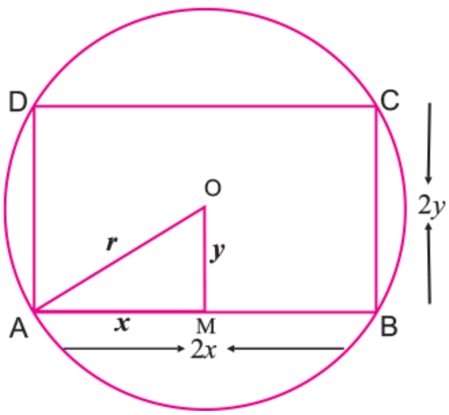
# Long Answer Questions-II (PYQ)

## [6 Marks]

**Q.1. Show that the rectangle of maximum perimeter which can be inscribed in a circle of radius r is the square of side $r\sqrt{2}$.**

**Ans.**




Let $ABCD$ be a rectangle inscribed in a circle of radius $r$ with centre at $O$. Let $AB = 2x$ and

$BC = 2y$ be the sides of the rectangle.

Then in right angled $\Delta OAM$, $AM^2 + OM^2 = OA^2$ (By Pythagoras theorem)

$$\Rightarrow x^2 + y^2 = r^2 \quad \Rightarrow \quad y = \sqrt{r^2 - x^2} \qquad \ldots(i)$$

Let $P$ be the perimeter of rectangle $ABCD$, then

$$P = 4x + 4y \Rightarrow \quad P = 4x + 4\sqrt{r^2 - x^2}$$

$$\Rightarrow \quad \frac{dP}{dx} = 4 - \frac{4x}{\sqrt{r^2 - x^2}}$$

For maximum or minimum value of $P$, we have $\frac{dP}{dx} = 0$

$$4 - \frac{4x}{\sqrt{r^2 - x^2}} = 0$$

$$\Rightarrow \quad 4 = \frac{4x}{\sqrt{r^2 - x^2}}$$

$$\sqrt{r^2 - x^2} = x \quad \Rightarrow \quad r^2 - x^2$$

$$\Rightarrow \quad 2x^2 = r^2 \quad \Rightarrow \quad x = \frac{r}{\sqrt{2}}$$

Now $\frac{d^2P}{dx^2} = \frac{-4\left\{\sqrt{r^2-x^2} - \frac{x.(-x)}{\sqrt{r^2-x^2}}\right\}}{(\sqrt{r^2-x^2})^2}$

$$= -\frac{4r^2}{(r^2-x^2)^{3/2}}$$

$$\therefore \left(\frac{d^2P}{dx^2}\right)_{x=\frac{r}{\sqrt{2}}} = \frac{-4r^2}{\left(r^2 - \frac{r^2}{2}\right)^{3/2}} = \frac{-8\sqrt{2}}{r} < 0$$

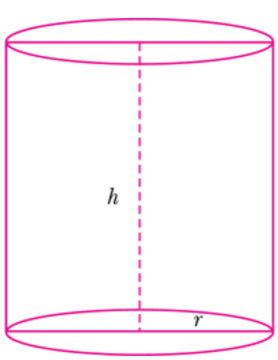
Thus, $P$ is maximum when $x = \frac{r}{\sqrt{2}}$. Putting $x = \frac{r}{\sqrt{2}}$ in $(i)$, we get $y = \frac{r}{\sqrt{2}}$

Therefore, $x = y \quad \Rightarrow 2x = 2y$

Hence, $P$ is maximum when rectangle is a square of side $2x = \sqrt{2}r$ .

**Q.2. Of all the closed right circular cylindrical cans of volume 128π cm$^3$, find the dimensions of the can which has minimum surface area.**

**Ans.**



Let $r$, $h$ be radius and height of closed right circular cylinder having volume $128\pi$ cm$^3$.

If $S$ be the surface area then

$$S = 2\pi rh + 2\pi r^2 \qquad \Rightarrow S = 2\pi(rh + r^2)$$

$$S = 2\pi\left(r.\frac{128}{r^2} + r^2\right) \qquad \left[\begin{array}{l} \because V = \pi r^2 h \\ \Rightarrow \ 128\pi = \pi r^2 h \\ \therefore \ h = \frac{128}{r^2} \end{array}\right]$$

$$S = 2\pi\left(\frac{128}{r} + r^2\right)$$

$$\Rightarrow \quad \frac{dS}{dr} = 2\pi\left(-\frac{128}{r^2} + 2r\right)$$

For extreme value of $S$

$$\frac{dS}{dr} = 0 \quad \Rightarrow \quad 2\pi\left(-\frac{128}{r^2} + 2r\right) = 0$$

$$\Rightarrow \qquad -\frac{128}{r^2} + 2r = 0$$

$$\Rightarrow \qquad 2r = \frac{128}{r^2}$$

$$\Rightarrow \qquad r^3 = \frac{128}{2}$$

$$\Rightarrow r^3 = 64 \quad \Rightarrow r = 4$$

Again

$$\frac{d^2S}{dr^2} = 2\pi\left(\frac{128\times2}{r^3} + 2\right)$$

$$\Rightarrow \quad \left.\frac{d^2S}{dr^2}\right]_{r=4} = +\text{ve}$$

Hence, for $r = 4$ cm, $S$ (surface area) is minimum.

Therefore, dimensions for minimum surface area of cylindrical can are

radius $r = 4$ cm and $h = \frac{128}{r^2} = \frac{128}{16} = 8\,\text{cm}$

**Q.3. Prove that the surface area of a solid cuboid, of square base and given volume, is minimum when it is a cube.**

**Ans.**

Let $x$ be the side of square base of cuboid and other side be $y$.

Then volume of cuboid with square base, $V = x \,.\, x \,.\, y = x^2y$

As volume of cuboid is given so volume is taken constant throughout the question, therefore,

$$y = \frac{V}{x^2} \qquad \ldots(i)$$

In order to show that surface area is minimum when the given cuboid is cube, we have to show $S'' > 0$ and $x = y$.

Let S be the surface area of cuboid, then

$$S = x^2 + xy + xy + xy + xy + x^2$$

$$S = 2x^2 + 4xy \qquad \ldots(ii)$$

$$\Rightarrow \qquad S = 2x^2 + 4x.\frac{V}{x^2}$$

$$\Rightarrow \qquad S = 2x^2 + \frac{4V}{x} \qquad ...(iii)$$

$$\Rightarrow \qquad \frac{dS}{dx} = 4x - \frac{4V}{x^2} \qquad ...(iv)$$

For maximum/minimum value of $S$, we have $\frac{dS}{dx} = 0$

$$\Rightarrow 4x - \frac{4V}{x^2} = 0 \Rightarrow 4V = 4x^3$$

$$\Rightarrow V = x^3 \qquad ...(v)$$

Putting $V = x^3$ in $(i)$, we have

$$y = \frac{x^3}{x^2} = x$$

Here, $y = x \Rightarrow$ cuboid is a cube.

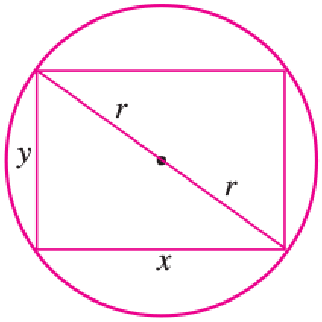
Differentiating $(iv)$ w.r.t $x$, we get

$$\frac{d^2S}{dx^2} = \left(4 + \frac{8V}{x^3}\right) > 0$$

Hence, surface area is minimum when given cuboid is a cube.

**Q.4. Show that the rectangle of maximum area that can be inscribed in a circle is a square.**

**Ans.**




Let $x$ and $y$ be the length and breadth of a rectangle inscribed in a circle of radius $r$.

Then $y^2 + x^2 = 4r^2$

$\Rightarrow \quad y = \sqrt{4r^2 - x^2}$ ...(i)

If A be the area of rectangle then $A = x \,.\, y$

$A = x.\sqrt{4r^2 - x^2}$ (using $(i)$)

$$\Rightarrow \quad \frac{\mathrm{dA}}{\mathrm{dx}} = x.\frac{1}{2\sqrt{4r^2 - x^2}}(-2x) + \sqrt{4r^2 - x^2}$$

$$= -\frac{x^2}{\sqrt{4r^2 - x^2}} + \sqrt{4r^2 - x^2} = \frac{-x^2 + 4r^2 - x^2}{\sqrt{4r^2 - x^2}}$$

$$\Rightarrow \quad \frac{\mathrm{dA}}{\mathrm{dx}} = \frac{4r^2 - 2x^2}{\sqrt{4r^2 - x^2}}$$

For maxima or minima,

$$\frac{dA}{dx} = 0 \quad \Rightarrow \quad \frac{4r^2 - 2x^2}{\sqrt{4r^2 - x^2}} = 0$$

$$\Rightarrow \quad 4r^2 - 2x^2 = 0$$

$$\Rightarrow 2x^2 = 4r^2 \Rightarrow \quad x = \sqrt{2}r$$

Now,

$$\frac{d^2A}{dx^2} = \frac{\sqrt{4r^2 - x^2}(-4x) - \frac{(4r^2 - 2x^2)(-2x)}{2\sqrt{4r^2 - x^2}}}{(4r^2 - x^2)}$$

$$= \frac{2x(x^2 - 6r^2)}{(4r^2 - x^2)^{3/2}}$$

$$\therefore \left[\frac{d^2A}{dx^2}\right]_{x=\sqrt{2}r} = \frac{2\sqrt{2}r\,(2r^2-4r^2)}{(4r^2-2r^2)^{3/2}}$$

$$= \frac{-4\sqrt{2}r^3}{(2r^2)^{3/2}} < 0$$

Hence, A is maximum when $x = \sqrt{2}r$

Putting $x = \sqrt{2}r$ in $(i)$ we get

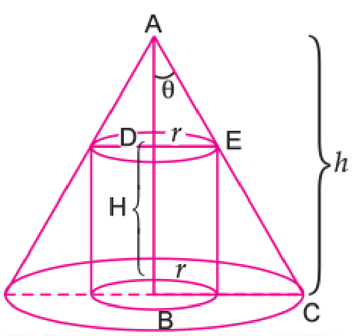
$$y = \sqrt{4r^2 - 2r^2} = \sqrt{2}r$$

*i.e.,* $\quad x = y = \sqrt{2}r$

Therefore, area of rectangle is maximum when $x = y$ *i.e.,* rectangle is a square.

**Q.5. Show that the height of the cylinder of maximum volume that can be inscribed in a cone of height h is $\frac{1}{3}$ h**

**Ans.**




Let $r$ and $H$ be the radius and height of inscribed cylinder respectively and θ be the semi-vertical angle of given cone.

If $V$ be the volume of cylinder.

then $V = \pi r^2 H$

$\therefore V = \pi r^2 \left(h - r\cot\theta\right)$

$\Rightarrow V = \pi\left(hr^2 - r^3\cot\theta\right)$

Differentiating with respect to $r$, we get

$$\begin{bmatrix} \text{In } \Delta ADE \\ \cot\,\theta = \frac{AD}{DE} \\ \cot\,\theta = \frac{h-H}{r} \\ \therefore\;\; H = h - r\cot\,\theta \end{bmatrix}$$

$$\Rightarrow \frac{dV}{dr} = \pi\left(2rh - 3r^2\,\cot\,\theta\right)$$

For maxima or minima,

$$\frac{dV}{dr} = 0$$

$$\Rightarrow \pi\left(2rh - 3r^2\,\cot\,\theta\right) = 0 \qquad \Rightarrow 2rh - 3r^2\,\cot\,\theta = 0 \qquad [\therefore \pi \neq 0]$$

$$\Rightarrow r\left(2h - 3r\cot\,\theta\right) = 0$$

$$2h - 3r\,\cot\,\theta = 0$$

$$\Rightarrow \quad r = \frac{2h}{3\cot\theta} = \frac{2h}{3}\tan\theta \qquad \Rightarrow [\because r \neq 0]$$

Now, $\frac{d^2V}{dr^2} = \pi\left(2h - 6r\,\cot\theta\right)$

$$\left.\frac{d^2V}{dr^2}\right]_{r=\frac{2h}{3}\tan\theta} = \pi\left(2h - 6\times\frac{2h}{3}\tan\theta.\;\cot\theta\right)$$

$$= \pi(2h - 4h) < 0$$

Hence, volume will be maximum when $r = \frac{2h}{3}\tan\theta$

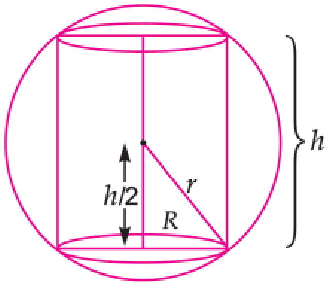
$\therefore H$ (height of cylinder) $= h - \frac{2h}{3}\tan\theta.\;\cot\theta = \frac{3h-2h}{3} = \frac{h}{3}$

**Q.6. Find the volume of the largest cylinder that can be inscribed in a sphere of radius *r*.**

**OR**

**Show that the height of the cylinder of maximum volume, that can be inscribed in a sphere of radius *R* is $\frac{2R}{\sqrt{3}}$ . Also find the maximum volume.**

**Ans.**




Let $R$, $h$ be the radius and height of inscribed cylinder respectively.

If $V$ be the volume of cylinder then

$$V = \pi R^2 h$$

$$\left[\begin{array}{l} \because\ R^2 + \left(\frac{h}{2}\right)^2 = r^2 \\ R^2 = r^2 - \frac{h^2}{4} \end{array}\right.$$

$$V = \pi\left(r^2 - \frac{h^2}{4}\right)h$$

$$V = \pi\left(r^2 h - \frac{h^3}{4}\right)$$

Differentiating with respect to $h$, we get

$$\frac{dV}{dh} = \pi\left(r^2 - \frac{3h^2}{4}\right) \qquad \ldots(\mathrm{i})$$

For maxima or minima

$$\frac{\mathrm{dV}}{\mathrm{dh}} = 0$$

$$\Rightarrow \pi\left(r^2 - \frac{3h^2}{4}\right) = 0 \qquad \Rightarrow r^2 - \frac{3h^2}{4} = 0$$

$$\Rightarrow r = \frac{h\sqrt{3}}{2} \qquad \Rightarrow h = \frac{2r}{\sqrt{3}}$$

Differentiating $(i)$ again with respect to $h$, we get

$$\Rightarrow \frac{d^2V}{dh^2} = -\frac{\pi 6h}{4} \qquad \Rightarrow \left.\frac{d^2V}{dh^2}\right]_{h=\frac{2r}{\sqrt{3}}} = -\frac{3\pi}{2}.\frac{2r}{\sqrt{3}} < 0$$

Hence, $V$ is maximum when $h = \frac{2r}{\sqrt{3}}$.

$$\therefore \text{Maximum volume} = \pi\left(r^2.\frac{2r}{\sqrt{3}} - \frac{8r^3}{4\times3\sqrt{3}}\right)$$

$$= \pi\left(\frac{24r^3 - 8r^3}{12\sqrt{3}}\right)$$

$$= \pi\frac{16r^3}{12\sqrt{3}} = \frac{4\pi r^3}{3\sqrt{3}}$$

**Q.7. A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth in 2 m and volume is 8 m$^3$. If building of tank costs ₹ 70 per sq. metre for the base and ₹ 45 per sq. metre for sides, what is the cost of least expensive tank?**

**Ans.**

Let $l$ and $b$ be the length and breadth of the tank.

If $C$ be the cost of constructing the tank then

$C = 70\ lb + 45 \times 2\ (2l + 2b)$ [depth = $2m$; area of 4 sides = $2(l \times 2 + b \times 2)$]

$= 70\ lb + 180\ l + 180\ b$

$$C = 70l \times \frac{4}{l} + 180l + 180 \times \frac{4}{l}$$
$$\Rightarrow \quad C = 280 + 180\left(l + \frac{4}{l}\right)$$

Differentiating with respect to $l$, we get

$$\frac{dC}{dl} = 180\left(1 - \frac{4}{l^2}\right) \qquad \ldots(i)$$

For maxima or minima

$$\frac{dC}{dl} = 0$$

$$\Rightarrow 180\left(1 - \frac{4}{l^2}\right) = 0 \qquad \Rightarrow l_2 = 4 \qquad \Rightarrow l = 2 \qquad [\because l \neq -2]$$

Differentiating $(i)$ again with respect to $l$, we get

$$\frac{d^2C}{dl^2} = 180 + \frac{8}{l^3}$$

$$\Rightarrow \left.\frac{d^2C}{dl^2}\right]_{l=2} = 181 > 0 \qquad [\because l = -2]$$

Here $C$ is minimum when $l = 2$

$$\therefore b = \frac{4}{2} = 2$$

Minimum cost $= 280 + 180\left(2 + \frac{4}{2}\right) = 280 + 720 =$ ₹1000.

**Q.8. If the sum of hypotenuse and a side of a right-angled triangle is given, show that the area of the triangle is maximum when the angle between them is $-\frac{\pi}{3}$.**

**Ans.**

Let $h$ and $x$ be the length of hypotenuse and one side of a right triangle and $y$ is length of the third side.

If $A$ be the area of triangle, then

$$A = \frac{1}{2}\text{xy} = \frac{1}{2}x\sqrt{h^2 - x^2}$$

$$\begin{bmatrix} \text{also given} \\ h + x = k \ (\text{constant}) \\ \therefore \quad h = k - x \end{bmatrix}$$

$$A = \frac{1}{2}x\sqrt{(k - x)^2 - x^2} = \frac{1}{2}x\sqrt{k^2 - 2\text{kx} + x^2 - x^2}$$

$$A^2 = \frac{x^2}{4}(k^2 - 2\,\text{kx})$$
$$\Rightarrow \quad A^2 = \frac{1}{4}(k^2x^2 - 2\,\text{kx}^3)$$

Differentiating with respect to $x$ we get

$$\frac{d(A^2)}{dx} = \frac{1}{4}(2k^2x - 6kx^2) \qquad \ldots(i)$$

For maxima or minima of $A^2$

$$\frac{d(A^2)}{dx} = 0 \quad \Rightarrow \quad \frac{1}{4}(2k^2x - 6kx^2) = 0$$

$$\Rightarrow 2k^2x - 6kx^2 = 0 \qquad \Rightarrow \quad 2kx\,(k - 3x) = 0$$

$\Rightarrow k - 3x = 0$ ; $\qquad 2kx \neq 0$

$$\begin{bmatrix} \because V = lbh \\ 8 = lb2 \\ \therefore\ b = \frac{8}{2l} = \frac{4}{l} \end{bmatrix}$$

$\Rightarrow \quad x = \frac{k}{3}$

Differentiating $(i)$ again with respect to $x$, we get

$$\frac{d^2(A^2)}{dx^2} = \frac{1}{4}(2k^2 - 12\,kx)$$

$$\left.\frac{d^2(A^2)}{dx^2}\right]_{x=k/3} = \frac{1}{4}\left(2k^2 - 12k.\frac{k}{3}\right)0$$

Hence, $A^2$ is maximum when $x = \frac{k}{3}$ and $h = k - \frac{k}{3} = \frac{2k}{3}$.

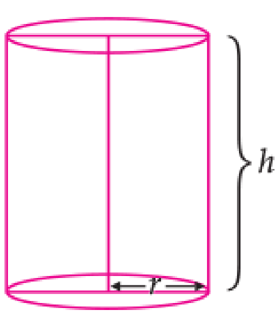
*i.e.*, $A$ is maximum when $x = \frac{k}{3}$ , $h = \frac{2k}{3}$

$\therefore\ \cos\theta = \frac{x}{h} = \frac{k}{3} \times \frac{3}{2k} = \frac{1}{2}$

$\Rightarrow \quad \cos\theta = \frac{1}{2} \qquad \Rightarrow \qquad \theta = \frac{\pi}{3}$

**Q.9. Show that the right circular cylinder, open at the top, and of given surface area and maximum volume is such that its height is equal to the radius of the base.**

**Ans.**



Let $r$, $h$ be the radius and height of given cylinder respectively, having surface area $S$.

If $V$ be the volume of cylinder, then

$V = \pi r^2 h$

$\therefore\ V = \pi r^2 . \left(\frac{S - \pi r^2}{2\pi r}\right)$

$\Rightarrow V = \frac{Sr - \pi r^3}{2}$

$\frac{dV}{dr} = \frac{1}{2}(S - 3\pi r^2)$

$$\begin{bmatrix} S = \pi r^2 + 2\pi rh \ \ ...(i) \\ \therefore h = \dfrac{S - \pi r^2}{2\pi r} \end{bmatrix}$$

For maxima or minima of $V$,

$\frac{dV}{dr} = 0 \qquad \Rightarrow \quad \frac{1}{2}(S - 3\pi r^2) = 0 \quad \Rightarrow \quad r = \sqrt{\frac{S}{3\pi}}$

Now, $\frac{d^2V}{dr^2} = \frac{1}{2}(-6\pi r) \qquad \Rightarrow \left.\frac{d^2V}{dr^2}\right]_{r=\sqrt{\frac{S}{3\pi}}} < 0$

$V$ is maximum when $r = \sqrt{\frac{S}{3\pi}} \quad \Rightarrow \quad 3\pi r^2 = S$

Putting it in $(i)$

$3\pi r^2 = \pi r^2 + 2\pi rh \qquad \Rightarrow 2\pi r^2 = 2\pi rh \qquad \Rightarrow r = h$

*i.e.*, $V$ is maximum when $r = h$ *i.e.*, height is equal to the radius of base.

**Q.10. The length of the sides of an isosceles triangle are $9 + x^2$, $9 + x^2$ and $18 - 2x^2$ units. Calculate the area of the triangle in terms of $x$ and find the value of $x$ which makes the area maximum.**

**Ans.**

Sides of isosceles triangles are $9 + x^2$, $9 + x^2$ and $18 - 2x^2$

$$S = \frac{9+x^2+9+x^2+18-2x^2}{2} = \frac{36}{2} = 18$$

If $A$ be area of triangle, then

$$A = \sqrt{S\,(S-a)\,(S-b)\,(S-c)}$$

$$A = \sqrt{18(18-9-x^2)\,(18-9-x^2)\,\,(18-18+2x^2)}$$

$$A = \sqrt{18\,(9-x^2)\,(9-x^2)\,\cdot 2x^2}$$

$$A = 6x\,(9-x^2) = 6\,(9x-x^3)$$

For maxima or minima of $A$

$$\frac{dA}{dx} = 6\,(9-3x^2) = 0 \qquad \Rightarrow 9-3x^2 = 0 \quad \Rightarrow \quad x = \pm\sqrt{3}$$

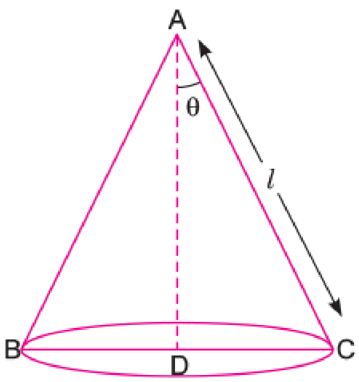
Again, $\frac{d^2A}{dx^2} = 6\,(-6x) = -36x$

Now, $\left.\frac{d^2A}{dx^2}\right]_{x=\sqrt{3}} =- 36\sqrt{3} < 0$ and $\left.\frac{d^2A}{dx^2}\right]_{x=-\sqrt{3}} =-36\,(-\sqrt{3}) > 0$

Hence, for $x = \sqrt{3}$, Area $(A)$ is maximum.

**Q.11. Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is .**

**Ans.**




Let $ABC$ be cone having slant height $l$ and semi-vertical angle $\theta$.

If $V$ be the volume of cone then.

$$V = \frac{1}{3}.\pi \times \mathrm{DC}^2 \times \mathrm{AD} \ = \frac{\pi}{3} \times l^2 \sin^2\theta \times l\cos\theta$$

$$\Rightarrow \qquad V = \frac{\pi l^3}{3}\sin^2\theta\cos\theta$$

$$\Rightarrow \qquad \frac{\mathrm{dV}}{d\theta} = \frac{\pi l^3}{3}[-\sin^3\theta + 2\sin\theta.\cos^2\theta]$$

For maximum value of $V$.

$$\Rightarrow \quad \frac{dV}{d\theta} = 0$$

$$\Rightarrow \quad \frac{\pi l^3}{3}\ [-\sin^3\theta + 2\sin\theta.\cos^2\theta] = 0$$

$$\Rightarrow \ -\sin^3\theta + 2\sin\theta.\cos^2\theta = 0$$

$$\Rightarrow -\sin\theta\left(\sin^2\theta - 2\cos^2\theta\right) = 0$$

$\Rightarrow \sin\theta = 0$ or $1 - \cos^2 q \ - 2\cos^2\theta = 0$

$\Rightarrow \theta = 0$ or $1 - 3\cos^2\theta = 0$

$\Rightarrow \theta = 0$ or $\cos\theta = \frac{1}{\sqrt{3}}$

Now $\frac{d^2V}{d\theta^2} = \frac{\pi l^3}{3}\{-3\sin^2\theta.\cos\theta - 4\sin^2\theta.\cos\theta + 2\cos^3\theta\}$

$\Rightarrow \frac{d^2V}{d\theta^2} = \frac{\pi l^3}{3}\{-7\sin^2\theta\cos\theta + 2\cos^3\theta\}$

$\Rightarrow \left.\frac{d^2V}{d\theta^2}\right]_{\theta=0} = +\text{ve}$

and

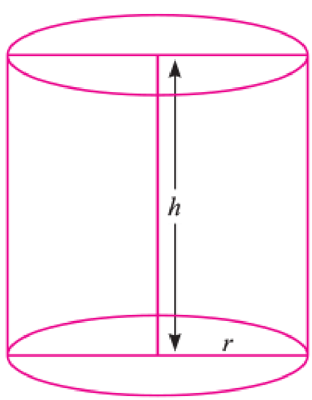
$\left.\frac{d^2V}{d\theta^2}\right]_{\cos\theta=\frac{1}{\sqrt{3}}} = -\text{ve}$ [Putting $\cos\theta$

$= \frac{1}{\sqrt{3}}$ and $\sin\theta = \sqrt{1-\left(\frac{1}{\sqrt{3}}\right)^2} = \frac{\sqrt{2}}{\sqrt{3}}$]

Hence, for $\cos\theta = \frac{1}{\sqrt{3}}$ or $\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$, $V$ is maximum.

**Q.12. Show that the height of a closed right circular cylinder of given surface and maximum volume, is equal to the diameter of its base.**

**Ans.**



Let $r$ and $h$ be radius and height of given cylinder of surface area $S$.

If $V$ be the volume of cylinder then

$V = \pi r^2 h$

$V = \frac{\pi r^2 . (S - 2\pi r^2)}{2\pi r}$

$[\because \quad S = 2\pi r^2 + 2\pi \mathrm{rh} \quad \Rightarrow \quad \frac{S - 2\pi r^2}{2\pi r} = h]$

$V = \frac{\mathrm{Sr} - 2\pi r^3}{2}$

$\Rightarrow \quad \frac{\mathrm{dV}}{\mathrm{dr}} = \frac{1}{2} \quad (S - 6\pi r^2)$

For maximum or minimum value of $V$

$\frac{\mathrm{dV}}{\mathrm{dr}} = 0$

$\Rightarrow \quad \frac{1}{2}\,(S - 6\pi r^2) = 0$

$\Rightarrow \quad S - 6\pi r^2 = 0$

$\Rightarrow \quad r^2 = \frac{S}{6\pi}$

$\Rightarrow \quad r = \sqrt{\frac{S}{6\pi}}$

Now $\frac{d^2V}{dr^2} = -\frac{1}{2} \times 12\pi r \qquad \Rightarrow \frac{d^2V}{dr^2} = -6\pi r \ \Rightarrow \quad \left[\frac{d^2V}{dr^2}\right]_{r=\sqrt{\frac{S}{6\pi}}} < 0$

Hence, for $r = \sqrt{\frac{S}{6\pi}}$ , volume $V$ is maximum.

$$\Rightarrow h = \frac{S - 2\pi . \frac{S}{6\pi}}{2\pi\sqrt{\frac{S}{6\pi}}} \quad \Rightarrow \quad h = \frac{3S - S}{3 \times 2\pi} \times \sqrt{\frac{6\pi}{S}}$$

$$\Rightarrow h = \frac{2S}{6\pi} . \frac{\sqrt{6\pi}}{\sqrt{S}} = 2\sqrt{\frac{S}{6\pi}}$$

$$\Rightarrow h = 2r \text{ (diameter)} \qquad \left[\because\ r = \sqrt{\frac{S}{6\pi}}\right]$$

Therefore, for maximum volume, height of cylinder is equal to diameter of its base.

**Q.13. Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.**

**Ans.**

Let $r$ and $h$ be the radius and height of right circular cylinder inscribed in a given cone of radius $R$ and height $H$. If $S$ be the curved surface area of cylinder then

$S = 2\pi rh$

$$\Rightarrow S = 2\pi r . \frac{(R - r)}{R} . H \Rightarrow S = \frac{2\pi H}{R}(\text{rR} - r^2)$$

$$\left[\begin{array}{l} \because \ \Delta AOC \sim \Delta FEC \\ \Rightarrow \quad \frac{OC}{EC} = \frac{AO}{FE} \\ \Rightarrow \quad \frac{R}{R - r} = \frac{H}{h} \\ \Rightarrow \quad h = \frac{(R - r) . H}{R} \end{array}\right]$$

Differentiating both sides

with respect to *r*, we get

$\frac{dS}{dr} = \frac{2\pi H}{R}(R - 2r)$

For maxima and minima

$$\Rightarrow \begin{array}{l} \frac{dS}{dr} = 0 \quad \Rightarrow \quad \frac{2\pi H}{R}(R - 2r) = 0 \\ R - 2r = 0 \quad \Rightarrow \quad r = \frac{R}{2} \end{array}$$

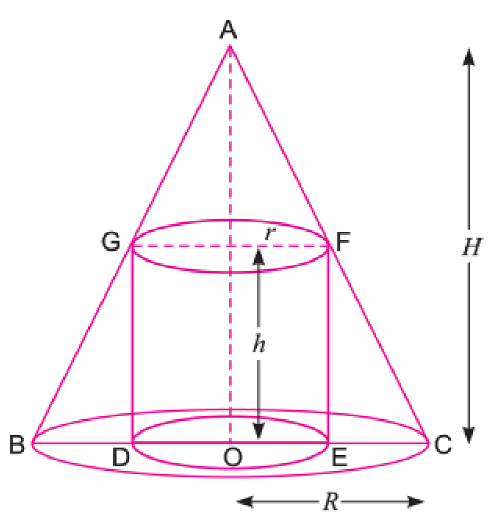
Now,

$\Rightarrow \frac{d^2S}{dr^2} = \frac{2\pi H}{R}(0 - 2)$

$\Rightarrow \left[\frac{d^2S}{dr^2}\right]_{r = R/2} = -\frac{4\pi H}{R} = -\text{ve}$

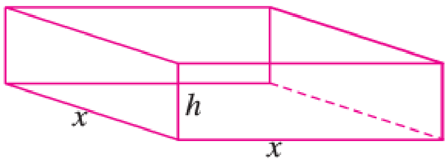
Hence, for $r = \frac{R}{2}, S$ is maximum.

*i.e.*, radius of cylinder is half of that of cone.



**Q.14. An open box with a square base is to be made out of a given quantity of cardboard of area $c^2$square units. Show that the maximum volume of the box is $\frac{c^3}{6\sqrt{3}}$ cubic units.**

**Ans.**




Let the length, breadth and height of open box with square be $x$, $x$ and $h$ unit respectively.

If $V$ be the volume of box then $V = x.x.\ h \quad \Rightarrow \quad V = x^2 h \qquad ....(i)$

Also $c^2 = x^2 + 4xh \quad \Rightarrow \quad h = \frac{c^2 - x^2}{4x}$

Putting it in $(i)$, we get

$$V = \frac{x^2(c^2 - x^2)}{4x} \quad \Rightarrow \quad V = \frac{c^2 x}{4} - \frac{x^3}{4}$$

Differentiating with respect to $x$, we get

$$\frac{dV}{dx} = \frac{c^2}{4} - \frac{3x^2}{4}$$

Now for maxima or minima $\frac{\mathrm{dV}}{\mathrm{dx}} = 0$

$$\Rightarrow \qquad \frac{c^2}{4} - \frac{3x^2}{4} = 0$$

$$\Rightarrow \qquad \frac{3x^2}{4} = \frac{c^2}{4}$$

$$\Rightarrow \quad x^2 = \frac{c^2}{3}$$

$$\Rightarrow \quad x = \frac{c}{\sqrt{3}}$$

Now, $\frac{d^2V}{\mathrm{dx}^2} = -\frac{6x}{4} = -\frac{3x}{2}$

$$\therefore \left[\frac{d^2V}{dx^2}\right]_{x={c}/{\sqrt{3}}} = -\frac{3c}{2\sqrt{3}} = -\text{ve}$$

Hence, for $x = \dfrac{c}{\sqrt{3}}$ volume of box is maximum.

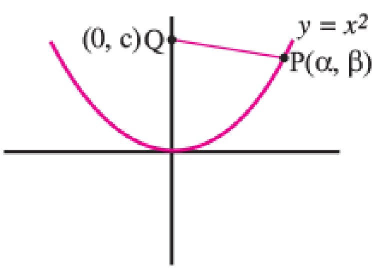
$$\therefore h = \frac{c^2 - x^2}{4x} = \frac{c^2 - \frac{c^2}{3}}{4\frac{c}{\sqrt{3}}} = \frac{2c^2}{3} \times \frac{\sqrt{3}}{4c} = \frac{c}{2\sqrt{3}}$$

Therefore maximum volume = $x^2 \,.\, h$

$$= \frac{c^2}{3} \cdot \frac{c}{2\sqrt{3}} = \frac{c^3}{6\sqrt{3}} \text{ cubic units}$$

**Q.15. Find the shortest distance of the point (0, $c$) from the parabola $y = x^2$, where $1 \le c \le 5$.**

**Ans.**




Let $P$ $(\alpha, \beta)$ be required point on parabola $y = x^2$ such that the distance of $P$ to given point $Q(0, c)$ is shortest.

Let $PQ = D$

$$\therefore \quad D = \sqrt{(\alpha - 0)^2 + (\beta - c)^2}$$

$$\Rightarrow \quad D^2 = \alpha^2 + (\beta - c)^2$$

$$\Rightarrow D^2 = \alpha^2 + (\alpha^2 - c)^2 \quad [\because (\alpha, \beta) \text{ lie on } y = x^2 \Rightarrow \beta = \alpha^2]$$

Now, $\frac{d(D^2)}{d\alpha} = 2\alpha + 2(\alpha^2 - c).2\alpha$

$= 2\alpha(1 + 2\alpha^2 - 2c) = 2\alpha + 4\alpha^3 - 4\alpha c$

For extremum value of $D$ or $D^2$

$\frac{d(D)^2}{d\alpha} = 0$

$\Rightarrow\ 2\alpha(1 + 2\alpha^2 - 2c) = 0$

$\Rightarrow$ $\alpha = 0,$ or $1 + 2\alpha^2 - 2c = 0$

$\Rightarrow\ \alpha = 0,$ or $\alpha = \pm\sqrt{\frac{2c-1}{2}}$

Again $\frac{d^2(D^2)}{d\alpha^2} = 2 + 12\alpha^2 - 4c$

$\Rightarrow\ \left.\frac{d^2(D^2)}{d\alpha^2}\right] = 2 - 4c = -\text{ve}$ $[\because 1 \leq c \leq 5]$

$\left[\frac{d^2(D^2)}{d\alpha^2}\right]_{\alpha=\pm\sqrt{\frac{2c-1}{2}}} = 2 + 12\left(\frac{2c-1}{2}\right) - 4c$

$= 2 + 12c - 6 - 4c = 8c - 4 > 0$ $[\because 1 \leq c \leq 5]$

*i.e.,* for $\alpha = \pm\sqrt{\frac{2c-1}{2}}D^2$ *i.e.,* $D$ is minimum (shortest)

Hence required points are $\left(\pm\sqrt{\frac{2c-1}{2}}, \frac{2c-1}{2}\right)$.

**Q.16. Show that the volume of the greatest cylinder that can be inscribed in a cone of height '*h*' and semi-vertical angle 'α' is.**

**Ans.**

Let a cylinder of base radius $r$ and height $h_1$ is included in a cone of height $h$ and semi-vertical angle $\alpha$.

Then $AB = r$, $OA = (h - h_1)$,

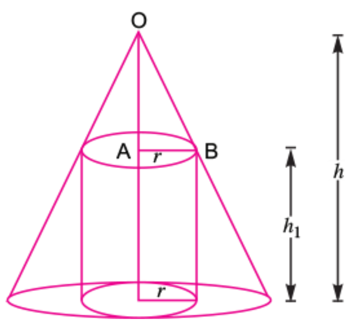
In right angle triangle $OAB$,

$\frac{AB}{OA} = \tan\alpha \quad \Rightarrow \quad \frac{r}{h-h_1}\tan\alpha$

or $r = (h - h_1)\tan\alpha$

$\therefore V = \pi\left[(h - h_1)\tan\alpha\right]^2 . h_1$ $\qquad$ ($\because$ Volume of cylinder $= \pi r^2 h$)

$V = \pi\tan^2\alpha . h_1 (h - h_1)^2$ $\qquad$ ...$(i)$




Differentiating with respect to $h_1$, we get

$$\frac{dV}{dh_1} = \pi\tan^2\alpha[h_1.2(h-h_1)(-1) + (h-h_1)^2 \times 1]$$

$$= \pi\tan^2\alpha\,(h - h_1)\left[-2h_1 + h - h_1\right]$$

$$= \pi\tan^2\alpha\,(h - h_1)(h - 3h_1)$$

For maximum volume $V, \frac{\mathrm{dV}}{\mathrm{dh_1}} = 0$

$\Rightarrow h - h_1 = 0 \quad \text{or} \quad h - 3h_1 = 0$

$\Rightarrow \quad h = h_1 \text{ or } h_1 = \frac{1}{3}h$

$\Rightarrow h_1 = \frac{1}{3}h$ $\qquad$ ($\because h = h_1$ is not possible)

Again differentiating with respect to $h_1$, we get

$$\frac{d^2V}{dh_1^2} = \pi \tan^2 \alpha[h - h_1)(-3) + (h - 3h_1)(-1)]$$

At $h_1 = \frac{1}{3}h$,

$$\frac{d^2V}{dh_1^2} = \pi \tan^2 \alpha \left[\left(h - \frac{1}{3}h\right)(-3) + 0\right]$$
$$= -2\pi h \tan^2 \alpha < 0$$

$\therefore$ Volume is maximum for $h_1 = \frac{1}{3}h$

$$V_{\max} = \pi \tan^2 \alpha . \left(\frac{1}{3}h\right)\left(h - \frac{1}{3}h\right)^2 \qquad \text{[Using (}i\text{)]}$$

$$= \frac{4}{27}\pi h^3 \tan^2 \alpha$$

**Q.17. The sum of the perimeter of a circle and a square is *k*, where *k* is some constant. Prove that the sum of their areas is least when the side of the square is double the radius of the circle.**

**Ans.**

Let side of square be $a$ units and radius of circle be $r$ units.

It is given that $4a + 2\pi r = k$, where $k$ is a constant

$$\Rightarrow r = \frac{k-4a}{2\pi}$$

Sum of areas, $A = a^2 + \pi r^2$

$$\Rightarrow A = a^2 + \pi\left[\frac{k-4a}{2\pi}\right]^2 = a^2 + \frac{1}{4\pi}(k-4a)^2$$

Differentiating with respect to $a$, we get

$$\frac{dA}{da} = 2a + \frac{1}{4\pi}.2(k - 4a).\ (-4) \qquad \ldots(i)$$
$$= 2a - \frac{2(k-4a)}{\pi}$$

For minimum area, $\frac{dA}{da} = 0$

$$\Rightarrow \quad 2a - \frac{2(k-4a)}{\pi} = 0$$
$$\Rightarrow \quad 2a = \frac{2(k-4a)}{\pi}$$

$$\Rightarrow \ 2a = \frac{2(2\pi r)}{\pi}$$

[As $k = 4a + 2\pi r$ given]

$\Rightarrow \ a = 2r$

Now, again differentiating equation $(i)$ with respect to $a$

$$\frac{d^2A}{da^2} = 2 - \frac{2}{\pi}(-4) = 2 + \frac{8}{\pi}$$

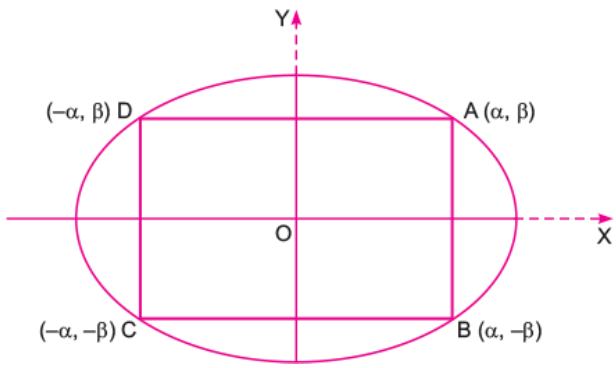
at $a = 2\pi$, $\frac{d^2A}{da^2} = 2 + \frac{8}{\pi} > 0$

$\therefore$ For $ax = 2r$, sum of areas is least.

Hence, sum of areas is least when side of the square is double the radius of the circle.

**Q.18. Find the area of the greatest rectangle that can be inscribed in an ellipse**
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

**Ans.**




Let $ABCD$ be rectangle having area $A$ inscribed in an ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad ...(i)$$

Let the coordinate of $A$ be $(\alpha, \beta)$

$\therefore$Coordinate of $B \equiv (\alpha, -\beta)$, $C \equiv (-\alpha, -\beta)$, $D \equiv (-\alpha, \beta)$

Now $A$ = Length × Breadth = $2\alpha \times 2\beta$

A = $4\alpha\beta$

$$\Rightarrow A=4\alpha.\sqrt{b^2\left(1-\frac{\alpha^2}{a^2}\right)}$$

$$\begin{bmatrix} \because\ (\alpha,\beta) \text{ lies on ellipse } (i) \\ \therefore\ \frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}=1\ i.e.,\ \beta=\sqrt{b^2\left(1-\frac{\alpha^2}{a^2}\right)} \end{bmatrix}$$

$$\Rightarrow A^2=16\alpha^2\left\{b^2\left(1-\frac{\alpha^2}{a^2}\right)\right\}$$

$$\Rightarrow \quad A^2=\frac{16b^2}{a^2}(a^2\alpha^2-\alpha^4)$$

$$\Rightarrow \frac{d(A^2)}{d\alpha}=\frac{16b^2}{a^2}(2a^2\alpha-4\alpha^3)$$

For maximum or minimum value

$$\frac{d(A^2)}{d\alpha} = 0$$

$$\Rightarrow 2a^2\alpha - 4\alpha^3 = 0 \qquad \Rightarrow \quad 2\alpha\left(a^2 - 2\alpha^2\right) = 0$$

$$\Rightarrow \alpha = 0,\ \alpha = \frac{a}{\sqrt{2}}$$

Again $\frac{d^2(A^2)}{d\alpha^2} = \frac{16b^2}{a^2}(2a^2 - 12\alpha^2)$

$$\Rightarrow \left.\frac{d^2(A^2)}{d\alpha^2}\right]_{\alpha=\frac{a}{\sqrt{2}}} = \frac{16b^2}{a^2}\left(2a^2 - 12 \times \frac{a^2}{2}\right) < 0$$

$\Rightarrow$ For $\alpha = \frac{a}{\sqrt{2}}, A^2 i.e., A$ is maximum.

*i.e.*, for greatest area $A$

$\alpha = \frac{a}{\sqrt{2}}$ and $\beta = \frac{b}{\sqrt{2}}$ (using $(i)$)

$\therefore$ Greatest area $= 4\alpha \,.\, \beta = 4\frac{a}{\sqrt{2}} \times \frac{b}{\sqrt{2}} = 2ab$

**Q.19. Tangent to the circle $x^2 + y^2 = 4$ at any point on it in the first quadrant makes intercepts *OA* and *OB* on *x* and *y* axes respectively, *O* being the centre of the circle. Find the minimum value of (*OA* + *OB*).**

**Ans.**

Let $AB$ be the tangent in the first quadrant to the circle $x^2 + y^2 = 4$ which make intercepts $OA$ and $OB$ on $x$ and $y$ axis respectively. Let $S = OA + OB$.

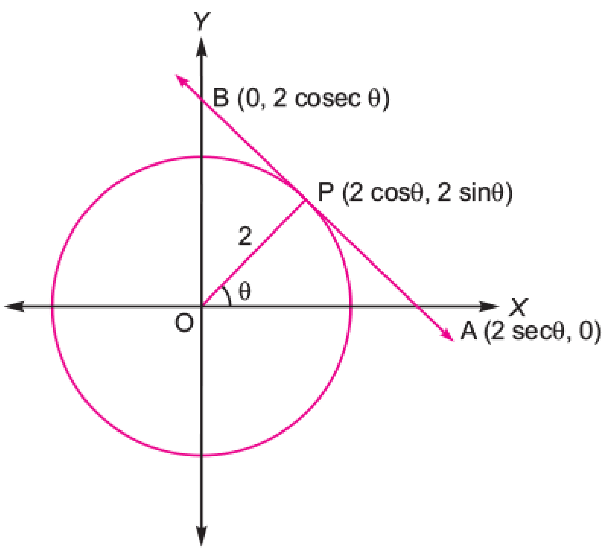
$$S = OA + OB \qquad \dots (i)$$

Let q be the angle made by $OP$ with positive direction of $x$-axis.

$\therefore$Coordinates of $P = (2 \cos \theta, 2 \sin \theta)$

Coordinates of $A = (2 \sec \theta, 0)$

Coordinates of $B = (0, 2 \operatorname{cosec} \theta)$



$$\Rightarrow \quad \frac{dS}{d\theta} = 2\left\{\sec\theta \tan\theta - \text{cosec}\,\theta \cot\theta\right\}$$

For extremum value of $V$

$$\Rightarrow \quad \frac{dS}{d\theta} = 0$$

$$\Rightarrow \quad 2\{\sec\theta \tan\theta - \text{cosec}\,\theta \cot\theta\} = 0$$

$$\Rightarrow \sec\theta \tan\theta - \text{cosec}\,\theta \cot\theta = 0$$

$$\Rightarrow \quad \frac{1}{\cos\theta}\frac{\sin\theta}{\cos\theta} = \frac{1}{\sin\theta}\frac{\cos\theta}{\sin\theta}$$

$$\Rightarrow \quad \frac{\sin\theta}{\cos^2\theta} = \frac{\cos\theta}{\sin^2\theta}$$

$$\Rightarrow \sin^3\theta = \cos^3\theta \qquad \Rightarrow \sin\theta = \cos\theta$$

$$\Rightarrow \theta = \frac{\pi}{4} \quad \left[\because \theta \text{ lies in first quadrant } \Rightarrow \; 0 \le \theta \le \frac{\pi}{4}\right]$$

Now, $\frac{d^2S}{d\theta^2} = 2\left\{(\sec^3\theta + \tan^2\theta\sec\theta) + (\text{cosec}^3\theta + \text{cosec}\,\theta\cot^2\theta)\right\}$

$\Rightarrow \left.\frac{d^2S}{d\theta^2}\right]_{\theta=\frac{\pi}{4}} = +\text{ve} \qquad \Rightarrow S$ is minimum when

$\therefore$ Minimum value of $S = OA + OB$ is $2\sec\frac{\pi}{4} + 2\text{cosec}\,\frac{\pi}{4} = 2\sqrt{2} + 2\sqrt{2} = 4\sqrt{2}$ units.

**Q.20. Find the absolute maximum and absolute minimum values of the function *f* given by $f(x) = \sin^2 x - \cos x$, $x \in [0, \pi]$.**

**Ans.**

Here, $f(x) = \sin^2 x - \cos x$

$f'(x) = 2\sin x.\cos x + \sin x$ $\Rightarrow f'(x) = \sin x(2\cos x + 1)$

**For critical point:** $f'(x) = 0$

$$\Rightarrow \quad \sin x(2\cos x + 1) = 0$$
$$\Rightarrow \quad \sin x = 0 \quad \text{or} \quad \cos x = -\frac{1}{2}$$
$$\Rightarrow \quad x = 0 \quad \text{or} \quad \cos x = \cos\frac{2\pi}{3}$$
$$\Rightarrow \quad x = 0 \quad \text{or} \quad x = 2n\pi \pm \frac{2\pi}{3}$$

where $n = 0, \pm 1, \pm 2$ ...

$\Rightarrow x = 0$ or $x = \frac{2\pi}{2}$ other values does not belong to $[0, \pi]$

**For absolute maximum or minimum values:**

$f(0) = \sin^2 0 - \cos 0 = 0 - 1 = -1$

$$f\left(\frac{2\pi}{3}\right) = \sin^2\frac{2\pi}{3} - \cos\frac{2\pi}{3}$$
$$= \left(\frac{\sqrt{3}}{2}\right)^2 - \left(-\frac{1}{2}\right) = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$$

$f(\pi) = \sin^2 \pi - \cos \pi = 0 - (-1) = 1$

Hence, absolute maximum value $= \frac{5}{4}$ and absolute minimum value $= -1$ .

**Q.21. If the function $f(x) = 2x^3 - 9mx^2 + 12m^2x + 1$, where $m > 0$ attains its maximum and minimum at $p$ and $q$ respectively such that $p^2 = q$, then find the value of $m$.**

**Ans.**

Given, $f(x) = 2x^3 - 9mx^2 + 12m^2x + 1$

$\Rightarrow f'(x) = 6x^2 - 18mx + 12m^2$

For extremum value of $f(x)$, $f'(x) = 0$

$\Rightarrow 6x^2 - 18mx + 12m^2 = 0 \qquad \Rightarrow x^2 - 3mx + 2m^2 = 0$

$\Rightarrow x^2 - 2mx - mx + 2m^2 = 0 \qquad \Rightarrow x(x - 2m) - m(x - 2m) = 0$

$\Rightarrow (x - m)(x - 2m) = 0 \qquad \Rightarrow x = m \text{ or } x = 2m$

Now, $\quad f'(x) = 12x - 18m$

$\Rightarrow f'(x)$ at $[x = m] = f'(m) = 12m - 18m = -6m < 0$

And, $f'(x)$ at $[x = 2m] = f'(2m) = 24m - 18m = 6m > 0$

Hence, $f(x)$ attains maximum and minimum value at $m$ and $2m$ respectively.

$\Rightarrow m = p$ and $2m = q$

But, $p^2 = q \qquad [\text{Given}]$

$\therefore m^2 = 2m \qquad \Rightarrow \qquad m^2 - 2m = 0$

$\Rightarrow m(m - 2) \qquad \Rightarrow \qquad m = 0 \text{ or } m = 2$

$\Rightarrow m = 2$ as $m > 0$

**Q.22. The sum of the surface areas of a cuboid with sides $x$, $2x$ and $\frac{x}{3}$ and a sphere is given to be constant. Prove that the sum of their volumes is minimum, if $x$ is equal to three times the radius of sphere. Also find the minimum value of the sum of their volumes.**

**Ans.**

Let $r$ be the radius of sphere and $S$, $V$ be the sum of surface area and volume of cuboid and sphere.

Now $V = \left(x.2x.\frac{x}{3}\right) + \frac{4}{3}\pi r^3$

$\Rightarrow V = \frac{2}{3}x^3 + \frac{4}{3}\pi r^3$

$\Rightarrow V = \frac{2}{3}(x^3 + 2\pi r^3)$

$\Rightarrow V = \frac{2}{3}\left\{\left(\frac{S - 4\pi r^2}{6}\right)^{\frac{3}{2}} + 2\pi r^3\right\}$

$\Rightarrow \frac{\mathrm{dV}}{\mathrm{dr}} = \frac{2}{3}\left\{\frac{3}{2}\left(\frac{S - 4\pi r^2}{6}\right)^{\frac{1}{2}}.\frac{1}{6}.(-8\pi r) + 6\pi r^2\right\}$

$$\left[\begin{array}{l} \because \quad S = 2\left[x.2x + x.\frac{x}{3} + \frac{x}{3}.2x\right] + 4\pi r^2 \\ \Rightarrow \quad S = \frac{18x^2}{3} + 4\pi r^2 = 6x^2 + 4\pi r^2 \\ \Rightarrow \quad x^2 = \frac{S - 4\pi r^2}{6} \quad \Rightarrow \quad x^3 = \left(\frac{S - 4\pi r^2}{6}\right)^{3/2} \end{array}\right.$$

For maximum or minimum value

$\frac{\mathrm{dV}}{\mathrm{dr}} = 0$

$\Rightarrow \quad \frac{2}{3}\left\{-2\pi r\left(\frac{S - 4\pi r^2}{6}\right)^{\frac{1}{2}} + 6\pi r^2\right\} = 0$

$\Rightarrow \quad \left(\frac{S - 4\pi r^2}{6}\right)^{\frac{1}{2}} = \frac{6\pi r^2}{2\pi r}$

$\Rightarrow \quad \left(\frac{S - 4\pi r^2}{6}\right)^{\frac{1}{2}} = 3r$

$\Rightarrow \quad r = \frac{1}{3}.\left(\frac{S - 4\pi r^2}{6}\right)^{\frac{1}{2}}$

Obviously, $\left.\frac{d^2V}{dr^2}\right]_{r=\frac{1}{3}\left(\frac{S-4\pi r^2}{6}\right)^{\frac{1}{2}}} = +\text{ve}$

$\therefore \quad V$ is minimum when $r = \frac{1}{3}\left(\frac{S-4\pi r^2}{6}\right)^{\frac{1}{2}}$

$\Rightarrow \quad 3r = \left(\frac{S-4\pi r^2}{6}\right)^{\frac{1}{2}}$

$\Rightarrow \quad 9r^2 = \left(\frac{S-4\pi r^2}{6}\right) \quad \Rightarrow \quad 54r^2 = S - 4\pi r^2$

$\Rightarrow \; 54r^2 = 6x^2 + 4\pi r^2 - 4\pi r^2 \; [\because S = 6x^2 + 4\pi r^2]$

$\Rightarrow \; x^2 = 9r^2 \qquad \Rightarrow \qquad x = 3r$

*i.e.,* $x$ is equal to three times the radius of sphere.

Now Minimum value of $V$ (sum of volume) $= \frac{2}{3}\left\{x^3 + 2\pi\left(\frac{x}{3}\right)^3\right\}$

$= \frac{2}{3}\left\{x^3 + \frac{2\pi}{27}x^3\right\} = \frac{2}{81}x^3(27 + 2\pi)$ cubic unit.

**Q.23. Find the maximum and minimum values of $f(x)$ = sec $x$ + log cos$^2$ $x$, 0 < $x$ < 2π.**

**Ans.**

We have

$f(x) = \sec x + \log \cos^2 x$

$f'(x) = \sec x.\tan x + \frac{1}{\cos^2 x}.2\cos x(-\sin x) = \sec x \,.\, \tan x - 2\tan x = \tan x\,(\sec x - 2)$

For critical point

$f'(x) = 0$

$\Rightarrow \tan x\,(\sec x - 0) = 0 \qquad \Rightarrow \; \tan x = 0 \;$ or $\; \sec x - 2 = 0$

$$\Rightarrow \quad x = n\pi \quad \text{or} \quad \sec\ x = 2$$
$$\Rightarrow \quad n\pi \quad \text{or} \quad \cos\ x = \tfrac{1}{2}$$
$$\Rightarrow \ x = n\pi \quad \text{or} \quad \cos\ x = \cos\tfrac{\pi}{3}$$
$$\Rightarrow \ x = n\pi \quad \text{or} \quad x = 2n\pi \pm \tfrac{\pi}{3}, \ \ n = 0, \ \pm\ 1, \pm\ 2.......$$

Thus possible value of $x$ in interval $0 < x < 2\Pi$ are

$x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$

Now, $f\left(\frac{\pi}{3}\right) = \sec\frac{\pi}{3} + \log\cos^2\frac{\pi}{3} = 2 + \log\left(\frac{1}{2}\right)^2$

$$= 2 + 2(\log 1 - \log 2) = 2 - 2\log 2 = 2(1 - \log 2) \qquad [\because \log 1 = 0]$$

$$f(\Pi) = \sec \Pi + \log \cos^2 \Pi = -1 + \log(-1)^2 = -1$$

$f\left(\frac{5\pi}{3}\right) = \sec\frac{5\pi}{3} + 2\log\ \cos\frac{5\pi}{3}$
$= \sec\ \left(2\pi - \frac{\pi}{3}\right) + 2\log\ \cos\left(2\pi - \frac{\pi}{3}\right)$

$= \sec\ \frac{\pi}{3} + 2\log\ \cos\frac{\pi}{3} = 2 + 2\log\frac{1}{2}$

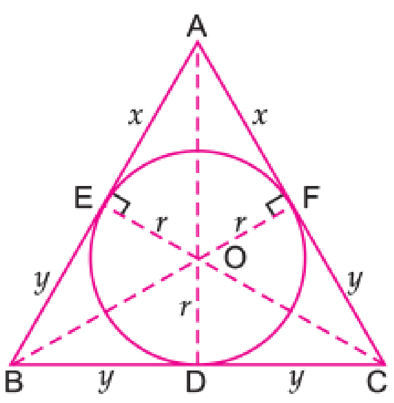
$= 2 + 2(\log 1 - \log 2) = 2 - 2\log 2 = 2(1 - \log 2)$

Hence,Maximum value of $f(x) = 2(1 - \log 2)$

Minimum value of $f(x) = -1$

**Q.24. Prove that the least perimeter of an isosceles triangle in which a circle of radius *r* can be inscribed is $6\sqrt{3}r$.**

**Ans.**



Let $\Delta ABC$ be isosceles triangle having $AB = AC$ in which a circle with centre $O$ and radius $r$ is inscribed touching sides $AB$, $BC$ and $AC$ at $E$, $D$ and $F$ respectively.

Let $AE = AF = x$, $BE = BD = y$

Obviously, $CF = CD = y$

Let $P$ be the perimeter of $\Delta ABC$.

$$\therefore \quad P = 2x + 4y \quad \Rightarrow P = \frac{4\,yr^2}{y^2 - r^2} + 4y$$

Differentiating w.r.t. $y$, we get

$$\Rightarrow \quad \frac{dP}{dy} = \frac{(y^2 - r^2).4r^2 - 4\,yr^2\,(2y - 0)}{(y^2 - r^2)^2} + 4$$

$$\Rightarrow \quad \frac{dP}{dy} = \frac{4y^2r^2 - 4r^4 - 8y^2r^2}{(y^2 - r^2)^2} + 4$$

$$\Rightarrow \quad \frac{dP}{dy} = \frac{-4r^2(r^2+y^2)}{(y^2 - r^2)^2} + 4$$

For critical point $\frac{dP}{dy} = 0$

$$\Rightarrow \quad \frac{-4r^2(r^2+y^2)}{(y^2-r^2)^2} + 4 = 0$$

$$\Rightarrow \quad -4r^2(r^2 + y^2) + 4(y^2 - r^2)^2 = 0$$

$$\Rightarrow \quad -r^4 - r^2y^2 + y^4 + r^4 - 2y^2r^2 = 0$$

$$\Rightarrow \quad y^4 - 3r^2y^2 = 0$$

$$\Rightarrow \quad y^2\left[y^2 - 3r^2\right] = 0$$

$$\Rightarrow \quad y = \sqrt{3}r \quad [\because y \neq 0]$$

Also $\left.\frac{d^2P}{dr^2}\right]_{\sqrt{3}r} = +\text{ve}$

$$
\begin{aligned}
& ar\,(\Delta ABC) = ar\,(\Delta BOC) + ar\,(\Delta AOC) + ar\,(\Delta AOB) \\
\Rightarrow\ & \tfrac{1}{2} AD.\,BC = \tfrac{1}{2}.\,BC.\,OD + \tfrac{1}{2}.\,AC\,.\,OF + \tfrac{1}{2}.\,AB.\,OE \\
\Rightarrow\ & 2y.\,(r + \sqrt{r^2 + x^2}) = 2y.\,r + (x + y).\ r + (x + y)\ .\ r \\
\Rightarrow\ & 2y.\,(r + \sqrt{r^2 + x^2}) = 2yr + 2(x + y)\ .\,r \\
\Rightarrow\ & yr + y\sqrt{r^2 + x^2} = yr + xr + yr \\
\Rightarrow\ & y\sqrt{r^2 + x^2} = xr + yr \\
\Rightarrow\ & y^2(r^2 + x^2) = x^2r^2 + y^2r^2 + 2xyr^2 \\
\Rightarrow\ & y^2r^2 + x^2y^2 = x^2r^2 + y^2r^2 + 2xyr^2 \\
\Rightarrow\ & x^2y^2 = x^2r^2 + 2xyr^2 \\
\Rightarrow\ & xy^2 = xr^2 + 2yr^2
\end{aligned}
$$

$\Rightarrow$ when $y = \sqrt{3}r$, the value of $P$ is minimum.

$\therefore$ Least perimeter $= 4y + \frac{4r^2y}{y^2 - r^2} = 4\sqrt{3}r + \frac{4r^2.\sqrt{3}r}{3r^2 - r^2}$

$= 4\sqrt{3}r + \frac{4\sqrt{3}r^3}{2r^2} = 6\sqrt{3}r$ units

**Q.25. Find the equations of tangents to the curve $3x^2 - y^2 = 8$, which pass through the point $\left(\frac{4}{3}, 0\right)$**

**Ans.**

Let the point of contact be $(x_0,\ y_0)$

Now given curve is $3x^2 - y^2 = 8$

Differentiating w.r.t. $x$ we get, $6x - 2y.\frac{\mathrm{dy}}{\mathrm{dx}} = 0$

$\Rightarrow \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{6x}{2y} = \frac{3x}{y} \quad \Rightarrow \quad \left.\frac{\mathrm{dy}}{\mathrm{dx}}\right]_{(x_0, y_0)} = \frac{3x_0}{y_0}$

Now, equation of required tangent is

$(y - y_0) = \frac{3x_0}{y_0}(x - x_0) \qquad \ldots(i)$

$\because$ (i) passes through $\left(\frac{4}{3},0\right)$

$\therefore (0 - y_0) = \frac{3x_0}{y_0}\left(\frac{4}{3} - x_0\right)$ $\Rightarrow - y_0^2 = 4x_0 - 3x_0^2$
(ii)

Also, $\because (x_0, y_0)$ lie on given curve $3x^2 - y^2 = 8$

$\Rightarrow 3x_0^2 - y_0^2 = 8 \Rightarrow y_0^2 = 3x_0^2 - 8$

Putting $y_0^2$ in (ii) we get

$-(3x_0^2 - 8) = 4x_0 - 3x_0^2$ $\Rightarrow 4x_0 = 8 \Rightarrow x_0 = 2$

$\therefore y_0 = \sqrt{3 \times 2^2 - 8} = \sqrt{4} = \pm 2$

Therefore, equations of required tangents are

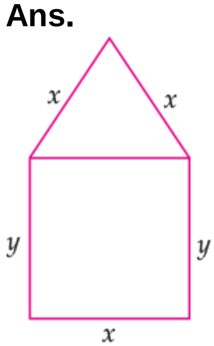
$(y - 2) = \frac{3 \times 2}{2}(x - 2)$ and $(y + 2) = \frac{3 \times 2}{-2}(x - 2)$

$\Rightarrow y - 2 = 3x - 6$ and $y + 2 =- 3x + 6$

$\Rightarrow 3x - y - 4 = 0$ and $3x + y - 4 = 0$

**Q.26. A window has the shape of a rectangle surmounted by an equilateral triangle. If the perimeter of the window is 12 m, find the dimensions of the rectangle that will produce the largest area of the window.**

**Ans.**



Let $x$ and $y$ be the dimensions of rectangular part of window and $x$ be side of equilateral part.

If $A$ be the total area of window, then $A = x.y + \frac{\sqrt{3}}{4}x^2$ ...$(i)$

Also, $x + 2y + 2x = 12$ $\Rightarrow 3x + 2y = 12$

$\Rightarrow y = \frac{12-3x}{2}$

$\therefore A = x.\frac{(12-3x)}{2} + \frac{\sqrt{3}}{4}x^2$ [From $(i)$]

$\Rightarrow A = 6x - \frac{3x^2}{2} + \frac{\sqrt{3}}{4}x^2$

$\Rightarrow A' = 6 - 3x + \frac{\sqrt{3}}{2}x$ [Differentiating with respect to $x$]

Now, for maxima or minima

$A' = 0 \Rightarrow 6 - 3x + \frac{\sqrt{3}}{2}x = 0 \Rightarrow x = \frac{12}{6-\sqrt{3}}$

Again $A'' = -3 + \frac{\sqrt{3}}{2} < 0$ (for any value of $x$)

$\Rightarrow A''\Big]_{x=\frac{12}{6-\sqrt{3}}} < 0$

*i.e.*, is maximum if $x = \frac{12}{6-\sqrt{3}}$ and $y = \frac{12-3\left(\frac{12}{6-\sqrt{3}}\right)}{2}$.

*i.e.*, For largest area of window, dimensions of rectangle are

$x = \frac{12}{6-\sqrt{3}}$ and $y = \frac{18-6\sqrt{3}}{6-\sqrt{3}}$.

**Q.27. Show that the normal at any point to the curve $x = a\cos\theta + a\theta\sin\theta, y = a\sin\theta - a\theta\cos\theta$ is at a constant distance from the origin.**

**Ans.**

Given $x = a\ \cos\theta + a\ \theta\sin\theta$

$y = a\sin\theta - \ a\ \theta\ \cos\theta$

$\therefore \quad \frac{\text{dx}}{d\theta} =- \ a\ \sin\theta + a\ (\theta\ \cos\theta + \ \sin\theta)$

$=- \ a\sin\theta + a\ \theta\ \cos\theta + a\sin\theta = a\ \theta\ \cos\theta$ and
$\frac{\text{dy}}{d\theta} = a\ \cos\theta - \ a(- \ \theta\sin\theta\ + \ \cos\theta)$

$= a\cos\theta + a\ \theta\sin\theta - \ a\cos\theta = a\ \theta\sin\theta$

$\Rightarrow \ \frac{\text{dy}}{\text{dx}} = \frac{\text{dy}/d\theta}{\text{dx}/d\theta} = \frac{a\ \theta\ \sin\theta}{a\ \theta\ \cos\theta} = \tan\theta$

$\therefore$ Slope of tangent at $\theta = \tan\theta$ $\quad\Rightarrow$ Slope of normal at $\theta =- \frac{1}{\tan\theta} =- \cot\theta$

Hence equation of normal at $\theta$ is

$\frac{y\ -\ (a\sin\theta\ -\ a\ \theta\ \cos\theta)}{x\ -\ (a\ \cos\theta + a\ \theta\ \sin\theta)} =- \ \cot\theta$

$\Rightarrow y - \ a\ \sin\theta + a\ \theta\ \cos\theta + x\cot\theta - \ \cot\theta\ (a\cos\theta + a\ \theta\sin\theta) = 0$

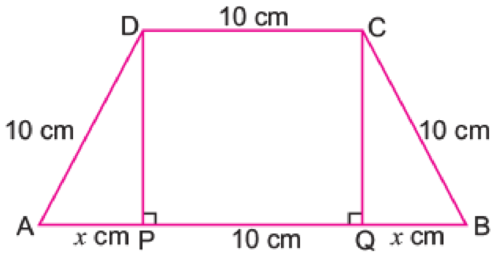
$\Rightarrow y - \ a\ \sin\theta + a\ \theta\ \cos\theta + x\frac{\cos\theta}{\sin\theta} - \ a\frac{\cos^2\theta}{\sin\theta} - \ a\theta\cos\theta = 0$

$\Rightarrow x\cos\theta + y\sin\theta - \ a = 0$

Distance from origin $(0,\ 0)$ to $(i) = \left|\frac{0.\cos\theta + 0.\sin\theta\ -\ a}{\sqrt{\cos^2\theta + \sin^2\theta}}\right| = a$

**Q.28. If length of three sides of a trapezium other than base are equal to 10 cm, then find the area of the trapezium when it is maximum.**

**Ans.**




The required trapezium is as given in figure. Draw perpendiculars $DP$ and $CQ$ on $AB$. Let $AP = x$ cm. Note that $\Delta APD \cong \Delta BQC$. Therefore, $QB = x$ cm. Also, by Pythagoras theorem $\mathrm{DP} = \mathrm{QC} = \sqrt{100 - x^2}$. Let $A$ be the area of the trapezium.

Then, $A \equiv A(x) = \frac{1}{2}$ (sum of parallel sides) × (height)

$$= \frac{1}{2}(2x + 10 + 10)(\sqrt{100 - x^2}) = (x + 10)\sqrt{100 - x^2}$$

or $A'(x) = (x + 10)\frac{(-2x)}{2\sqrt{100 - x^2}} + (\sqrt{100 - x^2})$

$$= \frac{-2x^2 - 10x + 100}{\sqrt{100 - x^2}}$$

Now $A'(x) = 0$ gives $2x^2 + 10x - 100 = 0$, *i.e.*, $x = 5$ and $x = -10$

Since $x$ represents distance, it cannot be negative.

So, $x = 5$.

$$A''(x) = \frac{\sqrt{100-x^2}(-4x-10)-(-2x^2-10x+100)\frac{(-2x)}{2\sqrt{100-x^2}}}{100-x^2}$$

$$= \frac{2x^3-300x-1000}{(100-x^2)^{\frac{3}{2}}} \quad \text{(on simplification)}$$

or $A''(5) = \frac{2(5)^3-300(5)-1000}{(100-(5)^2)^{\frac{3}{2}}} = \frac{-2250}{75\sqrt{75}} = \frac{-30}{\sqrt{75}} < 0$

Thus, area of trapezium is maximum at $x = 5$ and the maximum area is given by

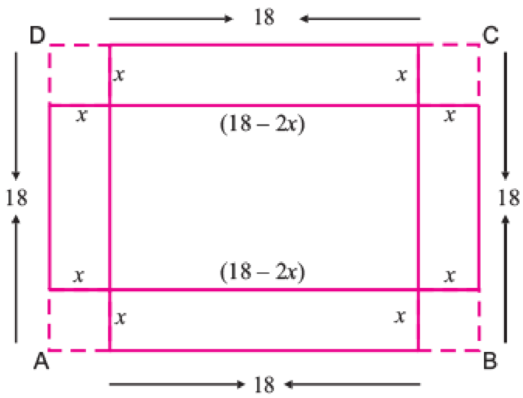
$$A(5) = (5 + 10)\sqrt{100 - (5)^2} = 15\sqrt{75} = 75\sqrt{3}\text{cm}^2$$

## Long Answer Questions-II (OIQ)

### [6 Marks]

**Q.1. A square piece of tin of side 18 cm is to be made into a box without top by cutting a square from each corner and folding up the flaps to form a box. Find the maximum volume of the box.**

**Ans.**




Let $x$ be the side of the square which is to be cut off from each corner.

Then dimensions of the box are $18 - 2x$, $18 - 2x$ and $x$.

Let $V$ be the volume of the box then

$$V = x(18 - 2x)(18 - 2x) = x(18 - 2x)^2$$

$$\Rightarrow V' = x\frac{d}{dx}(18 - 2x)^2 + (18 - 2x)^2\frac{d}{dx}x$$

$$= 2x(18 - 2x)\ (-2) + (18 - 2x)^2$$

$$= (18 - 2x)\ [- 4x + 18 - 2x]$$

$$\Rightarrow V' = (18 - 2x)\ (18 - 6x)$$

Also, $V'' = (18-2x)\frac{d}{dx}(18-6x) + (18-6x)\frac{d}{dx}(18-2x)$

$\Rightarrow V' = (18-2x)(-6) + (18-6x)(-2)$

For maximum or minimum value $V' = 0$

$\Rightarrow (18-2x)\ (18-6x) = 0 \Rightarrow x = 9$ or $x = 3$

Neglecting $x = 9$ $[\because$ For $x = 9$, length $= 18 - 2x = 18 - 2(9) = 0]$

Therefore, $x = 3$ is to be taken.

$V'(3) = (18-6)(-6) + (18-18)(-2) = 72 < 0$

Thus, volume is maximum when $x = 3$

$\therefore$ Length $= 18 - 2x = 18 - 6 = 12$ cm; Breadth $= 18 - 2x = 18 - 6 = 12$ cm; Height $= x = 3$ cm

Maximum volume of the box $= 12 \times 12 \times 3 = 432\ \text{cm}^3$.

**Q.2. A wire of length 28 cm is to be cut into two pieces. One of the two pieces is to be made into a square and the other into a circle. What should be the length of two pieces so that the combined area of them is minimum?**

**Ans.**

Let the length of one piece be $x$ cm, then the length of other piece will be $(28 - x)$ cm.

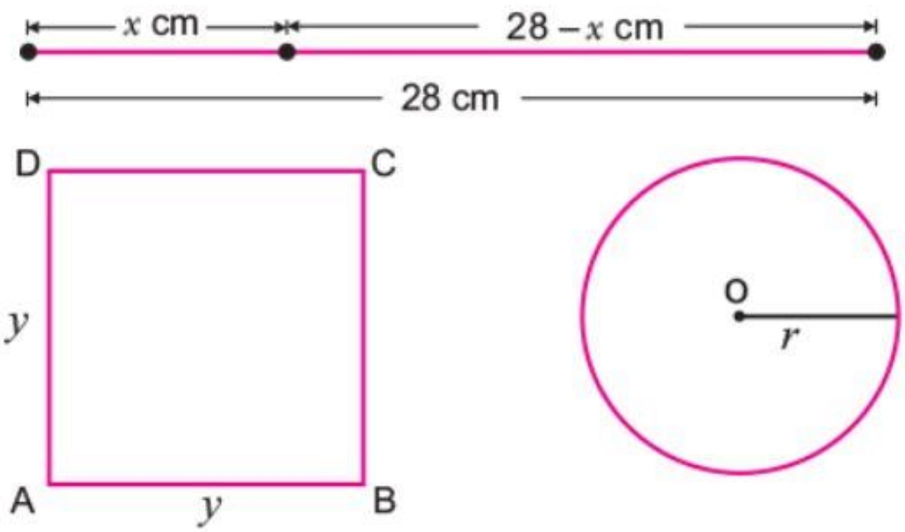
Let from the first piece we make a circle of radius $r$ and from the second piece we make a square of side $y$.

Then $\begin{cases} 2\pi r = x \Rightarrow r = \frac{x}{2\pi} \\ 4y = 28 - x \Rightarrow y = \frac{(28-x)}{4} \end{cases}$ ...$(i)$

Let $A$ be the combined area of the circle and square then

$A = \pi r^2 + y^2 \quad \Rightarrow A = \pi\left(\frac{x}{2\pi}\right)^2 + \left(\frac{28-x}{4}\right)^2 \qquad ...(ii)$

Differentiating $(ii)$ with respect to '$x$', we get




$A' = \frac{\pi}{4\pi^2}(2x) + \frac{1}{16}(2)(28-x)(-1)$

$\Rightarrow A' = \frac{x}{2\pi} - \frac{1}{8}(28 - x)$ and

$A' = \frac{1}{2\pi} - \frac{1}{8}(0 - 1) = \frac{1}{2\pi} + \frac{1}{8}$

For maximum and minimum $A' = 0$

$\Rightarrow \frac{x}{2\pi} - \frac{28}{8} + \frac{x}{8} = 0$

$\Rightarrow x\left(\frac{1}{2\pi} + \frac{1}{8}\right) = \frac{28}{8}$

$\Rightarrow \quad x\left(\frac{4+\pi}{8\pi}\right) = \frac{28}{8} \qquad \Rightarrow \quad x = \frac{28\pi}{4+\pi}$

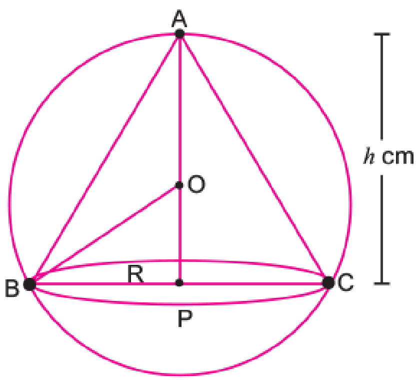
Since, $A' = +\text{ve}$ for $x = \frac{28\pi}{4+\pi}$ $\therefore A$ is min for $x = \frac{28\pi}{4+\pi}$

Thus, the required length of two pieces are

$$x = \frac{28\pi}{4+\pi}\text{cm and } 28 - x$$
$$= 28 - \frac{28\pi}{4+\pi} = \frac{192}{4+\pi}\text{cm}$$

**Q.3. Show that the height of the cone of maximum volume that can be inscribed in a sphere of radius 12 cm is 16 cm.**

**Ans.**




Let $O$ be the centre of a sphere of radius 12 cm and a cone $ABC$ of radius $R$ cm and height $h$ cm is inscribed in the sphere.

$AP = AO + OP \quad \Rightarrow h = 12 + OP \quad \Rightarrow \quad OP = (h - 12)$

Now in right angle $\Delta OBP$, by Pythagoras theorem, we get

$BO^2 = BP^2 + OP^2$

$$(12)^2 = R^2 + (h-12)^2 \Rightarrow 144 = R^2 + h^2 + 144 - 24h \Rightarrow R^2 = 24h - h^2$$

Volume of cone,

$$\Rightarrow \quad V = \frac{1}{3}\pi R^2 h = \frac{1}{3}\pi(24h - h^2)h$$

$$V = \frac{1}{3}\pi\,(24h^2 - h^3)$$

$$\Rightarrow \quad \frac{dV}{dh} = \frac{\pi}{3}(48h - 3h^2)$$

For maximum/minimum value of $V$, we have

$$\frac{dV}{dh} = 0 \Rightarrow \quad 48h - 3h^2 = 0$$

$$\Rightarrow \quad h(48 - 3h) = 0 \Rightarrow \quad \text{either } h = 0 \text{ or } h = 16$$

But height of cone cannot be zero.

Therefore $h = 16$ cm.

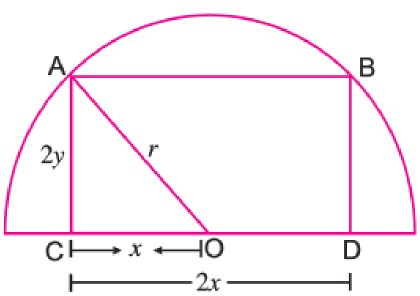
$$\text{Now, } \frac{d}{dh}\left(\frac{dV}{dh}\right) = \frac{\pi}{3}(48 - 6h)$$

$$\Rightarrow \quad \left(\frac{d^2V}{dh^2}\right)_{h=16} = \frac{\pi}{3}(48 - 6 \times 16) = -16\pi < 0$$

Hence, volume of cone is maximum when $h = 16$ cm.

**Q.4. A rectangle is inscribed in a semi-circle of radius *r* with one of its sides on diameter of semi-circle. Find the dimensions of the rectangle so that its area is maximum. Find the area also.**

**Ans.**



Let $ABCD$ be the rectangle which is inscribed in a semi-circle with centre $O$ and radius $r$.

Assume that the length of rectangle is $2x$ and breadth is $2y$.

$\Rightarrow \quad CD = AB = 2x$

In right angle $\Delta ACO$,

We get $x^2 + 4y^2 = r^2$ (by Pythagoras theorem) $\quad ...(i)$

Now area of rectangle $ABCD$ is given by,

$A = \text{length} \times \text{breadth} \quad \Rightarrow \quad A = 2x \times 2y = 4xy$

$\Rightarrow A = 4x\sqrt{\frac{(r^2 - x^2)}{4}} \qquad$ [from equation $(i)$]

$\Rightarrow \quad A^2 = 16x^2 \frac{(r^2 - x^2)}{4}$

Let $Z = A^2 = 4\left(r^2x^2 - x^4\right) \quad ...(ii)$

Then $Z$ is maximum or minimum according as $A$ is maximum or minimum.

Differentiating equation $(ii)$ with respect to $x$, we get

$\frac{\mathrm{dZ}}{\mathrm{dx}} = 4[r^2 .\, 2x - \, 4x^3]$

For maximum or minimum value of $Z$, we have $\frac{dZ}{dx} = 0 \; \Rightarrow \; 8x(r^2 - 2x^2) = 0$

$\Rightarrow \; r^2 - 2x^2 = 0 \qquad (\because x \text{ cannot be zero}) \quad \Rightarrow \quad x^2 = \frac{r^2}{2}$

Now, $\frac{d^2Z}{\mathrm{dx}^2} = 4[2r^2 - 12x^2]$

$$\frac{d^2Z}{dx^2}\left(\text{at } x=\frac{r}{\sqrt{2}}\right) = 4\left[2r^2 - 12 \times \frac{r^2}{2}\right]$$
$$= -16r^2 < 0 \text{ at } x = \frac{r}{\sqrt{2}}$$

Thus area will be maximum when $x = \frac{r}{\sqrt{2}} \quad \Rightarrow \quad 2x = \sqrt{2}\,r,$

Putting $\quad x = \frac{r}{\sqrt{2}}$ in equation $(i)$, we obtain $y = \frac{r}{2\sqrt{2}}$

$\Rightarrow \quad 2y = \frac{r}{\sqrt{2}}$

Now, $A = 2x \times 2y = \sqrt{2}r \times \frac{r}{\sqrt{2}} = r^2$ .

Hence, the dimensions are $\sqrt{2}r$ and $\frac{r}{\sqrt{2}}$ and area = $r^2$ sq units.

**Q.5. A wire of length 36 cm is cut into two pieces, one of the pieces is turned in the form of a square and other in the form of an equilateral triangle. Find the length of each piece so that the sum of the areas of the two be minimum.**

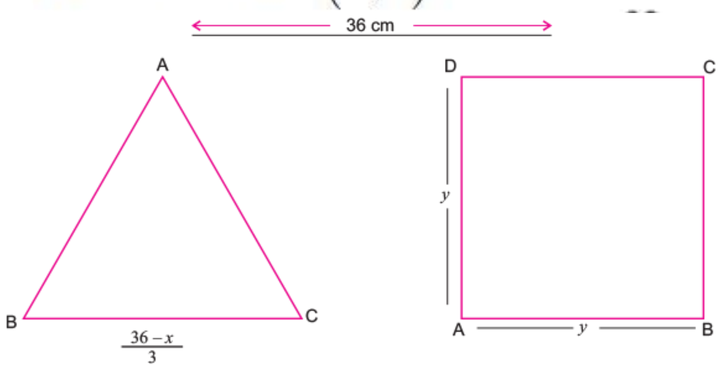
**Ans.**

Let the length of one piece be $x$, then the length of the other piece will be $36 - x$.

Let from first piece we make the square, then

$x = 4y \quad \Rightarrow \quad y = \frac{x}{4}$ , where $y$ is the side of the square ...$(i)$

From the second piece of length $(36 - x)$ we make an equilateral triangle, then side of the equilateral triangle $= \left(\frac{36-x}{3}\right)$



Now combined area of the two = $A = \left(\frac{x}{4}\right)^2 + \frac{\sqrt{3}}{4}\left(\frac{36-x}{3}\right)^2$

Differentiating with respect to $x$, we have

$$\Rightarrow \quad \frac{dA}{dx} = \frac{2x}{4}.\frac{1}{4} + \frac{\sqrt{3}}{4}.2.\frac{(36-x)}{3}.\left(-\frac{1}{3}\right)$$

$$\frac{dA}{dx} = \frac{x}{8} - \frac{\sqrt{3}}{18}(36-x) \qquad ...(ii)$$

For maximum/minimum, we have $\frac{dA}{dx} = 0$

$$\Rightarrow \quad \frac{x}{8} - \frac{\sqrt{3}}{18}(36-x) = 0$$

$$\Rightarrow \quad \frac{x}{8} = \frac{\sqrt{3}}{18}.(36-x) \quad \Rightarrow \quad \frac{x}{8} = 2\sqrt{3} - \frac{1}{6\sqrt{3}}x$$

$$\Rightarrow \quad x\left(\frac{1}{8} + \frac{1}{6\sqrt{3}}\right) = 2\sqrt{3} \quad \Rightarrow \quad x\left(\frac{3\sqrt{3}+4}{24\sqrt{3}}\right) = 2\sqrt{3}$$

$$\Rightarrow \quad x(4+3\sqrt{3}) = 144 \quad \Rightarrow \quad x = \frac{144}{4+3\sqrt{3}}$$

Thus, length of one piece is $x = \frac{144}{4+3\sqrt{3}}$ and the length of other piece is

$$36 - \frac{144}{(4+3\sqrt{3})} = \frac{144+108\sqrt{3}-144}{(4+3\sqrt{3})}$$

$$= \frac{108\sqrt{3}}{(4+3\sqrt{3})}\,\mathrm{cm}$$