Chapter

Numerical Methods

CONTENTS

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In problems related to science and technology, we generally come across various types of equations, simultaneous equations integrals for which we do not have standard methods for solutions. In such cases we find their approximate solutions through a series of numerical computations, using certain techniques which are generally known as numerical technequies or numerical computation or numerical methods. The numerical methods give approximate value (answer).

As a thumb rule, if we want the final answer to n places of decimal, we take the numbers involved and the intermediate computations results to (n + 2) places of decimal.

Numerical integration is the process of computing the value of a definite integral.

Assignment (Basic and Advance Level)
Answer Sheet of Assignment

Numerical Methods

5.1 Introduction

The limitations of analytical methods have led the engineers and scientists to evolve graphical and numerical methods. The graphical methods, though simple, give results to a low degree of accuracy. Numerical methods can, however, be derived which are more accurate.

5.2 Significant digits and Rounding off of Numbers

(1) **Significant digits :** The significant digits in a number are determined by the following rules :

(i) All non-zero digits in a number are significant.

(ii) All zeros between two non-zero digits are significant.

(iii) If a number having embedded decimal point ends with a non-zero or a sequences of zeros, then all these zeros are significant digits.

(iv) All zeros preceding a non-zero digit are non-significant.

Number	Number of significant digits
3.0450	5
0.0025	2
102.030070	9
35.9200	6
0.0002050	4
20.00	4
2000	1

(2) **Rounding off of numbers :** If a number is to be rounded off to n significant digits, then we follow the following rules :

(i) Discard all digits to the right of the nth digit.

(ii) If the $(n+1)$th digit is greater then 5 or it is 5 followed by a nonzero digit, then nth digit is increased by 1. If the $(n+1)$th digit is less then 5, then digit remains unchanged.

(iii) If the $(n+1)$th digit is 5 and is followed by zero or zeros, then nth digit is increased by 1 if it is odd and it remains unchanged if it is even.

5.3 Error due to Rounding off of Numbers

If a number is rounded off according to the rules, the maximum error due to rounding does not exceed the one half of the place value of the last retained digit in the number.

The difference between a numerical value X and its rounded value X_1 is called round off error is given by $E = X - X_1$.

5.4 Truncation and Error due to Truncation of Numbers

Leaving out the extra digits that are not required in a number without rounding off, is called truncation or chopping off.

The difference between a numerical value X and its truncated value X_1 is called truncation error and is given by $E = X - X_1$.

The maximum error due to truncation of a number cannot exceed the place value of the last retained digit in the number.

Remark **1 :** In truncation the numerical value of a positive number is decreased and that of a negative number is increased.

Remark **2 :** If we round off a large number of positive numbers to the same number of decimal places, then the average error due to rounding off is zero.

Remark **3 :** In case of truncation of a large number of positive numbers to the same number of decimal places the average truncation error is one half of the place value of the last retained digit.

Remark **4 :** If the number is rounded off and truncated to the same number of decimal places, then truncation error is greater than the round off error.

Remark **5 :** Round of error may be positive or negative but truncation error is always positive in case of positive numbers and negative in case of negative numbers.

Number	**Approximated number obtained by**	
	Chopping off	**Rounding off**
0.335217...	0.3352	0.3352
0.666666...	0.6666	0.6667
0.123451...	0.1234	0.1235
0.213450...	0.2134	0.2134
0.213950...	0.2139	0.2140
0.335750...	0.3357	0.3358
0.999999...	0.9999	1.0000
0.555555...	0.5555	0.5556

5.5 Relative and Percentage errors of Numbers

The difference between the exact value of a number X and its approximate value X_1, obtained by rounding off or truncation, is known as absolute error.

The quantity $\frac{X - X_1}{X}$ is called the relative error and is denoted by E_R.

Thus $E_R = \frac{X - X_1}{X} = \frac{\Delta X}{X}$. This is a dimensionless quantity.

The quantity $\frac{\Delta X}{X} \times 100$ is known as percentage error and is denoted by E_p, *i.e.* $E_p = \frac{\Delta X}{X} \times 100$.

Remark 1 : If a number is rounded off to n decimal digits, then $|E_R| < 0.5 \times 10^{-n+1}$

Remark 2 : If a number is truncated to n decimal places, then $|E_R| < 10^{-n+1}$

Example: 1 The number of significant digits in 0.0001 is

(a) 5 (b) 4 (c) 1 (d) None of these

Solution: (c) 0.0001 has only one significant digit 1.

Example: 2 When a number is rounded off to n decimal places, then the magnitude of relative error does not exceed **[DCE 1998]**

(a) 10^{-n} (b) 10^{-n+1} (c) $0.5 \times 10^{-n+1}$ (d) None of these

Solution: (c) When a number is rounded off to n decimal places, then the magnitude of relative error *i.e.* $|E_R|$ does not exceed $0.5 \times 10^{-n+1}$.

Example: 3 When the number 2.089 is rounded off to three significant digits, then the absolute error is

(a) 0.01 (b) –0.01 (c) 0.001 (d) –0.001

Solution: (d) When the number 2.089 is rounded off to three significant digits, it becomes 2.09. Hence, the absolute error that occurs is $2.089 - 2.09 = -0.001$

5.6 Algebraic and Transcendental Equation

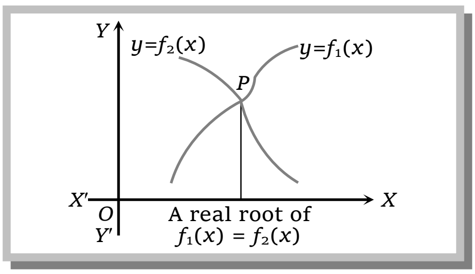
An equation of the form $f(x)$=0, is said to an algebraic or a transcendental equation according as $f(x)$ is a polynomial or a transcendental function respectively.

e.g. $ax^2 + bx + c = 0$, $ax^3 + bx^2 + cx + d = 0$ etc., where $a, b, c, d \in Q$, are algebraic equations whereas $ae^x + b \sin x = 0$; $a \log x + bx = 3$ *etc.* are transcendental equations.

5.7 Location of real Roots of an Equation

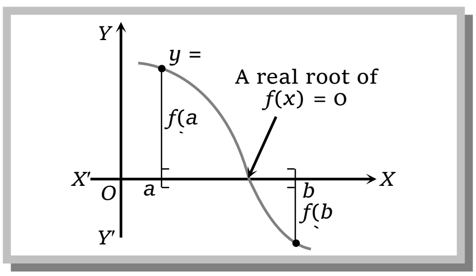
By location of a real root of an equation, we mean finding an approximate value of the root graphically or otherwise.

(1) **Graphical Method :** It is often possible to write $f(x) = 0$ in the form $f_1(x) = f_2(x)$ and then plot the graphs of the functions $y = f_1(x)$ and $y = f_2(x)$.



The abscissae of the points of intersection of these two graphs are the real roots of $f(x) = 0$.

(2) **Location Theorem :** Let $y = f(x)$ be a real-valued, continuous function defined on [a, b]. If $f(a)$ and $f(b)$



have opposite signs *i.e.* $f(a).f(b) < 0$, then the equation $f(x)=0$ has at least one real root between *a* and *b*.

5.8 Position of Real Roots

If $f(x)=0$ be a polynomial equation and $x_1, x_2 ....x_k$ are the consecutive real roots of $f(x)=0$, then positive or negative sign of the values of $f(-\infty), f(x)............. f(x_k), f(\infty)$ will determine the intervals in which the root of $f(x)=0$ will lie whenever there is a change of sign from $f(x_r)$ to $f(x_{r+1})$ the root lies in the interval $[x_r, x_{r+1}]$.

Example: 4 If all roots of equation $x^3 - 3x + k = 0$ are real, then range of value of k

(a) (–2, 2) (b) [–2, 2] (c) Both (d) None of these

Solution: (a) Let $f(x)=x^3 - 3x + k$, then $f(x)=3x^2 - 3$ and so $f(x)=0 \Rightarrow x = \pm 1$. The values of $f(x)$ at $x = -\infty, -1, 1, \infty$ are :

x :	$-\infty$	-1	1	∞
$f(x)$:	$-\infty$	$k+2$	$k-2$	∞

If all roots of given equation are real, then $k+2>0$ and $k-2<0 \Rightarrow -2<k<2$. Hence the range of k is (–2, 2)

Example: 5 For the smallest positive root of transcendental equation $x - e^{-x} = 0$, interval is **[MP PET 1996]**

(a) (0, 1) (b) (–1, 0) (c) (1, 2) (d) (2, 3)

Solution: (a) Let $f(x)=x - e^{-x} = 0 \Rightarrow xe^x - 1 = 0$ but $f(0) = -ive$ and $f(1) = +ive$. Therefore root lie in (0,1).

Example: 6 The maximum number of real roots of the equation $x^{2n} - 1 = 0$ is **[MP PET 2001]**

(a) 2 (b) 3 (c) n (d) $2n$

Solution: (a) Let $f(x)=x^{2n} - 1$, then $f'(x) = x^{2n-1} = 0 \Rightarrow x = 0$

Sign of $f(x)$ at $x = -\infty, 0, +\infty$ are

x :	$-\infty$	0	$+\infty$
$f(x)$:	$+ive$	$-ive$	$+ive$

This show that there are two real roots of $f(x)=0$ which lie in the interval $(-\infty, 0)$ and $(0, +\infty)$. Hence maximum number of real roots are 2.

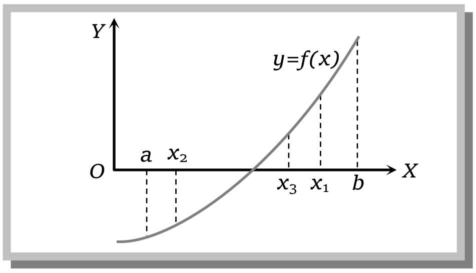
5.9 Solution of Algebraic and Transcendental Equations

There are many numerical methods for solving algebraic and transcendental equations. Some of these methods are given below. After locating root of an equation, we successively approximate it to any desired degree of accuracy.

(1) **Iterative method:** If the equation $f(x) = 0$ can be expressed as $x = g(x)$ (certainly $g(x)$ is non-constant), the value $g(x_0)$ of $g(x)$ at $x = x_0$ is the next approximation to the root α. Let $g(x_0) = x_1$, then $x_2 = g(x_1)$ is a third approximation to α. This process is repeated until a number, whose absolute difference from α is as small as we please, is obtained. This number is the required root of $f(x)=0$, calculated upto a desired accuracy.

Thus, if x_i is an approximation to α, then the next approximation $x_{i+1} = g(x_i)$ ....(i)

The relation (i) is known as Iterative formula or recursion formula and this method of approximating a real root of an equation $f(x)=0$ is called iterative method.



(2) **Successive bisection method :** This method consists in locating the root of the equation $f(x)=0$ between *a* and *b*. If $f(x)$ is continuous between *a* and *b*, and $f(a)$ and $f(b)$ are of opposite

signs *i.e.* $f(a).f(b)<0$, then there is (at least one) root between a and b. For definiteness, let $f(a)$ be negative and $f(b)$ be positive. Then the first approximation to the root $x_1 = \frac{1}{2}(a+b)$.

Working Rule : (i) Find $f(a)$ by the above formula.

(ii) Let $f(a)$ be negative and $f(b)$ be positive, then take $x_1 = \frac{a+b}{2}$.

(iii) If $f(x_1) = 0$, then c is the required root or otherwise if $f(x_1)$ is negative then root will be in (x_1, b) and if $f(x_1)$ is positive then root will be in (a, x_1).

(iv) Repeat it until you get the root nearest to the actual root.

Note : ❑ This method of approximation is very slow but it is reliable and can be applied to any type of algebraic or *transcendental equations*.

❑ This method may give a false root if $f(x)$ is discontinuous on [a, b].

Example: 7 By bisection method, the real root of the equation $x^3 - 9x + 1 = 0$ lying between $x = 2$ and $x = 4$ is nearer to

[MP PET 1997]

(a) 2.2 (b) 2.75 (c) 3.5 (d) 4.0

Solution: (b) Since $f(2) = 2^3 - 9(2) + 1 < 0$ and $f(4) = 4^3 - 9(4) + 1 > 0$

$\therefore$ Root will lie between 2 and 4.

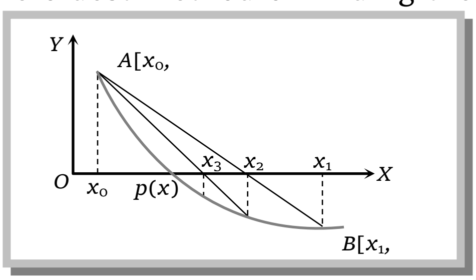
At $x = \frac{2+4}{2} = 3$, $f(3) = 3^3 - 9(3) + 1 > 0$

$\therefore$ Root lie between 2 and 3.

At $x = \frac{2+3}{2} = 2.5$, $f(2.5)$ is *–ive*, $\therefore$ root lies between 2.5 and 3

At $x = \frac{2.5+3}{2} = 2.75$ and $f(2.75) = (2.75)^3 - 9(2.75) + 1 < 0$. $\therefore$ Root is near to 2.75.

(3) **Method of false position or Regula-Falsi method :** This is the oldest method of finding the real root of an equation $f(x) = 0$ and closely resembles the bisection method. Here we choose two points x_0 and x_1 such that $f(x_0)$ and $f(x_1)$ are of opposite signs *i.e.* the graph of $y = f(x)$ crosses the x-axis between these points. This indicates that a root lies between x_0 and x_1 consequently $f(x_0)f(x_1) < 0$.



Equation of the chord joining the points $A[x_0, f(x_0)]$ and $B[x_1, f(x_1)]$ is

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) \qquad \text{........(i)}$$

The method consists in replacing the curve AB by means of the chord AB and taking the point of intersection of the chord with the x-axis as an approximation to the root. So the abscissa of the point where the chord cuts the x-axis $(y=0)$ is given by $x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_2)} f(x_0)$

.........(ii)

which is an approximation to the root.

If now $f(x_0)$ and $f(x_2)$ are of opposite signs, then the root lies between x_0 and x_2. So replacing x_1 by x_2 in (ii), we obtain the next approximation x_3. (The root could as well lie between x_1 and x_2 and we would obtain x_3 accordingly). This procedure is repeated till the root is found to desired accuracy. The iteration process based on (i) is known as the *method of false position*.

Working rule

(i) Calculate $f(x_0)$ and $f(x_1)$, if these are of opposite sign then the root lies between x_0 and x_1.

(ii) Calculate x_2 by the above formula.

(iii) Now if $f(x_2)=0$, then x_2 is the required root.

(iv) If $f(x_2)$ is negative, then the root lies in (x_2, x_1).

(v) If $f(x_2)$ is positive, then the root lies in (x_0, x_2).

(vi) Repeat it until you get the root nearest to the real root.

Note :
- This method is also known as the method of false position.
- The method may give a false root or may not converge if either *a* and *b* are not sufficiently close to each other or $f(x)$ is discontinuous on [*a*, *b*].
- Geometrically speaking, in this method, part of the curve between the points *P*(*a*, *f*(*a*)) and *Q* (*b*, *f*(*b*)) is replaced by the secant *PQ* and the point of intersection of this secant with *x*-axis gives an approximate value of the root.
- It converges more rapidly than bisection.

Example: 8 The root of the equation $x^3 + x - 3 = 0$ lies in interval (1, 2) after second iteration by false position method, it will be in

[MP PET 2003]

(a) (1.178, 2.00) (b) (1.25, 1.75) (c) (1.125, 1.375) (d) (1.875, 2.00)

Solution: (a) $f(x) = x^3 + x - 3$

$f(1) = -1$ and $f(2) = 7$

Therefore, root lie in (1, 2).

Now, take $x_0 = 1$, $x_1 = 2$

$\Rightarrow x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} . f(x_0) \Rightarrow x_2 = 1 - \frac{2-1}{+7-(-1)} . (-1) = 1.125$ and so $f(x_2) = -0.451$

Hence, roots lie in (1.125, 2)

$$\Rightarrow x_3 = 1.125 - \frac{2-1.125}{7-(-0.451)}(-0.451) = 1.178$$. So required root lie in (1.178, 2)

(4) **Newton-Raphson method :** Let x_0 be an approximate root of the equation $f(x)=0$. If $x_1 = x_0 + h$ be the exact root, then $f(x_1)=0$

$\therefore$ Expanding $f(x_0+h)$ by Taylor's series

$$f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + ... = 0$$

Since h is small, neglecting h^2 and higher powers of h, we get

$$f(x_0) + hf'(x_0) = 0$$

or $$h = -\frac{f(x_0)}{f'(x_0)} \qquad ....(i)$$

$\therefore$ A closer approximation to the root is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Similarly, starting with x_1, a still better approximation x_2 is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

In general, $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad ....(ii)$$

Which is known as the Newton-Raphson formula or Newton's iteration formula.

Working rule : (i) Find $|f(a)|$ and $|f(b)|$. If $|f(a)| < |f(b)|$, then let $a = x_0$, otherwise $b = x_0$.

(ii) Calculate $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

(iii) x_1 is the required root if $f(x_1)=0$.

(iv) To find nearest to the real root, repeat it.

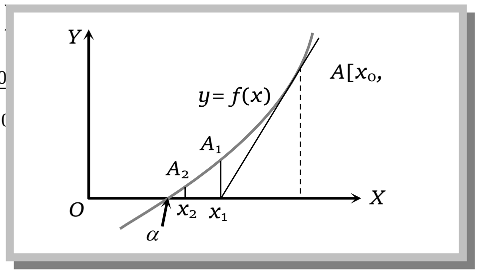
Note :
- ❑ Geometrically speaking, in Newton-Raphson method, the part of the graph of the function $y = f(x)$ between the point $P(a, f(a))$ and the x-axis is replaced by a tangent to the curve at the point at each step in the approximation process.
- ❑ This method is very useful for approximating isolated roots.
- ❑ The Newton-Raphson method fails if $f'(x)$ is difficult to compute or vanishes in a neighbourhood of the desired root. In such cases, the Regula-Falsi method should be used.
- ❑ The Newton-Raphson method is widely used since in a neighbourhood of the desired root, it converges more rapidly than the bisection method or the Regula-Falsi method.
- ❑ If the starting value a is not close enough to the desired root, the method may give a false root or may not converge.

- If $f(x_0)/f'(x_0)$ is not sufficiently small, this method does not work. Also if it work, it works faster.

Geometrical Interpretation

Let x_0 be a point near the root α of the equation $f(x)=0$. Then the equation of the tangent at $A_0[x_0, f(x_0)]$ is $y - f(x_0) = f'(x_0)(x - x_0$

It cuts the x-axis at $x_1 = x_0 - \frac{f(x_0}{f'(x_{}}$



Which is a first approximation to the root α. If A_1 is the point corresponding to x_1 on the curve, then the tangent at A_1 will cut the x-axis of x_2 which is nearer to α and is, therefore, a second approximation to the root. Repeating this process, we approach to the root α quite rapidly. Hence the method consists in replacing the part of the curve between the point A_0 and the x-axis by means of the tangent to the curve at A_0.

Example: 9 The value of the root nearest to the 2, after first iteration of the equation $x^4 - x - 10 = 0$ by Newton-Raphson method is

[MP PET 2003]

(a) 2.321 (b) 2.125 (c) 1.983 (d) 1.871

Solution: (d) Let $f(x) = x^4 - x - 10$, then $f(1) = -10$ and $f(2) = 4$

Thus roots lie in (1, 2). Also $|f(2)| < |f(1)|$, So take $x_0 = 2$

Also $f'(x) = 4x^3 - 1$

$f'(2) = 4(8) - 1 = 31$

$\therefore$ By Newton's rule, the first iteration,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{4}{31} = 1.871$$

Example: 10 For finding real roots of the equation $x^2 - x = 2$ by Newton-Raphson method, choose $x_0 = 1$, then the value of x_2 is

[MP PET 2000, 02]

(a) –1 (b) 3 (c) $\frac{11}{5}$ (d) None of these

Solution: (c) Let $f(x) = x^2 - x - 2$. Given $x_0 = 1$.

By Newton-Raphson method, $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$f'(x) = 2x - 1 = 2(1) - 1 = 1$ and $f(1) = -2$. Now $x_1 = 1 - \frac{-2}{1} = 3$

$f(x_1) = f(3) = 9 - 3 - 2 = 4$

$f'(x_1) = f'(3) = 2(3) - 1 = 5$; $x_2 = 3 - \frac{4}{5} = \frac{11}{5}$

Example: 11 The value of $\sqrt{12}$ correct to 3 decimal places by Newton-Raphson method is given by **[DCE 1999, 2000]**

(a) 3.463 (b) 3.462 (c) 3.467 (d) None of these

Solution: (a) Let $x=\sqrt{12} \Rightarrow x^2-12=0 \Rightarrow f(x)=x^2-12,\ f'(x)=2x$

$\therefore\ f(3)=-3$ *i.e. –ive* and $f(4)=+4$ *i.e. +ive*

Hence roots lie between 3 and 4. $\because\ |f(3)|<|f(4)|$, $\therefore\ x_0=3$

First iteration, $x_1=x_0-\dfrac{f(x_0)}{f'(x_0)}=3-\dfrac{-3}{6}=3.5$

Now second iteration, $f(3.5)=0.25,\ f'(3.5)=7$, $x_2=3.5-\dfrac{0.25}{7}=3.463$

5.10 Numerical Integration

It is the process of computing the value of a definite integral when we are given a set of numerical values of the integrand *f*(*x*) corresponding to some values of the independent variable *x*.

If $I=\int_a^b y.dx$. Then I represents the area of the region *R* under the curve $y=f(x)$ between the ordinates $x=a, x=b$ and the *x*-axis.

(1) **Trapezoidal rule**

Let $y=f(x)$ be a function defined on [*a*, *b*] which is divided into *n* equal sub-intervals each of width *h* so that $b-a=nh$.

Let the values of $f(x)$ for $(n+1)$ equidistant arguments $x_0=a, x_1=x_0+h,\ x_2=x_0+2h,......., x_n=x_0+nh=b$ be $y_0, y_1, y_2,.........y_n$ respectively.

$$\text{Then}\quad \int_a^b f(x)dx=\int_{x_0}^{x_0+nh} y\,dx=h\left[\frac{1}{2}(y_0+y_n)+(y_1+y_2+......+y_{n-1})\right]=\frac{h}{2}[(y_0+y_n)+2(y_1+y_2+......+y_{n-1})]$$

This rule is known as **Trapezoidal rule .**

The geometrical significance of this rule is that the curve $y=f(x)$ is replaced by *n* straight lines joining the points (x_0,y_0) and (x_1,y_1); (x_1,y_1) and (x_2,y_2); (x_{n-1},y_{n-1}) and (x_n,y_n). The area bounded by the curve $y=f(x)$. The ordinate $x=x_0$ and $x=x_n$ and the *x*-axis, is then approximately equivalent to the sum of the areas of the *n* trapeziums obtained.

Example: 12 If $e^0=1, e^1=2.72, e^2=7.39, e^3=20.09, e^4=54.60$, then by Trapezoidal rule $\int_0^4 e^x dx=$ **[MP PET 1995, 2001]**

(a) 53.87 (b) 53.60 (c) 58.00 (d) None of these

Solution: (c) $h=\dfrac{b-a}{n}=\dfrac{4-0}{4}=1$

x :	0	1	2	3	4
y :	1	2.72	7.39	20.09	54.60

By Trapezoidal rule, $\int_0^4 f(x)dx=\int_0^4 e^x\,dx=\dfrac{h}{2}[(y_0+y_4)+2(y_1+y_2+y_3)]$

$$=\frac{1}{2}[(1+54.6)+2(2.72+7.39+20.09)]=\frac{1}{2}[55.6+60.4]=58.00$$

Example: 13 By trapezoidal rule the value of the integral $\int_1^5 x^2\,dx$ on dividing the interval into four equal parts is **[MP PET**

(a) 42 (b) 41.3 (c) 41 (d) 40

Solution: (a) $h=\frac{5-1}{4}=1$

x :	1	2	3	4	5
y :	1	4	9	16	25

$$\int_1^5 x^2\,dx=\frac{h}{2}[(y_0+y_4)+2(y_1+y_2+y_3)]=\frac{1}{2}[(1+25)+2(4+9+16)]=\frac{1}{2}[26+58]=\frac{84}{2}=42$$

Example: 14 If for $n=4$, the approximate value of integral $\int_1^9 x^2\,dx$ by Trapezoidal rule is $2\left[\frac{1}{2}(1+9^2)+\alpha^2+\beta^2+7^2\right]$, then

[MP PET 2002]

(a) $\alpha=1,\ \beta=3$ (b) $\alpha=2,\ \beta=4$ (c) $\alpha=3,\ \beta=5$ (d) $\alpha=4,\ \beta=6$

Solution: (c) $h=\frac{b-a}{n}=\frac{9-1}{4}=2$

$x_0=1$, $x_1=x_0+nh=1+1.2=3$, $x_2=x_0+2.2=5$, $x_3=x_0+3.2=7$, $x_4=x_0+4.2=9$

$y_0=1,\ y_1=9,\ y_2=25,\ y_3=49,\ y_4=81$

By trapezoidal rule,

$$\int_a^b f(x)dx=\frac{h}{2}[(y_0+y_4)+2(y_1+y_2+y_3)]=\frac{2}{2}[(1+81)+2(9+25+49)]=2\left[\frac{1}{2}(1+9^2)+(3^2+5^2+7^2)\right]$$

Obvious from above equation, $\alpha=3,\ \beta=5$.

(2) **Simpson's one third rule :** Let $y=f(x)$ be a function defined on $[a, b]$ which is divided into n (an even number) equal parts each of width h so that $b-a=nh$.

Suppose the function $y=f(x)$ attains values $y_0, y_1, y_2, \ldots\ldots\ldots y_n$ at $n+1$ equidistant points $x_0=a,\ x_1=x_0+h,\ x_2=x_0+2h, \ldots\ldots\ldots, x_n=x_0+nh=b$ respectively. Then

$$\int_a^b f(x)dx=\int_{x_0}^{x_0+nh} y\,dx=\frac{h}{3}[(y_0+y_n)+4(y_1+y_3+y_5+\ldots.+y_{n-1})+2(y_2+y_4+\ldots+y_{n-2})]$$

= (one-third of the distance between two consecutive ordinates)

× [(sum of the extreme ordinates)+4(sum of odd ordinates)+2(sum of even ordinates)]

This formula is known as Simpson's one-third rule. Its geometric significance is that we replace the graph of the given function by $\frac{n}{2}$ arcs of second degree polynomials, or parabolas with vertical axes. It is to note here that the interval $[a, b]$ is divided into an even number of subinterval of equal width.

Simpson's rule yield more accurate results than the trapezoidal rule. Small size of interval gives more accuracy.

Example: 15 By Simpson's rule the value of the interval $\int_1^6 x\,dx$ on dividing, the interval into four equal parts is **[MP PET 2**

(a) 16 (b) 16.5 (c) 17 (d) 17.5

Solution: (d) $h=\frac{6-1}{4}=\frac{5}{4}=1.25$

$x_0=a=1,\ x_1=x_0+h=1+1.25=2.25$

$x_2=x_0+2h=1+2(1.25)=3.50$

$x_3 = x_0 + 3h = 1 + 3(1.25) = 4.75$

$x_4 = x_0 + 4h = 1 + 4(1.25) = 6.0$

By Simpson's rule, $\int_a^b f(x)dx = \int_1^6 xdx = \frac{1.25}{3}[(y_0 + y_4) + 4(y_1 + y_3) + 2(y_2)]$

$$= \frac{1.25}{3}[(1+6) + 4(2.25 + 4.75) + 2(3.5)] = \frac{1.25}{3}[7 + 28 + 7] = 17.5$$

Example: 16 Considering six sub-interval, the value of $\int_0^6 \frac{dx}{1+x^2}$ by Simpson's rule is

(a) 1.3562 (b) 1.3662 (c) 1.3456 (d) 1.2662

Solution: (b) $h = \frac{6-0}{6} = 1$

$x_0 = 0,\ x_1 = 0 + 1.1 = 1$

$x_2 = 0 + 2.1 = 2,\ x_3 = 3,\ x_4 = 4,\ x_5 = 5,\ x_6 = 6$

and $y_0 = \frac{1}{1+0} = 1,\ y_1 = \frac{1}{2},\ y_2 = \frac{1}{5},\ y_3 = \frac{1}{10},\ y_4 = \frac{1}{17},\ y_5 = \frac{1}{26},\ y_6 = \frac{1}{37}$

By Simpson's rule ,

$$\int_a^b f(x)dx = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)] = \frac{1}{3}\left[\left(1 + \frac{1}{37}\right) + 4\left(\frac{1}{2} + \frac{1}{10} + \frac{1}{26}\right) + 2\left(\frac{1}{5} + \frac{1}{17}\right)\right] = 1.3662$$

Example: 17 By Simpson's rule taking $n = 4$, the value of the integral $\int_0^1 \frac{1}{1+x^2}dx$ is equal to **[Kurukshetra CEE 1998; DCE 2**

(a) 0.785 (b) 0.788 (c) 0.781 (d) None of these

Solution: (a) $h = \frac{1-0}{4} = 0.25$

$x_0 = 0 \quad x_1 = 0 + 0.25 = 0.25$

$x_2 = 0 + 2(0.25) = 0.50,\ x_3 = 0.75,\ x_4 = 1$

and $y_0 = \frac{1}{1+x_0^2} = \frac{1}{1+0} = 1$, $y_1 = \frac{1}{1+(0.25)^2} = 0.941$, $y_2 = \frac{1}{1+(0.50)^2} = 0.8$, $y_3 = 0.64$ and $y_4 = 0.5$

$$\int_0^1 \frac{dx}{1+x^2} = \frac{0.25}{3}[(y_0 + y_4) + 4(y_1 + y_3) + 2(y_2)]$$

$$= \frac{0.25}{3}[(1 + 0.5) + 4(0.941 + 0.64) + 2(0.8)] = 0.785 \ .$$