* CHAPTER XXXI.

THE GENERAL THEORY OF CONTINUED FRACTIONS.

*436. In Chap. XXV. we have investigated the properties of Continued Fractions of the form $a_1 + \frac{1}{a_2 +}\,\frac{1}{a_3 +}\ldots$, where $a_2, a_3, \ldots$ are positive integers, and a_1 is either a positive integer or zero. We shall now consider continued fractions of a more general type.

*437. The most general form of a continued fraction is $\frac{b_1}{a_1 \pm}\,\frac{b_2}{a_2 \pm}\,\frac{b_3}{a_3 \pm}\ldots\ldots$, where $a_1, a_2, a_3, \ldots, b_1, b_2, b_3, \ldots$ represent any quantities whatever.

The fractions $\frac{b_1}{a_1}, \frac{b_2}{a_2}, \frac{b_3}{a_3}, \ldots$ are called *components* of the continued fraction. We shall confine our attention to two cases; (i) that in which the sign before each component is positive; (ii) that in which the sign is negative.

*438. *To investigate the law of formation of the successive convergents to the continued fraction*

$$\frac{b_1}{a_1 +}\,\frac{b_2}{a_2 +}\,\frac{b_3}{a_3 +}\ldots\ldots$$

The first three convergents are

$$\frac{b_1}{a_1}, \qquad \frac{a_2 b_1}{a_2 a_1 + b_2}, \qquad \frac{a_3 . a_2 b_1 + b_3 . b_1}{a_3 (a_2 a_1 + b_2) + b_3 . a_1}.$$

We see that the numerator of the third convergent may be formed by multiplying the numerator of the second convergent by a_3, and the numerator of the first by b_3 and adding the results together; also that the denominator may be formed in like manner.

Suppose that the successive convergents are formed in a similar way; let the numerators be denoted by p_1, p_2, $p_3 \dots$, and the denominators by q_1, q_2, q_3, ...

Assume that the law of formation holds for the n^{th} convergent; that is, suppose

$$p_n = a_n p_{n-1} + b_n p_{n-2}, \quad q_n = a_n q_{n-1} + b_n q_{n-2}.$$

The $(n+1)^{\text{th}}$ convergent differs from the n^{th} only in having $a_n + \dfrac{b_{n+1}}{a_{n+1}}$ in the place of a_n; hence

the $(n+1)^{\text{th}}$ convergent

$$= \frac{\left(a_n + \dfrac{b_{n+1}}{a_{n+1}}\right) p_{n-1} + b_n p_{n-2}}{\left(a_n + \dfrac{b_{n+1}}{a_{n+1}}\right) q_{n-1} + b_n q_{n-2}} = \frac{p_n + \dfrac{b_{n+1}}{a_{n+1}} p_{n-1}}{q_n + \dfrac{b_{n+1}}{a_{n+1}} q_{n-1}} = \frac{a_{n+1} p_n + b_{n+1} p_{n-1}}{a_{n+1} q_n + b_{n+1} q_{n-1}}.$$

If therefore we put

$$p_{n+1} = a_{n+1} p_n + b_{n+1} p_{n-1}, \quad q_{n+1} = a_{n+1} q_n + b_{n+1} q_{n-1},$$

we see that the numerator and denominator of the $(n+1)^{\text{th}}$ convergent follow the law which was supposed to hold in case of the n^{th}. But the law does hold in the case of the third convergent; hence it holds for the fourth; and so on; therefore it holds universally.

*439. In the case of the continued fraction

$$\frac{b_1}{a_1 -} \; \frac{b_2}{a_2 -} \; \frac{b_3}{a_3 -} \; \dots\dots,$$

we may prove that

$$p_n = a_n p_{n-1} - b_n p_{n-2}, \quad q_n = a_n q_{n-1} - b_n q_{n-2};$$

a result which may be deduced from that of the preceding article by changing the sign of b_n.

*440. In the continued fraction

$$\frac{b_1}{a_1 +} \; \frac{b_2}{a_2 +} \; \frac{b_3}{a_3 +} \; \dots\dots,$$

we have seen that

$$p_n = a_n p_{n-1} + b_n p_{n-2}, \quad q_n = a_n q_{n-1} + b_n q_{n-2}.$$

$$\therefore \quad \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} = \frac{(a_{n+1}p_n + b_{n+1}p_{n-1})\,q_n - (a_{n+1}q_n + b_{n+1}q_{n-1})\,p_n}{q_{n+1}q_n}$$

$$= -\frac{b_{n+1}q_{n-1}}{q_{n+1}}\left(\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right);$$

but $$\frac{b_{n+1}q_{n-1}}{q_{n+1}} = \frac{b_{n+1}q_{n-1}}{a_{n+1}q_n + b_{n+1}q_{n-1}},$$

and is therefore a proper fraction; hence $\dfrac{p_{n+1}}{q_{n+1}} - \dfrac{p_n}{q_n}$ is numerically less than $\dfrac{p_n}{q_n} - \dfrac{p_{n-1}}{q_{n-1}}$, and is of opposite sign.

By reasoning as in Art. 335, we may shew that every convergent of an odd order is greater than the continued fraction, and every convergent of an even order is less than the continued fraction; hence every convergent of an odd order is greater than every convergent of an even order.

Thus $\dfrac{p_{2n+1}}{q_{2n+1}} - \dfrac{p_{2n}}{q_{2n}}$ is positive and less than $\dfrac{p_{2n-1}}{q_{2n-1}} - \dfrac{p_{2n}}{q_{2n}}$; hence

$$\frac{p_{2n+1}}{q_{2n+1}} < \frac{p_{2n-1}}{q_{2n-1}}.$$

Also $\dfrac{p_{2n-1}}{q_{2n-1}} - \dfrac{p_{2n}}{q_{2n}}$ is positive and less than $\dfrac{p_{2n-1}}{q_{2n-1}} - \dfrac{p_{2n-2}}{q_{2n-2}}$; hence

$$\frac{p_{2n}}{q_{2n}} > \frac{p_{2n-2}}{q_{2n-2}}.$$

Hence the convergents of an odd order are all greater than the continued fraction but continually decrease, and the convergents of an even order are all less than the continued fraction but continually increase.

Suppose now that the number of components is infinite, then the convergents of an odd order must tend to some finite limit, and the convergents of an even order must also tend to some finite limit; if these limits are equal the continued fraction tends to *one* definite limit; if they are not equal, the odd convergents tend to one limit, and the even convergents tend to a different limit, and the continued fraction may be said to be *oscillating;* in this case the continued fraction is the symbolical representation of two quantities, one of which is the limit of the odd, and the other that of the even convergents.

*441. *To shew that the continued fraction* $\frac{b_1}{a_1+}\,\frac{b_2}{a_2+}\,\frac{b_3}{a_3+}\ldots$ *has a definite value if the limit of* $\frac{a_n a_{n+1}}{b_{n+1}}$ *when* n *is infinite is greater than zero.*

The continued fraction will have a definite value when n is infinite if the difference of the limits of $\frac{p_{n+1}}{q_{n+1}}$ and $\frac{p_n}{q_n}$ is equal to zero.

Now
$$\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} = -\frac{b_{n+1}q_{n-1}}{q_{n+1}}\left(\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right);$$

whence we obtain

$$\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} = (-1)^{n-1}\frac{b_{n+1}q_{n-1}}{q_{n+1}}\cdot\frac{b_n q_{n-2}}{q_n}\cdots\frac{b_4 q_2}{q_4}\cdot\frac{b_3 q_1}{q_3}\left(\frac{p_2}{q_2} - \frac{p_1}{q_1}\right).$$

But
$$\frac{b_{n+1}q_{n-1}}{q_{n+1}} = \frac{b_{n+1}q_{n-1}}{a_{n+1}q_n + b_{n+1}q_{n-1}} = \frac{1}{\dfrac{a_{n+1}q_n}{b_{n+1}q_{n-1}} + 1};$$

and
$$\frac{a_{n+1}q_n}{b_{n+1}q_{n-1}} = \frac{a_{n+1}(a_n q_{n-1} + b_n q_{n-2})}{b_{n+1}q_{n-1}} = \frac{a_n a_{n+1}}{b_{n+1}} + \frac{a_{n+1}b_n q_{n-2}}{b_{n+1}q_{n-1}};$$

also neither of these terms can be negative; hence if the limit of $\frac{a_n a_{n+1}}{b_{n+1}}$ is greater than zero so also is the limit of $\frac{a_{n+1}q_n}{b_{n+1}q_{n-1}}$; in which case the limit of $\frac{b_{n+1}q_{n-1}}{q_{n+1}}$ is less than 1; and therefore $\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}$ is the limit of the product of an infinite number of proper fractions, and must therefore be equal to zero; that is, $\frac{p_{n+1}}{q_{n+1}}$ and $\frac{p_n}{q_n}$ tend to the same limit; which proves the proposition.

For example, in the continued fraction

$$\frac{1^2}{3+}\,\frac{2^2}{5+}\,\frac{3^2}{7+}\cdots\cdots\frac{n^2}{2n+1+}\cdots,$$

$$Lim\,\frac{a_n a_{n+1}}{b_{n+1}} = Lim\,\frac{(2n+1)(2n+3)}{(n+1)^2} = 4;$$

and therefore the continued fraction tends to a definite limit.

*442. *In the continued fraction* $\frac{b_1}{a_1 -}\ \frac{b_2}{a_2 -}\ \frac{b_3}{a_3 -} \cdots,$ *if the denominator of every component exceeds the numerator by unity at least, the convergents are positive fractions in ascending order of magnitude.*

By supposition $\frac{b_1}{a_1}, \frac{b_2}{a_2}, \frac{b_3}{a_3}, \ldots$ are positive proper fractions in each of which the denominator exceeds the numerator by unity at least. The second convergent is $\cfrac{b_1}{a_1 - \cfrac{b_2}{a_2}}$, and since a_1 exceeds b_1 by unity at least, and $\frac{b_2}{a_2}$ is a proper fraction, it follows that $a_1 - \frac{b_2}{a_2}$ is greater than b_1; that is, the second convergent is a positive proper fraction. In like manner it may be shewn that $\cfrac{b_2}{a_2 - \cfrac{b_3}{a_3}}$ is a positive proper fraction; denote it by f_1, then the third convergent is $\frac{b_1}{a_1 - f_1}$, and is therefore a positive proper fraction. Similarly we may shew that $\frac{b_2}{a_2 -}\ \frac{b_3}{a_3 -}\ \frac{b_4}{a_4}$ is a positive proper fraction; hence also the fourth convergent

$$\frac{b_1}{a_1 -}\ \frac{b_2}{a_2 -}\ \frac{b_3}{a_3 -}\ \frac{b_4}{a_4}$$

is a positive proper fraction; and so on.

Again, $\quad p_n = a_n p_{n-1} - b_n p_{n-2}, \quad q_n = a_n q_{n-1} - b_n q_{n-2};$

$$\therefore \quad \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} = \frac{b_{n+1} q_{n-1}}{q_{n+1}} \left(\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} \right);$$

hence $\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}$ and $\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}$ have the same sign.

But $\frac{p_2}{q_2} - \frac{p_1}{q_1} = \frac{a_2 b_1}{a_1 a_2 - b_2} - \frac{b_1}{a_1} = \frac{b_1 b_2}{q_1 q_2}$, and is therefore positive; hence $\frac{p_2}{q_2} > \frac{p_1}{q_1}$, $\frac{p_3}{q_3} > \frac{p_2}{q_2}$, $\frac{p_4}{q_4} > \frac{p_3}{q_3}$; and so on; which proves the proposition.

Cor. If the number of the components is infinite, the convergents form an infinite series of proper fractions in ascending order of magnitude; and in this case the continued fraction must tend to a definite limit which cannot exceed unity.

*443. From the formula

$$p_n = a_n p_{n-1} + b_n p_{n-2}, \quad q_n = a_n q_{n-1} + b_n q_{n-2},$$

we may always determine in succession as many of the convergents as we please. In certain cases, however, a general expression can be found for the n^{th} convergent.

Example. To find the n^{th} convergent to $\frac{6}{5-}\,\frac{6}{5-}\,\frac{6}{5-}\ldots\ldots$

We have $p_n = 5p_{n-1} - 6p_{n-2}$; hence the numerators form a recurring series any three consecutive terms of which are connected by the relation

$$p_n - 5p_{n-1} + 6p_{n-2}.$$

Let $$S = p_1 + p_2 x + p_3 x^2 + \ldots + p_n x^{n-1} + \ldots;$$

then, as in Art. 325, we have $S = \dfrac{p_1 + (p_2 - 5p_1)\,x}{1 - 5x + 6x^2}$.

But the first two convergents are $\frac{6}{5}$, $\frac{30}{19}$;

$$\therefore\ S = \frac{6}{1 - 5x + 6x^2} = \frac{18}{1 - 3x} - \frac{12}{1 - 2x};$$

whence $$p_n = 18 \cdot 3^{n-1} - 12 \cdot 2^{n-1} = 6\,(3^n - 2^n).$$

Similarly if $$S' = q_1 + q_2 x + q_3 x^2 + \ldots + q_n x^{n-1} + \ldots,$$

we find $$S' = \frac{5 - 6x}{1 - 5x + 6x^2} = \frac{9}{1 - 3x} - \frac{4}{1 - 2x};$$

whence $$q_n = 9 \cdot 3^{n-1} - 4 \cdot 2^{n-1} = 3^{n+1} - 2^{n+1}.$$

$$\therefore\ \frac{p_n}{q_n} = \frac{6\,(3^n - 2^n)}{3^{n+1} - 2^{n+1}}.$$

This method will only succeed when a_n and b_n are constant for all values of n. Thus in the case of the continued fraction $\frac{b}{a+}\,\frac{b}{a+}\,\frac{b}{a+}\ldots$, we may shew that the numerators of the successive convergents are the coefficients of the powers of x in the expansion of $\dfrac{b}{1 - ax - bx^2}$, and the denominators are the coefficients of the powers of x in the expansion of $\dfrac{a + bx}{1 - ax - bx^2}$.

*444. For the investigation of the general values of p_n and q_n the student is referred to works on *Finite Differences;* it is only in special cases that these values can be found by *Algebra.* The following method will sometimes be found useful.

Example. Find the value of $\frac{1}{1+}\ \frac{2}{2+}\ \frac{3}{3+}\ldots\ldots$

The same law of formation holds for p_n and q_n; let us take u_n to denote either of them; then $\qquad u_n = nu_{n-1} + nu_{n-2}$,

or
$$u_n - (n+1)\,u_{n-1} = -(u_{n-1} - nu_{n-2}).$$

Similarly,
$$u_{n-1} - nu_{n-2} = -(u_{n-2} - \overline{n-1}\,u_{n-3}).$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$u_3 - 4u_2 = -(u_2 - 3u_1);$$

whence by multiplication, we obtain

$$u_n - (n+1)\,u_{n-1} = (-1)^{n-2}(u_2 - 3u_1).$$

The first two convergents are $\frac{1}{1}$, $\frac{2}{4}$; hence

$$p_n - (n+1)\,p_{n-1} = (-1)^{n-1}, \quad q_n - (n+1)\,q_{n-1} = (-1)^{n-2}.$$

Thus
$$\frac{p_n}{\lfloor n+1} - \frac{p_{n-1}}{\lfloor n} = \frac{(-1)^{n-1}}{\lfloor n+1}, \qquad \frac{q_n}{\lfloor n+1} - \frac{q_{n-1}}{\lfloor n} = \frac{(-1)^{n-2}}{\lfloor n+1},$$

$$\frac{p_{n-1}}{\lfloor n} - \frac{p_{n-2}}{\lfloor n-1} = \frac{(-1)^{n-2}}{\lfloor n}, \qquad \frac{q_{n-1}}{\lfloor n} - \frac{q_{n-2}}{\lfloor n-1} = \frac{(-1)^{n-3}}{\lfloor n},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{p_2}{\lfloor 3} - \frac{p_1}{\lfloor 2} = -\frac{1}{\lfloor 3}, \qquad \frac{q_2}{\lfloor 3} - \frac{q_1}{\lfloor 2} = \frac{1}{\lfloor 3},$$

$$\frac{p_1}{\lfloor 2} = \frac{1}{\lfloor 2}, \qquad \frac{q_1}{\lfloor 2} = \frac{1}{2} = 1 - \frac{1}{\lfloor 2};$$

whence, by addition

$$\frac{p_n}{\lfloor n+1} = \frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} - \ldots\ldots + \frac{(-1)^{n-1}}{\lfloor n+1};$$

$$\frac{q_n}{\lfloor n+1} = 1 - \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} - \frac{1}{\lfloor 4} + \ldots\ldots + \frac{(-1)^{n-2}}{\lfloor n+1}.$$

By making n infinite, we obtain

$$Lim\,\frac{p_n}{q_n} = \frac{1}{e} \div \left(1 - \frac{1}{e}\right) = \frac{1}{e-1},$$

which is therefore the value of the given expression.

*445. *If every component of* $\frac{b_1}{a_1+}\,\frac{b_2}{a_2+}\,\frac{b_3}{a_3+}\,\ldots$ *is a proper fraction with integral numerator and denominator, the continued fraction is incommensurable.*

For if possible, suppose that the continued fraction is commensurable and equal to $\frac{B}{A}$, where A and B are positive integers; then $\frac{B}{A}=\frac{b_1}{a_1+f_1}$, where f_1 denotes the infinite continued fraction $\frac{b_2}{a_2+}\,\frac{b_3}{a_3+}\,\ldots$; hence $f_1=\frac{Ab_1-Ba_1}{B}=\frac{C}{B}$ suppose. Now A, B, a_1, b_1 are integers and f_1 is positive, therefore C is a positive integer. Similarly $\frac{C}{B}=\frac{b_2}{a_2+f_2}$, where f_2 denotes the infinite continued fraction $\frac{b_3}{a_3+}\,\frac{b_4}{a_4+}\,\ldots$; hence $f_2=\frac{Bb_2-Ca_2}{C}=\frac{D}{C}$ suppose; and as before, it follows that D is a positive integer; and so on.

Again, $\frac{B}{A}$, $\frac{C}{B}$, $\frac{D}{C}$, ... are proper fractions; for $\frac{B}{A}$ is less than $\frac{b_1}{a_1}$, which is a proper fraction; $\frac{C}{B}$ is less than $\frac{b_2}{a_2}$; $\frac{D}{C}$ is less than $\frac{b_3}{a_3}$; and so on.

Thus A, B, C, D, ... form an *infinite* series of *positive integers* in *descending* order of magnitude; which is absurd. Hence the given fraction cannot be commensurable.

The above result still holds if some of the components are not proper fractions, provided that from and after a fixed component all the others are proper fractions.

For suppose that $\frac{b_n}{a_n}$ and all the succeeding components are proper fractions; thus, as we have just proved, the infinite continued fraction beginning with $\frac{b_n}{a_n}$ is incommensurable; denote it by k, then the complete quotient corresponding to $\frac{p_n}{q_n}$ is $\frac{k}{1}$, and therefore the value of the continued fraction is $\frac{p_{n-1}+kp_{n-2}}{q_{n-1}+kq_{n-2}}$.

This cannot be commensurable unless $\frac{p_{n-1}}{q_{n-1}}=\frac{p_{n-2}}{q_{n-2}}$; and this condition cannot hold unless $\frac{p_{n-2}}{q_{n-2}}=\frac{p_{n-3}}{q_{n-3}}$, $\frac{p_{n-3}}{q_{n-3}}=\frac{p_{n-4}}{q_{n-4}}$, ..., and finally $\frac{p_2}{q_2}=\frac{p_1}{q_1}$; that is $b_1b_2=0$, which is impossible; hence the given fraction must be incommensurable.

*446. *If every component of* $\frac{b_1}{a_1-}\,\frac{b_2}{a_2-}\,\frac{b_3}{a_3-}\,\ldots$ *is a proper fraction with integral numerator and denominator, and if the value of the infinite continued fraction beginning with any component is less than unity, the fraction is incommensurable.*

The demonstration is similar to that of the preceding article.

*EXAMPLES. XXXI. a.

1. Shew that in the continued fraction

$$\frac{b_1}{a_1-}\,\frac{b_2}{a_2-}\,\frac{b_3}{a_3-}\,\ldots\ldots,$$

$$p_n=a_np_{n-1}-b_np_{n-2},\quad q_n=a_nq_{n-1}-b_nq_{n-2}.$$

2. Convert $\left(\frac{2x+1}{2x}\right)^2$ into a continued fraction with unit numerators.

3. Shew that

$$(1)\quad \sqrt{a^2+b}=a+\frac{b}{2a+}\,\frac{b}{2a+}\,\ldots\ldots,$$

$$(2)\quad \sqrt{a^2-b}=a-\frac{b}{2a-}\,\frac{b}{2a-}\,\ldots\ldots$$

4. In the continued fraction $\frac{b_1}{a_1-}\,\frac{b_2}{a_2-}\,\frac{b_3}{a_3-}\,\ldots$, if the denominator of every component exceed the numerator by unity at least, shew that p_n and q_n increase with n.

5. If $a_1, a_2, a_3, \ldots a_n$ are in harmonical progression, shew that

$$\frac{a_n}{a_{n-1}}=\frac{1}{2-}\,\frac{1}{2-}\,\frac{1}{2-}\,\ldots\ldots\,\frac{1}{2-}\,\frac{a_2}{a_1}.$$

6. Shew that

$$\left(a+\frac{1}{2a+}\,\frac{1}{2a+}\cdots\right)^2+\left(a-\frac{1}{2a-}\,\frac{1}{2a-}\cdots\right)^2=2a^2,$$

and $\left(a+\frac{1}{2a+}\,\frac{1}{2a+}\cdots\right)\left(a-\frac{1}{2a-}\,\frac{1}{2a-}\cdots\right)=a^2-\frac{1}{2a^2-}\,\frac{1}{2a^2-}\cdots.$

7. In the continued fraction

$$\frac{b}{a+}\,\frac{b}{a+}\,\frac{b}{a+}\,\ldots\ldots,$$

shew that $\qquad p_{n+1}=bq_n, \quad bq_{n+1}-ap_{n+1}=b^2q_{n-1}.$

8. Shew that $\frac{b}{a+}\,\frac{b}{a+}\,\frac{b}{a+}\,\ldots\ldots=b.\frac{\alpha^x-\beta^x}{\alpha^{x+1}-\beta^{x+1}},$

x being the number of components, and α, β the roots of the equation

$$k^2-ak-b=0.$$

9. Prove that the product of the continued fractions

$$a+\frac{1}{b+}\,\frac{1}{c+}\,\frac{1}{d+}\,\frac{1}{a+}\cdots, \qquad -d+\frac{1}{-c+}\,\frac{1}{-b+}\,\frac{1}{-a+}\,\frac{1}{-d+}\cdots,$$

is equal to -1.

Shew that

10. $\frac{1}{1-}\,\frac{4}{5-}\,\frac{9}{13-}\,\frac{64}{25-}\ldots\ldots\frac{(n^2-1)^2}{n^2+(n+1)^2}=\frac{(n+1)(n+2)(2n+3)}{6}.$

11. $\frac{2}{1-}\,\frac{3}{5-}\,\frac{8}{7-}\ldots\ldots\frac{n^2-1}{2n+1}=\frac{n(n+3)}{2}.$

12. $\frac{2}{2-}\,\frac{3}{3-}\,\frac{4}{4-}\ldots\ldots\frac{n+1}{n+1-}\,\frac{n+2}{n+2}=1+1+\lfloor 2+\lfloor 3+\ldots+\lfloor n.$

13. $\frac{1}{1-}\,\frac{1}{3-}\,\frac{2}{4-}\,\frac{3}{5-}\ldots\ldots\frac{n-1}{n+1-}\ldots\ldots=e-1.$

14. $\frac{4}{1+}\,\frac{6}{2+}\,\frac{8}{3+}\ldots\ldots\frac{2n+2}{n+}\ldots\ldots=\frac{2(e^2-1)}{e^2+1}.$

15. $\frac{3.3}{1+}\,\frac{3.4}{2+}\,\frac{3.5}{3+}\ldots\ldots\frac{3(n+2)}{n+}\ldots\ldots=\frac{6(2e^3+1)}{5e^3-2}.$

16. If $u_1=\frac{a}{b}$, $u_2=\frac{b}{a+b}$, $u_3=\frac{a+b}{a+2b}$,......, each successive fraction being formed by taking the denominator and the sum of the numerator and denominator of the preceding fraction for its numerator and denominator respectively, shew that $u_\infty=\frac{\sqrt{5}-1}{2}$.

17. Prove that the n^{th} convergent to the continued fraction

$$\frac{r}{r+1-}\ \frac{r}{r+1-}\ \frac{r}{r+1-}\ \ldots\ldots \text{ is } \frac{r^{n+1}-r}{r^{n+1}-1}.$$

18. Find the value of $\dfrac{a_1}{a_1+1-}\ \dfrac{a_2}{a_2+1-}\ \dfrac{a_3}{a_3+1-}\ \ldots\ldots,$

$a_1, a_2, a_3, \ldots$ being positive and greater than unity.

19. Shew that the n^{th} convergent to $1-\dfrac{1}{4-}\ \dfrac{1}{4-}\ \ldots\ldots$ is equal to the $(2n-1)^{\text{th}}$ convergent to $\dfrac{1}{1+}\ \dfrac{1}{2+}\ \dfrac{1}{1+}\ \dfrac{1}{2+}\ \ldots\ldots$

20. Shew that the $3n^{\text{th}}$ convergent to

$$\frac{1}{5-}\ \frac{1}{2-}\ \frac{1}{1-}\ \frac{1}{5-}\ \frac{1}{2-}\ \frac{1}{1-}\ \frac{1}{5-}\ \ldots\ldots \text{ is } \frac{n}{3n+1}.$$

21. Shew that $\dfrac{1}{2+}\ \dfrac{2}{3+}\ \dfrac{3}{4+}\ \ldots\ldots = \dfrac{3-e}{e-2};$

hence shew that e lies between $2\frac{2}{3}$ and $2\frac{8}{11}$.

Conversion of Series into Continued Fractions.

*447. It will be convenient here to write the series in the form

$$\frac{1}{u_1}+\frac{1}{u_2}+\frac{1}{u_3}+\ldots\ldots+\frac{1}{u_n}.$$

Put
$$\frac{1}{u_r}+\frac{1}{u_{r+1}}=\frac{1}{u_r+x_r};$$

then
$$(u_r+x_r)(u_{r+1}+u_r)=u_r u_{r+1},$$

$$\therefore\ x_r=-\frac{u_r^{\,2}}{u_r+u_{r+1}}.$$

Hence
$$\frac{1}{u_1}+\frac{1}{u_2}=\cfrac{1}{u_1-\cfrac{u_1^{\,2}}{u_1+u_2}}=\frac{1}{u_1-}\ \frac{u_1^{\,2}}{u_1+u_2}.$$

Similarly,

$$\frac{1}{u_1}+\frac{1}{u_2}+\frac{1}{u_3}=\frac{1}{u_1}+\frac{1}{u_2+x_2}=\frac{1}{u_1-}\;\frac{u_1^{\,2}}{u_1+u_2+x_2}.$$

$$=\frac{1}{u_1-}\;\frac{u_1^{\,2}}{u_1+u_2-}\;\frac{u_2^{\,2}}{u_2+u_3};$$

and so on; hence generally

$$\frac{1}{u_1}+\frac{1}{u_2}+\frac{1}{u_3}+\ldots\ldots+\frac{1}{u_n}$$

$$=\frac{1}{u_1-}\;\frac{u_1^{\,2}}{u_1+u_2-}\;\frac{u_2^{\,2}}{u_2+u_3-}\;\ldots\ldots\;\frac{u^2_{n-1}}{u_{n-1}+u_n}.$$

Example 1. Express as a continued fraction the series

$$\frac{1}{a_0}-\frac{x}{a_0a_1}+\frac{x^2}{a_0a_1a_2}-\ldots\ldots+(-1)^n\frac{x^n}{a_0a_1a_2\ldots a_n}.$$

Put
$$\frac{1}{a_n}-\frac{x}{a_na_{n+1}}=\frac{1}{a_n+y_n};$$

then
$$(a_n+y_n)(a_{n+1}-x)=a_na_{n+1};$$

$$\therefore\;\; y_n=\frac{a_nx}{a_{n+1}-x}.$$

Hence
$$\frac{1}{a_0}-\frac{x}{a_0a_1}=\frac{1}{a_0+y_0}=\frac{1}{a_0+}\;\frac{a_0x}{a_1-x}.$$

Again,
$$\frac{1}{a_0}-\frac{x}{a_0a_1}+\frac{x^2}{a_0a_1a_2}=\frac{1}{a_0}-\frac{x}{a_0}\left(\frac{1}{a_1}-\frac{x}{a_1a_2}\right)=\frac{1}{a_0}-\frac{x}{a_0(a_1+y_1)}$$

$$=\frac{1}{a_0+}\;\frac{a_0x}{a_1+y_1-x}$$

$$=\frac{1}{a_0+}\;\frac{a_0x}{a_1-x+}\;\frac{a_1x}{a_2-x};$$

and generally
$$\frac{1}{a_0}-\frac{x}{a_0a_1}+\frac{x^2}{a_0a_1a_2}-\ldots+(-1)^n\frac{x^n}{a_0a_1a_2\ldots a_n}$$

$$=\frac{1}{a_0+}\;\frac{a_0x}{a_1-x+}\;\frac{a_1x}{a_2-x+}\;\ldots\ldots\;\frac{a_{n-1}x}{a_n-x}.$$

Example 2. Express $\log(1+x)$ as a continued fraction.

We have
$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\ldots.$$

The required expression is most simply deduced from the continued fraction equivalent to the series

$$\frac{x}{a_1}-\frac{x^2}{a_2}+\frac{x^3}{a_3}-\frac{x^4}{a_4}+\ldots\ldots.$$

By putting
$$\frac{1}{a_n} - \frac{x}{a_{n+1}} = \frac{1}{a_n + y_n},$$

we obtain
$$y_n = \frac{a_n^2 x}{a_{n+1} - a_n x};$$

hence we have

$$\frac{x}{a_1} - \frac{x^2}{a_2} + \frac{x^3}{a_3} - \frac{x^4}{a_4} + \ldots = \frac{x}{a_1 +}\ \frac{a_1^2 x}{a_2 - a_1 x +}\ \frac{a_2^2 x}{a_3 - a_2 x +}\ \frac{a_3^2 x}{a_4 - a_3 x +} \ldots;$$

$$\therefore\ \log(1+x) = \frac{x}{1+}\ \frac{1^2 x}{2 - x +}\ \frac{2^2 x}{3 - 2x +}\ \frac{3^2 x}{4 - 3x +} \ldots\ldots$$

*448. In certain cases we may simplify the components of the continued fraction by the help of the following proposition:

The continued fraction

$$\frac{b_1}{a_1 +}\ \frac{b_2}{a_2 +}\ \frac{b_3}{a_3 +}\ \frac{b_4}{a_4 +} \ldots\ldots$$

is equal to the continued fraction

$$\frac{c_1 b_1}{c_1 a_1 +}\ \frac{c_1 c_2 b_2}{c_2 a_2 +}\ \frac{c_2 c_3 b_3}{c_3 a_3 +}\ \frac{c_3 c_4 b_4}{c_4 a_4 +} \ldots\ldots;$$

where c_1, c_2, c_3, c_4,...... are any quantities whatever.

Let f_1 denote $\dfrac{b_2}{a_2 +}\ \dfrac{b_3}{a_3 +} \ldots\ldots$; then

the continued fraction $= \dfrac{b_1}{a_1 + f_1} = \dfrac{c_1 b_1}{c_1 a_1 + c_1 f_1}$.

Let f_2 denote $\dfrac{b_3}{a_3 +}\ \dfrac{b_4}{a_4 +} \ldots$; then

$$c_1 f_1 = \frac{c_1 b_2}{a_2 + f_2} = \frac{c_1 c_2 b_2}{c_2 a_2 + c_2 f_2}.$$

Similarly, $c_2 f_2 = \dfrac{c_2 c_3 b_3}{c_3 a_3 + c_3 f_3}$; and so on; whence the proposition is established.

*EXAMPLES. XXXI. b.

Shew that

1. $\dfrac{1}{u_0} - \dfrac{1}{u_1} + \dfrac{1}{u_2} - \dfrac{1}{u_3} + \ldots\ldots + (-1)^n \dfrac{1}{u_n}$

$$= \frac{1}{u_0+}\ \frac{u_0^2}{u_1 - u_0+}\ \frac{u_1^2}{u_2 - u_1+} \ldots\ldots \frac{u^2_{n-1}}{u_n - u_{n-1}}.$$

2. $\dfrac{1}{a_0} + \dfrac{x}{a_0a_1} + \dfrac{x^2}{a_0a_1a_2} + \ldots\ldots + \dfrac{x^n}{a_0a_1a_2\ldots a_n}$

$$= \frac{1}{a_0-}\ \frac{a_0x}{a_1+x-}\ \frac{a_1x}{a_2+x-} \ldots\ldots \frac{a_{n-1}x}{a_n+x}.$$

3. $\dfrac{r-1}{r-2} = \dfrac{r}{r-}\ \dfrac{r+1}{r+1-}\ \dfrac{r+2}{r+2-} \ldots\ldots$

4. $\dfrac{2n}{n+1} = \dfrac{1}{1-}\ \dfrac{1}{4-}\ \dfrac{1}{1-}\ \dfrac{1}{4-} \ldots\ldots$ to n quotients.

5. $1 + \dfrac{1}{2} + \dfrac{1}{3} + \ldots\ldots + \dfrac{1}{n+1} = \dfrac{1}{1-}\ \dfrac{1}{3-}\ \dfrac{4}{5-}\ \dfrac{9}{7-} \ldots\ldots \dfrac{n^2}{2n+1}.$

6. $\dfrac{1}{1^2} + \dfrac{1}{2^2} + \ldots\ldots + \dfrac{1}{(n+1)^2} = \dfrac{1}{1-}\ \dfrac{1^4}{1^2+2^2-} \ldots\ldots \dfrac{n^4}{n^2+(n+1)^2}.$

7. $e^x = 1 + \dfrac{x}{1-}\ \dfrac{x}{x+2-}\ \dfrac{2x}{x+3-}\ \dfrac{3x}{x+4-} \ldots\ldots$

8. $\dfrac{1}{a} - \dfrac{1}{ab} + \dfrac{1}{abc} - \dfrac{1}{abcd} + \ldots = \dfrac{1}{a+}\ \dfrac{a}{b-1+}\ \dfrac{b}{c-1+}\ \dfrac{c}{d-1+} \ldots\ldots$

9. $1 + \dfrac{1}{r} + \dfrac{1}{r^4} + \dfrac{1}{r^9} + \dfrac{1}{r^{16}} + \ldots = 1 + \dfrac{1}{r-}\ \dfrac{r}{r^3+1-}\ \dfrac{r^3}{r^5+1-}\ \dfrac{r^5}{r^7+1-} \ldots$

10. $\dfrac{a_1}{a_1+}\ \dfrac{a_2}{a_2+}\ \dfrac{a_3}{a_3+} \ldots\ldots \dfrac{a_n}{a_n} = \dfrac{1}{1+}\ \dfrac{1}{a_1+}\ \dfrac{a_1}{a_2+}\ \dfrac{a_2}{a_3+} \ldots\ldots \dfrac{a_{n-2}}{a_{n-1}}.$

11. If $P = \dfrac{a}{a+}\ \dfrac{b}{b+}\ \dfrac{c}{c+} \ldots\ldots,\quad Q = \dfrac{a}{b+}\ \dfrac{b}{c+}\ \dfrac{c}{d+} \ldots\ldots,$

shew that $$P(a+1+Q) = a+Q.$$

12. Shew that $\dfrac{1}{q_1} - \dfrac{x}{q_1q_2} + \dfrac{x^2}{q_2q_3} - \dfrac{x^3}{q_3q_4} + \ldots$ is equal to the continued fraction $\dfrac{1}{a_1+}\ \dfrac{x}{a_2+}\ \dfrac{x}{a_3+}\ \dfrac{x}{a_4+} \ldots$, where $q_1, q_2, q_3, \ldots$ are the denominators of the successive convergents.