CHAPTER 2

Trigonometric Ratios and Identities

- Measurement of Angles
- Trigonometric Functions
- Problems Based on Trigonometric Identities
- Trigonometric Ratios for Complementary and Supplementary Angles
- Trigonometric Ratios for Compound Angles
- Transformation Formulae
- Trigonometric Ratios of Multiples and Sub-Multiple Angles
- Values of Trigonometric Ratios of Standard Angles
- Sum of Sines or Cosines of *n* Angles in A.P.
- Conditional Identities
- Some Important Results and their Applications
- Important Inequalities

MEASUREMENT OF ANGLES

Angles in Trigonometry

In trigonometry, the idea of angle is more general; it may be positive or negative and has any magnitude (Fig. 2.1).

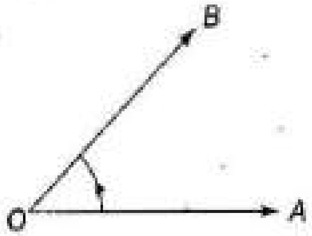


Fig. 2.1

In trigonometry, as in case of geometry, the measure of angle is the amount of rotation from the direction of one ray of the angle to the other. The initial and final positions of the revolving ray are respectively called the initial side (arm) and terminal side (arm), and the revolving line is called the generating line or the radius vector. For example, if OA and OB are the initial and final positions of the revolving ray, then the angle formed will be $\angle AOB$.

Angles Exceeding 360°

In geometry, we confine ourselves to angles from 0° to 360°. But there may be problems in which rotation involves more than one revolution, for example, the rotation of a flywheel. In trigonometry, we generalise the concept of angle to angles greater than 360°. This angle can be formed in the following way:

The revolving line (radius vector) starts from the initial position $\overline{OA}$ and makes n complete revolutions in anticlockwise direction and also a further angle α in the same direction. We then have a certain angle β_n given by $\beta_n = 360° \times n + \alpha$, where $0° < \alpha < 360°$ and n is a positive integer or zero.

Thus, there are infinitely many β_n angles with initial side $\overline{OA}$ and final side $\overline{OB}$.

For example, $\beta_0 = \alpha$, $\beta_1 = 360° + \alpha$, $\beta_2 = 720° + \alpha$, etc.

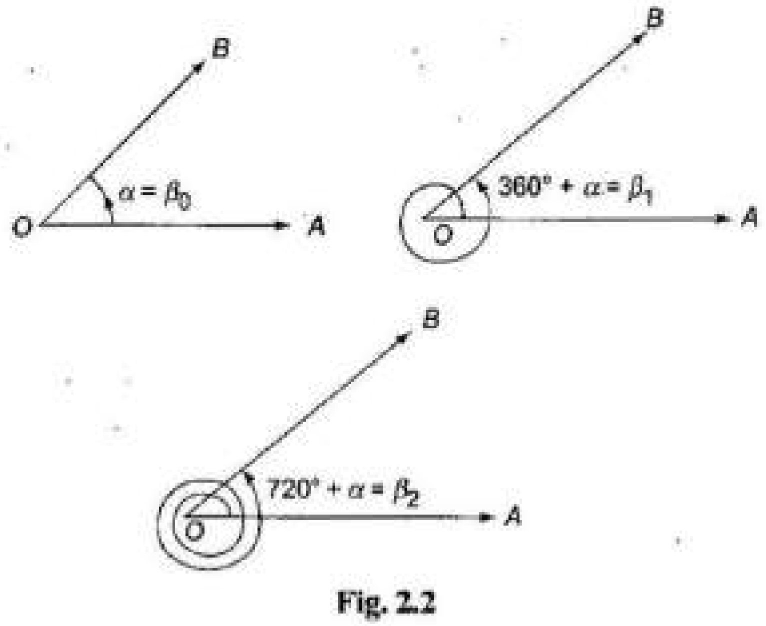


Fig. 2.2

Sign of Angles

Angles formed by anticlockwise rotation of the radius vector are taken as positive, whereas angles formed by clockwise rotation of the radius vector are taken as negative.

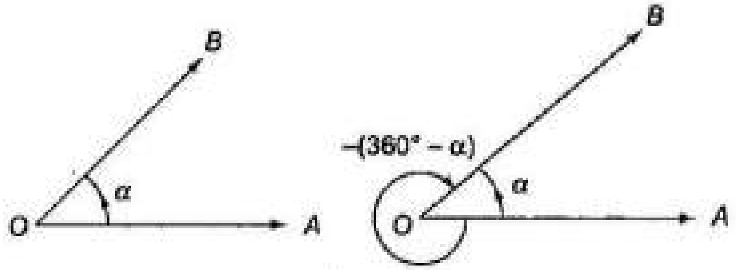


Fig. 2.3

Relation between Degree and Radian

Radian is a constant angle. One radian is the angle subtended by an arc of a circle at the centre. It is equal to arc/radius. It is expressed as rad.

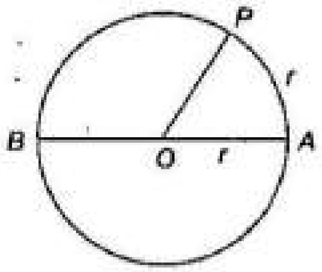


Fig. 2.4

Consider a circle with centre O and radius r. Let A be a point on the circle.
Join OA and cut off an arc of length equal to the radius of the circle.
Then, $\angle AOP = 1$ rad. Produce AO to meet the circle at B.
$\Rightarrow \quad \angle AOB =$ a straight angle $=$ 2 right angles
We know that the angles at the centre of a circle are proportional to the arcs subtending them.

$$\Rightarrow \frac{\angle AOP}{\angle AOB} = \frac{\text{arc}AP}{\text{arc}APB}$$

$$\Rightarrow \frac{\angle AOP}{2 \text{ right angles}} = \frac{r}{\pi r} \qquad \left[\because \text{arc } APB = \frac{1}{2}\text{(circumference)}\right]$$

$$\Rightarrow \angle AOP = \frac{2 \text{ right angles}}{\pi} \qquad \Rightarrow 1^R = \frac{180^\circ}{\pi}$$

Hence, $1 \text{ rad} = \dfrac{180^\circ}{\pi} \Rightarrow \pi \text{ rad} = 180^\circ$.

Note:

- *When an angle is expressed in radian, the word radian is generally omitted.*
- $1^\circ = 60'$ (60 min) and $1' = 60''$ (60 sec)
- *Since* $180^\circ = \pi$ *rad. Therefore,* $1^\circ = \pi/180$ *rad.*

Hence, $30^\circ = \dfrac{\pi}{180} \times 30 = \dfrac{\pi}{6}$ *rad,*

$45^\circ = \dfrac{\pi}{180} \times 45 = \dfrac{\pi}{4}$ *rad,*

$60^\circ = \dfrac{\pi}{180} \times 60 = \dfrac{\pi}{3}$ *rad,*

$$90^\circ = \frac{\pi}{180} \times 90 = \frac{\pi}{2} \text{ rad, etc.}$$

- *We have π rad = 180°*

$$\Rightarrow \quad 1 \text{ rad} = \frac{180^\circ}{\pi} = \left(\frac{180}{22} \times 7\right)^\circ = 57^\circ 16' 22'' \text{ (approx)}$$

- $180^\circ = \pi \text{ rad} \Rightarrow 1^\circ = \frac{\pi}{180} \text{ rad} = \left(\frac{22}{7 \times 180}\right) \text{rad} = 0.01746 \text{ rad.}$
- *Sum of interior angles on convex polygon of n sides is $(n-2)\pi$ rad.*

Example 2.1 **Express 45° 20′10″ in rad measure ($\pi = 3.1415$).**

Sol. $10'' = \frac{10}{60} \text{ min} = \frac{10}{60 \times 60} \text{ degrees} = \frac{1}{360} \text{ degrees}$

$$20' = \frac{20}{60} \text{ degrees} = \frac{1}{3} \text{degrees}$$

$$\therefore \quad 45^\circ 20' 10'' = \left(45 + \frac{1}{360} + \frac{1}{3}\right) \text{degrees} = \frac{16200 + 1 + 120}{360} = \frac{16321}{360}$$

$$\text{Now} \left(\frac{16321}{360}\right)^\circ = \frac{16321}{360} \times \frac{\pi}{180} \text{rad} = \frac{16321}{360} \times \frac{3.1416}{180} = \frac{51274.054}{64800} = 0.79 \text{ rad}$$

Example 2.2 **Express 1.2 rad in degree measure.**

Sol. $(1.2)^R = 1.2 \times \frac{180}{\pi} \text{degrees} = 1.2 \times \frac{180 \times 7}{22}$ $\quad \left[\because \pi = \frac{27}{7} \text{(approx)}\right]$

$$= 68.7272 = 68^\circ (.7272 \times 60)' = 68^\circ (43.63)' = 68^\circ 43' (.63 \times 60)'' = 68^\circ (43' 37.8'')$$

Example 2.3 **Find the length of an arc of a circle of radius 5 cm subtending a central angle measuring 15°.**

Sol. Let s be the length of the arc subtending an angle θ^R at the centre of a circle of radius r. Then, $\theta = s/r$.

Here, $r = 5$ cm and $\theta = 15^\circ = \left(15 \times \frac{\pi}{180}\right)^R = \left(\frac{\pi}{12}\right)^R$

$$\therefore \quad \theta = \frac{s}{r} \Rightarrow \frac{\pi}{12} = \frac{s}{5} \Rightarrow s = \frac{5\pi}{12} \text{cm}$$

Example 2.4 **Find in degrees the angle subtended at the centre of a circle of diameter 50 cm by an arc of length 11 cm.**

Sol. Here, $r = 25$ cm and $s = 11$ cm.

$$\therefore \quad \theta = \left(\frac{s}{r}\right)^R \Rightarrow \theta = \left(\frac{11}{25}\right)^R = \left(\frac{11}{25} \times \frac{180}{\pi}\right)^\circ$$

$$= \left(\frac{11}{25} \times \frac{180}{22} \times 7\right)^\circ$$

$$= \left(\frac{126}{5}\right)^\circ = \left(25\frac{1}{5}\right)^\circ = 25^\circ \left(\frac{1}{5} \times 60\right)' = 25^\circ 12'$$

Example 2.5 **If arcs of same length in two circles subtend angles of 60° and 75° at their centres, find the ratios of their radii.**

Sol. Let r_1 and r_2 be the radii of the given circles and let their arcs of same length s subtend angles of 60° and 75° at their centres.

Now, $60^R = \left(60\times\frac{\pi}{180}\right)^R = \left(\frac{\pi}{3}\right)^R$ and $75° = \left(75\times\frac{\pi}{180}\right)^R = \left(\frac{5\pi}{12}\right)^R$

$\therefore \quad \frac{\pi}{3} = \frac{s}{r_1}$ and $\frac{5\pi}{12} = \frac{s}{r_2} \Rightarrow \frac{\pi}{3}r_1 = s$ and $\frac{5\pi}{12}r_2 = s \Rightarrow \frac{\pi}{3}r_1 = \frac{5\pi}{12}r_2 \Rightarrow 4r_1 = 5r_2 \Rightarrow r_1 : r_2 = 5:4$

Hence, $r_1 : r_2 = 5:4$.

Example 2.6 **Assuming the distance of earth from the moon to be 38,400 km and the angle subtended by the moon at the eye of a person on earth to be 31′, find the diameter of the moon.**

Sol.

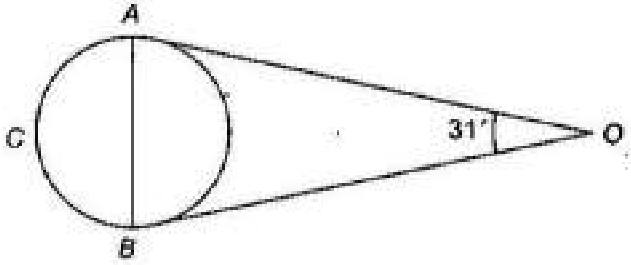


Fig. 2.5

Let AB be the diameter of the moon and O be the observer.

Given $\angle AOB = 31' = \frac{31}{60} \times \frac{\pi}{180}$ rad

Since the angle subtended by the moon is very small, its diameter will be approximately equal to the small arc of a circle whose centre is the eye of the observer and the radius is the distance of the earth from the moon. Also the moon subtends an angle of 31′ at the centre of this circle.

$\Rightarrow \quad \theta = \frac{l}{r}$, therefore $\frac{31}{60} \times \frac{\pi}{180} = \frac{AB}{38400}$

$\Rightarrow \quad AB = \frac{31}{60} \times \frac{22}{7\times180} \times 38,400 = 3464\frac{8}{63}$ km

Example 2.7 **Find the angle between the minute hand and the hour hand of a clock when the time is 7:20 AM.**

Sol. We know that the hour hand completes one rotation in 12 hr, while the minute hand completes one rotation in 60 min.

Therefore, the angle traced by the hour hand in 12 hr = 360°

Angle traced by the hour hand in 7 hr 20 min, i.e., $\frac{22}{3}$ hr $= \left(\frac{360}{12}\times\frac{22}{3}\right)^\circ = 220°$

Also, the angle traced by the minute hand in 60 min = 360°

The angle traced by the minute hand in 20 min $= \left(\frac{360}{60}\times 20\right)^\circ = 120°$

Hence, the required angle between the two hands = 220° – 120° = 100°.

Example 2.8 **For each natural number, k, let C_k denote a circle with radius k centimeters and centre at origin O. On the circle C_k a particle moves k centimetres in the counter-clockwise direction. After completing its motion on C_k, the particle moves to C_{k+1} in the radial direction. The motion of the particle continues in this manner. The particle starts at (1, 0). If the particle crosses the positive direction of x-axis for the first time on the circle C_n, then find the value of n.**

Sol.

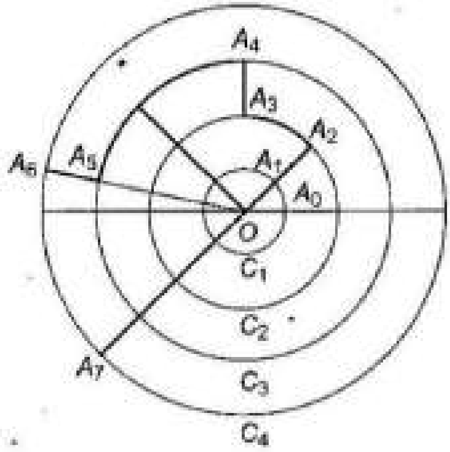


Fig. 2.6

The motion of the particle on the first four circles is shown with bold line in Fig. 2.6. Note that on every circle the particle travels just 1 rad. The particle crosses the positive direction of x-axis first time on C_n, where n is the least positive integer such that $n \geq 2\pi \Rightarrow n = 7$.

Concept Application Exercise 2.1

1. A horse is tied to a post by a rope. If the horse moves along a circular path always keeping the rope tight and describes 88 m when it has traced out 72° at the centre, find the length of the rope.
2. If the angular diameter of the moon is 30′, how far from the eye a coin of diameter 2.2 cm can be kept to hide the moon?
3. Find in degrees and radians the angle between the hour hand and the minute hand of a clock at half past three.
4. There is an equilateral triangle with side 4 and a circle with the centre on the one of the vertex of that triangle. The arc of that circle divides the triangle into two parts of equal area. How long is the radius of the circle?

TRIGONOMETRIC FUNCTIONS

Trigonometric Functions of Acute Angles

An angle whose measure is greater than 0° but less than 90° is called an acute angle. Consider a right-angled triangle ABC with right angle at B. The side opposite to the right angle is called the hypotenuse, side opposite to angle A is called the perpendicular for angle A and side opposite to the third angle is called the base for angle A.

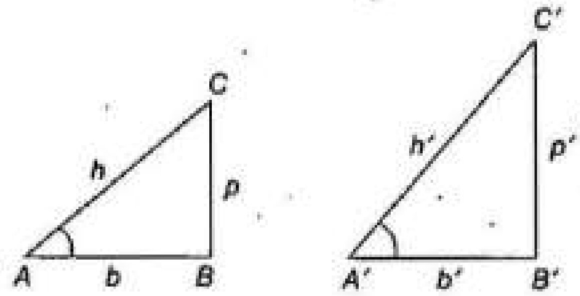


Fig. 2.7

The ratio of any two sides of the triangle depends only on measure of angle A, for if we take a larger and smaller right angle triangles as shown in Fig. 2.7, we have $\frac{h}{h'} = \frac{b}{b'} = \frac{p}{p'}$ (as these triangles are similar).

Thus, the ratio of the lengths of any two sides of a triangle is completely determined by angle A alone and is independent of the size of the triangle. There are six possible ratios that can be formed from the three sides of a right-angled triangle. Each of them has been given a name as follows.

Definitions

(i) $\sin A = \frac{p}{h}$ (ii) $\cos A = \frac{b}{h}$ (iii) $\tan A = \frac{p}{b}$

(iv) $\cot A = \frac{b}{p}$ (v) $\sec A = \frac{h}{b}$ (vi) $\operatorname{cosec} A = \frac{h}{p}$

The abbreviations stand for sine, cosine, tangent, cotangent, secant, and cosecant of A, respectively. These functions of angle A are called trigonometrical functions or trigonometrical ratios.

Example 2.9 **The circumference of a circle circumscribing an equilateral triangle is 24π units. Find the area of the circle inscribed in the equilateral triangle.**

Sol. $2\pi R = 24\pi$ (R is the radius of circumcircle)

$R = 12$

$\sin 30° = \frac{r}{R}$ (r is the radius of incircle)

$r = \frac{12}{2} = 6$

Therefore, area of incircle $= \pi r^2 = 36\pi$

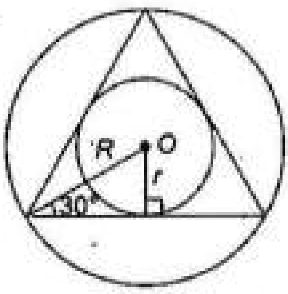


Fig. 2.8

Example 2.10 **In triangle ABC, $BC = 8$, $CA = 6$ and $AB = 10$. A line dividing the triangle ABC into two regions of equal area is perpendicular to AB at the point X. Then find the value of BX/$\sqrt{2}$.**

Sol. From the figure, $2\left(\frac{x \times y}{2}\right) = \frac{8 \times 6}{2} = 24$

$x \times x \tan B = 24$

$x^2 \times \frac{3}{4} = 24$

$x^2 = 32 \Rightarrow x = 4\sqrt{2}$

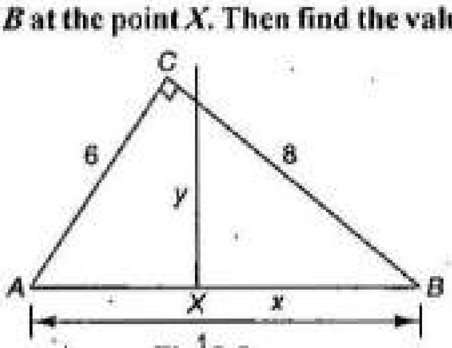


Fig. 2.9

Example 2.11 Let PQ and RS be tangents at the extremities of the diameter PR of a circle of radius r. If PS and RQ intersect at a point X on the circumference of the circle, then prove that $2r = \sqrt{PQ \times RS}$

Sol. From the figure, we have $\frac{PQ}{PR} = \tan(\pi/2 - \theta) = \cot\theta$.

and $\frac{RS}{PR} = \tan\theta$

$\Rightarrow \frac{PQ}{PR} \times \frac{RS}{PR} = 1$

$\Rightarrow (PR)^2 = PQ \times PS$

$\Rightarrow (2r)^2 = PQ \times PS$

$\Rightarrow 2r = \sqrt{PQ \times PS}$

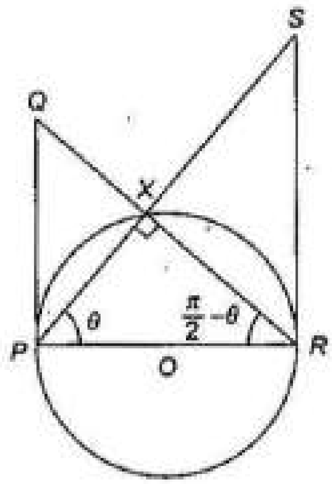


Fig. 2.10

Example 2.12 Two circles of radii 4 cm and 1 cm touch each other externally and θ is the angle contained by their direct common tangents. Then find $\sin\theta$.

Sol.

$\sin\frac{\theta}{2} = \frac{3}{5}$

$\cos\frac{\theta}{2} = \frac{4}{5}$

$\therefore \sin\theta = 2 \times \frac{3}{5} \times \frac{4}{5} = \frac{24}{25}$

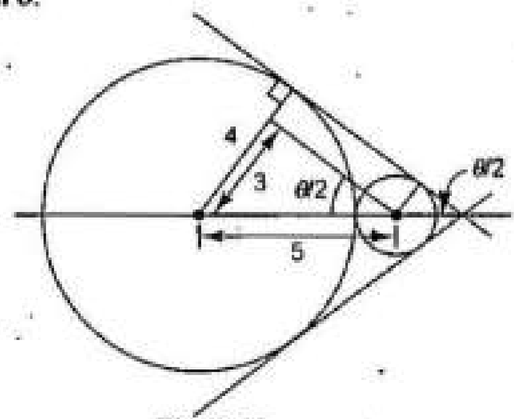


Fig. 2.11

Example 2.13 If angle C of triangle ABC is 90°, then prove that $\tan A + \tan B = \frac{c^2}{ab}$ (where, a, b, c are sides opposite to angles A, B, C respectively)

Sol. Draw ΔABC with $\angle C = 90°$

$\tan A + \tan B = \frac{a}{b} + \frac{b}{a}$

$= \frac{a^2 + b^2}{ab} = \frac{c^2}{ab}$

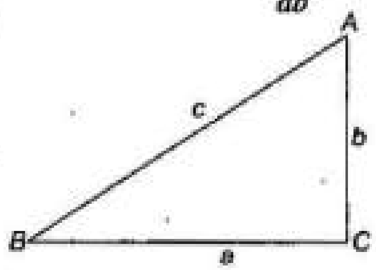


Fig. 2.12

Example 2.14 In the following diagram $\angle BAO = \tan^{-1}3$, then find the ratio $BC : CA$

Sol. $\therefore \tan\theta = 3$

$\therefore \frac{OC}{AC} = \tan\theta, \frac{OC}{BC} = \cot\theta$

$\Rightarrow \frac{BC}{AC} = \frac{\tan\theta}{\cot\theta} = \tan^2\theta = 9$

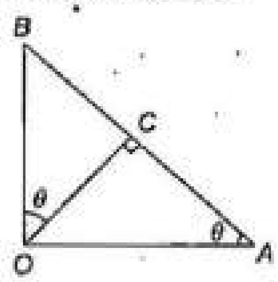


Fig. 2.13

Trigonometric Functions of Any Angle

Let A be a given angle with a specified initial ray. We introduce a rectangular coordinate system in the plane with the vertex of angle A as the origin and the initial ray of angle A as the positive ray of the x-axis (Fig. 2.14). We choose any point P on the terminal ray of angle A. Let the coordinates of P be (x, y) and its distance from the origin be r, then we define

(i) $\sin A = \frac{y}{r}$ (ii) $\cos A = \frac{x}{r}$ (iii) $\tan A = \frac{y}{x}$

(iv) $\cot A = \frac{x}{y}$ (v) $\sec A = \frac{r}{x}$ (vi) $\operatorname{cosec} A = \frac{r}{y}$

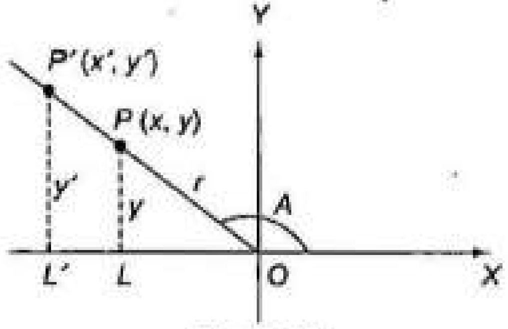


Fig. 2.14

These quantities are functions of angle A alone. They do not depend on the choice of point P and the terminal ray. If we choose a different point $P'(x', y')$ on the terminal ray of A at a distance r' from the origin, it is clear that x' and y' will have the same sign as that of x and y, respectively, because of similar triangles ΔOPL and $\Delta OP'L'$.

Also, any trigonometrical function of an angle A is equal to the same trigonometrical function of any angle $360n + A$, where n is any integer since all these angles will have the same terminal ray. For example, $\sin 60° = \sin 420° = \sin(-300°)$. After the coordinate system has been introduced, the plane is divided into four quadrants. An angle is said to be in that quadrant in which its terminal ray lies. For positive acute angles, this definition gives the same result as in case of a right-angled triangle since both x and y are positive for any point in the first quadrant. Consequently, they are the length of base and perpendicular of angle A.

Graphs and Other Useful Data of Trigonometric Functions

1. $y = f(x) = \sin x$

   Domain $\rightarrow R$, Range $\rightarrow [-1, 1]$

   Period $\rightarrow 2\pi$

   $\sin^2 x$, $|\sin x| \in [0, 1]$

   $\sin x = 0 \Rightarrow x = n\pi, n \in I$

   $\sin x = 1 \Rightarrow x = (4n+1)\pi/2, n \in I$

   $\sin x = -1 \Rightarrow x = (4n-1)\pi/2, n \in I$

   $\sin x = \sin\alpha \Rightarrow x = n\pi + (-1)^n\alpha, n \in I$

   $\sin x \geq 0 \Rightarrow x \in \bigcup_{n \in I} [2n\pi, \pi + 2n\pi]$

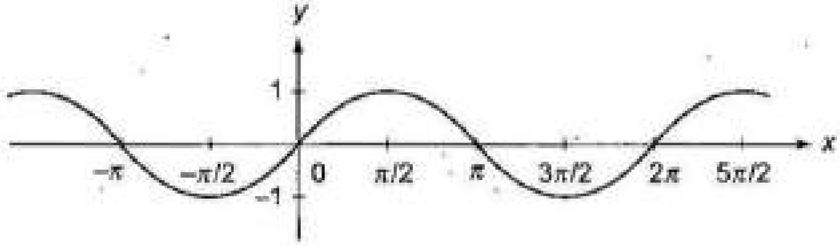


Fig. 2.15

2. $y = f(x) = \cos x$
   Domain → R, Range → $[-1, 1]$
   Period → 2π
   $\cos^2 x$, $|\cos x| \in [0, 1]$
   $\cos x = 0 \Rightarrow x = (2n+1)\pi/2, n \in I$
   $\cos x = 1 \Rightarrow x = 2n\pi, n \in I$
   $\cos x = -1 \Rightarrow x = (2n+1)\pi, n \in I$
   $\cos x = \cos\alpha \Rightarrow x = 2n\pi \pm \alpha, n \in I$
   $\cos x \ge 0 \Rightarrow x \in \bigcup_{n \in I} [2n\pi - \frac{\pi}{2}, 2n\pi + \pi/2]$

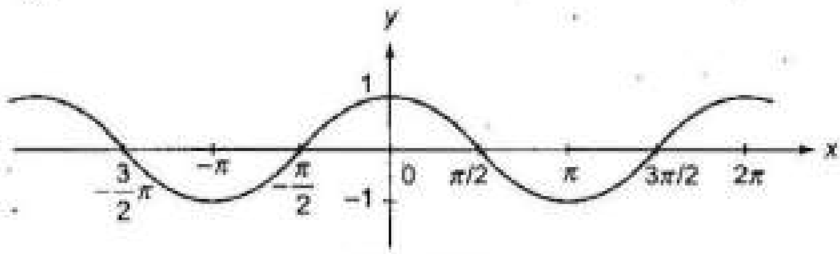


Fig. 2.16

3. $y = f(x) = \tan x$
   Domain → $R - (2n+1)\pi/2, n \in I$
   Range → $(-\infty, \infty)$
   Period → π
   Discontinuous at $x = (2n+1)\pi/2, n \in I$
   $\tan^2 x$, $|\tan x| \in [0, \infty)$
   $\tan x = 0 \Rightarrow x = n\pi, n \in I$
   $\tan x = \tan\alpha \Rightarrow x = n\pi + \alpha, n \in I$

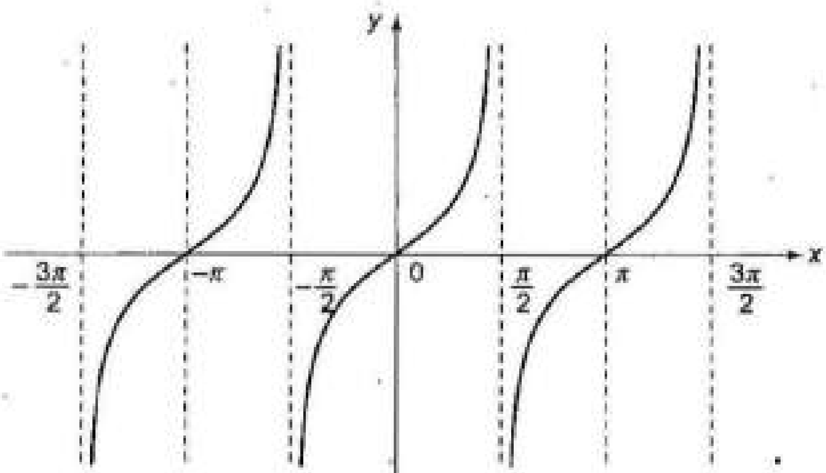


Fig. 2.17

4. $y = f(x) = \cot x$

Domain $\rightarrow R - n\pi, n \in I$; Range $\rightarrow (-\infty, \infty)$; Period $\rightarrow \pi$,

Discontinuous at $x = n\pi, n \in I$

$\cot^2 x$, $|\cot x| \in [0, \infty)$

$\cot x = 0 \Rightarrow x = (2n+1)\pi/2, n \in I$

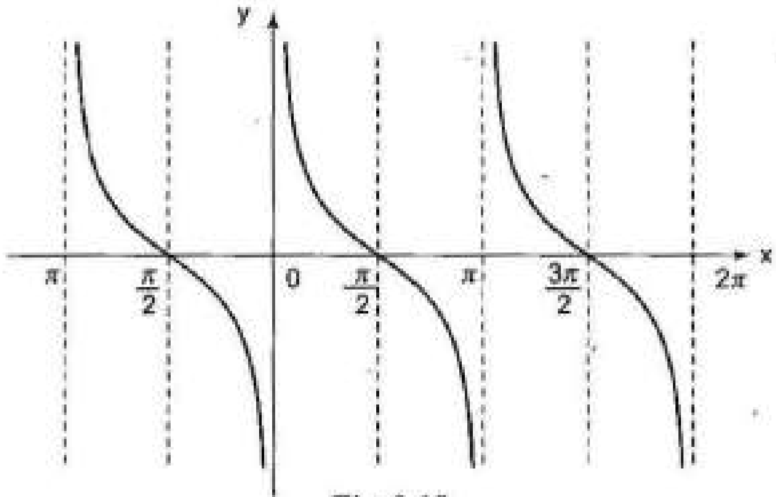


Fig. 2.18

5. $y = f(x) = \sec x$

Domain $\rightarrow R - (2n+1)\pi/2, n \in I$; Range $\rightarrow (-\infty, -1] \cup [1, \infty)$

Period $\rightarrow 2\pi$, $\sec^2 x$, $|\sec x| \in [1, \infty)$

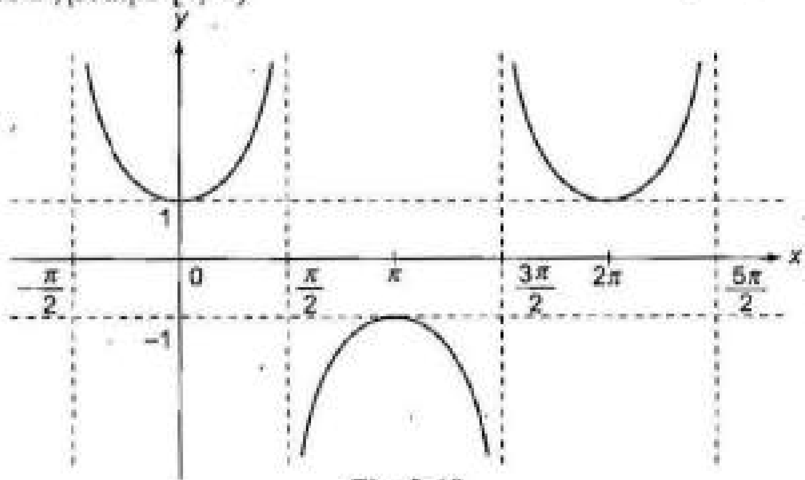


Fig. 2.19

6. $y = f(x) = \operatorname{cosec} x$

Domain $\rightarrow R - n\pi, n \in I$;

Range $\rightarrow (-\infty, -1] \cup [1, \infty)$

Period $\rightarrow 2\pi$, $\operatorname{cosec}^2 x$, $|\operatorname{cosec} x| \in [1, \infty)$

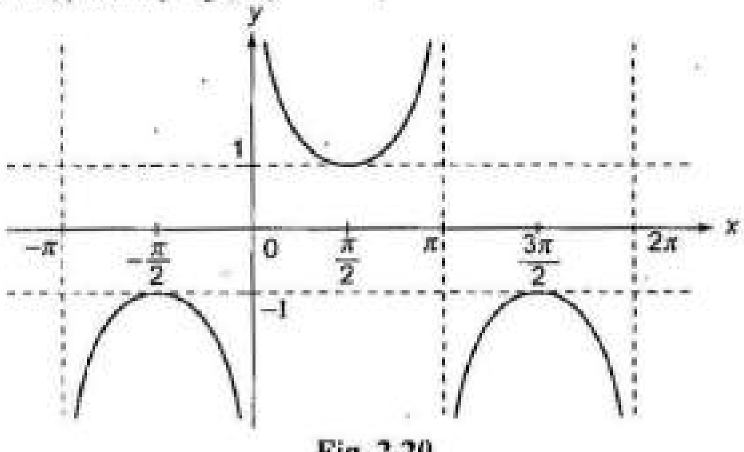


Fig. 2.20

Signs of the Trigonometric Ratios or Functions

The signs of trigonometric functions depend on the quadrant in which the terminal side of the angle lies. We always take the length $OP = r$ to be positive. Thus, $\sin\theta = y/r$ has the sign of y and $\cos\theta = x/r$ has the sign of x. The sign of $\tan\theta$ depends on the signs of x and y and similarly the signs of other trigonometric ratios are determined by the signs of x and/or y. Sign can also be determined by the graphs. Thus, we have the following:

Function	1st quadrant	2nd quadrant	3rd quadrant	4th quadrant
$\sin\theta$ $\operatorname{cosec}\theta$	+ve	+ve	−ve	−ve
$\cos\theta$ $\sec\theta$	+ve	−ve	−ve	+ve
$\tan\theta$ $\cot\theta$	+ve	−ve	+ve	−ve

Variations in the Values of Trigonometric Functions in Different Quadrants

	1st quadrant	2nd quadrant	3rd quadrant	4th quadrant
$\sin\theta$	↑ from 0 to 1	↓ from 1 to 0	↓ from 0 to −1	↑ from −1 to 0
$\cos\theta$	↓ from 1 to 0	↓ from 0 to −1	↑ from −1 to 0	↑ from 0 to 1
$\tan\theta$	↑ from 0 to ∞	↑ from −∞ to 0	↑ from 0 to ∞	↑ from −∞ to 0
$\cot\theta$	↓ from ∞ to 0	↓ from 0 to −∞	↓ from ∞ to 0	↓ from 0 to −∞
$\sec\theta$	↑ from 1 to ∞	↑ from −∞ to −1	↓ −1 to −∞	↓ from ∞ to 1
$\operatorname{cosec}\theta$	↓ from ∞ to 1	↑ from 1 to ∞	↑ from −∞ to −1	↓ from −1 to −∞

Note:

+∞ and −∞ are two symbols. These are not real numbers. When we say that $\tan\theta$ increases from 0 to ∞ as θ varies from 0 to $\pi/2$, it means that $\tan\theta$ increases in the interval $(0, \pi/2)$ and it attains arbitrarily large positive values as θ tends to $\pi/2$. Similarly, this happens for other trigonometrical functions as well.

Trigonometric Ratios of Standard Angles

Angle(θ) → T-Ratio ↓	30°	45°	60°
$\sin\theta$	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$
$\cos\theta$	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$
$\tan\theta$	$1/\sqrt{3}$	1	$\sqrt{3}$
$\operatorname{cosec}\theta$	2	$\sqrt{2}$	$2/\sqrt{3}$
$\sec\theta$	$2/\sqrt{3}$	$\sqrt{2}$	2
$\cot\theta$	$\sqrt{3}$	1	$1/\sqrt{3}$

Transformation of the Graphs of Trigonometric Functions

1. To draw the graph of $y = f(x + a)$; $(a > 0)$ from the graph of $y = f(x)$, shift the graph of $y = f(x)$, a units left along the x-axis.
   Consider the following illustration.

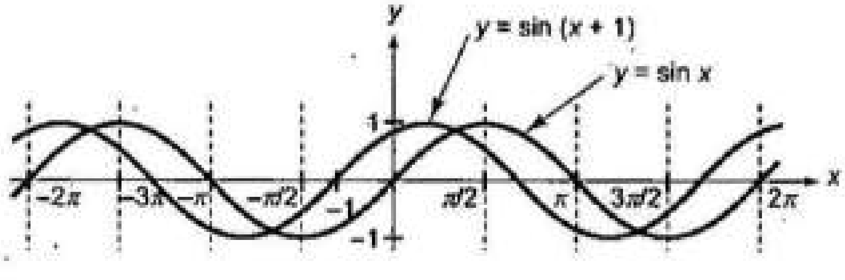


Fig. 2.21

To draw the graph of $y = f(x - a)$; $(a > 0)$ from the graph of $y = f(x)$, shift the graph of $y = f(x)$, a units right along the x-axis.
Consider the following illustration.

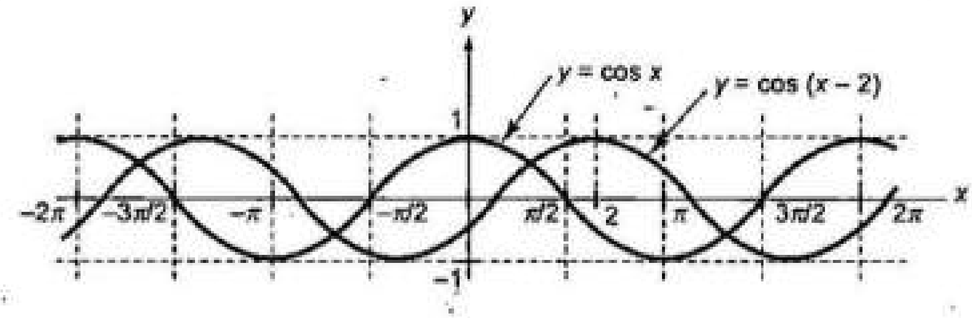


Fig. 2.22

2. To draw the graph of $y = f(x) + a$; $(a > 0)$ from the graph of $y = f(x)$, shift the graph of $y = f(x)$, a units upward along the y-axis.
   To draw the graph of $y = f(x) - a$; $(a > 0)$ from the graph of $y = f(x)$, shift the graph of $y = f(x)$, a units downward along the y-axis.

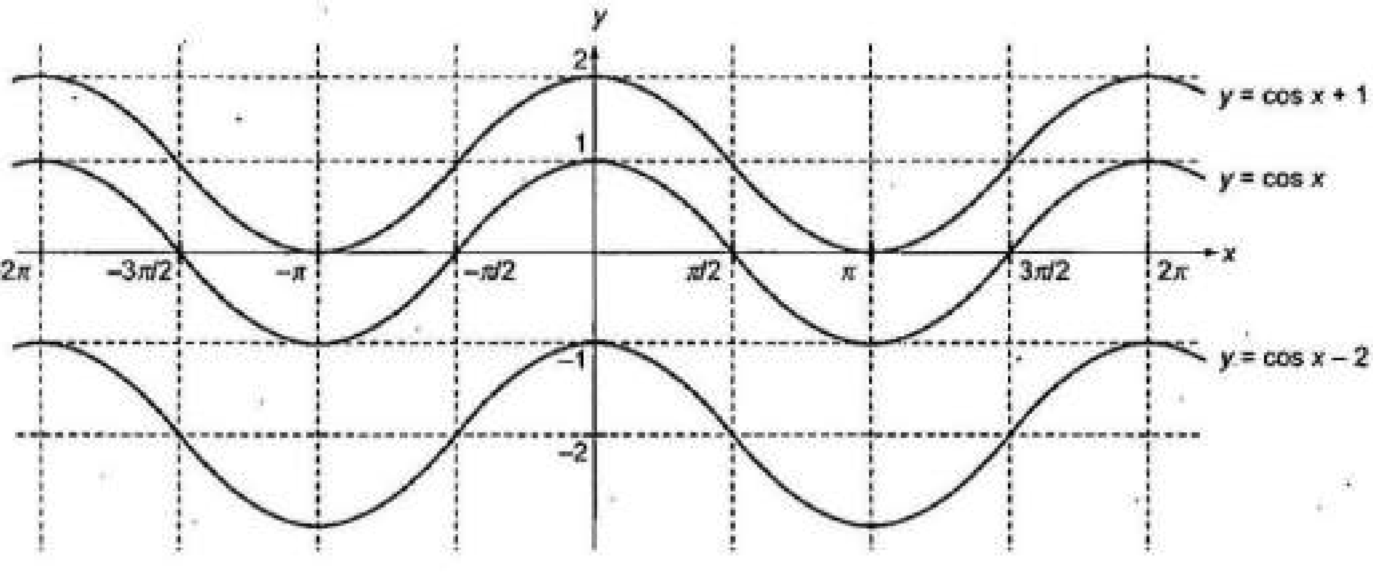


Fig. 2.23

3. If $y = f(x)$ has period T, then period of $y = f(ax)$ is $T/|a|$.

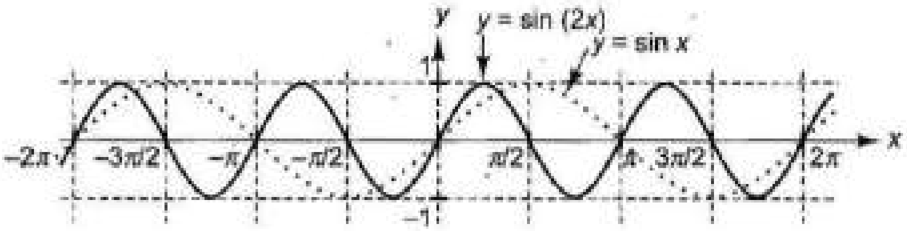


Fig. 2.24

Period of $y = \sin(2x)$ is $2\pi/2 = \pi$

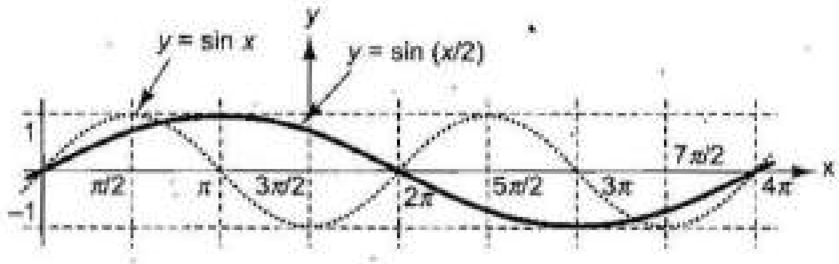


Fig. 2.25

Period of $y = \sin(x/2)$ is $2\pi/(1/2) = 4\pi$

4. Since $y = |f(x)| \geq 0$, to draw the graph of $y = |f(x)|$, take the mirror of the graph of $y = f(x)$ in x-axis for $f(x) < 0$, retaining the graph for $f(x) > 0$.
Consider the following illustrations.

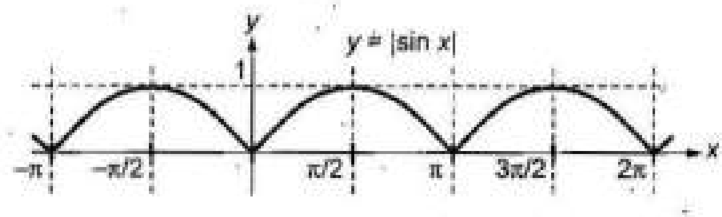


Fig. 2.26

Here period of $f(x) = |\sin x|$ is π.

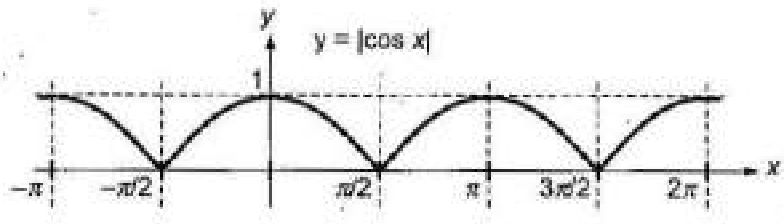


Fig. 2.27

Here period of $f(x) = |\cos x|$ is π

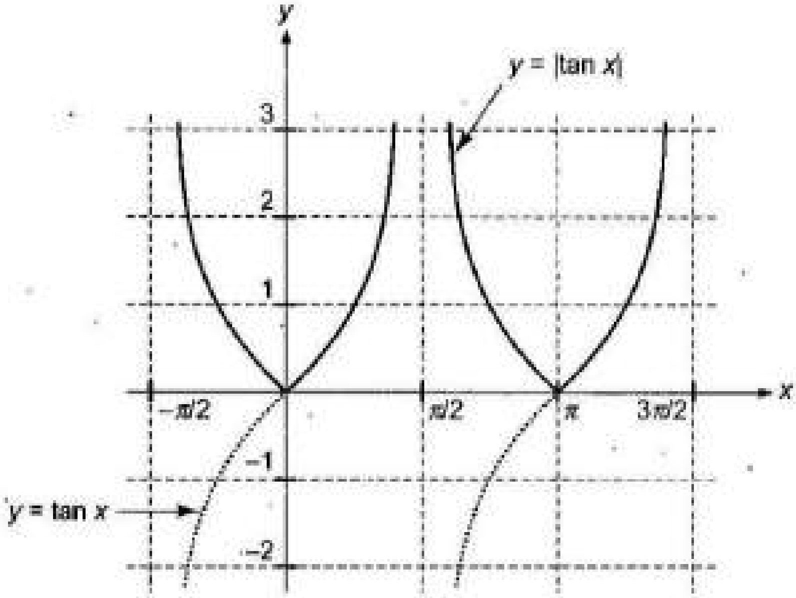


Fig. 2.28

5. Graph of $y = af(x)$ from the graph of $y f(x)$

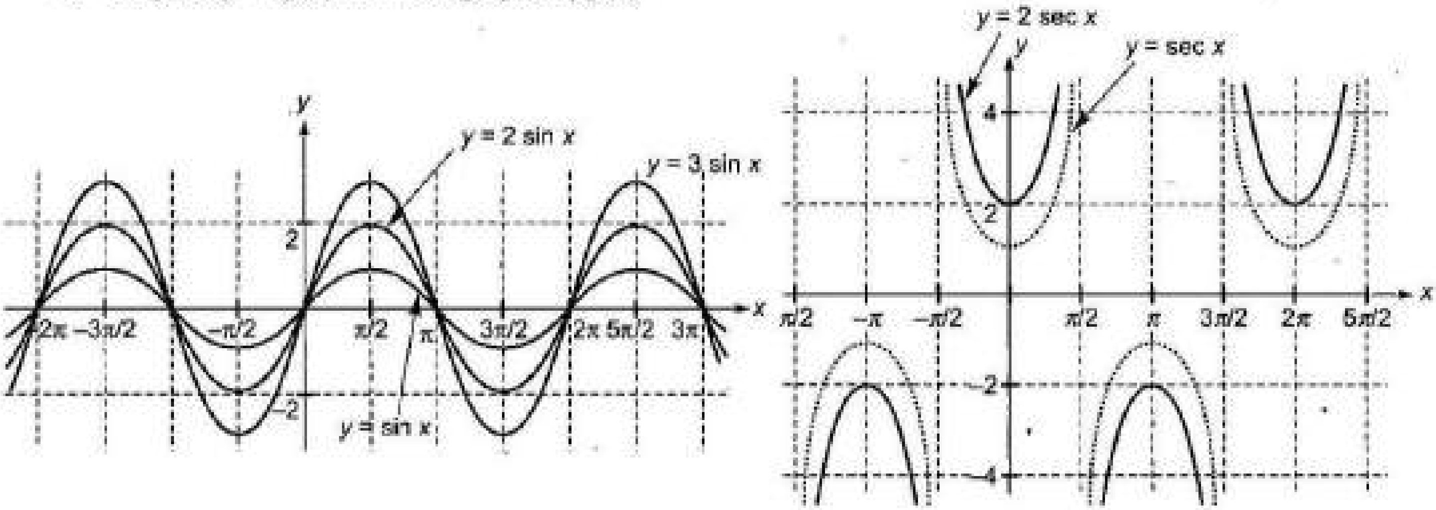


Fig. 2.29

Example 2.15 **Which of the following is possible?**

a. $\sin\theta = \frac{5}{3}$ **b.** $\tan\theta = 1002$

c. $\cos\theta = \frac{1+p^2}{1-p^2}, (p \neq \pm 1)$ **d.** $\sec\theta = \frac{1}{2}$

Sol. **b.** $\sin\theta = \frac{5}{3}$ is not possible as $-1 \le \sin\theta \le 1$.

$\cos\theta = \frac{1+p^2}{1-p^2}$ is not possible, as in $\frac{1+p^2}{1-p^2}$ numerator is always greater than the denominator for any value of p other than $p = 0$. Hence, $\frac{1+p^2}{1-p^2}$ does not lie in $[-1, 1]$.

$\sec\theta = \frac{1}{2}$ is not possible as $\sec\theta \in (-\infty, -1] \cup [1, \infty)$.

$\tan\theta = 1002$ is possible as $\tan\theta$ can take any real value.

Example 2.16 **Which of the following is greatest?**

a. $\tan 1$ **b.** $\tan 4$ **c.** $\tan 7$ **d.** $\tan 10$

Sol. **a.** $\tan 4 = \tan(\pi + (4 - \pi)) = \tan(4 - \pi) = \tan(0.86)$

$\tan 7 = \tan(2\pi + (7 - 2\pi)) = \tan(7 - 2\pi) = \tan(0.72)$

$\tan 10 = \tan(3\pi + (10 - 3\pi)) = \tan(10 - 3\pi) = \tan(0.58)$

Now $1 > 0.86 > 0.72 > 0.58$

$\Rightarrow$ $\tan 1 > \tan(0.86) > \tan(0.72) > \tan(0.58)$ [as 1, 0.86, 0.72, 0.58 lie in the first quadrant and tan functions increase in all the quadrant]

Hence, $\tan 1$ is greatest.

Example 2.17 **Which of the following is least?**

a. $\sin 3$ **b.** $\sin 2$ **c.** $\sin 1$ **d.** $\sin 7$

Sol. **d.** $\sin 3 = \sin[\pi - (\pi - 3)] = \sin(\pi - 3) = \sin(0.14)$

$\sin 2 = \sin[\pi - (\pi - 2)] = \sin(\pi - 2) = \sin(1.14)$

$\sin 7 = \sin[2\pi + (7 - 2\pi)] = \sin(7 - 2\pi) = \sin(0.72)$

Now $1.14 > 1 > 0.72 > 0.14$

$\Rightarrow$ $\sin(1.14) > \sin 1 > \sin(0.72) > \sin(0.14)$ [as 1.14, 1, 0.72, 0.14 lie in the first quadrant and sine functions increase in the first quadrant]

Hence, $\sin 3$ is least.

Alternative solution:

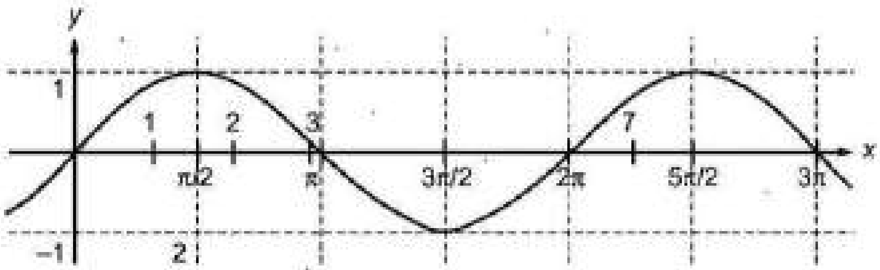


Fig. 2.30

From the graph, obviously $\sin 3$ is least.

Example 2.18 **If $A = 4\sin\theta + \cos^2\theta$, then which of the following is not true?**

a. **Maximum value of A is 5**
b. **Minimum value of A is -4**
c. **Maximum value of A occurs when $\sin\theta = 1/2$**
d. **Minimum value of A occurs when $\sin\theta = 1$**

Sol. **a, c, d.**

$$f(\theta) = 4\sin\theta + \cos^2\theta = 4\sin\theta + 1 - \sin^2\theta$$
$$= 5 - (4 - 4\sin\theta + \sin^2\theta) = 5 - (\sin\theta - 2)^2$$

Now maximum value of $f(\theta)$ occurs when $(\sin\theta - 2)^2$ is minimum.

Minimum value of $(\sin\theta - 2)^2$ occurs when $\sin\theta = 1$, then maximum value of $f(\theta)$ is $5 - (1 - 2)^2 = 4$.

Also minimum value of $f(\theta)$ occurs when $(\sin\theta - 2)^2$ is maximum.

Maximum value of $(\sin\theta - 2)^2$ occurs when $\sin\theta = -1$, then minimum value of $f(\theta)$ is $5 - (-1 - 2)^2 = -4$.

Example 2.19 **Is the equation $\sec^2\theta = \dfrac{4xy}{(x+y)^2}$ possible for real values of x and y?**

Sol. Given, $\sec^2\theta = \dfrac{4xy}{(x+y)^2}$

Since $\sec^2\theta \ge 1$, we get $\dfrac{4xy}{(x+y)^2} \ge 1$

$\Rightarrow\ (x+y)^2 \le 4xy$

$\Rightarrow\ (x+y)^2 - 4xy \le 0$ or $(x-y)^2 \le 0$

But for real values of x and y, $(x-y)^2 \ge 0$

Since $(x-y)^2 = 0$, $x = y$. Also $x + y \ne 0 \Rightarrow x \ne 0, y \ne 0$

Therefore, the given equation $\sec^2\theta = \dfrac{4xy}{(x+y)^2}$ is possible for real values of x and y only when $x = y\ (x \ne 0)$.

Example 2.20 **Show that the equation $\sin\theta = x + \dfrac{1}{x}$ is impossible if x is real.**

Sol. Given, $\sin\theta = x + \dfrac{1}{x}$

$\therefore\ \sin^2\theta = x^2 + \dfrac{1}{x^2} + 2x\,\dfrac{1}{x} = x^2 + \dfrac{1}{x^2} + 2 = \left(x - \dfrac{1}{x}\right)^2 + 4 \ge 4$

which is not possible since $\sin\theta^2 \le 1$.

Example 2.21 **If $\sin^2\theta_1 + \sin^2\theta_2 + \sin^2\theta_3 = 0$, then which of the following is not the possible value of $\cos\theta_1 + \cos\theta_2 + \cos\theta_3$.**

a. 3 b. –3 c. –1 d. –2

Sol. d.

$\sin^2\theta_1 + \sin^2\theta_2 + \sin^2\theta_3 = 0$

$\Rightarrow\ \sin^2\theta_1 = \sin^2\theta_2 = \sin^2\theta_3 = 0 \quad \Rightarrow\ \cos^2\theta_1, \cos^2\theta_2, \cos^2\theta_3 = 1 \quad \Rightarrow\ \cos\theta_1, \cos\theta_2, \cos\theta_3 = \pm 1$

$\cos\theta_1 + \cos\theta_2 + \cos\theta_3$ can be –3 (when all are –1)

or 3 (when all are +1)

or –1 (when any two are –1 and one +1)

or 1 (when any two are +1 and one –1)

but –2 is not a possible value.

Example 2.22 **For real values of θ, which of the following is/are positive?**

a. $\cos(\cos\theta)$ b. $\cos(\sin\theta)$ c. $\sin(\cos\theta)$ d. $\sin(\sin\theta)$

Sol. a, b. $\cos\theta, \sin\theta \in [-1, 1]$ or (value lies in 1st or 4th quadrant)

For which $\cos(\sin\theta)$ is always greater than 0.

$\sin(\cos\theta) < 0$, when $\cos\theta \in [-1, 0]$ and $\sin(\sin\theta) > 0$ when $\sin\theta \in [0, 1]$

Example 2.23 **Find the range of $f(x) = \dfrac{1}{4\cos x - 3}$.**

Sol. $-1 \le \cos x \le 1$

$\Rightarrow -4 \le 4\cos x \le 4$

$\Rightarrow -7 \le 4\cos x - 3 \le 1$

$$\Rightarrow -7 \le 4\cos x - 3 < 0 \text{ or } 0 < 4\cos x - 3 \le 1 \quad (\because 4\cos x - 3 \ne 0)$$

$$\Rightarrow -\frac{1}{7} \ge \frac{1}{4\cos x - 3} > -\infty \text{ or } \infty > \frac{1}{4\cos x - 3} \ge 1$$

$$\Rightarrow \frac{1}{4\cos x - 3} \in \left(-\infty, -\frac{1}{7}\right] \cup [1, \infty)$$

Example 2.24 **Find the range of $f(x) = \dfrac{1}{5\sin x - 6}$.**

Sol. $-1 \le \sin x \le 1$

$$\Rightarrow -5 \le 5\sin x \le 5$$

$$\Rightarrow -11 \le 5\sin x - 6 \le -1$$

$$\Rightarrow -1 \le \frac{1}{5\sin x - 6} \le -1/11$$

$$\Rightarrow \frac{1}{5\sin x - 6} \in [-1, -1/11]$$

Example 2.25 **Find the range of $f(x) = \cos^2 x + \sec^2 x$**

Sol. We have

$$f(x) = \cos^2 x + \sec^2 x$$

$$= (\cos x - \sec x)^2 + 2\cos x \sec x$$

$$= 2 + (\cos x - \sec x)^2 \ge 2$$

Example 2.26 **Find the range of $f(x) = \sin^2 x - 3\sin x + 2$**

Sol. $f(x) = \sin^2 x - 3\sin x + 2$

$$= (\sin x - 3/2)^2 + 2 - 9/4$$

$$= (\sin x - 3/2)^2 - 1/4$$

$$-1 \le \sin x \le 1$$

$$\Rightarrow -5/2 \le \sin x - 3/2 \le -1/2$$

$$\Rightarrow 1/4 \le (\sin x - 3/2)^2 \le 25/4$$

$$\Rightarrow 0 \le (\sin x - 3/2)^2 - 1/4 \le 6$$

Example 2.27 **Find the range of $f(x) = \sqrt{\sin^2 x - 6\sin x + 9} + 3$.**

Sol. $f(x) = \sqrt{\sin^2 x - 6\sin x + 9} + 3$

$$= \sqrt{(\sin x - 3)^2} + 3$$

$$= |\sin x - 3| + 3$$

Now $-1 \le \sin x \le 1$

$$\Rightarrow -4 \le \sin x - 3 \le -2$$

$$\Rightarrow 2 \le |\sin x - 3| \le 4$$

$$\Rightarrow 5 \le |\sin x - 3| + 3 \le 7$$

Example 2.28 **Find the range of $f(x) = \operatorname{cosec}^2 x + 25 \sec^2 x$.**

Sol. $f(x) = (1 + \cot^2 x) + 25(1 + \tan^2 x)$

$= 26 + \cot^2 x + 25 \tan^2 x$

$= 36 + 10 + (\cot^2 x + 25 \tan^2 x - 2 \cot x\, 5 \tan x)$

$= 36 + (\cot x - 5 \tan x)^2 \geq 36$

Example 2.29 **Find the value of x for which $f(x) = \sqrt{\sin x - \cos x}$ is defined, $x \in [0, 2\pi]$.**

Sol. $f(x) = \sqrt{\sin x - \cos x}$ is defined if $\sin x \geq \cos x$,

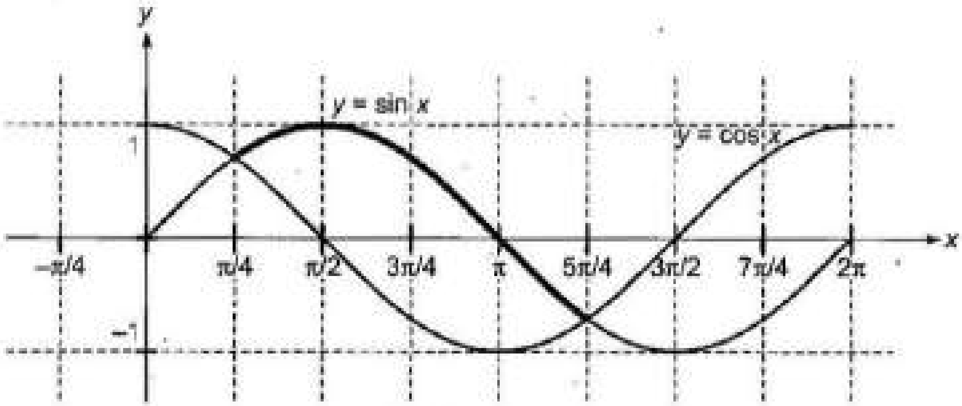


Fig. 2.31

From the graph, $\sin x \geq \cos x$, for $x \in \left[\dfrac{\pi}{4}, \dfrac{5\pi}{4}\right]$.

Example 2.30 **Which of the following is highest?**

a. cosec 1 **b.** cosec 2 **c.** cosec 4 **d.** cosec (–6)

Sol. **d.** Consider sin 1, sin 2 and –sin 6 (sin 4 is negative; hence, cosec 4 cannot be maximum).

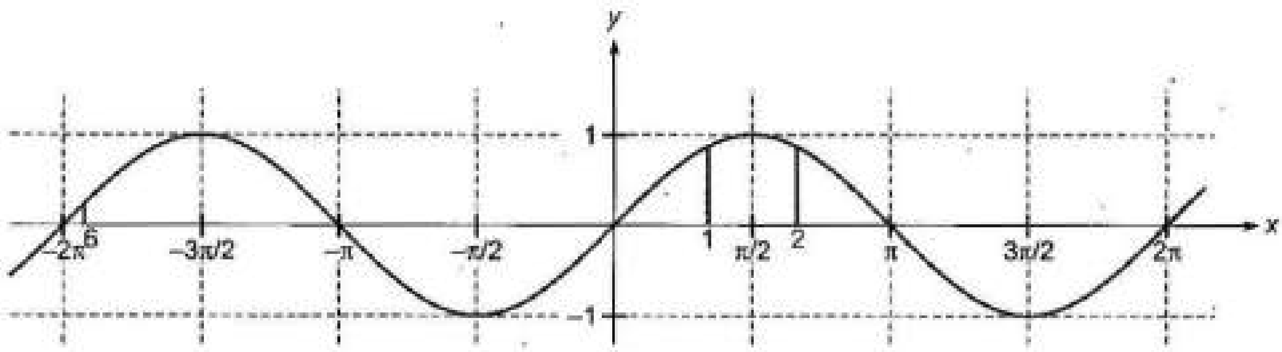


Fig. 2.32

From the graph, sin (–6) is least, hence cosec (–6) is maximum.

Example 2.31 **Solve $\tan x > \cot x$, where $x \in [0, 2\pi]$.**

Sol.

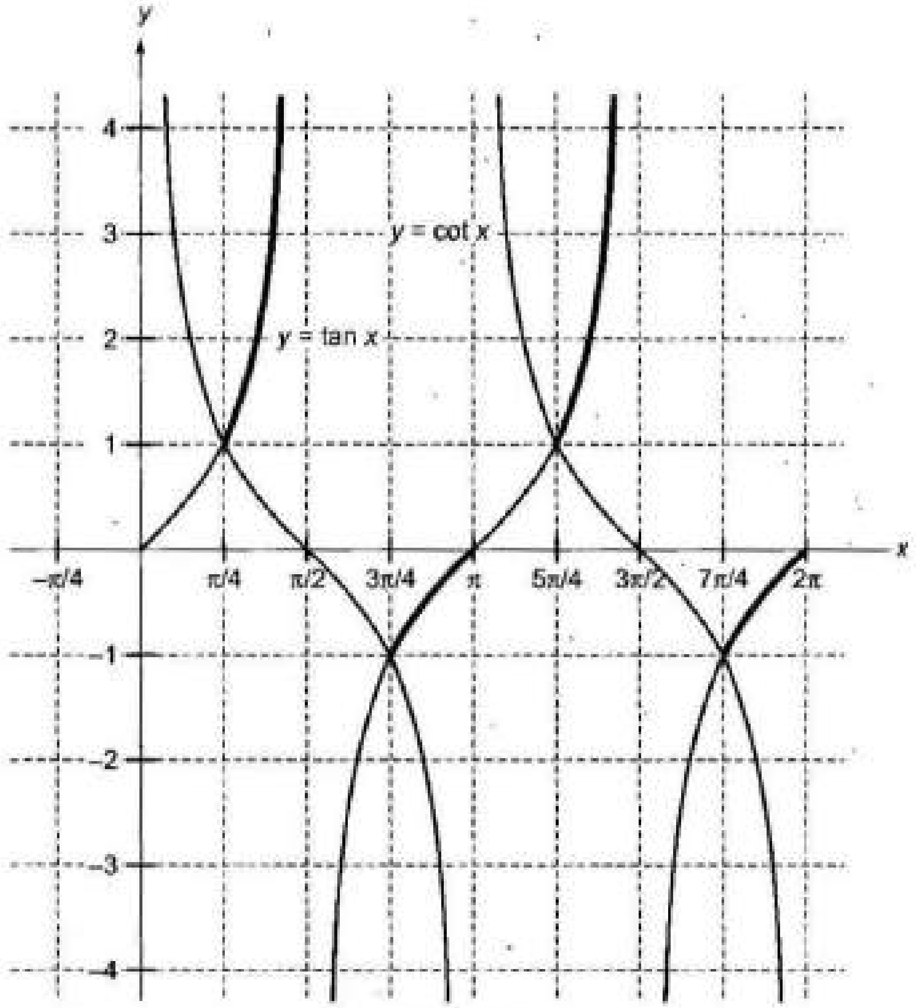


Fig. 2.33

We find that $\tan x \geq \cot x$

Therefore, values of $\tan x$ are more than the values of $\cot x$.

That is, values of x for which graph of $y = \tan x$ is above the graph of $y = \cot x$.

From the graph, it is clear that $x \in (\pi/4, \pi/2) \cup (3\pi/4, \pi) \cup (5\pi/4, 3\pi/2) \cup (7\pi/4, 2\pi)$.

Concept Application Exercise 2.2

1. Find the least value of $2\sin^2\theta + 3\cos^2\theta$.
2. Find the range of $f(x) = \sin(\cos x)$.
3. Find the range of $12\sin\theta - 9\sin^2\theta$.
4. Find the minimum value of $9\tan^2\theta + 4\cot^2\theta$.
5. Which of following is correct (where $n \in N$)?

   a. $\sin\theta = \frac{n+1}{n}$ **b.** $\sin\theta = \frac{n^2+1}{n+1}$ **c.** $\sec\theta = \frac{n+2}{n-1}$ **d.** $\sec\theta = \frac{n}{\sqrt{n^2+1}}$

6. If $\sin^2\theta_1 + \sin^2\theta_2 + \cdots + \sin^2\theta_n = 0$, then find the minimum value of $\cos\theta_1 + \cos\theta_2 + \cdots + \cos\theta_n$.
7. If $\sin^2\theta = x^2 - 3x + 3$ is meaningful, then find the values of x.
8. Find the range of $f(x) = \sqrt{4 - \sqrt{1 + \tan^2 x}}$.

9. Find the range of $f(x) = \dfrac{1}{2|\cos x| - 3}$.

10. Find the range of $f(x) = \cos^4 x + \sin^2 x - 1$.

11. Find the minimum value of the function $f(x) = (1 + \sin x)(1 + \cos x)$, $\forall x \in R$.

12. Prove that $(\sin\theta + \operatorname{cosec}\theta)^2 + (\cos\theta + \sec\theta)^2 \geq 9$.

PROBLEMS BASED ON TRIGONOMETRIC IDENTITIES

Example 2.32 **Show that $2(\sin^6 x + \cos^6 x) - 3(\sin^4 x + \cos^4 x) + 1 = 0$.**

Sol. $2(\sin^6 x + \cos^6 x) - 3(\sin^4 x + \cos^4 x) + 1$

$= 2[(\sin^2 x)^3 + (\cos^2 x)^3] - 3(\sin^4 x + \cos^4 x) + 1 = 2[(\sin^2 x + \cos^2 x)^3 - 3\sin^2 x\cos^2 x(\sin^2 x + \cos^2 x)]$

$- 3[(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x] + 1 = 2[1 - 3\sin^2 x\cos^2 x] - 3[1 - 2\sin^2 x\cos^2 x] + 1 = 0$

Example 2.33 **Prove that $\sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}} = \sec\theta + \tan\theta$, $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$.**

Sol. L.H.S. $= \sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}} = \sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}\,\dfrac{1+\sin\theta}{1+\sin\theta}}$

$= \sqrt{\dfrac{(1+\sin\theta)^2}{1-\sin^2\theta}} = \sqrt{\dfrac{(1+\sin\theta)^2}{\cos^2\theta}}$

$= \dfrac{1+\sin\theta}{\cos\theta} = \dfrac{1}{\cos\theta} + \dfrac{\sin\theta}{\cos\theta} = \sec\theta + \tan\theta$

= R.H.S.

Example 2.34 **Prove that $\dfrac{1}{\sec A - \tan A} - \dfrac{1}{\cos A} = \dfrac{1}{\cos A} - \dfrac{1}{\sec A + \tan A}$.**

Sol. To prove $\dfrac{1}{\sec A - \tan A} - \dfrac{1}{\cos A} = \dfrac{1}{\cos A} - \dfrac{1}{\sec A + \tan A}$

or, $\dfrac{1}{\sec A - \tan A} + \dfrac{1}{\sec A + \tan A} = \dfrac{1}{\cos A} + \dfrac{1}{\cos A} = \dfrac{2}{\cos A}$ (i)

Now L.H.S. $= \dfrac{1}{\sec A - \tan A} + \dfrac{1}{\sec A + \tan A} = \dfrac{\sec A + \tan A + \sec A - \tan A}{(\sec A - \tan A)(\sec A + \tan A)} = \dfrac{2}{\cos A}$

Example 2.35 **If $3\sin\theta + 5\cos\theta = 5$, then show that $5\sin\theta - 3\cos\theta = \pm 3$.**

Sol. Given, $3\sin\theta + 5\cos\theta = 5$ (i)

Let $5\sin\theta - 3\cos\theta = x$ (ii)

Squaring and adding, we get

$(9\sin^2\theta + 25\cos^2\theta + 30\sin\theta\cos\theta) + (25\sin^2\theta + 9\cos^2\theta - 30\sin\theta\cos\theta) = 25 + x^2$

$\Rightarrow$ $9(\sin^2\theta + \cos^2\theta) + 25(\sin^2\theta + \cos^2\theta) = 25 + x^2$.

$\Rightarrow$ $34 = 25 + x^2$ or $x^2 = 9$

$\Rightarrow \quad x = \pm 3$

Example 2.36 **If $(\sec A + \tan A)(\sec B + \tan B)(\sec C + \tan C) = (\sec A - \tan A)(\sec B - \tan B)(\sec C - \tan C)$, prove that the value of each side is ± 1.**

Sol. Let $(\sec A + \tan A)(\sec B + \tan B)(\sec C + \tan C) = x$ (i)

$(\sec A - \tan A)(\sec B - \tan B)(\sec C - \tan C) = x$. (ii)

Multiplying Eqs. (i) and (ii), we get

$(\sec^2 A - \tan^2 A)(\sec^2 B - \tan^2 B)(\sec^2 C - \tan^2 C) = x^2$

or $x^2 = 1$

$\therefore x = \pm 1$

Hence, each side is equal to ± 1.

Example 2.37 **If $\tan\theta + \sec\theta = 1.5$, find $\sin\theta$, $\tan\theta$, and $\sec\theta$.**

Sol. Given, $\sec\theta + \tan\theta = \dfrac{3}{2}$ (i)

Now, $\sec\theta - \tan\theta = \dfrac{1}{\sec\theta + \tan\theta} = \dfrac{2}{3}$ (see Fig. 2.34) (ii)

Adding Eqs. (i) and (ii), we get $2\sec\theta = \dfrac{3}{2} + \dfrac{2}{3} = \dfrac{13}{6}$

$\therefore \quad \sec\theta = \dfrac{13}{12}$

$\therefore \quad \tan\theta = \dfrac{5}{12}$

and $\sin\theta = \dfrac{5}{13}$

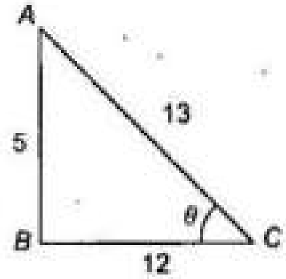


Fig. 2.34

Example 2.38 **If $\operatorname{cosec}\theta - \sin\theta = m$ and $\sec\theta - \cos\theta = n$, eliminate θ.**

Sol. Given $\operatorname{cosec}\theta - \sin\theta = m$, or $\dfrac{1}{\sin\theta} - \sin\theta = m$

or, $\dfrac{1 - \sin^2\theta}{\sin\theta} = m$, or $\dfrac{\cos^2\theta}{\sin\theta} = m$ (i)

Again $\sec\theta - \cos\theta = n$, or $\dfrac{1}{\cos\theta} - \cos\theta = n$

or, $\dfrac{1 - \cos^2\theta}{\cos\theta} = n$, or $\dfrac{\sin^2\theta}{\cos\theta} = n$ (ii)

From Eq. (i), $\sin\theta = \dfrac{\cos^2\theta}{m}$ (iii)

Putting in Eq. (ii), we get $\dfrac{\cos^4\theta}{m^2\cos\theta} = n$, or $\cos^3\theta = m^2 n$

$\therefore \quad \cos\theta = (m^2 n)^{\frac{1}{3}}$, or $\cos^2\theta = (m^2 n)^{\frac{2}{3}}$ (iv)

From Eq. (iii), $\sin\theta = \frac{\cos^2\theta}{m} = \frac{(m^2n)^{\frac{2}{3}}}{m} = \frac{m^{\frac{4}{3}}n^{\frac{2}{3}}}{m} = m^{\frac{1}{3}}n^{\frac{2}{3}} = (mn^2)^{\frac{1}{3}}$

$\therefore \quad \sin^2\theta = (mn^2)^{\frac{2}{3}}$ (v)

Adding Eqs. (iv) and (v), we get

$(m^2n)^{\frac{2}{3}} + (mn^2)^{\frac{2}{3}} = \cos^2\theta + \sin^2\theta$

or, $(m^2n)^{\frac{2}{3}} + (mn^2)^{\frac{2}{3}} = 1$

Example 2.39 If $\frac{\cos^4 A}{\cos^2 B} + \frac{\sin^4 A}{\sin^2 B} = 1$, then prove that

a. $\sin^4 A + \sin^4 B = 2\sin^2 A \sin^2 B$

b. $\frac{\cos^4 B}{\cos^2 A} + \frac{\sin^4 B}{\sin^2 A} = 1$

Sol. Given, $\frac{\cos^4 A}{\cos^2 B} + \frac{\sin^4 A}{\sin^2 B} = 1(\cos^2 A + \sin^2 A)$

$\Rightarrow \quad \frac{\cos^4 A}{\cos^2 B} - \cos^2 A = \sin^2 A - \frac{\sin^4 A}{\sin^2 B}$

$\Rightarrow \quad \frac{\cos^2 A(\cos^2 A - \cos^2 B)}{\cos^2 B} = \sin^2 A\frac{(\sin^2 B - \sin^2 A)}{\sin^2 B}$

$\Rightarrow \quad \frac{\cos^2 A}{\cos^2 B}(\cos^2 A - \cos^2 B) = \frac{\sin^2 A}{\sin^2 B}\left[(1 - \cos^2 B) - (1 - \cos^2 A)\right]$

$\Rightarrow \quad \frac{\cos^2 A}{\cos^2 B}(\cos^2 A - \cos^2 B) = \frac{\sin^2 A}{\sin^2 B}(\cos^2 A - \cos^2 B)$

$\Rightarrow \quad (\cos^2 A - \cos^2 B)\left(\frac{\cos^2 A}{\cos^2 B} - \frac{\sin^2 A}{\sin^2 B}\right) = 0$

when $\cos^2 A - \cos^2 B = 0$, $\cos^2 A = \cos^2 B$ (i)

when $\frac{\cos^2 A}{\cos^2 B} - \frac{\sin^2 A}{\sin^2 B} = 0$, $\cos^2 A \sin^2 B = \sin^2 A \cos^2 B$

$\Rightarrow \quad \cos^2 A(1 - \cos^2 B) = (1 - \cos^2 A)\cos^2 B$

$\Rightarrow \quad \cos^2 A - \cos^2 A\cos^2 B = \cos^2 B - \cos^2 A\cos^2 B$

$\Rightarrow \quad \cos^2 A = \cos^2 B$ (ii)

Thus, in both the cases, $\cos^2 A = \cos^2 B$.

$\therefore \quad 1-\sin^2 A = 1-\sin^2 B$, or $\sin^2 A = \sin^2 B$ (iii)

a. $\text{L.H.S.} = \sin^4 A + \sin^4 B$

$= (\sin^2 A - \sin^2 B)^2 + 2\sin^2 A \sin^2 B = 2\sin^2 A \sin^2 B = \text{R.H.S.}$ $[\because \sin^2 A = \sin^2 B]$

b. $\text{L.H.S.} = \dfrac{\cos^4 B}{\cos^2 A} + \dfrac{\sin^4 B}{\sin^2 A} = \dfrac{\cos^4 B}{\cos^2 B} + \dfrac{\sin^4 B}{\sin^2 B}$

$= \cos^2 B + \sin^2 B = 1$

Example 2.40 **If $x = \sec\theta - \tan\theta$ and $y = \operatorname{cosec}\theta + \cot\theta$, then prove that $xy + 1 = y - x$.**

Sol. $xy + 1 = \dfrac{1-\sin\theta}{\cos\theta}\dfrac{1+\cos\theta}{\sin\theta} + 1 = \dfrac{1-\sin\theta+\cos\theta}{\sin\theta\cos\theta}$

$= \dfrac{(\sin^2\theta+\cos^2\theta)}{\sin\theta\cos\theta} - \dfrac{(\sin\theta-\cos\theta)}{\sin\theta\cos\theta}$

$= (\tan\theta + \cot\theta) - (\sec\theta - \operatorname{cosec}\theta)$

$= (\operatorname{cosec}\theta + \cot\theta) - (\sec\theta - \tan\theta) = y - x$

Concept Application Exercise 2.3

1. Show that $3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x) = 13$.
2. If $\sec\theta + \tan\theta = p$, then find the value of $\tan\theta$.
3. If $(1 + \sin A)(1 + \sin B)(1 + \sin C) = (1 - \sin A)(1 - \sin B)(1 - \sin C)$, then find the value of $(1 + \sin A)(1 + \sin B)(1 + \sin C)$.
4. If $(\sec\theta + \tan\theta)(\sec\phi + \tan\phi)(\sec\psi + \tan\psi) = \tan\theta\tan\phi\tan\psi$, then $(\sec\theta - \tan\theta)(\sec\phi - \tan\phi)(\sec\psi - \tan\psi)$ is equal to

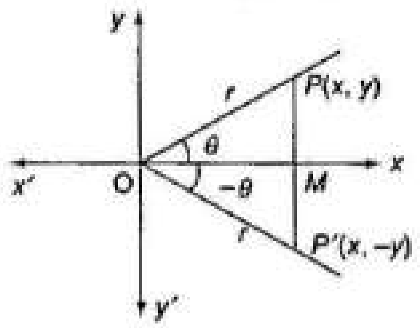
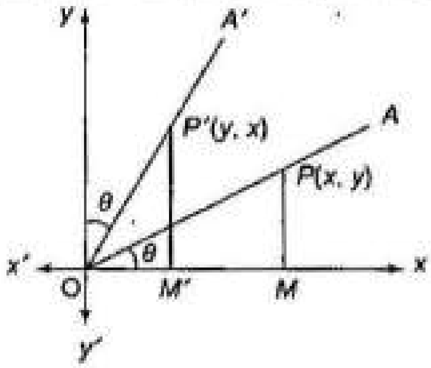
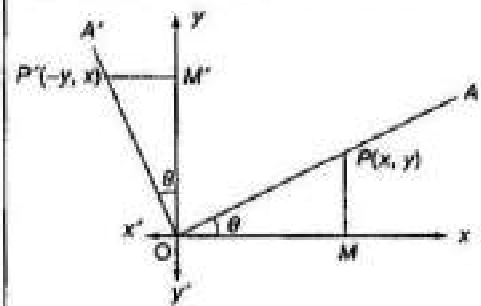
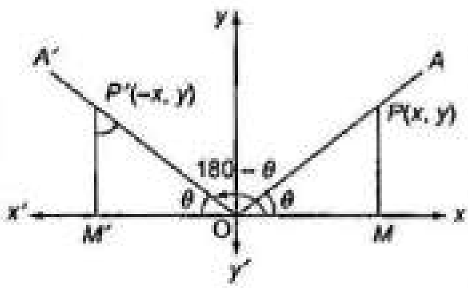
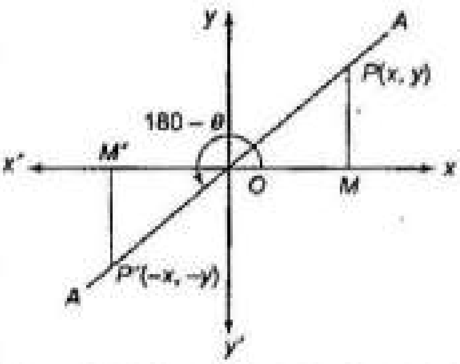
   a. $\cot\theta\cot\phi\cot\psi$ b. $\tan\theta\tan\phi\tan\psi$

   c. $\tan\theta + \tan\phi + \tan\psi$ d. $\cot\theta + \cot\phi + \cot\psi$

5. If $\dfrac{x}{a}\cos\theta + \dfrac{y}{b}\sin\theta = 1$, $\dfrac{x}{a}\sin\theta - \dfrac{y}{b}\cos\theta = 1$, then eliminate θ.
6. If $a + b\tan\theta = \sec\theta$ and $b - a\tan\theta = 3\sec\theta$, then find the value of $a^2 + b^2$.
7. If $a\sin^2 x + b\cos^2 x = c$, $b\sin^2 y + a\cos^2 y = d$ and $a\tan x = b\tan y$, then prove that $\dfrac{a^2}{b^2} = \dfrac{(d-a)(c-a)}{(b-c)(b-d)}$

TRIGONOMETRIC RATIOS FOR COMPLEMENTARY AND SUPPLEMENTARY ANGLES

In each of the following figures, x and y are positive. Also triangles OPM, $OP'M'$, or $OP'M$ are congruent.

$\sin(-\theta) = -\sin\theta$ $\cos(-\theta) = \cos\theta$	$\sin(-\theta) = \frac{-y}{r} = -\frac{y}{r} = -\sin\theta$, $\cos(-\theta) = \frac{x}{r} = \cos\theta$, $\tan(-\theta) = -\frac{y}{x} = \tan\theta$ Taking the reciprocals of these trigonometric ratios, we have $\text{cosec}(-\theta) = -\text{cosec}\,\theta$, $\sec(-\theta) = \sec\theta$ and $\cot(-\theta) = -\cot\theta$	
$\sin(90° - \theta) = \cos\theta$ $\cos(90° - \theta) = \sin\theta$	$\sin(90 - \theta) = \frac{x}{r} = \cos\theta$ $\cos(90 - \theta) = \frac{y}{r} = \sin\theta$ $\tan(90 - \theta) = \frac{x}{y} = \cot\theta$	
$\sin(90° + \theta) = \cos\theta$ $\cos(90° + \theta) = -\sin\theta$	$\sin(90 + \theta) = \frac{x}{r} = \cos\theta$ $\cos(90 + \theta) = \frac{-y}{r} = -\sin\theta$ $\tan(90 + \theta) = \frac{x}{-y} = \frac{-x}{y} = -\cot\theta$	
$\sin(180° - \theta) = \sin\theta$ $\cos(180° - \theta)$ $= -\cos\theta$	Now, $\sin(180 - \theta) = \frac{y}{r} = \sin\theta$ $\cos(180 - \theta) = -\frac{x}{r} = -\cos\theta$ and, $\tan(180 - \theta) = \frac{y}{-x} = -\tan\theta$	
$\sin(180° + \theta)$ $= -\sin\theta$ $\cos(180° + \theta)$ $= -\cos\theta$	$= \sin(180 + \theta) = \frac{-y}{r} = -\sin\theta$ $\cos(180 + \theta) = \frac{-x}{r} = -\cos\theta$ $\tan(180 + \theta) = \frac{-y}{-x} = \frac{y}{x} = \tan\theta$	

Since the terminal sides of co-terminal angles coincide, hence their trigonometrical ratios are same.

Clearly, $360° - \theta$ and $-\theta$ are coterminal angles.

Therefore, $\sin(360° - \theta) = \sin(-\theta) = -\sin\theta$, $\cos(360° - \theta) = \cos(-\theta) = \cos\theta$, and $\tan(360° - \theta) = \tan(-\theta) = -\tan\theta$. Similarly, $\operatorname{cosec}(360° - \theta) = -\operatorname{cosec}\theta$, $\sec(360° - \theta) = \sec\theta$ and $\cot(360° - \theta) = -\cot\theta$.

Also θ and $360° + \theta$ are co-terminal angles. Therefore, $\sin(360° + \theta) = \sin\theta$, $\cos(360° + \theta) = \cos\theta$, $\tan(360° + \theta) = \tan\theta$, $\sec(360° + \theta) = \sec\theta$, $\operatorname{cosec}(360° + \theta) = \operatorname{cosec}\theta$ and $\cot(360° + \theta) = \cot\theta$.

In fact, for any positive integer n, $(360° \times n + \theta)$ is co-terminal to θ. Therefore, for any positive integer n, we have $\sin(360° \times n + \theta) = \sin\theta$, $\cos(360° \times n + \theta) = \cos\theta$, $\tan(360° \times n + \theta) = \tan\theta$, $\operatorname{cosec}(360° \times n + \theta) = \operatorname{cosec}\theta$, $\sec(360° \times n + \theta) = \sec\theta$ and $\cot(360° \times n + \theta) = \cot\theta$.

Example 2.41 **Prove that $\sin(-420°)(\cos 390°) + \cos(-660°)(\sin 330°) = -1$.**

Sol. L.H.S. $= \sin(-420°)(\cos 390°) + \cos(-660°)(\sin 330°)$

$= -\sin 420° \cos 390° + \cos 660° \sin 330°$ $[\because \sin(-\theta) = -\sin\theta, \cos(-\theta) = \cos\theta]$

$= -\sin(90° \times 4 + 60°)\cos(90° \times 4 + 30°) + \cos(90° \times 7 + 30°)\sin(90° \times 3 + 60°)$

$= -(\sin 60°)(\cos 30°) + (\sin 30°)(-\cos 60°)$

$$= -\frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} + \frac{1}{2}\left(-\frac{1}{2}\right) = -1 = \text{R.H.S.}$$

Example 2.42 **Prove that $\dfrac{\cos(90° + \theta)\sec(-\theta)\tan(180° - \theta)}{\sec(360° - \theta)\sin(180° + \theta)\cot(90° - \theta)} = -1$.**

Sol. $$\text{L.H.S} = \frac{\cos(90° + \theta)\sec(-\theta)\tan(180° - \theta)}{\sec(360° - \theta)\sin(180° + \theta)\cot(90° - \theta)} = \frac{(-\sin\theta)(\sec\theta)(-\tan\theta)}{(\sec\theta)(-\sin\theta)(\tan\theta)} = -1 = \text{R.H.S.}$$

Example 2.43 **If A, B, C, D are angles of a cyclic quadrilateral, then prove that $\cos A + \cos B + \cos C + \cos D = 0$.**

Sol. We know that the opposite angles of a cyclic quadrilateral are supplementary, i.e., $A + C = \pi$ and $B + D = \pi$.

$\therefore\ A = \pi - C$ and $B = \pi - D$

$\Rightarrow\ \cos A = \cos(\pi - C) = -\cos C$

and $\cos B = \cos(\pi - D) = -\cos D$

$\therefore\ \cos A + \cos B + \cos C + \cos D = -\cos C - \cos D + \cos C + \cos D = 0$

Example 2.44 **Show that $\tan 1° \tan 2° \ldots \tan 89° = 1$.**

Sol. L.H.S. $= (\tan 1° \tan 89°)(\tan 2° \tan 88°) \ldots (\tan 44° \tan 46°)\tan 45°$

$= [\tan 1° \tan(90° - 1°)][\tan 2° \tan(90° - 2°)] \ldots [\tan 44° \tan(90° - 44°)]\tan 45°$

$= (\tan 1° \cot 1°)(\tan 2° \cot 2°) \ldots (\tan 44° \cot 44°)\tan 45°$

$= 1$ $[\because \tan\theta \cot\theta = 1 \text{ and } \tan 45° = 1]$

Example 2.45 **Show that $\sin^2 5° + \sin^2 10° + \sin^2 15° + \cdots + \sin^2 90° = 9\frac{1}{2}$.**

Sol. L.H.S. $= (\sin^2 5° + \sin^2 85°) + (\sin^2 10° + \sin^2 80°) + \cdots + (\sin^2 40° + \sin^2 50°) + \sin^2 45° + \sin^2 90°$

$= (\sin^2 5° + \cos^2 5°) + (\sin^2 10° + \cos^2 10°) + \cdots + (\sin^2 40° + \cos^2 40°) + \sin^2 45° + \sin^2 90°$

$$= (1 + 1 + 1 + 1 + 1 + 1 + 1 + 1) + \left(\frac{1}{\sqrt{2}}\right)^2 + 1 = 9\frac{1}{2}$$

Example 2.46 Find the value of $\cos^2\frac{\pi}{16}+\cos^2\frac{3\pi}{16}+\cos^2\frac{5\pi}{16}+\cos^2\frac{7\pi}{16}$.

Sol. $\text{L.H.S.}=\cos^2\frac{\pi}{16}+\cos^2\frac{3\pi}{16}+\cos^2\left(\frac{\pi}{2}-\frac{3\pi}{16}\right)+\cos^2\left(\frac{\pi}{2}-\frac{\pi}{16}\right)$

$=\cos^2\frac{\pi}{16}+\cos^2\frac{3\pi}{16}+\sin^2\frac{3\pi}{16}+\sin^2\frac{\pi}{16}$

$=\left(\cos^2\frac{\pi}{16}+\sin^2\frac{\pi}{16}\right)+\left(\cos^2\frac{3\pi}{16}+\sin^2\frac{3\pi}{16}\right)$

$=1+1=2$

Example 2.47 If $\sin(120°-\alpha)=\sin(120°-\beta)$, $0<\alpha,\beta<\pi$, then find the relation between α and β.

Sol. If $\sin A=\sin B$, where $A=120°-\alpha$ and $B=120°-\beta$

$\Rightarrow$ $A=B$ or $A=\pi-B$, i.e., $A+B=\pi$

$\Rightarrow$ $120°-\alpha=120°-\beta$, or $120°-\alpha+120°-\beta=180°$

$\Rightarrow$ $\alpha=\beta$ or $\alpha+\beta=60°$

Concept Application Exercise 2.4

1. In triangle ABC prove that
   a. $\sin A=\sin(B+C)$
   b. $\sin 2A=-\sin(2B+2C)$
   c. $\cos A=-\cos(A+B)$
   d. $\tan\left(\frac{A+B}{2}\right)=\cot\frac{C}{2}$
2. Prove that $\sin(-420°)(\cos 390°)+\cos(-660°)(\sin 330°)=-1$.
3. Prove that
   a. $\tan 720°-\cos 270°-\sin 150°\cos 120°=\frac{1}{4}$
   b. $\sin 780°\sin 480°+\cos 120°\sin 150°=\frac{1}{2}$
4. If $\alpha=\frac{\pi}{3}$, prove that $\cos\alpha\cos 2\alpha\cos 3\alpha\cos 4\alpha\cos 5\alpha\cos 6\alpha=-\frac{1}{16}$.
5. Find the value of $\tan\frac{\pi}{20}\tan\frac{3\pi}{20}\tan\frac{5\pi}{20}\tan\frac{7\pi}{20}\tan\frac{9\pi}{20}$.
6. Find the value of $\frac{\cot 54°}{\tan 36°}+\frac{\tan 20°}{\cot 70°}$.
7. Prove that $\sin^2\frac{\pi}{18}+\sin^2\frac{\pi}{9}+\sin^2\frac{7\pi}{18}+\sin^2\frac{4\pi}{9}=2$.
8. Prove that $\sec\left(\frac{3\pi}{2}-\theta\right)\sec\left(\theta-\frac{5\pi}{2}\right)+\tan\left(\frac{5\pi}{2}+\theta\right)\tan\left(\theta-\frac{3\pi}{2}\right)=-1$.
9. In any quadrilateral $ABCD$, prove that
   a. $\sin(A+B)+\sin(C+D)=0$
   b. $\cos(A+B)=\cos(C+D)$

TRIGONOMETRIC RATIOS FOR COMPOUND ANGLES

Cosine of the Difference and Sum of Two Angles

1. $\cos(A - B) = \cos A \cos B + \sin A \sin B$
2. $\cos(A + B) = \cos A \cos B - \sin A \sin B$

for all angles A and B.

Proof:

1. $\cos(A - B)$

Let $X'OX$ and YOY' be the coordinate axes. Consider a unit circle with O as the centre (Fig. 2.35).

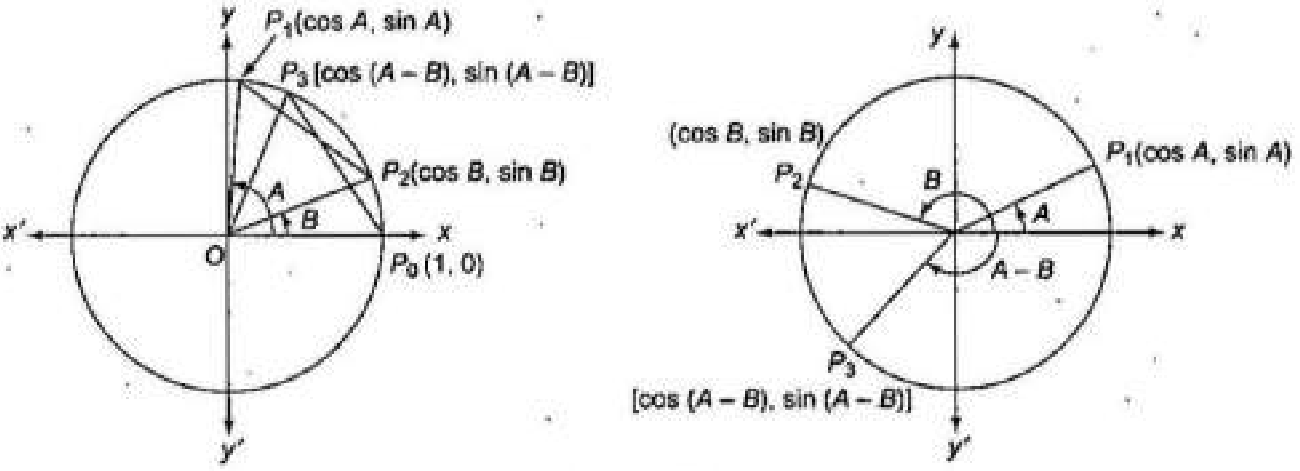


Fig. 2.35

Let P_1, P_2 and P_3 be the three points on the circle such that $\angle XOP_1 = A$, $\angle XOP_2 = B$ and $\angle XOP_3 = A - B$.

As we know that the terminal side of any angle intersects the circle with centre at O and unit radius at a point whose coordinates are the cosine and sine of the angle. Therefore, coordinates of P_1, P_2 and P_3 are $(\cos A, \sin A)$, $(\cos B, \sin B)$ and $(\cos(A - B), \sin(A - B))$, respectively.

We know that equal chords of a circle make equal angles at its centre and chords P_0P_3 and P_1P_2 subtend equal angles at O. Therefore,

Chord P_0P_3 = Chord P_1P_2

$$\Rightarrow \sqrt{[\cos(A-B)-1]^2 + [\sin(A-B)-0]^2} = \sqrt{(\cos B - \cos A)^2 + (\sin B - \sin A)^2}$$

$$\Rightarrow \{\cos(A-B)-1\}^2 + \sin^2(A-B) = (\cos B - \cos A)^2 + (\sin B - \sin A)^2$$

$$\Rightarrow \cos^2(A-B) - 2\cos(A-B) + 1 + \sin^2(A-B) = \cos^2 B + \cos^2 A - 2\cos A \cos B + \sin^2 B + \sin^2 A - 2\sin A \sin B$$

$$\Rightarrow 2 - 2\cos(A-B) = 2 - 2\cos A \cos B - 2\sin A \sin B$$

$$\Rightarrow \cos(A-B) = \cos A \cos B + \sin A \sin B$$

2. $\cos(A + B)$

$= \cos(A - (-B))$ (i)

$= \cos A \cos(-B) + \sin A \sin(-B)$ [Using Eq. (i)]

$= \cos A \cos B - \sin A \sin B$ [$\because \cos(-B) = \cos B$, $\sin(-B) = -\sin B$]

Hence, $\cos(A + B) = \cos A \cos B - \sin A \sin B$

> *Note:*
>
> *This method of proof of the above formula is true for all values of angles A and B whether positive, zero or negative.*

Sine of the Difference and Sum of Two Angles

1. $\sin(A-B) = \sin A \cos B - \cos A \sin B$
2. $\sin(A+B) = \sin A \cos B + \cos A \sin B$

Proof:

1. We have

$$\begin{aligned}\sin(A-B) &= \cos(90^\circ-(A-B)) && [\because \cos(90^\circ-\theta)=\sin\theta]\\ &= \cos((90^\circ-A)+B)\\ &= \cos(90^\circ-A)\cos B-\sin(90^\circ-A)\sin B\\ &= \sin A\cos B-\cos A\sin B && \text{(i)}\end{aligned}$$

2.

$$\begin{aligned}\sin(A+B) &= \sin(A-(-B))\\ &= \sin A\cos(-B)-\cos A\sin(-B) && [\text{Using equation (i)}]\\ &= \sin A\cos B+\cos A\sin B && [\because \sin(-B)=-\sin B]\end{aligned}$$

Tangent of the Difference and Sum of Two Angles

1. $\tan(A+B) = \dfrac{\tan A+\tan B}{1-\tan A\tan B}$
2. $\tan(A-B) = \dfrac{\tan A-\tan B}{1+\tan A\tan B}$

Proof:

1. We have

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}$$

$$= \frac{\tan A+\tan B}{1-\tan A\tan B} \quad \text{(i) [On dividing the numerator and denominator by } \cos A\cos B]$$

2. $\tan(A-B) = \tan(A+(-B))$

$$= \frac{\tan A+\tan(-B)}{1-\tan A\tan(-B)} \quad [\text{Using Eq. (i)}]$$

$$= \frac{\tan A-\tan B}{1+\tan A\tan B}$$

Similarly, it can be proved that

$$\cot(A+B) = \frac{\cot A\cot B-1}{\cot B+\cot A}$$

and $\cot(A-B) = \dfrac{\cot A\cot B+1}{\cot B-\cot A}$

Some More Results

1. $\sin(A+B)\sin(A-B) = \sin^2 A-\sin^2 B = \cos^2 B-\cos^2 A$
2. $\cos(A+B)\cos(A-B) = \cos^2 A-\sin^2 B = \cos^2 B-\sin^2 A$
3. $\sin(A+B+C) = \sin A\cos B\cos C+\cos A\sin B\cos C+\cos A\cos B\sin C-\sin A\sin B\sin C$
4. $\cos(A+B+C) = \cos A\cos B\cos C-\cos A\sin B\sin C-\sin A\cos B\sin C-\sin A\sin B\cos C$
5. $\tan(A+B+C) = \dfrac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}$

Proof:

1. $\sin(A+B)\sin(A-B) = (\sin A\cos B + \cos A\sin B)(\sin A\cos B - \cos A\sin B)$
$= \sin^2 A\cos^2 B - \cos^2 A\sin^2 B$
$= \sin^2 A(1-\sin^2 B) - (1-\sin^2 A)\sin^2 B$
$= \sin^2 A - \sin^2 A\sin^2 B - \sin^2 B + \sin^2 A\sin^2 B = \sin^2 A - \sin^2 B$
$= (1-\cos^2 A) - (1-\cos^2 B) = \cos^2 B - \cos^2 A$

2. $\cos(A+B)\cos(A-B) = (\cos A\cos B - \sin A\sin B)(\cos A\cos B + \sin A\sin B)$
$= \cos^2 A\cos^2 B - \sin^2 A\sin^2 B$
$= \cos^2 A(1-\sin^2 B) - (1-\cos^2 A)\sin^2 B = \cos^2 A - \sin^2 B$
$= (1-\sin^2 A) - (1-\cos^2 B) = \cos^2 B - \sin^2 A$

5. $\tan(A+B+C) = \tan((A+B)+C)$

$$= \frac{\tan(A+B)+\tan C}{1-\tan(A+B)\tan C} = \frac{\dfrac{\tan A+\tan B}{1-\tan A\tan B}+\tan C}{1-\left(\dfrac{\tan A+\tan B}{1-\tan A\tan B}\right)\tan C} = \frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}$$

Example 2.48 **Prove that** $\dfrac{\sin(B-C)}{\cos B\cos C} + \dfrac{\sin(C-A)}{\cos C\cos A} + \dfrac{\sin(A-B)}{\cos A\cos B} = 0.$

Sol. First term of L.H.S. is

$$\frac{\sin(B-C)}{\cos B\cos C} = \frac{\sin B\cos C - \cos B\sin C}{\cos B\cos C} = \frac{\sin B\cos C}{\cos B\cos C} - \frac{\cos B\sin C}{\cos B\cos C} = \tan B - \tan C$$

Similarly, second term of L.H.S. = $\tan C - \tan A$
and, third term of L.H.S. = $\tan A - \tan B$
Now L.H.S. = $(\tan B - \tan C) + (\tan C - \tan A) + (\tan A - \tan B) = 0$.

Example 2.49 **If $\sin\alpha\sin\beta - \cos\alpha\cos\beta + 1 = 0$, then prove that $1 + \cot\alpha\tan\beta = 0$.**

Sol. Given, $\sin\alpha\sin\beta - \cos\alpha\cos\beta + 1 = 0$
or $\cos\alpha\cos\beta - \sin\alpha\sin\beta = 1$
or $\cos(\alpha+\beta) = 1$ (i)

Now $1 + \cot\alpha\tan\beta = 1 + \dfrac{\cos\alpha}{\sin\alpha}\times\dfrac{\sin\beta}{\cos\beta}$

$$= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\sin\alpha\cos\beta}$$

$$= \frac{\sin(\alpha+\beta)}{\sin\alpha\cos\beta}$$

$$= \frac{0}{\sin\alpha\cos\beta} = 0 \qquad [\because \sin^2(\alpha+\beta) = 1-\cos^2(\alpha+\beta) = 1-1 = 0]$$

Example 2.50 **Show that $\cos^2\theta + \cos^2(\alpha+\theta) - 2\cos\alpha\cos\theta\cos(\alpha+\theta)$ is independent of θ.**

Sol. $\cos^2\theta + \cos^2(\alpha+\theta) - 2\cos\alpha\cos\theta\cos(\alpha+\theta) = \cos^2\theta + \cos(\alpha+\theta)[\cos(\alpha+\theta) - 2\cos\alpha\cos\theta]$
$= \cos^2\theta + \cos(\alpha+\theta)[\cos\alpha\cos\theta - \sin\alpha\sin\theta - 2\cos\alpha\cos\theta]$

$= \cos^2\theta - \cos(\alpha+\theta)[\cos\alpha\cos\theta + \sin\alpha\sin\theta]$
$= \cos^2\theta - \cos(\alpha+\theta)\cos(\alpha-\theta)$
$= \cos^2\theta - [\cos^2\alpha - \sin^2\theta] = \cos^2\theta + \sin^2\theta - \cos^2\alpha$
$= 1 - \cos^2\alpha$, which is independent of θ.

Example 2.51 **If $3\tan\theta\tan\varphi = 1$, then prove that $2\cos(\theta+\varphi) = \cos(\theta-\varphi)$.**

Sol. Given, $3\tan\theta\tan\varphi = 1$ or $\cot\theta\cot\varphi = 3$

or, $\dfrac{\cos\theta\cos\varphi}{\sin\theta\sin\varphi} = \dfrac{3}{1}$

By componendo and dividendo, we get

$$\frac{\cos\theta\cos\varphi + \sin\theta\sin\varphi}{\cos\theta\cos\varphi - \sin\theta\sin\varphi} = \frac{3+1}{3-1}$$

$$\Rightarrow \quad \frac{\cos(\theta-\varphi)}{\cos(\theta+\varphi)} = 2$$

$$\Rightarrow \quad 2\cos(\theta+\varphi) = \cos(\theta-\varphi)$$

Example 2.52 **If $\sin(A-B) = \dfrac{1}{\sqrt{10}}$, $\cos(A+B) = \dfrac{2}{\sqrt{29}}$, find the value of $\tan 2A$ where A and B lie between 0 and $\pi/4$.**

Sol. $\tan 2A = \tan[(A+B)+(A-B)] = \dfrac{\tan(A+B)+\tan(A-B)}{1-\tan(A+B)\tan(A-B)}$ (i)

Given that, $0 < A < \dfrac{\pi}{4}$ and $0 < B < \dfrac{\pi}{4}$

$\therefore \quad 0 < A+B < \dfrac{\pi}{2}$

Also, $-\dfrac{\pi}{4} < A-B < \dfrac{\pi}{4}$ and $\sin(A-B) = \dfrac{1}{\sqrt{10}} = (+)$ ve

$\therefore \quad 0 < A-B < \dfrac{\pi}{4}$

Now, $\sin(A-B) = \dfrac{1}{\sqrt{10}}$

$\Rightarrow \quad \tan(A-B) = \dfrac{1}{3}$ (ii)

$\cos(A+B) = \dfrac{2}{\sqrt{29}}$

$\Rightarrow \quad \tan(A+B) = \dfrac{5}{2}$ (iii)

From Eqs. (i), (ii) and (iii), we get

$$\tan 2A = \frac{\frac{5}{2}+\frac{1}{3}}{1-\frac{5}{2}\times\frac{1}{3}} = \frac{17}{6}\times\frac{6}{1} = 17$$

Example 2.53 Prove that $\dfrac{\cos 10° + \sin 10°}{\cos 10° - \sin 10°} = \tan 55°$.

Sol. $\dfrac{\cos 10° + \sin 10°}{\cos 10° - \sin 10°} = \dfrac{1 + \tan 10°}{1 - \tan 10°} = \dfrac{\tan 45° + \tan 10°}{1 - \tan 45° \tan 10°} = \tan(45° + 10°) = \tan 55°$ (dividing by $\cos 10°$)

Example 2.54 Prove that $\tan 70° = 2\tan 50° + \tan 20°$.

Sol. $\tan 70° = \tan(50° + 20°) = \dfrac{\tan 50° + \tan 20°}{1 - \tan 50° \tan 20°}$ (i)

$\Rightarrow \tan 70° (1 - \tan 50° \tan 20°) = \tan 50° + \tan 20°$

$\Rightarrow \tan 70° - \tan 50° \tan 20° \tan 70° = \tan 50° + \tan 20°$

$\Rightarrow \tan 70° = \tan 70° \tan 50° \tan 20° + \tan 50° + \tan 20°$

$= \tan(90° - 20°) \tan 50° \tan 20° + \tan 50° + \tan 20°$

$= \cot 20° \tan 50° \tan 20° + \tan 50° + \tan 20°$

$= \tan 50° + \tan 50° + \tan 20° = 2\tan 50° + \tan 20°$

Example 2.55 Let A, B, C be three angles such that $A + B + C = \pi$. If $\tan A \cdot \tan B = 2$. Then find the value of $\dfrac{\cos A \cos B}{\cos C}$.

Sol. Given $\tan A \cdot \tan B = 2$

Let $y = \dfrac{\cos A \cos B}{\cos C} = -\dfrac{\cos A \cdot \cos B}{\cos(A + B)} = \dfrac{\cos A \cdot \cos B}{\sin A \sin B - \cos A \cos B}$

$= \dfrac{1}{\tan A \tan B - 1} = \dfrac{1}{2 - 1} = 1$

Range of $f(\theta) = a\cos\theta + b\sin\theta$

Let $a = r\sin\alpha$ and $b = r\cos\alpha$, then $r^2 = a^2 + b^2$ and $\tan\alpha = a/b$

Now $f(\theta) = a\cos\theta + b\sin\theta = r\sin\alpha\cos\theta + r\cos\alpha\sin\theta = r\sin(\theta + \alpha) = \sqrt{a^2 + b^2}\sin\left(\theta + \tan^{-1}\dfrac{a}{b}\right)$

Now $-1 \le \sin\left(\theta + \tan^{-1}\dfrac{a}{b}\right) \le 1$

$\Rightarrow -\sqrt{a^2 + b^2} \le \sqrt{a^2 + b^2}\sin\left(\theta + \tan^{-1}\dfrac{a}{b}\right) \le \sqrt{a^2 + b^2}$

Hence, range is $\left[-\sqrt{a^2 + b^2}, \sqrt{a^2 + b^2}\right]$.

Example 2.56 **Find the maximum value of $\sqrt{3}\sin x + \cos x$ and x for which a maximum value occurs.**

Sol. $\sqrt{3}\sin x + \cos x = 2\left(\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\sin x\right) = 2\sin(x + \pi/6)$

which is maximum when $x + \pi/6 = \pi/2$ or $x = 60°$ and has a maximum value 2.

Example 2.57 **Find the maximum and minimum values of $\cos^2\theta - 6\sin\theta\cos\theta + 3\sin^2\theta + 2$.**

Sol. $\cos^2\theta - 6\sin\theta\cos\theta + 3\sin^2\theta + 2$

$$= \frac{1+\cos 2\theta}{2} - 3\sin 2\theta + 3\frac{(1-\cos 2\theta)}{2} + 2$$

$$= 4 - \cos 2\theta - 3\sin 2\theta$$

Now, $-\cos 2\theta - 3\sin 2\theta \in \left[-\sqrt{10}, \sqrt{10}\right]$

$$\Rightarrow \quad 4 - \cos 2\theta - 3\sin 2\theta \in \left[4-\sqrt{10}, 4+\sqrt{10}\right]$$

Concept Application Exercise 2.5

1. If $A + B = 225°$, then find the value of $\frac{\cot A}{1+\cot A} \times \frac{\cot B}{1+\cot B}$.
2. If $\tan A - \tan B = x$ and $\cot B - \cot A = y$, then find the value of $\cot(A - B)$.
3. If x is A.M. of $\tan \pi/9$ and $\tan 5\pi/18$ and y is A.M. of $\tan \pi/9$ and $\tan 7\pi/18$, then relate x and y.
4. Prove that $\frac{\tan^2 2\theta - \tan^2\theta}{1-\tan^2 2\theta\tan^2\theta} = \tan 3\theta \tan\theta$.
5. If $A + B = 45°$, show that $(1 + \tan A)(1 + \tan B) = 2$.
6. If $\tan A = 1/2$, $\tan B = 1/3$, then prove that $\cos 2A = \sin 2B$.
7. Find the maximum value of $1 + \sin\left(\frac{\pi}{4}+\theta\right) + 2\sin\left(\frac{\pi}{4}-\theta\right)$ for all real values of θ.
8. Find the maximum and minimum values of $6\sin x\cos x + 4\cos 2x$.
9. If $p(x) = \sin x(\sin^3 x + 3) + \cos x(\cos^3 x + 4) + \frac{1}{2}\sin^2 2x + 5$, then find the range of $p(x)$.
10. Find the value of $\cos\frac{\pi}{12}\left(\sin\frac{5\pi}{12} + \cos\frac{\pi}{4}\right) + \sin\frac{\pi}{12}\left(\cos\frac{5\pi}{12} - \sin\frac{\pi}{4}\right)$.
11. If $\cos(\alpha + \beta) + \sin(\alpha - \beta) = 0$ and $\tan\beta \neq 1$, then find the value of $\tan\alpha$.
12. If $\sin A + \cos 2A = 1/2$ and $\cos A + \sin 2A = 1/3$, then find the value of $\sin 3A$.
13. If $\sin x + \sin y + \sin z = 0 = \cos x + \cos y + \cos z$, then find the value of expression $\cos(\theta - x) + \cos(\theta - y) + \cos(\theta - z)$.

TRANSFORMATION FORMULAE

Formulae to Transform the Product into Sum or Difference

We know that

$$\sin A \cos B + \cos A \sin B = \sin(A+B) \quad \text{(i)}$$

$$\sin A \cos B - \cos A \sin B = \sin(A-B) \quad \text{(ii)}$$

$$\cos A \cos B - \sin A \sin B = \cos(A+B) \quad \text{(iii)}$$

$$\cos A \cos B + \sin A \sin B = \cos(A-B) \quad \text{(iv)}$$

Adding Eqs. (i) and (ii), we obtain

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B) \quad \text{(v)}$$

Subtracting Eqs. (ii) from (i), we get

$$2 \cos A \sin B = \sin(A+B) - \sin(A-B) \quad \text{(vi)}$$

Adding Eqs. (iii) and (iv), we get

$$2 \cos A \cos B = \cos(A+B) + \cos(A-B) \quad \text{(vii)}$$

Subtracting Eqs. (iii) from (iv), we get

$$2 \sin A \sin B = \cos(A-B) - \cos(A+B) \quad \text{(viii)}$$

Formulae to Transform the Sum or Difference into Product

Let $A + B = C$ and $A - B = D$. Then, $A = \dfrac{C+D}{2}$ and $B = \dfrac{C-D}{2}$

Substituting the values of A, B, C, and D in Eqs. (v), (vi), (vii), (viii), we get

$$\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right) \quad \text{(ix)}$$

$$\sin C - \sin D = 2\sin\left(\frac{C-D}{2}\right)\cos\left(\frac{C+D}{2}\right) \quad \text{(x)}$$

$$\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right) \quad \text{(xi)}$$

$$\left.\begin{aligned} &\cos D - \cos C = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right) \\ \text{or}\quad &\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right) \\ \text{or}\quad &\cos C - \cos D = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right) \end{aligned}\right\} \quad \text{(xii)}$$

These four formulae are used to convert the sum or difference of two sines or two cosines into the product of sines and cosines.

Example 2.58 **If $\sin A = \sin B$ and $\cos A = \cos B$, then prove that $\sin \dfrac{A-B}{2} = 0$.**

Sol. We have $\sin A = \sin B$ and $\cos A = \cos B$

$\Rightarrow \quad \sin A - \sin B = 0$ and $\cos A - \cos B = 0$

$$\Rightarrow \quad 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)=0 \text{ and } -2\sin\left(\frac{A-B}{2}\right)\sin\left(\frac{A+B}{2}\right)=0$$

$\Rightarrow \quad \sin\frac{A-B}{2}=0$, which is common for both the equations.

Example 2.59 **Prove that $\cos 55° + \cos 65° + \cos 175° = 0$.**

Sol. L.H.S. $= \cos 55° + \cos 65° + \cos 175°$

$$= 2\cos\frac{55°+65°}{2}\cos\frac{55°-65°}{2} + \cos 175°$$

$$= 2\cos 60° \cos(-5°) + \cos 175° = 2\times\frac{1}{2}\cos 5° + \cos(180° - 5°) = \cos 5° - \cos 5° = 0$$

Example 2.60 **Prove that $\cos 18° - \sin 18° = \sqrt{2}\sin 27°$.**

Sol. L.H.S. $= \cos 18° - \sin 18° = \cos 18° - \sin(90° - 72°) = \cos 18° - \cos 72°$

$$= 2\sin\frac{18°+72°}{2}\sin\frac{72°-18°}{2}$$

$$= 2\sin 45° \sin 27° = 2\frac{1}{\sqrt{2}}\sin 27° = \sqrt{2}\sin 27°$$

Example 2.61 **Prove that**

a. $\dfrac{\sin 5A - \sin 3A}{\cos 5A + \cos 3A} = \tan A$

b. $\dfrac{\sin A + \sin 3A}{\cos A + \cos 3A} = \tan 2A$

Sol.

a. L.H.S. $= \dfrac{\sin 5A - \sin 3A}{\cos 5A + \cos 3A} = \dfrac{2\sin\left(\frac{5A-3A}{2}\right)\cos\left(\frac{5A+3A}{2}\right)}{2\cos\left(\frac{5A+3A}{2}\right)\cos\left(\frac{5A-3A}{2}\right)} = \dfrac{2\sin A\cos 4A}{2\cos 4A\cos A} = \tan A =$ R.H.S.

b. L.H.S. $= \dfrac{\sin 3A + \sin A}{\cos 3A + \cos A} = \dfrac{2\sin\left(\frac{3A+A}{2}\right)\cos\left(\frac{3A-A}{2}\right)}{2\cos\left(\frac{3A+A}{2}\right)\cos\left(\frac{3A-A}{2}\right)} = \dfrac{\sin 2A\cos A}{\cos 2A\cos A} = \tan 2A =$ R.H.S.

Example 2.62 **Prove that $\cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha+\beta+\gamma) = 4\cos\frac{\alpha+\beta}{2}\cos\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}$.**

Sol. L.H.S. $= \cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha+\beta+\gamma)$

$$= (\cos\alpha + \cos\beta) + [\cos\gamma + \cos(\alpha+\beta+\gamma)]$$

$$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) + 2\cos\left(\frac{\alpha+\beta+\gamma+\gamma}{2}\right)\cos\left(\frac{\alpha+\beta+\gamma-\gamma}{2}\right)$$

$$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) + 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha+\beta+2\gamma}{2}\right)$$

$$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\left\{\cos\left(\frac{\alpha-\beta}{2}\right) + \cos\left(\frac{\alpha+\beta+2\gamma}{2}\right)\right\}$$

$$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\left\{2\cos\left(\frac{\frac{\alpha-\beta}{2}+\frac{\alpha+\beta+2\gamma}{2}}{2}\right)\cos\left(\frac{\frac{\alpha+\beta+2\gamma}{2}-\frac{\alpha-\beta}{2}}{2}\right)\right\}$$

$$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\left\{2\cos\left(\frac{\alpha+\gamma}{2}\right)\cos\left(\frac{\beta+\gamma}{2}\right)\right\} = 4\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\beta+\gamma}{2}\right)\cos\left(\frac{\gamma+\alpha}{2}\right) = \text{R.H.S.}$$

Example 2.63 **Prove that** $\dfrac{\sin A+\sin 2A+\sin 4A+\sin 5A}{\cos A+\cos 2A+\cos 4A+\cos 5A} = \tan 3A.$

Sol. $\dfrac{\sin A+\sin 2A+\sin 4A+\sin 5A}{\cos A+\cos 2A+\cos 4A+\cos 5A}$

$$= \frac{(\sin 5A+\sin A)+(\sin 4A+\sin 2A)}{(\cos 5A+\cos A)+(\cos 4A+\cos 2A)}$$

$$= \frac{2\sin 3A\cos 2A+2\sin 3A\cos A}{2\cos 3A\cos 2A+2\cos 3A\cos A}$$

$$= \frac{2\sin 3A(\cos 2A+\cos A)}{2\cos 3A(\cos 2A+\cos A)} = \tan 3A$$

Example 2.64 **Prove that** $\left(\dfrac{\cos A+\cos B}{\sin A-\sin B}\right)^n + \left(\dfrac{\sin A+\sin B}{\cos A-\cos B}\right)^n = 2\cot^n\dfrac{A-B}{2}$ **or 0, accordingly as n is even or odd.**

Sol. $\text{L.H.S.} = \left(\dfrac{2\cos\frac{A+B}{2}\cos\frac{A-B}{2}}{2\cos\frac{A+B}{2}\sin\frac{A-B}{2}}\right)^n + \left(\dfrac{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}{2\sin\frac{A+B}{2}\sin\frac{B-A}{2}}\right)^n$

$$= \left(\cot\frac{A-B}{2}\right)^n + \left(-\cot\frac{A-B}{2}\right)^n \qquad [\because \sin(-\theta) = -\sin\theta]$$

$$= \cot^n\frac{A-B}{2} + (-1)^n\cot^n\frac{A-B}{2} = \cot^n\frac{A-B}{2}\left[1+(-1)^n\right]$$

$$= \begin{cases} 0, \text{ if } n \text{ is odd} \\ 2\cot^n\dfrac{A-B}{2}, \text{ if } n \text{ is even.} \end{cases}$$

Example 2.65 **Prove that** $(\cos\alpha+\cos\beta)^2+(\sin\alpha+\sin\beta)^2 = 4\cos^2\left(\dfrac{\alpha-\beta}{2}\right).$

Sol. $\text{L.H.S.} = (\cos\alpha+\cos\beta)^2+(\sin\alpha+\sin\beta)^2$

$$= \left\{2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\right\}^2 + \left\{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\right\}^2$$

$$= 4\cos^2\left(\frac{\alpha-\beta}{2}\right)\left\{\cos^2\frac{\alpha+\beta}{2}+\sin^2\frac{\alpha+\beta}{2}\right\}$$

$$= 4\cos^2\left(\frac{\alpha-\beta}{2}\right) \quad \left[\because \cos^2\frac{\alpha+\beta}{2}+\sin^2\frac{\alpha+\beta}{2}=1\right]$$

= R.H.S.

Example 2.66 If $\sec(\theta+\alpha)+\sec(\theta-\alpha)=2\sec\theta$, then show that $\cos^2\theta=1+\cos\alpha$.

Sol. $\sec(\theta+\alpha)+\sec(\theta-\alpha)=2\sec\theta$

$$\Rightarrow \frac{1}{\cos(\theta+\alpha)}+\frac{1}{\cos(\theta-\alpha)}=\frac{2}{\cos\theta}$$

$$\Rightarrow \frac{\cos(\theta-\alpha)+\cos(\theta+\alpha)}{\cos(\theta+\alpha)\cos(\theta-\alpha)}=\frac{2}{\cos\theta}$$

$$\Rightarrow \frac{2\cos\theta\cos\alpha}{\cos^2\theta-\sin^2\alpha}=\frac{2}{\cos\theta}$$

$$\Rightarrow \cos^2\theta\cos\alpha=\cos^2\theta-\sin^2\alpha$$

$$\Rightarrow \sin^2\alpha=\cos^2\theta(1-\cos\alpha)$$

$$\Rightarrow 1-\cos^2\alpha=\cos^2\theta(1-\cos\alpha) \quad \Rightarrow 1+\cos\alpha=\cos^2\theta$$

Example 2.67 In quadrilateral $ABCD$ if $\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)+\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)=2$, then find the value of $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{D}{2}$.

Sol. $\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)+\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)=2$

$$\Rightarrow \frac{1}{2}[\sin A+\sin B+\sin C+\sin D]=2$$

$$\Rightarrow \sin A+\sin B+\sin C+\sin D=4$$

$$\Rightarrow A=B=C=D=90°$$

$$\Rightarrow \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{D}{2}=1/4$$

Concept Application Exercise 2.6

1. a. Prove that $\sin 65°+\cos 65°=\sqrt{2}\cos 20°$.
   b. Prove that $\sin 47°+\cos 77°=\cos 17°$.
2. Prove that $\cos 80°+\cos 40°-\cos 20°=0$.
3. Prove that $\sin 10°+\sin 20°+\sin 40°+\sin 50°=\sin 70°+\sin 80°$.
4. Prove that $\cos\frac{\pi}{5}+\cos\frac{2\pi}{5}+\cos\frac{6\pi}{5}+\cos\frac{7\pi}{5}=0$.
5. If $\sin\alpha-\sin\beta=\frac{1}{3}$ and $\cos\beta-\cos\alpha=\frac{1}{2}$, show that $\cot\frac{\alpha+\beta}{2}=\frac{2}{3}$.
6. If $\operatorname{cosec} A+\sec A=\operatorname{cosec} B+\sec B$, prove that $\tan A\tan B=\cot\frac{A+B}{2}$.
7. Prove that $\sin 25°\cos 115°=\frac{1}{2}(\sin 40°-1)$.

8. If $\cos A = \frac{3}{4}$, then find the value of $32 \sin\left(\frac{A}{2}\right)\sin\left(\frac{5A}{2}\right)$.

9. If $x \cos \theta = y \cos\left(\theta + \frac{2\pi}{3}\right) = z \cos\left(\theta + \frac{4\pi}{3}\right)$, prove that $xy + yz + zx = 0$.

10. If $y \sin \varphi = x \sin(2\theta + \varphi)$, show that $(x + y) \cot(\theta + \varphi) = (y - x) \cot \theta$.

11. If $\cos(A + B) \sin(C + D) = \cos(A - B) \sin(C - D)$, prove that $\cot A \cot B \cot C = \cot D$.

12. If $\tan(A + B) = 3 \tan A$, prove that

  a. $\sin(2A + B) = 2 \sin B$

  b. $\sin 2(A + B) + \sin 2A = 2 \sin 2B$

TRIGONOMETRIC RATIOS OF MULTIPLES AND SUB-MULTIPLE ANGLES

Formulae for Multiple Angles

1. $\cos 2A = \cos(A + A) = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1$

   Also $\sin^2 A = \frac{1}{2}(1 - \cos 2A)$, $\cos^2 A = \frac{1}{2}(1 + \cos 2A)$

2. $\sin 2A = \sin(A + A) = \sin A \cos A + \sin A \cos A = 2\sin A \cos A$

3. $\tan 2A = \tan(A + A) = \frac{\tan A + \tan A}{1 - \tan A \tan A} = \frac{2 \tan A}{1 - \tan^2 A}$

4. $\sin 3A = \sin(2A + A)$

$$= \sin 2A \cos A + \cos 2A \sin A = 2 \sin A \cos A \cos A + (1 - 2 \sin^2 A) \sin A$$
$$= 2 \sin A \cos^2 A + \sin A - 2 \sin^3 A$$
$$= 2 \sin A (1 - \sin^2 A) + \sin A - 2 \sin^3 A$$
$$= 2 \sin A - 2 \sin^3 A + \sin A - 2 \sin^3 A$$
$$= 3 \sin A - 4 \sin^3 A$$

5. $\cos 3A = \cos(2A + A)$

$$= \cos 2A \cos A - \sin 2A \sin A = (2 \cos^2 A - 1) \cos A - 2 \sin A \cos A \sin A$$
$$= 2 \cos^3 A - \cos A - 2 \cos A (1 - \cos^2 A)$$
$$= 2 \cos^3 A - \cos A - 2 \cos A + 2 \cos^3 A$$
$$= 4 \cos^3 A - 3 \cos A$$

6. $\sin 2A$ and $\cos 2A$ in terms of $\tan A$

$$\sin 2A = 2 \sin A \cos A = \frac{2 \sin A \cos A}{\cos^2 A + \sin^2 A} = \frac{2 \tan A}{1 + \tan^2 A}$$

[dividing numerator and denominator by $\cos^2 A$]

$$\cos 2A = \cos^2 A - \sin^2 A = \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A} = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

[dividing numerator and denominator by $\cos^2 A$]

Also $\tan^2 A = \frac{1 - \cos 2A}{1 + \cos 2A}$

7. In the formula of $\tan(A+B+C)$, putting $B=A$ and $C=A$, we get

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1-3\tan^2 A}$$

Similarly, we can prove that $\cot 3A = \dfrac{\cot^3 A - 3\cot A}{3\cot^2 A - 1}$

8. $\tan(A_1 + A_2 + \cdots A_n) = \dfrac{S_1 - S_3 + S_5 - S_7 + \cdots}{1 - S_2 + S_4 - S_6 + \cdots}$,

where

$S_1 = \tan A_1 + \tan A_2 + \cdots + \tan A_n$ = Sum of the tangents of the separate angles,

$S_2 = \tan A_1 \tan A_2 + \tan A_1 \tan A_3 + \cdots$ = Sum of the product of tangents taken two at a time,

$S_3 = \tan A_1 \tan A_2 \tan A_3 + \tan A_2 \tan A_3 \tan A_4 + \cdots$ = Sum of the product of tangents taken three at a time, and so on.

If $A_1 = A_2 = \cdots = A_n = A$, then we have

$S_1 = n\tan A$, $S_2 = {}^nC_2 \tan^2 A$, $S_3 = {}^nC_3 \tan^3 A, \cdots$

Example 2.68 **Prove that**

a. $\dfrac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$

b. $\dfrac{\sin 2\theta}{1-\cos 2\theta} = \cot\theta$

c. $\dfrac{1+\sin 2\theta + \cos 2\theta}{1+\sin 2\theta - \cos 2\theta} = \cot\theta$

d. $\dfrac{1+\sin\theta - \cos\theta}{1+\sin\theta+\cos\theta} = \tan\theta/2$

e. $\dfrac{\cos 2\theta}{1+\sin 2\theta} = \tan(\pi/4 - \theta)$

f. $\dfrac{\cos\theta}{1+\sin\theta} = \tan\left(\dfrac{\pi}{4} - \dfrac{\theta}{2}\right)$

Sol.

a. $\text{L.H.S.} = \dfrac{\sin 2\theta}{1+\cos 2\theta} = \dfrac{2\sin\theta\cos\theta}{2\cos^2\theta} = \tan\theta = \text{R.H.S.}$

b. $\text{L.H.S.} = \dfrac{\sin 2\theta}{1-\cos 2\theta} = \dfrac{2\sin\theta\cos\theta}{2\sin^2\theta} = \cot\theta = \text{R.H.S.}$

c. $\text{L.H.S.} = \dfrac{1+\sin 2\theta+\cos 2\theta}{1+\sin 2\theta - \cos 2\theta} = \dfrac{(1+\cos 2\theta)+\sin 2\theta}{(1-\cos 2\theta)+\sin 2\theta}$

$$= \frac{2\cos^2\theta + 2\sin\theta\cos\theta}{2\sin^2\theta + 2\sin\theta\cos\theta}$$

$$= \frac{2\cos\theta(\cos\theta+\sin\theta)}{2\sin\theta(\cos\theta+\sin\theta)} = \frac{\cos\theta}{\sin\theta} = \cot\theta = \text{R.H.S.}$$

d. $\text{L.H.S.} = \dfrac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} = \dfrac{(1-\cos\theta)+\sin\theta}{(1+\cos\theta)+\sin\theta}$

$$= \frac{2\sin^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \frac{2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)}{2\cos\frac{\theta}{2}\left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)}$$

$$= \tan\frac{\theta}{2} = \text{R.H.S.}$$

e. $$\text{L.H.S.} = \frac{\cos 2\theta}{1+\sin 2\theta}$$

$$= \frac{\sin\left(\frac{\pi}{2}-2\theta\right)}{1+\cos\left(\frac{\pi}{2}-2\theta\right)}$$

$$= \frac{2\sin\left(\frac{\pi}{4}-\theta\right)\cos\left(\frac{\pi}{4}-\theta\right)}{2\cos^2\left(\frac{\pi}{4}-\theta\right)}$$

$$= \tan\left(\frac{\pi}{4}-\theta\right) = \text{R.H.S.}$$

f. $$\text{L.H.S.} = \frac{\cos\theta}{1+\sin\theta} = \frac{\sin\left(\frac{\pi}{2}-\theta\right)}{1+\cos\left(\frac{\pi}{2}-\theta\right)} = \frac{2\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)} = \tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right) = \text{R.H.S.}$$

Example 2.69 **Prove that** $\dfrac{1+\sin 2\theta}{1-\sin 2\theta} = \left(\dfrac{1+\tan\theta}{1-\tan\theta}\right)^2$.

Sol. $$\text{L.H.S.} = \frac{1+\sin 2\theta}{1-\sin 2\theta}$$

$$= \frac{\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta}{\sin^2\theta+\cos^2\theta-2\sin\theta\cos\theta}$$

$$= \left(\frac{\sin\theta+\cos\theta}{\sin\theta-\cos\theta}\right)^2 = \left(\frac{1+\tan\theta}{1-\tan\theta}\right)^2$$ (dividing numerator and denominator by $\cos\theta$)

Example 2.70 **If $\alpha+\beta=90°$, find the maximum value of $\sin\alpha\sin\beta$.**

Sol. Let $y = \sin\alpha\sin\beta = \sin\alpha\sin(90° - \alpha) = \sin\alpha\cos\alpha = \frac{1}{2}\sin 2\alpha$

which has the maximum value 1/2 when $\sin 2\alpha = 1$.

Example 2.71 **Prove that** $\dfrac{1-\tan^2\left(\frac{\pi}{4}-A\right)}{1+\tan^2\left(\frac{\pi}{4}-A\right)} = \sin 2A$.

Sol. $$\frac{1-\tan^2\left(\frac{\pi}{4}-A\right)}{1+\tan^2\left(\frac{\pi}{4}-A\right)} = \frac{1-\tan^2\theta}{1+\tan^2\theta}\ \left(\text{where } \frac{\pi}{4}-A=\theta\right) = \cos 2\theta = \cos\left(\frac{\pi}{2}-2A\right) = \sin 2A$$

Example 2.72 **Prove that** $(\cos A - \cos B)^2 + (\sin A - \sin B)^2 = 4\sin^2\dfrac{A-B}{2}$.

Sol. $\text{L.H.S.} = (\cos A - \cos B)^2 + (\sin A - \sin B)^2$

$= \cos^2 A + \cos^2 B - 2\cos A\cos B + \sin^2 A + \sin^2 B - 2\sin A\sin B$

$$= (\cos^2 A + \sin^2 A) + (\cos^2 B + \sin^2 B) - 2(\cos A \cos B + \sin A \sin B)$$

$$= 2 - 2\cos(A - B) = 2(1 - \cos(A - B)) = 4\sin^2 \frac{A-B}{2}$$

Example 2.73 **If $\sin A = \frac{3}{5}$ and $0° < A < 90°$, find the values of $\sin 2A$, $\cos 2A$, $\tan 2A$, and $\sin 4A$.**

Sol. Given $\sin A = \frac{3}{5}$ and A is an acute angle.

$\therefore \quad \cos A = \frac{4}{5}$ [$\because A$ is acute]

and $\tan A = \frac{3}{4}$

Now, $\sin 2A = 2\sin A \cos A = 2 \times \frac{3}{5} \times \frac{4}{5} = \frac{24}{25}$

$$\cos 2A = 1 - 2\sin^2 A = 1 - 2 \times \frac{9}{25} = \frac{7}{25}$$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A} = \frac{2 \times \frac{3}{4}}{1 - \left(\frac{3}{4}\right)^2} = \frac{\frac{6}{4}}{1 - \frac{9}{16}} = \frac{\frac{6}{4}}{\frac{7}{16}} = \frac{6}{4} \times \frac{16}{7} = \frac{24}{7}$$

$$\sin 4A = 2\sin 2A \cos 2A = 2 \times \frac{24}{25} \times \frac{7}{25} = \frac{336}{625}$$

Example 2.74 **If $\tan \alpha = \frac{1}{7}$, $\sin \beta = \frac{1}{\sqrt{10}}$, prove that $\alpha + 2\beta = \frac{\pi}{4}$, where $0 < \alpha < \frac{\pi}{2}$ and $0 < \beta < \frac{\pi}{2}$.**

Sol. $\tan(\alpha + 2\beta) = \frac{\tan\alpha + \tan 2\beta}{1 - \tan\alpha \tan 2\beta} = \frac{\frac{1}{7} + \tan 2\beta}{1 - \frac{1}{7}\tan 2\beta}$ (i)

Now, $\tan 2\beta = \frac{2\tan\beta}{1-\tan^2\beta} = \frac{2 \times \frac{1}{3}}{1 - \frac{1}{9}} = \frac{3}{4}$ [$\tan\beta > 0$ as $0 < \beta < \pi/2$]

Substituting the value of $\tan 2\beta$ in Eq. (i), we get

$$\tan(\alpha + 2\beta) = \frac{\frac{1}{7} + \frac{3}{4}}{1 - \frac{1}{7} \times \frac{3}{4}} = \frac{25}{25} = 1$$

Now, $0 < \alpha < \frac{\pi}{2}$ and $0 < \beta < \frac{\pi}{2}$

$\therefore \quad 0 < 2\beta < \pi$, but $\tan 2\beta = \frac{3}{4} > 0$

$\Rightarrow \quad 0 < 2\beta < \frac{\pi}{2}$

Hence, $0 < \alpha + 2\beta < \pi$.

In the interval $(0, \pi)$, $\tan\theta$ takes value 1 at $\pi/4$ only

$$\therefore \quad \alpha + 2\beta = \frac{\pi}{4}$$

Example 2.75 **Show that $\sqrt{2+\sqrt{2+\sqrt{2+2\cos 8\theta}}} = 2\cos\theta,\ \ 0<\theta<\pi/16$.**

Sol. L.H.S. $= \sqrt{2+\sqrt{2+\sqrt{2(1+\cos 8\theta)}}}$

$$= \sqrt{2+\sqrt{2+\sqrt{2(2\cos^2 4\theta)}}} \qquad \left[\because 1+\cos 8\theta = 2\cos^2\frac{8\theta}{2}\right]$$

$$= \sqrt{2+\sqrt{2+\sqrt{(4\cos^2 4\theta)}}}$$

$$= \sqrt{2+\sqrt{2+2\cos 4\theta}}$$

$$= \sqrt{2+\sqrt{2(1+\cos 4\theta)}}$$

$$= \sqrt{2+\sqrt{2(2\cos^2 2\theta)}} \qquad [\because 1+\cos 4\theta = 2\cos^2 2\theta]$$

$$= \sqrt{2+2\cos 2\theta} = \sqrt{2(1+\cos 2\theta)} = \sqrt{2(2\cos^2\theta)} = 2\cos\theta = \text{R.H.S.}$$

Example 2.76 **Prove that $\dfrac{\sec 8\theta - 1}{\sec 4\theta - 1} = \dfrac{\tan 8\theta}{\tan 2\theta}$.**

Sol. L.H.S. $= \dfrac{\sec 8\theta - 1}{\sec 4\theta - 1} = \dfrac{\dfrac{1}{\cos 8\theta} - 1}{\dfrac{1}{\cos 4\theta} - 1} = \dfrac{1-\cos 8\theta}{\cos 8\theta} \times \dfrac{\cos 4\theta}{1-\cos 4\theta}$

$$= \frac{2\sin^2 4\theta}{\cos 8\theta} \times \frac{\cos 4\theta}{2\sin^2 2\theta} \qquad \left[\begin{aligned}&\because 1-\cos 8\theta = 2\sin^2\frac{8\theta}{2} = 2\sin^2 4\theta\\ &\text{and } 1-\cos 4\theta = 2\sin^2\frac{4\theta}{2} = 2\sin^2 2\theta\end{aligned}\right]$$

$$= \frac{(2\sin 4\theta\cos 4\theta)}{\cos 8\theta} \times \frac{\sin 4\theta}{2\sin^2 2\theta}$$

$$= \left(\frac{2\sin 4\theta\cos 4\theta}{\cos 8\theta}\right) \times \left(\frac{2\sin 2\theta\cos 2\theta}{2\sin^2 2\theta}\right)$$

$$= \left(\frac{\sin 2(4\theta)}{\cos 8\theta}\right) \times \left(\frac{\cos 2\theta}{\sin 2\theta}\right) = \left(\frac{\sin 8\theta}{\cos 8\theta}\right) \times \left(\frac{\cos 2\theta}{\sin 2\theta}\right)$$

$$= \tan 8\theta \cot 2\theta = \frac{\tan 8\theta}{\tan 2\theta} = \text{R.H.S.}$$

Example 2.77 **Show that $\sqrt{3}\ \text{cosec}\ 20° - \sec 20° = 4$.**

Sol. L.H.S. $= \dfrac{\sqrt{3}}{\sin 20°} - \dfrac{1}{\cos 20°} = \dfrac{\sqrt{3}\cos 20° - \sin 20°}{\sin 20°\cos 20°}$

$$= \frac{2\left(\frac{\sqrt{3}}{2}\cos 20^\circ - \frac{1}{2}\sin 20^\circ\right)}{\sin 20^\circ \cos 20^\circ}$$

$$= \frac{2\,(\sin 60^\circ \cos 20^\circ - \cos 60^\circ \sin 20^\circ)}{\sin 20^\circ \cos 20^\circ}$$

$$= \frac{2\sin(60^\circ - 20^\circ)}{\sin 20^\circ \cos 20^\circ} = \frac{2\sin 40^\circ}{\sin 20^\circ \cos 20^\circ}$$

$$= \frac{4\sin 40^\circ}{2\sin 20^\circ \cos 20^\circ} = \frac{4\sin 40^\circ}{\sin 40^\circ} = 4$$

Example 2.78 **Prove that** $\left(1 + \cos\frac{\pi}{8}\right)\left(1 + \cos\frac{3\pi}{8}\right)\left(1 + \cos\frac{5\pi}{8}\right)\left(1 + \cos\frac{7\pi}{8}\right) = \frac{1}{8}$.

Sol. We have $\cos\frac{7\pi}{8} = \cos\left(\pi - \frac{\pi}{8}\right) = -\cos\frac{\pi}{8}$

and $\cos\frac{5\pi}{8} = \cos\left(\pi - \frac{3\pi}{8}\right) = -\cos\frac{3\pi}{8}$

$\therefore$ L.H.S. $= \left(1 + \cos\frac{\pi}{8}\right)\left(1 + \cos\frac{3\pi}{8}\right)\left(1 - \cos\frac{3\pi}{8}\right)\left(1 - \cos\frac{\pi}{8}\right)$

$$= \left(1 - \cos^2\frac{\pi}{8}\right)\left(1 - \cos^2\frac{3\pi}{8}\right)$$

$$= \sin^2\frac{\pi}{8}\,\sin^2\frac{3\pi}{8}$$

$$= \frac{1}{4}\left(2\sin^2\frac{\pi}{8}\right)\left(2\sin^2\frac{3\pi}{8}\right)$$

$$= \frac{1}{4}\left[\left(1 - \cos\frac{\pi}{4}\right)\left(1 - \cos\frac{3\pi}{4}\right)\right] \qquad \left[\because 1 - \cos\theta = 2\sin^2\frac{\theta}{2}\right]$$

$$= \frac{1}{4}\left[\left(1 - \frac{1}{\sqrt{2}}\right)\left(1 + \frac{1}{\sqrt{2}}\right)\right] = \frac{1}{4}\left(1 - \frac{1}{2}\right) = \frac{1}{8} = \text{R.H.S.}$$

Example 2.79 **Prove that** $\cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{5\pi}{8} + \cos^4\frac{7\pi}{8} = \frac{3}{2}$.

Sol. We have $\frac{7\pi}{8} = \pi - \frac{\pi}{8}$ and $\frac{5\pi}{8} = \pi - \frac{3\pi}{8}$

$\Rightarrow$ $\cos\frac{7\pi}{8} = -\cos\frac{\pi}{8}$ and $\cos\frac{5\pi}{8} = -\cos\frac{3\pi}{8}$

$\Rightarrow$ $\cos^4\frac{7\pi}{8} = \cos^4\frac{\pi}{8}$ and $\cos^4\frac{5\pi}{8} = \cos^4\frac{3\pi}{8}$

$\therefore$ L.H.S. $= 2\cos^4\frac{\pi}{8} + 2\cos^4\frac{3\pi}{8}$

$$= 2\left[\left(\cos^2\frac{\pi}{8}\right)^2 + \left(\cos^2\frac{3\pi}{8}\right)^2\right]$$

$$= 2\left\{\frac{1+\cos\frac{\pi}{4}}{2}\right\}^2 + \left\{\frac{1+\cos\frac{3\pi}{4}}{2}\right\}^2 \qquad \left[\because \frac{1+\cos 2\theta}{2} = \cos^2\theta\right]$$

$$= \frac{1}{2}\left\{\left(1+\cos\frac{\pi}{4}\right)^2 + \left(1+\cos\frac{3\pi}{4}\right)^2\right\}^2$$

$$= \frac{1}{2}\left\{\left(1+\frac{1}{\sqrt{2}}\right)^2 + \left(1-\frac{1}{\sqrt{2}}\right)^2\right\}$$

$$= \frac{1}{2}\left\{\left(1+\frac{1}{2}+\sqrt{2}\right) + \left(1+\frac{1}{2}-\sqrt{2}\right)\right\} = \frac{3}{2} = \text{R.H.S.}$$

Example 2.80 **If $\pi < x < 2\pi$, prove that $\dfrac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}} = \cot\left(\dfrac{x}{2}+\dfrac{\pi}{4}\right)$.**

Sol. L.H.S. $= \dfrac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}} = \dfrac{\sqrt{2\cos^2\frac{x}{2}}+\sqrt{2\sin^2\frac{x}{2}}}{\sqrt{2\cos^2\frac{x}{2}}-\sqrt{2\sin^2\frac{x}{2}}}$

$$= \frac{\sqrt{2}\left|\cos\frac{x}{2}\right| + \sqrt{2}\left|\sin\frac{x}{2}\right|}{\sqrt{2}\left|\cos\frac{x}{2}\right| - \sqrt{2}\left|\sin\frac{x}{2}\right|}$$

$$= \frac{\left|\cos\frac{x}{2}\right| + \left|\sin\frac{x}{2}\right|}{\left|\cos\frac{x}{2}\right| - \left|\sin\frac{x}{2}\right|}$$

$$= \frac{-\cos\frac{x}{2}+\sin\frac{x}{2}}{-\cos\frac{x}{2}-\sin\frac{x}{2}} \qquad \left[\because \pi < x < 2\pi,\ \therefore \frac{\pi}{2} < \frac{x}{2} < \pi\right]$$

$\Rightarrow$ $\cos x/2$ is negative and $\sin x/2$ is positive.

$$= \frac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}} = \frac{\cot\frac{x}{2}-1}{\cot\frac{x}{2}+1} \qquad \text{[dividing numerator and denominator by } \sin x/2]$$

$$= \cot\left(\frac{x}{2}+\frac{\pi}{4}\right) = \text{R.H.S.}$$

Example 2.81 **If $\sin\alpha+\sin\beta = a$ and $\cos\alpha+\cos\beta = b$, prove that $\tan\dfrac{\alpha-\beta}{2} = \pm\sqrt{\dfrac{4-a^2-b^2}{a^2+b^2}}$.**

Sol. Given, $\sin\alpha+\sin\beta = a$ (i)

and $\cos\alpha+\cos\beta = b$ (ii)

Now $(\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2 = b^2 + a^2$

$\Rightarrow \quad \cos^2\alpha + \cos^2\beta + 2\cos\alpha\cos\beta + \sin^2\alpha + \sin^2\beta + 2\sin\alpha\sin\beta = b^2 + a^2$

$\Rightarrow \quad (\cos^2\alpha + \sin^2\alpha) + (\cos^2\beta + \sin^2\beta) + 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) = a^2 + b^2$

$\Rightarrow \quad 2 + 2\cos(\alpha - \beta) = a^2 + b^2$

or $\cos(\alpha - \beta) = \dfrac{a^2 + b^2 - 2}{2}$

Now, $\tan\dfrac{\alpha - \beta}{2} = \pm\sqrt{\dfrac{1 - \cos(\alpha - \beta)}{1 + \cos(\alpha - \beta)}}$

$$= 4 \pm \sqrt{\frac{1 - \frac{a^2 + b^2 - 2}{2}}{1 + \frac{a^2 + b^2 - 2}{2}}} = \pm\sqrt{\frac{4 - a^2 - b^2}{a^2 + b^2}}$$

Example 2.82 **If $\tan\dfrac{\theta}{2} = \sqrt{\dfrac{a-b}{a+b}}\tan\dfrac{\varphi}{2}$, prove that $\cos\alpha = \dfrac{a\cos\varphi + b}{a + b\cos\varphi}$.**

Sol. Given, $\tan\dfrac{\theta}{2} = \sqrt{\dfrac{a-b}{a+b}}\tan\dfrac{\varphi}{2}$ (i)

Now, $\cos\theta = \dfrac{1 - \tan^2\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}} = \dfrac{1 - \frac{a-b}{a+b}\tan^2\frac{\varphi}{2}}{1 + \frac{a-b}{a+b}\tan^2\frac{\varphi}{2}}$

$$= \frac{1 - \frac{a-b}{a+b}\,\frac{\sin^2\frac{\varphi}{2}}{\cos^2\frac{\varphi}{2}}}{1 + \frac{a-b}{a+b}\,\frac{\sin^2\frac{\varphi}{2}}{\cos^2\frac{\varphi}{2}}}$$

$$= \frac{(a+b)\cos^2\frac{\varphi}{2} - (a-b)\sin^2\frac{\varphi}{2}}{(a+b)\cos^2\frac{\varphi}{2} + (a-b)\sin^2\frac{\varphi}{2}}$$

$$= \frac{a\left(\cos^2\frac{\varphi}{2} - \sin^2\frac{\varphi}{2}\right) + b\left(\cos^2\frac{\varphi}{2} + \sin^2\frac{\varphi}{2}\right)}{a\left(\cos^2\frac{\varphi}{2} + \sin^2\frac{\varphi}{2}\right) + b\left(\cos^2\frac{\varphi}{2} - \sin^2\frac{\varphi}{2}\right)}$$

$$= \frac{a\cos\varphi + b}{a + b\cos\varphi}$$

Example 2.83 If $\cos\theta = \cos\alpha\cos\beta$, prove that $\tan\frac{\theta+\alpha}{2}\tan\frac{\theta-\alpha}{2} = \tan^2\frac{\beta}{2}$.

Sol. Given, $\cos\theta = \cos\alpha\cos\beta$, we have $\cos\beta = \frac{\cos\theta}{\cos\alpha}$ (i)

$$\text{Now, } \tan^2\frac{\beta}{2} = \frac{1-\cos\beta}{1+\cos\beta} = \frac{1-\frac{\cos\theta}{\cos\alpha}}{1+\frac{\cos\theta}{\cos\alpha}}$$

$$= \frac{\cos\alpha - \cos\theta}{\cos\alpha + \cos\theta}$$

$$= \frac{2\sin\frac{\alpha+\theta}{2}\sin\frac{\theta-\alpha}{2}}{2\cos\frac{\theta+\alpha}{2}\cos\frac{\theta-\alpha}{2}} = \tan\frac{\theta+\alpha}{2}\tan\frac{\theta-\alpha}{2}$$

Concept Application Exercise 2.7

1. Prove that $\cot\theta - \tan\theta = 2\cot 2\theta$.
2. Prove that $\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta} = \sec 2\theta - \tan 2\theta$.
3. Prove that $\tan\left(\frac{\pi}{4}+\theta\right) - \tan\left(\frac{\pi}{4}-\theta\right) = 2\tan 2\theta$.
4. Prove that $1 + \tan\theta\tan 2\theta = \sec 2\theta$.
5. Prove that $\frac{1+\sin 2A-\cos 2A}{1+\sin 2A+\cos 2A} = \tan A$.
6. Show that $\frac{1}{\sin 10^\circ} - \frac{\sqrt{3}}{\cos 10^\circ} = 4$.
7. Prove that $\operatorname{cosec} A - 2\cot 2A\cos A = 2\sin A$.
8. Prove that $\frac{1+\sin 2A}{\cos 2A} = \frac{\cos A+\sin A}{\cos A-\sin A} = \tan\left(\frac{\pi}{4}+A\right)$.
9. Prove that $\cos^3\theta\sin 3\theta + \sin^3\theta\cos 3\theta = \frac{3}{4}\sin 4\theta$.
10. Prove that $\tan\theta + \tan(60^\circ+\theta) + \tan(120^\circ+\theta) = 3\tan 3\theta$.
11. If α and β are the two different roots of equation $a\cos\theta + b\sin\theta = c$, prove that

    a. $\tan(\alpha+\beta) = \frac{2ab}{a^2-b^2}$ b. $\cos(\alpha+\beta) = \frac{a^2-b^2}{a^2+b^2}$
12. If $\cos\theta = \frac{\cos\alpha-\cos\beta}{1-\cos\alpha\cos\beta}$, prove that one of the values of $\tan\frac{\theta}{2}$ is $\tan\frac{\alpha}{2}\cot\frac{\beta}{2}$.

13. If $\tan\theta\tan\phi = \sqrt{\frac{(a-b)}{(a+b)}}$, prove that $(a - b\cos 2\theta)(a - b\cos 2\phi)$ is independent of θ and ϕ.

14. If θ is an acute angle and $\sin\frac{\theta}{2} = \sqrt{\frac{x-1}{2x}}$, find $\tan\theta$ in terms of x.

15. Prove that $(1+\sec 2\theta)(1+\sec 4\theta)(1+\sec 8\theta) = \frac{\tan 8\theta}{\tan\theta}$.

16. Prove that $\frac{\sin^2 3A}{\sin^2 A} - \frac{\cos^2 3A}{\cos^2 A} = 8\cos 2A$.

17. If $A = 110°$, then prove that $\frac{1+\sqrt{1+\tan^2 2A}}{\tan 2A} = -\tan A$.

18. In triangle ABC, $a = 3$, $b = 4$ and $c = 5$. Then find the value of $\sin A + \sin 2B + \sin 3C$.

VALUES OF TRIGONOMETRIC RATIOS OF STANDARD ANGLES

1. **Value of $\sin 15°, \cos 15°, \sin 75°, \cos 75°, \tan 15°, \tan 75°$:**

$$\sin 15° = \sin(45° - 30°) = \sin 45°\cos 30° - \sin 30°\cos 45° = \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} - \frac{1}{2}\frac{1}{\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

Also, $\sin 15° = \cos 75° = -\cos 105°$

Similarly, we can prove that $\cos 15° = \frac{\sqrt{3}+1}{2\sqrt{2}}$

Also, $\cos 15° = \sin 75° = \sin 105°$

$$\tan 15° = \tan(60° - 45°) = \frac{\tan 60° - \tan 45°}{1+\tan 60°\tan 45°} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3}$$

$$\tan 75° = \tan(60° + 45°) = \frac{\tan 60° + \tan 45°}{1-\tan 60°\tan 45°} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3}$$

2. **Value of $\sin 18°, \cos 18°$:**

Let $\theta = 18°$, then $5\theta = 90°$

$\Rightarrow\ 2\theta + 3\theta = 90°$

$\Rightarrow\ 2\theta = 90° - 3\theta$

$\Rightarrow\ \sin 2\theta = \sin(90° - 3\theta)$

$\Rightarrow\ \sin 2\theta = \cos 3\theta$

$\Rightarrow\ 2\sin\theta\cos\theta = 4\cos^3\theta - 3\cos\theta$

$\Rightarrow\ 2\sin\theta = 4\cos^2\theta - 3$ [dividing by $\cos\theta$]

$\Rightarrow\ 2\sin\theta = 4(1-\sin^2\theta) - 3 = 1 - 4\sin^2\theta$

$\Rightarrow\ 4\sin^2\theta + 2\sin\theta - 1 = 0$

$\Rightarrow\ \sin\theta = \frac{-2\pm\sqrt{4+16}}{8} = \frac{-2\pm 2\sqrt{5}}{8} = \frac{-1\pm\sqrt{5}}{4}$

$\because\ \theta = 18°$

$\therefore$ $\sin\theta = \sin 18° > 0$, for 18° lies in the first quadrant.

$\therefore$ $\sin\theta$, i.e., $\sin 18° = \dfrac{\sqrt{5}-1}{4}$

Value of cos 18°:

$$\cos^2 18° = 1 - \sin^2 18° = 1 - \left(\frac{\sqrt{5}-1}{4}\right)^2 = 1 - \frac{5+1-2\sqrt{5}}{16} = \frac{10+2\sqrt{5}}{16}$$

$$\Rightarrow \quad \cos 18° = \frac{1}{4}\sqrt{10+2\sqrt{5}} \qquad [\because \cos 18° > 0]$$

3. **Value of cos 36°, sin36°:**

$$\cos 36° = 1 - 2\sin^2 18° = 1 - 2\left(\frac{\sqrt{5}-1}{4}\right)^2 = \frac{\sqrt{5}+1}{4}.$$

Value of sin 36°:

$$\sin^2 36° = 1 - \cos^2 36° = 1 - \left(\frac{\sqrt{5}+1}{4}\right)^2 = 1 - \frac{6+2\sqrt{5}}{16} = \frac{16-6-2\sqrt{5}}{16} = \frac{10-2\sqrt{5}}{16}$$

$$\therefore \quad \sin 36° = \frac{1}{4}\sqrt{10-2\sqrt{5}} \qquad [\because \sin 36° > 0]$$

Note:

- $\sin 54° = \sin(90° - 36°) = \cos 36° = \dfrac{\sqrt{5}+1}{4}$
- $\cos 54° = \cos(90° - 36°) = \sin 36° = \dfrac{1}{4}(\sqrt{10-2\sqrt{5}})$

4. **Value of tan $7\frac{1°}{2}$, cot $7\frac{1°}{2}$:**

Let $\theta = 7\frac{1}{2}°$, then $2\theta = 15°$

$$\tan\theta = \frac{1-\cos 2\theta}{\sin 2\theta} \qquad [\because 1-\cos 2\theta = 2\sin^2\theta \text{ and } \sin 2\theta = 2\sin\theta\cos\theta]$$

$$= \frac{1-\cos 15°}{\sin 15°} = \frac{1-\dfrac{\sqrt{3}+1}{2\sqrt{2}}}{\dfrac{\sqrt{3}-1}{2\sqrt{2}}} = \frac{2\sqrt{2}-\sqrt{3}-1}{\sqrt{3}-1} = (\sqrt{3}-\sqrt{2})(\sqrt{2}-1)$$

Value of cot $82\frac{1°}{2}$:

$$\cot 82\frac{1}{2}° = \cot(90° - 7\frac{1°}{2}) = \tan 7\frac{1°}{2} = (\sqrt{3}-\sqrt{2})(\sqrt{2}-1)$$

Value of cot $7\frac{1°}{2}$:

Let $\theta = 7\frac{1°}{2}$, then $2\theta = 15°$

$$\text{Now, } \cot\theta = \frac{1+\cos 2\theta}{\sin 2\theta} = \frac{1+\cos 15^\circ}{\sin 15^\circ} = \frac{1+\frac{\sqrt{3}+1}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}} = \frac{2\sqrt{2}+\sqrt{3}+1}{\sqrt{3}-1} = (\sqrt{3}+\sqrt{2})(\sqrt{2}+1)$$

Value of $\tan 82\frac{1^\circ}{2}$:

$$\tan 82\frac{1^\circ}{2} = \tan\left(90^\circ - 7\frac{1^\circ}{2}\right) = \cot 7\frac{1^\circ}{2} = (\sqrt{3}+\sqrt{2})(\sqrt{2}+1)$$

All these values are tabulated as follows:

	7.5°	15°	18°	22.5°	36°	67.5°	75°
sin	$\frac{\sqrt{8-2\sqrt{6}-2\sqrt{2}}}{4}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{2-\sqrt{2}}}{2}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$
cos	$\frac{\sqrt{8+2\sqrt{6}+2\sqrt{2}}}{4}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$	$\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{2-\sqrt{2}}}{2}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$
tan	$(\sqrt{3}-\sqrt{2})(\sqrt{2}-1)$	$2-\sqrt{3}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\sqrt{2}-1$	$\sqrt{5-2\sqrt{5}}$	$\sqrt{2}+1$	$2+\sqrt{3}$
cot	$(\sqrt{3}+\sqrt{2})(\sqrt{2}+1)$	$2+\sqrt{3}$	$\sqrt{(5+2\sqrt{5})}$	$\sqrt{2}+1$	$\sqrt{\left(1+\frac{2}{\sqrt{5}}\right)}$	$\sqrt{2}-1$	$2-\sqrt{3}$

Example 2.84 **Find the angle θ whose cosine is equal to its tangent.**

Sol. Given, $\cos\theta = \tan\theta \quad \Rightarrow \quad \cos^2\theta = \sin\theta$

$\Rightarrow \quad 1-\sin^2\theta = \sin\theta$ or $\sin^2\theta + \sin\theta - 1 = 0$

$$\Rightarrow \quad \sin\theta = \frac{-1\pm\sqrt{5}}{2} = 2\ \frac{\sqrt{5}-1}{4} = 2\sin 18^\circ$$

$\Rightarrow \quad \theta = \sin^{-1}(2\sin 18^\circ)$

Example 2.85 **Find the value of $\cos 12^\circ + \cos 84^\circ + \cos 156^\circ + \cos 132^\circ$.**

Sol. $\cos 12^\circ + \cos 84^\circ + \cos 156^\circ + \cos 132^\circ$

$= (\cos 12^\circ + \cos 132^\circ) + (\cos 84^\circ + \cos 156^\circ)$

$$= 2\cos\left(\frac{12^\circ+132^\circ}{2}\right)\cos\left(\frac{132^\circ-12^\circ}{2}\right) + 2\cos\left(\frac{84^\circ+156^\circ}{2}\right)\cos\left(\frac{156^\circ-84^\circ}{2}\right)$$

$= 2\cos 72^\circ \cos 60^\circ + 2\cos 120^\circ \cos 36^\circ$

$= 2\sin 18^\circ \cos 60^\circ + 2\cos 120^\circ \cos 36^\circ$

$$= 2\left(\frac{\sqrt{5}-1}{4}\right)\frac{1}{2} + 2\left(-\frac{1}{2}\right)\left(\frac{\sqrt{5}+1}{4}\right) = -\frac{1}{2}$$

Example 2.86 **Prove that $\cos 36° \cos 72° \cos 108° \cos 144° = \frac{1}{16}$.**

Sol. $\cos 36° \cos 72° \cos 108° \cos 144°$

$$= \cos 36° \sin 18° (-\sin 18°)(-\cos 36°)$$

$$= \cos^2 36° \sin^2 18° = \left(\frac{\sqrt{5}+1}{4}\right)^2 \left(\frac{\sqrt{5}-1}{4}\right)^2$$

$$= \left[\left(\frac{\sqrt{5}+1}{4}\right)\left(\frac{\sqrt{5}-1}{4}\right)\right]^2 = \frac{1}{16}$$

Concept Application Exercise 2.8

1. Prove that $\sin^2 48° - \cos^2 12° = \frac{\sqrt{5}+1}{8}$.
2. Prove that $4(\sin 24° + \cos 6°) = \sqrt{3} + \sqrt{15}$.
3. Find the value of $\sin 47° + \sin 61° - \sin 11° - \sin 25°$.

SUM OF SINES OR COSINES OF *N* ANGLES IN A.P.

$$\sin \alpha + \sin(\alpha + \alpha\beta) + \sin(\alpha + 2\beta) + \cdots + \sin(\alpha + \overline{n-1}\beta) = \frac{\sin \frac{n\beta}{2}}{\sin \frac{\beta}{2}} \times \sin\left[\alpha + (n-1)\frac{\beta}{2}\right]$$

Proof:

Let $S = \sin \alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \cdots + \sin(\alpha + \overline{n-1}\beta)$

Here angle are in A.P. and common difference of angles $= \beta$

Multiplying both sides by $2 \sin \frac{\beta}{2}$, we get

$$2S \sin \frac{\beta}{2} = 2\sin\alpha \sin\frac{\beta}{2} + 2\sin(\alpha+\beta)\sin\frac{\beta}{2} + \cdots + 2\sin(\alpha + \overline{n-1}\beta)\sin\frac{\beta}{2} \quad \text{(i)}$$

Now, $2 \sin \alpha \sin \frac{\beta}{2} = \cos\left(\alpha - \frac{\beta}{2}\right) - \cos\left(\alpha + \frac{\beta}{2}\right)$

$$2\sin(\alpha+\beta)\sin\frac{\beta}{2} = \cos\left(\alpha + \frac{\beta}{2}\right) - \cos\left(\alpha + \frac{3\beta}{2}\right)$$

$$2\sin(\alpha+2\beta)\sin\frac{\beta}{2} = \cos\left(\alpha + \frac{3\beta}{2}\right) - \cos\left(\alpha + \frac{5\beta}{2}\right)$$

$\vdots$

$$2\sin(\alpha + \overline{n-1}\beta)\sin\frac{\beta}{2} = \cos\left[\alpha + (2n-3)\frac{\beta}{2}\right] - \cos\left[\alpha(2n-1)\frac{\beta}{2}\right]$$

Adding, we get R.H.S. of Eq. (*i*) $= \cos\left(\alpha - \frac{\beta}{2}\right) - \cos\left[\alpha + (2n-1)\frac{\beta}{2}\right]$

$$\text{or } 2\sin\frac{\beta}{2}S = 2\sin\left[\alpha+(n-1)\frac{\beta}{2}\right]\sin\frac{n\beta}{2}$$

$$\Rightarrow \quad S = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\sin\left[\alpha+(n-1)\frac{\beta}{2}\right]$$

In the above result replacing α by $\pi/2+\alpha$, we get

$$\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\cdots+\cos(\alpha+\overline{n-1}\beta) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\cos\left[\alpha+(n-1)\frac{\beta}{2}\right]$$

Example 2.87 **Find the value of $\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}$.**

Sol. $S = \cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}$

$$= \frac{\sin\left(3\frac{\pi}{7}\right)}{\sin\left(\frac{\pi}{7}\right)}\cos\left(\frac{\pi}{7}+\frac{3\pi}{7}\right)$$

$$= \frac{2\sin\left(\frac{3\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)}{2\sin\left(\frac{2\pi}{7}\right)} = \frac{\sin\left(\frac{7\pi}{7}\right)-\sin\left(\frac{\pi}{7}\right)}{2\sin\left(\frac{2\pi}{7}\right)} = -\frac{1}{2}$$

Example 2.88 **Prove that $\sin\theta+\sin 3\theta+\sin 5\theta+\cdots+\sin(2n-1)\theta = \frac{\sin^2 n\theta}{\sin\theta}$.**

Sol. $$\sin\theta+\sin 3\theta+\sin 5\theta+\cdots+\sin(2n-1)\theta = \frac{\sin\left[n\left(\frac{2\theta}{2}\right)\right]}{\sin\left(\frac{2\theta}{2}\right)}\sin\left(\frac{\theta+(2n-1)\theta}{2}\right)$$

$$= \frac{\sin^2 n\theta}{\sin\theta}$$

Concept Application Exercise 2.9

1. Find the value of $\cos\frac{\pi}{11}+\cos\frac{3\pi}{11}+\cos\frac{5\pi}{11}+\cos\frac{7\pi}{11}+\cos\frac{9\pi}{11}$.
2. Find the average value of $\sin 2°, \sin 4°, \sin 6°, \ldots, \sin 180°$.
3. Find the value of $\sum_{r=1}^{n-1}\sin^2\frac{r\pi}{n}$.

CONDITIONAL IDENTITIES

Some Standard Identities in Triangle

1. $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

Proof:

In ΔABC, we have $A + B + C = \pi$

$\Rightarrow \quad A + B = \pi - C$

$\Rightarrow \quad \tan (A + B) = \tan (\pi - C)$

$$\Rightarrow \quad \frac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C$$

$\Rightarrow \quad \tan A + \tan B = -\tan C + \tan A \tan B \tan C$

$\Rightarrow \quad \tan A + \tan B + \tan C = \tan A \tan B \tan C$

2. $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{C}{2}\tan\frac{B}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$

Proof:

Since $A + B + C = \pi$, we have $\frac{A}{2} + \frac{B}{2} = \frac{\pi}{2} - \frac{C}{2}$

$$\Rightarrow \tan\left(\frac{A}{2} + \frac{B}{2}\right) = \tan\left(\frac{\pi}{2} - \frac{C}{2}\right) = \cot\frac{C}{2}$$

$$\Rightarrow \frac{\tan\frac{A}{2} + \tan\frac{B}{2}}{1 - \tan\frac{A}{2}\tan\frac{B}{2}} = \frac{1}{\tan\frac{C}{2}}$$

$$\Rightarrow \tan\frac{A}{2}\tan\frac{C}{2} + \tan\frac{B}{2}\tan\frac{C}{2} = 1 - \tan\frac{A}{2}\tan\frac{B}{2}$$

$$\Rightarrow \tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$$

3. $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

Proof:

$$\begin{aligned}(\sin 2A + \sin 2B) + \sin 2C &= 2 \sin (A + B) \cos (A - B) + \sin 2C \\ &= 2 \sin (\pi - C) \cos (A - B) + \sin 2C \\ &= 2 \sin C \cos (A - B) + 2 \sin C \cos C \\ &= 2 \sin C [\cos (A - B) + \cos C] \\ &= 2 \sin C [\cos (A - B) + \cos \{\pi - (A + B)\}] \\ &= 2 \sin C [\cos (A - B) - \cos (A + B)] \\ &= 2 \sin C \times 2 \sin A \sin B = 4 \sin A \sin B \sin C\end{aligned}$$

4. $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$

Proof:

$$
\begin{aligned}
&(\cos 2A + \cos 2B) + \cos 2C \\
&= 2 \cos (A+B) \cos (A-B) + 2\cos^2 C - 1 \\
&= 2 \cos (\pi - C) \cos (A-B) + 2\cos^2 C - 1 \\
&= -2 \cos C \cos (A-B) + 2\cos^2 C - 1 \\
&= -2 \cos C [\cos (A-B) - \cos C] - 1 \\
&= -2\cos C [\cos (A-B) - \cos \{\pi - (A+B)\}] - 1 \\
&= -2\cos C [\cos (A-B) + \cos (A+B)] - 1 \\
&= -1 - 4 \cos A \cos B \cos C
\end{aligned}
$$

5. $\cos A + \cos B + \cos C = 1 + 4 \sin \dfrac{A}{2} \sin \dfrac{B}{2} \sin \dfrac{C}{2}$

Proof:

$$
\begin{aligned}
&(\cos A + \cos B) + \cos C - 1 \\
&= 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2} + \cos C - 1 \\
&= 2\cos\left(\frac{\pi}{2} - \frac{C}{2}\right)\cos \frac{A-B}{2} + \cos C - 1 \\
&= 2\sin \frac{C}{2} \cos \frac{A-B}{2} + 1 - 2\sin^2 \frac{C}{2} - 1 \\
&= 2 \sin \frac{C}{2} \cos \frac{A-B}{2} - 2\sin^2 \frac{C}{2} \\
&= 2\sin \frac{C}{2}\left[\cos \frac{A-B}{2} - \sin \frac{C}{2}\right] \\
&= 2\cos \frac{C}{2}\left[\cos \frac{A-B}{2} - \sin \left(\frac{\pi}{2} - \frac{A+B}{2}\right)\right] \\
&= 2\sin \frac{C}{2}\left[\cos \frac{A-B}{2} - \cos \frac{A+B}{2}\right] \\
&= 2 \sin \frac{C}{2}\left(2 \sin \frac{A}{2} \sin \frac{B}{2}\right) = 4\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}
\end{aligned}
$$

6. $\sin A + \sin B + \sin C = 4 \cos \dfrac{A}{2} \cos \dfrac{B}{2} \cos \dfrac{C}{2}$

Proof:

$$
\begin{aligned}
&(\sin A + \sin B) + \sin C \\
&= 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} + \sin C \\
&= 2\sin\left(\frac{\pi}{2} - \frac{C}{2}\right)\cos \frac{A-B}{2} + \sin C \\
&= 2\cos \frac{C}{2} \cos \frac{A-B}{2} + 2\sin \frac{C}{2} \cos \frac{C}{2}
\end{aligned}
$$

$$= 2\cos\frac{C}{2}\left[\cos\frac{A-B}{2} + \sin\frac{C}{2}\right]$$

$$= 2\cos\frac{C}{2}\left[\cos\frac{A-B}{2} + \sin\left(\frac{\pi}{2} - \frac{A+B}{2}\right)\right]$$

$$= 2\cos\frac{C}{2}\left[\cos\frac{A-B}{2} + \cos\frac{A+B}{2}\right]$$

$$= 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

Note:

$\tan A + \tan B + \tan C = \tan A \tan B \tan C$ *is true for* $A + B + C = n\pi$, *where* $n \in N$.

Example 2.89 **If $A + B + C = 180°$, prove that $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A\cos B\cos C$.**

Sol. $\cos^2 A + \cos^2 B + \cos^2 C = \frac{1+\cos 2A}{2} + \frac{1+\cos 2B}{2} + \frac{1+\cos 2C}{2}$

$$= \frac{1}{2}(\cos 2A + \cos 2B + \cos 2C) + \frac{3}{2}$$

$$= \frac{1}{2}(-1 - 4\cos A\cos B\cos C) + \frac{3}{2}$$

$$= 1 - 2\cos A\cos B\cos C$$

Example 2.90 **Prove that in triangle ABC, $\cos^2 A + \cos^2 B - \cos^2 C = 1 - 2\sin A\sin B\cos C$.**

Sol. $\cos^2 A + \cos^2 B - \cos^2 C = \cos^2 A + \sin^2 C - \sin^2 B$

$$= \cos^2 A + \sin(C+B)\sin(C-B)$$

$$= 1 - \sin^2 A + \sin A\sin(C-B)$$

$$= 1 - \sin A[\sin A - \sin(C-B)]$$

$$= 1 - \sin A[\sin(B+C) - \sin(C-B)]$$

$$= 1 - 2\sin A\sin B\cos C$$

Example 2.91 **In triangle ABC, prove that**

$\sin(B+C-A) + \sin(C+A-B) + \sin(A+B-C) = 4\sin A\sin B\sin C$.

Sol. $\sin(B+C-A) + \sin(C+A-B) + \sin(A+B-C) = \sin(\pi-2A) + \sin(\pi-2B) + \sin(\pi-2C)$

$$= \sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C$$

Example 2.92 If $x+y+z=xyz$, prove that $\frac{2x}{1-x^2}+\frac{2y}{1-y^2}+\frac{2z}{1-z^2}=\frac{2x}{1-x^2}\frac{2y}{1-y^2}\frac{2z}{1-z^2}$.

Sol. Let $x=\tan A$, $y=\tan B$, $z=\tan C$

Now $x+y+z=xyz$

$\Rightarrow$ $\tan A+\tan B+\tan C=\tan A\tan B\tan C$

$\Rightarrow$ $A+B+C=n\pi$

$\Rightarrow$ $2A+2B+2C=2n\pi$

$\Rightarrow$ $\tan 2A+\tan 2B+\tan 2C=\tan 2A\tan 2B\tan 2C$

$$\Rightarrow \frac{2\tan A}{1-\tan^2 A}+\frac{2\tan B}{1-\tan^2 B}+\frac{2\tan C}{1-\tan^2 C}=\frac{2\tan A}{1-\tan^2 A}\frac{2\tan B}{1-\tan^2 B}\frac{2\tan C}{1-\tan^2 C}$$

$$\Rightarrow \frac{2x}{1-x^2}+\frac{2y}{1-y^2}+\frac{2z}{1-z^2}=\frac{2x}{1-x^2}\frac{2y}{1-y^2}\frac{2z}{1-z^2}$$

Example 2.93 If $A+B+C=\pi$, prove that $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}-\sin^2\frac{C}{2}=1-2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$.

Sol.
$$\sin^2\frac{A}{2}-\sin^2\frac{C}{2}+\sin^2\frac{B}{2}=\sin\left(\frac{A+C}{2}\right)\sin\left(\frac{A-C}{2}\right)+1-\cos^2\frac{B}{2}$$
$$=\cos\left(\frac{B}{2}\right)\sin\left(\frac{A-C}{2}\right)-\cos^2\frac{B}{2}+1$$
$$=\cos\left(\frac{B}{2}\right)\left[\sin\left(\frac{A-C}{2}\right)-\cos\frac{B}{2}\right]+1$$
$$=\cos\left(\frac{B}{2}\right)\left[\sin\left(\frac{A-C}{2}\right)-\sin\left(\frac{A+C}{2}\right)\right]+1$$
$$=1-2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$$

Example 2.94 **The product of the sines of the angles of a triangle is p and the product of their cosines is q. Show that the tangents of the angles are the roots of the equation $qx^3-px^2+(1+q)x-p=0$.**

Sol. From the question, $\sin A\sin B\sin C=p$ and $\cos A\cos B\cos C=q$

$$\therefore \tan A\tan B\tan C=\frac{p}{q} \qquad \text{(i)}$$

$$\text{Also, } \tan A+\tan B+\tan C=\tan A\tan B\tan C=\frac{p}{q} \qquad \text{(ii)}$$

Now, $\tan A\tan B+\tan B\tan C+\tan C\tan A$

$$=\frac{\sin A\sin B\cos C+\sin B\sin C\cos A+\sin C\sin A\cos B}{\cos A\cos B\cos C}$$

$$=\frac{1}{2q}[(\sin^2A+\sin^2B-\sin^2C)+(\sin^2B+\sin^2C\ \sin^2A)+\sin^2C+\sin^2A-\sin^2B)]$$

[$\because A+B+C=\pi$ and $2\sin A\sin B\cos C=\sin^2A+\sin^2B-\sin^2C$]

$$= \frac{1}{2q}[\sin^2 A + \sin^2 B + \sin^2 C] = \frac{1}{4q}[3 - (\cos 2A + \cos 2B + \cos 2C)] = \frac{1}{q}[1 + \cos A \cos B \cos C] = \frac{1}{q}(1 + q)$$

The equation whose roots are $\tan A$, $\tan B$ $\tan C$ will be given by

$$x^3 - (\tan A + \tan B + \tan C)x^2 + (\tan A \tan B + \tan B \tan C + \tan C \tan A)x - \tan A \tan B \tan C = 0$$

or $x^3 - \frac{p}{q}x^2 + \frac{1+q}{q}x - \frac{p}{q} = 0$, or $qx^3 - px^2 + (1+q)x - p = 0$

Concept Application Exercise 2.10

1. In triangle ABC, prove that
   a. $\cos^2\frac{A}{2} + \cos^2\frac{B}{2} - \cos^2\frac{C}{2} = 2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$
   b. $\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} = 2 + 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$
2. If $A + B + C = \pi/2$, show that
   a. $\sin^2 A + \sin^2 B + \sin^2 C = 1 - 2\sin A \sin B \sin C$
   b. $\cos^2 A + \cos^2 B + \cos^2 C = 2 + 2\sin A \sin B \sin C$
3. a. If $A + B = C$, prove that $\cos^2 A + \cos^2 B + \cos^2 C = 1 + 2\cos A \cos B \cos C$.
   b. If $\alpha + \beta = 60°$, prove that $\cos^2\alpha + \cos^2\beta - \cos\alpha\cos\beta = 3/4$.
4. Prove that $\cos^2(\beta - \gamma) + \cos^2(\gamma - \alpha) + \cos^2(\alpha - \beta) = 1 + 2\cos(\beta - \alpha)\cos(\gamma - \alpha)\cos(\alpha - \beta)$.
5. If $A + B + C = \pi/2$, show that
   a. $\cot A + \cot B + \cot C = \cot A \cot B \cot C$
   b. $\tan A \tan B + \tan B \tan C + \tan C \tan A = 1$
6. If $A + B + C = \pi$, prove that
   a. $\tan 3A + \tan 3B + \tan 3C = \tan 3A \tan 3B \tan 3C$
   b. $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$
7. If $A + B + C = \pi$, prove that $\frac{\cos A}{\sin B \sin C} + \frac{\cos B}{\sin C \sin A} + \frac{\cos C}{\sin A \sin B} = 2$.

SOME IMPORTANT RESULTS AND THEIR APPLICATIONS

Result 1. $\cos A \cos(60 - A)\cos(60 + A) = \frac{1}{4}\cos 3A$

Proof:

We have

L.H.S. $= \cos A \cos(60 - A)\cos(60 + A)$

$= \cos A(\cos^2 60° - \sin^2 A)$ $\qquad [\because \cos(A + B)\cos(A - B) = \cos^2 A - \sin^2 B]$

$= \cos A\left(\frac{1}{4} - \sin^2 A\right) = \cos A\left(\frac{1}{4} - (1 - \cos^2 A)\right) = \cos A\left(-\frac{3}{4} + \cos^2 A\right)$

$= \frac{1}{4}\cos A\left(-3 + 4\cos^2 A\right) = \frac{1}{4}\left(4\cos^3 A - 3\cos A\right)$

$= \frac{1}{4}\cos 3A = $ R.H.S.

Result 2. $\sin A \sin(60-A)\sin(60+A) = \frac{1}{4}\sin 3A$

Proof:

We have

$$\text{L.H.S.} = \sin A \sin(60-A)\sin(60+A)$$

$$= \sin A(\sin^2 60^\circ - \sin^2 A) \qquad [\because \sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B]$$

$$= \sin A\left(\frac{3}{4} - \sin^2 A\right) = \frac{1}{4}\sin A\left(3 - 4\sin^2 A\right)$$

$$= \frac{1}{4}\left(3\sin A - 4\sin^3 A\right)$$

$$= \frac{1}{4}\sin 3A = \text{R.H.S.}$$

Result 3. $\tan\alpha\tan(60^\circ-\alpha)\tan(60^\circ+\alpha) = \tan 3\alpha$

Using the above two results, we can prove this result

Example 2.95 **Prove that $\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ = \frac{1}{16}$.**

Sol. $\cos 20^\circ \cos 40^\circ \cos 80^\circ \cos 60^\circ = \cos 20^\circ \cos(60^\circ - 20^\circ)\cos(60^\circ + 20^\circ)\cos 60^\circ$

$$= \frac{1}{4}\cos(3\times 20^\circ)\ \cos 60^\circ = \frac{1}{4}\cos^2 60^\circ = \frac{1}{4}\times\frac{1}{4} = \frac{1}{16}$$

Example 2.96 **Prove that $\sin 10^\circ \sin 30^\circ \sin 50^\circ \sin 70^\circ = \frac{1}{16}$.**

Sol. $\sin 10^\circ \sin 30^\circ \sin 50^\circ \sin 70^\circ = \sin 10^\circ \sin(60^\circ - 10^\circ)\sin(60^\circ + 10^\circ)\sin 30^\circ$

$$= \frac{1}{4}\sin(3\times 10^\circ)\ \sin 30^\circ = \frac{1}{4}\sin^2 30^\circ = \frac{1}{16}$$

Example 2.97. **Prove that $\tan 20^\circ \tan 40^\circ \tan 80^\circ = \tan 60^\circ$.**

Sol. $\tan 20^\circ \tan 40^\circ \tan 80^\circ$

$= \tan 20^\circ \tan(60^\circ - 20^\circ)\tan(60^\circ + 20^\circ) = \tan(3\times 20^\circ) = \tan 60^\circ$

Result 4. $\cos A \cos 2A \cos 2^2 A \cos 2^3 A \cdots \cos 2^{n-1} A = \dfrac{\sin 2^n A}{2^n \sin A}$

Proof: $\text{L.H.S.} = \cos A \cos 2A \cos 2^2 A \cos 2^3 A \cdots \cos 2^{n-1} A$

$$= \frac{1}{2\sin A}\left[(2\sin A\cos A)\cos 2A\cos 2^2A\cos 2^3A\cdots\cos 2^{n-1}A\right]$$

$$= \frac{1}{2\sin A}\left[(\sin 2A\cos 2A\cos 2^2A\cos 2^3A\cdots\cos 2^{n-1}A\right]$$

$$= \frac{1}{2^2\sin A}\left[(2\sin 2A\cos 2A)\cos 2^2A\cos 2^3A\cdots\cos 2^{n-1}A\right]$$

$$= \frac{1}{2^2\sin A}\left[\sin 2(2A).\ \cos 2^2A\cos 2^3A\cdots\cos 2^{n-1}A\right]$$

$$= \frac{1}{2^3\sin A}\left[(2\sin 2^2A\cos 2^2A)\cos 2^3A\cdots\cos 2^{n-1}A\right]$$

$$= \frac{1}{2^3 \sin A}[\sin(2\times 2^2 A)\cos 2^3 A \cdots \cos 2^{n-1} A]$$

$$= \frac{1}{2^3 \sin A}[(\sin 2^3 A \cos 2^3 A \cos 2^4 A \cdots \cos 2^{n-1} A]$$

...

...

$$= \frac{1}{2^{n-1} \sin A}[\sin 2^{n-1} A \cos 2^{n-1} A]$$

$$= \frac{1}{2^n \sin A}[2 \sin 2^{n-1} A \cos 2^{n-1} A]$$

$$= \frac{1}{2^n \sin A}\sin(2\times 2^{n-1} A)$$

$$= \frac{1}{2^n \sin A}\sin 2^n A = \text{R.H.S.}$$

Example 2.98 **If $\theta = \dfrac{\pi}{2^n + 1}$, show that $\cos\theta \cos 2\theta\, 2\cos 2^2\theta \cdots \cos 2^{n-1}\theta = \dfrac{1}{2^n}$.**

Sol. In the above result, put $\theta = \dfrac{\pi}{2^n + 1}$

$$\Rightarrow \text{R.H.S.} = \frac{\sin 2^n\theta}{2^n \sin\theta} = \frac{\sin\left(\frac{\pi}{2^n+1}\right)2^n}{2^n \sin\left(\frac{\pi}{2^n+1}\right)} = \frac{\sin\left(\frac{2^n+1-1}{2^n+1}\right)\pi}{2^n \sin\left(\frac{\pi}{2^n+1}\right)}$$

$$= \frac{\sin\left(\pi - \frac{\pi}{2^n+1}\right)}{2^n \sin\left(\frac{\pi}{2^n+1}\right)}$$

$$= \frac{\sin\left(\frac{\pi}{2^n+1}\right)}{2^n \sin\left(\frac{\pi}{2^n+1}\right)}.$$

$$= \frac{1}{2^n}.$$

Example 2.99 **Prove that $\cos\dfrac{2\pi}{15}\cos\dfrac{4\pi}{15}\cos\dfrac{8\pi}{15}\cos\dfrac{14\pi}{15} = \dfrac{1}{16}$.**

Sol. We have

$$\text{L.H.S.} = \cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\left(\pi - \frac{\pi}{15}\right)$$

$$= \left(\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\right)\left(-\cos\frac{\pi}{15}\right)$$

$$= -\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}$$

$$= -\cos A\cos 2A\cos 2^2 A\cos 2^3 A, \text{ where } A = \pi/15$$

$$= -\left[\frac{\sin 2^4 A}{2^4\sin A}\right] = -\frac{\sin 16A}{2^4\sin A}$$

$$= -\frac{\sin(15A + A)}{16\sin A} = \frac{-\sin(\pi + A)}{16\sin A} \qquad [\because 15A = \pi]$$

$$= \frac{\sin A}{16\sin A} = \frac{1}{16} = \frac{1}{16} = \text{R.H.S.}$$

Example 2.100 **Prove that $\sin 6° \sin 42° \sin 66° \sin 78° = \frac{1}{16}$.**

Sol. $\sin 6° \sin 42° \sin 66° \sin 78°$

$= \sin 6° \cos 48° \cos 24° \cos 12°$

$$= \sin 6° \frac{2^3 \sin 12° \cos 12° \cos 24° \cos 48°}{2^3 \sin 12°}$$

$$= \sin 6° \frac{\sin 96°}{2^3 \sin 12°}$$

$$= \frac{2\sin 6° \cos 6°}{2^4 \sin 12°} = \frac{\sin 12°}{2^4 \sin 12°} = \frac{1}{16}$$

Concept Application Exercise 2.11

1. If $\alpha = \frac{\pi}{15}$, prove that $\cos 2\alpha \cos 4\alpha \cos 8\alpha \cos 14\alpha = \frac{1}{16}$.
2. Prove that $\sin 20° \sin 40° \sin 60° \sin 80° = \frac{3}{16}$.
3. Prove that $\cos 10° \cos 30° \cos 50° \cos 70° = \frac{3}{16}$.

IMPORTANT INEQUALITIES

Example 2.101 **In ΔABC, $\tan A + \tan B + \tan C \geq 3\sqrt{3}$, where A, B, C are acute angles.**

Sol. In ΔABC,

$\tan A + \tan B + \tan C = \tan A \tan B \tan C$

Also, $\frac{\tan A + \tan B + \tan C}{3} \geq \sqrt[3]{\tan A \tan B \tan C}$ [since A.M. $\geq$ G.M.]

$\Rightarrow \quad \tan A \tan B \tan C \geq \sqrt[3]{\tan A \tan B \tan C}$

$\Rightarrow \quad \tan^2 A \tan^2 B \tan^2 C \geq 27$

$\Rightarrow \quad \tan A \tan B \tan C \geq 3\sqrt{3}$ [cubing both sides]

$\Rightarrow \quad \tan A + \tan B + \tan C \geq 3\sqrt{3}$

Example 2.102 **In ΔABC, prove that $\cos A + \cos B + \cos C \leq 3/2$.**

Sol. Let $\cos A + \cos B + \cos C = x$

$$\Rightarrow \quad 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)+1-2\sin^2\frac{C}{2}=x$$

$$\Rightarrow \quad 2\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right)+1-2\sin^2\frac{C}{2}=x$$

$$\Rightarrow \quad 2\sin^2\frac{C}{2}-2\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right)+x-1=0$$

This is quadratic in $\sin C/2$ which is real. So, discriminant $D \geq 0$.

$$4\cos^2\left(\frac{A-B}{2}\right)-4\times 2(x-1)\geq 0$$

$$\Rightarrow \quad 2(x-1)\leq \cos^2\left(\frac{A-B}{2}\right)$$

$\Rightarrow \quad 2(x-1)\leq 1$

$\Rightarrow \quad x \leq 3/2$

Thus, $\cos A + \cos B + \cos C \leq 3/2$

Note: Since $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

We have $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\leq\frac{1}{8}$.

Students are advised to remember this as a standard result.

Example 2.103 **Find the least value of $\sec A + \sec B + \sec C$ in an acute angle triangle.**

Sol. In an acute angle triangle, $\sec A$, $\sec B$ and $\sec C$ are positive.

Now A.M. $\geq$ H.M.

$$\Rightarrow \quad \frac{\sec A+\sec B+\sec C}{3}\geq\frac{3}{\cos A+\cos B+\cos C}$$

But in ΔABC, $\cos A + \cos B + \cos C \leq 3/2$

$$\Rightarrow \quad \frac{\sec A+\sec B+\sec C}{3}\geq 2$$

$\Rightarrow \quad \sec A + \sec B + \sec C \geq 6$

Example 2.104 In ΔABC, prove that $\operatorname{cosec}\frac{A}{2}+\operatorname{cosec}\frac{B}{2}+\operatorname{cosec}\frac{C}{2}\ge 6$.

Sol. In ΔABC, we know that $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\le\frac{1}{8}$

Now A.M. ≥ G.M.

$$\Rightarrow \frac{\operatorname{cosec}\frac{A}{2}+\operatorname{cosec}\frac{B}{2}+\operatorname{cosec}\frac{C}{2}}{3}\ge\left(\operatorname{cosec}\frac{A}{2}\operatorname{cosec}\frac{B}{2}\operatorname{cosec}\frac{C}{2}\right)^{1/3}$$

$$\Rightarrow \frac{\operatorname{cosec}\frac{A}{2}+\operatorname{cosec}\frac{B}{2}+\operatorname{cosec}\frac{C}{2}}{3}\ge\left(\frac{1}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}\right)^{1/3}$$

$$\Rightarrow \frac{\operatorname{cosec}\frac{A}{2}+\operatorname{cosec}\frac{B}{2}+\operatorname{cosec}\frac{C}{2}}{3}\ge(8)^{1/3}$$

$$\Rightarrow \operatorname{cosec}\frac{A}{2}+\operatorname{cosec}\frac{B}{2}+\operatorname{cosec}\frac{C}{2}\ge 6$$

EXERCISES

Subjective Type

Solutions on page 2.85

1. Are the set of angles α and β given by $\alpha=\left(2n+\frac{1}{2}\right)\pi\pm A$ and $\beta=m\pi+(-1)^m\left(\frac{\pi}{2}-A\right)$ same, where $n, m\in I$?
2. If ABC is a triangle and $\tan\frac{A}{2}$, $\tan\frac{B}{2}$, $\tan\frac{C}{2}$ are in H.P., then find the minimum value of $\cot B/2$.
3. Find the sum of the series $\operatorname{cosec}\theta+\operatorname{cosec}2\theta+\operatorname{cosec}4\theta+\cdots$ to n terms.
4. In ΔABC, if $\sin^3\theta=\sin(A-\theta)\sin(B-\theta)\sin(C-\theta)$, prove that $\cot\theta=\cot A+\cot B+\cot C$.
5. In triangle ABC, prove that $\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\le\frac{3}{2}$. Hence, deduce that

   $\cos\frac{\pi+A}{4}\cos\frac{\pi+B}{4}\cos\frac{\pi+C}{4}\le\frac{1}{8}$.
6. If $\frac{x}{\tan(\theta+\alpha)}=\frac{y}{\tan(\theta+\beta)}=\frac{z}{\tan(\theta+\gamma)}$, then show that $\sum\frac{x+y}{x-y}\sin^2(\alpha-\beta)=0$.
7. If $\tan 6\theta=p/q$, find the value of $\frac{1}{2}(p\operatorname{cosec}2\theta-q\sec 2\theta)$ in terms of p and q.
8. If $0<\alpha<\pi/2$ and $\sin\alpha+\cos\alpha+\tan\alpha+\cot\alpha+\sec\alpha+\operatorname{cosec}\alpha=7$, then prove that $\sin 2\alpha$ is a root of the equation $x^2-44x-36=0$.
9. Prove that $1+\cot\theta\le\cot\frac{\theta}{2}$ for $0<\theta<\pi$. Find θ when equality sign holds.
10. Show that $2^{\sin x}+2^{\cos x}\ge 2^{1-1/\sqrt{2}}$.
11. If A, B and C are the angles of a triangle, show that $\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\ge 1$.

12. Let A, B, C be three angles such that $A = \pi/4$ and $\tan B \tan C = p$. Find all possible values of p such that A, B, C are the angles of a triangle.

13. Eliminate x from the equations, $\sin(a + x) = 2b$ and $\sin(a - x) = 2c$.

14. If $\tan\beta = \dfrac{n\sin\alpha\cos\alpha}{1 - n\sin^2\alpha}$, prove that $\tan(\alpha - \beta) = (1 - n)\tan\alpha$.

15. Show that $\dfrac{\sin x}{\cos 3x} + \dfrac{\sin 3x}{\cos 9x} + \dfrac{\cos 9x}{\cos 27x} = \dfrac{1}{2}[\tan 27x - \tan x]$.

16. Prove that $\dfrac{2\cos 2^n\theta + 1}{2\cos\theta + 1} = (2\cos\theta - 1)(2\cos 2\theta - 1)(2\cos 2^2\theta - 1)\cdots(2\cos 2^{n-1}\theta - 1)$.

17. Prove that $\dfrac{\tan 2^n\theta}{\tan\theta} = (1 + \sec 2\theta)(1 + \sec 2^2\theta)(1 + \sec 2^3\theta)\cdots(1 + \sec 2^n\theta)$.

Objective Type

Solutions on page 2.92

Each question has four choices a, b, c, and d, out of which *only one* answer is correct. Find the correct answer.

1. Which of the following is correct?

   a. $\sin 1° > \sin 1$ b. $\sin 1° < \sin 1$ c. $\sin 1° = \sin 1$ d. $\sin 1° = \dfrac{\pi}{180}\sin 1$

2. The equation $\sin^2\theta = \dfrac{x^2 + y^2}{2xy}$ is possible if

   a. $x = y$ b. $x = -y$ c. $2x = y$ d. none of these

3. If $1 + \sin x + \sin^2 x + \sin^3 x + \cdots \infty$ is equal to $4 + 2\sqrt{3}$, $0 < x < \pi$, then x is equal to

   a. $\dfrac{\pi}{6}$ b. $\dfrac{\pi}{4}$ c. $\dfrac{\pi}{3}$ or $\dfrac{\pi}{6}$ d. $\dfrac{\pi}{3}$ or $\dfrac{2\pi}{3}$

4. If $\dfrac{\sin x}{a} = \dfrac{\cos x}{b} = \dfrac{\tan x}{c} = k$, then $bc + \dfrac{1}{ck} + \dfrac{ak}{1 + bk}$ is equal to

   a. $k\left(a + \dfrac{1}{a}\right)$ b. $\dfrac{1}{k}\left(a + \dfrac{1}{a}\right)$ c. $\dfrac{1}{k^2}$ d. $\dfrac{a}{k}$

5. If A, B, C are angles of a triangle, then $2\sin\dfrac{A}{2}\operatorname{cosec}\dfrac{B}{2}\sin\dfrac{C}{2} - \sin A\cot\dfrac{B}{2} - \cos A$ is

   a. independent of A, B, C b. function of A, B
   c. function of C d. none of these

6. The least value of $6\tan^2\phi + 54\cot^2\phi + 18$ is

   I: 54 when A.M. ≥ G.M. is applicable for $6\tan^2\phi$, $54\cot^2\phi$, 18.
   II: 54 when A.M. ≥ G.M is applicable for $6\tan^2\phi$, $54\cot^2\phi$ and 18 added further.
   III: 78 when $\tan^2\phi = \cot^2\phi$.

   a. I is correct b. I and II are correct
   c. III is correct d. none of the above is correct

7. If 5 tan $\theta = 4$, then $\frac{5\sin\theta - 3\cos\theta}{5\sin\theta + 2\cos\theta}$ is equal to

a. 0 b. 1 c. 1/6 d. 6

8. If 2 sec $2\theta = \tan\phi + \cot\phi$, then one of the values of $\theta + \phi$ is

a. $\pi/2$ b. $\pi/4$ c. $\pi/3$ d. none of these

9. If $\sin x + \operatorname{cosec} x = 2$, then $\sin^n x + \operatorname{cosec}^n x$ is equal to

a. 2 b. 2^n c. 2^{n-1} d. 2^{n-2}

10. A quadratic equation whose roots are $\operatorname{cosec}^2\theta$ and $\sec^2\theta$ can be

a. $x^2 - 5x + 2 = 0$ b. $x^2 - 3x + 6 = 0$ c. $x^2 - 5x + 5 = 0$ d. none of these

11. If $\pi < \alpha < \frac{3\pi}{2}$, then $\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} + \sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}$ is equal to

a. $\frac{2}{\sin\alpha}$ b. $-\frac{2}{\sin\alpha}$ c. $\frac{1}{\sin\alpha}$ d. $-\frac{1}{\sin\alpha}$

12. The value of $\cos\frac{\pi}{7} + \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{5\pi}{7} + \cos\frac{6\pi}{7} + \cos\frac{7\pi}{7}$ is

a. 1 b. –1 c. 0 d. none of these

13. The least value of $2\sin^2\theta + 3\cos^2\theta$ is

a. 1 b. 2 c. 3 d. 5

14. The greatest value of $\sin^4\theta + \cos^4\theta$ is

a. 1/2 b. 1 c. 2 d. 3

15. If $f(x) = \cos^2\theta + \sec^2\theta$, then

a. $f(x) < 1$ b. $f(x) = 1$ c. $2 > f(x) > 1$ d. $f(x) \geq 2$

16. If $f(x) = \sin^6 x + \cos^6 x$, then range of $f(x)$ is

a. $\left[\frac{1}{4}, 1\right]$ b. $\left[\frac{1}{4}, \frac{3}{4}\right]$ c. $\left[\frac{3}{4}, 1\right]$ d. none of these

17. If $a \leq 3\cos x + 5\sin(x - \pi/6) \leq b$ for all x, then (a, b) is

a. $(-\sqrt{19}, \sqrt{19})$ b. $(-17, 17)$ c. $(-\sqrt{21}, \sqrt{21})$ d. none of these

18. The equation $\sin x(\sin x + \cos x) = k$ has real solutions if and only if k is a real number such that

a. $0 \leq k \leq \frac{1+\sqrt{2}}{2}$ b. $2-\sqrt{3} \leq k \leq 2+\sqrt{3}$ c. $0 \leq k \leq 2-\sqrt{3}$ d. $\frac{1-\sqrt{2}}{2} \leq k \leq \frac{1+\sqrt{2}}{2}$

19. If $\cot(\alpha + \beta) = 0$, then $\sin(\alpha + 2\beta)$ can be

a. $-\sin\alpha$ b. $\sin\beta$ c. $\cos\alpha$ d. $\cos\beta$

20. If $\frac{x}{\cos\theta} = \frac{y}{\cos\left(\theta - \frac{2\pi}{3}\right)} = \frac{z}{\cos\left(\theta + \frac{2\pi}{3}\right)}$, then $x + y + z$ is equal to

a. 1 b. 0 c. –1 d. none of these

21. $\sin^{2n} x + \cos^{2n} x$ lies between

a. – 1 and 1 b. 0 and 1 c. 1 and 2 d. none of these

22. The roots of the equation $4x^2 - 2\sqrt{5}\,x + 1 = 0$ are

a. sin 36°, sin 18° b. sin 18°, cos 36° c. sin 36°, cos 18° d. cos 18°, cos 36°

23. If $\frac{3\pi}{4} < \alpha < \pi$, then $\sqrt{2\cot\alpha + \frac{1}{\sin^2\alpha}}$ is equal to

a. $1 + \cot\alpha$ b. $-1 - \cot\alpha$ c. $1 - \cot\alpha$ d. $-1 + \cot\alpha$

24. If $f(\theta) = 5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3$, then range of $f(\theta)$ is

a. $[-5, 11]$ b. $[-3, 9]$ c. $[-2, 10]$ d. $[-4, 10]$

25. If $\alpha, \beta, \gamma, \delta$ are the smallest positive angles in ascending order of magnitude which have their sines equal to the positive quantity k, then the value of $4\sin\frac{\alpha}{2} + 3\sin\frac{\beta}{2} + 2\sin\frac{\gamma}{2} + \sin\frac{\delta}{2}$ is equal to

a. $2\sqrt{1-k}$ b. $2\sqrt{1+k}$ c. $\frac{\sqrt{1+k}}{2}$ d. none of these

26. Let $A_0A_1A_2A_3A_4A_5$ be a regular hexagon inscribed in a circle of unit radius. Then the product of the lengths of the line segments A_0A_1, A_0A_2 and A_0A_4 is

a. 3/4 b. $3\sqrt{3}$ c. 3 d. $3\sqrt{3}/2$

27. If $\sin\theta_1 + \sin\theta_2 + \sin\theta_3 = 3$, then $\cos\theta_1 + \cos\theta_2 + \cos\theta_3$ is equal to

a. 3 b. 2 c. 1 d. 0

28. If $\sin^2\theta = \frac{x^2 + y^2 + 1}{2x}$, then x must be

a. -3 b. -2 c. 1 d. none of these

29. If $\sin(\alpha+\beta) = 1$, $\sin(\alpha-\beta) = \frac{1}{2}$, then $\tan(\alpha+2\beta)\tan(2\alpha+\beta)$ is equal to, $\alpha, \beta \in (0, \pi/2)$

a. 1 b. -1 c. 0 d. none of these

30. Which of the following is not the value of $\sin 27° - \cos 27°$?

a. $-\frac{\sqrt{3-\sqrt{5}}}{2}$ b. $-\frac{\sqrt{5-\sqrt{5}}}{2}$ c. $-\frac{\sqrt{5}-1}{2\sqrt{2}}$ d. none of these

31. If $\operatorname{cosec}\theta - \cot\theta = q$, then the value of $\operatorname{cosec}\theta$ is

a. $q + \frac{1}{q}$ b. $q - \frac{1}{q}$ c. $\frac{1}{2}\left(q + \frac{1}{q}\right)$ d. none of these

32. If $\sin\theta + \cos\theta = \frac{1}{5}$ and $0 \le \theta < \pi$, then $\tan\theta$ is

a. $-4/3$ b. $-3/4$ c. 3/4 d. 4/3

33. If $x = \frac{2\sin\theta}{1+\cos\theta+\sin\theta}$, then $\frac{1-\cos\theta+\sin\theta}{1+\sin\theta}$ is equal to

a. $1+x$ b. $1-x$ c. x d. $1/x$

34. If $\theta = \pi/4n$, then the value of $\tan\theta\tan 2\theta \cdots \tan(2n-2)\theta\tan(2n-1)\theta$ is

a. -1 b. 1 c. 0 d. 2

35. The value of the expression $\frac{2(\sin 1° + \sin 2° + \sin 3° + \cdots + \sin 89°)}{2(\cos 1° + \cos 2° + \cdots + \cos 44°) + 1}$ equals

a. $\sqrt{2}$ b. $1/\sqrt{2}$ c. 1/2 d. 1

36. If $\sec\alpha$ and $\operatorname{cosec}\alpha$ are the roots of $x^2 - px + q + 0$, then

a. $p^2 = q(q-2)$ b. $p^2 = q(q+2)$ c. $p^2 + q^2 = 2q$ d. none of these

37. If $\sin x + \sin^2 x = 1$, then the value of $\cos^{12} x + 3\cos^{10} x + 3\cos^8 x + \cos^6 x - 2$ is equal to

a. 0 b. 1 c. −1 d. 2

38. If $\cos(A - B) = 3/5$ and $\tan A \tan B = 2$, then

a. $\cos A \cos B = 1/5$ b. $\sin A \sin B = -2/5$ c. $\cos A \cos B = -1/5$ d. $\sin A \sin B = -1/5$

39. If $(1 + \tan\alpha)(1 + \tan 4\alpha) = 2$, $\alpha \in (0, \pi/16)$ then α is equal to

a. $\frac{\pi}{20}$ b. $\frac{\pi}{30}$ c. $\frac{\pi}{40}$ d. $\frac{\pi}{60}$

40. If $A = \sin 45° + \cos 45°$ and $B = \sin 44° + \cos 44°$, then

a. $A > B$ b. $A < B$ c. $A = B$ d. none of these

41. $\frac{1}{4}\left[\sqrt{3}\cos 23° - \sin 23°\right]$ is equal to

a. $\cos 43°$ b. $\cos 7°$ c. $\cos 53°$ d. none of these

42. If $\cos\theta_1 = 2\cos\theta_2$, then $\tan\frac{\theta_1 - \theta_2}{2}\tan\frac{\theta_1 + \theta_2}{2}$ is equal to

a. $\frac{1}{3}$ b. $-\frac{1}{3}$ c. 1 d. −1

43. Value of $\frac{3 + \cot 80° \cot 20°}{\cot 80° + \cot 20°}$ is equal to

a. $\cot 20°$ b. $\tan 50°$ c. $\cot 50°$ d. $\cot\sqrt{20°}$

44. If $\tan\beta = 2\sin\alpha\sin\gamma\,\text{cosec}(\alpha + \gamma)$, then $\cot\alpha$, $\cot\beta$, $\cot\gamma$ are in

a. A.P. b. GP. c. H.P. d. none of these

45. In triangle ABC, if $\sin A\cos B = \frac{1}{4}$ and $3\tan A = \tan B$, then $\cot^2 A$ is equal to

a. 2 b. 3 c. 4 d. 5

46. $\tan 100° + \tan 125° + \tan 100° \tan 125°$ is equal to

a. 0 b. 1/2 c. −1 d. 1

47. $\tan 20° + \tan 40° + \sqrt{3}\tan 20° \tan 40°$ is equal to

a. $\frac{1}{\sqrt{3}}$ b. $\sqrt{3}$ c. $-\frac{1}{\sqrt{3}}$ d. $-\sqrt{3}$

48. $\frac{\sqrt{2} - \sin\alpha - \cos\alpha}{\sin\alpha - \cos\alpha}$ is equal to

a. $\sec\left(\frac{\alpha}{2} - \frac{\pi}{8}\right)$ b. $\cos\left(\frac{\pi}{8} - \frac{\alpha}{2}\right)$ c. $\tan\left(\frac{\alpha}{2} - \frac{\pi}{8}\right)$ d. $\cot\left(\frac{\alpha}{2} - \frac{\pi}{2}\right)$

49. If $\sin\theta_1 - \sin\theta_2 = a$ and $\cos\theta_1 + \cos\theta_2 = b$, then

a. $a^2 + b^2 \ge 4$ b. $a^2 + b^2 \le 4$ c. $a^2 + b^2 \ge 3$ d. $a^2 + b^2 \le 2$

50. If $\frac{1 + \sin 2x}{1 - \sin 2x} = \cot^2(a + x)\ \forall x \in R \sim \left(n\pi + \frac{\pi}{4}\right)$, $n \in N$, then a can be

a. $\frac{\pi}{4}$ b. $\frac{\pi}{2}$ c. $\frac{3\pi}{4}$ d. none of these

51. If $\tan\alpha$ is equal to the integral solution of the inequality $4x^2-16x+15<0$ and $\cos\beta$ is equal to the slope of the bisector of the first quadrant, then $\sin(\alpha+\beta)\sin(\alpha-\beta)$ is equal to

a. $\frac{3}{5}$ b. $\frac{3}{5}$ c. $\frac{2}{\sqrt{5}}$ d. $\frac{4}{5}$

52. If $\frac{\cos(x-y)}{\cos(x+y)}+\frac{\cos(z+t)}{\cos(z-t)}=0$, then the value of $\tan x\tan y\tan z\tan t$ is equal to

a. 1 b. -1 c. 2 d. -2

53. Let $f(n)=2\cos nx\ \forall n\in N$, then $f(1)f(n+1)-f(n)$ is equal to

a. $f(n+3)$ b. $f(n+2)$ c. $f(n+1)f(2)$ d. $f(n+2)f(2)$

54. If in triangle ABC, $\sin A\cos B=1/4$ and $3\tan A=\tan B$, then the triangle is

a. right angled b. equilateral c. isosceles d. none of these

55. If A and B are acute positive angles satisfying the equations $3\sin^2 A+2\sin^2 B=1$ and $3\sin 2A-2\sin 2B=0$, then $A+2B$ is equal to

a. π b. $\frac{\pi}{2}$ c. $\frac{\pi}{4}$ d. $\frac{\pi}{6}$

56. Let $f(\theta)=\frac{\cot\theta}{1+\cot\theta}$ and $\alpha+\beta=\frac{5\pi}{4}$, then the value $f(\alpha)f(\beta)$ is

a. $\frac{1}{2}$ b. $-\frac{1}{2}$ c. 2 d. none of these

57. If $y=(1+\tan A)(1-\tan B)$ where $A-B=\frac{\pi}{4}$, then $(y+1)^{y+1}$ is equal to

a. 9 b. 4 c. 27 d. 81

58. If $\sin(y+z-x)$, $\sin(z+x-y)$, $\sin(x+y-z)$ are in A.P., then $\tan x$, $\tan y$, $\tan z$ are in

a. A.P. b. G.P. c. H.P. d. none of these

59. If $\cos\alpha+\cos\beta=0=\sin\alpha+\sin\beta$, then $\cos 2\alpha+\cos 2\beta$ is equal to

a. $-2\sin(\alpha+\beta)$ b. $-2\cos(\alpha+\beta)$ c. $2\sin(\alpha+\beta)$ d. $2\cos(\alpha+\beta)$

60. If $x_1, x_2, x_3, \ldots, x_n$ are in A.P. whose common difference is α, then the value of $\sin\alpha(\sec x_1\sec x_2+\sec x_2\sec x_3+\cdots+\sec x_{n-1}\sec x_n)$ is

a. $\frac{\sin(n-1)\alpha}{\cos x_1\cos x_n}$ b. $\frac{\sin n\alpha}{\cos x_1\cos x_n}$ c. $\sin(n-1)\alpha\cos x_1\cos x_n$ d. $\sin n\alpha\cos x_1\cos x_n$

61. If $\tan\frac{\pi}{9}$, x and $\tan\frac{5\pi}{18}$ are in A.P. and $\tan\frac{\pi}{9}$, y and $\tan\frac{7\pi}{18}$ are also in A.P., then

a. $2x=y$ b. $x>2$ c. $x=y$ d. none of these

62. Let $x=\sin 1°$, then the value of the expression

$$\frac{1}{\cos 0°\cdot\cos 1°}+\frac{1}{\cos 1°\cdot\cos 2°}+\frac{1}{\cos 2°\cdot\cos 3°}+\cdots+\frac{1}{\cos 44°\cdot\cos 45°}$$ is equal to

a. x b. $1/x$ c. $\sqrt{2}/x$ d. $x/\sqrt{2}$

63. Let α and, β be such that $\pi<\alpha-\beta<3\pi$. If $\sin\alpha+\sin\beta=-\frac{21}{65}$ and $\cos\alpha+\cos\beta=-\frac{17}{65}$, then the value of $\cos\frac{\alpha-\beta}{2}$ is

a. $-\frac{3}{\sqrt{130}}$ b. $\frac{3}{\sqrt{130}}$ c. $\frac{6}{65}$ d. $-\frac{6}{65}$

64. If $\frac{\sin(x+y)}{\sin(x-y)}=\frac{a+b}{a-b}$, then $\frac{\tan x}{\tan y}$ is equal to

a. $\frac{b}{a}$ b. $\frac{a}{b}$ c. ab d. none of these

65. $\frac{\sin 3\theta+\sin 5\theta+\sin 7\theta+\sin 9\theta}{\cos 3\theta+\cos 5\theta+\cos 7\theta+\cos 9\theta}$ is equal to

a. $\tan 3\theta$ b. $\cot 3\theta$ c. $\tan 6\theta$ d. $\cot 6\theta$

66. If x, y, z are in A.P, then $\frac{\sin x-\sin z}{\cos z-\cos x}$ is equal to

a. $\tan y$ b. $\cot y$ c. $\sin y$ d. $\cos y$

67. If $\cos 25° + \sin 25° = p$, then $\cos 50°$ is

a. $\sqrt{2-p^2}$ b. $-\sqrt{2-p^2}$ c. $p\sqrt{2-p^2}$ d. $-p\sqrt{2-p^2}$

68. $\frac{\sin^2 A-\sin^2 B}{\sin A\cos A-\sin B\cos B}$ is equal to

a. $\tan(A-B)$ b. $\tan(A+B)$ c. $\cot(A-B)$ d. $\cot(A+B)$

69. If $\tan A=\frac{1-\cos B}{\sin B}$, then $\tan 2A$ is

a. $\tan 2A=\tan B$ b. $\tan 2A=\tan^2 B$

c. $\tan 2A=\tan^2 B+2\tan B$ d. none of these

70. If $a+b=3-\cos 4\theta$ and $a-b=4\sin 2\theta$, then ab is always less than or equal to

a. $\frac{1}{2}$ b. 1 c. $\frac{2}{3}$ d. $\frac{3}{4}$

71. The value of $\cos^2 10° - \cos 10° \cos 50° + \cos^2 50°$ is equal to

a. $\frac{4}{3}$ b. $\frac{1}{3}$ c. $\frac{3}{4}$ d. 3

72. The numerical value of $\tan 20° \tan 80° \cot 50°$ is equal to

a. $\sqrt{3}$ b. $\frac{1}{\sqrt{3}}$ c. $2\sqrt{3}$ d. $\frac{1}{2\sqrt{3}}$

73. If $\tan^2\theta=2\tan^2\phi+1$, then $\cos 2\theta+\sin^2\phi$ equals

a. -1 b. 0 c. 1 d. none of these

74. The value of $\cot 70° + 4\cos 70°$ is

a. $\frac{1}{\sqrt{3}}$ b. $\sqrt{3}$ c. $2\sqrt{3}$ d. $\frac{1}{2}$

75. If x_1 and x_2 are two distinct roots of the equation $a\cos x+b\sin x=c$, then $\tan\frac{x_1+x_2}{2}$ is equal to

a. $\frac{a}{b}$ b. $\frac{b}{a}$ c. $\frac{c}{a}$ d. $\frac{a}{c}$

76. Given that $(1+\sqrt{1+x})\tan y = 1+\sqrt{1-x}$. Then $\sin 4y$ is equal to

a. $4x$ b. $2x$ c. x d. none of these

77. If $\cos x = \tan y$, $\cos y = \tan z$, $\cos z = \tan x$, then the value of $\sin x$ is

a. $2\cos 18°$ b. $\cos 18°$ c. $\sin 18°$ d. $2\sin 18°$

78. If $\sin 2\theta = \cos 3\theta$ and θ is an acute angle, then $\sin\theta$ equals

a. $\frac{\sqrt{5}-1}{4}$ b. $-\left(\frac{\sqrt{5}-1}{4}\right)$ c. $\frac{\sqrt{5}+1}{4}$ d. $\frac{-\sqrt{5}-1}{4}$

79. If θ_1 and θ_2 are two values lying in $[0, 2\pi]$ for which $\tan\theta = \lambda$, then $\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}$ is equal to

a. 0 b. -1 c. 2 d. 1

80. If $\tan\theta = \sqrt{n}$ where $n \in N, \geq 2$, then $\sec 2\theta$ is always

a. a rational number b. an irrational number c. a positive integer d. a negative integer

81. If $\sin x + \cos x = \frac{\sqrt{7}}{2}$ where $x \in A$, then $\tan\frac{x}{2}$ is equal to

a. $\frac{3-\sqrt{7}}{3}$ b. $\frac{\sqrt{7}-2}{3}$ c. $\frac{4-\sqrt{7}}{4}$ d. none of these

82. The value of $\sin^2\frac{\pi}{8}+\sin^2\frac{3\pi}{8}+\sin^2\frac{5\pi}{8}+\sin^2\frac{7\pi}{8}$ is

a. 1 b. 2 c. $1\frac{1}{8}$ d. $2\frac{1}{8}$

83. If $x \in \left(\pi, \frac{3\pi}{2}\right)$, then $4\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)+\sqrt{4\sin^4 x+\sin^2 2x}$ is always equal to

a. 1 b. 2 c. -2 d. none of these

84. $\cos^3 x \sin 2x = \sum_{x=0}^{n} a_r \sin(rx)\ \forall x \in R$, then

a. $n=5, a_1=1/2$ b. $n=5, a_1=1/4$ c. $n=5, a_2=1/8$ d. $n=5, a_2=1/4$

85. The value of $\cos 2(\theta+\phi)+4\cos(\theta+\phi)\sin\theta\sin\phi+2\sin^2\phi$ is

a. independent of θ only b. independent of ϕ only

c. independent of both θ and ϕ d. dependent on θ and ϕ

86. If $\cos 2B = \frac{\cos(A+C)}{\cos(A-C)}$, then $\tan A$, $\tan B$, $\tan C$ are in

a. A.P. b. GP. c. H.P. d. none of these

87. If $\cos x = \frac{2\cos y-1}{2-\cos y}$, where $x, y \in (0, \pi)$, then $\tan\frac{x}{2}\cot\frac{y}{2}$ is equal to

a. $\sqrt{2}$ b. $\sqrt{3}$ c. $\frac{1}{\sqrt{2}}$ d. $\frac{1}{\sqrt{3}}$

88. If $\tan x = b/a$, then $\sqrt{(a+b)/(a-b)}+\sqrt{(a-b)/(a+b)}$ is equal to

a. $2\sin x/\sqrt{\sin 2x}$ b. $2\cos x/\sqrt{\cos 2x}$ c. $2\cos x/\sqrt{\sin 2x}$ d. $2\sin x/\sqrt{\cos 2x}$

89. If α is a root of $25\cos^2\theta + 5\cos\theta - 12 = 0$, $\frac{\pi}{2} < \alpha < \pi$, then $\sin 2\alpha$ is equal to

a. $\frac{24}{25}$ b. $-\frac{24}{25}$ c. $\frac{13}{18}$ d. $-\frac{13}{18}$

90. The value of $\tan 9° - \tan 27° - \tan 63° + \tan 81°$ is

a. 2 b. 3 c. 4 d. none of these

91. If $\sin^{-1} a + \sin^{-1} b + \sin^{-1} c = \pi$, then $a\sqrt{1-a^2} + b\sqrt{1-b^2} + c\sqrt{1-c^2}$ is equal to

a. $a+b+c$ b. $a^2b^2c^2$ c. $2abc$ d. $4abc$

92. If $A + B + C = 3\pi/2$, then $\cos 2A + \cos 2B + \cos 2C$ is equal to

a. $1 - 4\cos A\cos B\cos C$ b. $4\sin A\sin B\sin C$

c. $1 + 2\cos A\cos B\cos C$ d. $1 - 4\sin A\sin B\sin C$

93. In triangle ABC, $\tan\frac{A}{2}, \tan\frac{B}{2}, \tan\frac{C}{2}$ are in H.P., then the value of $\cot\frac{A}{2} \times \cot\frac{C}{2}$ is equal to

a. 1 b. 2 c. 3 d. 4

94. In any triangle ABC, $\sin^2 A - \sin^2 B + \sin^2 C$ is always equal to

a. $2\sin A\sin B\cos C$ b. $2\sin A\cos B\sin C$

c. $2\sin A\cos B\cos C$ d. $2\sin A\sin B\sin C$

95. If $\tan^2\alpha\tan^2\beta + \tan^2\beta\tan^2\gamma + \tan^2\gamma\tan^2\alpha + 2\tan^2\alpha\tan^2\beta\tan^2\gamma = 1$, then the value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$ is

a. 3 b. 2 c. 1 d. none of these

96. In triangle ABC, $\frac{\sin A + \sin B + \sin C}{\sin A + \sin B - \sin C}$ is equal to

a. $\tan\frac{A}{2}\cot\frac{B}{2}$ b. $\cot\frac{A}{2}\tan\frac{B}{2}$ c. $\cot\frac{A}{2}\cot\frac{B}{2}$ d. $\tan\frac{A}{2}\tan\frac{B}{2}$

97. $\frac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C}$ is equal to

a. $8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ b. $8\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

c. $8\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}$ d. $8\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

98. If $\cos^2 A + \cos^2 B + \cos^2 C = 1$, then ΔABC is

a. equilateral b. isosceles c. right angled d. none of these

99. In triangle ABC, $\tan A + \tan B + \tan C = 6$ and $\tan A\tan B = 2$, then the values of $\tan A$, $\tan B$, $\tan C$ are

a. 1, 2, 3 b. 3, 2/3, 7/3 c. 4, 1/2, 3/2 d. none of these

100. The value of $\tan 6° \tan 42° \tan 66° \tan 78°$ is

a. 1 b. 1/2 c. 1/4 d. 1/8

101. If $0 < \alpha < \frac{\pi}{6}$, then $\alpha(\text{cosec}\,\alpha)$ is

a. less than $\pi/6$ b. greater than $\pi/6$ c. less than $\pi/3$ d. greater than $\pi/3$

102. If θ is eliminated from the equations $x = a\cos(\theta - \alpha)$ and $y = b\cos(\theta - \beta)$, then $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\cos(\alpha - \beta)$ is equal to

a. $\sec^2(\alpha - \beta)$ b. $\operatorname{cosec}^2(\alpha - \beta)$ c. $\cos^2(-\beta)$ d. $\sin^2(\alpha - \beta)$

103. If $\left|\cos\theta\left\{\sin\theta + \sqrt{\sin^2\theta + \sin^2\alpha}\right\}\right| \le k$, then the value of k is

a. $\sqrt{1+\cos^2\alpha}$ b. $\sqrt{1+\sin^2\alpha}$ c. $\sqrt{2+\sin^2\alpha}$ d. $\sqrt{2+\cos^2\alpha}$

104. If $\sin\theta_1 \sin\theta_2 - \cos\theta_1 \cos\theta_2 + 1 = 0$, then the value of $\tan(\theta_1/2)\cot(\theta_2/2)$ is always equal to

a. -1 b. 1 c. 2 d. -2

105. The numerical value of $\tan\frac{\pi}{3} + 2\tan\frac{2\pi}{3} + 4\tan\frac{4\pi}{3} + 8\tan\frac{8\pi}{3}$ is equal to

a. $-5\sqrt{3}$ b. $-5/\sqrt{3}$ c. $5\sqrt{3}$ d. $5/\sqrt{3}$

106. $\tan^6\frac{\pi}{9} - 33\tan^4\frac{\pi}{9} + 27\tan^2\frac{\pi}{9}$ is equal to

a. 0 b. $\sqrt{3}$ c. 3 d. 9

107. If $\cos x + \cos y - \cos(x+y) = \frac{3}{2}$, then

a. $x+y=0$ b. $x=2y$ c. $x=y$ d. $2x=y$

108. If $a\sin x + b\cos(x+\theta) + b\cos(x-\theta) = d$, then the minimum value of $|\cos\theta|$ is equal to

a. $\frac{1}{2|b|}\sqrt{d^2-a^2}$ b. $\frac{1}{2|a|}\sqrt{d^2-a^2}$ c. $\frac{1}{2|d|}\sqrt{d^2-a^2}$ d. none of these

109. If $\frac{\sin x}{\sin y} = \frac{1}{2}, \frac{\cos x}{\cos y} = \frac{3}{2}$ where $x, y \in \left(0, \frac{\pi}{2}\right)$, then the value of $\tan(x+y)$ is equal to

a. $\sqrt{13}$ b. $\sqrt{14}$ c. $\sqrt{17}$ d. $\sqrt{15}$

110. If $x \in (\pi, 2\pi)$ and $\frac{\sqrt{1+\cos x} + \sqrt{1-\cos x}}{\sqrt{1+\cos x} - \sqrt{1-\cos x}} = \cot\left(a + \frac{x}{2}\right)$, then a is equal to

a. $\frac{\pi}{4}$ b. $\frac{\pi}{2}$ c. $\frac{\pi}{3}$ d. none of these

111. If $\tan x = n\tan y, n \in R^+$, then the maximum value of $\sec^2(x-y)$ is equal to

a. $\frac{(n+1)^2}{2n}$ b. $\frac{(n+1)^2}{n}$ c. $\frac{(n+1)^2}{2}$ d. $\frac{(n+1)^2}{4n}$

112. If $\cot^2 x = \cot(x-y)\cot(x-z)$, then $\cot 2x$ is equal to (where $x \ne \pm\pi/4$)

a. $\frac{1}{2}(\tan y + \tan z)$ b. $\frac{1}{2}(\cot y + \cot z)$ c. $\frac{1}{2}(\sin y + \sin z)$ d. none of these

113. If A, B, C are acute positive angles such that $A + B + C = \pi$ and $\cot A \cot B \cot C = k$, then

a. $K \le \frac{1}{3\sqrt{3}}$ b. $K \ge \frac{1}{3\sqrt{3}}$ c. $K < \frac{1}{9}$ d. $K > \frac{1}{3}$

114. If $u=\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}+\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}$, then the difference between the maximum and minimum values of u^2 is given by

a. $2(a^2+b^2)$ b. $2\sqrt{a^2+b^2}$ c. $(a+b)^2$ d. $(a-b)^2$

115. If $(\sin x+\cos x)^2+k\sin x\cos x=1$ holds $\forall x\in R$, then the value of k equals

a. 2 b. 2 c. –2 d. 3

116. The range of k for which the inequality $k\cos^2x-k\cos x+1\geq 0\ \forall x\in(-\infty,\infty)$, is

a. $k<\frac{-1}{2}$ b. $\frac{-1}{2}\leq k\leq 4$ c. $k>4$ d. $\frac{1}{2}\leq k\leq 5$

117. The minimum vertical distance between the graphs of $y=2+\sin x$ and $y=\cos x$ is

a. 2 b. 1 c. $\sqrt{2}$ d. $2-\sqrt{2}$

118. If $\theta=3\alpha$ and $\sin\theta=\frac{a}{\sqrt{a^2+b^2}}$ The value of the expression $a\operatorname{cosec}\alpha-b\sec\alpha$ is

a. $\frac{a}{\sqrt{a^2+b^2}}$ b. $2\sqrt{a^2+b^2}$ c. $a+b$ d. none of these

119. If the equation $\cot^4x-2\operatorname{cosec}^2x+a^2=0$ has at least one solution, then the sum of all possible integral values of 'a' is equal to

a. 4 b. 3 c. 2 d. 0

120. If the inequality $\sin^2x+a\cos x+a^2>1+\cos x$ holds for any $x\in R$ then the largest negative integral value of 'a' is

a. –4 b. –3 c. –2 d. –1

121. In triangle ABC if angle C is 90° and area of triangle is 30 sq. units, then the minimum possible value of the hypotenuse c is equal to

a. $30\sqrt{2}$ b. $60\sqrt{2}$ c. $120\sqrt{2}$ d. $\sqrt{30}$

122. The distance between the two parallel lines is 1 unit. A point 'A' is chosen to lie between the lines at a distance 'd' from one of them. Triangle ABC is equilateral with B on one line and C on the other parallel line. The length of the side of the equilateral triangle is

a. $\frac{2}{3}\sqrt{d^2+d+1}$ b. $2\sqrt{\frac{d^2-d+1}{3}}$ c. $2\sqrt{d^2-d+1}$ d. $\sqrt{d^2-d+1}$

123. Given that a, b, c are the sides of a triangle ABC which is right angled at C, then the minimum value of $\left(\frac{c}{a}+\frac{c}{b}\right)^2$ is

a. 0 b. 4 c. 6 d. 8

124. Let $y=(\sin x+\operatorname{cosec}x)^2+(\cos x+\sec x)^2+(\cos x+\sec x)^2$, then the minimum value of y, $\forall x\in R$, is

a. 7 b. 3 c. 9 d. 0

Multiple Correct Answers Type

Solutions on page 2.116

Each question has four choices a, b, c, and d, out of which one or more answers are correct.

1. If $\cos\beta$ is the geometric mean between $\sin\alpha$ and $\cos\alpha$, where $0<\alpha,\beta<\pi/2$, then $\cos 2\beta$ is equal to

a. $-2\sin^2\left(\frac{\pi}{4}-\alpha\right)$ b. $-2\cos^2\left(\frac{\pi}{4}+\alpha\right)$ c. $2\sin^2\left(\frac{\pi}{4}+\alpha\right)$ d. $2\cos^2\left(\frac{\pi}{4}-\alpha\right)$

2. If $0\leq\theta\leq\pi$ and $81^{\sin^2\theta}+81^{\cos^2\theta}=30$, then θ is

a. 30° b. 60° c. 120° d. 150°

3. Suppose $ABCD$ (in order) is a quadrilateral inscribed in a circle. Which of the following is/are always true?

a. $\sec B = \sec D$ b. $\cot A + \cot C = 0$ c. $\operatorname{cosec} A = \operatorname{cosec} C$ d. $\tan B + \tan D = 0$

4. Which of the following statements are always correct (where Q denotes the set of rationals)?

a. $\cos 2\theta \in Q$ and $\sin 2\theta \in Q \Rightarrow \tan \theta \in Q$ (if defined)

b. $\tan \theta \in Q \Rightarrow \sin 2\theta, \cos 2\theta$ and $\tan 2\theta \in Q$ (if defined)

c. if $\sin \theta \in Q$ and $\cos \theta \in Q \Rightarrow \tan 3\theta \in Q$ (if defined)

d. if $\sin \theta \in Q \Rightarrow \cos 3\theta \in Q$

5. Which of the following quantities are rational?

a. $\sin\left(\frac{11\pi}{12}\right)\sin\left(\frac{5\pi}{12}\right)$ b. $\operatorname{cosec}\left(\frac{9\pi}{10}\right)\sec\left(\frac{4\pi}{5}\right)$

c. $\sin^4\left(\frac{\pi}{8}\right)+\cos^4\left(\frac{\pi}{8}\right)$ d. $\left(1+\cos\frac{2\pi}{9}\right)\left(1+\cos\frac{4\pi}{9}\right)\left(1+\cos\frac{8\pi}{9}\right)$.

6. In which of the following sets the inequality $\sin^6 x + \cos^6 x > 5/8$ holds good?

a. $(-\pi/8, \pi/8)$ b. $(3\pi/8, 5\pi/8)$ c. $(\pi/4, 3\pi/4)$ d. $(7\pi/8, 9\pi/8)$

7. Which of the following inequalities hold true in any triangle ABC?

a. $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le \frac{1}{8}$ b. $\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \le \frac{3\sqrt{3}}{8}$

c. $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2} < \frac{3}{4}$ d. $\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2} \le \frac{9}{4}$

8. For $\alpha = \pi/7$ which of the following hold(s) good?

a. $\tan\alpha\tan 2\alpha\tan 3\alpha = \tan 3\alpha - \tan 2\alpha - \tan\alpha$

b. $\operatorname{cosec}\alpha = \operatorname{cosec} 2\alpha + \operatorname{cosec} 4\alpha$

c. $\cos\alpha - \cos 2\alpha + \cos 3\alpha = 1/2$

d. $8\cos\alpha\cos 2\alpha\cos 4\alpha = 1$

9. Which of the following is/are correct?

a. $(\tan x)^{\ln(\sin x)} > (\cot x)^{\ln(\sin x)}, \forall x \in (0, \pi/4)$ b. $4^{\ln \operatorname{cosec} x} < 5^{\ln \operatorname{cosec} x}, \forall x \in (0, \pi/2)$

c. $(1/2)^{\ln(\cos x)} < (1/3)^{\ln(\cos x)}, \forall x \in (0, \pi/2)$ d. $2^{\ln(\tan x)} > 2^{\ln(\sin x)}, \forall x \in (0, \pi/2)$

10. Which of the following do/does not reduce to unity?

a. $\frac{\sin(180^\circ + A)}{\tan(180^\circ + A)}\frac{\cot(90^\circ + A)}{\tan(90^\circ + A)}\frac{\cos(360^\circ - A)\operatorname{cosec} A}{\sin(-A)}$

b. $\frac{\sin(-A)}{\sin(180^\circ + A)} - \frac{\tan(90^\circ + A)}{\cot A} + \frac{\cos A}{\sin(90^\circ + A)}$

c. $\frac{\sin 24^\circ\cos 6^\circ - \sin 6^\circ\cos 24^\circ}{\sin 21^\circ\cos 39^\circ - \cos 51^\circ\sin 69^\circ}$

d. $\frac{\cos(90^\circ + A)\sec(-A)\tan(180^\circ - A)}{\sec(360^\circ + A)\sin(180^\circ + A)\cot(90^\circ - A)}$

11. Which of the following identities, wherever defined, hold(s) good?

a. $\cot\alpha - \tan\alpha = 2\cot 2\alpha$ b. $\tan(45° + \alpha) - \tan(45° - \alpha) = 2\,\text{cosec}\,2\alpha$

c. $\tan(45° + \alpha) + \tan(45° - \alpha) = 2\sec 2\alpha$ d. $\tan\alpha + \cot\alpha = 2\tan 2\alpha$

12. A circle centred at O has radius 1 and contains the point A. Segment AB is tangent to the circle at A and $\angle AOB = \theta$. If point C lies on OA and BC bisects the angle ABO, then OC equals

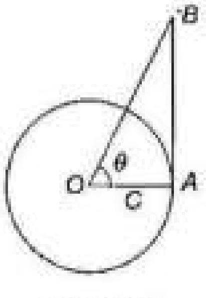


Fig. 2.36

a. $\sec\theta(\sec\theta - \tan\theta)$ b. $\dfrac{\cos^2\theta}{1+\sin\theta}$ c. $\dfrac{1}{1+\sin\theta}$ d. $\dfrac{1-\sin\theta}{\cos^2\theta}$

13. The expression $(\tan^4 x + 2\tan^2 x + 1)\cos^2 x$ when $x = \pi/12$ can be equal to

a. $4(2-\sqrt{3})$ b. $4(\sqrt{2}+1)$ c. $16\cos^2\pi/12$ d. $16\sin^2\pi/12$

14. Let α, β and γ be some angles in the first quadrant satisfying $\tan(\alpha+\beta) = 15/8$ and $\text{cosec}\,\gamma = 17/8$, then which of the following hold(s) good?

a. $\alpha + \beta + \gamma = \pi$ b. $\cot\alpha\cot\beta\cot\gamma = \cot\alpha + \cot\beta + \cot\gamma$

c. $\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma$ d. $\tan\alpha\tan\beta + \tan\beta\tan\gamma + \tan\gamma\tan\alpha = 1$

15. $(a+2)\sin\alpha + (2a-1)\cos\alpha = (2a+1)$ if $\tan\alpha$ is

a. 3/4 b. 4/3 c. $2a/(a^2+1)$ d. $2a/(a^2-1)$

16. Let $f(x) = \log(\log_{1/3}(\log_7(\sin x + a)))$ be defined for every real value of x, then the possible value of a is

a. 3 b. 4 c. 5 d. 6

17. If $b > 1$, $\sin t > 0$, $\cos t > 0$ and $\log_b(\sin t) = x$, then $\log_b(\cos t)$ is equal to

a. $\dfrac{1}{2}\log_b(1-b^{2x})$ b. $2\log(1-b^{x/2})$

c. $\log_b\sqrt{1-b^{2x}}\ \log_b(1-b^{2x})$ d. $\sqrt{1-x^2}$

18. The equation $x^3 - \dfrac{3}{4}x = -\dfrac{\sqrt{3}}{8}$ is satisfied by

a. $x = \cos\left(\dfrac{5\pi}{18}\right)$ b. $x = \cos\left(\dfrac{7\pi}{18}\right)$ c. $x = \cos\left(\dfrac{23\pi}{18}\right)$ d. $x = \cos\left(\dfrac{17\pi}{18}\right)$

19. If $\sin(x+20°) = 2\sin x \cos 40°$ where $x \in (0, \pi/2)$ then which of the following hold(s) good?

a. $\cos 2x = 1/2$ b. $\text{cosec } 4x = 2$ c. $\sec\frac{x}{2} = \sqrt{6} - \sqrt{2}$ d. $\tan\frac{x}{2} = (2-\sqrt{3})$

Reasoning Type

Solutions on page 2.122

Each question has four choices a, b, c, and d, out of which *only one* is correct. Each question contains STATEMENT 1 and STATEMENT 2.

a. Both the statements are TRUE and STATEMENT 2 is the correct explanation of STATEMENT 1
b. Both the statements are TRUE but STATEMENT 2 is NOT the correct explanation of STATEMENT 1
c. STATEMENT 1 is TRUE and STATEMENT 2 is FALSE
d. STATEMENT 1 is FALSE and STATEMENT 2 is TRUE

1. **Statement 1:** If $x + y + z = xyz$, then at most one of the numbers can be negative.
Statement 2: In a triangle ABC, $\tan A + \tan B + \tan C = \tan A \tan B \tan C$ and there can be at most one obtuse angle in a triangle.

2. **Statement 1:** $\cos 1 < \cos 7$.
Statement 2: $1 < 7$.

3. **Statement 1:** $\tan 4 < \tan 7.5$.
Statement 2: $\tan x$ is always an increasing function.

4. **Statement 1:** $\cos 1 < \sin 1$.
Statement 2: In the first quadrant, cosine decreases but sine increases.

5. **Statement 1:** If $f(\theta) = (\sin\theta + \text{cosec }\theta)^2 + (\cos\theta + \sec\theta)^2$, then the minimum value of $f(\theta)$ is 9.
Statement 2: Maximum value of $\sin 2\theta$ is 1.

6. **Statement 1:** If $\sin^2\theta_1 + \sin^2\theta_2 + \cdots + \sin^2\theta_n = 0$, then the different sets of values of $(\theta_1, \theta_2, \cdots, \theta_n)$ for which $\cos\theta_1 + \cos\theta_2 + \cdots + \cos\theta_n = n - 4$ is $n(n-1)$.
Statement 2: If $\sin^2\theta_1 + \sin^2\theta_2 + \cdots + \sin^2\theta_n = 0$, then $\cos\theta_1, \cos\theta_2, \cdots, \cos\theta_n = \pm 1$.

7. **Statement 1:** The minimum value of $27^{\cos 2x} 81^{\sin 2x}$ is $\frac{1}{243}$.
Statement 2: The minimum value of $a\cos\theta + b\sin\theta$ is $-\sqrt{a^2+b^2}$.

8. **Statement 1:** If A, B, C are the angles of a triangle such that angle A is obtuse, then $\tan B \tan C > 1$.
Statement 2: In any triangle, $\tan A = \frac{\tan B + \tan C}{\tan B \tan C - 1}$.

9. **Statement 1:** $\tan 5°$ is an irrational number.
Statement 2: $\tan 15°$ is an irrational number.

10. **Statement 1:** $\sin \pi/18$ is a root of $8x^3 - 6x + 1 = 0$.
Statement 2: For any $\theta \in R$, $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

11. Let f be any one of the six trigonometric functions. Let $A, B \in R$ satisfying $f(2A) = f(2B)$.
Statement 1: $A = n\pi + B$, for some $n \in Z$.
Statement 2: 2π is one of the period of f.

12. **Statement 1:** $\sin 3 < \sin 1 < \sin 2$.
Statement 2: $\sin x$ is positive in first and second quadrants.

13. **Statement 1:** The maximum and minimum values of the function $f(x) = \dfrac{1}{3\sin x + 4\cos x - 2}$ do not exist.
Statement 2: The given function is an unbounded function.

14. **Statement 1:** The minimum value of the expression $\sin\alpha + \sin\beta + \sin\gamma$ is negative, where α, β, γ are real numbers such that $\alpha + \beta + \gamma = \pi$.
Statement 2: If α, β, γ are the angles of a triangle, then $\sin\alpha + \sin\beta + \sin\gamma = 4\cos\alpha/2 \cos\beta/2 \cos\gamma/2$.

15. **Statement 1:** If in a triangle, $\sin^2 A + \sin^2 B + \sin^2 C = 2$ then one of the angles must be 90°.
Statement 2: In any triangle $\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2\cos A \cos B \cos C$.

16. **Statement 1:** In a triangle, the least value of the sum of cosines of its angles is unity.
Statement 2: $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$, if A, B, C are the angles of a triangle.

17. Let α, β, and γ satisfy $0 < \alpha < \beta < \gamma < 2\pi$ and $\cos(x+\alpha) + \cos(x+\beta) + \cos(x+\gamma) = 0 \ \forall x \in R$.
Statement 1: $\gamma - \alpha = \dfrac{2\pi}{3}$.
Statement 2: $\cos\alpha + \cos\beta + \cos\gamma = 0$ and $\sin\alpha + \sin\beta + \sin\gamma = 0$.

18. If $A + B + C = \pi$, then
Statement 1: $\cos^2 A + \cos^2 B + \cos^2 C$ has its minimum value $\dfrac{3}{4}$.
Statement 2: Maximum value of $\cos A \cos B \cos C$ is $\dfrac{1}{8}$.

19. **Statement 1:** If $xy + yz + zx = 1$ where $x, y, z \in R^+$,
then $\dfrac{x}{1+x^2} + \dfrac{y}{1+y^2} + \dfrac{z}{1+z^2} = \dfrac{2}{\sqrt{(1+x^2)(1+y^2)(1+z^2)}}$.
Statement 2: In a triangle ABC, $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$.

20. **Statement 1:** In any triangle ABC,
$$\ln\left(\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}\right) = \ln\cot\frac{A}{2} + \ln\cot\frac{B}{2} + \ln\cot\frac{C}{2}.$$
Statement 2: $\ln\left(1 + \sqrt{3} + (2+\sqrt{3})\right) = \ln 1 + \ln\sqrt{3} + \ln(2+\sqrt{3})$.

Linked Comprehension Type

Solutions on page 2.126

Based upon each paragraph, three multiple choice questions have to be answered. Each question has 4 choices a, b, c, and d, out of which only one is correct.

For Problems 1 – 3

If $\sin\alpha = A\sin(\alpha+\beta)$, $A \neq 0$, then

1. The value of $\tan\alpha$ is

a. $\dfrac{A\sin\beta}{1 - A\cos\beta}$ b. $\dfrac{A\sin\beta}{1 + A\cos\beta}$ c. $\dfrac{A\cos\beta}{1 - A\sin\beta}$ d. $\dfrac{A\sin\beta}{1 + A\cos\beta}$

2. The value of $\tan\beta$ is

a. $\dfrac{\sin\alpha(1+A\cos\beta)}{A\cos\alpha\cos\beta}$ b. $\dfrac{\sin\alpha(1-A\cos\beta)}{A\cos\alpha\cos\beta}$ c. $\dfrac{\cos\alpha(1-A\sin\beta)}{A\cos\alpha\cos\beta}$ d. $\dfrac{\cos\alpha(1+A\sin\beta)}{A\cos\alpha\cos\beta}$

3. Which of the following is not the value of $\tan(\alpha+\beta)$?

a. $\dfrac{\sin\beta}{\cos\beta - A}$ b. $\dfrac{\sin\alpha\cos\alpha}{A\cos\beta-\sin^2\alpha}$ c. $\dfrac{\sin\alpha\cos\alpha}{A\cos\beta+\sin^2\alpha}$ d. none of these

For Problems 4 – 6

If $\alpha, \beta, \gamma, \delta$ are the solutions of the equation $\tan\left(\theta+\dfrac{\pi}{4}\right)=3\tan 3\theta$, no two of which have equal tangents.

4. The value of $\tan\alpha+\tan\beta+\tan\gamma+\tan\delta$ is

a. 1/3 b. 8/3 c. –8/3 d. 0

5. The value of $\tan\alpha\tan\beta\tan\gamma\tan\delta$ is

a. –1/3 b. –2 c. 0 d. none of these

6. The value of $\dfrac{1}{\tan\alpha}+\dfrac{1}{\tan\beta}+\dfrac{1}{\tan\gamma}+\dfrac{1}{\tan\delta}$ is

a. –8 b. 8 c. 2/3 d. 1/3

For Problems 7 – 9

$\sin\alpha+\sin\beta=\dfrac{1}{4}$ and $\cos\alpha+\cos\beta=\dfrac{1}{3}$

7. The value of $\sin(\alpha+\beta)$ is

a. $\dfrac{24}{25}$ b. $\dfrac{13}{25}$ c. $\dfrac{12}{13}$ d. none of these

8. The value of $\cos(\alpha+\beta)$ is

a. $\dfrac{12}{25}$ b. $\dfrac{7}{25}$ c. $\dfrac{12}{13}$ d. none of these

9. The value of $\tan(\alpha+\beta)$ is

a. $\dfrac{25}{7}$ b. $\dfrac{25}{12}$ c. $\dfrac{25}{13}$ d. $\dfrac{24}{7}$

For Problems 10 – 12

To find the sum $\sin^2\dfrac{2\pi}{7}+\sin^2\dfrac{4\pi}{7}+\sin^2\dfrac{8\pi}{7}$ we follow the following method.

Put $7\theta = 2n\pi$, where n is any integer.

Then $\sin 4\theta = \sin(2n\pi - 3\theta) = -\sin 3\theta$. (i)

This means that $\sin\theta$ takes the values $0, \pm\sin(2\pi/7), \pm\sin(4\pi/7)$ and $\pm\sin(8\pi/7)$.

Since $\sin(6\pi/7) = \sin(8\pi/7)$, from equation (1), we now get

$2\sin 2\theta\cos 2\theta = 4\sin^3\theta - 3\sin\theta$

$\Rightarrow 4\sin\theta\cos\theta(1-2\sin^2\theta) = \sin\theta(4\sin^2\theta - 3)$

Rejecting the value $\sin\theta = 0$, we get

$4\cos\theta(1-2\sin^2\theta) = 4\sin^2\theta - 3$

$\Rightarrow \quad 16\cos^2\theta(1-2\sin^2\theta)^2=(4\sin^2\theta-3)^2$

$\Rightarrow \quad 16(1-\sin^2\theta)(1-4\sin^2\theta+4\sin^4\theta)=16\sin^4\theta-24\sin^2\theta+9$

$\Rightarrow \quad 64\sin^6\theta-112\sin^4\theta-56\sin^2\theta-7=0$

This is cubic in $\sin^2\theta$ with the roots $\sin^2(2\pi/7)$, $\sin^2(4\pi/7)$ and $\sin^2(8\pi/7)$.

The sum of these roots is $\sin^2\frac{2\pi}{7}+\sin^2\frac{4\pi}{7}+\sin^2\frac{8\pi}{7}=\frac{112}{64}=\frac{7}{4}$.

Now answer the following questions.

10. The value of $\left(\tan^2\frac{\pi}{7}+\tan^2\frac{2\pi}{7}+\tan^2\frac{3\pi}{7}\right)\left(\cot^2\frac{\pi}{7}+\cot^2\frac{2\pi}{7}+\cot^2\frac{3\pi}{7}\right)$ is

a. 105 b. 35 c. 210 d. none of these

11. The value of $\dfrac{\tan^2\frac{\pi}{7}+\tan^2\frac{2\pi}{7}+\tan^2\frac{3\pi}{7}}{\cot^2\frac{\pi}{7}+\cot^2\frac{2\pi}{7}+\cot^2\frac{3\pi}{7}}$ is

a. 7 b. 35/3 c. 21/5 d. none of these

12. The value of $\tan^2\frac{\pi}{7}\tan^2\frac{2\pi}{7}\tan^2\frac{3\pi}{7}$ is

a. -3 b. -7 c. -5 d. none of these

For Problems 13 – 15

An altitude BD and a bisector BE are drawn in the triangle ABC from the vertex B. It is known that the length of side $AC=1$, and the magnitudes of the angles BEC, ABD, ABE, BAC form an arithmetic progression:

13. The area of circle circumscribing ΔABC is

a. $\frac{\pi}{8}$ b. $\frac{\pi}{4}$ c. $\frac{\pi}{2}$ d. π

14. Let 'O' be the circumcentre of ΔABC, the radius of the circle inscribed in ΔBOC is

a. $\frac{1}{8\sqrt{3}}$ b. $\frac{1}{4\sqrt{3}}$ c. $\frac{1}{2\sqrt{3}}$ d. $\frac{1}{2}$

15. Let B' be the image of point B with respect to side AC of ΔABC, then the length BB' is equal to

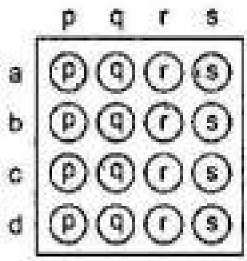
a. $\frac{\sqrt{3}}{4}$ b. $\frac{\sqrt{2}}{4}$ c. $\frac{1}{\sqrt{2}}$ d. $\frac{\sqrt{3}}{2}$

Matrix-Match Type

Solutions on page 2.129

Each question contains statements given in two columns which have to be matched.

Statements (a, b, c, d) in column I have to be matched with statements (p, q, r, s) in column II. If the correct matches are a-p, a-s, b-q, b-r, c-p, c-q and d-s, then the correctly bubbled 4 × 4 matrix should be as follows:



1. If $\cos\theta - \sin\theta = \frac{1}{5}$ where $0 < \theta < \frac{\pi}{2}$

Column I	Column II
a. $(\cos\theta + \sin\theta)/2$	p. $\frac{4}{5}$
b. $\sin 2\theta$	q. $\frac{7}{10}$
c. $\cos 2\theta$	r. $\frac{24}{25}$
d. $\cos\theta$	s. $\frac{7}{25}$

2. For all real values of θ

Column I	Column II
a. $A = \sin^2\theta + \cos^4\theta$	p. $A \in [-1, 1]$
b. $A = 3\cos^2\theta + \sin^4\theta$	q. $A \in \left[\frac{3}{4}, 1\right]$
c. $A = \sin^2\theta - \cos^4\theta$	r. $A \in [2\sqrt{2}, \infty)$
d. $A = \tan^2\theta + 2\cot^2\theta$	s. $A \in [1, 3]$

3. If $\cos\alpha + \cos\beta = 1/2$ and $\sin\alpha + \sin\beta = 1/3$

Column I	Column II
a. $\cos\left(\frac{\alpha+\beta}{2}\right)$	p. $\pm\frac{\sqrt{13}}{12}$
b. $\cos\left(\frac{\alpha-\beta}{2}\right)$	q. $\frac{2}{3}$
c. $\tan\left(\frac{\alpha+\beta}{2}\right)$	r. $\pm\frac{3}{\sqrt{13}}$
d. $\tan\left(\frac{\alpha-\beta}{2}\right)$	s. $\pm\sqrt{\frac{131}{13}}$

4.

Column I	Column II
a. $\sin(410° - A)\cos(400° + A) + \cos(410° - A)\sin(400° + A)$ has the value equal to	p. -1
b. $\dfrac{\cos^2 1° - \cos^2 2°}{2\sin 3° \sin 1°}$ is equal to	q. 0
c. $\sin(-870°) + \text{cosec}(-660°) + \tan(-855°) + 2\cot(840°) + \cos(480°)$ $+ \sec(900°)$	r. $\dfrac{1}{2}$
d. If $\cos\theta = \dfrac{4}{5}$ where $\theta \in \left(\dfrac{3\pi}{2}, 2\pi\right)$ and $\cos\phi = \dfrac{3}{5}$ where $\phi \in \left(0, \dfrac{\pi}{2}\right)$, then $\cos(\theta - \phi)$ has the value equal to	s. 1

5.

Column I	Column II
a. The maximum value of $\{\cos(2A + \theta) + \cos(2B + \theta)\}$, where A, B are constants, is	p. $2\sin(A + B)$
b. The maximum value of $\{\cos 2A + \cos 2B\}$, where $(A + B)$ is constant and $A, B \in (0, \pi/2)$, is	q. $2\sec(A + B)$
c. The minimum value of $\{\sec 2A + \sec 2B\}$, where $(A + B)$ is constant and $A, B \in (0, \pi/4)$, is	r. $2\cos(A + B)$
d. The minimum value of $\sqrt{\{\tan\theta + \cot\theta - 2\cos 2(A + B)\}}$, where A, B are constants and $\theta \in (0, \pi/2)$, is	s. $2\cos(A - B)$

6.

Column I	Column II
a. $\cos 20° + \cos 80° - \sqrt{3}\cos 50°$	p. -1
b. $\cos 0° + \cos\dfrac{\pi}{7} + \cos\dfrac{2\pi}{7} + \cos\dfrac{3\pi}{7} + \cos\dfrac{4\pi}{7} + \cos\dfrac{5\pi}{7} + \cos\dfrac{6\pi}{7}$	q. $-\dfrac{3}{4}$
c. $\cos 20° + \cos 40° + \cos 60° - 4\cos 10° \cos 20° \cos 30°$	r. 1
d. $\cos 20° \cos 100° + \cos 100° \cos 140° - \cos 140° \cos 200°$	s. 0

7.

Column I	Column II
a. Suppose ABC is a triangle with three acute angles A, B and C. The point whose coordinates are $(\cos B - \sin A, \sin B - \cos A)$ can be in the	p. 1^{st} quadrant
b. If $2^{\sin\theta} > 1$ and $3^{\cos\theta} < 1$, then $\theta \in$	q. 2^{nd} quadrant
c. $\|\cos x + \sin x\| = \|\sin x\| + \|\cos x\|$	r. 3^{rd} quadrant
d. If $\sqrt{\dfrac{1-\sin A}{1+\sin A}} + \dfrac{\sin A}{\cos A} = \dfrac{1}{\cos A}$, for all permissible values of A, then A can belong to	s. 4^{th} quadrant

8.

Column I	Column II
a. If $x^2 + y^2 = 1$ and $P = (3x - 4x^3)^2 + (3y - 4y^3)^2$, then P is equal to	p. 1
b. If $a + b = 3 - \cos 4\theta$ and $a - b = 4\sin 2\theta$, then the maximum value of (ab) is	q. 4
c. The least positive integral value of x for which $3\cos\theta = x^2 - 8x + 19$ holds good is	r. 5
d. If $x = \dfrac{4\lambda}{1+\lambda^2}$ and $y = \dfrac{2-2\lambda^2}{1+\lambda^2}$, where λ is a real parameter, then $x^2 - xy + y^2$ lies between $[a, b]$ then $(a + b)$ is	s. 8

9.

Column I	Column II
a. In triangle ABC, $3\sin A + 4\cos B = 6$ and $3\cos A + 4\sin B = 1$, then $\angle C$ can be	p. $60°$
b. In any triangle, if $(\sin A + \sin B + \sin C)(\sin A + \sin B - \sin C) = 3\sin A \sin B$, then the angle C	q. $30°$
c. If $8\sin x \cos^5 x - 8\sin^5 x \cos x = 1$, then $x =$	r. $165°$
d. 'O' is the centre of the inscribed circle in a $30° - 60° - 90°$ triangle ABC with right angled at C. If the circle is tangent to AB at D, then the angle $\angle COD$ is	s. $7.5°$

Integer Type

Solutions on page 2.135

1. If $f(\theta) = \dfrac{1 - \sin 2\theta + \cos 2\theta}{2\cos 2\theta}$ then value of $8f(11°)\cdot f(34°)$ is ________.
2. If $f(x) = 2(7\cos x + 24\sin x)(7\sin x - 24\cos x)$, for every $x \in R$, then maximum value of $(f(x))^{1/4}$ is ________.
3. In a triangle ABC, $\angle C = \dfrac{\pi}{2}$. If $\tan\left(\dfrac{A}{2}\right)$ and $\tan\left(\dfrac{B}{2}\right)$ are the roots of the equation $ax^2 + bx + c = 0$ $(a \neq 0)$, then the value of $\dfrac{a+b}{c}$ (where, a, b, c are sides of Δ opposite to angles A, B, C resp.) is ________.

4. If $(1 + \tan 5°)(1 + \tan 10°)(1 + \tan 15°)...(1 + \tan 45°) = 2^k$, then the value of '$k$' is ________.

5. The value of $\sqrt{3}\left|\dfrac{\dfrac{2\sin(140°)\sec(280°)}{\sec(220°)} + \dfrac{\sec(340°)}{\text{cosec}(20°)}}{\dfrac{\cot(200°) - \tan(280°)}{\cot(200°)}}\right|$ is ________.

6. If $x, y \in R$ satisfy $(x + 5)^2 + (y - 12)^2 = (14)^2$, then the minimum value of is ________.

7. Suppose x and y are real numbers such that $\tan x + \tan y = 42$ and $\cot x + \cot y = 49$. Then the prime number by which the value of $\tan(x + y)$ is not divisible by 5 is ________.

8. Let $0 \le a, b, c, d \le \pi$ where b and c are not complementary, such that $2 \cos a + 6 \cos b + 7 \cos c + 9 \cos d = 0$ and $2 \sin a - 6 \sin b + 7 \sin c - 9 \sin d = 0$, then the value of $3\dfrac{\cos(a + d)}{\cos(b + c)}$ is ________.

9. Suppose A and B are two angles such that $A, B \in (0, \pi)$, and satisfy $\sin A + \sin B = 1$ and $\cos A + \cos B = 0$. Then the value of $12 \cos 2A + 4 \cos 2B$ is ________.

10. α and β are the positive acute angles and satisfying equations $5 \sin 2\beta = 3 \sin 2\alpha$ and $\tan \beta = 3 \tan \alpha$ simultaneously. Then the value of $\tan \alpha + \tan \beta$ is ________.

11. The absolute value of the expression $\tan \dfrac{\pi}{16} + \tan \dfrac{5\pi}{16} + \tan \dfrac{9\pi}{16} + \tan \dfrac{13\pi}{16}$ is ________.

12. The greatest integer less than or equal to $\dfrac{1}{\cos 290°} + \dfrac{1}{\sqrt{3}\sin 250°}$ is ________.

13. The maximum value of $y = \dfrac{1}{\sin^6 x + \cos^6 x}$ is ________.

14. The maximum value of $\cos^2(45° + x) + (\sin x - \cos x)^2$ is ________.

15. The value of $9\dfrac{\sin^4 t + \cos^4 t - 1}{\sin^6 t + \cos^6 t - 1}$ is ________.

16. The value of $\text{cosec}10° + \text{cosec}50° - \text{cosec}70°$ is ________.

17. The minimum value of $\sqrt{(3\sin x - 4\cos x - 10)(3\sin x + 4\cos x - 10)}$ is ________.

18. Number of triangles ABC if $\tan A = x$, $\tan B = x + 1$ and $\tan C = 1 - x$ is ________.

19. If $\log_{10}\sin x + \log_{10}\cos x = -1$ and $\log_{10}(\sin x + \cos x) = \dfrac{(\log_{10} n) - 1}{2}$, then the value of '$n/3$' is ________.

20. The value of $\dfrac{\sin 1° + \sin 3° + \sin 5° + \sin 7°}{\cos 1° \cdot \cos 2° \cdot \sin 4°}$ is ________.

21. In a triangle ABC, if $A - B = 120°$ and $\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2} = \dfrac{1}{32}$ then, the value of $8\cos C$ is ________.

22. In a triangle ABC if $\tan A = \dfrac{1}{2}$, $\tan B = k + \dfrac{1}{2}$ and $\tan C = 2k + \dfrac{1}{2}$, then the possible value of $[k]$, where $[\cdot]$ represents greatest integer function is ________.

23. If $\sin^3 x \cos 3x + \cos^3 x \sin 3x = 3/8$, then the value of $8\sin 4x$ is ________.

Archives

Solutions on page 2.141

Subjective

1. If $\tan\alpha = \frac{m}{m+1}$ and $\tan\beta = \frac{1}{2m+1}$, find the possible values of $(\alpha+\beta)$. **(IIT-JEE, 1978)**

2. a. Draw the graph of $y = \frac{1}{\sqrt{2}}(\sin x + \cos x)$ from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$. **(IIT-JEE, 1979)**

   b. If $\cos(\alpha+\beta) = \frac{4}{5}$, $\sin(\alpha-\beta) = \frac{5}{13}$, and α, β lie between 0 and $\pi/4$, find $\tan 2\alpha$.

3. Prove that $5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3$ lies between -4 and 10.

4. Given $\alpha + \beta - \gamma = \pi$, prove that $\sin^2\alpha + \sin^2\beta - \sin^2\gamma = 2\sin\alpha\sin\beta\cos\gamma$. **(IIT-JEE, 1980)**

5. For all θ in $[0, \pi/2]$ show that $\cos(\sin\theta) \ge \sin(\cos\theta)$. **(IIT-JEE, 1981)**

6. Without using tables, prove that $(\sin 12°)(\sin 48°)(\sin 54°) = 1/8$. **(IIT-JEE, 1980)**

7. Show that $16\cos\left(\frac{2\pi}{15}\right)\cos\left(\frac{4\pi}{15}\right)\cos\left(\frac{8\pi}{15}\right)\cos\left(\frac{16\pi}{15}\right) = 1$. **(IIT-JEE, 1983)**

8. Prove that $\tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha = \cot\alpha$. **(IIT-JEE, 1988)**

9. ABC is a triangle such that $\sin(2A+B) = \sin(C-A) = -\sin(B+2C) = 1/2$. If A, B and C are in A.P. determine the values of A, B, and C. **(IIT-JEE, 1990)**

10. Show that the value of $\frac{\tan x}{\tan 3x}$, wherever defined, never lies between $\frac{1}{3}$ and 3. **(IIT-JEE, 1992)**

11. Prove that $\sum_{k=1}^{n-1}(n-k)\cos\frac{2k\pi}{n} = -\frac{n}{2}$, where $n \ge 3$ is an integer. **(IIT-JEE, 1997)**

12. Find the range of values of t for which $2\sin t = \frac{1-2x+5x^2}{3x^2-2x-1}$, $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. **(IIT-JEE, 2005)**

13. Find the maximum value of the expression $\frac{1}{\sin^2\theta + 3\sin\theta\cos\theta + 5\cos^2\theta}$. **(IIT-JEE, 2010)**

Objective

Fill in the blanks

1. Suppose $\sin^3 x \sin 3x = \sum_{m=0}^{n} C_m \cos mx$ is an identity in x, where $C_0, C_1, \ldots, C_n$ are constants, and $C_n \ne 0$, then the value of n is______. **(IIT-JEE, 1981)**

2. The side of a triangle inscribed in a given circle subtends angles α, β and γ at the centre. The minimum value of the arithmetic mean of $\cos\left(\alpha + \frac{\pi}{2}\right)$, $\cos\left(\beta + \frac{\pi}{2}\right)$ and $\cos\left(\gamma + \frac{\pi}{2}\right)$ is equal to______.

   (IIT-JEE, 1987)

3. The value of $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{7\pi}{14}\sin\frac{9\pi}{14}\sin\frac{11\pi}{14}\sin\frac{13\pi}{14}$ is equal to______. **(IIT-JEE, 1991)**

4. If $K = \sin(\pi/18)\sin(5\pi/18)\sin(7\pi/18)$, then the numerical value of K is______. **(IIT-JEE, 1993)**

5. If $A > 0$, $B > 0$ and $A + B = \pi/3$, then the maximum value of $\tan A \tan B$ is______. **(IIT-JEE, 1993)**

6. If $\cos(x - y)$, $\cos x$ and $\cos(x + y)$ are in H.P., then $\cos x \sec\left(\frac{y}{2}\right) =$ ______. **(IIT-JEE, 1997)**

True or false

1. If $\tan A = \frac{1 - \cos B}{\sin B}$, then $\tan 2A = \tan B$. **(IIT-JEE, 1983)**

Multiple choice questions with one correct answer

1. If $\tan\theta + \sin\theta = m$ and $\tan\theta - \sin\theta = n$, then
   a. $m^2 - n^2 = 4mn$
   b. $m^2 + n^2 = 4mn$
   c. $m^2 - n^2 = m^2 + n^2$
   d. $m^2 - n^2 = 4\sqrt{mn}$ **(IIT-JEE, 1970)**

2. If $\tan\theta = -\frac{4}{3}$, then $\sin\theta$ is **(IIT-JEE, 1979)**
   a. $-\frac{4}{5}$ but not $\frac{4}{5}$
   b. $-\frac{4}{5}$ or $\frac{4}{5}$
   c. $\frac{4}{5}$ but not $-\frac{4}{5}$
   d. none of these

3. If $\alpha + \beta + \gamma = 2\pi$, then
   a. $\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2} = \tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$
   b. $\tan\frac{\alpha}{2}\tan\frac{\beta}{2} + \tan\frac{\beta}{2}\tan\frac{\gamma}{2} + \tan\frac{\gamma}{2}\tan\frac{\alpha}{2} = 1$
   c. $\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2} = -\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$
   d. none of these **(IIT-JEE, 1979)**

4. Given $A = \sin^2\theta + \cos^4\theta$, then for all real θ,
   a. $1 \le A \le 2$
   b. $3/4 \le A \le 1$
   c. $13/16 \le A \le 1$
   d. $3/4 \le A \le 13/16$ **(IIT-JEE, 1980)**

5. The value of $\left(1 + \cos\frac{\pi}{8}\right)\left(1 + \cos\frac{3\pi}{8}\right)\left(1 + \cos\frac{5\pi}{8}\right)\left(1 + \cos\frac{7\pi}{8}\right)$ is
   a. 1/4
   b. 3/4
   c. 1/8
   d. 3/8 **(IIT-JEE, 1984)**

6. The value of the expression $\sqrt{3}$ cosec $20° - \sec 20°$ is equal to
   a. 2
   b. $2\sin 20°/\sin 40°$
   c. 4
   d. $4\sin 20°/\sin 40°$ **(IIT-JEE, 1988)**

7. $3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x)$ is equal to
   a. 11
   b. 12
   c. 13
   d. 14 **(IIT-JEE, 1995)**

8. $\sec^2\theta = \frac{4xy}{(x+y)^2}$ is true if and only if
   a. $x + y \ne 0$
   b. $x = y, x \ne 0$
   c. $x = y$
   d. $x \ne 0, y \ne 0$ **(IIT-JEE, 1996)**

9. Let $f(\theta) = \sin\theta(\sin\theta + \sin 3\theta)$. Then $f(\theta)$ is
   a. ≥ 0 only when $\theta \ge 0$
   b. ≤ 0 for all real θ
   c. ≥ 0 for all real θ
   d. ≤ 0 only when $\theta \le 0$ **(IIT-JEE, 2000)**

10. The maximum value of $(\cos\alpha_1)(\cos\alpha_2)\cdots(\cos\alpha_n)$, under the restrictions $0 \le \alpha_1, \alpha_2, \ldots, \alpha_n \le \pi/2$ and $(\cot\alpha_1)(\cot\alpha_2)\cdots(\cot\alpha_n) = 1$ is

a. $1/2^{n/2}$ b. $1/2^n$ c. $1/2n$ d. 1

(IIT-JEE, 2001)

11. If $\alpha + \beta = \pi/2$ and $\beta + \gamma = \alpha$, then $\tan\alpha$ equals

a. $2(\tan\beta + \tan\gamma)$ b. $\tan\beta + \tan\gamma$ c. $\tan\beta + 2\tan\gamma$ d. $2\tan\beta + \tan\gamma$

(IIT-JEE, 2001)

12. Given both θ and ϕ are acute angles and $\sin\theta = 1/2$, $\cos\phi = 1/3$, then the value of $\theta + \phi$ belongs to

a. $\left(\frac{\pi}{3}, \frac{\pi}{2}\right]$ b. $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right]$ c. $\left(\frac{2\pi}{3}, \frac{5\pi}{6}\right]$ d. $\left(\frac{5\pi}{6}, \pi\right]$

(IIT-JEE, 2004)

13. Let $0 < x < \pi/4$, then $(\sec 2x - \tan 2x)$ equals

a. $\tan\left(x - \frac{\pi}{4}\right)$ b. $\tan\left(\frac{\pi}{4} - x\right)$ c. $\tan\left(x + \frac{\pi}{4}\right)$ d. $\tan^2\left(x + \frac{\pi}{4}\right)$

(IIT-JEE, 1994)

14. Let n be a positive integer such that $\sin\frac{\pi}{2n} + \cos\frac{\pi}{2n} = \frac{\sqrt{x}}{2}$. Then

a. $6 \le n \le 8$ b. $4 < n \le 8$ c. $4 \le n \le 8$ d. $4 < n < 8$

(IIT-JEE, 1994)

15. Let $\theta \in (0, \pi/4)$ and $t_1 = (\tan\theta)^{\tan\theta}$, $t_2 = (\tan\theta)^{\cot\theta}$, $t_3 = (\cot\theta)^{\tan\theta}$ and $t_4 = (\cot\theta)^{\cot\theta}$, then

a. $t_1 > t_2 > t_3 > t_4$ b. $t_4 > t_3 > t_1 > t_2$ c. $t_3 > t_1 > t_2 > t_4$ d. $t_2 > t_3 > t_1 > t_2$

(IIT-JEE, 2006)

Multiple choice questions with one or more than one correct answers

1. The expression $3\left[\sin^4\left(\frac{3}{2}\pi - \alpha\right) + \sin^4(3\pi + \alpha)\right] - 2\left[\sin^6\left(\frac{1}{2}\pi + \alpha\right) + \sin^6(5\pi - \alpha)\right]$ is equal to

a. 0 b. 1 c. 3 d. none of these

(IIT-JEE, 1984)

2. For $0 < \phi \le \pi/2$, if $x = \sum_{n=0}^{\infty} \cos^{2n}\phi$, $y = \sum_{n=0}^{\infty} \sin^{2n}\phi$, $z = \sum_{n=0}^{\infty} \cos^{2n}\phi \sin^{2n}\phi$, then

a. $xyz = xz + y$ b. $xyz = xy + z$ c. $xyz = x + y + z$ d. $xyz = yz + x$

(IIT-JEE, 1992)

3. Which of the following number(s) is/are rational?

a. $\sin 15°$ b. $\cos 15°$ c. $\sin 15° \cos 15°$ d. $\sin 15° \cos 75°$

(IIT-JEE, 1998)

4. For a positive integer n, let $f_n(\theta) = (\tan\theta/2)(1 + \sec\theta)(1 + \sec 2\theta)(1 + \sec 4\theta)\cdots(1 + \sec 2^n\theta)$. Then

a. $f_2(\pi/16) = 1$ b. $f_3(\pi/32) = 1$ c. $f_4(\pi/64) = 1$ d. $f_5(\pi/128) = 1$

(IIT-JEE, 1999)

5. The minimum value of the expression $\sin\alpha + \sin\beta + \sin\gamma$, where α, β, γ are real numbers satisfying $\alpha + \beta + \gamma = \pi$ is

   a. positive b. zero c. negative d. -3 (IIT-JEE, 1995)

ANSWERS AND SOLUTIONS

Subjective Type

1. Let $m = 2k$, i.e., m is even where $k \in I$

   Now, $\beta = 2k\pi + \frac{\pi}{2} - A = \left(2k + \frac{1}{2}\right)\pi - A$ (i)

   If $m = 2k + 1$, i.e., m is odd, then

   $\beta = (2k+1)\pi - \left(\frac{\pi}{2} - A\right) = \left(2k + \frac{1}{2}\right)\pi + A$ (ii)

   From Eqs. (i) and (ii), β can be expressed as

   $\beta = \left(2k + \frac{1}{2}\right)\pi \pm A, k \in I$

   which is same as α.

2. $A + B + C = \pi$

   $\Rightarrow \frac{A}{2} + \frac{B}{2} = \frac{\pi}{2} - \frac{C}{2}$

   $\Rightarrow \cot\left(\frac{A}{2} + \frac{B}{2}\right) = \cot\left(\frac{\pi}{2} - \frac{C}{2}\right)$

   $\Rightarrow \frac{\cot\frac{A}{2}\cot\frac{B}{2} - 1}{\cot\frac{A}{2} + \cot\frac{B}{2}} = \tan\frac{C}{2} = \frac{1}{\cot\frac{C}{2}}$

   $\Rightarrow \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} = \cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}$ (i)

   But $\tan\frac{A}{2}, \tan\frac{B}{2}, \tan\frac{C}{2}$ are in H.P.

   $\therefore \cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ are in A.P.

   So, $\cot\frac{A}{2} + \cot\frac{C}{2} = 2\cot\frac{B}{2}$.

   Hence, Eq. (i) becomes $\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} = 3\cot\frac{B}{2} \Rightarrow \cot\frac{A}{2}\cot\frac{C}{2} = 3$

   $\Rightarrow$ G.M. of $\cot\frac{A}{2}$ and $\cot\frac{C}{2} = \sqrt{\cot\frac{A}{2}\cot\frac{C}{2}} = \sqrt{3}$

   and A.M. of $\cot\frac{A}{2}$ and $\cot\frac{C}{2} = \frac{\cot\frac{A}{2} + \cot\frac{C}{2}}{2} = \cot\frac{B}{2}$

But A.M. ≥ G.M.

$\Rightarrow \quad \cot \dfrac{B}{2} \geq \sqrt{3}$

Therefore, the minimum value of $\cot B/2$ is $\sqrt{3}$.

3. $\operatorname{cosec} \theta = \dfrac{1}{\sin\theta}$

$$= \frac{1}{\sin\theta} \frac{\sin\theta/2}{\sin\theta/2} = \frac{\sin\left(\theta - \frac{\theta}{2}\right)}{\sin\theta \sin\left(\frac{\theta}{2}\right)} = \frac{\sin\theta\cos\frac{\theta}{2} - \cos\theta\sin\frac{\theta}{2}}{\sin\theta\sin\frac{\theta}{2}}$$

$\therefore \operatorname{cosec} \theta = \cot\dfrac{\theta}{2} - \cot\theta$

Similarly, $\operatorname{cosec} 2\theta = \cot\theta - \cot 2\theta$

$\operatorname{cosec} 4\theta = \cot 2\theta - \cot 4\theta$

$\vdots$

$\operatorname{cosec} 2^{n-1}\theta = \cot 2^{n-2}\theta - \cot 2^{n-1}\theta$

Therefore, sum $= \cot\dfrac{\theta}{2} - \cot 2^{n-1}\theta$

4. Let $\cot\theta = \cot A + \cot B + \cot C$

$\Rightarrow \cot\theta - \cot A = \cot B + \cot C$

$$\Rightarrow \quad \frac{\sin(A-\theta)}{\sin A\sin\theta} = \frac{\sin(B+C)}{\sin B\sin C}$$

$$\Rightarrow \quad \sin(A-\theta) = \frac{\sin^2 A\sin\theta}{\sin B\sin C} \qquad \text{(i)}$$

Similarly, $\sin(B-\theta) = \dfrac{\sin^2 B\sin\theta}{\sin A\sin C}$ (ii)

and $\sin(C-\theta) = \dfrac{\sin^2 C\sin\theta}{\sin A\sin B}$ (iii)

By multiplying corresponding sides of Eqs. (i), (ii), and (iii), we have

$\sin^3\theta = \sin(A-\theta)\sin(B-\theta)\sin(C-\theta)$

5. Let $\sin\dfrac{A}{2} + \sin\dfrac{B}{2} + \sin\dfrac{C}{2} = k$

or $2\sin\dfrac{A+B}{4}\cos\dfrac{A-B}{4} + \cos\dfrac{A+B}{2} = k$

$$\Rightarrow \quad 2\sin^2\frac{A+B}{4} - 2\cos\frac{A-B}{4}\sin\frac{A+B}{4} + k - 1 = 0$$

Since $\sin\dfrac{A+B}{4}$ is real, $4\cos^2\dfrac{A-B}{4} - 8(k-1) \geq 0$

$$\Rightarrow \quad 2(k-1) \leq \cos^2\frac{A-B}{4} \leq 1 \Rightarrow k \leq 3/2$$

Hence, $2\sin\frac{A+B}{4}\left[\cos\frac{A-B}{4}-\sin\frac{\pi-C}{4}\right]\le\frac{1}{2}$

$\Rightarrow\quad 2\sin\frac{A+B}{4}\left[\cos\frac{A-B}{4}-\cos\frac{\pi+C}{4}\right]\le\frac{1}{2}$

$\Rightarrow\quad 4\sin\frac{A+B}{4}\sin\frac{\pi+C+A-B}{8}\sin\frac{\pi+C-A+B}{8}\le\frac{1}{2}$

$\Rightarrow\quad 4\sin\frac{\pi-C}{4}\sin\frac{\pi-B}{4}\sin\frac{\pi-A}{4}\le\frac{1}{2}\qquad\Rightarrow\cos\frac{\pi+C}{4}\cos\frac{\pi+B}{4}\cos\frac{\pi+A}{4}\le\frac{1}{8}$

6. Here $\frac{x}{y}=\frac{\tan(\theta+\alpha)}{\tan(\theta+\beta)}$. By componendo and dividendo, we get

$$\frac{x+y}{x-y}=\frac{\tan(\theta+\alpha)+\tan(\theta+\beta)}{\tan(\theta+\alpha)-\tan(\theta+\beta)}=\frac{\sin(2\theta+\alpha+\beta)}{\sin(\alpha-\beta)}$$

$$\therefore\ \frac{x+y}{x-y}\sin^2(\alpha-\beta)=\sin(2\theta+\alpha+\beta)\ \sin(\alpha-\beta)$$

$$=\frac{1}{2}[\cos 2(\theta+\beta)-\cos 2(\theta+\alpha)] \qquad \text{(i)}$$

Similarly, $\frac{y+z}{y-z}\sin^2(\beta-\gamma)=\frac{1}{2}[\cos 2(\theta+\gamma)-\cos 2(\theta+\beta)]$ (ii)

and $\frac{z+x}{z-x}\sin^2(\gamma-\alpha)=\frac{1}{2}[\cos 2(\theta+\alpha)-\cos 2(\theta+\gamma)]$ (iii)

Adding Eqs. (i), (ii), and (iii), we get L.H.S. = 0.

7. Here, we have $\tan 6\theta=p/q$

$\Rightarrow\quad \frac{\sin 6\theta}{\cos 6\theta}=\frac{p}{q}\qquad\Rightarrow\frac{p}{\sin 6\theta}=\frac{q}{\cos 6\theta}=\frac{\sqrt{p^2+q^2}}{\sqrt{1}}=\sqrt{p^2+q^2}=k$ (say)

Now $y=\frac{1}{2}(p\operatorname{cosec}\theta-q\sec 2\theta)=\frac{1}{2}\left(\frac{p}{\sin 2\theta}-\frac{q}{\cos 2\theta}\right)$

$\Rightarrow\quad y=\frac{1}{2}\left[\frac{p\cos 2\theta-q\sin 2\theta}{\sin 2\theta\cos 2\theta}\right]$

$$=\left[\frac{2k\sin 6\theta\cos 2\theta-2k\cos 6\theta\sin 2\theta}{4\sin 2\theta\cos 2\theta}\right]=k\frac{\sin(6\theta-2\theta)}{\sin 4\theta}=k=\sqrt{p^2+q^2}$$

8. $\sin\alpha+\cos\alpha+(\tan\alpha+\cot\alpha)+(\sec\alpha+\operatorname{cosec}\alpha)=7$

$\Rightarrow\quad (\sin\alpha+\cos\alpha)+\frac{1}{\sin\alpha\cos\alpha}+\frac{\sin\alpha+\cos\alpha}{\sin\alpha\cos\alpha}=7$

$\Rightarrow\quad (\sin\alpha+\cos\alpha)\left(1+\frac{1}{\sin\alpha\cos\alpha}\right)=7-\frac{1}{\sin\alpha\cos\alpha}$

$\Rightarrow\quad (1+\sin 2\alpha)\left(1+\frac{4}{\sin 2\alpha}+\frac{4}{\sin^2 2\alpha}\right)=49-\frac{28}{\sin 2\alpha}+\frac{4}{\sin^2 2\alpha}$

Let $\sin 2\alpha = x$

$$\Rightarrow \quad (1+x)\left(1+\frac{4}{x}+\frac{4}{x^2}\right)=49-\frac{28}{x}+\frac{4}{x^2} \quad \Rightarrow (1+x)\left(x^2+4x+4\right)=49x^2-28x+4$$

$$\Rightarrow \quad x^3-44x^2-36x=0$$

$$\Rightarrow \quad x^2-44x-36=0. \text{ (as } x=\sin 2a \neq 0)$$

9. We have

$$1+\cot\theta-\cot\frac{\theta}{2}=1+\frac{\cot^2\frac{\theta}{2}-1}{2\cot\frac{\theta}{2}}-\cot\frac{\theta}{2}$$

$$=\frac{2\cot\frac{\theta}{2}+\cot^2\frac{\theta}{2}-1-2\cot^2\frac{\theta}{2}}{2\cot\frac{\theta}{2}}$$

$$=\frac{-\left(\cot\frac{\theta}{2}-1\right)^2}{2\cot\frac{\theta}{2}}\leq 0 \text{ for } 0<\theta<\pi$$

$$\Rightarrow \quad 1+\cot\theta\leq\cot\frac{\theta}{2}$$

Equality holds when $\cot\frac{\theta}{2}-1=0\Rightarrow\theta=\frac{\pi}{2}$.

10. Since A.M. of two positive quantities $\geq$ their G.M., we have

$$\frac{2^{\sin x}+2^{\cos x}}{2}\geq\sqrt{2^{\sin x}\,2^{\cos x}}=\sqrt{2^{\sin x+\cos x}}=\sqrt{2^{\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)}}\geq\sqrt{2^{-\sqrt{2}}}$$

$$\Rightarrow \quad 2^{\sin x}+2^{\cos x}\geq 2\times 2^{\frac{-1}{\sqrt{2}}}=2^{1-\frac{1}{\sqrt{2}}}$$

11. We have $\frac{A}{2}+\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2}$, so that

$$\tan\left(\frac{A}{2}+\frac{B}{2}\right)=\tan\left(\frac{\pi}{2}-\frac{C}{2}\right)$$

$$\Rightarrow \quad \frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{1-\tan\frac{A}{2}\tan\frac{B}{2}}=\frac{1}{\tan\frac{C}{2}}$$

$$\Rightarrow \quad \tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1$$

Now $\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}-1$

$$=\frac{1}{2}\left[\sum 2\tan^2\frac{A}{2}-\sum 2\tan\frac{A}{2}\tan\frac{B}{2}\right]$$

$$=\frac{1}{2}\left[\left(\tan\frac{A}{2}-\tan\frac{B}{2}\right)^2+\left(\tan\frac{B}{2}-\tan\frac{C}{2}\right)^2+\left(\tan\frac{C}{2}-\tan\frac{A}{2}\right)^2\right]\geq 0.$$

12. $A+B+C=\pi$

$$\Rightarrow\quad B+C=\frac{3\pi}{4}\Rightarrow 0<B,C<\frac{3\pi}{4}\text{ Also }\tan B\tan C=p$$

$$\Rightarrow\quad \frac{\sin B\sin C}{\cos B\cos C}=\frac{p}{1}$$

$$\Rightarrow\quad \frac{\cos B\cos C-\sin B\sin C}{\cos B\cos C+\sin B\sin C}=\frac{1-p}{1+p}$$

$$\Rightarrow\quad \frac{\cos(B+C)}{\cos(B-C)}=\frac{1-p}{1+p}$$

$$\Rightarrow\quad \frac{1+p}{\sqrt{2}(p-1)}=\cos(B-C) \qquad \text{(i)}$$

Since B or C can vary from 0 to $3\pi/4$, we get

$$0\leq B-C<\frac{3\pi}{4}\Rightarrow -\frac{1}{\sqrt{2}}<\cos(B-C)\leq 1.$$

Equation (i) will now lead to $-\frac{1}{\sqrt{2}}<\frac{p+1}{\sqrt{2}(p-1)}\leq 1$

$$\text{For } 0<1+\frac{p+1}{p-1}\qquad\Rightarrow\frac{2p}{(p-1)}>0\qquad\Rightarrow p<0\text{ or }p>1 \qquad \text{(ii)}$$

$$\text{Also } \frac{p+1-\sqrt{2}(p-1)}{\sqrt{2}(p-1)}\leq 0$$

$$\Rightarrow\quad \frac{\left(p-\left(\sqrt{2}+1\right)^2\right)}{(p-1)}\geq 0$$

$$\Rightarrow\quad p<1\text{ or }p\geq\left(\sqrt{2}+1\right)^2 \qquad \text{(iii)}$$

Combining Eqs. (ii) and (iii), we get $p<0$ or $p\geq\left(\sqrt{2}+1\right)^2$.

13. Adding $\sin(a+x)+\sin(a-x)=2(b+c)$

$\Rightarrow 2\sin a\cos x=2(b+c)$

$$\Rightarrow \cos x=\frac{b+c}{\sin a} \qquad \text{(i)}$$

Subtracting, we get

$\sin(a+x)-\sin(a-x)=2(b-c)$

$\Rightarrow 2\cos a \sin x = 2(b-c)$

$$\Rightarrow \sin x = \frac{b-c}{\cos a} \qquad \text{(ii)}$$

Squaring and adding Eq. (i) and Eq. (ii), we get

$$\frac{(b+c)^2}{\sin^2 a}+\frac{(b-c)^2}{\cos^2 a}=1$$

14. $\tan\beta = \dfrac{n\sin\alpha\cos\alpha}{1-n\sin^2\alpha}$

$$=\frac{\dfrac{n\sin\alpha\cos\alpha}{\cos^2\alpha}}{\dfrac{1}{\cos^2\alpha}-\dfrac{n\sin^2\alpha}{\cos^2\alpha}} \qquad \text{[dividing numerator and denominator by } \cos^2\alpha\text{]}$$

$$=\frac{n\tan\alpha}{\sec^2\alpha-n\tan^2\alpha}=\frac{n\tan\alpha}{1+\tan^2\alpha-n\tan^2\alpha}=\frac{n\tan\alpha}{1+(1-n)\tan^2\alpha} \qquad \text{(i)}$$

Now, L.H.S. $=\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

$$=\frac{\tan\alpha-\dfrac{n\tan\alpha}{1+(1-n)\tan^2\alpha}}{1+\tan\alpha\dfrac{n\tan\alpha}{1+(1-n)\tan^2\alpha}} \qquad \text{[From Eq. (i)]}$$

$$=\frac{\tan\alpha+(1-n)\tan^3\alpha-n\tan\alpha}{1+(1-n)\tan^2\alpha+n\tan^2\alpha}=\frac{(1-n)\tan\alpha+(1-n)\tan^3\alpha}{1+\tan^2\alpha}=(1-n)\tan\alpha$$

15. L.H.S. contains x, $3x$, $9x$ and $27x$, whereas R.H.S contains $27x$ and x only. So, we will manipulate terms as shown below

R.H.S. $=\dfrac{1}{2}[\tan 27x-\tan x]$

$$=\frac{1}{2}[(\tan 27x-\tan 9x)+(\tan 9x-\tan 3x)+(\tan 3x-\tan x)]$$

$$=\frac{1}{2}\left[\left(\frac{\sin 27x}{\cos 27x}-\frac{\sin 9x}{\cos 9x}\right)+\left(\frac{\sin 9x}{\cos 9x}-\frac{\sin 3x}{\cos 3x}\right)+\left(\frac{\sin 3x}{\cos 3x}-\frac{\sin x}{\cos x}\right)\right]$$

$$=\frac{1}{2}\left[\frac{\sin(27x-9x)}{\cos 27x\cos 9x}+\frac{\sin(9x-3x)}{\cos 9x\cos 3x}+\frac{\sin(3x-x)}{\cos 3x\cos x}\right]$$

$$=\frac{1}{2}\left[\frac{\sin 18x}{\cos 27x\cos 9x}+\frac{\sin 6x}{\cos 9x\cos 3x}+\frac{\sin 2x}{\cos 3x\cos x}\right]$$

$$= \frac{1}{2}\left[\frac{2\sin 9x\cos 9x}{\cos 27x\cos 9x} + \frac{2\sin 3x\cos 3x}{\cos 9x\cos 3x} + \frac{2\sin x\cos x}{\cos 3x\cos x}\right]$$

$$= \frac{\sin 9x}{\cos 27x} + \frac{\sin 3x}{\cos 9x} + \frac{\sin x}{\cos 3x} = \frac{\sin x}{\cos 3x} + \frac{\sin 3x}{\cos 9x} + \frac{\sin 9x}{\cos 27x} = \text{L.H.S.}$$

16. We have to prove $\dfrac{2\cos 2^n\theta+1}{2\cos\theta+1} = (2\cos\theta-1)(2\cos 2\theta-1)(2\cos 2^2\theta-1)\ldots(2\cos 2^{n-1}\theta-1)$

or $2\cos 2^n\theta+1 = [(2\cos\theta+1)(2\cos\theta-1)](2\cos 2\theta-1)(2\cos 2^2\theta-1)\ldots(2\cos 2^{n-1}\theta-1)$

Now $[(2\cos\theta+1)(2\cos\theta-1)](2\cos 2\theta-1)(2\cos 2^2\theta-1)\ldots(2\cos 2^{n-1}\theta-1)$

$= (4\cos^2\theta-1)(2\cos 2\theta-1)(2\cos 2^2\theta-1\ldots(2\cos 2^{n-1}\theta-1)$

$= (2\cos 2\theta+1)(2\cos 2\theta-1)(2\cos 2^2\theta-1)\ldots(2\cos 2^{n-1}\theta-1)$ [using $\cos 2\theta = 2\cos^2\theta-1$]

$= (4\cos^2 2\theta-1)(2\cos 2^2\theta-1)\ldots(2\cos 2^{n-1}\theta-1)$

$= (2\cos 2^2\theta+1)(2\cos 2^2\theta-1)\ldots(2\cos 2^{n-1}\theta-1)$

$= (4\cos^2 2^2\theta-1)(2\cos 2^3\theta-1)\ldots(2\cos 2^{n-1}\theta-1)$

$\vdots$

$= (2\cos 2^{n-1}\theta+1)(2\cos 2^{n-1}\theta-1)$

$= 4\cos^2 2^{n-1}\theta-1$

$= 2\cos 2^n\theta+1$

17. We have to prove that $\dfrac{\tan 2^n\theta}{\tan\theta} = (1+\sec 2\theta)(1+\sec 2^2\theta)(1+\sec 2^3\theta)\ldots(1+\sec 2^n\theta)$

or $\tan 2^n\theta = \tan\theta(1+\sec 2\theta)(1+\sec 2^2\theta)(1+\sec 2^3\theta)\ldots(1+\sec 2^n\theta)$

Now $\tan\theta(1+\sec 2\theta)(1+\sec 2^2\theta)(1+\sec 2^3\theta)\ldots(1+\sec 2^n\theta)$

$$= \tan\theta\left(\frac{1+\cos 2\theta}{\cos 2\theta}\right)(1+\sec 2^2\theta)(1+\sec 2^3\theta)\ldots(1+\sec 2^n\theta)$$

$$= \frac{\sin\theta}{\cos\theta}\left(\frac{2\cos^2\theta}{\cos 2\theta}\right)(1+\sec 2^2\theta)(1+\sec 2^3\theta)\ldots(1+\sec 2^n\theta)$$

$= (\tan 2\theta)(1+\sec 2^2\theta)(1+\sec 2^3\theta)\ldots(1+\sec 2^n\theta)$

$$= (\tan 2\theta)\left(\frac{1+\cos 2^2\theta}{\cos 2^2\theta}\right)(1+\sec 2^3\theta)\ldots(1+\sec 2^n\theta)$$

$$= (\tan 2\theta)\left(\frac{2\cos^2 2\theta}{\cos 2^2\theta}\right)(1+\sec 2^3\theta)\ldots(1+\sec 2^n\theta)$$

$= (\tan 2^2\theta)(1+\sec 2^3\theta)\ldots(1+\sec 2^n\theta)$

$\vdots$

$= \tan 2^{n-1}\theta(1+\sec 2^n\theta)$

$$= \tan 2^{n-1}\theta\left(\frac{1+\cos 2^n\theta}{\cos 2^n\theta}\right)$$

$$= \tan 2^{n-1}\theta\left(\frac{2\cos^2 2^{n-1}\theta}{\cos 2^n\theta}\right)$$

$= \tan 2^n\theta$

Objective Type

1. b. Since $f(x) = \sin x$ is an increasing function for $0 < x < \pi/2$ and 1 radian is approximately 57°. Therefore, $1° < 1^R \Rightarrow \sin 1° < \sin 1$.

2. a. We have $\dfrac{x^2+y^2}{2xy} = \dfrac{1}{2}\left(\dfrac{x}{y} + \dfrac{y}{x}\right)$

Now, $\sin^2\theta = \dfrac{x^2+y^2}{2xy} \quad \Rightarrow \dfrac{x^2+y^2}{2xy} \geq 0$ $\quad [\because \sin^2\theta \geq 0]$

Therefore, x and y have the same sign.

Now, $\dfrac{x^2+y^2}{2xy} = \dfrac{1}{2}\left(\dfrac{x}{y} + \dfrac{y}{x}\right) \quad \Rightarrow \dfrac{x^2+y^2}{2xy} \geq 1$ ($\because$ A.M. $\geq$ G.M.)

But $\sin^2\theta \leq 1$. Therefore, $\dfrac{x^2+y^2}{2xy} = 1 \Rightarrow x = y$.

3. d. Since $0 < x < \pi$. Therefore, $\sin x > 0$

We have $1 + \sin x + \sin^2 x + \cdots \infty = 4 + 2\sqrt{3}$

$$\Rightarrow \quad \frac{1}{1-\sin x} = 4 + 2\sqrt{3}$$

$$\Rightarrow \quad \sin x = 1 - \frac{1}{4+2\sqrt{3}}$$

$$= \frac{3+2\sqrt{3}}{4+2\sqrt{3}} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad x = \frac{\pi}{3} \text{ or } \frac{2\pi}{3}$$

4. b. $\dfrac{\cos x \tan x}{k^2} + \dfrac{1}{\tan x} + \dfrac{\sin x}{1+\cos x}$

$$= \frac{\sin x}{k^2} + \frac{\cos x\,(1+\cos x) + \sin^2 x}{\sin x\,(1+\cos x)} = \frac{\cos x(1+\cos x) + (1-\cos^2 x)}{\sin x(1+\cos x)} = \frac{a}{k} + \frac{1}{\sin x} = \frac{a}{k} + \frac{1}{ak}$$

5. a. $2\sin\dfrac{A}{2}\operatorname{cosec}\dfrac{B}{2}\left(\sin\dfrac{C}{2} - \cos\dfrac{A}{2}\cos\dfrac{B}{2}\right) - \cos A$

$$= 2\sin\frac{A}{2}\operatorname{cosec}\frac{B}{2}\left(\cos\frac{A+B}{2} - \cos\frac{A}{2}\cos\frac{B}{2}\right) - \cos A$$

$$= 2\sin\frac{A}{2}\operatorname{cosec}\frac{B}{2}\left(-\sin\frac{A}{2}\sin\frac{B}{2}\right) - \cos A = -2\sin^2\frac{A}{2} - \cos A = -1$$

6. b. Applying A.M. ≥ G.M. in $6\tan^2\phi$, $54\cot^2\phi$, 18, we get

$$\frac{6\tan^2\phi+54\cot^2\phi+18}{3}\geq(6\times54\times18)^{1/3}\geq18$$

This is true if $6\tan^2\phi=54\cot^2\phi=18$

$\Rightarrow\quad \tan^2\phi=3$ and $\cot^2\phi=1/3$

Therefore, I and II are correct.

7. c. $5\tan\theta=4\Rightarrow\tan\theta=\frac{4}{5}$

Now $\frac{5\sin\theta-3\cos\theta}{5\sin\theta+2\cos\theta}=\frac{5\frac{\sin\theta}{\cos\theta}-3}{5\frac{\sin\theta}{\cos\theta}+2}=\frac{5\tan\theta-3}{5\tan\theta+2}=\frac{5\times\frac{4}{5}-3}{5\times\frac{4}{5}+2}=\frac{1}{6}$

8. b. $2\sec2\theta=\tan\phi+\cot\phi$

$$\Rightarrow\quad \frac{2}{\cos2\theta}=\frac{\sin^2\phi+\cos^2\phi}{\sin\phi\cos\phi}$$

$$\Rightarrow\quad \frac{2}{\cos2\theta}=\frac{1}{\sin\phi\cos\phi}$$

$$\Rightarrow\quad \cos2\theta=\sin2\phi$$

$$\Rightarrow\quad 2\theta=90°-2\phi$$

$$\Rightarrow\quad \theta+\phi=\frac{\pi}{4}$$

9. a. $\sin x+\operatorname{cosec}x=2$

$\Rightarrow\quad (\sin x-1)^2=0$

$\Rightarrow\quad \sin x=1$

$\Rightarrow\quad \sin^n x+\operatorname{cosec}^n x=1+1=2$

10. c. $\sec^2\theta+\operatorname{cosec}^2\theta=\frac{1}{\cos^2\theta}+\frac{1}{\sin^2\theta}=\frac{4}{\sin^2 2\theta}\geq4$

Also, $\sec^2\theta\operatorname{cosec}^2\theta=\frac{4}{\sin^2 2\theta}\geq4$

Hence, the only equation which can have roots $\operatorname{cosec}^2\theta$ and $\sec^2\theta$ is $x^2-5x+5=0$.

11. b. $\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}+\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}=\frac{1-\cos\alpha+1+\cos\alpha}{\sqrt{1-\cos^2\alpha}}$

$$=\frac{2}{|\sin x|}=\frac{2}{-\sin\alpha}\ (\text{since } \pi<\alpha<3\pi/2)$$

12. b. We have $\cos\frac{\pi}{7}+\cos\frac{2\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{5\pi}{7}+\cos\frac{6\pi}{7}+\cos\frac{7\pi}{7}$

$$=\left(\cos\frac{\pi}{7}+\cos\frac{6\pi}{7}\right)+\left(\cos\frac{2\pi}{7}+\cos\frac{5\pi}{7}\right)+\left(\cos\frac{3\pi}{7}+\cos\frac{4\pi}{7}\right)+\cos\pi$$

$$=\left(\cos\frac{\pi}{7}-\cos\frac{\pi}{7}\right)+\left(\cos\frac{2\pi}{7}-\cos\frac{2\pi}{7}\right)+\left(\cos\frac{3\pi}{7}-\cos\frac{3\pi}{7}\right)+\cos\pi$$

$=\cos\pi=-1$

13. b. $2\sin^2\theta+3\cos^2\theta=2(\sin^2\theta+\cos^2\theta)+\cos^2\theta=2+\cos^2\theta\geq 2$
$[\because \cos^2\theta>0]$

14. b. $\sin^4\theta+\cos^4\theta=(\sin^2\theta+\cos^2\theta)^2-2\sin^2\theta\cos^2\theta=1-2\sin^2\theta\cos^2\theta\leq 1$

15. d. We have
$f(x)=\cos^2\theta+\sec^2\theta=(\cos\theta-\sec\theta)^2+2\cos\theta\sec\theta=2+(\cos\theta-\sec\theta)^2\geq 2$

16. a. $f(x)=\cos^6 x+\sin^6 x$

$=(\cos^2 x+\sin^2 x)(\sin^4 x+\cos^4 x-\cos^2 x\sin^2 x)$

$=((\sin^2 x+\cos^2 x)^2-3\sin^2 x\cos^2 x)$

$=1-\frac{3}{4}\sin^2 2x$

$\Rightarrow\ f(x)\in\left[\frac{1}{4},1\right]$

17. a. $f(x)=3\cos x+5\sin(x-\pi/6)$

$=\frac{1}{2}\cos x+5\times\frac{\sqrt{3}}{2}\sin x$

Then, $-\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{5\sqrt{3}}{2}\right)^2}\leq f(x)\leq\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{5\sqrt{3}}{2}\right)^2}$

$\Rightarrow\ -\sqrt{19}\leq f(x)\leq\sqrt{19}$

18. d. $1-\cos 2x+\sin 2x=2k$

$\Rightarrow\ \sin 2x-\cos 2x=2k-1$

$\Rightarrow\ \sin(2x-\alpha)=\frac{2k-1}{\sqrt{2}}$

$\Rightarrow\ -1\leq\frac{2k-1}{\sqrt{2}}\leq 1$

$\Rightarrow\ \frac{1-\sqrt{2}}{2}\leq k\leq\frac{1+\sqrt{2}}{2}$

19. d. $\cot(\alpha+\beta)=0\Rightarrow\alpha+\beta=\pi/2+n\pi,\ n\in I$

$\Rightarrow\ \sin(\alpha+2\beta)=\sin(90°+\beta)=\cos\beta$ (for $n=0$).

20. b. We have

$$\frac{x}{\cos\theta}=\frac{y}{\cos\left(\theta-\frac{2\pi}{3}\right)}=\frac{z}{\cos\left(\theta+\frac{2\pi}{3}\right)}$$

Therefore, each ratio is equal to

$$\frac{x+y+z}{\cos\theta+\cos\left(\theta-\frac{2\pi}{3}\right)+\cos\left(\theta+\frac{2\pi}{3}\right)}=\frac{x+y+z}{0}$$

$\Rightarrow \quad x+y+z=0.$

21. b. Since $0 \le \sin^{2n} x \le \sin^2 x$

$0 \le \cos^{2n} x \le \cos^2 x$ [as $\sin^4 x = \sin^2 x \sin^2 x \le \sin^2 x$, $\sin^4 x \le \sin^2 x$ and so on]

$\Rightarrow \quad 0 \le \sin^{2n} x + \cos^{2n} x \le \sin^2 x + \cos^2 x = 1$

$\Rightarrow \quad 0 \le \sin^{2n} x + \cos^{2n} x \le 1$

22. b. $4x^2 - 2\sqrt{5}\,x + 1 = 0$

Let α and β be the roots, we have

$$\alpha+\beta=\frac{2\sqrt{5}}{4}=\frac{\sqrt{5}}{2},\ \alpha\beta=\frac{1}{4}$$

Since $\sin 18° = \frac{\sqrt{5}-1}{4}$, $\cos 36° = \frac{\sqrt{5}+1}{4}$

$\therefore \quad \sin 18° + \cos 36° = \frac{2\sqrt{5}}{4} = \frac{\sqrt{5}}{2}$ $\sin 18° \cos 36° = \frac{5-1}{16} = \frac{4}{16} = \frac{1}{4}$

Here the required roots are sin 18°, cos 36°.

23. b. $\sqrt{2\cot\alpha + \frac{1}{\sin^2\alpha}} = \sqrt{2\cot\alpha + \operatorname{cosec}^2\alpha}$

$$= \sqrt{2\cot\alpha + 1 + \cot^2\alpha} = |1+\cot\alpha| = -1-\cot\alpha$$

[since $\cot\alpha < -1$ when $3\pi/4 < \alpha < \pi$, $|1+\cot\alpha| = -1-\cot\alpha$]

24. d. $f(\theta) = 5\cos\theta + 3\cos\left(\theta+\frac{\pi}{3}\right) + 3 = 5\cos\theta + \frac{3}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta + 3 = \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta + 3$

$= \sqrt{\left(\frac{169}{4}+\frac{27}{4}\right)}\sin(\theta-\alpha)+3$. Thus, the range of $f(\theta)$ is $[-4, 10]$.

25. b. Since $\alpha < \beta < \gamma < \delta$ and $\sin\alpha = \sin\beta = \sin\gamma = \sin\delta = K$, therefore $\beta = \pi - \alpha$, $\gamma = 2\pi + \alpha$, $\delta = 3\pi - \alpha$

$\Rightarrow \quad 4\sin\frac{\alpha}{2} + 3\sin\frac{\beta}{2} + 2\sin\frac{\gamma}{2} + \sin\frac{\delta}{2} = 4\sin\frac{\alpha}{2} + 3\cos\frac{\alpha}{2} - 2\sin\frac{\alpha}{2} - \cos\frac{\alpha}{2}$

$$= 2\sin\frac{\alpha}{2} + 2\cos\frac{\alpha}{2} = 2\sqrt{1+\sin\alpha} = 2\sqrt{1+K}$$

26. c. Let O be the centre of the circle

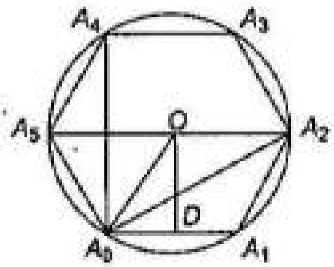


Fig. 2.37

Since $\angle A_0OA_1 = \dfrac{360^\circ}{6} = 60^\circ$

A_0OA_1 is an equilateral triangle, we get $A_0A_1 = 1$ [radius of circle = 1]

Also $A_0A_2 = A_0A_4 = 2OD = 2[OA_0]\sin 60^\circ = 2(1)\dfrac{\sqrt{3}}{2} = \sqrt{3}$

$\therefore \quad (A_0A_1)(A_0A_2)(A_0A_4) = (1)(\sqrt{3})(\sqrt{3}) = 3$

27. d. The given relation is satisfied only when $\sin\theta_1 = \sin\theta_2 = \sin\theta_3 = 1$

$\Rightarrow \quad \cos\theta_1 = \cos\theta_2 = \cos\theta_3 = 0$

$\Rightarrow \quad \cos\theta_1 + \cos\theta_2 + \cos\theta_3 = 0$

28. c. $\sin^2\theta \le 1$

$\Rightarrow \quad \dfrac{x^2+y^2+1}{2x} \le 1 \qquad \Rightarrow x^2+y^2-2x+1 \le 0$ [as $x > 0$]

$\Rightarrow \quad (x-1)^2 + y^2 \le 0$

It is possible, iff $x - 1$ and $y = 0$.

29. a. $\sin(\alpha+\beta) = 1 \Rightarrow \alpha+\beta = \dfrac{\pi}{2}$ $\sin(\alpha-\beta) = \dfrac{1}{2} \Rightarrow \alpha - \beta = \dfrac{\pi}{6}$

Solving, we get $\alpha = \pi/3$ and $\beta = \pi/6$

Now $\tan(\alpha+2\beta)\tan(2\alpha+\beta) = \tan\left(\dfrac{\pi}{3}+\dfrac{\pi}{3}\right)\tan\left(\dfrac{2\pi}{3}+\dfrac{\pi}{6}\right) = \tan\dfrac{2\pi}{3}\tan\dfrac{5\pi}{6} = \left(-\cot\dfrac{\pi}{3}\right)\left(-\cot\dfrac{\pi}{6}\right)$

$= \left(-\dfrac{1}{\sqrt{3}}\right)\left(-\sqrt{3}\right) = 1$

30. a. $\sin 27^\circ - \sin 63^\circ = -2\cos 45^\circ \sin 18^\circ$

$$= -\sqrt{2}\left(\frac{\sqrt{5}-1}{4}\right) = -\frac{\sqrt{5}-1}{2\sqrt{2}} = -\frac{\sqrt{3-\sqrt{5}}}{2}$$

31. c. $\operatorname{cosec}\theta - \cot\theta = q$ (i)

$\therefore \quad \operatorname{cosec}\theta + \cot\theta = \dfrac{1}{q}$

$\therefore \quad \operatorname{cosec}\theta = \dfrac{1}{2}[q + (1/q)]$ (on addition).

32. a. Squaring both the sides, we get

$1 + \sin 2\theta = \dfrac{1}{25}$

$\Rightarrow \sin 2\theta = -\dfrac{24}{25}$

Let $t = \tan\theta$, we get $\dfrac{2t}{1+t^2} = -\dfrac{24}{25}$

$\Rightarrow 50t + 24 + 24t^2 = 0$

$\Rightarrow 12t^2 + 25t + 12 = 0$

$\Rightarrow (4t+3)(3t+4) = 0$

$\Rightarrow t = -4/3$ (as for $t = -3/4$ (rejected) as if $\tan\theta = -3/4$, then $\theta \in [\pi/2, \pi)$ and $\sin\theta + \cos\theta = -1/5$)

33. c. Multiplying x above and below by $1 - \cos\theta + \sin\theta$, we get

$$x = \frac{2\sin\theta(1-\cos\theta+\sin\theta)}{(1+\sin\theta)^2 - \cos^2\theta} = \frac{2\sin\theta(1-\cos\theta+\sin\theta)}{(1+\sin\theta)^2 - (1-\sin^2\theta)}$$

Putting $1 - \sin^2\theta = (1+\sin\theta)(1-\sin\theta)$, we get $\dfrac{2\sin\theta}{2\sin\theta}\dfrac{1-\cos\theta+\sin\theta}{1+\sin\theta} = x$.

34. b. $2n\theta = \pi/2$

$\therefore \quad \theta, (2n-1)\theta = (\pi/2) - \theta; 2\theta, (2n-2)\theta = (\pi/2) - 2\theta, \ldots$

They form complementary angles A and B so that $\tan A \tan B = \tan A \cot A = 1$ for each pair.

35. a. $N^r = 2[(\sin 1° + \sin 89°) + (\sin 2° + \sin 88°) + \cdots + (\sin 44° + \sin 46°) + \sin 45°]$

$$\Rightarrow \frac{N^r}{D^r} = 2\{\sin 45°[2(\cos 44° + \cos 43° + \cdots + \cos 1°)] + 1\}$$

$$= 2\sin 45°$$

$$= \sqrt{2}$$

36. b. $\sec\alpha + \operatorname{cosec}\alpha = p$, $\sec\alpha \operatorname{cosec}\alpha = q$

$$\Rightarrow \quad \frac{\sin\alpha + \cos\alpha}{\sin\alpha\cos\alpha} = p \text{ and } \frac{1}{\sin\alpha\cos\alpha} = q$$

$$\Rightarrow \quad \frac{1 + 2\sin\alpha\cos\alpha}{\sin^2\alpha\cos^2\alpha} = p^2$$

$$\Rightarrow \quad \frac{1 + \frac{2}{q}}{\frac{1}{q^2}} = p^2$$

$$\Rightarrow \quad q^2\left(1 + \frac{2}{q}\right) = p^2 \Rightarrow q(q+2) = p^2$$

37. c. We have $\sin x + \sin^2 x = 1$

$\Rightarrow \quad \sin x = 1 - \sin^2 x \Rightarrow \sin x = \cos^2 x$

Now $\cos^{12} x + 3\cos^{10} x + 3\cos^8 x + \cos^6 x - 2 = \sin^6 x + 3\sin^5 x + 3\sin^4 x + \sin^3 x - 2$

$= (\sin^2 x)^3 + 3(\sin^2 x)^2 \sin x + 3(\sin^2 x)(\sin x)^2 + (\sin x)^3 - 2$

$= (\sin^2 x + \sin x)^3 - 2 = (1)^3 - 2 = -1$

38. a. $\cos(A - B) = \dfrac{3}{5}$

$\Rightarrow \quad 5\cos A\cos B + 5\sin A\sin B = 3$ (i)

From 2nd relation, we have

$\sin A\sin B = 2\cos A\cos B$ (ii)

$\Rightarrow \quad \cos A\cos B = \dfrac{1}{5}$ and $\sin A\sin B = \dfrac{2}{5}$

39. a. $(1 + \tan A)(1 + \tan B) = 2$

$\Rightarrow \quad \tan A + \tan B = 1 - \tan A\tan B$

$\Rightarrow \quad \tan(A + B) = 1$, i.e., $A + B = \dfrac{\pi}{4}$

or $\alpha + 4\alpha = \dfrac{\pi}{4}$, i.e., $\alpha = \dfrac{\pi}{20}$

40. a. $A = \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}} = \sqrt{2}\sin 90°$

$B = \sqrt{2}\left[\dfrac{1}{\sqrt{2}}\sin 44° + \dfrac{1}{\sqrt{2}}\cos 44°\right] = \sqrt{2}\sin(45° + 44°)$

$= \sqrt{2}\sin 89° < \sqrt{2}\sin 90° = \sqrt{2} \qquad \therefore A > B \Rightarrow$ (a)

41. d. $\dfrac{1}{4}(\sqrt{3}\cos 23° - \sin 23°) = \dfrac{1}{2}(\cos 30°\cos 23° - \sin 30°\sin 23°) = \dfrac{1}{2}\cos(30° + 23°) = \dfrac{1}{2}\cos 53°$

42. b. $$\tan\left(\frac{\theta_1 - \theta_2}{2}\right)\tan\left(\frac{\theta_1 + \theta_2}{2}\right) = \frac{\sin\left(\frac{\theta_1 + \theta_2}{2}\right)\sin\left(\frac{\theta_1 - \theta_2}{2}\right)}{\cos\left(\frac{\theta_1 + \theta_2}{2}\right)\cos\left(\frac{\theta_1 - \theta_2}{2}\right)}$$

$$= \frac{\cos\theta_2 - \cos\theta_1}{\cos\theta_1 + \cos\theta_2} = \frac{-1}{3}$$

43. b. $$\frac{3 + \dfrac{\cos 80°\cos 20°}{\sin 80°\sin 20°}}{\dfrac{\cos 80°}{\sin 80°} + \dfrac{\cos 20°}{\sin 20°}} = \frac{2\sin 80°\sin 20° + (\cos 80°\cos 20° + \sin 80°\sin 20°)}{\sin 20°\cos 80° + \cos 20°\sin 80°}$$

$$= \frac{-\cos 100° + \cos 60° + \cos 60°}{\sin 100°} = \frac{1 - \cos 100°}{\sin 100} = \tan 50°$$

44. a. $\tan\beta = 2\sin\alpha\sin\gamma\,\text{cosec}(\alpha + \gamma) = \dfrac{2\sin\alpha\sin\gamma}{\sin(\alpha + \gamma)}$

$\Rightarrow \quad \cot\beta = \dfrac{\sin(\alpha + \gamma)}{2\sin\alpha\sin\gamma}$

$\Rightarrow \quad 2\cot\beta = \dfrac{\sin\alpha\cos\gamma + \cos\alpha\sin\gamma}{\sin\alpha\sin\gamma} = \cot\alpha + \cot\gamma$

$\Rightarrow$ $\cot\alpha$, $\cot\beta$, $\cot\gamma$ are in A.P.

45. b. $3\sin A\cos B = \sin B\cos A$

$\Rightarrow \quad \cos A\sin B = \dfrac{3}{4}$

$\Rightarrow \quad \sin(A+B) = 1$

$\Rightarrow \quad C = \dfrac{\pi}{2},\ B = \dfrac{\pi}{2} - A$

$\Rightarrow \quad 3\tan A = \tan\left(\dfrac{\pi}{2} - A\right)$

$\Rightarrow \quad 3 = \cot^2 A$

46. d. $\tan(100° + 125°) = \dfrac{\tan 100° + \tan 125°}{1 - \tan 100° \tan 125°}$

$\therefore \quad \tan 225° = \dfrac{\tan 100° + \tan 125°}{1 - \tan 100° \tan 125°}$, i.e., $1 = \dfrac{\tan 100° + \tan 125°}{1 - \tan 100° \tan 125°}$

i.e., $\tan 100° + \tan 125° + \tan 100° \tan 125° = 1$

47. b. We know that $\tan(20° + 40°) = \dfrac{\tan 20° + \tan 40°}{1 - \tan 20° \tan 40°}$

$\Rightarrow \quad \sqrt{3} = \dfrac{\tan 20° + \tan 40°}{1 - \tan 20° \tan 40°}$

$\Rightarrow \quad \sqrt{3} - \sqrt{3}\tan 20° \tan 40° = \tan 20° + \tan 40°$

$\Rightarrow \quad \tan 20° + \tan 40° + \sqrt{3}\tan 20° \tan 40° = \sqrt{3}$

48. c. $\dfrac{\sqrt{2} - \sin\alpha - \cos\alpha}{\sin\alpha - \cos\alpha}$

$$= \frac{\sqrt{2} - \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin\alpha + \frac{1}{\sqrt{2}}\cos\alpha\right)}{\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin\alpha - \frac{1}{\sqrt{2}}\cos\alpha\right)}$$

$$= \frac{\sqrt{2} - \sqrt{2}\cos\left(\alpha - \frac{\pi}{4}\right)}{\sqrt{2}\sin\left(\alpha - \frac{\pi}{4}\right)}$$

$= \dfrac{\sqrt{2}(1 - \cos\theta)}{\sqrt{2}\sin\theta}$, where $\theta = \alpha - \dfrac{\pi}{4} = \dfrac{2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)} = \tan\dfrac{\theta}{2} = \tan\left(\dfrac{\alpha}{2} - \dfrac{\pi}{8}\right)$

49. b. $\sin\theta_1 - \sin\theta_2 = a$, $\cos\theta_1 + \cos\theta_2 = b$

$\Rightarrow \quad a^2 + b^2 = 2 + 2\cos(\theta_1 + \theta_2)$

$\Rightarrow \quad 0 \le a^2 + b^2 \le 4$

50. c. $\dfrac{1+\sin 2x}{1-\sin 2x}=\dfrac{(\sin x+\cos x)^2}{(\sin x-\cos x)^2}=\left(\dfrac{1+\tan x}{1-\tan x}\right)^2=\left(\tan\left(\dfrac{\pi}{4}+x\right)\right)^2=\tan^2\left(\dfrac{\pi}{4}+x\right)$

$$=\cot^2\left(\frac{\pi}{2}+\frac{\pi}{4}+x\right)=\cot^2\left(\frac{3\pi}{4}+x\right)$$

$\Rightarrow \quad a=\dfrac{3\pi}{4}$

51. d. We have $4x^2-16x+15<0 \Rightarrow \dfrac{3}{2}<x<\dfrac{5}{2}$

Therefore, the integral solution of $4x^2-16x+15<0$ is $x=2$

Thus, $\tan\alpha=2$. It is given that $\cos\beta=\tan 45°=1$.

$\therefore \quad \sin(\alpha+\beta)\sin(\alpha-\beta)=\sin^2\alpha-\sin^2\beta=\dfrac{1}{1+\cot^2\alpha}-(1-\cos^2\beta)=\dfrac{1}{1+\frac{1}{4}}-0=\dfrac{4}{5}$

52. b. $\dfrac{\cos(x-y)}{\cos(x+y)}+\dfrac{\cos(z+t)}{\cos(z-t)}=0$

$\Rightarrow \quad \dfrac{1+\tan x\tan y}{1-\tan x\tan y}+\dfrac{1-\tan z\tan t}{1+\tan z\tan t}=0$

$\Rightarrow \quad 1+\tan z\tan t+\tan x\tan y+\tan x\tan y\tan z\tan t+1-\tan z\tan t-\tan x\tan y$
$+\tan x\tan y\tan z\tan t=0$

$\Rightarrow \quad \tan x\tan y\tan z\tan t=-1.$

53. b. $f(n)=2\cos nx$

$\Rightarrow \quad f(1)f(n+1)-f(n)=4\cos x\cos(n+1)x-2\cos nx=2[2\cos(n+1)x\cos x-\cos nx]$

$=2[\cos(n+2)x+\cos nx-\cos nx]=2\cos(n+2)x=f(n+2).$

54. a. $\dfrac{\tan A}{\tan B}=\dfrac{1}{3} \Rightarrow \dfrac{\sin A\cos B}{\cos A\sin B}=\dfrac{1}{3}$

Put $\sin A\cos B=\dfrac{1}{4}$

$\Rightarrow \quad \cos A\sin B=\dfrac{3}{4}$

$\Rightarrow \quad \sin(A+B)=\dfrac{1}{4}+\dfrac{3}{4}=1$

$\Rightarrow \quad \sin C=1=\sin\pi/2$

$\Rightarrow \quad C=\pi/2$. Hence, the triangle is right angled.

55. b. $3\sin^2 A+2\sin^2 B=1$

$\Rightarrow \quad 3\sin^2 A=\cos 2B$

Also $3\sin 2A-2\sin 2B=0$

$\Rightarrow \quad \sin 2B=\dfrac{3}{2}\sin 2A$

Now, $\cos(A+2B) = \cos A \cos 2B - \sin A \sin 2B = \cos A\, 3\sin^2 A - \sin A\, \frac{3}{2}\sin 2A$

$$= 3\sin^2 A \cos A - 3\sin^2 A \cos A = 0$$

$\therefore A + 2B = \pi/2$

56. a. $$f(\beta) = f\left(\frac{5\pi}{4} - \alpha\right) = \frac{\cot\left(\frac{5\pi}{4} - \alpha\right)}{1 + \cot\left(\frac{5\pi}{4} - \alpha\right)}$$

$$= \frac{1}{1 + \tan\left(\frac{5\pi}{4} - \alpha\right)}$$

$$= \frac{1}{1 + \frac{1 - \tan\alpha}{1 + \tan\alpha}} = \frac{1 + \tan\alpha}{2}$$

As $f(\alpha) = \frac{\cot\alpha}{1 + \cot\alpha} = \frac{1}{1 + \tan\alpha}$, we have $f(\alpha) f(\beta) = \frac{1}{2}$

57. c. $A - B = \frac{\pi}{4} \Rightarrow \tan(A - B) = \tan\frac{\pi}{4}$

$\Rightarrow \quad \frac{\tan A - \tan B}{1 + \tan A \tan B}$

$\Rightarrow \quad \tan A - \tan B - \tan A \tan B = 1$

$\Rightarrow \quad \tan A - \tan B - \tan A \tan B + 1 = 2$

$\Rightarrow \quad (1 + \tan A)(1 - \tan B) = 2 \Rightarrow y = 2$

Hence, $(y+1)^{y+1} = (2+1)^{2+1} = (3)^3 = 27.$

58. a. Applying $b - a = c - b$ for A.P., we get $2\cos z \sin(x - y) = 2\cos x \sin(y - z)$

Dividing by $2\cos x \cos y \cos z$, etc., we get $\tan x - \tan y = \tan y - \tan z$.

59. b. $(\cos\alpha + \cos\beta)^2 - (\sin\alpha + \sin\beta)^2 = 0$

$\Rightarrow \quad (\cos^2\alpha + \cos^2\beta + 2\cos\alpha\cos\beta) - (\sin^2\alpha + \sin^2\beta + 2\sin\alpha\sin\beta) = 0$

$\Rightarrow \quad \cos 2\alpha + \cos 2\beta = -2(\cos\alpha\cos\beta - \sin\alpha\sin\beta)$

$$= -2\cos(\alpha + \beta)$$

60. a. We have

$\sin\alpha \sec x_1 \sec x_2 + \sin\alpha \sec x_2 \sec x_3 + \cdots + \sin\alpha \sec x_{n-1} \sec x_n$

$$= \frac{\sin(x_2 - x_1)}{\cos x_1 \cos x_2} + \frac{\sin(x_3 - x_2)}{\cos x_2 \cos x_3} + \cdots + \frac{\sin(x_n - x_{n-1})}{\cos x_{n-1} \cos x_n}$$

$= (\tan x_2 - \tan x_1) + (\tan x_3 - \tan x_2) + \cdots + (\tan x_n - \tan x_{n-1})$

$= \tan x_n - \tan x_1 = \frac{\sin(x_n - x_1)}{\cos x_n \cos x_1} = \frac{\sin(n-1)\alpha}{\cos x_n \cos x_1}.$ $\quad [\because x_n = x_1 + (n-1)\alpha]$

61. a. By the given conditions $\tan\dfrac{\pi}{9}+\tan\dfrac{5\pi}{18}=2x$

$$\tan\frac{\pi}{9}+\tan\frac{7\pi}{18}=2y$$

$$\Rightarrow\quad 2x=\tan 20^\circ+\tan 50^\circ$$

$$=\frac{\sin 20^\circ}{\cos 20^\circ}+\frac{\sin 50^\circ}{\cos 50^\circ}$$

$$=\frac{\sin 20^\circ\cos 50^\circ+\cos 20^\circ\sin 50^\circ}{\cos 20^\circ\cos 50^\circ}$$

$$=\frac{\sin 70^\circ}{\cos 20^\circ\cos 50^\circ}$$

$$=\frac{\cos 20^\circ}{\cos 20^\circ\cos 50^\circ}=\frac{1}{\cos 50^\circ}=\frac{1}{\sin 40^\circ}=\operatorname{cosec} 40^\circ$$

$$2y=\tan 20^\circ+\tan 70^\circ$$

$$=\frac{\sin 20^\circ}{\cos 20^\circ}+\frac{\sin 70^\circ}{\cos 70^\circ}$$

$$=\frac{\sin 90^\circ}{\cos 20^\circ\cos 70^\circ}$$

$$=\frac{1}{\cos 20^\circ\cos 70^\circ}=\frac{1}{\cos 20^\circ\sin 20^\circ}$$

$$=\frac{2}{2\sin 20^\circ\cos 20^\circ}=\frac{2}{\sin 40^\circ}=2\operatorname{cosec} 40^\circ$$

$$\therefore\quad 2y=2(2x)\Rightarrow y=2x$$

62. b. $$\frac{1}{\sin 1^\circ}\left[\frac{\sin(1^\circ-0^\circ)}{\cos 0^\circ\cos 1^\circ}+\frac{\sin(2^\circ-1^\circ)}{\cos 1^\circ\cos 2^\circ}+\frac{\sin(3^\circ-2^\circ)}{\cos 2^\circ\cos 3^\circ}+\cdots+\frac{\sin(45^\circ-44^\circ)}{\cos 44^\circ\cos 45^\circ}\right]$$

$$=\frac{1}{\sin 1^\circ}[\tan 1^\circ+(\tan 2^\circ-\tan 1^\circ)+(\tan 3^\circ-\tan 2^\circ)+(\tan 4^\circ-\tan 3^\circ)+\cdots+(\tan 45^\circ-\tan 44^\circ)]$$

$$=\frac{1}{\sin 1^\circ}=\frac{1}{x}$$

63. a. We have $\sin\alpha+\sin\beta=-\dfrac{21}{65}$ (i)

$\cos\alpha+\cos\beta=-\dfrac{17}{65}$ (ii)

Squaring Eq. (i), we get $\sin^2\alpha+\sin^2\beta+2\sin\alpha\sin\beta=\left(\dfrac{21}{65}\right)^2$ (iii)

Squaring Eq. (ii), we get $\cos^2\alpha+\cos^2\beta+2\cos\alpha\cos\beta=\left(\dfrac{27}{65}\right)^2$ (iv)

Adding Eqs. (iii) and (iv), we get $2+2\cos(\alpha-\beta)=\frac{1}{(65)^2}\left[(27)^2+(21)^2\right]=\frac{1}{(65)^2}(729+441)$

$$\Rightarrow \quad 2+2\cos(\alpha-\beta)=\frac{1}{(65)^2}(1170)=\frac{18}{65}$$

$$\Rightarrow \quad 1+\cos(\alpha-\beta)=\frac{9}{65}$$

$$\Rightarrow \quad 2\cos^2\frac{\alpha-\beta}{2}=\frac{9}{65}$$

$$\Rightarrow \quad \cos\frac{\alpha-\beta}{2}=-\frac{3}{\sqrt{130}} \qquad \left[\because \pi<\alpha-\beta<3\pi \Rightarrow \frac{\pi}{2}<\frac{\alpha-\beta}{2}<\frac{3\pi}{2} \Rightarrow \cos\left(\frac{\alpha-\beta}{2}\right)<0\right]$$

64. b. $\frac{\sin(x+y)}{\sin(x-y)}=\frac{a+b}{a-b}$

$$\Rightarrow \quad \frac{\sin(x+y)+\sin(x-y)}{\sin(x+y)-\sin(x-y)}=\frac{(a+b)+(a-b)}{(a+b)-(a-b)}$$

$$\Rightarrow \quad \frac{2\sin x\cos y}{2\cos x\sin y}=\frac{2a}{2b}$$

$$\Rightarrow \quad \frac{\tan x}{\tan y}=\frac{a}{b}$$

65. c. $\frac{\sin 3\theta+\sin 5\theta+\sin 7\theta+\sin 9\theta}{\cos 3\theta+\cos 5\theta+\cos 7\theta+\cos 9\theta}=\frac{(\sin 3\theta+\sin 9\theta)+(\sin 5\theta+\sin 7\theta)}{(\cos 3\theta+\cos 9\theta)+(\cos 5\theta+\cos 7\theta)}$

$$=\frac{2\sin 6\theta\cos 3\theta+2\sin 6\theta\cos\theta}{2\cos 6\theta\cos 3\theta+2\cos 6\theta\cos\theta}=\frac{2\sin 6\theta(\cos 3\theta+\cos\theta)}{2\cos 6\theta(\cos 3\theta+\cos\theta)}=\tan 6\theta$$

66. b. $\frac{\sin x-\sin z}{\cos z-\cos x}=\frac{2\cos\left(\frac{x+z}{2}\right)\sin\left(\frac{x-z}{z}\right)}{2\sin\left(\frac{x+z}{z}\right)\sin\left(\frac{x-z}{z}\right)}=\cot\left(\frac{x+z}{2}\right)=\cot(y)$

67. c. $\cos 50°=\cos^2 25°-\sin^2 25°=(\cos 25°+\sin 25°)(\cos 25°-\sin 25°)=p(\cos 25°-\sin 25°)$ (i)

Now $(\cos 25°-\sin 25°)^2+(\cos 25°+\sin 25°)^2=1+1$

$\therefore \quad \cos 25°-\sin 25°=\sqrt{2-p^2}$ (ii)

We have taken +ve sign as $\cos 25°>\sin 25°$, therefore $\cos 50°=p\sqrt{2-p^2}$, by Eqs. (i) and (ii).

68. b. $\frac{\sin^2 A-\sin^2 B}{\sin A\cos A-\sin B\cos B}=\frac{2\sin(A+B)\sin(A-B)}{\sin 2A-\sin 2B}=\frac{2\sin(A+B)\sin(A-B)}{2\sin(A-B)\cos(A+B)}$

$$=\tan(A+B)$$

69. a. $\tan A=\frac{1-\cos B}{\sin B}=\frac{2\sin^2(B/2)}{2\sin(B/2)\cos(B/2)}=\tan\frac{B}{2} \qquad \Rightarrow \tan 2A=\tan B$

70. b. On adding, we get $a=\frac{3-\cos 4\theta+4\sin 2\theta}{2}=(1+\sin 2\theta)^2$

On subtracting, we get $b=(1-\sin 2\theta)^2 \Rightarrow ab=\cos^4 2\theta\le 1$

71. c. $\cos^2 10° - \cos 10° \cos 50° + \cos^2 50°$

$$= \frac{1}{2}\left[1 + \cos 20° - (\cos 60° + \cos 40°) + (1 + \cos 100°)\right]$$

$$= \frac{1}{2}[1 + \cos 20° - \frac{1}{2} - \cos 40° + 1 - \cos 80°]$$

$$= \frac{1}{2}\left[\frac{3}{2} + \cos 20° - (2 \cos 60° \cos 20°)\right] = \frac{3}{4}$$

72. a. $\tan 20° \tan 80° \cot 50° = \tan 20° \tan 80° \tan 40°$

$$= \tan 20° \tan(60° - 20°) \tan(60° + 20°) = \tan 60° = \sqrt{3}$$

73. b. $\tan^2 \theta = 2 \tan^2 \phi + 1$

$\Rightarrow \quad 1 + \tan^2 \theta = 2(1 + \tan^2 \phi)$

$\Rightarrow \quad \sec^2 \theta = 2 \sec^2 \phi$

$\Rightarrow \quad \cos^2 \phi = 2 \cos^2 \theta$

$= 1 + \cos 2\theta$

$\Rightarrow \quad \cos 2\theta = \cos^2 \phi - 1$

$= -\sin^2 \phi$

$\Rightarrow \quad \sin^2 \phi + \cos 2\theta = 0$

74. b. $\cot 70° + 4 \cos 70° = \dfrac{\cos 70° + 4 \sin 70° \cos 70°}{\sin 70°}$

$$= \frac{\cos 70° + 2 \sin 140°}{\sin 70°}$$

$$= \frac{\cos 70° + 2 \sin(180° - 40°)}{\sin 70°}$$

$$= \frac{\sin 20° + \sin 40° + \sin 40°}{\sin 70°}$$

$$= \frac{2 \sin 30° \cos 10° + \sin 40°}{\sin 70°}$$

$$= \frac{\sin 80° + \sin 40°}{\sin 70°}$$

$$= \frac{2 \sin 60° \cos 20°}{\sin 70°} = \sqrt{3}$$

75. b. $a \cos x + b \sin x = c$

$$\Rightarrow \quad \frac{a\left(1 - \tan^2 \frac{x}{2}\right)}{1 + \tan^2 \frac{x}{2}} + \frac{2b \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} = c$$

$\Rightarrow \quad (c+a)\tan^2\dfrac{x}{2} - 2b\tan\dfrac{x}{2} + c - a = 0$

$\Rightarrow \quad \tan\dfrac{x_1}{2} + \tan\dfrac{x_2}{2} = \dfrac{2b}{c+a}$ and $\tan\dfrac{x_1}{2}\tan\dfrac{x_2}{2} = \dfrac{c-a}{c+a}$

$$\Rightarrow \quad \tan\left(\frac{x_1+x_2}{2}\right) = \frac{\dfrac{2b}{c+a}}{1-\dfrac{c-a}{c+a}} = \frac{2b}{2a} = \frac{b}{a}$$

76. c. $\tan y = \dfrac{1+\sqrt{1-x}}{1+\sqrt{1+x}}$

If $x = \cos\theta$, then $\sqrt{1-x} = \sqrt{2}\sin(\theta/2)$, $\sqrt{1+x} = \sqrt{2}\cos(\theta/2)$

$$\Rightarrow \quad \tan y = \frac{\sqrt{2}\left[\dfrac{1}{\sqrt{2}}+\sin\dfrac{\theta}{2}\right]}{\sqrt{2}\left[\dfrac{1}{\sqrt{2}}+\cos\dfrac{\theta}{2}\right]} = \frac{\sin\dfrac{\pi}{4}+\sin\dfrac{\theta}{2}}{\cos\dfrac{\pi}{4}+\cos\dfrac{\theta}{2}}$$

$$= \frac{2\sin\left(\dfrac{\pi}{8}+\dfrac{\theta}{4}\right)\cos\left(\dfrac{\pi}{8}-\dfrac{\theta}{4}\right)}{2\cos\left(\dfrac{\pi}{8}+\dfrac{\theta}{4}\right)\cos\left(\dfrac{\pi}{8}-\dfrac{\theta}{4}\right)}$$

$$= \tan\left(\frac{\pi}{8}+\frac{\theta}{4}\right)$$

$\Rightarrow \quad 4y = \dfrac{\pi}{2} + \theta$

$\Rightarrow \quad \sin 4y = \cos\theta = x$

77. d. We have $\cos x = \tan y$

$\Rightarrow \quad \cos^2 x = \tan^2 y$

$= \sec^2 y - 1$

$= \cot^2 z - 1 \qquad [\because \cos y = \tan z, \sec y = \cot z]$

$\Rightarrow \quad 1 + \cos^2 x = \cot^2 z$

$= \dfrac{\tan^2 x}{1-\tan^2 x} \qquad [\because \cos z = \tan x]$

$= \dfrac{\sin^2 x}{\cos^2 x - \sin^2 x}$

$\Rightarrow \quad 2\sin^4 x - 6\sin^2 x + 2 = 0$

$\Rightarrow \quad \sin^2 x = \dfrac{3-\sqrt{5}}{2}$

$$\Rightarrow \quad \sin^2 x = \left(\frac{\sqrt{5}-1}{2}\right)^2$$

$$\Rightarrow \quad \sin x = \frac{\sqrt{5}-1}{2} = 2\sin 18^\circ$$

78. a. $\sin 2\theta = \cos 3\theta \qquad \Rightarrow 2\sin\theta\cos\theta = 4\cos^3\theta - 3\cos\theta$

$$\Rightarrow \quad 2\sin\theta = 4(1-\sin^2\theta) - 3 \Rightarrow 4\sin^2\theta + 2\sin\theta - 1 = 0$$

$$\Rightarrow \quad \sin\theta = \frac{\sqrt{5}-1}{4}$$

79. b. $\tan\theta = \lambda$, we get $\dfrac{2\tan\theta/2}{1-\tan^2\theta/2} = \lambda$

$$\Rightarrow \quad \lambda\tan^2\frac{\theta}{2} + 2\tan\frac{\theta}{2} - \lambda = 0$$

$$\Rightarrow \quad \tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2} = -1$$

80. a. $\tan\theta = \sqrt{n} \qquad \Rightarrow \cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta} = \dfrac{1-n}{1+n}$ = rational.

81. b. $\sin x + \cos x = \dfrac{\sqrt{7}}{2}$

$$\Rightarrow \quad \frac{2\tan\frac{x}{2}}{\left(1+\tan^2\frac{x}{2}\right)} + \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{\sqrt{7}}{2}$$

$$\Rightarrow \quad (\sqrt{7}+2)\tan^2\frac{x}{2} - 4\tan\frac{x}{2} + (\sqrt{7}-2) = 0$$

$$\Rightarrow \quad \tan\frac{x}{2} = \frac{4 \pm \sqrt{16-4(7-4)}}{2(\sqrt{7}+2)} = \frac{1}{(\sqrt{7}+2)} \text{ as } \frac{x}{2} < \frac{\pi}{8}$$

$$= \frac{\sqrt{7}-2}{3}$$

82. b. $\sin\dfrac{7\pi}{8} = \sin\left(\pi - \dfrac{\pi}{8}\right) = \sin\dfrac{\pi}{8}$; $\sin\dfrac{5\pi}{8} = \sin\left(\pi - \dfrac{3\pi}{8}\right) = \sin\dfrac{3\pi}{8}$

Therefore, the given value $= 2\left[\sin^2\dfrac{\pi}{8} + \sin^2\dfrac{3\pi}{8}\right] = 2\left[\sin^2\dfrac{\pi}{8} + \cos^2\dfrac{\pi}{8}\right]$

$$= 2(1) = 2 \quad \left[\because \sin\frac{3\pi}{8} = \sin\left(\frac{\pi}{2} - \frac{\pi}{8}\right) = \cos\frac{\pi}{8}\right]$$

83. b. $4\cos^2\left(\dfrac{\pi}{4} - \dfrac{x}{2}\right) + \sqrt{4\sin^4 x + \sin^2 2x} = 4\cos^2\left(\dfrac{\pi}{4} - \dfrac{x}{2}\right) + \sqrt{4\sin^2 x(\cos^2 x + \sin^2 x)}$

$$= 2\left(1 + \cos\left(\frac{\pi}{2} - x\right)\right) + 2|\sin x| = 2 + 2\sin x - 2\sin x \text{ as } x \in \left(\pi, \frac{3\pi}{2}\right) = 2.$$

84. b. $\cos^3 x \sin 2x = \cos^2 x \cos x \sin 2x$

$$= \left(\frac{1-\cos 2x}{2}\right)\left(\frac{2\sin 2x \cos x}{2}\right)$$

$$= \frac{1}{4}(1-\cos 2x)(\sin 3x + \sin x)$$

$$= \frac{1}{4}\left[\sin 3x + \sin x - \frac{1}{2}(2\sin 3x.\cos 2x) - \frac{1}{2}(2\cos 2x \sin x)\right]$$

$$= \frac{1}{4}\left[\sin 3x + \sin x - \frac{1}{2}(\sin 5x + \sin x) - \frac{1}{2}(\sin 3x - \sin x)\right]$$

$$= \frac{1}{4}\left[\sin x + \frac{1}{2}\sin 3x - \frac{1}{2}\sin 5x\right]$$

$\Rightarrow \quad a_1 = 1/4, a_3 = 1/8, n = 5$

85. b. Given expression is $2\sin^2\phi + 4\cos(\theta+\phi)\sin\theta\sin\phi + \cos 2(\theta+\phi)$

$= (1-\cos 2\phi) + 4\cos(\theta+\phi)\sin\theta\sin\phi + 2\cos^2(\theta+\phi) - 1$

$= -\cos 2\phi + 4\cos(\theta+\phi)\sin\theta\sin\phi + 2\cos^2(\theta+\phi)$

$= -\cos 2\phi + 2\cos(\theta+\phi)[\cos(\theta+\phi) + 2\sin\theta\sin\phi]$

$= -\cos 2\phi + 2\cos(\theta+\phi)[\cos\theta\cos\phi + \sin\theta\sin\phi]$

$= -\cos 2\phi + 2\cos(\theta+\phi)\cos(\theta-\phi)$

$= -\cos 2\phi + \cos 2\theta + \cos 2\phi = \cos 2\theta$

86. b. $\dfrac{\cos 2B}{1} = \dfrac{\cos(A+C)}{\cos(A-C)}$

Applying componendo and dividendo, we get

$$\Rightarrow \quad \frac{1-\cos 2B}{1+\cos 2B} = \frac{\cos(A-C)-\cos(A+C)}{\cos(A-C)+\cos(A+C)}$$

$$\Rightarrow \quad \frac{2\sin^2 B}{2\cos^2 B} = \frac{2\sin A\sin C}{2\cos A\cos C}$$

$\Rightarrow \quad \tan^2 B = \tan A \tan C$

$\Rightarrow \quad \tan A, \tan B, \tan C$ are in G.P.

87. b. $\cos x = \dfrac{2\cos y - 1}{2-\cos y}$

$$\Rightarrow \quad \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{\frac{2(1-\tan^2 y/2)}{1+\tan^2 y/2} - 1}{2 - \frac{1-\tan^2 y/2}{1+\tan^2 y/2}}$$

$\Rightarrow \quad 6\tan^2 y/2 = 2\tan^2 \frac{x}{2}$

$\Rightarrow \quad \tan\frac{x}{2}\cot\frac{y}{2} = \sqrt{3}$

88. b. We have $\sqrt{\left(\frac{a+b}{a-b}\right)} + \sqrt{\left(\frac{a-b}{a+b}\right)} = \frac{a+b+a-b}{\sqrt{(a^2-b^2)}}$

$$= \frac{2a}{\sqrt{(a^2-b^2)}} = \frac{2}{\sqrt{[1-(b/a)^2]}}$$

$$= \frac{2}{\sqrt{(1-\tan^2 x)}} = \frac{2\cos x}{\sqrt{(\cos^2 x - \sin^2 x)}}$$

$$= \frac{2\cos x}{\sqrt{(\cos 2x)}}$$

89. b. Since α is a root of $25\cos^2\theta + 5\cos\theta - 12 = 0$

$\therefore \quad 25\cos^2\alpha + 5\cos\alpha - 12 = 0 \qquad \Rightarrow \cos\alpha = \frac{-5 \pm \sqrt{25+1200}}{50} = -\frac{4}{5} \left[\because \frac{\pi}{2} < \alpha < \pi\right]$

and $\sin\alpha = \sqrt{1-\frac{16}{25}} = \frac{3}{5}$; therefore, $\sin 2\alpha = 2\sin\alpha\cos\alpha = 2\left(\frac{3}{5}\right)\left(\frac{-4}{5}\right) = \frac{-24}{25}$

90. c. We have $\tan 9° - \tan 27° - \tan 63° + \tan 81° = (\tan 9° + \tan 81°) - (\tan 27° + \tan 63°)$

$$= \frac{1}{\sin 9°\cos 9°} - \frac{1}{\sin 27°\cos 27°}$$

$$= \frac{2}{\sin 18°} - \frac{2}{\sin 54°}$$

$$= 2\left[\frac{\sin 54° - \sin 18°}{\sin 54°\sin 18°}\right]$$

$$= 2\left[\frac{2\cos 36°\sin 18°}{\sin 18°\cos 36°}\right] = 4$$

91. c. Let $A = \sin^{-1}a$, $B = \sin^{-1}b$ and $C = \sin^{-1}c$, we have $A + B + C = \pi$.

$$a\sqrt{1-a^2} + b\sqrt{1-b^2} + c\sqrt{1-c^2} = \frac{1}{2}(\sin 2A + \sin 2B + \sin 2C) = \frac{1}{2}[4\sin A\sin B\sin C] = 2\,abc$$

92. d. $\cos 2A + \cos 2B + \cos 2C = 2\cos(A+B)\cos(A-B) + \cos 2C$

$$= 2\cos\left(\frac{3\pi}{2} - C\right)\cos(A-B) + \cos 2C$$
$$= -2\sin C\cos(A-B) + 1 - 2\sin^2 C$$
$$= 1 - 2\sin C(\cos(A-B) + \sin C)$$
$$= 1 - 2\sin C\{\cos(A-B) + \sin[3\pi/2 - (A+B)]\}$$
$$= 1 - 2\sin C[\cos(A-B) - \cos(A+B)]$$

$= 1 - 4\sin A\sin B\sin C$

93. c. $\dfrac{2}{\tan\frac{B}{2}} = \dfrac{1}{\tan\frac{A}{2}} + \dfrac{1}{\tan\frac{C}{2}}$

$\Rightarrow \quad 2\tan\frac{A}{2}\tan\frac{C}{2} = \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{B}{2}\tan\frac{A}{2} = 1 - \tan\frac{A}{2}\tan\frac{C}{2}$

$\Rightarrow \quad \tan\frac{A}{2}\tan\frac{C}{2} = \frac{1}{3}$

$\Rightarrow \quad \cot\frac{A}{2}\cot\frac{C}{2} = 3$

94. b. $\sin^2 A - \sin^2 B + \sin^2 C = \sin(A+B)\sin(A-B) + \sin^2 C = \sin C(\sin(A-B) + \sin C)$

$= \sin C(\sin(A-B) + \sin(A+B)) = 2\sin A\cos B\sin C$

95. c. $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = \dfrac{\tan^2\alpha}{1+\tan^2\alpha} + \dfrac{\tan^2\beta}{1+\tan^2\beta} + \dfrac{\tan^2\gamma}{1+\tan^2\gamma}$

$= \dfrac{x}{1+x} + \dfrac{y}{1+y} + \dfrac{z}{1+z}$ [where $x = \tan^2\alpha, y = \tan^2\beta, z = \tan^2\gamma$]

$= \dfrac{(x+y+z) + (xy+yz+zx+2xyz) + xy+yz+zx+xyz}{(1+x)(1+y)(1+z)}$

$= \dfrac{1+x+y+z+xy+yz+zx+xyz}{(1+x)(1+y)(1+z)} = 1$ [$\because xy+yz+zx+2xyz = \gamma$]

96. c. $D' = \sin A + \sin B - \sin C$

$= 2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2} - 2\sin\dfrac{C}{2}\cos\dfrac{C}{2}$

$= 2\cos\dfrac{C}{2}\left[\cos\left(\dfrac{A-B}{2}\right) - \sin\dfrac{C}{2}\right]$

$= 2\cos\dfrac{C}{2}\left[\cos\dfrac{A-B}{2} - \cos\dfrac{A+B}{2}\right]$

$= 2\cos\dfrac{C}{2}\left[2\sin\dfrac{A}{2}\sin\dfrac{B}{2}\right]$

$= 4\sin\dfrac{A}{2}\sin\dfrac{B}{2}\cos\dfrac{C}{2}$

Also $\sin A + \sin B + \sin C = 4\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$

$\Rightarrow \quad \dfrac{\sin A + \sin B + \sin C}{\sin A + \sin B - \sin C} = \cot\dfrac{A}{2}\cot\dfrac{B}{2}$

97. a. $\dfrac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C} = \dfrac{4\sin A\sin B\sin C}{4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = 8\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$ $\left[\because \sin A = 2\sin\dfrac{A}{2}\cos\dfrac{A}{2}\right]$

98. c. We have $\cos^2 A + \cos^2 B - (1 - \cos^2 C) = 0$

$\Rightarrow \quad \cos^2 A + \cos^2 B - \sin^2 C = 0$

$\Rightarrow \quad \cos^2 A + \cos(B + C)\cos(B - C) = 0$

$\Rightarrow \quad 2\cos A \cos B \cos C = 0$

Hence, either A or B or C is 90°.

99. a. In a triangle, $\tan A + \tan B + \tan C = \tan A \tan B \tan C$ (i)

$\Rightarrow \quad 6 = 2\tan C$

$\Rightarrow \quad \tan C = 3$

Also $\tan A + \tan B = 6 - 3 = 3$ (ii)

$\Rightarrow \quad \tan A$ and $\tan B$ are roots $x^2 - 3x + 2 = 0$ by Eqs. (i) and (ii).

$\Rightarrow \quad \tan A, \tan B = 2, 1$ or $1, 2$ and $\tan C = 3$.

100. a. We have $\tan 6° \tan 42° \tan 66° \tan 78°$

$= \tan 6° \tan(60° - 18°) \tan(60° + 6°) \tan(60° + 18°)$

$$= \frac{\tan 6° \tan(60° + 6°) \tan 18° \tan(60° - 18°) \tan(60° + 18°)}{\tan 18°} = \frac{\tan 6° \tan(60° + 6°) \tan(3 \times 18°)}{\tan 18°}$$

$$= \frac{\tan 6° \tan(60° - 6) \tan(60° + 6)}{\tan 18°} = \frac{\tan 18°}{\tan 18°} = 1$$

101. c.

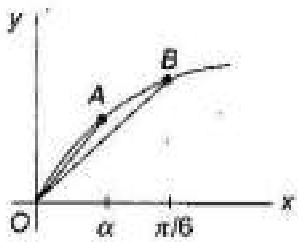


Fig. 2.38

In the graph of $y = \sin x$, let $A \equiv (\alpha, \sin \alpha)$, $B = \left(\frac{\pi}{6}, \sin\frac{\pi}{6}\right)$

Clearly, slope of $OA >$ slope of OB, so $\frac{\sin \alpha}{\alpha} > \frac{\sin \pi/6}{\pi/6} = \frac{3}{\pi} \Rightarrow \frac{\alpha}{\sin \alpha} < \frac{\pi}{3}$.

102.d. $(\alpha - \beta) = (\theta - \beta) - (\theta - \alpha)$

$\Rightarrow \cos(\alpha - \beta) = \cos(\theta - \beta)\cos(\theta - \alpha) + \sin(\theta - \beta)\sin(\theta - \alpha)$

$$= \frac{y}{b} \times \frac{x}{a} + \sqrt{1 - \frac{x^2}{a^2}}\sqrt{1 - \frac{y^2}{b^2}}$$

$$\Rightarrow \left[\frac{xy}{ab} - \cos(\alpha - \beta)\right]^2 = \left(1 - \frac{x^2}{a^2}\right)\left(1 - \frac{y^2}{b^2}\right)$$

$$\Rightarrow \frac{x^2y^2}{a^2b^2}+\cos^2(\alpha-\beta)-\frac{2xy}{ab}\cos(\alpha-\beta) = 1-\frac{y^2}{b^2}-\frac{x^2}{a^2}+\frac{x^2y^2}{a^2b^2}$$

$$\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2xy}{ab}\cos(\alpha-\beta)= \sin^2(\alpha-\beta)$$

103. b. Let $u = \cos\theta\left\{\sin\theta+\sqrt{\sin^2\theta+\sin^2\alpha}\right\}$

$\Rightarrow \quad (u-\sin\theta\cos\theta)^2 = \cos^2\theta(\sin^2\theta+\sin^2\alpha)$

$\Rightarrow \quad u^2\tan^2\theta-2u\tan\theta+u^2-\sin^2\alpha=0$

Since $\tan\theta$ is real, $4u^2-4u^2(u^2-\sin^2\alpha)\geq 0$.

$\Rightarrow \quad u^2\leq 1+\sin^2\alpha$

$\Rightarrow \quad |u|\leq\sqrt{1+\sin^2\alpha}$

104. a. $\sin\theta_1\sin\theta_2-\cos\theta_1\cos\theta_2=-1$

$\Rightarrow \quad \cos(\theta_1+\theta_2)=1$

$\Rightarrow \quad \theta_1+\theta_2=2n\pi,\, n\in I$

$\Rightarrow \quad \frac{\theta_1}{2}+\frac{\theta_2}{2}=n\pi$

Thus, $\tan\frac{\theta_1}{2}\cot\frac{\theta_2}{2}=\tan\frac{\theta_1}{2}\cot\left(n\pi-\frac{\theta_1}{2}\right)=-\tan\frac{\theta_1}{2}\cot\frac{\theta_1}{2}=-1$

105. a. $\tan\frac{\pi}{3}+2\tan\frac{2\pi}{3}+4\tan\frac{4\pi}{3}+8\tan\frac{8\pi}{3}$

$=\tan\frac{\pi}{3}+2\tan\left(\pi-\frac{\pi}{3}\right)+4\tan\left(\pi+\frac{\pi}{3}\right)+8\tan\left(3\pi-\frac{\pi}{3}\right)$

$=\tan\frac{\pi}{3}-2\tan\frac{\pi}{3}+4\tan\frac{\pi}{3}-8\tan\frac{\pi}{3}=-5\tan\frac{\pi}{3}=-5\sqrt{3}$

106. c. Since $\tan 3\theta=\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$

Putting $\theta=\frac{\pi}{9}$, we get $\tan\frac{\pi}{3}=\frac{3\tan\frac{\pi}{9}-\tan^3\frac{\pi}{9}}{1-3\tan^2\frac{\pi}{9}}$

$\Rightarrow \quad 3\left(1-3\tan^2\frac{\pi}{9}\right)^2=\left(3\tan\frac{\pi}{9}-\tan^3\frac{\pi}{9}\right)^2$

$\Rightarrow \quad \tan^6\frac{\pi}{9}-33\tan^4\frac{\pi}{9}+27\tan^2\frac{\pi}{9}=3$

107. c. $\cos x+\cos y-\cos(x+y)=\frac{3}{2}$

$\Rightarrow \quad 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)-2\cos^2\left(\frac{x+y}{2}\right)+1=\frac{3}{2}$

$\Rightarrow \quad 2\cos^2\left(\frac{x+y}{2}\right)-2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)+\frac{1}{2}=0$

Now $\cos\left(\frac{x+y}{2}\right)$ is always real, then discriminant ≥ 0.

$\Rightarrow \quad 4\cos^2\left(\frac{x-y}{2}\right)-4\geq 0$

$\Rightarrow \quad \cos^2\left(\frac{x-y}{2}\right)\geq 1$

$\Rightarrow \quad \cos^2\left(\frac{x-y}{2}\right)=1$

$\Rightarrow \quad \frac{x-y}{2}=0 \Rightarrow x=y$

108. a. $a\sin x+b\cos(x+\theta)+b\cos(x-\theta)=d$

$\Rightarrow \quad a\sin x+2b\cos x\cos\theta=d$

$\Rightarrow \quad |d|\leq\sqrt{a^2+4b^2\cos^2\theta}$

$\Rightarrow \quad \frac{d^2-a^2}{4b^2}\leq\cos^2\theta$

$\Rightarrow \quad |\cos\theta|\geq\frac{\sqrt{d^2-a^2}}{2|b|}$

109. d. $\frac{\sin x}{\sin y}=\frac{1}{2}, \frac{\cos x}{\cos y}=\frac{3}{2} \Rightarrow \frac{\tan x}{\tan y}=\frac{1}{3} \Rightarrow \tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}=\frac{4\tan x}{1-3\tan^2 x}$

Also $\sin y=2\sin x, \cos y=\frac{2}{3}\cos x$

$\Rightarrow \quad \sin^2 y+\cos^2 y=4\sin^2 x+\frac{4\cos^2 x}{9}=1$

$\Rightarrow \quad 36\tan^2 x+4=9\sec^2 x=9(1+\tan^2 x)$

$\Rightarrow \quad 27\tan^2 x=5$

$\Rightarrow \quad \tan x=\frac{\sqrt{5}}{3\sqrt{3}}$

$\Rightarrow \quad \tan(x+y)=\frac{\frac{4\sqrt{5}}{3\sqrt{3}}}{1-\frac{15}{27}}=\frac{4\sqrt{5}\times 27}{12\times 3\sqrt{3}}=\sqrt{15}$

110. a. $\sqrt{1+\cos x}=\sqrt{2\cos^2\frac{x}{2}}=\sqrt{2}\left|\cos\frac{x}{2}\right|$ and $\sqrt{1-\cos x}=\sqrt{2\sin^2\frac{x}{2}}=\sqrt{2}\left|\sin\frac{x}{2}\right|$

$\Rightarrow \frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}=\frac{\left|\cos\frac{x}{2}\right|+\left|\sin\frac{x}{2}\right|}{\left|\cos\frac{x}{2}\right|-\left|\sin\frac{x}{2}\right|}$

$$= \frac{-\cos\frac{x}{2}+\sin\frac{x}{2}}{-\cos\frac{x}{2}-\sin\frac{x}{2}} \qquad \left(\because \ \frac{\pi}{2}<\frac{x}{2}<\pi\right)$$

$$= \frac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}}$$

$$= \frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}$$

$$= \tan\left(\frac{\pi}{4}-\frac{x}{2}\right) = \cot\left(\frac{\pi}{2}-\left(\frac{\pi}{4}-\frac{x}{2}\right)\right) = \cot\left(\frac{\pi}{4}+\frac{x}{2}\right)$$

111. d. $\tan x = n\tan y$, $\cos(x-y) = \cos x\cos y + \sin x\sin y$

$\Rightarrow \quad \cos(x-y) = \cos x\cos y\,(1+\tan x\,.\,\tan y) = \cos x\cos y\,(1+n\tan^2 y)$

$$\Rightarrow \quad \sec^2(x-y) = \frac{\sec^2 x\sec^2 y}{(1+n\tan^2 y)^2}$$

$$= \frac{(1+\tan^2 x)(1+\tan^2 y)}{(1+n\tan^2 y)^2}$$

$$= \frac{(1+n^2\tan^2 y)(1+\tan^2 y)}{(1+n\tan^2 y)^2}$$

$$= 1+\frac{(n-1)^2\tan^2 y}{(1+n\tan^2 y)^2}$$

Now, $\left(\dfrac{1+n\tan^2 y}{2}\right)^2 \geq n\tan^2 y \qquad (\because \text{A.M.} \geq \text{G.M.})$

$$\Rightarrow \quad \frac{\tan^2 y}{(1+n\tan^2 y)^2} \leq \frac{1}{4n}$$

$$\Rightarrow \quad \sec^2(x-y) \leq 1+\frac{(n-1)^2}{4n} = \frac{(n+1)^2}{4n}$$

112. b. $\cot^2 x = \cot(x-y)\cot(x-z)$

$$\Rightarrow \quad \cot^2 x = \left(\frac{\cot x\cot y+1}{\cot y-\cot x}\right)\left(\frac{\cot x\cot z+1}{\cot z-\cot x}\right)$$

$\Rightarrow \quad \cot^2 x\cot y\cot z - \cot^3 x\cot y - \cot^3 x\cot z + \cot^4 x$

$\qquad\qquad = \cot^2 x\cot y\cot z + \cot x\cot y + \cot x\cot z + 1$

$\Rightarrow \quad \cot^3 x(\cot y+\cot z) + \cot x(\cot y+\cot z) + 1 - \cot^4 x = 0$

$\Rightarrow \quad \cot x(\cot y+\cot z)(1+\cot^2 x) + (1-\cot^2 x)(1+\cot^2 x) = 0$

$\Rightarrow \quad [\cot x(\cot y+\cot z) + (1-\cot^2 x)] = 0$

$$\Rightarrow \quad \frac{\cot^2 x-1}{2\cot x} = \frac{1}{2}(\cot y+\cot z) = \cot 2x$$

113. a. $A+B+C=\pi \quad \Rightarrow \quad \tan A+\tan B+\tan C=\tan A\tan B\tan C$

Now, A.M. ≥ G.M.

$$\Rightarrow \quad \frac{\tan A+\tan B+\tan C}{3} \geq (\tan A\tan B\tan C)^{1/3}$$

$$\Rightarrow \quad \frac{\tan A+\tan B+\tan C}{3} \geq (\tan A\tan B\tan C)^{1/3}$$

$$\Rightarrow \quad (\tan A\tan B\tan C)^{2/3} \geq 3$$

$$\Rightarrow \quad \left(\frac{1}{K}\right)^{2/3} \geq 3$$

$$\Rightarrow \quad K \leq \frac{1}{3\sqrt{3}}$$

114. d. $u^2=(a^2\cos^2\theta+b^2\sin^2\theta)+(a^2\sin^2\theta+b^2\sin^2\theta)$

$$=a^2+b^2+2\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}\,\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}$$

$$=a^2+b^2+2\sqrt{\sin^2\theta\cos^2\theta\left(a^4+b^4\right)+a^2b^2\left(\sin^4\theta+\cos^4\theta\right)}$$

$$=a^2+b^2+2\sqrt{a^2b^2\left(1-2\sin^2\theta\cos^2\theta\right)+\left(a^4+b^4\right)\sin^2\theta\cos^2\theta}$$

$$=(a^2+b^2)+2\sqrt{a^2b^2+\left(a^2-b^2\right)^2\sin^2\theta\cos^2\theta}$$

$$=(a^2+b^2)+2\sqrt{a^2b^2+\frac{\left(a^2-b^2\right)^2}{4}\sin^2 2\theta}$$

$$\text{Max. } u^2=(a^2+b^2)=2\sqrt{a^2b^2+\frac{\left(a^2-b^2\right)^2}{4}}$$

Min. $u^2=(a^2+b^2)+2ab$

$$\text{Therefore, the difference}=2\sqrt{a^2b^2+\frac{\left(a^2-b^2\right)^2}{4}}-2ab=\sqrt{4a^2b^2+a^4+b^4-2a^2b^2}-2ab$$

$$=\sqrt{\left(a^2+b^2\right)^2}-2ab=a^2+b^2-2ab=(a-b)^2$$

115. c. $(\sin x+\cos x)^2+k\sin x\cos x=1$

$$\Rightarrow 1+\sin 2x+\frac{k}{2}\sin 2x=1$$

$$\Rightarrow \sin 2x\left[1+\frac{k}{2}\right]=0$$

For this to be an identity, $1+\frac{k}{2}=0 \Rightarrow k=-2$

116. b. $k\cos^2 x - k\cos x + 1 \ge 0\ \forall x \in (-\infty, \infty)$

$\Rightarrow k(\cos^2 x - \cos x) + 1 \ge 0$

Now $\cos^2 x - \cos x = \left(\cos x - \frac{1}{2}\right)^2 - \frac{1}{4}$

$\Rightarrow -\frac{1}{4} \le \cos^2 x - \cos x \le 2$

We have $2k + 1 \ge 0$ and $-\frac{k}{4} + 1 \ge 0$

Hence, $-\frac{1}{2} \le k \le 4$.

117. d. $\min(2 + \sin x - \cos x) = \min\left[2 + \sqrt{2}\sin\left(x - \frac{\pi}{4}\right)\right] = 2 - \sqrt{2}$

118. b. $a \operatorname{cosec} \alpha - b \sec \alpha = \frac{a}{\sin \alpha} + \frac{b}{\cos \alpha}$

$$= \frac{\sqrt{a^2 + b^2}}{\sin \alpha \cos \alpha}\left[\frac{a}{\sqrt{a^2 + b^2}}\cos \alpha \frac{b}{\sqrt{a^2 + b^2}}\sin \alpha\right]$$

Now $\sin 3\alpha = \frac{a}{\sqrt{a^2 + b^2}}$ gives $\sqrt{a^2 + b^2}\left[\frac{\sin 3\alpha \cos \alpha - \cos 3\alpha \sin \alpha}{\sin \alpha \cos \alpha}\right] = 2\sqrt{a^2 + b^2}$

119. d. $\cot^4 x - 2(1 + \cot^2 x) + a^2 = 0$

$\Rightarrow \cot^4 x - 2\cot^2 x + a^2 - 2 = 0$

$\Rightarrow (\cot^2 x - 1)^2 = 3 - a^2$

To have at least one solution, $3 - a^2 \ge 0$

$\Rightarrow a^2 - 3 \le 0$

$a \in [-\sqrt{3}, \sqrt{3}]$

Integral values are $-1, 0, 1$; therefore, the sum is 0.

120. b. $\sin^2 x + a\cos x + a^2 > 1 + \cos x$

Putting $x = 0$, we have

$a + a^2 > 2$

$\Rightarrow a^2 + a - 2 > 0$

$\Rightarrow (a + 2)(a - 1) > 0$

$\Rightarrow a < -2$ or $a > 1$

Therefore, the largest negative integral value of 'a' $= -3$.

121. d. $\Delta = \frac{1}{2}ab \Rightarrow ab = 60$

$b = c\cos\theta;\ a = c\sin\theta$

$\Rightarrow \Delta = \frac{1}{2}c^2 \sin\theta\cos\theta = \frac{c^2\sin 2\theta}{4} = 30$

$\Rightarrow c^2 = 120\,\text{cosec}\,2\theta$

$\Rightarrow c^2_{\min} = 120$

$\Rightarrow c_{\min} = 2\sqrt{30}$

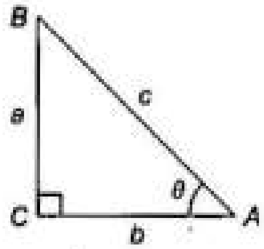


Fig. 2.39

122. b. From the figure,

$x\cos(\theta + 30°) = \theta$

and $x\sin\theta = 1 - d$ (i)

Dividing $\sqrt{3}\cos\theta = \frac{1+d}{1-d}$, squaring Eq. (ii) and (ii)

putting the value of $\cot\theta$, we get $x^2 = \frac{1}{3}(4d^2 - 4d + 4)$

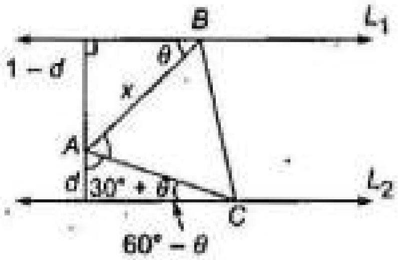


Fig. 2.40

$\Rightarrow x = 2\frac{1}{2}$

123. d. $a = c\sin\theta,\ b = c\cos\theta$

$\Rightarrow \left(\frac{c}{a} + \frac{c}{b}\right)^2 = \left(\frac{1}{\sin\theta} + \frac{1}{\cos\theta}\right)^2 = \frac{4(1+\sin 2\theta)}{\sin^2 2\theta}$

$= 4\left(\frac{1}{\sin^2 2\theta} + \frac{1}{\sin 2\theta}\right)$ where $0 < \theta < \frac{\pi}{2}$

$\Rightarrow \left(\frac{c}{a} + \frac{c}{b}\right)^2_{\min} = 8$ when $2\theta = 90°$.

$\Rightarrow \theta = 45°$

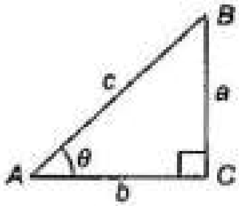


Fig. 2.41

124. c. $y = (\sin^2 x + \cos^2 x) + 2(\sin x\,\text{cosec}\,x + \cos x\sec x) + \sec^2 x + \text{cosec}^2 x$

$= 5 + 2 + \tan^2 x + \cot^2 x$

$= 7 + (\tan x - \cot x)^2 + 2$

$\therefore y_{\min} = 9$

Multiple Correct Answers Type

1. a, b. $2\sin\alpha\cos\alpha = 2\cos^2\beta$

$\sin 2\alpha = 1 + \cos 2\beta$

$\therefore\ \cos 2\beta = -(1 - \sin 2\alpha) = -\left(1 - \cos\left(\frac{\pi}{2} - 2\alpha\right)\right) = -2\sin^2\left(\frac{\pi}{4} - \alpha\right) = -2\cos^2\left(\frac{\pi}{4} + \alpha\right)$

2. a, b, c, d. $\cos^2\theta = 1 - \sin^2\theta$. Let $81^{\sin^2\theta} = z$, we get $z + \frac{81}{z} = 30$ or $z^2 - 30z + 81 = 0$

$\Rightarrow \quad (z-27)(z-3) = 0$, i.e., $81^{\sin^2\theta} = 3^{4\sin^2\theta} = 3^3$ or 3^1

$\therefore \quad \sin^2\theta = \frac{3}{4}, \frac{1}{4}$ or $\sin\theta = \pm\frac{\sqrt{3}}{2}, \pm\frac{1}{2}$

$\therefore \quad \theta = 30°, 60°, 120°, 150°$

3. b, c, d. Opposite angles of a cyclic quadrilateral are supplementary.

4. a, b, c.

a. $\tan\theta = \frac{1-\cos 2\theta}{\sin 2\theta} \Rightarrow$ (a) is correct.

b. $\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}$

$\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$

$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

$\Rightarrow$ (b) is correct.

c. $\tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta} \Rightarrow$ (c) is correct.

d. $\sin\theta = 1/3$ which is rational but $\cos 3\theta = \cos\theta(4\cos^2\theta - 3)$ which is irrational.
$\Rightarrow$ (d) is incorrect.

5. a, b, c, d.

a. $\sin\left(\frac{11\pi}{12}\right)\sin\left(\frac{5\pi}{12}\right) = \sin\left(\frac{\pi}{12}\right)\cos\left(\frac{\pi}{12}\right)$

$$= \frac{1}{2}\sin\left(\frac{\pi}{6}\right) = \frac{1}{4} \in Q$$

b. $\operatorname{cosec}\left(\frac{9\pi}{10}\right)\sec\left(\frac{4\pi}{5}\right) = -\operatorname{cosec}\left(\frac{\pi}{10}\right)\sec\left(\frac{\pi}{5}\right)$

$$= \frac{1}{\sin 18° \cos 36°}$$

$$= -\frac{16}{(\sqrt{5}-1)\ (\sqrt{5}+1)} = -4 \in Q$$

c. $\sin^4\left(\frac{\pi}{8}\right) + \cos^4\left(\frac{\pi}{8}\right) = 1 - 2\sin^2\left(\frac{\pi}{8}\right)\cos^2\left(\frac{\pi}{8}\right)$

$$= 1 - \frac{1}{2}\sin^2\left(\frac{\pi}{4}\right) = 1 - \frac{1}{4} = \frac{3}{4} \in Q$$

d. $2\cos^2\frac{\pi}{9}\; 2\cos^2\frac{2\pi}{9}\; 2\cos^2\frac{4\pi}{9} = 8(\cos 20°\cos 40°\cos 80°) = \frac{1}{8} \in Q$

6. **a, b, d.** $(\sin^2 x + \cos^2 x)^3 - 3\sin^2 x\cos^2 x(\sin^2 x + \cos^2 x) > \frac{5}{8}$

$\Rightarrow \quad 1 - 3\sin^2 x\cos^2 x > \frac{5}{8}$

$\Rightarrow \quad \frac{3}{8} > 3\sin^2 x\cos^2 x$

$\Rightarrow \quad 1 - 2\sin^2 2x > 0$

$\Rightarrow \quad \cos 4x > 0$

$\Rightarrow \quad 4x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\Rightarrow \quad 4x \in \left(2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}\right)$, $n \in Z$, generalizing now verify.

7. **a, b, d.**

c. $\sum \sin^2\frac{A}{2} = \frac{1}{2}[3 - (\cos A + \cos B + \cos C)]$

$$= \frac{3}{2} - \frac{1}{2}(\cos A + \cos B + \cos C)$$

but $[\cos A + \cos B + \cos C]_{\max} = \frac{3}{2}$

$\therefore \quad \left.\sum \sin^2\frac{A}{2}\right]_{\min} = \frac{3}{2} - \frac{3}{4} = \frac{3}{4}$

$\therefore \quad \sum \sin^2\frac{A}{2} \ge \frac{3}{4}$

$\Rightarrow$ (c) is incorrect

a, b, d are correct and hold good in an equilateral triangle as the maximum values.

8. **a, b, c.**

a. $\tan\alpha\tan 2\alpha\tan 3\alpha = \tan 3\alpha - \tan 2\alpha - \tan\alpha$ always holds good.

b. R.H.S. $= \frac{\sin 4\alpha + \sin 2\alpha}{\sin 2\alpha\sin 4\alpha} = \frac{2\sin 3\alpha\cos\alpha}{\sin 2\alpha\sin 4\alpha} = \frac{1}{\sin\alpha} = \operatorname{cosec}\alpha$ (using $\pi/7 = \alpha$)

$\Rightarrow$ (b) is correct

c. $\cos\alpha + \cos 3\alpha + \cos 5\alpha = \dfrac{\sin 3\alpha}{\sin\alpha}\cos 3\alpha = \dfrac{\sin 6\alpha}{2\sin\alpha} = \dfrac{\sin\frac{6\pi}{7}}{2\sin\frac{\pi}{7}} = \dfrac{\sin\left(\pi+\frac{\pi}{7}\right)}{2\sin\frac{\pi}{7}} = \dfrac{1}{2}$

Also $\cos 2\alpha = \cos\frac{2\pi}{7} = -\cos\left(\pi - \frac{5\pi}{7}\right) = -\cos\left(\frac{5\pi}{7}\right) = -\cos 5\alpha$

d. $8\cos\alpha\cos 2\alpha\cos 4\alpha = \dfrac{\sin 8\alpha}{\sin\alpha} = \dfrac{\sin\frac{8\pi}{7}}{\sin\frac{\pi}{7}} = -\dfrac{\sin\frac{\pi}{7}}{\sin\frac{\pi}{7}} = -1$

9. a, b, c, d.

a. For $x \in \left(0, \frac{\pi}{4}\right)$, $\tan x < \cot x$

Also $\ln(\sin x) < 0$

$\Rightarrow (\tan x)^{\ln(\sin x)} > (\cot x)^{\ln(\sin x)}$

b. For $x \in \left(0, \frac{\pi}{2}\right)$, $\operatorname{cosec} x \geq 1$

$\Rightarrow \ln(\operatorname{cosec} x) \geq 0$

$\Rightarrow 4^{\ln \operatorname{cosec} x} < 5^{\ln(\operatorname{cosec} x)}$

c. $x \in \left(0, \frac{\pi}{2}\right) \Rightarrow \cos x \in (0, 1)$

$\Rightarrow \ln(\cos x) < 0$

Also $\frac{1}{2} > \frac{1}{3}$

$\Rightarrow \left(\frac{1}{2}\right)^{\ln(\cos x)} < \left(\frac{1}{3}\right)^{\ln(\cos x)}$

d. For $x \in \left(0, \frac{\pi}{2}\right)$

Since $\sin x < \tan x$, we get $\ln(\sin x) < \ln(\tan x)$

$\Rightarrow 2^{\ln(\sin x)} < 2^{\ln(\tan x)}$

10. a, c, d.

a. 1

b. 3

c. $\dfrac{\sin 24^\circ\cos 6^\circ - \cos 24^\circ\sin 6^\circ}{\sin 21^\circ\cos 39^\circ - \sin 39^\circ\cos 21^\circ} = \dfrac{\sin(18^\circ)}{\sin(-18^\circ)} = -1$

d. -1

11. a, c.

a. $\dfrac{\cos^2\alpha - \sin^2\alpha}{\sin\alpha\cos\alpha} = 2\cot 2\alpha$

b. $\frac{1+t}{1-t}-\frac{1-t}{1+t}$ where $t=\tan\alpha$

$$=\frac{(1+t)^2-(1-t)^2}{1-t^2}=\frac{4t}{1-t^2}=\frac{2\times 2\tan\alpha}{1-\tan^2\alpha}=2\tan 2\alpha$$

$\Rightarrow$ (b) is incorrect.

c. $\frac{1+t}{1-t}+\frac{1-t}{1+t}=\frac{(1+t)^2+(1-t)^2}{1-t^2}=\frac{2(1+t^2)}{1-t^2}$ (where $t=\tan\alpha$)

$$=\frac{2}{\cos 2\alpha}=2\sec 2\alpha$$

$\Rightarrow$ (c) is correct.

d. $\tan\alpha+\cot\alpha=\frac{1}{\cos\alpha\sin\alpha}=\frac{2}{\sin 2\alpha}=2\operatorname{cosec} 2\alpha$

$\Rightarrow$ (d) is incorrect.

12. a, c, d.

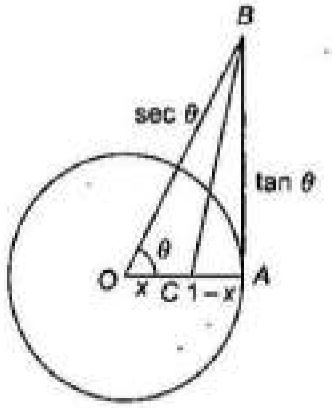


Fig. 2.42

Using property of angle bisector, we get

$$\frac{\sec\theta}{\tan\theta}=\frac{x}{1-x}$$

$$\Rightarrow\quad x=\frac{\sec\theta}{\sec\theta+\tan\theta}=\sec\theta(\sec\theta-\tan\theta)=\frac{1}{1+\sin\theta}$$

13. a, d. $y=\frac{(1+\tan^2 x)^2}{1+\tan^2 x}=1+\tan^2 x$

$$=1+\left(2-\sqrt{3}\right)^2$$

$$=8-4\sqrt{3}=4(2-\sqrt{3})$$

$$=4\left[\left(\sqrt{\frac{3}{2}}-\frac{1}{\sqrt{2}}\right)^2\right]$$

$$= 4\left[4\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)^2\right]$$

$$= 16\sin^2\frac{\pi}{12}$$

14. **b, d.** $\tan(\alpha+\beta) = \frac{15}{8}$ and cosec $\gamma = \frac{17}{8} \Rightarrow \tan\gamma = \frac{8}{15}$

$\therefore \quad \alpha+\beta+\gamma = \frac{\pi}{2} \Rightarrow$ (b) is true.

Also $\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

$\Rightarrow \quad \cot\gamma = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

$\Rightarrow \quad \tan\alpha\tan\beta+\tan\beta\tan\gamma+\tan\gamma\tan\alpha = 1$

15. **b, d.** Divide by $\cos\alpha$ and square both sides and let $\tan\alpha = t$ so that $\sec^2\alpha = 1+t^2$

$\Rightarrow \quad [(a+2)t+(2a-1)]^2 = [(2a+1)^2(1+t^2)]$

$\Rightarrow \quad t^2[(a+2)^2-(2a+1)^2]+2(a+2)(2a-1)t+[(2a-1)^2-(2a+1)^2] = 0$

$\Rightarrow \quad 3(1-a^2)t^2+2(2a^2+3a-2)t-4\times 2a = 0$

$\Rightarrow \quad 3(1-a^2)t^2-4(1-a^2)t+6at-8a = 0$

$\Rightarrow \quad t(1-a^2)(3t-4)+2a(3t-4) = 0$

$\Rightarrow \quad (3t-4)[(1-a^2)t+2a] = 0$

$\Rightarrow \quad t = \tan\alpha = \frac{4}{3}$ or $\frac{2a}{a^2-1}$

16. **a, b, c.** $\log_{1/3}\log_7(\sin x+a) > 0$

$\Rightarrow 0 < \log_7(\sin x+a) < 1$

$\Rightarrow 1 < (\sin x+a) < 7, \forall x \in R$

It is found that 'a' should be less than the minimum value of $7-\sin x$ and 'a' must be greater than the maximum value of $1-\sin x$

$\Rightarrow 1-\sin x < a < 7-\sin x \; \forall x \in R$

$\Rightarrow 2 < a < 6$

17. **a, c.** $\log_b \sin t = x \Rightarrow \sin t = b^x$

Let $\log_b(\cos t) = y$, then $b^y = \cos t$

$\Rightarrow b^{2y} = \cos^2 t = 1-\sin^2 t = 1-b^{2x}$

$\Rightarrow 2y = \log_b(1-b^{2x})$

$\Rightarrow y = \frac{1}{2}\log_b(1-b^{2x}) = \log_b\sqrt{1-b^{2x}}$

10. a. Statement 2 is true as it is one of the standard results of multiple angles.
Putting $A = \pi/18$ in the formula $\sin 3A = 3\sin A - 4\sin^3 A$, we get $8x^3 - 6x + 1 = 0$, where $x = \sin \pi/18$. Hence, statement 1 is also true because of statement 2.

11. a. Statement 2 is true, because each trigonometric function has a principle period of π or 2π and hence 2π is one of the periods of every trigonometric function.
Thus $f(2A) = f(2B)$
$\Rightarrow$ $2A = 2n\pi + 2B$, for some $n \in Z$
$\Rightarrow$ $A = n\pi + B$
$\Rightarrow$ Statement 1 is true because of statement 2.

12. b. From Fig. 2.43, $\sin 3 < \sin 1 < \sin 2$
But statement 2 is not sufficient to ensure this.
Hence, answer is (b).

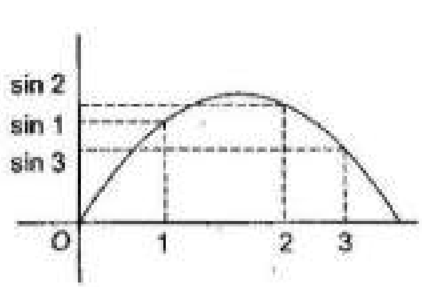


Fig. 2.43

13. a. Let $g(x) = 3\sin x + 4\cos x - 2$

Maximum value of $g(x) = \sqrt{3^2 + 4^2} - 2 = 3$

Minimum value of $g(x) = -\sqrt{3^2 + 4^2} - 2 = -7$

Therefore, the range of $f(x) = \dfrac{1}{g(x)}$ is $R - \left(-\dfrac{1}{7}, \dfrac{1}{3}\right)$

Hence, it is an unbounded function, and $f(x)$ has no maximum and no minimum values.

14. b. Statement 2 is true as it is one of the identities in triangle.
R.H.S. in statement 2 is always positive as $\alpha, \beta, \gamma \in (0, \pi/2)$
Statement 1 is true as select $\alpha = 2\pi, \beta = -\pi/2, \gamma = -\pi/2$
Then $\sin\alpha + \sin\beta + \sin\gamma = 0 - 1 - 1 = -2$, which shows that minimum value will be negative.
But statement 2 is not the correct explanation of statement 1, as $\alpha + \beta + \gamma = \pi$ does not follow that α, β, γ are angles of a triangle.

15. a. $\sin^2 A + \sin^2 B + \sin^2 C = \dfrac{1-\cos 2A}{2} + \dfrac{1-\cos 2B}{2} + \dfrac{1-\cos 2C}{2}$

$$= \frac{3-(\cos 2A + \cos 2B + \cos 2C)}{2}$$

$$= \frac{3-(-1-4\cos A\cos B\cos C)}{2}$$

$$= 2 + 2\cos A\cos B\cos C \quad \text{(i)}$$

Hence, statement 2 is true.

From statement 1 using Eq. (i), we get $\cos A \cos B \cos C = 0$, then either A, B or C is $90°$.

Both statement 1 and statement 2 are true and statement 2 is the correct explanation of statement 1.

16. d. Statement 2 is true as it is one of the conditional identities in the triangle. Since R.H.S. > 1 in statement 2, statement 1 is false.

17. d. Given $\cos x \sum \cos \alpha - \sin x \sum \sin \alpha = 0 \; \forall \; x \in R$

Hence, $\cos \alpha + \cos \beta + \cos \gamma = 0$

and $\quad \sin \alpha + \sin \beta + \sin \gamma = 0$

Hence, statement 2 is true.

Now $(\cos \alpha + \cos \beta)^2 = (-\cos \gamma)^2$

and $(\sin \alpha + \sin \beta)^2 = (-\sin \gamma)^2$

Adding, we get

$2 + 2\cos(\alpha - \beta) = 1$

$\Rightarrow \quad \cos(\alpha - \beta) = -1/2$

Similarly, $\cos(\beta - \gamma) = -1/2$ and $\cos(\gamma - \alpha) = -1/2$

Now $0 < \alpha < \beta < \gamma < 2\pi$

$\Rightarrow \quad \beta - \alpha < \gamma - \alpha$

Hence, $\beta - \alpha = \dfrac{2\pi}{3}$ and $\gamma - \alpha = \dfrac{4\pi}{3}$.

Statement 1 is false.

18. a. $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A \cos B \cos C$

$$\therefore \quad \sum \cos^2 A\Big|_{\min} = 1 - 2 \times \frac{1}{8} = 1 - \frac{1}{4} = \frac{3}{4}$$

19. d. Let $x = \cot A$; $y = \cot B$ and $z = \cot C$

$\Rightarrow \quad \sum \cot A \cot B = 1$

$\Rightarrow \quad A + B + C = n\pi$

In statement 1

$$\therefore \quad \text{L.H.S.} = \frac{1}{2}[\sin 2A + \sin 2B + \sin 2C]$$

$$= 2 \sin A \sin B \sin C$$

$$\text{R.H.S.} = \frac{2}{|\operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C|}$$

$$= \pm 2 \sin A \sin B \sin C = \text{L.H.S.}$$

Statement 2 is true as it is one of the conditional identities in the triangle.

20. b. In triangle ABC, $A + B + C = \pi$

and $\cot \dfrac{A}{2} + \cot \dfrac{B}{2} + \cot \dfrac{C}{2} = \cot \dfrac{A}{2} \cot \dfrac{B}{2} \cot \dfrac{C}{2}$

Therefore, $\ln\left(\cot \dfrac{A}{2} + \cot \dfrac{B}{2} + \cot \dfrac{C}{2}\right) = \ln \cot \dfrac{A}{2} + \ln \cot \dfrac{B}{2} + \ln \cot \dfrac{C}{2}$

Hence, statement 1 is true.

In statement 2, R.H.S. $= \ln 1 + \ln \sqrt{3} + \ln(2 + \sqrt{3}) = \ln(1\sqrt{3}(2 + \sqrt{3})) = \ln(2\sqrt{3} + 3) =$ R.H.S.

But statement 2 does not explain statement 1.

18. **a, b, d.** Let $x = \cos\theta$, then $4\cos^3\theta - 3\cos\theta = -\dfrac{\sqrt{3}}{2}$

$$\Rightarrow \cos 3\theta = \cos\frac{5\pi}{6}$$

$$\Rightarrow 3\theta = 2n\pi \pm \frac{5\pi}{6}$$

$$\Rightarrow \theta = \frac{2n\pi}{3} \pm \frac{5\pi}{18}$$

Putting $n = 0$, we get $\theta = 5\pi/18$

$$n = 1 \Rightarrow \theta = \frac{2\pi}{3} \pm \frac{5\pi}{18} = \frac{17\pi}{18}$$

$$\text{and } \theta = \frac{2\pi}{3} - \frac{5\pi}{18} = \frac{17\pi}{18}$$

19. **a, c, d.** $\sin x\cos 20° + \cos x\sin 20° = 2\sin x\cos 40°$

$\Rightarrow \sin 20°\cos x = \sin x(2\cos 40° - \cos 20°)$

$$\Rightarrow \tan x = \frac{\sin 20°}{2\cos 40° - \cos 20°} = \frac{\sin 20°}{\cos 40° + \cos 40° - \cos 20°} = \frac{\sin 20°}{\cos 40° + 2\sin 30°\sin(-10°)}$$

$$= \frac{\sin 20°}{\sin 50° - \sin 10°} = \frac{\sin 20°}{2\cos 30°\sin 20°}$$

$$\Rightarrow \tan x = \frac{1}{\sqrt{3}} \Rightarrow x = 30°$$

Reasoning Type

1. d. Statement 1 is wrong as z can be written as $\dfrac{-(x+y)}{1-xy}$.

It implies that for any values of xy ($xy \neq 1$), we get a value of z and statement 2 is correct.

2. b. $\cos 7 = \cos(2\pi + 7 - 2\pi) = \cos(7 - 2\pi) = \cos(0.72)$

Now 1 rad and 0.72 rad angles are for first quadrant where $\cos x$ is decreasing, hence $\cos 1 < \cos 0.72$ or $\cos 1 < \cos 7$.

But statement 2 is not the correct explanation for $\cos 1 < \cos 7$.

Note that $\cos 0.5 > \cos 7$.

3. b. $\tan 4 = \tan(\pi + (4 - \pi)) = \tan(4 - \pi) = \tan 0.86$

$\tan 7.5 = \tan(2\pi + (7.5 - 2\pi)) = \tan(7.5 - 2\pi) = \tan 1.22$

Now both angles, i.e., 0.86 rad and 1.22 rad are for first quadrant, hence $\tan 0.86 < \tan 1.22$ as $\tan x$ is an increasing function. But this is not always true, as $3 > 1$ but $\tan 3 < \tan 1$.

Hence, both statements are true but statement 2 is not the correct explanation of statement 1.

4. b. In first quadrant, $\cos\theta > \sin\theta$ for $\theta \in (\pi/4, \pi/2)$.
Hence, $\cos 1 < \sin 1$.
Also in first quadrant, cosine is decreasing and sine is increasing, but this is not the correct reason for which $\cos 1 < \sin 1$. Thus, the correct answer is (b).

5. a. $f(\theta) = (\sin\theta + \operatorname{cosec}\theta)^2 + (\cos\theta + \sec\theta)^2$
$= \sin^2\theta + \cos^2\theta + \operatorname{cosec}^2\theta + \sec^2\theta + 4$
$$= 5 + \frac{1}{\sin^2\theta} + \frac{1}{\cos^2\theta} = 5 + \frac{4}{\sin^2 2\theta} \geq 9$$
Hence, the correct answer is (a).

6. d. $\sin^2\theta_1 + \sin^2\theta_2 + \cdots + \sin^2\theta_n = 0$
$\Rightarrow \quad \sin\theta_1 = \sin\theta_2 = \cdots = \sin\theta_n = 0$
$\Rightarrow \quad \cos^2\theta_1, \cos^2\theta_2, \ldots, \cos^2\theta_n = 1$
$\Rightarrow \quad \cos\theta_1, \cos\theta_2, \ldots, \cos\theta_n = \pm 1$
Now $\cos\theta_1 + \cos\theta_2 + \cdots + \cos\theta_n = n - 4$ means two of $\cos\theta_1, \cos\theta_2, \ldots, \cos\theta_n$ must be -1 and the others are 1. Now two values from $\cos\theta_1, \cos\theta_2, \ldots, \cos\theta_n$ can be selected in ${}^nC_2$ ways. Hence, the number of solutions is ${}^nC_2 = \frac{n(n-1)}{2}$.
Hence, statement 1 is false, but statement 2 is correct.

7. a. Let $y = 27^{\cos 2x} \times 81^{\sin 2x} = 3^{3\cos 2x + 4\sin 2x}$
Now $-\sqrt{3^2 + 4^2} \leq 3\cos 2x + 4\sin 2x \leq \sqrt{3^2 + 4^2}$
or $-5 \leq 3\cos 2x + 4\sin 2x \leq 5$
$\Rightarrow \quad 3^{-5} \leq 3^{3\cos 2x + 4\sin 2x} \leq 3^5$
Hence, the correct answer is (a).

8. d. Obviously in triangle ABC,
$$\tan A = \tan(\pi - (B + C)) = -\tan(B + C) = \frac{\tan B + \tan C}{\tan B \tan C - 1}$$
If angle A is obtuse, then $\tan A < 0$
$$\Rightarrow \quad \frac{\tan B + \tan C}{\tan B \tan C - 1} < 0$$
$\Rightarrow \quad \tan B \tan C < 1$ (as B and C will be acute)
Thus statement 1 is false and statement 2 is true.

9. a. We know that $\tan 15° = 2 - \sqrt{3}$ which is an irrational number. Hence, statement 2 is true.
Statement 1 is also true as if $\tan 5°$ is a rational number, then $\tan 15° = \frac{3\tan 5° - \tan^3 5°}{1 - 3\tan^2 5°}$ should be a rational number, which is not true.
Hence, $\tan 5°$ is an irrational number.
Obviously, statement 2 is the correct reasoning for statement 1.

Linked Comprehension Type

For Problems 1 – 3

1. a., 2. b., 3. c.

Sol **1. a.** $\sin\alpha = A\sin(\alpha+\beta) = A(\sin\alpha\cos\beta + \sin\beta\cos\alpha)$

$$\Rightarrow \quad \sin\alpha(1 - A\cos\beta) = A\sin\beta\cos\alpha \qquad \text{(i)}$$

$$\Rightarrow \quad \tan\alpha = \frac{A\sin\beta}{(1 - A\cos\beta)} \qquad \text{(ii)}$$

2. b. $\tan\beta = \dfrac{\sin\beta}{\cos\beta} = \dfrac{(1-A\cos\beta)\tan\alpha}{A\cos\beta} = \dfrac{(1-A\cos\beta)\sin\alpha}{A\cos\alpha\cos\beta}$ [from Eqs. (i) and (ii)]

3. c. $\tan(\alpha+\beta) = \dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

$$= \frac{\dfrac{A\sin\beta}{1-A\cos\beta} + \dfrac{\sin\beta}{\cos\beta}}{1 - \dfrac{A\sin\beta\sin\beta}{(1-A\cos\beta)\cos\beta}}$$

$$= \frac{A\sin\beta\cos\beta + \sin\beta - A\sin\beta\cos\beta}{\cos\beta - A\cos^2\beta - A\sin^2\beta} = \frac{\sin\beta}{\cos\beta - A}$$

Also $\tan(\alpha+\beta) = \dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

$$= \frac{\dfrac{\sin\alpha}{\cos\alpha} + \dfrac{\sin\alpha(1-A\cos\beta)}{A\cos\alpha\cos\beta}}{1 - \dfrac{\sin^2\alpha(1-A\cos\beta)}{A\cos^2\alpha\cos\beta}} \qquad \text{[from Eq. (ii)]}$$

$$= \frac{[A\sin\alpha\cos\beta + \sin\alpha - A\sin\alpha\cos\beta]\cos\alpha}{A\cos^2\alpha\cos\beta - \sin^2\alpha + A\sin^2\alpha\cos\beta}$$

$$= \frac{\sin\alpha\cos\alpha}{A\cos\beta - \sin^2\alpha}$$

For Problems 4 – 6

4. d, 5. a, 6. b.

Sol. We have $\tan\left(\theta + \dfrac{\pi}{4}\right) = 3\tan 3\theta$

$$\Rightarrow \quad \frac{1+\tan\theta}{1-\tan\theta} = 3 \times \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$$

$\Rightarrow \quad \dfrac{1+t}{1-t} = 3\left(\dfrac{3t-t^3}{1-3t^2}\right)$ (putting $t = \tan\theta$)

$\Rightarrow \quad 3t^4 - 6t^2 + 8t - 1 = 0$

Hence,

S_1 = sum of roots $= t_1 + t_2 + t_3 + t_4 = 0$

S_2 = sum of product of roots taken two at a time $= -2$

S_3 = sum of product of roots taken three at time $= -8/3$

S_4 = product of all roots $= -1/3$

$$\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4} = \frac{\sum t_1t_2t_3}{t_1t_2t_3t_4} = \frac{-8}{3}$$

For Problems 7 – 9

7. a, 8. b, 9. d.

Sol. $\sin\alpha + \sin\beta = 3$ (i)

$$\Rightarrow \quad 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) = 3 \quad \text{(ii)}$$

$\cos\alpha + \cos\beta = 4$ (iii)

$$\Rightarrow \quad 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) = 4 \quad \text{(iv)}$$

Dividing Eq. (ii) by Eq. (iv), we have

$$\tan\left(\frac{\alpha+\beta}{2}\right) = \frac{3}{4}$$

$$\Rightarrow \quad \sin(\alpha+\beta) = \frac{2\tan\left(\frac{\alpha+\beta}{2}\right)}{1+\tan^2\left(\frac{\alpha+\beta}{2}\right)} = \frac{2\times\frac{3}{4}}{1+\left(\frac{3}{4}\right)^2} = \frac{24}{25}$$

$$\text{and} \quad \cos(\alpha+\beta) = \frac{1-\tan^2\left(\frac{\alpha+\beta}{2}\right)}{1+\tan^2\left(\frac{\alpha+\beta}{2}\right)} = \frac{1-\left(\frac{3}{4}\right)^2}{1+\left(\frac{3}{4}\right)^2} = \frac{7}{25}$$

For Problems 10 – 12

10. a, 11. c, 12. b.

Sol. Let $\theta = \dfrac{n\pi}{7}$ (so that $7\theta = n\pi$)

$\Rightarrow \quad 4\theta + 3\theta = n\pi$

$\Rightarrow \quad \tan 4\theta = \tan(n\pi - 3\theta) = -\tan 3\theta$

$$\Rightarrow \quad \frac{4\tan\theta - 4\tan^3\theta}{1-6\tan^2\theta + \tan^4\theta} = -\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$$

$$\Rightarrow \quad \frac{4z-4z^3}{1-6z^2+z^4} = -\frac{3z-z^3}{1-3z^2} \qquad \text{[where } \tan\theta = z \text{ (say)]}$$

$$\Rightarrow \quad (4-4z^2)(1-3z^2) = -(3-z^2)(1-6z^2+z^4)$$

$$\Rightarrow \quad z^6 - 21z^4 + 35z^2 - 7 = 0 \qquad \text{(i)}$$

This is a cubic equation in z^2, i.e., in $\tan^2\theta$.

The roots of this equation are therefore $\tan^2 \pi/7$, $\tan^2 2\pi/7$ and $\tan^2 3\pi/7$ from Eq. (i), sum of the

$$\text{roots} = \frac{-(-21)}{1} = 21$$

$$\Rightarrow \quad \tan^2\frac{\pi}{7} + \tan^2\frac{2\pi}{7} + \tan^2\frac{3\pi}{7} = 21 \qquad \text{(ii)}$$

Putting $1/y$ in place of z in Eq. (i), we get $-7y^6 + 35y^4 - 21y^2 + 1 = 0$

or $\quad 7y^6 - 35y^4 = 21y^2 - 1 = 0 \qquad$ (iii)

This is a cubic equation in y^2, i.e., in $\cot^2\theta$.

The roots of this Eq. are therefore $\cot^2 \pi/7$, $\cot^2 2\pi/7$ and $\cot^2 3\pi/7$.

Sum of the roots of Eq. (iii) = 35/7 = 5

$$\Rightarrow \quad \cot^2\frac{\pi}{7} + \cot^2\frac{2\pi}{7} + \cot^2\frac{3\pi}{7} = 5 \qquad \text{(iv)}$$

By multiplying Eqs. (ii) and (iv), we get

$$\left(\tan^2\frac{\pi}{7} + \tan^2\frac{2\pi}{7} + \tan^2\frac{3\pi}{7}\right)\left(\cot^2\frac{\pi}{7} + \cot^2\frac{2\pi}{7} + \cot^2\frac{3\pi}{7}\right) = 21 \times 5 = 105$$

For Problems 13 – 15

13. b, 14. b, 15. d.

Sol. Angles BEC, ABD, ABE and BAC are in A.P.

Let $\angle BEC = \alpha - 3\beta$, $\angle ABD = \alpha - \beta$, $\angle ABE = \alpha + \beta$ and $\angle BAC = \alpha + 3\beta$

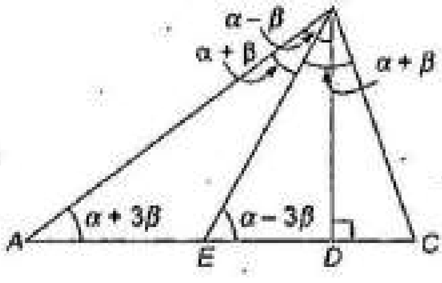


Fig. 2.44

From ΔABD, $\alpha - \beta + \alpha + 3\beta = \dfrac{\pi}{2}$

$\Rightarrow \quad 2\beta + 2\beta = \dfrac{\pi}{2} \Rightarrow \alpha + \beta = \dfrac{\pi}{4}$

Now, $\alpha - 3\beta = (\alpha + 3\beta) + (\alpha + \beta)$ [using exterior angle theorem]

$\Rightarrow \quad \alpha = -7\beta$

$\therefore \quad \beta = -\dfrac{\pi}{24},\ \alpha = \dfrac{7\pi}{24}$

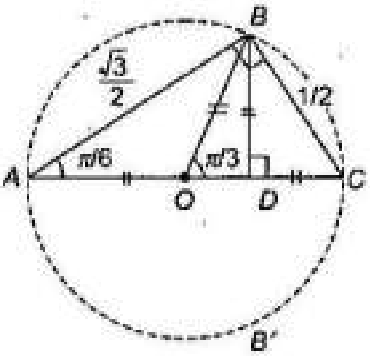


Fig. 2.45

$\therefore \quad \angle B = 2(\alpha + \beta) = \dfrac{\pi}{2},\ \angle A = \dfrac{\pi}{6},\ \angle C = \dfrac{\pi}{3}$

$\Rightarrow \quad ABC$ is a 30°–90°–60° triangle.

13. Area of the circle circumscribing $\Delta ABC = \pi\left(\dfrac{1}{2}\right)^2 = \dfrac{\pi}{4}$.

14. ΔBOC is equilateral $\Rightarrow r = \dfrac{\Delta}{s} = \dfrac{\dfrac{\sqrt{3}}{2}\left(\dfrac{1}{2}\right)^2}{\dfrac{1}{2}\left(\dfrac{3}{2}\right)} = \dfrac{1}{4\sqrt{3}}$

15. $BD = OB \sin\dfrac{\pi}{3} = \dfrac{1}{2}\sin\dfrac{\pi}{3} = \dfrac{\sqrt{3}}{4}$

$\therefore \quad BB' = 2BD = \dfrac{\sqrt{3}}{2}$

Matrix-Match Type

1. a → q; b → r; c → s; d → p

$\cos\theta - \sin\theta = \dfrac{1}{5}$, where $0 < \theta < \dfrac{\pi}{2}$. (i)

Squaring both sides of Eq. (i), we get

$1 - \sin 2\theta = \dfrac{1}{25}$

$\Rightarrow \quad \sin 2\theta = \dfrac{24}{25} \quad \Rightarrow \quad \cos 2\theta = \dfrac{7}{25}$

Also, $(\cos\theta + \sin\theta)^2 = (\cos\theta - \sin\theta)^2 + 4\cos\theta\sin\theta = \dfrac{1}{25} + 2\sin 2\theta = \dfrac{1}{25} + \dfrac{48}{25} = \dfrac{49}{25}$

$$\Rightarrow \quad \cos\theta + \sin\theta = \frac{7}{5} \qquad \text{(ii)}$$

$$\Rightarrow \quad (\cos\theta + \sin\theta)/2 = \frac{7}{10}$$

Also solving Eqs. (i) and (ii), we get $\cos\theta = 4/5$.

2. **a → q; b → s; c → p; d → r**

a. $A = \sin^2\theta + \cos^4\theta$

$$= \frac{1-\cos 2\theta}{2} + \left(\frac{1+\cos 2\theta}{2}\right)^2$$

$$= \frac{1}{2} - \frac{1}{2}\cos 2\theta + \frac{1}{4} + \frac{1}{2}\cos 2\theta + \frac{1}{4}\cos^2 2\theta$$

$$= \frac{3}{4} + \frac{1}{4}\left(\frac{\cos 4\theta + 1}{2}\right) = \frac{3}{4} + \frac{1}{8} + \frac{1}{8}\cos 4\theta$$

Now, $-1 \le \cos 4\theta \le 1$

$$\Rightarrow \quad \frac{-1}{8} \le \frac{\cos 4\theta}{8} \le \frac{1}{8}$$

$$\Rightarrow \quad \frac{3}{4} + \frac{1}{8} - \frac{1}{8} \le \frac{3}{4} + \frac{1}{4}\left(\frac{1+\cos 4\theta}{2}\right) \le \frac{3}{4} + \frac{1}{8} + \frac{1}{8}$$

$$\Rightarrow \quad \frac{3}{4} \le A \le 1$$

b. $A = 3\cos^2\theta + \sin^4\theta = 3\dfrac{1+\cos 2\theta}{2} + \left(\dfrac{1-\cos 2\theta}{2}\right)^2$

$$= \frac{3+3\cos 2\theta}{2} + \frac{1-2\cos 2\theta + \cos^2 2\theta}{4}$$

$$= \frac{7+4\cos 2\theta + \cos^2 2\theta}{4} = \frac{(\cos 2\theta + 2)^2 + 3}{4}$$

Now, $1 \le \cos 2\theta + 2 \le 3$

$$\Rightarrow \quad 1 \le \frac{(\cos 2\theta + 2)^2 + 3}{4} \le 3$$

c. $A = \sin^2\theta - \cos^4\theta$

$$= \frac{1-\cos 2\theta}{2} - \left(\frac{1+\cos 2\theta}{2}\right)^2$$

$$= \frac{1}{2} - \frac{1}{2}\cos 2\theta - \frac{1}{4} - \frac{1}{2}\cos 2\theta - \frac{1}{4}\cos^2 2\theta$$

$$= \frac{1}{4} - \cos 2\theta - \frac{1}{4}\cos^2 2\theta$$

$$= -\left(\frac{1}{4}\cos^2 2\theta + \cos 2\theta - \frac{1}{4}\right)$$

$$= \frac{5}{4} - \left(\frac{1}{2}\cos 2\theta + 1\right)^2$$

Now, $-\frac{1}{2} \le \frac{1}{2}\cos 2\theta \le \frac{1}{2}$

$$\Rightarrow \quad \frac{1}{2} \le \frac{1}{2}\cos 2\theta + 1 \le \frac{3}{2}$$

$$\Rightarrow \quad \frac{1}{4} \le \left(\frac{1}{2}\cos 2\theta + 1\right)^2 \le \frac{9}{4}$$

$$\Rightarrow \quad -\frac{9}{4} \le -\left(\frac{1}{2}\cos 2\theta + 1\right)^2 \le -\frac{1}{4}$$

$$\Rightarrow \quad -1 \le \frac{5}{4} - \left(\frac{1}{2}\cos 2\theta + 1\right)^2 \le 1$$

d. $A = \tan^2\theta + 2\cot^2\theta = (\tan\theta - \sqrt{2}\cot\theta)^2 + 2\sqrt{2} \ge 2\sqrt{2}$

3. **a → r; b → p; c → q; d → s**

$\cos\alpha + \cos\beta = 1/2$

$$\Rightarrow \quad 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) = \frac{1}{2} \tag{i}$$

$\sin\alpha + \sin\beta = 1/3$

$$\Rightarrow \quad 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) = \frac{1}{3} \tag{ii}$$

Dividing Eq. (ii) by Eq. (i), we get

$$\tan\left(\frac{\alpha+\beta}{2}\right) = \frac{2}{3}$$

$$\Rightarrow \quad \cos\left(\frac{\alpha+\beta}{2}\right) = \pm\frac{3}{\sqrt{13}}$$

Squaring and adding the given results, we have

$$2 + 2\cos(\alpha - \beta) = \frac{13}{36}$$

$$\Rightarrow \quad \cos(\alpha - \beta) = -\frac{59}{72}$$

Now, $2\cos^2\left(\frac{\alpha-\beta}{2}\right) - 1 = \cos(\alpha - \beta)$

$$\Rightarrow \quad 2\cos^2\left(\frac{\alpha-\beta}{2}\right) = 1 - \frac{59}{72} = \frac{13}{72}$$

$$\Rightarrow \cos\left(\frac{\alpha-\beta}{2}\right) = \pm\frac{\sqrt{13}}{12}$$

$$\Rightarrow \tan\left(\frac{\alpha-\beta}{2}\right) = \pm\sqrt{\frac{131}{13}}$$

4. **a → s; b → r; c → p; d → q**

a. $\sin(410° + 400°) = \sin 810° = \sin(720° + 90°) = \sin 90° = 1$

b. $\dfrac{\sin^2 2° - \sin^2 1°}{2\sin 3° \sin 1°} = \dfrac{\sin 3° \sin 1°}{2\sin 3° \sin 1°} = \dfrac{1}{2}$

c. $\sin(-870°) + \operatorname{cosec}(-660°) + \tan(-855°) + 2\cot(840°) + \cos(480°) + \sec(900°)$

$= -\sin(810° + 60°) - \operatorname{cosec}(720° - 60°) - \tan(810° + 45°) + 2\cot 120° + \cos 120° + \sec 180°$

$= -\dfrac{1}{2} + \dfrac{2}{\sqrt{3}} + 1 - \dfrac{2}{\sqrt{3}} - \dfrac{1}{2} - 1 = -1$

d. $\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi = \dfrac{4}{5} \times \dfrac{3}{5} - \dfrac{3}{5} \times \dfrac{4}{5} = 0$

5. **a → s; b → r; c → q; d → p**

a. $\{\cos(2A + \theta) + \cos(2B + \theta)\} = 2\cos(A - B)\cos(A + B + \theta)$

Maximum value is $2\cos(A - B)$ when $\cos(A + B + \theta) = 1$

b. $\{\cos 2A + \cos 2B\}$

$2\cos(A + B)\cos(A - B)$

Maximum value is $2\cos(A - B)$ when $\cos(A + B) = 1$

c. For $y = \sec x$, $x \in (0, \pi/2)$, tangent drawn to it at any point lies completely below the graph of $y = \sec x$, thus $\dfrac{\sec 2A + \sec 2B}{2} \geq \sec(A + B)$

$\Rightarrow \sec 2A + \sec 2B \geq \sec(A + B)$

Hence, the minimum value is $2\sec(A + B)$.

d. $\sqrt{|\tan\theta + \cot\theta - 2\cos 2(A + B)|} = \sqrt{\left(\sqrt{\tan\theta} - \sqrt{\cot\theta}\right)^2 + 2 - 2\cos 2(A + B)}$

$$= \sqrt{\left(\sqrt{\tan\theta} - \sqrt{\cot\theta}\right)^2 + 4\sin^2(A + B)}$$

Minimum value occurs when $\sqrt{\tan\theta} = \sqrt{\cot\theta}$ and

minimum value is $\sqrt{4\sin^2(A + B)} = 2\sin(A + B)$

6. **a → s; b → r; c → p; d → q**

a. $\cos 20° + \cos 80° - \sqrt{3}\cos 50° = 2\cos 30° \cos 50° - \sqrt{3}\cos 50°$

$$= \sqrt{3}\cos 50° - \sqrt{3}\cos 50° = 0$$

b. $\cos 0° + \cos\dfrac{\pi}{7} + \cos\dfrac{2\pi}{7} + \cos\dfrac{3\pi}{7} + \cos\dfrac{4\pi}{7} + \cos\dfrac{5\pi}{7} + \cos\dfrac{6\pi}{7}$

$$= 1 + \left(\cos\frac{\pi}{7} + \cos\frac{6\pi}{7}\right) + \left(\cos\frac{2\pi}{7} + \cos\frac{5\pi}{7}\right) + \left(\cos\frac{3\pi}{7} + \cos\frac{4\pi}{7}\right)$$

$$=1+\left(\cos\frac{\pi}{7}+\cos\left(\pi-\frac{\pi}{7}\right)\right)+\left(\cos\frac{2\pi}{7}+\cos\left(\pi-\frac{2\pi}{7}\right)\right)+\left(\cos\frac{3\pi}{7}+\cos\left(\pi-\frac{3\pi}{7}\right)\right)$$

$=1+0+0+0=1$

c. $\cos 20° + \cos 40° + \cos 60° - 4\cos 10° \cos 20° \cos 30°$
$= \cos 20° + \cos 40° + \cos 60° - 4\cos 10° \cos 20° \cos 30°$
$= 2\cos 30° \cos 10° + 2\cos^2 30° - 1 - 4\cos 10° \cos 20° \cos 30°$
$= 2\cos 30° (\cos 10° + \cos 30°) - 1 - 4\cos 10° \cos 20° \cos 30°$
$= 2\cos 30° (2\cos 10° \cos 20°) - 1 - 4\cos 10° \cos 20° \cos 30° = -1$

d. $\cos 20° \cos 100° + \cos 100° \cos 140° - \cos 140° \cos 200°$

$$=\frac{1}{2}(\cos 120° + \cos 80° + \cos 240° + \cos 40° - \cos 340° - \cos 60°)$$

$$=\frac{1}{2}(-\frac{1}{2}+\cos 80° - \frac{1}{2} + \cos 40° - \cos 340° - \frac{1}{2}) = \frac{1}{2}(-\frac{3}{2}+\cos 80° + \cos 40° - \cos 20°)$$

$$=\frac{1}{2}(-\frac{3}{2}+2\cos 60° \cos 20° - \cos 20°) = \frac{1}{2}(-\frac{3}{2}) = -\frac{3}{4}$$

7. **a → q; b → q; c → p, r; d → p, s**

a. Since angles, A, B and C are acute angles
$\therefore A + B > \pi/2$
$A > \frac{\pi}{2} - B$
$\sin A - \cos B > 0$
$\Rightarrow \cos B - \sin A < 0$ (i)
Again, $B > \frac{\pi}{2} - A$
$\sin B > \cos A$
$\sin B - \cos A > 0$ (ii)
From Eq. (i) and (ii), we get that x-coordinates is –ve and y-coordinate is +ve.
Therefore, line is in 2nd quadrant only

b. $2^{\sin\theta} > 1 \Rightarrow \sin\theta > 0 \Rightarrow \theta \in$ 1st or 2nd quadrant
$3^{\cos\theta} < 1 \Rightarrow \cos\theta < 0 \Rightarrow \theta \in$ 2nd or 3rd quadrant
Hence, $\theta \in$ 2nd quadrant

c. $|\cos x + \sin x| = |\sin x| + |\cos x|$
$\Rightarrow \cos x$ and $\sin x$ must have same sign or at least one is zero.
$\Rightarrow x \in$ 2nd or 4th quadrant

d. L.H.S $= \dfrac{1-\sin A}{|\cos A|} + \dfrac{\sin A}{\cos A} = \dfrac{1}{\cos A}$ which is true only if $|\cos A| = \cos A$

8. **a → p; b → p; c → q; d → s**

a. $x = \sin\theta$, $y = \cos\theta$
$P = (3\sin\theta - 4\sin^3\theta)^2 + (3\cos\theta - 4\cos^3\theta)^2 = \sin^2 3\theta + \cos^2 3\theta = 1$

b. On adding, we get $a = \dfrac{3-\cos 4\theta + 4\sin 2\theta}{2} = (1+\sin 2\theta)^2$
On subtracting, we get $b = (1-\sin 2\theta)^2 \Rightarrow ab = \cos^4 2\theta \le 1$

c. $3\cos\theta = x^2 - 8x + 19$
$\Rightarrow 3\cos\theta = (x-4)^2 + 3$

Now L.H.S. = $3\cos\theta \le 3$ or L.H.S. has the greatest value 3.

But R.H.S. $(x-4)^2+3 \ge 3$ or R.H.S. has the least value 3.

Hence, L.H.S. = R.H.S. when $3\cos\theta = (x-4)^2+3=3$

$\Rightarrow \cos\theta = 1$ and $x-4=0$

$\Rightarrow \theta = 2n\pi$ and $x=4$, where $n \in Z$

d. $\lambda = \tan\theta$

$x = 2\sin 2\theta$ and $y = 2\cos 2\theta$

$E = x^2 - xy + y^2 = 4 - 4\sin 2\theta \cos 2\theta = 4 - 2\sin 4\theta$

$E \in [2, 6] \Rightarrow a + b = 8$

9. **a → q; b → p; c → s; d → r**

a. $9 + 16 + 24\sin(A+B) = 37$ (on squaring and adding)

$24\sin(A+B) = 12$

$\sin(A+B) = \frac{1}{2} \Rightarrow \sin C = \frac{1}{2}$

$C = 30°$ or $150°$

$\Rightarrow C = 30°$ ($\because$ for $C = 150°$)

b. $(\sin A + \sin B)^2 - \sin^2 C = 3\sin A \sin B$

$\Rightarrow \sin^2 A - \sin^2 C + \sin^2 B = \sin A \sin B$

$\Rightarrow \sin(A+C)\sin(A-C) + \sin^2 B = \sin A \sin B$

$\Rightarrow \sin B[\sin(A-C) + \sin(A+C)] = \sin A \sin B$

$\Rightarrow 2\sin A\cos C = \sin A$ (as $\sin B \ne 0$)

$\Rightarrow \cos C = 1/2$

$\Rightarrow C = 60°$

c. $2\sin x\cos x[4\cos^4 x - 4\sin^4 x] = 1$

$\Rightarrow (\sin 2x)[2(\cos^2 x + \sin^2 x)][2\cos^2 x - 2\sin^2 x] = 1$

$\Rightarrow (\sin 2x) 2 \times 2\cos 2x = 1$

$\Rightarrow 2\sin 4x = 1$

$\Rightarrow \sin 4x = \frac{1}{2} \Rightarrow 4x = 30° \Rightarrow x = 7.5°$

d.

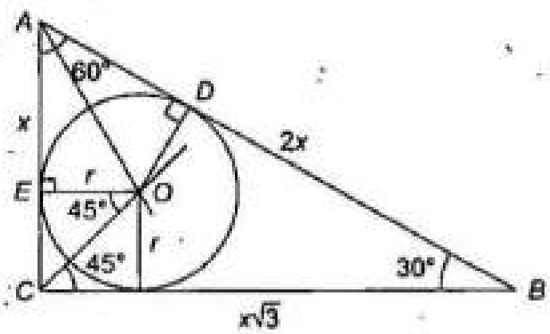


Fig. 2.46

Obviously, $AEOD$ is a cyclic quadrilateral, we have

$\angle COD = 120° + 45° = 165°$

Integer Type

1. (4) $f(\theta)=\dfrac{1-\sin 2\theta+\cos 2\theta}{2\cos 2\theta}$

$$=\frac{(\cos\theta-\sin\theta)^2+(\cos^2\theta-\sin^2\theta)}{2(\cos\theta-\sin\theta)(\cos\theta+\sin\theta)}$$

$$=\frac{\cos\theta}{\cos\theta+\sin\theta}$$

$$=\frac{1}{1+\tan\theta}$$

$$f(11^\circ)\cdot f(34^\circ)=\frac{1}{(1+\tan 11^\circ)}\times\frac{1}{(1+\tan 34^\circ)}$$

$$=\frac{1}{(1+\tan 11^\circ)}\times\frac{1}{(1+\tan(45^\circ-11^\circ))}$$

$$=\frac{1}{(1+\tan 11^\circ)}\times\frac{1}{\left(1+\frac{1-\tan 11^\circ}{1+\tan 11^\circ}\right)}$$

$$=\frac{1}{(1+\tan 11^\circ)}\times\frac{(1+\tan 11^\circ)}{2}=\frac{1}{2}$$

2. (5) $f(x)=2(7\cos x+24\sin x)(7\sin x-24\cos x)$

$r\cos\theta=7;\ r\sin\theta=24$

$r^2=625;\ \tan\theta=\dfrac{24}{7}$

$f(x)=2r\cos(x-\theta)\times r\sin(x-\theta)$

$=r^2(\sin 2(x-\theta))$

$f(x)_{max}=25^2\Rightarrow(f(x))^{1/4}=5$

3. (1) $\tan\left(\dfrac{A}{2}\right)+\tan\left(\dfrac{B}{2}\right)=-\dfrac{b}{a};\ \tan\left(\dfrac{A}{2}\right)\times\tan\left(\dfrac{B}{2}\right)=\dfrac{c}{a}$

$A+B=90^\circ\Rightarrow\dfrac{A+B}{2}=45^\circ$

$$\Rightarrow\tan\left(\frac{A+B}{2}\right)=1=\frac{-\frac{b}{a}}{1-\frac{c}{a}}$$

$$\Rightarrow 1-\frac{c}{a}=-\frac{b}{a}$$

$\Rightarrow a+b=c$

$$\Rightarrow\frac{a+b}{c}=1$$

4. (5) $(1+\tan\theta)[1+\tan(45°-\theta)] = (1+\tan\theta)\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right)$

$$= (1+\tan\theta)\left(\frac{2}{1+\tan\theta}\right) = 2$$

Hence, L.H.S. is equal to

$2(1+\tan 5°)(1+\tan 40°)(1+\tan 10°)(1+\tan 35°)(1+\tan 15°)(1+\tan 30°)(1+\tan 20°)(1+\tan 25°)$

$= 2\times 2^4 = 2^5$

5. (3) $$\sqrt{3}\left|\frac{\frac{-2\sin(40°)\cos(40°)}{\cos(80°)}+\frac{\sin(20°)}{\cos(20°)}}{\frac{\cot(20°)+\tan(80°)}{\cot(20°)}}\right| = \sqrt{3}\left|\frac{\tan(20°)-\tan(80°)}{1+\tan 20°\tan 80°}\right|$$

$$= \sqrt{3}\tan(60°) = 3$$

6. (1) Let $x+5 = 14\cos\theta$ and $y-12 = 14\sin\theta$

$\therefore x^2+y^2 = (14\cos\theta-5)^2+(14\sin\theta+12)^2$

$= 196+25+144+28(12\sin\theta-5\cos\theta)$

$= 365+28(12\sin\theta-5\cos\theta)$

$\therefore \sqrt{x^2+y^2}\Big|_{\min} = \sqrt{365-28\times 13} = \sqrt{365-364} = 1$

7. (5) $\cot x+\cot y = 49$

$\Rightarrow \frac{1}{\tan x}+\frac{1}{\tan y} = 49$

$\Rightarrow \frac{\tan y+\tan x}{\tan x\tan y} = 49$

$\Rightarrow \tan x\tan y = \frac{\tan x+\tan y}{49} = \frac{42}{49} = \frac{6}{7}$

$\Rightarrow \tan(x+y) = \frac{42}{1-(6/7)} = \frac{42}{1/7} = 294$ which is divisible by 2, 3 and 7 but not by 5.

8. (7) From the given equations, we have

$(2\cos a+9\cos d)^2 = (6\cos b+7\cos c)^2$

And $(2\sin a-9\sin d)^2 = (6\sin b-7\sin c)^2$

Adding, we have $36\cos(a+d) = 84\cos(b+c)$

$\Rightarrow \frac{\cos(a+d)}{\cos(b+c)} = \frac{7}{3}$

9. (8) Since $\cos A + \cos B = 0$

$\Rightarrow A + B = \pi$,

$\therefore B = \pi - A$

$\Rightarrow \sin A + \sin(\pi - A) = 1$

$\Rightarrow \sin A = \frac{1}{2}$

$\Rightarrow A = 30°$ and $B = 150°$ or $A = 150°$ and $B = 30°$

$\Rightarrow 12 \cos 60° + 4 \cos 300° = 8$

10. (4) $5\frac{2\tan\beta}{1+\tan^2\beta} = 3\frac{2\tan\alpha}{1+\tan^2\alpha}$

$$\Rightarrow \frac{5\tan\beta}{1+\tan^2\beta} = \frac{3\tan\alpha}{1+\tan^2\alpha} \quad \text{(i)}$$

Substituting $\tan\beta = 3\tan\alpha$, we have

$$\frac{5\times 3\tan\alpha}{1+9\tan^2\alpha} = \frac{3\tan\alpha}{1+\tan^2\alpha}$$

$\Rightarrow 5 + 5\tan^2\alpha = 1 + 9\tan^2\alpha$

$\Rightarrow 4\tan^2\alpha = 4$

$\Rightarrow \tan\alpha = 1$, i.e., $\tan\beta = 3$

$\therefore \tan\alpha + \tan\beta = 4$

11. (4) Let $\theta = \frac{\pi}{16} \Rightarrow 8\theta = \frac{\pi}{2}$

$y = \tan\theta + \tan 5\theta + \tan 9\theta + \tan 13\theta$

$\therefore y = (\tan\theta - \cot\theta) + (\tan 5\theta - \cot 5\theta)$

[as $\tan 13\theta = \tan(8\theta + 5\theta) = -\cot 5\theta$ and $\tan 9\theta = \tan(8\theta + \theta) = -\cot\theta$]

$= (\tan\theta - \cot\theta) + (\cot 3\theta - \tan 3\theta)$

$$= \frac{\sin^2\theta - \cos^2\theta}{\sin\theta\cos\theta} + \frac{\cos^2 3\theta - \sin^2 3\theta}{\sin 3\theta\cos 3\theta}$$

$$\Rightarrow y = 2\left[\frac{\cos 6\theta}{\sin 6\theta} - \frac{\cos 2\theta}{\sin 2\theta}\right]$$

$$= 2\left[\frac{\sin 2\theta\cos 6\theta - \cos 2\theta\sin 6\theta}{\sin 6\theta\sin 2\theta}\right]$$

$$= -2\left[\frac{\sin 4\theta}{\cos 2\theta\sin 2\theta}\right] = -4 \quad \left(\because 6\theta = \frac{\pi}{2} - 2\theta\right)$$

Hence, absolute value = 4.

12. (2) $\cos 290° = \sin 20°$; $\sin 250° = -\sin 70° = -\cos 20°$

$$\Rightarrow \frac{1}{\sin 20°} - \frac{1}{\sqrt{3}\cos 20°}$$

$$= \frac{\sqrt{3}\cos 20° - \sin 20°}{\sqrt{3}\sin 20°\cos 20°}$$

$$= \frac{2[\sin 60° \cos 20° - \sin 20° \cos 60°]}{\sqrt{3}\sin 20° \cos 20°}$$

$$= \frac{4\sin 40°}{\sqrt{3}\sin 40°} = \frac{4\sqrt{3}}{3}$$

Hence, the greatest integer less than or equal to is 2

13. (4) $\sin^6 x + \cos^6 x = (\sin^2 x + \cos^2 x)(\sin^4 x + \cos^4 x - \sin^2 x \cos^2 x)$

$$= 1 - 3\sin^2 x \cos^2 x = 1 - \frac{3(\sin 2x)^2}{4}$$

$$\Rightarrow y = \frac{4}{4 - 3(\sin 2x)^2}$$

$$\Rightarrow y_{max} = \frac{4}{4 - 3(1)} = 4$$

14. (3) $\cos^2(45° + x) + (\sin x - \cos x)^2 = \left[\frac{\cos x}{\sqrt{2}} - \frac{\sin x}{\sqrt{2}}\right]^2 + (\sin x - \cos x)^2$

$$= \frac{3}{2}(1 - \sin 2x) = \frac{3}{2}(1 - (-1))$$

Hence, the maximum value is $= \frac{3}{2}(1-(-1)) = 3$

15. (6) Nr. $= (\sin^2 t + \cos^2 t)^2 - 2\sin^2 t \cos^2 t - 1 = -2\sin^2 t \cos^2 t$

Dr. $= (\sin^2 t + \cos^2 t)^3 - 3\sin^2 t \cos^2 t - 1 = -3\sin^2 t \cos^2 t$

16. (6) $\frac{1}{\sin 10°} + \frac{1}{\sin 50°} - \frac{1}{\sin 70°} = \frac{1}{\cos 80°} + \frac{1}{\cos 40°} - \frac{1}{\cos 20°}$

$$= \frac{\cos 40° \cos 20° + \cos 80° \cos 20° - \cos 40° \cos 80°}{\cos 20° \cos 40° \cos 80°}$$

$$= 8[\cos 20°(\cos 40° + \cos 80°) - \cos 40° \cos 80°]$$

$$= 8[2\cos 20° \cos 60° \cos 20° - \cos 40° \cos 80°]$$

$$= 4[2\cos^2 20° - 2\cos 40° \cos 80°]$$

$$= 4[1 + \cos 40° - (\cos 120° + \cos 40°)]$$

$$= 4 \times \frac{3}{2} = 6$$

17. (7) $f(x) = 9\sin^2 x - 16\cos^2 x - 10(3\sin x - 4\cos x) - 10(3\sin x + 4\cos x) + 100$

$= 25\sin^2 x - 60\sin x + 84$

$= (5\sin x - 6)^2 + 48$

The minimum value of $f(x)$ occurs when $\sin x = 1$.

Therefore, the minimum value of $\sqrt{f(x)}$ is 7.

18. (0) In ΔABC, $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

$\Rightarrow x + x + 1 + 1 - x = x(1 + x)(1 - x)$

$\Rightarrow 2 + x = x - x^3$

$\Rightarrow x^3 = -2 \Rightarrow x = -2^{1/3}$

$\Rightarrow \tan A = x < 0 \Rightarrow$ A is obtuse

$\Rightarrow \tan B = x + 1 = 1 - 2^{1/3} < 0$

Hence, A and B are obtuse, which is not possible in a triangle.

Hence, no such triangle can exist.

19. (4) Given $\log_{10}\left(\frac{\sin 2x}{2}\right) = -1$

$\Rightarrow \frac{\sin 2x}{2} = \frac{1}{10}$

$\Rightarrow \sin 2x = \frac{1}{5}$ (i)

Also $\log_{10}(\sin x + \cos x) = \frac{\log_{10}\left(\frac{n}{10}\right)}{2}$

$\Rightarrow \log_{10}(\sin x + \cos x)^2 = \log_{10}\left(\frac{n}{10}\right)$

$\Rightarrow 1 + \sin 2x = \frac{n}{10}$

$\Rightarrow 1 + \frac{1}{5} = \frac{n}{10}$

$\Rightarrow \frac{6}{5} = \frac{n}{10}$

$\Rightarrow \frac{n}{3} = 4$

20. (4) $$\frac{2\sin 4^\circ \cos 3^\circ + 2\sin 4^\circ \cos 1^\circ}{\cos 1^\circ \cos 2^\circ \sin 4^\circ} = \frac{2\sin 4^\circ[\cos 3^\circ + \cos 1^\circ]}{\cos 1^\circ \cos 2^\circ \sin 4^\circ}$$

$$= \frac{4\cos 2^\circ \cos 1^\circ}{\cos 1^\circ \cos 2^\circ} = 4$$

21. (7) $2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{1}{16}$

$\Rightarrow \left(\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right)\sin\frac{C}{2} = \frac{1}{16}$

$\Rightarrow \sin\frac{C}{2}\left(\frac{1}{2} - \sin\frac{C}{2}\right) = \frac{1}{16}$

$\Rightarrow \sin^2\frac{C}{2} - \frac{1}{2}\sin\frac{C}{2} + \frac{1}{16} = 0$

$\Rightarrow \left(\frac{1}{4} - \sin\frac{C}{2}\right)^2 = 0$

$\Rightarrow \sin\frac{C}{2} = \frac{1}{4}$

$\Rightarrow \cos C = 1 - 2\sin^2\frac{C}{2} = 1 - \frac{1}{8} = \frac{7}{8}$

22. (2) In the triangle,

$\tan A + \tan B + \tan C = \tan A \tan B \tan C$

$\Rightarrow \frac{1}{2}\frac{(2k+1)}{2}\frac{(4k+1)}{2} = \frac{3}{2} + 3k$

$\Rightarrow \frac{8k^2+6k+1}{8} = \frac{3+6k}{2}$

$\Rightarrow 8k^2 + 6k + 1 = 12 + 24k$

$\Rightarrow 8k^2 - 18k - 11 = 0$

$\Rightarrow 8k^2 - 22k + 4k - 11 = 0$

$\Rightarrow (2k+1)(4k-11) = 0$

$\Rightarrow k = -1/2$ or $11/4$

For $k = -1/2$, $\tan B = 0$ (not possible)

$\therefore k = 11/4$

23. (4) $4\sin^3 x\cos 3x + 4\cos^3 x\sin 3x = \frac{3}{2}$

$\Rightarrow (3\sin x - \sin 3x)\cos 3x + (3\cos x + \cos 3x)\sin 3x = \frac{3}{2}$

$\Rightarrow 3[\sin x\cos 3x + \cos x\sin 3x] = \frac{3}{2}$

$\Rightarrow \sin 4x = \frac{1}{2}$

Archives

Subjective

1. We have $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$

$$= \frac{\frac{m}{m+1}+\frac{1}{2m+1}}{1-\frac{m}{m+1}\times\frac{1}{2m+1}}$$

$$= \frac{2m^2+2m+1}{2m^2+2m+1} = 1$$

$\Rightarrow \quad \alpha+\beta = n\pi + \pi/4$, where $n \in Z$.

2. a. To draw the graph of $y = \dfrac{1}{\sqrt{2}}(\sin x + \cos x)$ from $x = -\dfrac{\pi}{2}$ to $x = \dfrac{\pi}{2}$

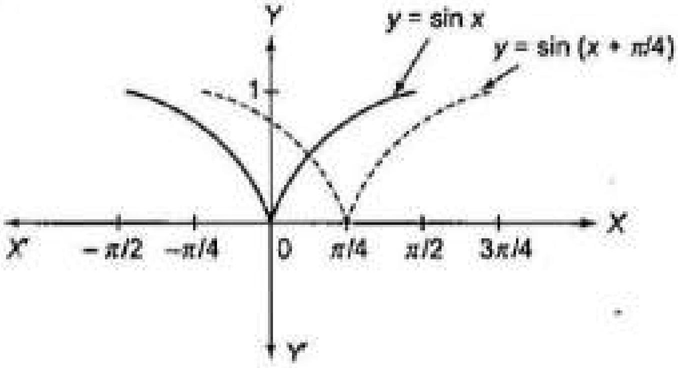


Fig. 2.47

$$y = \frac{1}{\sqrt{2}}(\sin x + \cos x) = \sin\left(x+\frac{\pi}{4}\right)$$

b. We have $\cos(\alpha+\beta) = \dfrac{4}{5}$ and $\sin(\alpha-\beta) = \dfrac{5}{13}$

$\Rightarrow \quad \tan(\alpha+\beta) = \dfrac{3}{4}$ and $\tan(\alpha-\beta) = \dfrac{5}{12}$

$$\tan 2\alpha = \tan[(\alpha+\beta)+(\alpha-\beta)] = \frac{\tan(\alpha-\beta)+\tan(\alpha+\beta)}{1-\tan(\alpha-\beta)\tan(\alpha+\beta)} = \frac{\frac{5}{12}+\frac{3}{4}}{1-\left(\frac{5}{12}\right)\left(\frac{3}{4}\right)} = \frac{56}{33}$$

3. We have,

$$5\cos\theta + 3\cos\left(\theta+\frac{\pi}{3}\right)+3 = 5\cos\theta + 3\cos\theta\cos\frac{\pi}{3} - 3\sin\theta\sin\frac{\pi}{3}+3 = \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta + 3$$

Now, $-\sqrt{\left(\frac{13}{2}\right)^2+\left(\frac{3\sqrt{3}}{2}\right)^2} \le \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta \le \sqrt{\left(\frac{13}{2}\right)^2+\left(\frac{3\sqrt{3}}{2}\right)^2}$

$\Rightarrow \quad -7 \le \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta \le 7$

$\Rightarrow \quad -4 \le \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta + 3 \le 10$

4. Given $\alpha+\beta-\gamma=\pi$ and to prove that

$\sin^2\alpha+\sin^2\beta-\sin^2\gamma = 2\sin\alpha\sin\beta\cos\gamma$

L.H.S. $= \sin^2\alpha+\sin^2\beta-\sin^2\gamma$

$= \sin^2\alpha+\sin(\beta+\gamma)\sin(\beta-\gamma)$

$= \sin^2\alpha+\sin(\beta+\gamma)\sin(\pi-\alpha)$ $\quad (\because \alpha+\beta-\gamma=\pi)$

$= \sin^2\alpha+\sin(\beta+\gamma)\sin\alpha = \sin\alpha[\sin\alpha+\sin(\beta+\gamma)]$

$= \sin\alpha[\sin[\pi-(\beta-\gamma)]+\sin(\beta+\gamma)]$

$= \sin\alpha[\sin(\beta-\gamma)+\sin(\beta+\gamma)] = \sin\alpha[2\sin\beta\cos\gamma]$

$= 2\sin\alpha\sin\beta\cos\gamma$

= R.H.S.

5. We have,

$\cos\theta+\sin\theta = \sqrt{2}\left[\frac{1}{\sqrt{2}}\cos\theta+\frac{1}{\sqrt{2}}\sin\theta\right] = \sqrt{2}\sin(\pi/4+\theta)$

$\therefore \quad \cos\theta+\sin\theta \le \sqrt{2} < \pi/2$ $\quad \left(\begin{matrix}\because \sqrt{2}=1.414\\ \pi/2=1.57\end{matrix}\right)$

$\therefore \quad \cos\theta+\sin\theta<\pi/2 \Rightarrow \quad \cos\theta<\pi/2-\sin\theta$ (i)

As $\theta\in[0,\pi/2]$ in which $\sin\theta$ increases, taking sin on both the sides of Eq. (i) we get

$\sin(\cos\theta)<\sin(\pi/2-\sin\theta) \Rightarrow \sin(\cos\theta)<\cos(\sin\theta)$

$\Rightarrow \quad \cos(\sin\theta)>\sin(\cos\theta)$ (ii)

6. L.H.S. $= \sin 12°\sin 48°\sin 54°$

$= \frac{1}{2}[2\sin 12°\cos 42°]\sin 54°$

$= \frac{1}{2}\left[\sin^2 54° - \frac{1}{2}\sin 54°\right]$

$= \frac{1}{4}\left[2\sin^2 54° - \sin 54°\right]$

$= \frac{1}{4}\left[2\left(\frac{1+\sqrt{5}}{4}\right)^2 - \left(\frac{1+\sqrt{5}}{4}\right)\right]$

$= \frac{1}{4}\left[2\left(\frac{1+5+2\sqrt{5}}{16}\right) - \left(\frac{1+\sqrt{5}}{4}\right)\right]$

$= \frac{1}{4}\times\frac{1}{8}[6+2\sqrt{5}-2-2\sqrt{5}] = \frac{1}{32}\times 4 = \frac{1}{8} =$ R.H.S.

7. We know that

$$\cos A \cos 2A \cos 4A \cdots \cos 2^n A = \frac{1}{2^{n+1} \sin A} \sin\left(2^{n+1} A\right)$$

$$\therefore \quad 16 \cos \frac{2\pi}{15} \cos 2\left(\frac{2\pi}{15}\right) \cos 2^2 \left(\frac{2\pi}{15}\right) \cos 2^3 \left(\frac{2\pi}{15}\right)$$

$$= 16 \frac{\sin\left(2^4 A\right)}{2^4 \sin A} \qquad \text{(where } A = 2\pi/15\text{)}$$

$$= 16 \frac{\sin(32\pi/15)}{16 \sin 2\pi/15} = \frac{\sin(32\pi/15)}{\sin(2\pi + 2\pi/15)} = \frac{\sin(32\pi/15)}{\sin(32\pi/15)} = 1$$

8. We know that

$$\tan 2\alpha = \frac{2\tan \alpha}{1 - \tan^2 \alpha}$$

$$\Rightarrow \quad \frac{1 - \tan^2 \alpha}{\tan \alpha} = 2 \cot 2\alpha \qquad \Rightarrow \quad \cot \alpha - \tan \alpha = 2 \cot 2\alpha \qquad \text{(i)}$$

Now we have to prove $\tan \alpha + 2 \tan 2\alpha + 4 \tan 4\alpha + 8 \cot 8\alpha = \cot \alpha$

L.H.S.

$= \tan \alpha + 2 \tan 2\alpha + 4 \tan 4\alpha + 4 (2 \cot 8\alpha)$

$= \tan \alpha + 2 \tan 2\alpha + 4 \tan 4\alpha + 4 (\cot 4\alpha - \tan 4\alpha)$ [using Eq. (i)]

$= \tan \alpha + 2 \tan 2\alpha + 4 \tan 4\alpha + 4 \cot 4\alpha - 4 \tan 4\alpha = \tan \alpha + 2 \tan 2\alpha + 2 (2 \cot 4\alpha)$

$= \tan \alpha + 2 \tan 2\alpha + 2 (\cot 2\alpha - \tan 2\alpha)$ [using Eq. (i)]

$= \tan \alpha + 2 \cot 2\alpha = \tan \alpha + (\cot \alpha - \tan \alpha)$ [using Eq. (i)]

$= \cot \alpha = $ R.H.S.

9. Given that in ΔABC, A, B and C are in A.P.

$\therefore \quad A + B = 2B$

Also $A + B + C = 180° \quad \Rightarrow \quad B + 2B = 180° \quad \Rightarrow \quad B = 60°$

Also given that $\sin(2A + B) = \sin(C - A) = -\sin(B + 2C) = \dfrac{1}{2}$

$$\Rightarrow \quad \sin(2A + 60°) = \sin(C - A) = -\sin(60 + 2C) = \frac{1}{2} \qquad \text{(i)}$$

From Eq. (i), we have

$$\sin(2A + 60°) = \frac{1}{2}$$

$\Rightarrow \quad 2A + 60° = 150°$

$\Rightarrow \quad 2A = 90°$

$\Rightarrow \quad A = 45°$

$\Rightarrow \quad C = \pi - A - B = 75°$

10. Let $y = \dfrac{\tan x}{\tan 3x}$

$$= \frac{\tan x\left(1 - 3\tan^2 x\right)}{3\tan x - \tan^3 x}$$

$$= \frac{1 - 3\tan^2 x}{3 - \tan^2 x}$$

$\Rightarrow \quad 3y-(\tan^2 x)y = 1-3\tan^2 x$

$\Rightarrow \quad (y-3)\tan^2 x = 3y-1$

$$\Rightarrow \quad \tan^2 x = \frac{3y-1}{y-3}$$

$$\Rightarrow \quad \frac{3y-1}{y-3} \geq 0 \qquad \text{(L.H.S. is a prefect square)}$$

Fig. 2.48

$\Rightarrow \quad y < \frac{1}{3}$ or $y \geq 3$

Thus, y never lies between 1/3 and 3.

11. $S = \sum_{k=1}^{n-1} (n-k)\cos\frac{2k\pi}{n}$

$$= (n-1)\cos\frac{2\pi}{n} + (n-2)\cos 2\frac{2\pi}{n} + \cdots + 1\cos(n-1)\frac{2\pi}{n} \qquad \text{(i)}$$

We know that $\cos\theta = \cos(2\pi-\theta)$. Replacing each angle θ by $2\pi-\theta$ in Eq. (i), we get

$$S = (n-1)\cos(n-1)\frac{2\pi}{n} + (n-2)\cos(n-2)\frac{2\pi}{n} + \cdots + 1\cos\frac{2\pi}{n} \quad \text{[using Eq. (i)]} \qquad \text{(ii)}$$

Adding terms having the same angle and taking n common, we have

$$2S = n\left[\cos\frac{2\pi}{n} + \cos\frac{4\pi}{n} + \cos\frac{6\pi}{n} + \cdots + \cos(n-1)\frac{2\pi}{n}\right]$$

$$= \frac{2\pi}{n} = n\left[\frac{\sin(n-1)\frac{\pi}{n}}{\sin\frac{\pi}{n}}\cos\frac{\frac{2\pi}{n}+(n-1)\frac{2\pi}{n}}{2}\right]$$

$= n\,1\cos\pi = -n \quad (\because \sin(\pi-\theta) = \sin\theta)$

$\therefore \quad S = -n/2$

12. Given that

$$2\sin t = \frac{1-2x+5x^2}{3x^2-2x-1},\ t \in [-\pi/2, \pi/2]$$

This can be written as

$(6\sin t-5)x^2 + 2(1-2\sin t)x - (1+2\sin t) = 0$

For the given equation to hold, x should be a real number, therefore the above equation should have real roots, i.e., $D \geq 0$

$\Rightarrow \quad 4(1-2\sin t)^2 + 4(6\sin t-5)(1+2\sin t) \geq 0$

$\Rightarrow \quad 16\sin^2 t - 8\sin t - 4 \geq 0$

$\Rightarrow \quad (4\sin^2 t - 2\sin t - 1) \geq 0$

$$\Rightarrow \quad 4\left(\sin t-\frac{\sqrt{5}+1}{4}\right)\left(\sin t+\frac{\sqrt{5}-1}{4}\right)\geq 0$$

$$\Rightarrow \quad \sin t \leq -\left(\frac{\sqrt{5}-1}{4}\right) \qquad \Rightarrow \quad \sin t \geq \frac{\sqrt{5}+1}{4}$$

$$\Rightarrow \quad \sin t \leq \sin(-\pi/10) \text{ or } \sin t \geq \sin(3\pi/10) \qquad \Rightarrow \quad t \leq -\pi/10 \text{ or } t \geq 3\pi/10$$

(Note that $\sin x$ is an increasing function from $-\pi/2$ to $\pi/2$.

Therefore, the range of t is $[-\pi/2, -\pi/10] \cup [3\pi/10, \pi/2]$.

13. $$\frac{1}{4\cos^2\theta+1+\frac{3}{2}\sin 2\theta} = \frac{1}{2[1+\cos 2\theta]+1+\frac{3}{2}\sin 2\theta}$$

$$= \frac{1}{2\cos 2\theta+\frac{3}{2}\sin 2\theta+3}$$

Now $-\sqrt{2^2+\left(\frac{3}{2}\right)^2} \leq 2\cos 2\theta+\frac{3}{2}\sin 2\theta \leq \sqrt{2^2+\left(\frac{3}{2}\right)^2}$

or $-\frac{5}{2} \leq 2\cos 2\theta+\frac{3}{2}\sin 2\theta \leq \frac{5}{2}$

$\Rightarrow \frac{1}{2} \leq 2\cos 2\theta+\frac{3}{2}\sin 2\theta+3 \leq \frac{11}{2}$

$\Rightarrow \frac{2}{11} \leq \frac{1}{2\cos 2\theta+\frac{3}{2}\sin 2\theta+3} \leq 2$

Hence, the maximum value is 2.

Objective

Fill in the blanks

1. According to the given question, we have expressed L.H.S. in the form $C_0+C_1\cos x+C_2\cos 2x+\cdots+C_n\cos nx$.

   Now,

   $$\sin^3 x\sin 3x = \frac{3\sin x-\sin 3x}{4}\sin 3x = \frac{3\sin x\sin 3x-\sin^2 3x}{4} = \frac{3(\cos 2x-\cos 4x)-(1-\cos 6x)}{8}$$

   Hence, $n = 6$.

2. We know that A.M. $\geq$ G.M.

   It implies that the minimum value of A.M. is obtained when A.M. = G.M.

   Therefore, the quantities whose A.M. is being taken are equal.

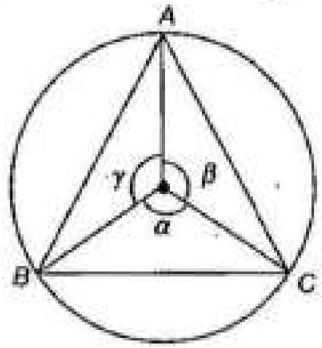


Fig. 2.49

That is, $\cos\left(\alpha+\frac{\pi}{2}\right)=\cos\left(\beta+\frac{\pi}{2}\right)=\cos\left(\gamma+\frac{\pi}{2}\right)$

$\Rightarrow\quad \sin\alpha=\sin\beta=\sin\gamma$

Also, $\alpha+\beta+\gamma=360^\circ$

$\Rightarrow\quad \alpha=\beta=\gamma=120^\circ=2\pi/3$

Therefore, the minimum value of A.M. $=\dfrac{\cos\left(\frac{2\pi}{3}+\frac{\pi}{2}\right)+\cos\left(\frac{2\pi}{3}+\frac{\pi}{2}\right)+\cos\left(\frac{2\pi}{3}+\frac{\pi}{2}\right)}{3}$

$$=\frac{-3\sin\frac{2\pi}{3}}{3}=-\frac{\sqrt{3}}{2}$$

3. $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{7\pi}{14}\sin\frac{9\pi}{14}\sin\frac{11\pi}{14}\sin\frac{13\pi}{14}$

$$=\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{\pi}{2}\sin\left(\pi-\frac{5\pi}{14}\right)\sin\left(\pi-\frac{3\pi}{14}\right)\sin\left(\pi-\frac{\pi}{14}\right)=\sin^2\frac{\pi}{14}\sin^2\frac{3\pi}{14}\sin^2\frac{5\pi}{14}$$

$$=\left(\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\right)^2$$

$$=\left[\cos\left(\frac{\pi}{2}-\frac{\pi}{14}\right)\cos\left(\frac{\pi}{2}-\frac{3\pi}{14}\right)\cos\left(\frac{\pi}{2}-\frac{5\pi}{14}\right)\right]^2$$

$$=\left[\cos\frac{3\pi}{7}\cos\frac{2\pi}{7}\cos\frac{\pi}{7}\right]^2$$

$$=\left[\frac{1}{2\sin\pi/7}\left\{2\cos\frac{\pi}{7}\sin\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}\right\}\right]^2$$

$$=\left[\frac{1}{2^2\sin\pi/7}\left\{2\sin\frac{2\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}\right\}\right]^2$$

$$=\left[\frac{1}{2^3\sin\pi/7}\left(2\sin\frac{4\pi}{7}\cos\left(\frac{\pi-3\pi}{7}\right)\right)\right]^2$$

$$= \left(\frac{1}{8\sin\pi/7}\sin\frac{8\pi}{7}\right)^2$$

$$= \left(\frac{\sin(\pi+\pi/7)}{8\sin\pi/7}\right)^2 = \left(\frac{-\sin\pi/7}{8\sin\pi/7}\right)^2 = \left(\frac{1}{8}\right)^2 = \frac{1}{64}$$

4. $k = \sin\frac{\pi}{14}\sin\frac{5\pi}{18}\sin\frac{7\pi}{18}$

$$= \cos\left(\frac{\pi}{2}-\frac{\pi}{18}\right)\cos\left(\frac{\pi}{2}-\frac{5\pi}{18}\right)\cos\left(\frac{\pi}{2}-\frac{7\pi}{18}\right)$$

$$= \cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}$$

$$= \frac{1}{2^3\sin\frac{\pi}{9}}\sin\frac{8\pi}{9} = \frac{1}{8\sin\pi/9}\sin\pi/9 = \frac{1}{8} \qquad \left[\because \sin\frac{8\pi}{9} = \sin(\pi-\pi/9) = \sin\pi/9\right]$$

5. $A+B=\pi/3 \quad \Rightarrow \quad \tan(A+B)=\sqrt{3}$

$\Rightarrow \quad \frac{\tan A+\tan B}{1-\tan A\tan B} = \sqrt{3}$

$\Rightarrow \quad \frac{\tan A+\frac{y}{\tan A}}{1-y} = \sqrt{3}$ [where $y=\tan A\tan B$]

$\Rightarrow \quad \tan^2 A+\sqrt{3}(y-1)\tan A+y=0$

For real value of $\tan A$,

$3(y-1)^2-4y\ge 0$

$\Rightarrow \quad 3y^2-10y+3\ge 0$

$\Rightarrow \quad (y-3)\left(y-\frac{1}{3}\right)\ge 0$

$\Rightarrow \quad y\le\frac{1}{3}$ or $y\ge 3$

But $A, B>0$ and $A+B=\pi/3 \Rightarrow A, B<\pi/3$

$\Rightarrow \quad \tan A\tan B<3$

$\therefore \quad y\le\frac{1}{3}$, i.e., the maximum value of y is 1/3.

6. We have $\frac{2}{\cos x} = \frac{1}{\cos(x-y)}+\frac{1}{\cos(x+y)} = \frac{2\cos x\cos y}{\cos^2 x-\sin^2 y}$

$\Rightarrow \quad \cos^2 x-2\sin^2 y=\cos^2 x\cos y$

$\Rightarrow \quad \cos^2 x(1 - \cos y) = \sin^2 y$

$\Rightarrow \quad \cos^2 x \, 2\sin^2 \frac{y}{2} = 4\sin^2 \frac{y}{2} \cos^2 \frac{y}{2}$

$\Rightarrow \quad \cos^2 x = 2\cos^2 \frac{y}{2}$

$\Rightarrow \quad \cos^2 x \sec^2 \frac{y}{2} = 2$

$\Rightarrow \quad \cos x \sec \frac{y}{2} = \pm\sqrt{2}$

True or false

1. $\tan A = \dfrac{1-\cos B}{\sin B} = \dfrac{2\sin^2 \frac{B}{2}}{2\sin\frac{B}{2}\cos\frac{B}{2}} = \tan\dfrac{B}{2}$

Hence, $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A} = \dfrac{2\tan\frac{B}{2}}{1-\tan^2\frac{B}{2}} = \tan B$

Therefore, the statement is true.

Multiple choice questions with one correct answer

1. d. From the given relations, $m + n = 2\tan\theta$, $m - n = 2\sin\theta$

$\Rightarrow \quad m^2 - n^2 = 4\tan\theta\sin\theta$ (i)

Also $4\sqrt{mn} = 4\sqrt{\tan^2\theta - \sin^2\theta} = 4\sin\theta\tan\theta$ (ii)

From Eqs. (i) and (ii), we get $m^2 - n^2 = 4\sqrt{mn}$.

2. b. $\tan\theta = \dfrac{-4}{3} \quad \Rightarrow \quad \theta \in$ II quadrant or IV quadrant $\quad \Rightarrow \quad \sin\theta = \pm 4/5$

If $\theta \in$ II quadrant, $\sin\theta = 4/5$

If $\theta \in$ IV quadrant, $\sin\theta = -4/5$

3. a. $\alpha + \beta + \gamma = 2\pi \Rightarrow \dfrac{\alpha}{2} + \dfrac{\beta}{2} + \dfrac{\gamma}{2} = \pi$

$\therefore \quad \tan\left(\dfrac{\alpha}{2} + \dfrac{\beta}{2}\right) = \tan\left(\pi - \dfrac{\gamma}{2}\right) = -\tan\dfrac{\gamma}{2}$

$\Rightarrow \quad \dfrac{\tan\alpha/2 + \tan\beta/2}{1 - \tan\alpha/2\tan\beta/2} = \tan\gamma/2$

$\Rightarrow \quad \tan\alpha/2 + \tan\beta/2 + \tan\gamma/2 = \tan\alpha/2\tan\beta/2\tan\gamma/2$

4. b. We have $\sin^2\theta + \cos^4\theta = \sin^2\theta + \cos^2\cos^2\theta \le \sin^2\theta + \cos^2\theta$ $[\because \cos^2\theta \le 1]$

Thus, $\quad A = \sin^2\theta + \cos^4\theta \le 1$

Again, $A = \sin^2\theta + \cos^4\theta = 1 - \cos^2\theta + \cos^4\theta = 1 + (\cos^4\theta - \cos^2\theta)$

$$= 1 + \left(\cos^2\theta - \frac{1}{2}\right)^2 - \frac{1}{4} = \frac{3}{4} + \left(\cos^2\theta - \frac{1}{2}\right)^2 \geq \frac{3}{4}$$

Hence, $3/4 \leq A \leq 1$.

5. c. We have $\cos\frac{7\pi}{8} = \cos\left(\pi - \frac{\pi}{8}\right) = -\cos\frac{\pi}{8}$

and $\cos\frac{5\pi}{8} = \cos\left(\pi - \frac{3\pi}{8}\right) = -\cos\frac{3\pi}{8}$

$$\therefore \quad \text{L.H.S.} = \left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1-\cos\frac{3\pi}{8}\right)\left(1-\cos\frac{\pi}{8}\right)$$

$$= \left(1-\cos^2\frac{\pi}{8}\right)\left(1-\cos^2\frac{3\pi}{8}\right)$$

$$= \sin^2\frac{\pi}{8}\sin^2\frac{3\pi}{8}$$

$$= \frac{1}{4}\left(2\sin^2\frac{\pi}{8}\right)\left(2\sin^2\frac{3\pi}{8}\right)$$

$$= \frac{1}{4}\left[\left(1-\cos\frac{\pi}{4}\right)\left(1-\cos\frac{3\pi}{4}\right)\right] \qquad \left[\because 1-\cos\theta = 2\sin^2\frac{\theta}{2}\right]$$

$$= \frac{1}{4}\left[\left(1-\frac{1}{\sqrt{2}}\right)\left(1+\frac{1}{\sqrt{2}}\right)\right] = \frac{1}{4}\left(1-\frac{1}{2}\right) = \frac{1}{8} = \frac{1}{4}\left(1-\frac{1}{2}\right) = \frac{1}{8} = \text{R.H.S.}$$

6. c. The given expression is $\sqrt{3}\,\text{cosec}\,20° - \sec 20° = \dfrac{\sqrt{3}}{\sin 20°} - \dfrac{1}{\cos 20°}$

$$= \frac{\sqrt{3}\cos 20° - \sin 20°}{\sin 20°\cos 20°}$$

$$= 4\left[\frac{\frac{\sqrt{3}}{2}\cos 20° - \frac{1}{2}\sin 20°}{2\sin 20°\cos 20°}\right]$$

$$= 4\left[\frac{\sin 60°\cos 20° - \cos 60°\sin 20°}{\sin 2\times 20°}\right]$$

$$= \frac{4\sin(60°-20°)}{\sin 40°} = \frac{4\sin 40°}{\sin 40°} = 4$$

7. c. $3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x)$

$= 3(1-\sin 2x)^2 + 6(1+\sin 2x) + 4[(\sin^2 x + \cos^2 x)^3 - 3\sin^2 x\cos^2 x(\sin^2 x + \cos^2 x)]$

$$= 3(1 - 2\sin 2x + \sin^2 2x) + (6 + 6\sin 2x) + 4\left[1 - \frac{3}{4}\sin^2 2x\right]$$

$$= 13 + 3\sin^2 2x - 3\sin^2 2x = 13$$

8. b. Given, $\sec^2\theta = \dfrac{4xy}{(x+y)^2}$

Now, $\sec^2\theta \ge 1 \Rightarrow \dfrac{4xy}{(x+y)^2} \ge 1$

$\Rightarrow \quad (x+y)^2 \le 4xy$

$\Rightarrow \quad (x+y)^2 - 4xy \le 0$

$\Rightarrow \quad (x-y)^2 \le 0$

But for real values of x and y, $(x-y)^2 \ge 0 \quad \Rightarrow \quad (x-y)^2 = 0$

$\therefore \quad x = y$

Also $\quad x + y \ne 0 \Rightarrow x \ne 0, y \ne 0$

9. c. $f(\theta) = \sin\theta(\sin\theta + \sin 3\theta) = (\sin\theta + 3\sin\theta - 4\sin^3\theta)\sin\theta = (4\sin\theta - 4\sin^3\theta)\sin\theta$

$= 4\sin^2\theta(1-\sin^2\theta) = 4\sin^2\theta\cos^2\theta = (2\sin\theta\cos\theta)^2 = (\sin 2\theta)^2 \ge 0$

which is true for all θ.

10. a. We are given that

$(\cot\alpha_1)(\cot\alpha_2)\cdots(\cot\alpha_n) = 1$

$\Rightarrow \quad (\cos\alpha_1)(\cos\alpha_2)\cdots(\cos\alpha_n) = (\sin\alpha_1)(\sin\alpha_2)\cdots(\sin\alpha_n)$ (i)

Let $y = (\cos\alpha_1)(\cos\alpha_2)\cdots(\cos\alpha_n)$ (to be maximum)

Squaring both sides, we get

$y^2 = (\cos^2\alpha_1)(\cos^2\alpha_2)\cdots(\cos^2_n)$

$= \cos\alpha_1\sin\alpha_1\cos\alpha_2\sin\alpha_2\cdots\cos\alpha_n\sin\alpha_n$ [using Eq. (i)]

$= \dfrac{1}{2^n}[\sin 2\alpha_1 \sin 2\alpha_2 \cdots \sin 2\alpha_n]$

As $0 \le \alpha_1, \alpha_2, \cdots, \alpha_n \le \pi/2$

$\therefore \quad 0 \le 2\alpha_1, 2\alpha_2, \cdots, 2\alpha_2 \le \pi$

$\Rightarrow \quad 0 \le \sin 2\alpha_1, \sin 2\alpha_2, \cdots, \sin 2\alpha_n \le 1$

$\therefore \quad y^2 \le \dfrac{1}{2^n}1 \Rightarrow y \le \dfrac{1}{2^{n/2}}$

Therefore, the maximum value of y is $1/2^{n/2}$.

11. c. $\alpha + \beta = \dfrac{\pi}{2} \quad \Rightarrow \quad \alpha = \dfrac{\pi}{2} - \beta \quad \Rightarrow \quad \tan\alpha = \cot\beta \quad \Rightarrow \quad \tan\alpha\tan\beta = 1$ (i)

Again, $\beta + \gamma = \alpha \Rightarrow \gamma = \alpha - \beta$

$\Rightarrow \quad \tan\gamma = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} = \dfrac{\tan\alpha - \tan\beta}{2}$ [using Eq. (i)]

$\Rightarrow \quad \tan\alpha = \tan\beta + 2\tan\gamma$

12. b. Given that $\sin\theta = 1/2$ and $\cos\phi = 1/3$, and θ and ϕ are acute angles.

$\therefore \quad \theta = \pi/6$ and $0 < \dfrac{1}{3} < \dfrac{1}{2}$

or $\cos\pi/2 < \cos\phi < \cos\pi/3$ or $\pi/3 < \phi < \pi/2$

$\therefore \quad \dfrac{\pi}{3} + \dfrac{\pi}{6} < \theta + \phi < \dfrac{\pi}{2} + \dfrac{\pi}{6}$ or $\dfrac{\pi}{2} < \theta + \phi < \dfrac{2\pi}{3} \quad \Rightarrow \quad \theta + \phi \in \left(\dfrac{\pi}{2}, \dfrac{2\pi}{3}\right)$

13. b. $\sec 2x - \tan 2x = \dfrac{1-\sin 2x}{\cos 2x} = \dfrac{1-\cos 2\left(\frac{\pi}{4}-x\right)}{\sin 2\left(\frac{\pi}{4}-x\right)} = \dfrac{2\sin^2\left(\frac{\pi}{4}-x\right)}{2\sin\left(\frac{\pi}{4}-x\right)\cos\left(\frac{\pi}{4}-x\right)} = \tan\left(\frac{\pi}{4}-x\right)$

14. d. $\sin\dfrac{\pi}{2n}+\cos\dfrac{\pi}{2n}=\dfrac{\sqrt{n}}{2} \quad \Rightarrow \quad \sin^2\dfrac{\pi}{2n}+\cos^2\dfrac{\pi}{2n}+2\sin\dfrac{\pi}{2n}\cos\dfrac{\pi}{2n}=\dfrac{n}{4}$

$\Rightarrow \quad 1+\sin\dfrac{\pi}{n}=\dfrac{n}{4} \quad \Rightarrow \quad \sin\dfrac{\pi}{n}=\dfrac{n-4}{4}$

For $n=2$, the given equation is not satisfied.

Considering that $n>1$ and $n\neq 2$, $0<\sin\dfrac{\pi}{n}<1 \Rightarrow 0<\dfrac{n-4}{4}<1 \quad \Rightarrow \quad 4<n<8$

15. b. $\theta\in\left(0,\dfrac{\pi}{4}\right) \Rightarrow \tan\theta<1$ and $\cot\theta>1$.

Let $\tan\theta=1-x$ and $\cot\theta=1+y$, where $x, y>0$ and are very small, then

$t_1=(1-x)^{1-x}, t_2=(1-x)^{1+y}, t_3=(1+y)^{1-x}, t_4=(1+y)^{1+y}$

Clearly, $t_4>t_3$ and $t_1>t_2$. Also $t_3>t_1$.

Thus, $t_4>t_3>t_1>t_2$.

Multiple choice questions with one or more than one correct answers

1. b. $3\left[\sin^4\left(\dfrac{3}{2}\pi-\alpha\right)+\sin^4(3\pi+\alpha)\right]-2\left[\sin^6\left(\dfrac{1}{2}\pi+\alpha\right)+\sin^6(5\pi-\alpha)\right]$

$=3(\cos^4\alpha+\sin^4\alpha)-2(\cos^6\alpha+\sin^6\alpha)$

$=3(1-2\sin^2\alpha\cos^2\alpha)-2[(\sin^2\alpha\cos^2\alpha)^3-3\sin^2\alpha\cos^2\alpha(\sin^2\alpha+\cos^2\alpha)]$

$=3(1-2\sin^2\alpha\cos^2\alpha)-2[1-3\sin^2\alpha\cos^2\alpha]=1$

2. b, c. All are infinite G.P.'s with common ratio < 1

$x=\dfrac{1}{1-\cos^2\phi}=\dfrac{1}{\sin^2\phi}, y=\dfrac{1}{1-\sin^2\phi}=\dfrac{1}{\cos^2\phi}, z=\dfrac{1}{1-\cos^2\phi\sin^2\phi}$

Now, $xy+z=\dfrac{1}{\sin^2\phi\cos^2\phi}+\dfrac{1}{1-\sin^2\phi\cos^2\phi}=\dfrac{1}{\sin^2\phi\cos^2\phi(1-\sin^2\phi\cos^2\phi)}$

or $xy+z=xyz$ (i)

Clearly, $x+y=\dfrac{\sin^2\phi+\cos^2\phi}{\sin^2\phi\cos\phi}=xy$

$\therefore \quad x+y+z=xyz$ [using Eq. (i)]

3. c. We know that $\sin 15°=\dfrac{\sqrt{3}-1}{2\sqrt{2}}$ (irrational)

$\cos 15° \dfrac{\sqrt{3}+1}{2\sqrt{2}}$ (irrational)

$$\sin 15° \cos 15° = \frac{1}{2}(2\sin 15° \cos 15°)$$

$$= \frac{1}{2}\sin 30° = \frac{1}{4} \text{ (rational)}$$

$$\sin 15° \cos 75° = \sin 15° \cos (90 - 15°)$$

$$= \sin 15° \sin 15° = \sin^2 15° = \frac{1}{2}(1 + \cos 30°) = \frac{1}{2}\left(1 - \frac{\sqrt{3}}{2}\right) \text{ (irrational)}$$

4. a, b, c, d.

$$f_n(\theta) = \frac{\sin(\theta/2)}{\cos(\theta/2)}\left[\frac{2\cos^2(\theta/2)}{\cos\theta}\frac{2\cos^2\theta}{\cos 2\theta}\frac{2\cos^2 2\theta}{\cos 4\theta}\cdots\right]$$

$$= \frac{\sin\theta}{\cos\theta}\left[\frac{2\cos^2\theta}{\cos 2\theta}\frac{2\cos^2 2\theta}{\cos 4\theta}\cdots\right]$$

$$= \frac{\sin 2\theta}{\cos 2\theta}\left[\frac{2\cos^2 2\theta}{\cos 4\theta}\cdots\right] = \tan 2^n\theta$$

$$f_2\left(\frac{\pi}{16}\right) = \tan 4\ \frac{\pi}{16} = \tan\frac{\pi}{4} = 1$$

Similarly, $f_3\left(\frac{\pi}{32}\right)$; $f_4\left(\frac{\pi}{64}\right)$ and $f_5\left(\frac{\pi}{128}\right)$ are found to be $\tan\frac{\pi}{4} = 1$

5. c. For $\theta = -\pi/2$, $\beta = -\pi/2$ and $\gamma = 2\pi$

$\sin\alpha + \sin\beta + \sin\gamma = -2$

Hence, the minimum value of the expression is negative.