, the tank must have a square base if side $10m$ and depth $2.5m$.

31. A square piece of tin of side 18 cm is to be made into a box without the top, by cutting a square piece from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is maximum. Also, find the maximum volume of the box.

Sol. Let the side of the square piece cut from each corner of the given square piece of side 18 cm be x cm.

Then, the dimensions of the open box are $(18-2x)$ cm, $(18-2x)$ cm and x cm.



$\therefore\ V = (18-2x)^2 \times x \ \Rightarrow V = (324 - 72x + 4x^2)x$

$\Rightarrow V = 4x^3 - 72x^2 + 324x$ ...(1)

$\Rightarrow \frac{dV}{dx} = 12x^2 - 144x + 324$

$\Rightarrow \frac{dV}{dx} = 12(x^2 - 12x + 27) \ \Rightarrow \frac{d^2V}{dx^2} = 12(2x - 12)$

Now, $\frac{dV}{dx} = 0 \ \Rightarrow x^2 - 12x + 27 = 0 \ \Rightarrow (x-3)(x-9) = 0 \ \Rightarrow x = 3 \quad [\because x \neq 9]$

$\left[\frac{d^2V}{dx^2}\right]_{x=3} = 12(2\times 3 - 12) = -72 < 0$

$\therefore\ V$ is maximum at $x = 3$ cm and maximum volume

$= \left[4\times 3^3 \ \ 72\times 3^2 + 324\times 3\right]$

$= \left[4\times 27 - 72\times 9 + 324\times 3\right] = 108 - 648 + 972 = 432 \text{ cm}^3$

32. An open tank with a square base and vertical sides is to be constructed from a metal sheet so as to hold a given quantity of water. Show that the cost of the material will be least when the depth of the tank is half of its width.

Sol. Let x be the side of the square and h be the depth. Then, $V = x^2h$

Let the cost per square be Rs. P.

Then, $C = (x^2 + 4xh)P \ \Rightarrow C = \left(x^2 + 4x \times \frac{V}{x^2}\right)P \ \Rightarrow C = \left(x^2 + \frac{4V}{x}\right)P$

$\therefore\ \frac{dC}{dx} = \left(2x - \frac{4V}{x^2}\right)P$ and $\frac{d^2C}{dx^2} = \left(2 + \frac{8V}{x^3}\right)P$

Now, $\frac{dC}{dx} = 0 \ \Rightarrow 2x - \frac{4V}{x^2} = 0,\ 2x = \frac{4V}{x^2} \ \Rightarrow x = (2V)^{\frac{1}{3}}$

and $\left[\frac{d^2C}{dx^2}\right]_{x=(2V)^{\frac{1}{3}}} = P\left(2 + \frac{8V}{2V}\right) = 6P > 0$

$\therefore\ C$ is minimum, when $x = (2V)^{\frac{1}{3}}$. Then $h = \frac{V}{x^2} = \frac{1}{2}(2v)^{\frac{1}{3}} = \frac{1}{2}x$

Hence, for minimum lost, the depth of the tank should be equal to half of the side as its square base.

33. A wire of length 36 cm is cut into two pieces. One of the pieces is turned in the form of a square and the other in the form of an equilateral triangle. Find the length of each piece so that the sum of the areas of the two be minimum.

Sol. Let the perimeter of the square be x cm. then the perimeter of the triangle is $(36-x)$ cm.

$\therefore$ side of the square $= \dfrac{x}{4}$ cm and side of the triangle $= \dfrac{1}{3}(36-x)$ cm

$$\therefore \quad A = \frac{x^2}{16} + \frac{\sqrt{3}}{4}\left(12 - \frac{x}{3}\right)^2 = \frac{x^2}{16} + \frac{\sqrt{3}}{4}\left(144 + \frac{x^2}{9} - 8x\right)$$

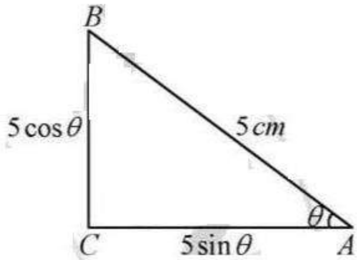
$$\Rightarrow A = \left(\frac{\sqrt{3}}{36} + \frac{1}{16}\right)x^2 - 2\sqrt{3}x + 36\sqrt{3} \quad \Rightarrow \frac{dA}{dx} = \frac{\left(4\sqrt{3}+9\right)}{144} \times 2x - 2\sqrt{3} \text{ and } \frac{d^2A}{dx^2} = \frac{4\sqrt{3}+9}{72} > 0$$

$$\therefore \quad \frac{dA}{dx} = 0 \quad \Rightarrow x = \frac{144\sqrt{3}}{\left(4\sqrt{3}+9\right)} \text{ cm and } 36 - \frac{144\sqrt{3}}{4\sqrt{3}+9} = \frac{324}{4\sqrt{3}+9} \text{ cm}$$

34. Find the largest possible area of a right angled triangle whose hypotenuse 5 cm.

Sol. Consider a right-angle ΔABC in which hypotenuse $AC = 5$ cm and let $\angle BAC = \theta$

Then $AB = 5\cos\theta$ and $BC = 5\sin\theta$



$$\therefore \quad \text{area of triangle} = \frac{1}{2} AB \times BC = \frac{1}{2} \times 5\cos\theta \times 5\sin\theta$$

$$\Rightarrow A = \frac{25}{4}\sin 2\theta \quad \Rightarrow \frac{dA}{d\theta} = \frac{25}{2}\cos 2\theta \text{ and } \frac{d^2A}{d\theta^2} = -25\sin 2\theta$$

Now, $\dfrac{dA}{d\theta} = 0 \Rightarrow \cos 2\theta = 0 \Rightarrow 2\theta = \dfrac{\pi}{2} \Rightarrow \theta = \dfrac{\pi}{4}$

$$\Rightarrow \left[\frac{d^2A}{d\theta^2}\right]_{\theta=\frac{\pi}{4}} = -25\sin\left(\frac{\pi}{2}\right) = -25 < 0$$

$\therefore \theta = \dfrac{\pi}{4}$ is a point of maxima. Maximum area $= \dfrac{25}{4} \times \sin\left(2 \times \dfrac{\pi}{4}\right) = \dfrac{25}{4} \times 1 = \dfrac{25}{4}$ sq. cm.

INCREASING AND DECREASING FUNCTIONS (XII, R.S. AGGARWAL)

EXERCISE 11 G (Pg.No.: 549)

1. Show that the function $f(x)=5x-2$ is a strictly increasing function on R.

Sol. $f(x)=5x-2 \Rightarrow f'(x)=5 \quad \Rightarrow f'(x)>0$, for all $x>0$

Hence, $f(x)$ is strictly increasing function for all $x>0$

2. Show that the function $f(x)=-2x+7$ is a strictly decreasing function on R.

Sol. $f(x)=-2x+7 \quad \Rightarrow f'(x)=-2 \quad \Rightarrow f'(x)<0$ for all value of x

Hence, $f(x)$ is strictly decreasing function on R.

3. Prove that $f(x)=ax+b$, where a and b are constants and $a>0$, is a strictly increasing function on R.

Sol. $f(x)=ax+b \quad \Rightarrow f'(x)=a \quad \Rightarrow f'(x)>0$, for all x

Hence, $f(x)$ is strictly increasing function .

4. Prove that the function $f(x)=e^{2x}$ is strictly increasing on R.

Sol. $f(x)=e^{2x} \quad \Rightarrow f'(x)=e^{2x}\times 2 \quad \Rightarrow f'(x)>0$

Hence, $f(x)$ is strictly increasing function.

5. Show that the function $f(x)=x^2$ is

(a) strictly increasing on $]0,\infty[$ (b) strictly decreasing on $]-\infty,0[$

(c) neither strictly increasing nor strictly decreasing on R

Sol. (a) Let $f'(x)=2x \quad \therefore 2x>0,\ x\in(0,\infty) \quad \Rightarrow f'(x)>0 \quad \Rightarrow 2x>0 \quad \Rightarrow x>0$

(b) $f'(x)<0 \quad \Rightarrow 2x<0$

(c) Since $f(x)=x^2$ is strictly increasing on $(0,\infty[$ *and strictly decreasing on* $]-\infty,0)$.

If is neither strictly increasing nor strictly decreasing on the whole real line.

6. Show that the function $f(x)=|x|$ is :

(a) strictly increasing on $]0,\infty[$ (b) strictly decreasing on $]-\infty,0[$

Sol. Let $f(x)=|x| \quad \Rightarrow f(x)=\begin{cases} x, & x\in(0,\infty) \\ -x, & x\in(-\infty,0)\end{cases} \quad \Rightarrow f'(x)=\begin{cases} 1, & x\in(0,\infty) \\ -1, & x\in(-\infty,0)\end{cases}$

Hence, $f'(x)>0,\ x\in(0,\infty)$

(a) So, $f(x)$ is strictly increasing at $x\in(0,\infty) \quad \therefore f'(x)<0,\ x\in(-\infty,0)$

(b) So, $f'(x)$ is strictly decreasing at $x\in(-\infty,0)$

7. Prove that the function $f(x)=\log_e x$ is strictly increasing on $]0,\infty[$.

Sol. $f(x)=\log_e x \quad \Rightarrow f'(x)=\frac{1}{x}$.

As $\frac{1}{x}>0 \quad \therefore \; x\in(0,\infty)$.

So, $f'(x)>0,\ x\in(0,\infty)$. So, $f'(x)$ strictly increasing are $(0,\infty)$.

8. Prove that the function $f(x)=\log_a x$ is strictly increasing on $]0,\infty[$ when $a>1$ and strictly decreasing on $]0,\infty[$ when $0<a<1$.

Sol. Let, $f(x)=\log_a x$

(a) $f'(x)=\frac{1}{x\log a} \quad \therefore \; \frac{1}{x\log a}>0,\ x\in(0,\infty)$ and $a>1$.

So, $f'(x)>0 \quad \Rightarrow x\in(0,\infty)$ and $a>1$. Hence, $f'(x)$ is strictly increasing function.

(b) $\frac{1}{x\log a}<0,\ x\in(0,\infty),\ 0<a<1 \quad \therefore \; f'(x)<0$

Hence, $f'(x)$ is strictly decreasing are $(0,\infty)$ and $0<a<1$

9. Prove that $f(x)=3^x$ is strictly increasing on R.

Sol. $f(x)=3^x \quad \Rightarrow f'(x)=(3^x\log 3)$

$f'(x)>0$ for all value of R. Hence, $f(x)$ is strictly increasing function on R

10. Show that $f(x)=x^3-15x^2+75x-50$ is increasing on R.

Sol. $f(x)=x^3-15x^2+75x-50 \quad \Rightarrow f'(x)=3x^2-30x+75$

$\Rightarrow f'(x)=3(x^2-10x+25) \quad \Rightarrow f'(x)=3(x-5)^2$

$f'(x)\geq 0$ for all $x\geq 0$. Hence, $f(x)$ is increasing function for all

11. Show that $f(x)=\left(x-\frac{1}{x}\right)$ is increasing for all $x\in R$, where $x\neq 0$.

Sol. $f(x)=x-\frac{1}{x} \quad \Rightarrow f'(x)=1+\frac{1}{x^2} \quad \Rightarrow f'(x)=\frac{x^2+1}{x^2}$

$\Rightarrow f'(x)\geq 0$ for all $x\neq 0$. Hence, $f(x)$ is increasing function for all $x\neq 0$.

12. Show that $f(x)=\left(\frac{3}{x}+5\right)$ is decreasing for all $x\in R$, where $x\neq 0$.

Sol. $f(x)=\frac{3}{x}+5 \quad \Rightarrow f'(x)=3\left(-\frac{1}{x^2}\right) \quad \Rightarrow f'(x)=\frac{-3}{x^2} \quad \Rightarrow f'(x)<0$ for all value of R.

Hence, $f(x)$ is decreasing function on R.

13. Show that $f(x)=\frac{1}{(1+x^2)}$ is increasing for all $x\leq 0$

Sol. $f(x)=\frac{1}{1+x^2} \quad \Rightarrow f'(x)=\frac{-2x}{(1+x^2)^2} \quad \Rightarrow f'(x)\leq 0$

Hence, $f(x)$ is increasing function for all $x\leq 0$.

14. Show that $f(x)=\left(x^3+\frac{1}{x^3}\right)$ is decreasing on $]-1, 1[$.

Sol. $f(x)=x^3+\frac{1}{x^3} \Rightarrow f'(x)=3x^2-3x^{-4} \Rightarrow f'(x)=3\left(x^2-\frac{1}{x^4}\right)$

$\Rightarrow f'(x)=3\left(\frac{x^6-1}{x^4}\right) \Rightarrow f'(x)=3\left\{\frac{(x^2)^3-1}{x^4}\right\}$

$\Rightarrow f'(x)=\frac{3(x^2-1)(x^4+x^2+1)}{x^4}=\frac{3(x-1)(x+1)(x^4+x^2+1)}{x^4}<0$ if $x\in(-1,1)$

Hence, $f(x)$ is decreasing function on $]-1, 1[$.

15. Show that $f(x)=\frac{x}{\sin x}$ is increasing on $\left]0,\frac{\pi}{2}\right[$.

Sol. $f(x)=\frac{x}{\sin x} \Rightarrow f'(x)=\frac{\sin x.1-x.\cos x}{(\sin x)^2} \Rightarrow f'(x)=\frac{\sin x-x\cos x}{(\sin x)^2}$

Now $f'(x)>0 \Rightarrow \frac{\sin x-x\cos x}{(\sin x)^2}>0 \Rightarrow (\sin x-x\cos x)>0 \Rightarrow \tan x>x$, which is true on $\left]0,\frac{\pi}{2}\right[$.

16. Prove that the function $f(x)=\log(1+x)-\frac{2x}{(x+2)}$ is increasing for all $x>-1$.

Sol. $f(x)=\log(1+x)-\frac{2x}{x+2} \Rightarrow f'(x)=\frac{1}{1+x}-\left\{\frac{(x+2).2-2x.1}{(x+2)^2}\right\} \Rightarrow f'(x)=\frac{1}{1+x}-\left\{\frac{2x+4-2x}{(x+2)^2}\right\}$

$\Rightarrow f'(x)=\frac{1}{1+x}-\frac{4}{(x+2)^2} \Rightarrow f'(x)=\frac{x^2+4x+4-4x-4}{(1+x)(x+2)^2} \Rightarrow f'(x)=\frac{x^2}{(1+x)(x+2)^2}$

$\Rightarrow f'(x)>0 \Rightarrow \frac{x^2}{(1+x)(x+2)^2}>0 \quad [\because (1+x)>0]$

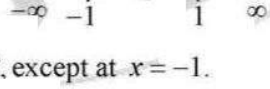
Hence, $f(x)$ is increasing function for all $x>-1$.

17. Let I be an interval disjoint from $]-1, 1[$. prove that the function $f(x)=\left(x+\frac{1}{x}\right)$ is strictly increasing on I.

Sol. $f(x)=x+\frac{1}{x} \Rightarrow f'(x)=1-\frac{1}{x^2} \Rightarrow f'(x)=\frac{x^2-1}{x^2} \Rightarrow f'(x)=\frac{(x-1)(x+1)}{x^2}$

So, $f'(x)$ is positive in $(-\infty, -1)\cup(1, \infty)$.

Hence, $f(x)$ is strictly increasing except $(-1, 1)$.



18. Show that $f(x)=\frac{(x-2)}{(x+1)}$ is increasing for all $x\in R$, except at $x=-1$.

Sol. $f(x)=\frac{x-2}{x+1} \Rightarrow f'(x)=\frac{(x+1).1-(x-2).1}{(x+1)^2} \Rightarrow f'(x)=\frac{x+1-x+2}{(x+1)^2}$

$\Rightarrow f'(x)=\frac{3}{(x+1)^2} \quad [\therefore (x+1)>0]$, when $x\neq-1$

Hence, $f(x)$ is increasing function for all $x \in R$, except at $x = -1$.

19. Find the intervals on which the function $f(x) = (2x^2 - 3x)$ is

(a) strictly increasing (b) strictly decreasing.

Sol. $f(x) = 2x^2 - 3x \quad \Rightarrow f'(x) = 4x - 3 \qquad \ldots\text{(i)}$

(a) $f(x)$ is strictly increasing $\Rightarrow f'(x) > 0 \quad \Rightarrow 4x - 3 > 0 \quad \Rightarrow x > \frac{3}{4}$

$\therefore$ $f(x)$ is strictly increasing on the interval $\left]\frac{3}{4}, \infty\right[$.

(b) $f(x)$ is strictly decreasing $\Rightarrow f'(x) < 0 \quad \Rightarrow 4x - 3 < 0 \quad \Rightarrow x < \frac{3}{4}$

$\therefore$ $f(x)$ is strictly decreasing on the interval $\left]-\infty, \frac{3}{4}\right[$.

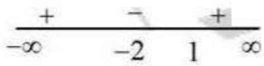
20. Find the intervals on which the function $f(x) = 2x^3 - 3x^2 - 36x + 7$ is

(a) strictly increasing (b) strictly decreasing.

Sol. (a) $f(x) = 2x^3 - 3x^2 - 36x + 7 \Rightarrow f'(x) = 6x^2 - 6x - 36 \Rightarrow f'(x) = 6(x^2 - x - 6) = 6(x+2)(x-3)$

$\therefore$ $f(x)$ is strictly increasing on $]-\infty, -2[\cup]3, \infty[$

(b) $f(x)$ is decreasing $f'(x) < 0$



$\therefore$ $f(x)$ is strictly decreasing on $]-2, 3[$.

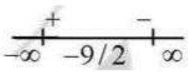
21. Find the intervals on which the function $f(x) = 6 - 9x - x^2$ is

(a) strictly increasing (b) strictly decreasing.

Sol. $f(x) = 6 - 9x - x^2 \quad \Rightarrow f'(x) = -9 - 2x \quad \Rightarrow f'(x) = -(2x + 9)$.

Now, $f'(x) > 0 \Rightarrow -(2x+9) > 0 \Rightarrow 2x + 9 < 0 \Rightarrow x < \frac{-9}{2}$

(a) Hence, $f(x)$ is strictly increasing on $\left]-\infty, \frac{-9}{2}\right[$.



(b) Hence, $f(x)$ is strictly decreasing on $\left]\frac{-9}{2}, \infty\right[$

Find the intervals on which each of the following functions is (a) increasing (b) decreasing.

22. $f(x) = \left(x^4 - \frac{x^3}{3}\right)$

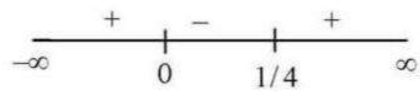
Sol. Here, $f(x) = x^4 - \frac{x^3}{3} \quad \Rightarrow f'(x) = 4x^3 - \frac{1}{3}.3x^2 \quad \Rightarrow f'(x) = 4x^3 - x^2 \quad \Rightarrow f'(x) = x^2(4x - 1)$

Now $f'(x) = 0 \quad \Rightarrow x^2(4x-1) = 0 \quad \Rightarrow x = 0, \frac{1}{4}$

These points divided the real line into three disjoint interval namely $(-\infty, 0), \left(0, \frac{1}{4}\right), \left(\frac{1}{4}, \infty\right)$.

For $x \in \left(\frac{1}{4}, \infty\right)$, $f'(x) > 0 \quad \Rightarrow f$ is increasing for $x \in \left(\frac{1}{4}, \infty\right)$

For $x \in \left(0, \frac{1}{4}\right)$, $f'(x) < 0 \Rightarrow f$ is decreasing for $x \in \left(0, \frac{1}{4}\right)$



For $x \in (-\infty, 0)$, $f'(x) > 0 \Rightarrow f$ is increasing for $x \in (-\infty, 0)$

$\therefore f$ is increasing $x \in (-\infty, 0) \cup (3, \infty)$ and deceasing for $x \in \left(0, \frac{1}{4}\right)$.

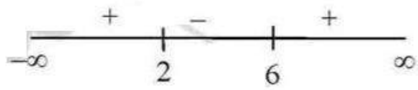
23. $f(x) = x^3 - 12x^2 + 36x + 17$

Sol. Here, $f(x) = x^3 - 12x^2 + 36x + 17 \Rightarrow f'(x) = 3x^2 - 24x + 36 \Rightarrow f'(x) = 3(x^2 - 8x + 12)$

Now, $f'(x) = 0 \Rightarrow (x-6)(x-2) = 0 \Rightarrow x = 2$ or 6

These points divided the real line into three disjoint interval namely $(-\infty, 2), (2, 6), (6, \infty)$

For $x \in (6, \infty)$, $f'(x) > 0 \Rightarrow f$ is increasing for $x \in [6, \infty)$



For $x \in (2, 6)$, $f'(x) < 0 \Rightarrow f$ is decreasing for $x \in (2, 6)$

For $x \in (-\infty, 2)$, $f'(x) > 0 \Rightarrow f$ is increasing for $x \in (-\infty, 2)$

$\therefore f$ is increasing $x \in (-\infty, 2) \cup (6, \infty)$ and deceasing for $x \in (2, 6)$.

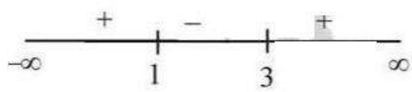
24. $f(x) = (x^3 - 6x^2 + 9x + 10)$

Sol. Here $f(x) = x^3 - 6x^2 + 9x + 10 \Rightarrow f'(x) = 3x^2 - 12x + 9 \Rightarrow f'(x) = 3(x^2 - 4x + 3)$

$\Rightarrow f'(x) = 3(x^2 - 3x - x + 3) \Rightarrow f'(x) = 3\{(x-3)(x-1)\}$

$\Rightarrow f'(x) = 3(x-1)(x-3)$. Now $f'(x) = 0 \Rightarrow x = 1$ or 3

These points divided the real line into three disjoint interval namely $(-\infty, 1), (1, 3), (3, \infty)$



For $x \in (3, \infty)$, $f'(x) > 0 \Rightarrow f$ is increasing for $x \in (3, \infty)$

For $x \in (1, 3)$, $f'(x) < 0 \Rightarrow f$ is decreasing for $x \in (1, 3)$

For $x \in (-\infty, 1)$, $f'(x) > 0 \Rightarrow f$ is increasing for $x \in (-\infty, 1)$

$\therefore f$ is increasing $x \in (-\infty, 1) \cup (3, \infty)$ and deceasing for $x \in (1, 3)$.

25. $f(x) = (6 + 12x + 3x^2 - 2x^3)$

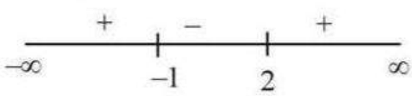
Sol. Here, $f(x) = 6 + 12x + 3x^2 - 2x^3 \Rightarrow f'(x) = 12 + 6x - 6x^2$

$\Rightarrow f'(x) = 6(2 + x - x^2) \Rightarrow f'(x) = -6(x^2 - x - 2)$

$\Rightarrow f'(x) = -6(x^2 - 2x + x - 2) \Rightarrow f'(x) = -6\{x(x-2) + 1(x-2)\}$

$\Rightarrow f'(x) = -6(x+1)(x-2)$. Now $f'(x) = 0 \Rightarrow x = 1$ or 2

These points divided the real line into three disjoint interval namely $(-\infty, -1), (-1, 2), (2, \infty)$



For $x \in (2, \infty)$, $f'(x) < 0 \Rightarrow f$ is decreasing for $x \in (2, \infty)$

For $x \in (-1, 2)$, $f'(x) > 0 \Rightarrow f$ is increasing for $x \in (-1, 2)$

For $x \in (-\infty, -1)$, $f'(x) < 0 \Rightarrow f$ is decreasing for $x \in (-\infty, -1)$

Hence f is increasing for $(-1, 2)$ and deceasing for $(-\infty, -1] \cup [2, \infty)$

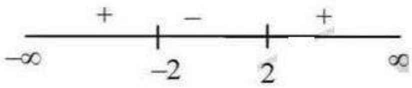
26. $f(x) = 2x^3 - 24x + 5$

Sol. Here, $f(x) = 2x^3 - 24x + 5 \Rightarrow f'(x) = 6x^2 - 24$

$\Rightarrow f'(x) = 6(x^2 - 4) \Rightarrow f'(x) = 6(x-2)(x+2)$

Now $f'(x) = 0 \Rightarrow x = -2$ or 2

These points divided the real line into three disjoint interval namely $(-\infty, -2), (-2, 2), (2, \infty)$



For $x \in (2, \infty)$, $f'(x) > 0 \Rightarrow f$ is increasing for $x \in (2, \infty)$

For $x \in (-2, 2)$, $f'(x) < 0 \Rightarrow f$ is decreasing for $x \in (-2, 2)$

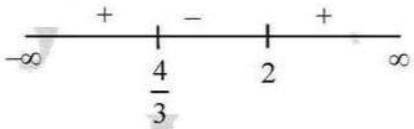
For $x \in (-\infty, -2)$, $f'(x) > 0 \Rightarrow f$ is increasing for $x \in (-\infty, -2)$

Hence f is increase for $(-\infty, -2] \cup [2, \infty)$ and deceasing for $(-2, 2)$.

27. $f(x) = (x-1)(x-2)^2$

Sol. $f'(x) = (x-2)(3x-4)$

Strictly increasing on $\left(-\infty, \frac{4}{3}\right] \cup [2, \infty)$.



Strictly decreasing on $\left(\frac{4}{3}, 2\right)$

28. $f(x) = (x^4 - 4x^3 + 4x^2 + 15)$

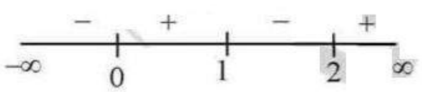
Sol. Here, $f(x) = x^4 - 4x^3 + 4x^2 + 15 \Rightarrow f'(x) = 4x^3 - 12x^2 + 8x$

$\Rightarrow f'(x) = 4x(x^2 - 3x + 2) \Rightarrow f'(x) = 4x(x^2 - 2x - x + 2)$

$\Rightarrow f'(x) = 4x\{(x-2)(x-1)\} \Rightarrow f'(x) = 4x(x-1)(x-2)$

Now $f'(x) = 0 \Rightarrow x = 0, 1, 2$

These points divided the real line into four disjoint interval namely $(-\infty, 0), (0, 1), (1, 2), (2, \infty)$.



For $x \in (2, \infty)$, $f'(x) > 0 \Rightarrow f$ is increasing for $x \in [2, \infty)$

For $x \in (1, 2)$, $f'(x) < 0 \Rightarrow f$ is decreasing for $x \in (1, 2)$

For $x \in (0, 1)$, $f'(x) > 0 \Rightarrow f$ is increasing for $x \in [0, 1)$

For $x \in (-\infty, 0)$, $f'(x) < 0$, f is decreasing for $x \in (-\infty, \infty)$

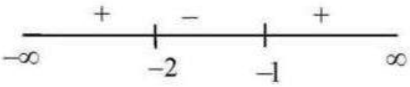
Hence f is increase for $x \in [0, 1) \cup (2, \infty]$ and deceasing for $(-\infty, 0] \cup (1, 2]$.

29. $f(x) = 2x^3 + 9x^2 + 12x + 15$

Sol. Here, $f(x) = 2x^3 + 9x^2 + 12x + 15$

$\Rightarrow f'(x) = 6x^2 + 18x + 12$

$\Rightarrow f'(x) = 6(x^2 + 3x + 2) \quad \Rightarrow f(x) = 6(x+1)(x+2)$



Hence *f* is increase for $x \in (-\infty, -2) \cup (-1, \infty)$ and decreasing for $x \in (-2, -1)$.

30. $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$

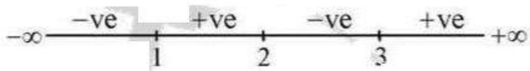
Sol. Given $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$, $D_f = R$

$f'(x) = 4x^3 - 24x^2 + 44x - 24 = 4(x^3 - 6x^3 + 11x - 6) = 4(x-1)(x-2)(x-3)$

Now, $f'(x) = 0 \quad \Rightarrow (x-1)(x-2)(x-3) - 0 \quad \Rightarrow x = 1, 2, 3$

$\therefore$ Critical value of $f(x)$ are $x = 1, 2, 3$

Sign scheme for $f'(x)$ at $x = 1, 2, 3$ is



Here $f'(x) \geq 0$ for $x \in [1, 2] \cup [3, \infty)$

Hence, $f(x)$ is increasing in $[1, 2] \cup [3, \infty)$

$f'(x) \leq 0$ for $(-\infty, 1] \cup [2, 3]$, hence $f(x)$ is decreasing in $(-\infty, 1] \cup [2, 3]$

31. $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is (a) strictly increasing (b) strictly increasing

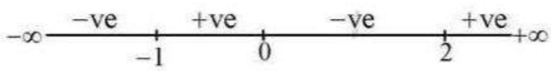
Sol. Given $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$, $D_f = R$

$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) = 12x(x-2)(x+1)$

Now, $f'(x) = 0 \Rightarrow \quad 12x(x-2)(x+1) = 0 \Rightarrow x = -1, 0, 2$

$\therefore$ Critical value of $f(x)$ are $x = -1, 0, 2$

Sign scheme for $f'(x)$ at $x = 1, 2, 3$ is



Here $f'(x) > 0$ for $x \in (-1, 0) \cup (2, \infty)$

Hence, $f(x)$ is strictly increasing in $(-1, 0) \cup (2, \infty)$

$f'(x) < 0$ for $(-\infty, -1) \cup (0, 2)$, hence $f(x)$ is strictly decreasing in $(-\infty, -1) \cup (0, 2)$

32. $f(x) = \frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x^2 + \frac{36}{5}x + 11$ is (a) strictly increasing (b) strictly decreasing

Sol. $\frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x^2 + \frac{36}{5}x + 11$

Finding $f'(x)$

$$f'(x) = \frac{3}{10} \times 4x^3 - \frac{4}{5} \times 3x^2 - 3 \times 2x + \frac{36}{5} + 0$$

$$f'(x) = \frac{12}{10}x^3 - \frac{12}{5}x^2 - 6x + \frac{36}{5}$$

$$f'(x) = \frac{6}{5}x^3 - \frac{12}{5}x^2 - 6x + \frac{36}{5}$$

$$f'(x)=6\left(\frac{x^3}{5}-\frac{2x^2}{5}-x+\frac{6}{5}\right)$$

$$f'(x)=6\left(\frac{x^3-2x^2-5x+6}{5}\right)$$

$$=\frac{6}{5}\left(x^3-2x^2-5x+6\right)$$

$$=\frac{6}{5}(x-1)\left(x^2-x-6\right)$$

$$=\frac{6}{5}(x-1)\left(x^2-3x+2x-6\right)$$

$$=\frac{6}{5}(x-1)\left[x(x-3)+2(x-3)\right]$$

$$=\frac{6}{5}(x-1)(x+2)(x-3)$$

Hence $f'(x)=\frac{6}{5}(x-1)(x+2)(x-3)$

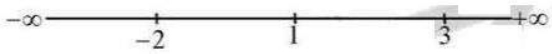
Putting $f'(x)=0$

$$\frac{6}{5}(x-1)(x+2)(x-3)=0$$

$$(x-1)(x+2)(x-3)=0$$

Hence $x=-2,1\,\&\,3$

Plotting point on real line



Thus we get four disjoing intervals

i.e. $(-\infty,-2),(-2,1),(1,3)(3,\infty)$

Value of x	Interval	Sign of $f(x)$ $=\frac{6}{5}(x-1)(x+2)(x+3)$	Nature of $f(x)$
$-\infty<x<-2$	$x\in(-\infty,-2)$	$=(-)(-)(-)$	Strictly decreasing
$-2<x<1$	$x\in(-2,1)$	$=(-)(+)(-)=(+)$	Strictly increasing
$-1<x<3$	$x\in(1,3)$	$=(+)(+)(-)=(-)$	Strictly decreasing
$3<x<\infty$	$x\in(3,\infty)$	$=(+)(+)(+)=(+)$	Strictly increasing

$\Rightarrow f(x)$ is strictly decreasing on the interval $x\in(-\infty,-2)\,\&\,(1,3)$

$f(x)$ is strictly increasing on the interval $x\in(-2,1)\,\&\,(3,\infty)$

TANGENTS & NORMALS (R. S. AGGARWAL)

EXERCISE 11H (Pg. No.: 564)

1. Find the slope of the tangent to the curve

(i) $y=\left(x^3-x\right)$ at $x=2$ (ii) $y=\left(2x^2+3\sin x\right)$ at $x=0$ (iii) $y=\left(\sin 2x+\cot x+2\right)^2$ at $x=\frac{\pi}{2}$

Sol. **(i)** $y=\left(x^3-x\right)$ at $x=2$

Let the equation of curve, $y=x^3-x$

Differentiating both sides w.r.t. x, $\frac{dy}{dx}=3x^2-1 \Rightarrow \left(\frac{dy}{dx}\right)_{x=2}=3\times(2)^2-1=12-1=11$

(ii) $y=\left(2x^2+3\sin x\right)$ at $x=0$

Let be the equation of curve, $y=\left(2x^2+3\sin x\right)$

Differentiating both sides w.r.t. x, $\frac{dy}{dx}=4x+3\cos x \Rightarrow \left(\frac{dy}{dx}\right)_{x=0}=4(0)+3\cos 0=3(1)=3$

(iii) $y=\left(\sin 2x+\cot x+2\right)^2$ at $x=\frac{\pi}{2}$

Let the given equation curve, $y=\left(\sin 2x+\cot x+2\right)^2$

Differentiating both sides w.r.t. x, $\frac{dy}{dx}=2\left(\sin 2x+\cot x+2\right)\left(\cos 2x.2-\text{cosec}^2 x+0\right)$

$\Rightarrow \frac{dy}{dx}=2\left(\sin 2x+\cot x+2\right)\left(2\cos 2x-\text{cosec}^2 x\right)$

$\Rightarrow \frac{dy}{dx}=2\left(\sin 2.\frac{\pi}{2}+\cot\frac{\pi}{2}+2\right)\left(2\cos 2.\frac{\pi}{2}-\text{cosec}^2\frac{\pi}{2}\right) \quad \therefore \left(\frac{dy}{dx}\right)_{x=\frac{\pi}{2}}=2(0+2)(-2-1)=-12$

Find the equations of the tangent and the normal to the given curve at the indicated point :

2. $y=x^3-2x+7$ at $(1,6)$

Sol. Let be the equation of curve, $y=x^3-2x+7$

Differentiating both sides w.r.t. x, $\frac{dy}{dx}=3x^2-2$

$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,6)}=3(1)^2-2=3-2=1 \qquad \therefore m=1$

$\therefore$ The required equation of the tangent at point $(1,6)$, $(y-y_1)=m(x-x_1)$.

$\Rightarrow \frac{y-y_1}{(x-x_1)}=m \Rightarrow \frac{y-6}{x-1}=1 \Rightarrow y-6=x-1 \Rightarrow y-6+1=x$

$\Rightarrow y-5=x \quad \Rightarrow x-y+5=0$

And the required equation of the normal at point $(1,6)$,

$$(y-y_1)=\frac{-1}{m}(x-x_1) \Rightarrow \frac{y-y_1}{x-x_1}=\frac{-1}{m} \Rightarrow \frac{y-6}{x-1}=\frac{-1}{1}$$

$$\Rightarrow y-6=-x+1 \Rightarrow x+2y-6-1=0 \Rightarrow x+y-7=0$$

3. $y^2=4ax$ at $\left(\frac{a}{m^2},\frac{2a}{m}\right)$

Sol. The equation of curve, $y^2=4ax$

Differentiating both sides w.r.t. x, $2y.\frac{dy}{dx}=4a$

$$\Rightarrow \frac{dy}{dx}=\frac{4a}{2y} \Rightarrow \left(\frac{dy}{dx}\right)_{\left(\frac{a}{m^2},\frac{2a}{m}\right)}=\frac{4a}{2\left(\frac{2a}{m}\right)} \quad \therefore \frac{dy}{dx}=m$$

$\therefore$ The required equation of the tangent at point $\left(\frac{a}{m^2},\frac{2a}{m}\right)$, $(y-y_1)=\frac{dy}{dx}(x-x_1)$

$$\Rightarrow \frac{\left(y-\frac{2a}{m}\right)}{\left(x-\frac{a}{m^2}\right)}=m \Rightarrow \frac{m(my-2a)}{(m^2x-a)}=m \Rightarrow m^2x-a-my+2a=0 \Rightarrow m^2x-my+a=0$$

And, the required equation of the normal at point $\left(\frac{a}{m^2},\frac{2a}{m}\right)$, $y-y_1=\frac{-1}{dy/dx}(x-x_1)$

$$\Rightarrow \frac{y-y_1}{x-x_1}=\frac{-1}{m} \Rightarrow \frac{\left(y-\frac{2a}{m}\right)}{\left(x-\frac{a}{m^2}\right)}=\frac{-1}{m} \Rightarrow \frac{m(ym-2a)}{m^2x-a}=-1$$

$$\Rightarrow m(ym^2-2am)=-m^2x+a \Rightarrow ym^3-2am^2=-m^2x+a \Rightarrow m^2x+ym^3-2am^2-a=0$$

4. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at $(a\cos\theta, b\sin\theta)$

Sol. Let the given equation of curve, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Differentiating both sides w.r.t. x, $\frac{1}{a^2}.2x+\frac{1}{b^2}.2y\frac{dy}{dx}=0$

$$\Rightarrow \frac{2y}{b^2}\frac{dy}{dx}=\frac{-2x}{a^2} \Rightarrow \frac{dy}{dx}=\frac{-xb^2}{ya^2} \Rightarrow \left(\frac{dy}{dx}\right)_{(a\cos\theta,\ b\sin\theta)}=-\frac{a\cos\theta.b^2}{b\sin\theta.a^2} \Rightarrow \frac{dy}{dx}=-\frac{b}{a}\cot\theta$$

Equation of the tangent at point $(a\cos\theta, b\sin\theta)$

$$\Rightarrow y-b\sin\theta=-\frac{b}{a}\cot\theta(x-a\cos\theta) \Rightarrow y-b\sin\theta=\frac{-b}{a}.\frac{\cos\theta}{\sin\theta}(x-a\cos\theta)$$

$$\Rightarrow ay\sin\theta-ab\sin^2\theta=-xb\cos\theta+ab\cos^2\theta$$

$$\Rightarrow ay\sin\theta+bx\cos\theta=ab(\cos^2\theta+\sin^2\theta) \quad \therefore bx.\cos\theta+ay\sin\theta=ab$$

Now, equation of the normal at point $(a\cos\theta, b\sin\theta)$

$$\Rightarrow y-b\sin\theta=\frac{-1}{-\left(\frac{b}{a}\cot\theta\right)}(x-a\cos\theta) \Rightarrow y-b\sin\theta=\frac{a}{b}.\frac{\sin\theta}{\cos\theta}(x-a\cos\theta)$$

$\Rightarrow\ yb\cos\theta - b^2\sin\theta\cos\theta = ax\sin\theta - a^2\sin\theta\cos\theta$

$\Rightarrow\ a^2\sin\theta\cos\theta - b^2\sin\theta\cos\theta = ax\sin\theta - by\cos\theta$

$\Rightarrow\ \sin\theta\cos\theta\left(a^2 - b^2\right) = ax\sin\theta - by\cos\theta \quad \Rightarrow\ ax\sec\theta - by\,\text{cosec}\,\theta = a^2 - b^2$

5. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ at $\left(a\sec\theta,\ b\tan\theta\right)$

Sol. Let the given equation is, $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

Differentiating both sides w.r.t. x, $\dfrac{1}{a^2}.2x - \dfrac{1}{b^2}.2y\dfrac{dy}{dx} = 0$

$\Rightarrow\ \dfrac{dy}{dx} = \dfrac{xb^2}{ya^2} \quad \Rightarrow\ \left(\dfrac{dy}{dx}\right)_{(a\sec\theta,\ b\tan\theta)} = \dfrac{a\sec\theta.b^2}{b\tan\theta.a^2} \quad \Rightarrow\ \dfrac{dy}{dx} = \dfrac{b}{a}.\dfrac{\sec\theta}{\tan\theta}$

The equation of the tangent at point $\left(a\sec\theta,\ b\tan\theta\right)$.

$\Rightarrow\ y - b\tan\theta = \dfrac{b}{a}.\dfrac{\sec\theta}{\tan\theta}\left(x - a\sec\theta\right) \quad \Rightarrow\ ay\tan\theta - ab\tan^2\theta = bx\sec\theta - ab\sec^2\theta$

$\Rightarrow\ bx\sec\theta - ay\tan\theta = ab\left(\sec^2\theta - \tan^2\theta\right) \quad \therefore\ bx\sec\theta - ay\tan\theta = ab$

Now, the equation of the normal at point $\left(a\sec\theta,\ b\tan\theta\right)$.

$\Rightarrow\ y - b\tan\theta = -\dfrac{a\tan\theta}{b\sec\theta}\left(x - a\sec\theta\right) \quad \Rightarrow\ by\sec\theta - b^2\tan\theta\sec\theta = -ax\tan\theta + a^2\sec\theta\tan\theta$

$\Rightarrow\ ax\tan\theta + by\sec\theta = \sec\theta\tan\theta\left(a^2 + b^2\right) \quad \therefore\ ax\cos\theta + by\cot\theta = a^2 + b^2$

6. $y = x^3$ at $P(1,\ 1)$

Sol. Let be the equation of curve, $y = x^3$

Differentiating both sides w.r.t. x, $\dfrac{dy}{dx} = 3x^2 \quad \Rightarrow\ \left(\dfrac{dy}{dx}\right)_{(1,\ 1)} = 3(1)^2 = 3$

The required equation of the tangent at point $(1,\ 1)$, $y - y_1 = m\left(x - x_1\right)$

$\Rightarrow\ \dfrac{y - y_1}{x - x_1} = m \quad \Rightarrow\ \dfrac{y-1}{x-1} = 3 \quad \Rightarrow\ y - 1 = 3x - 3 \quad \Rightarrow\ 3x - y - 2 = 0$

And, the required equation of the normal at point $(1,\ 1)$, $y - y_1 = \dfrac{-1}{m}\left(x - x_1\right)$.

$\Rightarrow\ \dfrac{y - y_1}{x - x_1} = \dfrac{-1}{m} \quad \Rightarrow\ \dfrac{y-1}{x-1} = \dfrac{-1}{3} \quad \Rightarrow\ 3y - 3 = -x + 1 \quad \Rightarrow\ x + 3y - 3 - 1 = 0\ \therefore\ x + 3y - 4 = 0$

7. $y^2 = 4ax$ at $\left(at^2,\ 2at\right)$

Sol. The given equation of the curve, $y^2 = 4ax$

Differentiating both side w.r.t. x, $2y\dfrac{dy}{dx} = 4a \quad \Rightarrow\ \dfrac{dy}{dx} = \dfrac{2a}{y} \quad \Rightarrow\ \left(\dfrac{dy}{dx}\right)_{\left(at^2,2at\right)} = \dfrac{2a}{2at} = \dfrac{1}{t}$

Equation of the tangent at point $\left(at^2, 2at\right)$.

$\Rightarrow\ y - 2at = \dfrac{1}{t}\left(x - at^2\right) \quad \Rightarrow\ yt - 2at^2 = x - at^2 \quad \Rightarrow\ x - ty + at^2 = 0$

Also equation if the normal at point $\left(at^2, 2at\right)$

$\Rightarrow y-2at=-t\left(x-at^2\right) \Rightarrow y-2at=-tx+at^3 \Rightarrow y+tx=at^3+2at \therefore tx+y=at^3+2at$

8. $y=\cot^2 x-2\cot x+2$ at $x=\frac{\pi}{4}$

Sol. The given equation of the curve, $y=\cot^2 x-2\cot x+2 \quad \ldots(1)$

Putting the value of $x=\frac{\pi}{4}$ in equation (1), we get, $y=1$

Differentiating both sides w.r.t. x in (1), $\frac{dy}{dx}=2\cot x\left(-\operatorname{cosec}^2 x\right)-2\left(-\operatorname{cosec}^2 x\right)$

$\Rightarrow \frac{dy}{dx}=-2\cot x\operatorname{cosec}^2 x+2\operatorname{cosec}^2 x \quad \Rightarrow \left(\frac{dy}{dx}\right)_{\left(x=\frac{\pi}{4}\right)}=0$

Equation of the tangent at point $\left(\frac{\pi}{4},1\right)$; $y-1=0\left(x-\frac{\pi}{4}\right) \Rightarrow y-1=0$

Now, equation of the normal at point $\left(\frac{\pi}{4},1\right)$; $y-\frac{\pi}{4}=\frac{1}{0}\left(x-\frac{\pi}{4}\right) \Rightarrow 0=x-\frac{\pi}{4} \quad \therefore x=\frac{\pi}{4}$

9. $16x^2+9y^2=144$ at $\left(2, y_1\right)$ where $y_1>0$.

Sol. The given equation of the curve, $16x^2+9y^2=144 \quad \ldots(1)$

Putting the value of $x_1=2$ in equation (1), we get, $y_1=\frac{4\sqrt{5}}{3}$

On differentiating both sides of (1), w.r.t. x, $32x+18y\frac{dy}{dx}=0$

$\Rightarrow 18y\frac{dy}{dx}=-32x \quad \Rightarrow \frac{dy}{dx}=\frac{-16x}{9y} \quad \Rightarrow \left(\frac{dy}{dx}\right)_{\left(2,\frac{4\sqrt{5}}{3}\right)}=\frac{-16\times 2}{\left(9\times\frac{4\sqrt{5}}{3}\right)}=\frac{-8}{3\sqrt{5}}$

Equation of the tangent at point $\left(2, \frac{4\sqrt{5}}{3}\right)$

$$y-\frac{4\sqrt{5}}{3}=\frac{-8}{3\sqrt{5}}(x-2) \Rightarrow 3\sqrt{5}y-20=-8x+16 \Rightarrow 8x+3\sqrt{5}y-36=0$$

Now, equation of the normal at point $\left(2, \frac{4\sqrt{5}}{3}\right)$

$$y-\frac{4\sqrt{5}}{3}=\frac{3\sqrt{5}}{8}(x-2) \Rightarrow 8y-\frac{32\sqrt{5}}{3}=3\sqrt{5}x-6\sqrt{5}$$

$\Rightarrow 24y-32\sqrt{5}=9\sqrt{5}x-18\sqrt{5} \quad \therefore 9\sqrt{5}x-24y+14\sqrt{5}=0$

10. $y=x^4-6x^3+13x^2-10x+5$ at the point where $x=1$.

Sol. When $x=1$, than $y=x^4-6x^3+13x^2-10x+5$

$\Rightarrow y=(1)^4-6(1)^3+13(1)^2-10(1)+5 \quad \Rightarrow y=1-6+13-10+5$

$\Rightarrow y=19-16 \quad \Rightarrow y=3$ So, the point of contact is $(1, 3)$.

The given equation of curve is $y=x^4-6x^3+13x^2-10x+5$

Differentiating both sides w.r.t. x, $\frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10$

$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,3)} = 4(1)^3 - 18(1)^2 + 26(1) - 10 = 4 - 18 + 26 - 10 = 30 - 28 = 2 \quad \therefore\ m = 2$

$\therefore$ Required equation of the tangent is $y - y_1 = m(x - x_1)$.

$\Rightarrow y - 3 = 2x - 2 \quad \Rightarrow 2x - y - 2 + 3 = 0 \quad \Rightarrow 2x - y + 1 = 0 \qquad \therefore\ 2x - y + 1 = 0$

And, the required equation of the normal is $y - y_1 = \frac{-1}{m}(x - x_1)$

$\Rightarrow y - 3 = \frac{-1}{2}(x - 1) \quad \Rightarrow 2y - 6 = -x + 1 \quad \Rightarrow 2y - 6 + x - 1 = 0 \quad \Rightarrow x + 2y - 7 = 0$

11. Find the equation of the tangent to the curve $\sqrt{x} + \sqrt{y} = a$ at $\left(\frac{a^2}{4}, \frac{a^2}{4}\right)$.

Sol. Let be the equation of the curve be the $\sqrt{x} + \sqrt{y} = a$ ... (i)

Differentiating both side of (1), w.r.t. x, $\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$

$\Rightarrow \frac{1}{2\sqrt{y}}\frac{dy}{dx} = -\frac{1}{2\sqrt{x}} \quad \Rightarrow \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}} \quad \Rightarrow \frac{dy}{dx} = -\sqrt{\frac{y}{x}} \quad \Rightarrow \left(\frac{dy}{dx}\right)_{\left(\frac{a^2}{4}, \frac{a^2}{4}\right)} = -1$

Equation of the tangent at point $\left(\frac{a^2}{4}, \frac{a^2}{4}\right)$

$\Rightarrow y - \frac{a^2}{4} = -1\left(x - \frac{a^2}{4}\right) \quad \Rightarrow y - \frac{a^2}{4} = -x + \frac{a^2}{4} \quad \Rightarrow y + x = \frac{a^2}{4} + \frac{a^2}{4} \quad \Rightarrow 2(x + y) = a^2$

12. Show that the equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

Sol. Let be the equation of curve, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Differentiating both sides w.r.t. x, $\frac{1}{a^2}.2x - \frac{1}{b^2}.2y.\frac{dy}{dx} = 0$

$\Rightarrow \frac{-2y}{b^2}.\frac{dy}{dx} = \frac{-2x}{a^2} \quad \Rightarrow \frac{dy}{dx} = \frac{x\,b^2}{y\,a^2} \quad \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{x_1 b^2}{y_1 a^2}$

Equation of tangent at point (x_1, y_1), $y - y_1 = m(x - x_1) \quad \Rightarrow y - y_1 = \frac{x_1 b^2}{y_1 a^2}(x - x_1)$

$\Rightarrow yy_1a^2 - y_1^2a^2 = xx_1b^2 - x_1^2b^2 \quad \Rightarrow x_1^2b^2 - y_1^2a^2 = xx_1b^2 - yy_1a^2$

Dividing both side by a^2b^2, $\frac{x_1^2b^2}{a^2b^2} - \frac{y_1^2a^2}{a^2b^2} = \frac{xx_1b^2}{a^2b^2} - \frac{yy_1a^2}{a^2b^2}$

$\Rightarrow \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = \frac{xx_1}{a^2} - \frac{yy_1}{b^2} \quad \Rightarrow \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ [From equation (1)] $\quad \therefore\ \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$

13. Find the equation of the tangent to the curve $y = (\sec^4 x - \tan^4 x)$ at $x = \frac{\pi}{3}$.

Sol. Put $x=\frac{\pi}{3}$, then, $y=\sec^4\left(\frac{\pi}{3}\right)-\tan^4\left(\frac{\pi}{3}\right)=(2)^4-\left(\sqrt{3}\right)^4=16-9=7$

Differentiating both sides w.r.t. x, $\frac{dy}{dx}=4\sec^3 x.\sec x.\tan x-4\tan^3 x\sec^2 x$

$\Rightarrow \frac{dy}{dx}=4\sec^2 x\tan x\left(\sec^2 x-\tan^2 x\right) \quad \Rightarrow \frac{dy}{dx}=4\sec^2 x\tan x$

$\Rightarrow \left(\frac{dy}{dx}\right)_{x=\frac{\pi}{3}}=4\sec^2\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{3}\right)=4.4.\sqrt{3}=16\sqrt{3}$

Equation of tangent at point $\left(\frac{\pi}{3},7\right)$, $y-7=16\sqrt{3}\left(x-\frac{\pi}{3}\right)$

$\Rightarrow y-7=16\sqrt{3}x-\frac{16\sqrt{3}}{3}\pi \quad \Rightarrow y-7=\frac{48\sqrt{3}x-16\sqrt{3}\pi}{3} \quad \Rightarrow 3y-21=48\sqrt{3}x-16\sqrt{3}\pi$

$\Rightarrow 48\sqrt{3}x-3y-16\sqrt{3}\pi+21=0 \qquad \therefore\ 3y-48\sqrt{3}x+16\sqrt{3}\pi-21=0$

14. Find the equation of the normal to the curve $y=\left(\sin 2x+\cot x+2\right)^2$ at $x=\frac{\pi}{2}$.

Sol. Let the given equation of the curve, $y=\left(\sin 2x+\cot x+2\right)^2 \qquad \ldots(1)$

Putting the value of x in equation (1), we get, $y=4$

Differentiating both side of (1), w.r.t. x,

$\frac{dy}{dx}=2\left(\sin 2x+\cot x+2\right)\left(2\cos 2x-\operatorname{cosec}^2 x\right)$

$\Rightarrow \left(\frac{dy}{dx}\right)_{\left(x=\frac{\pi}{2}\right)}=-12$. Equation of the normal at point $\left(\frac{\pi}{2},4\right)$,

$$y-4=\frac{1}{12}\left(x-\frac{\pi}{2}\right) \quad \Rightarrow 12y-48=x-\frac{\pi}{2} \quad \Rightarrow 24y-96=2x-\pi$$

$\Rightarrow 24y-96-2x+\pi=0 \qquad \therefore\ 24y-2x+\pi-96=0$

15. Show that the tangents to the curve $y=2x^3-4$ at the points $x=2$ and $x=-2$ are parallel.

Sol. The given equation of curve, $y=2x^3-4$.

Differentiating both sides w.r.t. x, $\frac{dy}{dx}=6x^2-0 \quad \Rightarrow \frac{dy}{dx}=6x^2$

$\Rightarrow \left(\frac{dy}{dx}\right)_{x=2}=6(2)^2=6\times 4=24$

$\therefore\ m_1=24$, $\frac{dy}{dx}=6x^2 \Rightarrow \left(\frac{dy}{dx}\right)_{x=2}=6(2)^2=6\times 4=24$

$\therefore\ m_2=24$, $\frac{dy}{dx}=6x^2 \quad \Rightarrow \left(\frac{dy}{dx}\right)_{x=-2}=6(-2)^2=6\times 4=24$

$\because$ Tangents to the curve at the points $x=2$ and $x=-2$ are parallel. Hence, $m_1=m_2$.

16. Find the equation of the tangent to the curve $x^2+3y=3$, which is parallel to the line $y-4x+5=0$.

Sol. The given equation of curve, $x^2+3y=3 \qquad \ldots(1)$

Differentiating both sides w.r.t. x, $2x+3\frac{dy}{dx}=0$

$\Rightarrow 3\frac{dy}{dx} = -2x \quad \Rightarrow \frac{dy}{dx} = -\frac{2x}{3} \quad \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{-2x_1}{3}$

$\therefore \ m_1 = \frac{-2x_1}{3}$ and $y - 4x + 5 = 0 \quad \Rightarrow y - 4x = -5$

Differentiating both sides w.r.t. x, $\frac{dy}{dx} - 4 = 0$

$\Rightarrow \frac{dy}{dx} = 4 \quad \Rightarrow -\frac{2x_1}{3} = 4 \quad \Rightarrow -2x_1 = 12 \quad \Rightarrow x_1 = -6$

Putting the value of x_1 in equation (1), then $x_1^2 + 3y_1 = 3$

$\Rightarrow (-6)^2 + 3y_1 = 3 \quad \Rightarrow 36 + 3y_1 = 3 \quad \Rightarrow 3y_1 = 3 - 36 \quad \Rightarrow 3y_1 = -33 \quad \Rightarrow y_1 = -11$

The equation of the tangent at $(-6, -11)$.

$y - y_1 = m(x - x_1) \quad \Rightarrow y - (-11) = 4\{x - (-6)\} \quad \Rightarrow y + 11 = 4x + 24$

$\Rightarrow 4x - y + 24 - 11 = 0 \quad \Rightarrow 4x - y + 13 = 0$

17. At what points on the curve $x^2 + y^2 - 2x - 4y + 1 = 0$, is the tangent parallel to the y-axis?

Sol. The given equation of curve, $x^2 + y^2 - 2x - 4y + 1 = 0 \quad \ldots(1)$

Differentiating both sides w.r.t. x, $2x + 2y\frac{dy}{dx} - 2 - 4\frac{dy}{dx} = 0$

$\Rightarrow 2y\frac{dy}{dx} - 4\frac{dy}{dx} = 2 - 2x \quad \Rightarrow \frac{dy}{dx}(2y - 4) = 2 - 2x \quad \Rightarrow \frac{dy}{dx} = \frac{2(1-x)}{2(y-2)} \quad \Rightarrow \frac{dy}{dx} = \frac{1-x}{y-2}$

When the tangent is parallel to y-axis, then $\frac{1}{\frac{dy}{dx}} = 0$

$\Rightarrow \frac{dx}{dy} = 0 \quad \Rightarrow \frac{y-2}{1-x} = 0 \quad \Rightarrow y - 2 = 0 \quad \Rightarrow y = 2$

Putting the value of y in equation (1), then $x^2 + y^2 - 2x - 4y + 1 = 0$

$\Rightarrow x^2 + (2)^2 - 2(x) - 4(2) + 1 = 0 \quad \Rightarrow x^2 + 4 - 2x - 8 + 1 = 0$

$\Rightarrow x^2 - 2x - 3 = 0 \quad \Rightarrow x^2 - 3x + x - 3 = 0 \quad \Rightarrow x(x-3) + 1(x-3) = 0$

$\Rightarrow (x-3)(x+1) = 0 \quad \Rightarrow x = 3$ and $x = -1 \quad \therefore$ Points are $(3, 2)$ and $(-1, 2)$.

18. Find the points on the curve $2a^2y = x^3 - 3ax^2$ where the tangent is parallel to the x-axis.

Sol. The given equation of curve, $2a^2y = x^3 - 3ax^2 \quad \ldots(1)$

Differentiating both sides w.r.t. x, $2a^2\frac{dy}{dx} = 3x^2 - 3a.2x$

$\Rightarrow 2a^2\frac{dy}{dx} = 3x^2 - 6ax \quad \Rightarrow \frac{dy}{dx} = \frac{3x^2 - 6ax}{2a^2}$

When the tangent is parallel to x-axis, than $\frac{dy}{dx} = 0$

$\Rightarrow \frac{3x^2 - 6ax}{2a^2} = 0 \quad \Rightarrow 3x^2 - 6ax = 0 \quad \Rightarrow 3x(x - 2a) = 0 \quad \Rightarrow x - 2a = 0 \quad \therefore x = 2a$

Putting the value of x in equation (1), than $2a^2y = x^3 - 3ax^2$

$\Rightarrow 2a^2y=(2a)^3-3a(2a)^2 \quad \Rightarrow 2a^2y=8a^3-12a^3 \quad \Rightarrow 2a^2y=2a^2(4a-6a)$

$\Rightarrow y=\dfrac{2a^2(4a-6a)}{2a^2} \quad \Rightarrow y=-2a$. Hence, required points $(2a, -2a), (0, 0)$.

19. Prove that the tangents to the curve $y=x^2-5x+6$ at the point $(2, 0)$ and $(3, 0)$ are at right angles.

Sol. The given equation of curve, $y=x^2-5x+6$

Differentiating both sides w.r.t. x, $\dfrac{dy}{dx}=2x-5$

$\Rightarrow \left(\dfrac{dy}{dx}\right)_{(2,\,0)}=2(2)-5=4-5=-1 \quad \therefore m_1=-1.$

$\Rightarrow \left(\dfrac{dy}{dx}\right)_{(3,\,0)}=2(3)-5=6-5=1 \quad \therefore m_2=1.$

If two slopes are perpendicular than $m_1\times m_2=-1 \quad \Rightarrow -1\times 1=-1$.

20. Find the points on the curve $y=x^2+3x+4$ at which the tangent passes through the origin.

Sol. The given equation of curve. $y=x^2+3x+4$.

Differentiating both sides w.r.t. x, $\dfrac{dy}{dx}=2x+3 \Rightarrow \left(\dfrac{dy}{dx}\right)_{(x_1,\,y_1)}=2x_1+3 \quad \therefore m=2x_1+3$

The equation of the tangent is passes through the origin, $y-y_1=m(x-x_1)$

$\Rightarrow \dfrac{y-y_1}{x-x_1}=m \Rightarrow \dfrac{0-y_1}{0-x_1}=2x_1+3 \Rightarrow \dfrac{-y_1}{-x_1}=2x_1+3 \quad \Rightarrow y_1=2x_1^2+3x_1$ ....(1)

Also, (x_1, y_1) lies on the given curve

$\therefore y_1=x_1^2+3x_1+4 \quad \therefore 2x_1^2+3x_1=x_1^2+3x_1+4$

$\Rightarrow 2x_1^2-x_1^2+3x_1-3x_1-4=0 \quad \Rightarrow x_1^2=4 \quad \therefore x_1=\pm 2$

Putting the value of x_1 in equation (1), than $y_1=2x_1^2+3x_1$

$\Rightarrow x_1=-2,\ y_1=2(-2)^2+3(-2)=8-6=2 \quad \Rightarrow x_1=2, y_1=2(2)^2+3(2)=8+6=14$

Hence, points are $(-2, 2)$ and $(2, 14)$.

21. Find the points on the circle $y=x^3-11x+5$ at which the equation of tangent is $y=x-11$

Sol. Equation of curve is $y=x^3-11x+5$

We know that

Slope of tangent is $\dfrac{dy}{dx}$

$\dfrac{dy}{dx}=\dfrac{e(x^3-11x+5)}{dx}, \quad \dfrac{dy}{dx}=3x^2-11$ .... (i)

Also

Given tangent is $y=x-12$

Comparing with $y=mx+c$, when m is the slope

Slope of tangent = 1 .... (ii)

From (i) and (ii)

$\frac{dy}{dx}=1$ 3

$3x^2-11=1$

$3x^2=1+11$

$3x^2=12$

$x^2=\frac{12}{3}$

$x^2=4$

$x=\pm 2$

When $x=2$

$y=(2)^3-11(2)+5$

$y=8-22+5$

$y=-9$

So point $(2,-9)$

When $x=-2$

$y=(-2)^3-11(-2)+5$

$y=-8+22+5$

$y=19$

so point is $(-2,19)$

Hence points $(2,-9)\&(-2,19)$ But $(-2,19)$ does not satisfy line $y=x-11$

As $19\neq -2-11$ $\therefore$ only point is $(2,-9)$

22. Find the equation of the tangents to the curve $2x^2+3y^2=14$, parallel to the line $x+3y=4$

Sol. The equation of the given curve is $2x^2+3y^2=14$ ...(i)

Differentiating both sides w.r.t. x, we get, $4x+6y\frac{dy}{dx}=0$ $\Rightarrow \frac{dy}{dx}=-\frac{2x}{3y}$

Since tangent is parallel to the line $x+3y=4$.

Therefore, Slope of the tangent = Slope of the line $x+3y=4$

$\Rightarrow \frac{dy}{dx}=\frac{-1}{3} \Rightarrow -\frac{2x}{3y}=\frac{-1}{3} \Rightarrow y=2x$ ..... (ii)

From (i) and (ii) we have $2x^2+3\times(2x)^2=14$

$\Rightarrow 2x^2+12x^2=14$

$\Rightarrow 14x^2=14 \Rightarrow x=\pm 1$

Putting $x=\pm 1$ in equation (i) we have

If $x=1$ then $y=2$ and if $x=-1$ then $y=-2$

So, the points of contact are (1,2) and $(-1,-2)$

The respective tangents are $\frac{y-2}{x-1}=\frac{-1}{3}$ and $\frac{y+2}{x+1}=\frac{-1}{3}$

$\Rightarrow 3y-6=-x+1$ and $3y+6=-x-1$ $\Rightarrow 3y+x=7$ and $3y+x=-7$

23. Find the equation of the tangent to the curve $x^2+2y=8$ which is perpendicular to the line $x-2y+1=0$.

Sol. Let the given equation of the curve, $x^2+2y=8$ ...(1)

On differentiating both side of equation (1), w.r.t. x, $\frac{dy}{dx}=-x_1$. Also $x-2y+1=0$ $\Rightarrow \frac{dy}{dx}=\frac{1}{2}$

Two slope are perpendicular than $\left(-x_1\right)\left(\frac{1}{2}\right)=-1$

$\Rightarrow x_1=2$, Putting the value of x_1 in equation (1), then we get, $y_1=2$

$\therefore$ Slope $=-2$. Equation of the tangent at point $(2,\,2)$.

$\Rightarrow y-2=-2(x-2) \quad \Rightarrow y-2=-2x+4 \quad \therefore y+2x-6=0$

24. Find the point on the curve $y=2x^2-6x-4$ at which the tangent is parallel to the x-axis.

Sol. The given equation of curve, $y=2x^2-6x-4 \quad \ldots(1)$

Differentiating w.r.t. x, $\frac{dy}{dx}=4x-6$

When the tangent is parallel to the x-axis, $\frac{dy}{dx}=0$

$\Rightarrow 4x-6=0 \quad \Rightarrow 4x=6 \quad \Rightarrow x=\frac{3}{2}$

Putting the value of x in equation (1), then $y=2x^2-6x-4$

$\Rightarrow y=2\left(\frac{3}{2}\right)^2-6\left(\frac{3}{2}\right)-4 \quad \Rightarrow y=\frac{18}{4}-9-4 \quad \Rightarrow y=\frac{9-18-8}{2}$

$\Rightarrow y=\frac{-17}{2}$ Hence, point $\left(\frac{3}{2},\frac{-17}{2}\right)$.

25. Find a point on the parabola $y=(x-3)^2$, where the tangent is parallel to the chord joining the points $(3,\,0)$ and $(4,\,1)$.

Sol. The given equation of curve, $y=(x-3)^2$

$\Rightarrow y=x^2+9-6x \quad \Rightarrow y=x^2-6x+9 \quad \ldots(1)$

Differentiating w.r.t. x, $\frac{dy}{dx}=2x-6 \quad \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,\,y_1)}=2x_1-6$

The equations of the line through the points $(3,0)$ and $(4,1)$

$$y-0=\frac{1-0}{4-3}(x-3) \quad \Rightarrow y=x-3 \quad \Rightarrow \frac{dy}{dx}=1$$

Now, $1=2x_1-6 \quad \Rightarrow 2x_1=1+6 \quad \Rightarrow x_1=\frac{7}{2}$

Putting the value of x, in equation (1), than $y_1=(x_1-3)^2$

$\Rightarrow y_1=\left(\frac{7}{2}-3\right)^2 \quad \Rightarrow y_1=\left(\frac{7-6}{2}\right)^2 \quad \Rightarrow y_1=\left(\frac{1}{2}\right)^2 \quad \Rightarrow y_1=\frac{1}{4}$ Hence, point $\left(\frac{7}{2},\frac{1}{4}\right)$.

26. Show that the curves $x=y^2$ and $xy=k$ cut at right angles if $8k^2=1$.

Sol. The given curve, $y^2=x$

Differentiating both sides w.r.t. x, $2y.\frac{dy}{dx}=1$

$\Rightarrow \frac{dy}{dx}=\frac{1}{2y} \quad \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,\,y_1)}=\frac{1}{2y_1} \quad \Rightarrow m_1=\frac{dy}{dx}=\frac{1}{2y_1}$

And, $xy = k \Rightarrow x\frac{dy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = \frac{-y}{x} \Rightarrow m_2 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{-y_1}{x_1}$

Two slopes are perpendicular then $m_1 \times m_2 = -1$.

$\Rightarrow \frac{1}{2y_1}\left(\frac{-y_1}{x_1}\right) = -1 \Rightarrow \frac{1}{2x_1} = 1 \Rightarrow \frac{1}{2} = x_1$

Putting the value of x_1 in equation (1), than $y_1^2 = x_1$

$\Rightarrow y_1^2 = \frac{1}{2} \Rightarrow y_1 = \pm\frac{1}{\sqrt{2}}$

The given curve, $xy = k \Rightarrow \frac{1}{2}.\frac{1}{\sqrt{2}} = k \Rightarrow \frac{1}{2\sqrt{2}} = k \Rightarrow \frac{1}{2\sqrt{2}} = k \Rightarrow 1 = 2\sqrt{2}\,k$

Squaring both side, $(1)^2 = \left(2\sqrt{2}k\right)^2 \quad \therefore 1 = 8k^2$

27. Show that the curve $xy = a^2$ and $x^2 + y^2 = 2a^2$ touch each other.

Sol. The given equation of curve be, $xy = a^2$ ...(1)

On differentiating both side (1), w.r.t. x, $x\frac{dy}{dx} + y.1 = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-y}{x} \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{-y_1}{x_1} \Rightarrow m_1 = -\frac{y_1}{x_1}$

Again the given equation of curve be,

$x^2 + y^2 = 2a^2$ ...(2)

On differentiating both side of (2), w.r.t. x, we get, $2x + 2y\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-x}{y} \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{-x_1}{y_1}$

Two slope are perpendicular to each than $m_1 \times m_2 = -1$

$\Rightarrow \frac{-y_1}{x_1} \times \frac{x_1}{y_1} = -1$. $\therefore$ Hence two slope are perpendicular to each other.

28. Show that the curves $x^3 - 3xy^2 + 2 = 0$ and $3x^2y - y^3 - 2 = 0$ cut orthogonally.

Sol. The given equation of curve is $x^3 - 3xy^2 + 2 = 0$.

Differentiating both sides w.r.t. x, $3x^2 - 3\left\{x.2y\frac{dy}{dx} + y^2\right\} = 0$

$\Rightarrow 3x^2 - 6xy\frac{dy}{dx} - 3y^2 = 0 \Rightarrow -6xy\frac{dy}{dx} = 3y^2 - 3x^2$

$\Rightarrow \frac{dy}{dx} = \frac{3\left(y^2 - x^2\right)}{-6xy} \Rightarrow \frac{dy}{dx} = \frac{-1}{2}\left\{\frac{y^2}{xy} - \frac{x^2}{xy}\right\} \Rightarrow \frac{dy}{dx} = \frac{-1}{2}\left\{\frac{y}{x} - \frac{x}{y}\right\}$

$\Rightarrow \frac{dy}{dx} = \frac{-y}{2x} + \frac{x}{2y} \Rightarrow \frac{dy}{dx} = \frac{-y^2 + x^2}{2xy} \Rightarrow \left(\frac{dy}{dx}\right)_{(x, y)} = \frac{-y_1^2 + x_1^2}{2x_1y_1} = m_1$

And the equation of another curve is $3x^2y - y^3 - 2 = 0$.

Differentiating both sides w.r.t. x, $3\left\{x^2\frac{dy}{dx} + y.2x\right\} - 3y^2\frac{dy}{dx} = 0$

$$\Rightarrow 3x^2\frac{dy}{dx}+6xy-3y^2\frac{dy}{dx}=0 \quad \Rightarrow \frac{dy}{dx}=\frac{-6xy}{\left(3x^2-3y^2\right)} \quad \Rightarrow \frac{dy}{dx}=\frac{-6xy}{\left(3x^2-3y^2\right)}$$

$$\Rightarrow \frac{dy}{dx}=\frac{-6xy}{3\left(x^2-y^2\right)} \quad \Rightarrow \frac{dy}{dx}=\frac{-2xy}{x^2-y^2} \quad \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,\,y_1)}=\frac{-2x_1y_1}{x_1^2-y_1^2}=m_2$$

If two slopes are perpendicular than $m_1\times m_2=-1$. Now, $\left(\frac{-y_1^2+x_1^2}{2x_1y_1}\right)\times\left\{-\left(\frac{2x_1y_1}{x_1^2-y_1^2}\right)\right\}=-1.$

29. Find the equation of tangent to the curve $x=\theta+\sin\theta,\ y=1+\cos\theta$ at $\theta=\frac{\pi}{4}$

Sol. Let the given equation of the curve, $x=\theta+\sin\theta$ ...(1)

On differentiating both side of (1), w.r.t. θ, $\frac{dx}{d\theta}=1+\cos\theta$

The given another equation of the curve, $y=1+\cos\theta$ ...(2)

On differentiating both side of (2), w.r.t. θ, $\frac{dy}{dx}=-\sin\theta$

$$\therefore \frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=-\frac{\sin\theta}{1+\cos\theta}=-\left(\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}\right)=-\tan\left(\frac{\theta}{2}\right)$$

Equation of the tangent at point $(\theta+\sin\theta,\ 1+\cos\theta)$

$$\Rightarrow y-(1+\cos\theta)=-\tan\frac{\theta}{2}\{x-(\theta+\sin\theta)\} \text{ at } \left(\theta=\frac{\pi}{4}\right)$$

$$\Rightarrow y-\left(1+\frac{1}{\sqrt{2}}\right)=-\frac{1}{\sqrt{2}+1}\left(x-\frac{\pi}{4}-\frac{1}{\sqrt{2}}\right) \quad \therefore y=\left(1-\sqrt{2}\right)x+\frac{\left(\sqrt{2}-1\right)}{4}\pi+2$$

30. Find the equation of the tangent at $t=\frac{\pi}{4}$, for the curve $x=\sin 3t,\ y=\cos 2t$.

Sol. Let the given equation of the curve, $x=\sin 3t$ ...(1)

On differentiating both side (1), w.r.t. t, $\frac{dx}{dt}=\cos 3t.3 \quad \Rightarrow \frac{dx}{dt}=3\cos 3t$

The given another equation of curve, $y=\cos 2t$ ...(2)

On differentiating both side of (2), w.r.t. t, $\frac{dy}{dt}=-\sin 2t.2 \quad \Rightarrow \frac{dy}{dt}=-2\sin 2t$

$$\Rightarrow \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{-2\sin 2t}{3\cos 3t} \quad \Rightarrow \left(\frac{dy}{dx}\right)_{\left(t=\frac{\pi}{4}\right)}=\frac{-2}{3\times\left(-\frac{1}{2}\right)} \quad \Rightarrow \frac{dy}{dx}=\frac{4}{3} \quad \Rightarrow m=\frac{4}{3}$$

$$\Rightarrow x=\sin 3t \text{ at } t=\frac{\pi}{4} \quad \therefore \quad x=\frac{1}{\sqrt{2}} \ \&\ y=\cos 2t \text{ at } t=\frac{\pi}{4} \therefore y=0$$

Equation of the tangent at $\left(\frac{1}{\sqrt{2}},\ 0\right)$

$$\Rightarrow y-0=\frac{4}{3}\left(x-\frac{1}{\sqrt{2}}\right) \quad \Rightarrow 3y=4x-\frac{4}{\sqrt{2}} \quad \therefore \quad 4x-3\sqrt{2}y-2\sqrt{2}=0$$