10

Definite Integral

10.01 Definite Integral

The definite integral is a powerful tool in mathematics, physics, mechanics, and other disciplines. Calculation of areas bounded by curves of arc lengths, volumes, work, velocity, path length, moments of inertia and so forth reduce to the evaluations of a definite integral. The definite integral has a unique value. A definite integrals is given by a function $f(x)$ in the interval $[a, b]$ and denoted by $\int_a^b f(x)dx$ where a is called the lower limit of the integral and b is called the upper limit of the integral. The definite integral is introduced either as the limit of a sum or if it has an anti derivative F in the interval $[a, b]$, then its value is the difference between the values of F at the end points, i.e., $F(b) - F(a)$.

(i) Definite Integral as a limit of a sum

(ii) Fundamental theorem of Integral Calculus

(iii) To find the value of common definite Integral

(iv) Basic properties of definite Integral

10.02 Definite integral as a limit of sum

In a series if the number of terms approaches to infinity and each term approches to zero, then definite integral is defined as limit of sum.

Definition : Let $f(x)$ be a continuous function defined on close interval $[a, b]$ and interval $[a, b]$ is divided into n equal parts by the points $a+h, a+2h, a+3h, ..., a+(n-1)h$ (where h is the length of each part), then

$$\int_a^b f(x)dx = \lim_{h\to 0}\left[h\{f(a)+f(a+h)+...+f(a+\overline{n-1}h)\}\right] \text{ (where } n\to\infty \text{ and } nh=b-a\text{)}$$

$$= \lim_{h\to 0}\left[h\{f(a+h)+f(a+2h)+...+f(a+nh)\}\right]$$

This method of finding the definite Integral is called ab-initio method.

Proof : Let $f(x)$ be real and continuous function in the interval $[a, b]$

Dividing the interval $[a, b]$ into n equal sub-intervals with h width $AA_n = OA_n - OA$

or $$AA_1 + A_1A_2 + A_2A_3 + ... + A_{n-1}A_n = b-a$$

or $$\underbrace{h+h+h+....+h}_{n \text{ times}} = b-a$$

or $$nh = b-a \qquad \Rightarrow h = \frac{b-a}{n}$$

let $y = f(x)$ when $x = a$, $y = f(a)$

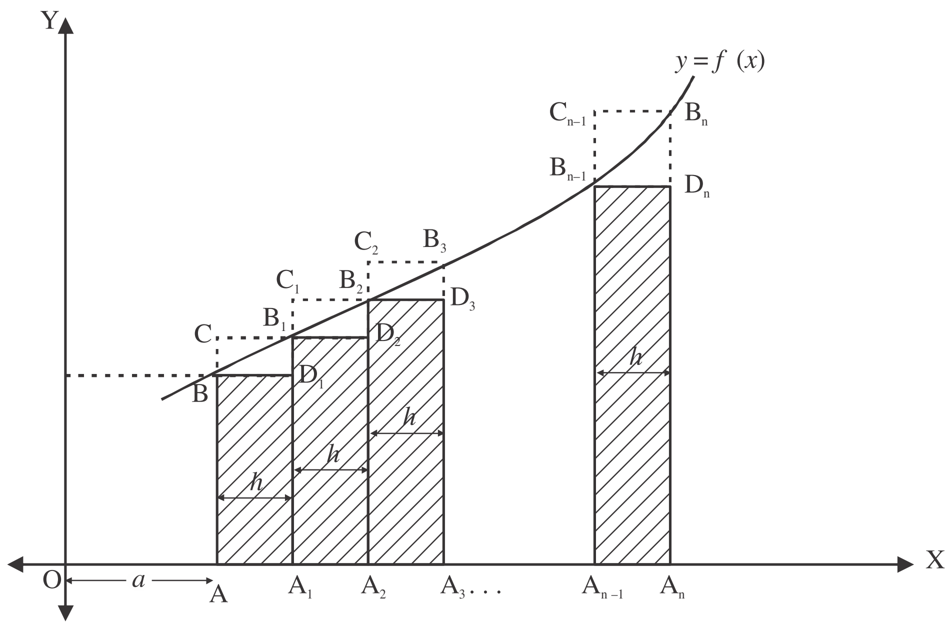
$\therefore$ According to figure, coordinates of B will be $(a, f(a))$

i.e. $AB = f(a)$

similarly

$$A_1B_1 = f(a+h),\ A_2B_2 = f(a+2h), \ldots,\ A_nB_n = f(a+nh)$$

Let the area of rectangular blocks below the curve in the given figure be Δ_1 then–



$$\Delta_1 = \text{Rectangle } AA_1D_1B + \text{Rectangle } A_1A_2D_2B_1 + \ldots. + \text{Rectaangle } A_{n-1}A_nD_nB_{n-1}$$

$$= AB \times AA_1 + A_1B_1 \times A_1A_2 + \ldots. + A_{n-1}B_{n-1} \times A_{n-1}A_n$$

$$= f(a) \times h + f(a+h) \times h + f(a+2h) \times h + \ldots. + f(a+\overline{n-1}h) \times h$$

$$= h\left[f(a) + f(a+h) + f(a+2h) + \ldots. + f(a+\overline{n-1}h)\right]$$

and if we denote $y = f(x)$, x - axis and two ordinates $x = a$, $x = b$ and the area bounded by AA_nB_nBA Δ then the vlaue of Δ_1 will be less then Δ again let

$$\Delta_2 = \text{Rectangle } AA_1B_1C + \text{ Rectangle } A_1A_2B_2C_1 + \ldots. + \text{ Rectangle } A_{n-1}A_nB_nC_{n-1}$$

$$= A_1B_1 \times AA_1 + A_2B_2 \times A_1A_2 + \ldots. + A_nB_n \times A_{n-1}A_n$$

$$= f(a+h) \times h + f(a+2h) \times h + \ldots. + f(a+nh) \times h$$

$$= h\left[f(a+h) + f(a+2h) + \ldots. + f(a+nh)\right]$$

This area will be greater then Δ therefore the vlaue of Δ will be greater than Δ_1 and less than Δ_2 i.e.

$$\Delta_1 < \Delta < \Delta_2$$

again $$\Delta_2 - \Delta_1 = h\,f(a+nh) - h\,f(a)$$

$$= h\left[f(b) - f(a)\right] \qquad (\because a + nh = b)$$

clearly as the rectangular strips become narrower and narrower, h will be minimum and $h \to 0$ then the value of Δ_1 and Δ_2 will be close to Δ

i.e. $\lim_{h\to 0} \Delta_1 = \lim_{h\to 0} \Delta_2 = \Delta$

$\therefore$ $$\Delta = \int_a^b f(x)\,dx = \lim_{h\to 0} h\left[f(a) + f(a+h) + ... + f(a+\overline{n-1}h)\right]$$

$\Rightarrow$ $$\Delta = \int_a^b f(x)dx = \lim_{h\to 0} h\left[f(a+h) + f(a+2h) + ... + f(a+nh)\right]$$

Conclusion : Definite Integral can be expressed as a limit of a sum

NOTE : We can define the formula as

(i) $$\int_a^b f(x)\,dx = \lim_{n\to\infty} h\left[f(a) + f(a+h) + ... + f(a+\overline{n-1}h)\right],$$

where $h = \dfrac{b-a}{n}$ clearly $n \to \infty$ then $h \to 0$

(ii) $$\int_a^b f(x)dx = \lim_{h\to 0} h\left[f(a+h) + f(a+2h) + ... + f(a+nh)\right], \text{ where } h = \frac{b-a}{n}$$

Any of the above given formula can be used to find the integration.

Some Important Results:

(i) $\sum r = 1 + 2 + 3 + ... + n = \dfrac{n(n+1)}{2}$

(ii) $\sum r^2 = 1^2 + 2^2 + 3^2 + ... + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

(iii) $\sum r^3 = 1^3 + 2^3 + 3^3 + ... + n^3 = \left[\dfrac{n(n+1)}{2}\right]^2$

(v) $\sum (2r-1) = 1 + 3 + 5 + ... + (2n-1) = n^2$

(vi) $a + (a+d) + (a+2d) + ... + (a+\overline{n-1}\,d) = \dfrac{n}{2}\left[2a + (n-1)d\right]$

(vii) $a + ar + ar^2 + ... + ar^{n-1} = \dfrac{a(r^n - 1)}{(r-1)}, r \neq 1$

Illustrative Examples

Example 1. Find $\int_0^2 (2x+1)\,dx$ as the limit of a sum.

Solution : By definition $$\int_a^b f(x)\,dx = \lim_{h\to 0} h\left[f(a+h) + f(a+2h) + f(a+3h) + ... + f(a+nh)\right],$$

where $nh = b - a$

$$a = 0,\ b = 2,\ f(x) = 2x + 1,\ \ nh = 2 - 0 = 2$$

$$\therefore \quad \int_0^2 (2x+1)dx = \lim_{h\to 0} h\left[f(0+h) + f(0+2h) + f(0+3h) + ... + f(0+nh)\right]$$

$$= \lim_{h\to 0} h\left[f(h) + f(2h) + f(3h) + ... + f(nh)\right]$$

$$= \lim_{h\to 0} h\left[(2h+1) + (4h+1) + (6h+1) + ... + (2nh+1)\right]$$

$$= \lim_{h\to 0} h\left[(2h + 4h + 6h + ... + 2nh) + (1+1+1+...+n \text{ बार})\right]$$

$$= \lim_{h\to 0} h\left[2h(1+2+3+...+n) + n\right]$$

$$= \lim_{h\to 0} h\left[2h\frac{n(n+1)}{2} + n\right] = \lim_{h\to 0}\left[h^2 n(n+1) + nh\right]$$

$$= \lim_{h\to 0} h\left[nh(nh+h) + nh\right] = \lim_{h\to 0}\left[2(2+h) + 2\right] \qquad (\because nh = 2)$$

$$= \left[2(2+0) + 2\right] = 4 + 2 = 6.$$

Example 2. Find $\int_{-1}^{1} e^x\, dx$ as the limit of a sum.

Solution : Here $f(x) = e^x, \quad a = -1, \quad b = 1 \qquad (\therefore nh = 1 + 1 = 2)$

$$\int_{-1}^{1} e^x dx = \lim_{h\to 0} h\left[f(-1+h) + f(-1+2h) + f(-1+3h) + ... + f(-1+nh)\right]$$

$$= \lim_{h\to 0} h\left[e^{-1+h} + e^{-1+2h} + e^{-1+3h} + ... + e^{-1+nh}\right]$$

$$= \lim_{h\to 0} h\left[e^{-1}.e^{h} + e^{-1}.e^{2h} + e^{-1}.e^{3h} + ... + e^{-1}.e^{nh}\right]$$

$$= \lim_{h\to 0} he^{-1}\left[e^{h} + e^{2h} + e^{3h} + ... + e^{nh}\right]$$

$$= \frac{1}{e}\lim_{h\to 0} h.e^{h}.\frac{(e^h)^n - 1}{e^n - 1}$$

$$= \frac{1}{e}\lim_{h\to 0} e^{h}.h\frac{e^{nh} - 1}{e^h - 1} = \frac{1}{e}\lim_{h\to 0} he^{h}\,\frac{e^2 - 1}{e^h - 1} \qquad [\because nh = 2]$$

$$= \frac{e^2 - 1}{e}\lim_{h\to 0} e^{h}.\lim_{h\to 0}\frac{h}{e^h - 1} = \left(e - 1/e\right)e^{o}.\lim_{h\to 0}\frac{1}{\left((e^h - 1)/h\right)}$$

$$= \left(e - \frac{1}{e}\right)\times 1 \times \frac{1}{1} = e - \frac{1}{e}.$$

Example 3. Find $\int_0^1 x^2\, dx$ as the limit of a sum.

Solution : Here $f(x)=x^2,\ a=0,\ b=1$ $\quad\therefore\quad nh=b-a=1-0=1$

$$\therefore \quad \int_0^1 x^2\, dx=\lim_{h\to 0} h\left[f(0+h)+f(0+2h)+f(0+3h)+\ldots+f(0+nh)\right]$$

$$=\lim_{h\to 0} h\left[f(h)+f(2h)+f(3h)+\ldots+f(nh)\right]$$

$$=\lim_{h\to 0} h[h^2+4h^2+9h^2+\ldots+n^2h^2]$$

$$=\lim_{h\to 0} h.h^2[1^2+2^2+3^2+\ldots+n^2]$$

$$=\lim_{h\to 0} h^3\frac{n(n+1)(2n+1)}{6}$$

$$=\lim_{h\to 0}\frac{nh(nh+h)(2nh+h)}{6}$$

$$=\lim_{h\to 0}\frac{1(1+h)(2\times 1+h)}{6}$$

$$=\frac{(1+0)(2+0)}{6}=\frac{2}{6}=\frac{1}{3}.$$

Exercise 10.1

Evaluate the following definite integrals as a limit of sums

1. $\int_3^5 (x-2)\,dx$
2. $\int_a^b x^2\, dx$
3. $\int_1^3 (x^2+5x)\,dx$
4. $\int_a^b e^{-x}dx$
5. $\int_0^2 (x+4)\,dx$
6. $\int_1^3 (2x^2+5)\,dx$

10.03 Fundamental theorem of integral calculus

Statement : If $f(x)$ is a continuous function defined on an interval $[a, b]$ and

$\frac{d}{dx}[F(x)]=f(x)$, i.e., the anti derivative of $f(x)$ is $F(x)$ then

$$\int_a^b f(x)dx=[F(x)]_a^b=F(b)-F(a)$$

$$=\lim_{h\to 0} h\left[f(a+h)+f(a+2h)+\ldots+f(a+nh)\right],\qquad h=\frac{b-a}{n}$$

where $F(b)-F(a)$, gives the value of the definite integral and it is unique.

10.04 Definition

If $f(x)$ is a continuous function defined on an interval $[a, b]$ and the integration of $f(x)$ is $F(x)$ then

$$\int_a^b f(x)\,dx=\left[F(x)\right]_a^b=F(b)-F(a),$$

where a is called the lower limit of the integral and b is called the upper limit of the integral. The definite integrals is introduced either as the limit of a sum or if it has an anti derivative $F(x)$ in the interval $[a, b]$, then its value is the difference between the values of $F(x)$ at the end points, i.e. $F(b) - F(a)$.

10.05 To Find the value of definite integrals

To find the definite Integral, firstly we find the integration by the known method and then the limits are substituted in place of variable. The following examples show the procedure:-

(i) $$\int_0^{\pi/2} \cos x \, dx = \left[\sin x\right]_0^{\pi/2} = \sin\frac{\pi}{2} - \sin 0 = 1 - 0 = 1$$

(ii) $$\int_1^2 x^3 \, dx = \left[\frac{x^4}{4}\right]_1^2 = \frac{2^4}{4} - \frac{1^4}{4} = 4 - \frac{1}{4} = \frac{15}{4}.$$

(iii) $$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \left[\sin^{-1} x\right]_0^1 = \sin^{-1}(1) - \sin^{-1}(0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

We can find the vlaue of definite integral by the methods used to solve the indefinite integral, usually the methods are used:

(i) Using standard formula
(ii) Substitution
(iii) Partial fraction
(iv) Integration by Parts

10.06 Evaluation of definite integral by substitution

To evaluate $\int_a^b f(x)$, by substitution, the steps could be as follows:

(i) Consider the integral without limits and substitute, the independent variable (say x) with new variable t to convert the given integral to a known form.

(ii) Integrate the new integrand with respect to the new variable t without mentioning the constant of integration.

(iii) Resubstitute for the new variable and write the integration in terms of the original variable and solve it for given limit.

Illustrative Examples

Example 4. Evaluate the following definite integrals

(i) $\int_{-1}^{2} \frac{dx}{3x-2}$ (ii) $\int_{\pi/4}^{\pi/2} \frac{dx}{1-\cos 2x}$ (iii) $\int_0^{\infty} \frac{\sin(\tan^{-1} x)}{1+x^2} dx$ (iv) $\int_0^1 \frac{2x}{1+x^4} dx.$

Solution : (i) Let $$I = \int_{-1}^{2} \frac{dx}{(3x-2)} = \frac{1}{3}[\log|3x-2|]_{-1}^{2} = \frac{1}{3}[\log 4 - \log|-5|]$$

$$= \frac{1}{3}\left[\log 4 - \log 5\right] = \frac{1}{3}\log\frac{4}{5}.$$

(ii) Let $I = \int_{\pi/4}^{\pi/2} \frac{dx}{1-\cos 2x} = \int_{\pi/4}^{\pi/2} \frac{dx}{2\sin^2 x} = \frac{1}{2}\int_{\pi/4}^{\pi/2} \operatorname{cosec}^2 x\, dx$

$$= \frac{1}{2}[-\cot x]_{\pi/4}^{\pi/2} = \frac{1}{2}[-\cot \pi/2 + \cot \pi/4] = \frac{1}{2}[0+1] = \frac{1}{2}$$

(iii) Let $I = \int_0^{\infty} \frac{\sin\left(\tan^{-1} x\right)}{1+x^2} dx$

Let $\tan^{-1} x = t \Rightarrow \frac{1}{1+x^2} dx = dt$ when $x=0$ then $t=0$; $x=\infty$, $t=\pi/2$

$\therefore$ $I = \int_0^{\pi/2} \sin t\, dt = [-\cos t]_0^{\pi/2} = -\cos \pi/2 + \cos 0 = 0+1 = 1.$

(iv) Let $I = \int_0^1 \frac{2x}{1+x^4} dx$, Let $x^2 = t \Rightarrow 2x\, dx = dt$

when $x=0$ then $t=0$; $x=1$, $t=1$

$\therefore$ $I = \int_0^1 \frac{dt}{1+t^2} = [\tan^{-1} t]_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}.$

Example 5. Evaluate the following definite integrals.

(i) $\int_0^{\pi/4} (2\sec^2 x + x^3 + 1) dx$ (ii) $\int_0^1 \frac{e^x}{1+e^{2x}} dx$ (iii) $\int_0^1 x\, e^x dx$

Solution : (i) Let $I = \int_0^{\pi/4} (2\sec^2 x + x^3 + 1)\, dx$

$$= \left[2\tan x + \frac{x^4}{4} + x\right]_0^{\pi/4} = \left[2\tan\frac{\pi}{4} + \frac{1}{4}\left(\frac{\pi}{4}\right)^4 + \frac{\pi}{4}\right] - (0+0+0)$$

$$= 2\times 1 + \frac{1}{4}\times\frac{\pi^4}{256} + \frac{\pi}{4} = 2 + \frac{\pi^4}{1024} + \frac{\pi}{4}.$$

(ii) Let $I = \int_0^1 \frac{e^x}{1+e^{2x}} dx$ Let $e^x = t \Rightarrow e^x dx = dt$

when $x=0$ then $t = e^0 = 1$

when $x=1$ then $t = e^1 = e$

$\therefore$ $I = \int_1^e \frac{dt}{1+t^2} = [\tan^{-1} t]_1^e = \tan^{-1} e - \tan^{-1}(1) = \tan^{-1} e - \frac{\pi}{4}$

(iii) Let $I = \int_0^1 \underset{\text{I}}{x}\, \underset{\text{II}}{e^x}\, dx$ (Integralsting by parts taking e^x as second function)

$$= [xe^x]_0^1 - \int_0^1 1\times e^x dx = [1.e^1 - 0] - [e^x]_0^1$$

$$= e - [e^1 - e^0] = e - e + e^0 = e^0 = 1$$

Example 6. Evaluate the following definite integrals:

(i) $\int_0^{\pi/2} \frac{\cos x\, dx}{(1+\sin x)(2+\sin x)}$ (ii) $\int_1^e e^x \left(\frac{1+x\log x}{x}\right) dx$

Solution : (i) Let $I = \int_0^{\pi/2} \frac{\cos x\, dx}{(1+\sin x)(2+\sin x)}$ Let $\sin x = t \quad \therefore \cos x\, dx = dt$

when $x = 0, t = 0$ and when $x = \pi/2,\ t = 1$

$$\therefore \quad I = \int_0^1 \frac{dt}{(1+t)(2+t)} = \int_0^1 \left[\frac{1}{1+t} - \frac{1}{2+t}\right] dt$$

$$= [\log|(1+t)| - \log|2+t|]_0^1$$

$$= \left[\log\left|\frac{1+t}{2+t}\right|\right]_0^1 = \log\frac{2}{3} - \log\frac{1}{2} = \log\left(\frac{2}{3}\times\frac{2}{1}\right) = \log\frac{4}{3}.$$

(ii) Let $I = \int_1^e e^x \left(\frac{1+x\log x}{x}\right) dx$

$$= \int_1^e e^x \left[\frac{1}{x} + \log x\right] dx$$

$$= \left[e^x \log x\right]_1^e \qquad \left[\because \int e^x[f(x) + f'(x)]dx = e^x f(x)\right]$$

$$= e^e \log e - e^1 \log 1 = e^e \times 1 - e \times 0 = e^e$$

Example 7. Evaluate the following definite integrals.

(i) $\int_0^{\pi/4} \frac{\sin 2x}{\sin^4 x + \cos^4 x} dx$ (ii) $\int_a^{\infty} \frac{dx}{x^4\sqrt{a^2+x^2}}$

Solution : (i) Let $I = \int_0^{\pi/4} \frac{\sin 2x}{\sin^4 x + \cos^4 x} dx$

$$= \int_0^{\pi/4} \frac{2\sin x \cos x}{\sin^4 x + \cos^4 x} dx$$

Dividing Nr and Dr by $\cos^4 x$, we get

$$I = \int_0^{\pi/4} \frac{2\tan x \sec^2 x}{1+\tan^4 x} dx$$

Let $\tan^2 x = t \Rightarrow 2\tan x \sec^2 x\, dx = dt$

as $x = 0$ then $t = 0$ and when $x = \pi/4$ then $t = 1$

$$\therefore \quad I = \int_0^1 \frac{dt}{1+t^2} = [\tan^{-1} t]_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

(ii) $$I = \int_a^{\infty} \frac{dx}{x^4\sqrt{a^2+x^2}}$$

Let $x = a\tan\theta \Rightarrow dx = a\sec^2\theta d\theta$

when $x = a$ then $\theta = \pi/4$ and $x = \infty$ then $\theta = \pi/2$

$$\therefore \quad I = \int_{\pi/4}^{\pi/2} \frac{a\sec^2\theta d\theta}{a^4\tan^4\theta\sqrt{a^2 + a^2\tan^2\theta}}$$

$$= \int_{\pi/4}^{\pi/2} \frac{a\sec^2\theta d\theta}{a^4\tan^4\theta \times a\sec\theta}$$

$$= \int_{\pi/4}^{\pi/2} \frac{\sec\theta d\theta}{a^4\tan^4\theta} = \frac{1}{a^4}\int_{\pi/4}^{\pi/2} \frac{1/\cos\theta}{\sin^4\theta/\cos^4\theta} d\theta$$

$$= \frac{1}{a^4}\int_{\pi/4}^{\pi/2} \frac{\cos^3\theta}{\sin^4\theta} d\theta = \frac{1}{a^4}\int_{\pi/4}^{\pi/2} \frac{(1-\sin^2\theta)\cos\theta d\theta}{\sin^4\theta} d\theta$$

Let $\sin\theta = t \Rightarrow \cos\theta d\theta = dt$

as $\theta = \pi/4$ then $t = 1/\sqrt{2}$ and $\theta = \pi/2$ then $t = 1$

$$\therefore \quad I = \frac{1}{a^4}\int_{1/\sqrt{2}}^{1} \frac{(1-t^2)dt}{t^4} = \frac{1}{a^4}\int_{1/\sqrt{2}}^{1} \left(\frac{1}{t^4} - \frac{1}{t^2}\right) dt$$

$$= \frac{1}{a^4}\left[-\frac{1}{3t^3} + \frac{1}{t}\right]_{1/\sqrt{2}}^{1} = \frac{1}{a^4}\left[\left(-\frac{1}{3} + \frac{1}{1}\right) - \left(-\frac{1}{3\times 1/2\sqrt{2}} + \frac{1}{1/\sqrt{2}}\right)\right]$$

$$= \frac{1}{a^4}\left[\frac{2}{3} - \left(-\frac{2\sqrt{2}}{3} + \sqrt{2}\right)\right] = \frac{1}{a^4}\left[\frac{2}{3} + \frac{2\sqrt{2}}{3} - \sqrt{2}\right]$$

$$= \frac{1}{a^4}\left[\frac{2 + 2\sqrt{2} - 3\sqrt{2}}{3}\right] = \frac{1}{a^4}\left(\frac{2-\sqrt{2}}{3}\right) = \frac{2-\sqrt{2}}{3a^4}$$

Example 8. Evaluate $\int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$

Solution : Let $I = \int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$

Dividing Nr and Dr by $\cos^2 x$, we get

$$I = \int_0^{\pi/2} \frac{\sec^2 x\, dx}{a^2 + b^2\tan^2 x}$$

Let $b\tan x = t \Rightarrow b\sec^2 x\, dx = dt$, when $x = 0$ then $t = 0$, $x = \pi/2$ then $t = \infty$

$$\therefore \quad I = \frac{1}{b}\int_0^{\infty} \frac{dt}{a^2 + t^2} = \frac{1}{b} \times \frac{1}{a}\left[\tan^{-1}\left(\frac{t}{a}\right)\right]_0^{\infty}$$

$$= \frac{1}{ab}[\tan^{-1}\infty - \tan^{-1}0] = \frac{1}{ab}[\pi/2 - 0] = \frac{\pi}{2ab}.$$

Example 9. Evaluate $\int_0^{\pi/2} (\sqrt{\tan x} + \sqrt{\cot x})\, dx$

Solution : Let $I = \int_0^{\pi/2} (\sqrt{\tan x} + \sqrt{\cot x})\, dx$

$$= \int_0^{\pi/2} \left[\frac{\sqrt{\sin x}}{\sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\sin x}} \right] dx$$

$$= \int_0^{\pi/2} \frac{\sin x + \cos x}{\sqrt{\sin x \cos x}}\, dx$$

$$= \sqrt{2} \int_0^{\pi/2} \frac{(\sin x + \cos x)\, dx}{\sqrt{2 \sin x \cos x}}$$

$$= \sqrt{2} \int_0^{\pi/2} \frac{(\sin x + \cos x) dx}{\sqrt{1 - (1 - 2\sin x \cos x)}} = \sqrt{2} \int_0^{\frac{\pi}{2}} \frac{(\sin x + \cos x) dx}{\sqrt{1 - (\sin x - \cos x)^2}}$$

Let $\sin x - \cos x = t \Rightarrow (\cos x + \sin x)\, dx = dt$, Also when $x = 0$ then $t = -1$, $x = \pi/2$ then $t = 1$

$\therefore$ $$I = \sqrt{2} \int_{-1}^{1} \frac{dt}{\sqrt{1 - t^2}} = \sqrt{2} \left[\sin^{-1} t \right]_{-1}^{1}$$

$$= \sqrt{2} \left[\sin^{-1}(1) - \sin^{-1}(-1) \right] = \sqrt{2} \left[\frac{\pi}{2} - \left(\frac{-\pi}{2} \right) \right]$$

$$= \sqrt{2} \left(\frac{\pi}{2} + \frac{\pi}{2} \right) = \pi \sqrt{2}$$

Exercise 10.2

Evaluate the following definite integrals:

1. $\int_1^3 (2x+1)^3\, dx$
2. $\int_0^{\pi/2} \frac{\sin x}{1 + \cos^2 x}\, dx$
3. $\int_1^3 \frac{\cos(\log x)}{x}\, dx$
4. $\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$
5. $\int_0^{\pi/2} \sqrt{1 + \sin x}$
6. $\int_o^c \frac{y}{\sqrt{y + c}}\, dy$
7. $\int_o^\infty \frac{e^{\tan^{-1} x}}{1 + x^2}\, dx$
8. $\int_1^2 \frac{(1 + \log x)^2}{x}\, dx$
9. $\int_\alpha^\beta \frac{dx}{(x - \alpha)(\beta - x)}, \beta > \alpha$
10. $\int_0^{\pi/4} \frac{(\sin x + \cos x)}{9 + 16 \sin 2x}$
11. $\int_{1/e}^{e} \frac{dx}{x (\log x)^{1/3}}$
12. $\int_0^{\pi/4} \sin 2x \cos 3x dx$
13. $\int_e^{e^2} \left[\frac{1}{\log x} - \frac{1}{(\log x)^2} \right] dx$
14. $\int_0^1 \frac{x^3}{\sqrt{1 - x^2}}\, dx$
15. $\int_{\pi/2}^{\pi} \frac{1 - \sin x}{1 - \cos x}\, dx$

16. $\int_0^{\pi/4} \frac{dx}{4\sin^2 x + 5\cos^2 x}$ 17. $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx$ 18. $\int_{-1}^{1} x \tan^{-1} x dx$

19. $\int_0^1 \frac{x \sin^{-1} x}{\sqrt{1-x^2}}$ 20. $\int_0^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx$ 21. $\int_1^2 \log x \, dx$

22. $\int_{4/\pi}^{2/\pi} \left(-\frac{1}{x^3}\right) \cos\left(\frac{1}{x}\right) dx$ 23. $\int_0^{\pi/2} \frac{\sin x \cos x \, dx}{\cos^2 x + 3\cos x + 2}$ 24. $\int_0^3 \sqrt{\frac{x}{3-x}} dx$

25. $\int_0^1 \frac{x^2}{1+x^2} dx$ 26. $\int_1^2 \frac{1}{(x+1)(x+2)} dx$

10.07 Basic properties of definite integral

Property-I If the limits are not changed then by changing the variable in definite integral the vlaue of the integral does not change.

i.e. $$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

Proof : Let $\int f(x)\,dx = F(x)$ $\therefore \int f(t)\,dt = F(t)$

$\therefore$ $$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

and $$\int_a^b f(t)\,dt = [F(t)]_a^b = F(b) - F(a) = \int_a^b f(x)\,dx$$

$\therefore$ $$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

Property-II If the limits are interchanged then the sign of the integral changes while value remain same.

i.e. $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Proof : Let $\int f(x)\,dx = F(x)$

$\therefore$ $$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

and $$\int_b^a f(x)\,dx = [F(x)]_b^a = F(a) - F(b) = -[F(b) - F(a)] = -\int_b^a f(x)\,dx$$

similarly $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Property-III If $a < c < b$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Proof : Let $\int f(x)\,dx = F(x)$

$\therefore$ $$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a) \qquad (1)$$

again $$\int_a^c f(x)\,dx+\int_c^b f(x)\,dx=\left[F(x)\right]_a^c+\left[F(x)\right]_c^b$$

$$=F(c)-F(a)+F(b)-F(c)$$

$$=F(b)-F(a) \qquad (2)$$

from (1) and (2) $$\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx$$

Generalization

If $a<c_1<c_2<....<c_n<b$,

$$\int_a^b f(x)\,dx=\int_a^{c_1} f(x)\,dx+\int_{c_1}^{c_2} f(x)\,dx+....+\int_{c_n}^b f(x)\,dx$$

Note: This property is used when integrand is obtained from more than one rule for given interval of integration say [a, b].

Property-IV $$\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx$$

Proof: $$\text{LHS}=\int_a^b f(a+b-x)\,dx$$

Let $a+b-x=y\Rightarrow -dx=dy$

when $x=a$ then $y=b$ and when $x=b$ then $y=a$

$\therefore$ $$\text{LHS}=\int_b^a f(y)\cdot(-dy)=\int_a^b f(y)\,dy \qquad \text{(by property-II)}$$

$$=\int_a^b f(x)\,dx=\text{RHS} \qquad \text{(by property-I)}$$

i.e. $$\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx$$

Special condition : If $a=0$ then

$$\int_0^b f(x)\cdot dx=\int_0^b f(b-x)\,dx$$

If a fucntion $f(x)$ does not change by putting $(b-x)$ in place of x then this property is used. For using this property the lower limit has to be zero.

Illustrative Examples

Example 10. Evaluate $\int_0^{\pi/2}\frac{1}{1+\sqrt{\cot x}}dx$.

Solution : Let $$I=\int_0^{\pi/2}\frac{1}{1+\sqrt{\cot x}}dx$$

or, $$I=\int_0^{\pi/2}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx \qquad (1)$$

$$= \int_0^{\pi/2} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}} dx$$

or $$I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx \qquad (2)$$

adding (1) and (2)

$$2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx + \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx$$

or, $$2I = \int_0^{\pi/2} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \frac{\pi}{2}$$

$\therefore$ $$I = \frac{\pi}{4} \qquad \text{or} \quad \int_0^{\pi/2} \frac{1}{1+\sqrt{\cot x}} dx = \frac{\pi}{4}$$

Note: Similarly using property IV, the value of the following integrals will also be $\pi/4$.

(i) $\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x} dx$ (ii) $\int_0^{\pi/2} \frac{\cos^n x}{\sin^n x + \cos^n x} dx$ (iii) $\int_0^{\pi/2} \frac{1}{1+\tan^n x} dx$

(iv) $\int_0^{\pi/2} \frac{1}{1+\cot^n x}$ (v) $\int_0^{\pi/2} \frac{\sec^n x}{\sec^n x + \operatorname{cosec}^n x} dx$ (vi) $\int_0^{\pi/2} \frac{\operatorname{cosec}^n x}{\sec^n x + \operatorname{cosec}^n x} dx$

Example 11. Prove that: $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{a} f(-x)\,dx$.

Solution : Let $I = \int_{-a}^{a} f(x)\,dx$

By Property-IV, $I = \int_{-a}^{a} f(-a+a-x)dx = \int_{-a}^{a} f(-x)\,dx$

Example 12. Evaluate $\int_1^4 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}} dx$

Solution : Let $$I = \int_1^4 \frac{\sqrt{x}dx}{\sqrt{5-x}+\sqrt{x}} \qquad (1)$$

or, $$I = \int_1^4 \frac{\sqrt{5-x}}{\sqrt{5-(5-x)}+\sqrt{5-x}} dx$$

or, $$I = \int_1^4 \frac{\sqrt{5-x}}{\sqrt{x}+\sqrt{5-x}} dx \qquad (2)$$

Adding (1) and (2),

$$2I = \int_1^4 \frac{\sqrt{x}+\sqrt{5-x}}{\sqrt{x}+\sqrt{5-x}}\,dx$$

$$= \int_1^4 dx = [x]_1^4 = 4-1 = 3$$

$\therefore \qquad I = 3/2.$

Property V : $\int_o^{na} f(x)\,dx = n\int_o^a f(x)\,dx$, and $f(a+x) = f(x)$, where $f(x)$ is periodic function with period a.

Proof : By property III

$$\int_o^{na} f(x)\,dx = \int_o^a f(x)\,dx + \int_a^{2a} f(x)\,dx + \int_{2a}^{3a} f(x)\,dx + \ldots + \int_{(n-1)a}^{na} f(x)\,dx$$

Now in integral $\int_o^{2a} f(x)\,dx$ putting $x = a+t \Rightarrow dx = dt$ when $x = a$, $t = 0$ and $x = 2a$, $t = a$

$\therefore \quad \int_a^{2a} f(x)\,dx = \int_o^a f(a+t)\,dt = \int_o^a f(a+x)\,dx = \int_o^a f(x)\,dx \qquad [\because f(a+x) = f(x)]$

Now

$$f(x) = f(x+a) = f(x+2a) = \ldots.. = f(x+na)$$

$$\therefore \quad \int_o^{na} f(x)\,dx = \underbrace{\int_o^a f(x)\,dx + \int_o^a f(x)\,dx + \ldots + \int_o^a f(x)\,dx}_{n \text{ times}} = n\int_o^a f(x)\,dx$$

Property-VI $\qquad \int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & ; \text{ If } f(x) \text{ is an even function i.e. } f(-x) = f(x) \\ 0 & ; \text{ If } f(x) \text{ if an odd function i.e. } f(-x) = -f(x) \end{cases}$

Proof : By property III

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{o} f(x)\,dx + \int_o^a f(x)\,dx \qquad (\because -a < 0 < a)$$

$$= I_1 + \int_o^a f(x)\,dx \qquad (1)$$

where $\qquad I_1 = \int_{-a}^{o} f(x)\,dx$

Let $\qquad x = -y \Rightarrow dx = -dy$

when $x = -a$ then $y = a$, $x = 0$ then $y = 0$

$\therefore \qquad I_1 = \int_a^o -f(-y)\,dy = \int_o^a f(-y)\,dy \qquad$ (Property II)

$$= \int_o^a f(-x)\,dx \qquad \text{(Property I)}$$

from eq. (1)

$$\int_{-a}^{a} f(x)\,dx = \int_{o}^{a} f(-x)\,dx + \int_{o}^{a} f(x)\,dx \qquad (2)$$

Case (i): when $f(x)$ is an even function if $f(-x) = f(x)$

then $$\int_{-a}^{a} f(x)\,dx = \int_{o}^{a} f(x)\,dx + \int_{o}^{a} f(x)\,dx = 2\int_{o}^{a} f(x)\,dx$$

Case (ii): when $f(x)$ is an odd function if $f(-x) = -f(x)$

then $$\int_{-a}^{a} f(x)\,dx = -\int_{o}^{a} f(x)\,dx + \int_{o}^{a} f(x)\,dx = 0$$

$$\therefore \quad \int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_{0}^{a} f(x)\,dx & ; \text{ If } f(x) \text{ is an even function then } f(-x) = f(x) \\ 0 & ; \text{ If } f(x) \text{ is an odd function then } f(-x) = -f(x) \end{cases}$$

Property-VII: $$\int_{0}^{2a} f(x)\,dx = \begin{cases} 2\int_{0}^{a} f(x)\,dx & ; \text{ If } f(2a-x) = f(x) \\ 0 & ; \text{ If } f(2a-x) = -f(x) \end{cases}$$

Proof : $$\int_{o}^{2a} f(x)\,dx = \int_{o}^{a} f(x)\,dx + \int_{a}^{2a} f(x)\,dx \qquad [\text{property III } \therefore o < a < 2a]$$

$$= \int_{o}^{a} f(x)\,dx + I_1 \qquad (1)$$

here $$I_1 = \int_{a}^{2a} f(x)\,dx$$

Let $x = 2a - y \Rightarrow dx = -dy$ when $x = a$ then $y = a$ and $x = 2a$ then $y = o$

$$\therefore \quad I_1 = \int_{a}^{o} -f(2a-y)\,dy = \int_{o}^{a} f(2a-y)\,dy \qquad (\text{property II})$$

$$= \int_{o}^{a} f(2a-x)\,dx \qquad (\text{property I})$$

substituting the vlaue of I_1 in (1)

$$\int_{o}^{2a} f(x)\,dx = \int_{o}^{a} f(x)\,dx + \int_{o}^{a} f(2a-x)\,dx$$

Case (i): when $f(2a-x) = f(x)$

then $$\int_{o}^{2a} f(x)\,dx = \int_{o}^{a} f(x)\,dx + \int_{o}^{a} f(x)\,dx = 2\int_{o}^{a} f(x)\,dx$$

Case (ii): when $f(2a-x) = -f(x)$

then $$\int_{o}^{2a} f(x)\,dx = \int_{o}^{a} f(x)\,dx - \int_{o}^{a} f(x)\,dx = 0$$

$$\therefore \quad \int_{0}^{2a} f(x)\,dx = \begin{cases} 2\int_{0}^{a} f(x)\,dx & ; \text{ If } f(2a-x) = f(x) \\ 0 & ; \text{ If } f(2a-x) = -f(x) \end{cases}$$

Note: (i) when $f(2a - x) = f(x)$ then $f(x)$ should not be considered as even function $f(x)$ is even function only when $f(-x) = f(x)$.

(ii) If the lower limit is zero then we use property-IV i.e. we substitute x with $f(a + b - x)$ but some time $f(x)$ doesn't change then we use property VII.

10.08 Special property (Eliminating x)

If $f(a + b - x) = f(x)$ then eliminating x from $\int_a^b x\, f(x)\, dx$

$$\int_a^b x\, f(x)\, dx = \frac{a+b}{2}\int_a^b f(x)\, dx$$

Proof : Let $$I = \int_a^b f(x)\, dx$$

Using Property IV

$$\int_a^b (a+b-x)\, f(a+b-x)\, dx$$

but given $$f(a+b-x) = f(x)$$

$\therefore$ $$I = \int_a^b (a+b-x)\, f(x)\, dx$$

$$= (a+b)\int_a^b f(x) dx - \int_a^b x\, f(x)\, dx$$

or $$I = (a+b)\int_a^b f(x) dx - I$$

or $$2I = (a+b)\int_a^b f(x)\, dx \Rightarrow I = \frac{a+b}{2}\int_a^b f(x)\, dx$$

Illustrative Examples

Example 13. Evaluate $\int_0^\pi \frac{x \sin x}{1+\cos^2 x} dx$

Solution : Let $$I = \int_0^\pi \frac{x \sin x}{1+\cos^2 x} dx$$

or, $$I = \int_0^\pi x.\left(\frac{\sin x}{1+\cos^2 x}\right) dx$$

here, $$f(x) = \frac{\sin x}{1+\cos^2 x}$$

$$\therefore \quad f(\pi - x) = \frac{\sin(\pi - x)}{1+\cos^2(\pi - x)} = \frac{\sin x}{1+\cos^2 x} = f(x)$$

$\therefore$ Eliminating x,

$$I = \frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\cos^2 x} dx$$

Let $\cos x = t \Rightarrow \sin x\, dx = -dt \quad x = 0$ then $t = 1$ and $x = \pi$ then $t = -1$

$\therefore$
$$I = \frac{\pi}{2}\int_1^{-1} \frac{-dt}{1+t^2} = \frac{\pi}{2}\int_{-1}^{1} \frac{1}{1+t^2}dt = \frac{\pi}{2}(\tan^{-1} t)_{-1}^{1}$$

$$= \frac{\pi}{2}\left[\tan^{-1}(1) - \tan^{-1}(-1)\right] = \frac{\pi}{2}\left[\frac{\pi}{4} - \left(\frac{-\pi}{4}\right)\right] = \frac{\pi}{2}\left[\frac{\pi}{2}\right] = \frac{\pi^2}{4}.$$

Important standard integral

$$I = \int_0^{\pi/2} \log \sin x\, dx = -\frac{\pi}{2}\log 2 = \int_0^{\pi/2} \log \cos x\, dx$$

Solution : Let $I = \int_0^{\pi/2} \log \sin x\, dx$ (1)

Using property IV,

$$I = \int_0^{\pi/2} \log\left[\sin\left(\pi/2 - x\right)\right]dx$$

or $I = \int_0^{\pi/2} \log \cos x\, dx$ (2)

Adding (1) and (2)

$$2I = \int_0^{\pi/2} [\log \sin x + \log \cos x]\, dx$$

$$= \int_0^{\pi/2} \log(\sin x \cos x) dx$$

$$= \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right) dx = \int_0^{\pi/2} (\log \sin 2x - \log 2) dx$$

$$= \int_0^{\pi/2} \log \sin 2x\, dx - \log 2\int_0^{\pi/2} dx$$

$$= \int_0^{\pi/2} \log \sin 2x\, dx - (\log 2)[x]_0^{\pi/2}$$

or $2I = I_1 - \frac{\pi}{2}(\log 2)$ (3)

when $I_1 = \int_0^{\pi/2} \log \sin 2x\, dx$

Let $2x = t \Rightarrow dx = \frac{dt}{2}$

when $x = 0$ then $t = 0$ and $x = \pi/2$ then $t = \pi$

$\therefore$ $I_1 = \frac{1}{2}\int_0^{\pi} \log(\sin t)\, dt = \frac{1}{2} \times 2\int_0^{\pi/2} \log \sin t\, dt$ (Property VII)

$= \int_0^{\pi/2} \log \sin x\, dx$ (Property I) (Using equation (1))

$\therefore$ from equation (3) $$2I = I - \frac{\pi}{2}\log_e 2 \Rightarrow I = -\frac{\pi}{2}(\log_e 2)$$

or $$\int_0^{\pi/2} \log \sin x \, dx = \int_0^{\pi/2} \log \cos x \, dx = -\frac{\pi}{2}\log 2.$$

$$\int_0^{\pi/2} \log \operatorname{cosec} x \, dx = \int_0^{\pi/2} \log \sec x \, dx = \frac{\pi}{2}\log 2.$$

Illustrative Examples

Example 14. Evaluate the following definite Integrals

(i) $\int_1^4 f(x)\,dx$ when $f(x) = \begin{bmatrix} 4x+3, 1 \le x \le 2 \\ 3x+5, 2 \le x \le 4 \end{bmatrix}$ (ii) $\int_0^2 |1-x|\,dx$ (iii) $\int_{-1}^1 e^{|x|}dx$

Solution : (i) $$\int_1^4 f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^4 f(x)\,dx$$

$$= \int_1^2 (4x+3)\,dx + \int_2^4 (3x+5)dx \left[\because f(x) = \begin{bmatrix} 4x+3 & ; & 1 \le x \le 2 \\ 3x+5 & ; & 2 \le x \le 4 \end{bmatrix}\right]$$

$$= \left[2x^2 + 3x\right]_1^2 + \left[\frac{3x^2}{2} + 5x\right]_2^4$$

$$= \left[(8+6)-(2+3)\right] + \left[(24+20)-(6+10)\right] = 9 + 28 = 37.$$

(ii) $$\int_0^2 |1-x|\,dx = \int_0^1 |1-x|\,dx + \int_1^2 |1-x|\,dx$$

$$\left[\because \quad |1-x| = \begin{cases} 1-x & x < 1 \\ -(1-x), & x > 1 \end{cases}\right]$$

$$= \int_0^1 (1-x)\,dx + \int_1^2 (1-x)\,dx$$

$$= \left[x - x^2/2\right]_0^1 - \left[x - x^2/2\right]_1^2$$

$$= \left[(1-1/2)-0\right] - \left[(2-2)-(1-1/2)\right] = (1/2) + (1/2) = 1.$$

(iii) $$\int_{-1}^1 e^{|x|}dx = \int_{-1}^0 e^{|x|}dx + \int_0^1 e^{|x|}dx \qquad \left[\because |x| = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}\right]$$

$$= \int_{-1}^0 e^{-x}dx + \int_0^1 e^x dx$$

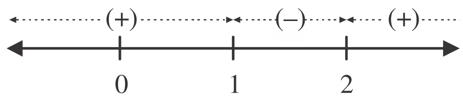
$$= [-e^{-x}]_{-1}^0 + [e^x]_0^1 = (-e^o + e^1) + (e - e^o) = 2e - 2.$$

Example 15. Evaluate the following definite integrals

(i) $\int_0^2 \left|x^2-3x+2\right| dx$ (ii) $\int_{1/e}^{e} \left|\log_e x\right| dx$ (iii) $\int_0^{\pi} \left|\cos x\right| dx$

Solution : (i) Here $x^2-3x+2=(x-1)(x-2)$

The sign of x^2-3x+2 will be different for various vlaues of x



$$\therefore \quad |x^2-3x+2| = \begin{cases} x^2-3x+2, & 0 \le x \le 1 \\ -(x^2-3x+2), & 1 \le x \le 2 \end{cases}$$

$$\therefore \quad \int_0^2 |x^2-3x+2|\,dx = \int_0^1 |x^2-3x+2|\,dx + \int_1^2 |x^2-3x+2|\,dx$$

$$= \int_0^1 \left(x^2-3x+2\right) dx + \int_1^2 -\left(x^2-3x+2\right) dx$$

$$= \left[\frac{x^3}{3}-\frac{3x^2}{2}+2x\right]_0^1 - \left[\frac{x^3}{3}-\frac{3x^2}{2}+2x\right]_1^2$$

$$= \left[(1/3-3/2+2)-(0)\right] - \left[(8/3-6+4)-(1/3-3/2+2)\right]$$

$$= \frac{5}{6}-\frac{2}{3}+\frac{5}{6} = \frac{5}{3}-\frac{2}{3} = 1$$

(ii) $$\int_{1/e}^{e} |\log_e x|\,dx = \int_{1/e}^{1} |\log_e x|\,dx + \int_1^e |\log_e x|\,dx$$

$$= \int_{1/e}^{1} -\log_e x\,dx + \int_1^e \log_e x\,dx \qquad \left[\because |\log_e x| = \begin{cases} -\log_e x, & \text{If } \ 1/e < x < 1 \\ \log_e x, & \text{If } \ 1 \le x < e \end{cases}\right]$$

$$= -\left[x(\log_e x - 1)\right]_{1/e}^{1} + \left[x(\log_e x - 1)\right]_1^e \qquad \left[\because \int \log_e x dx = x\left(\log_e x - 1\right)\right]$$

$$= -\left[(0-1)-1/e(-1-1)\right] + \left[e(1-1)-(0-1)\right]$$

$$= 1-2/e+1 = 2-2/e$$

(iii) $$\int_0^{\pi} |\cos x|\,dx = \int_0^{\pi/2} |\cos x|\,dx + \int_{\pi/2}^{\pi} |\cos x|\,dx$$

$$= \int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{\pi} (-\cos x)\,dx \qquad \because \ |\cos x| = \begin{cases} \cos x & ; \ 0 < x \le \pi/2 \\ -\cos x & ; \ \pi/2 < x \le \pi \end{cases}$$

$$= [\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{\pi}$$

$$= (\sin \pi/2 - \sin 0) - (\sin \pi - \sin \pi/2) = (1-0)-(0-1) = 2$$

Example 16. Evaluate the following definite integrals:

(i) $\int_0^{\pi/2} \log \cot x \, dx$ (ii) $\int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx$

Solution : (i) Let $I = \int_0^{\pi/2} \log \cot x \, dx$ (1)

or, $I = \int_0^{\pi/2} \log[\cot(\pi/2 - x)] \, dx$ (using property IV)

or, $I = \int_0^{\pi/2} \log \tan x \, dx$ (2)

adding (1) and (2)

$$2I = \int_0^{\pi/2} \log \cot x \, dx + \int_0^{\pi/2} \log \tan x \, dx$$

$$= \int_0^{\pi/2} \left[\log(\cot x) + \log(\tan x)\right] dx$$

$$= \int_0^{\pi/2} \log(\cot x \times \tan x) \, dx$$

$$= \int_0^{\pi/2} \log(1) \, dx = \int_0^{\pi/2} (0) \, dx$$

or, $2I = 0$ $\therefore I = 0$

(ii) Let $I = \int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx$ (1)

using property IV

$$I = \int_0^{\pi/2} \frac{\sin(\frac{\pi}{2} - x) - \cos(\frac{\pi}{2} - x)}{1 + \sin(\frac{\pi}{2} - x)\cos(\frac{\pi}{2} - x)} dx$$

or, $I = \int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \sin x \cos x} dx$ (2)

adding (1) and (2)

$$2I = 0 \Rightarrow I = 0$$

Example 17. Evaluate the following definite Integrals:

(i) $\int_0^8 \frac{\sqrt{x}}{\sqrt{x} + \sqrt{8 - x}} dx$ (ii) $\int_0^a \frac{dx}{x + \sqrt{a^2 - x^2}}$

Solution : (i) Let $I = \int_0^8 \frac{\sqrt{x}}{\sqrt{x} + \sqrt{8 - x}} dx$ (1)

using property IV,

$$I = \int_0^8 \frac{\sqrt{8 - x}}{\sqrt{8 - x} + \sqrt{8 - (8 - x)}} dx$$

or $$I=\int_0^8 \frac{\sqrt{8-x}}{\sqrt{8-x}+\sqrt{x}}dx \quad (2)$$

adding (1) and (2)

$$2I=\int_0^8 \frac{\sqrt{x}+\sqrt{8-x}}{\sqrt{8-x}+\sqrt{x}}dx=\int_0^8 dx=[x]_0^8=8, \qquad \therefore \quad I=4$$

(ii) $$I=\int_0^a \frac{dx}{x+\sqrt{a^2-x^2}}$$

Let $x=a\sin\theta \Rightarrow dx=a\cos\theta\, d\theta$

when $x=0$ then $\theta=0$ and $x=a$ then $\theta=\pi/2$

$\therefore$ $$I=\int_0^{\pi/2} \frac{a\cos\theta\, d\theta}{a\sin\theta+a\cos\theta}=\int_0^{\pi/2}\frac{\cos\theta}{\sin\theta+\cos\theta}d\theta \quad (1)$$

Property-(IV) $$I=\int_0^{\pi/2} \frac{\cos(\frac{\pi}{2}-\theta)\,d\theta}{\sin(\frac{\pi}{2}-\theta)+\cos(\frac{\pi}{2}-\theta)}$$

$$I=\int_0^{\pi/2} \frac{\sin\theta\, d\theta}{\cos\theta+\sin\theta} \quad (2)$$

adding (1) and (2)

$$2I=\int_0^{\pi/2}\left(\frac{\sin\theta+\cos\theta}{\sin\theta+\cos\theta}\right)d\theta$$

$$=\int_0^{\pi/2} d\theta=[\theta]_0^{\pi/2}=\frac{\pi}{2}-0$$

$\therefore$ $$I=\frac{\pi}{4}$$

Example 18. Evaluate $\int_0^{\pi/2}\frac{\sin^2 x}{\sin x+\cos x}dx$

Solution : Let $$I=\int_0^{\pi/2}\frac{\sin^2 x}{\sin x+\cos x}dx \quad (1)$$

using property IV $$I=\int_0^{\pi/2}\frac{\sin^2\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}dx$$

or, $$I=\int_0^{\pi/2}\frac{\cos^2 x}{\cos x+\sin x}\,dx \qquad (2)$$

Adding (1) and (2),

$$2I=\int_0^{\pi/2}\frac{\sin^2 x+\cos^2 x}{\sin x+\cos x}\,dx$$

$$I=\frac{1}{2}\int_0^{\pi/2}\frac{1}{\sin x+\cos x}\,dx$$

$$=\frac{1}{2}\int_0^{\pi/2}\frac{1}{\left(\frac{2\tan(x/2)}{1+\tan^2(x/2)}\right)+\left(\frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}\right)}$$

(converting $\sin x$ and $\cos x$ into $\tan x/2$)

$$=\frac{1}{2}\int_0^{\pi/2}\frac{1+\tan^2(x/2)}{2\tan(x/2)+1-\tan^2(x/2)}\,dx$$

or $$I=\frac{1}{2}\int_0^{\pi/2}\frac{\sec^2(x/2)}{1+2\tan(x/2)-\tan^2(x/2)}\,dx$$

Let $\tan\frac{x}{2}=t\Rightarrow\frac{1}{2}\sec^2\frac{x}{2}\,dx=dt$

when $x=0$ then $t=0$; when $x=\pi/2$ then $t=1$

$\therefore$ $$I=\int_0^1\frac{dt}{1+2t-t^2}=\int_0^1\frac{dt}{2-(t-1)^2}$$

$$=\frac{1}{2\sqrt{2}}\left[\log\left|\frac{\sqrt{2}+(t-1)}{\sqrt{2}-(t-1)}\right|\right]_0^1$$

$$=\frac{1}{2\sqrt{2}}\left[\log\frac{\sqrt{2}}{\sqrt{2}}-\log\frac{\sqrt{2}-1}{\sqrt{2}+1}\right]$$

$$=\frac{1}{2\sqrt{2}}\left[0+\log\frac{\sqrt{2}+1}{\sqrt{2}-1}\right]=\frac{1}{2\sqrt{2}}\log\left[\frac{\left(\sqrt{2}+1\right)}{\left(\sqrt{2}-1\right)}\times\frac{\left(\sqrt{2}+1\right)}{\left(\sqrt{2}+1\right)}\right]$$

$$=\frac{1}{2\sqrt{2}}\log\frac{\left(\sqrt{2}+1\right)^2}{(2-1)}=\frac{2}{2\sqrt{2}}\log\left(\sqrt{2}+1\right)=\frac{1}{\sqrt{2}}\log\left(\sqrt{2}+1\right).$$

Example 19. Evaluate the followng Integral

$$\int_{-a}^{a}\sqrt{\frac{a-x}{a+x}}dx$$

Solution : Let $I=\int_{-a}^{a}\sqrt{\frac{a-x}{a+x}}dx$

$$=\int_{-a}^{a}\frac{a-x}{\sqrt{a^2-x^2}}dx$$

$$=\int_{-a}^{a}\frac{a}{\sqrt{a^2-x^2}}dx-\int_{-a}^{a}\frac{x}{\sqrt{a^2-x^2}}dx \quad (1)$$

or, $=I_1-I_2$

where $I_1=\int_{-a}^{a}\frac{a}{\sqrt{a^2-x^2}}dx=2a\int_{0}^{a}\frac{1}{\sqrt{a^2-x^2}}dx$ ($\because f(x)$ is an even function)

using property VI

$$=2a\left[\sin^{-1}x/a\right]_0^a=2a\left(\sin^{-1}(1)-\sin^{-1}(0)\right)=2a\times\left(\pi/2-0\right)=\pi a$$

and $I_2=\int_{-a}^{a}\frac{x}{\sqrt{a^2-x^2}}dx=0$

(property VI when $f\left(x\right)$ is an odd function $\int_{-a}^{a}f\left(x\right)dx=0$)

$\therefore$ from (1), $I=\pi a-0=\pi a$

Example 20. Prove that:

$$\int_{0}^{\pi/4}\log_e(1+\tan x)dx=\frac{\pi}{8}\log_e 2.$$

Solution : Let $I=\int_{0}^{\pi/4}\log_e(1+\tan x)\,dx$

Using Property IV,

$$I=\int_{0}^{\pi/4}\log_e\left[1+\tan\left(\frac{\pi}{4}-x\right)\right]dx$$

$$=\int_{0}^{\pi/4}\log_e\left[1+\frac{\tan\left(\pi/4\right)-\tan x}{1+\tan(\pi/4)\tan x}\right]dx$$

$$=\int_{0}^{\frac{\pi}{4}}\log_e\left[1+\frac{1-\tan x}{1+\tan x}\right]dx$$

$$=\int_{0}^{\pi/4}\log_e\left(\frac{2}{1+\tan x}\right)dx$$

$$= \int_0^{\pi/4} [\log_e 2 - \log_e (1 + \tan x)]\, dx$$

$$= \int_0^{\pi/4} (\log_e 2)\, dx - \int_0^{\pi/4} \log_e (1 + \tan x)\, dx$$

or $$I = (\log_e 2)[x]_0^{\pi/4} - I$$

or $$2I = \frac{\pi}{4} \log_e 2 \Rightarrow I = \frac{\pi}{8} \log_e 2$$

$\Rightarrow$ $$\int_0^{\pi/4} \log(1 + \tan x)\, dx = \frac{\pi}{8} \log_e 2,$$ Hence proved.

Example 21. Prove that: $I = \int_0^{\pi} \log(1 + \cos x)\, dx = \pi \log_e (1/2)$.

Solution : Let $$I = \int_0^{\pi} \log(1 + \cos x)\, dx \quad \text{. . . (1)}$$

Using property IV,

$$I = \int_0^{\pi} \log[1 + \cos(\pi - x)]\, dx$$

or, $$I = \int_0^{\pi} \log(1 - \cos x)\, dx \quad \text{. . . (2)}$$

Adding (1) and (2),

$$2I = \int_0^{\pi} \log(1 + \cos x) + \log(1 - \cos x)\, dx$$

$$= \int_0^{\pi} \log\{(1 + \cos x)(1 - \cos x)\}\, dx$$

$$= \int_0^{\pi} \log(1 - \cos^2 x)\, dx$$

or $$2I = \int_0^{\pi} \log \sin^2 x\, dx = 2\int_0^{\pi} \log \sin x\, dx$$

or $$I = \int_0^{\pi} \log \sin x\, dx$$

or $$I = 2\int_0^{\pi/2} \log \sin x\, dx \quad \text{(property VII)}$$

or $$I = 2I_1, \text{ and } I_1 = \int_0^{\pi/2} \log \sin x\, dx \quad \text{. . . (3)}$$

or $$I_1 = \int_0^{\pi/2} \log \cos x\, dx \quad \text{(Using property IV)} \quad \text{. . . (4)}$$

Adding equations (3) and (4),

$$2I_1 = \int_0^{\pi/2} (\log \sin x + \log \cos x)\, dx$$

$$= \int_0^{\pi/2} \log(\sin x \cos x)\, dx$$

or $$2I_1 = \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right) dx$$

or $$2I_1 = \int_0^{\pi/2} \log(\sin 2x)\, dx - \int_0^{\pi/2} (\log 2)\, dx$$

or $$2I_1 = I_2 - (\log 2)[x]_0^{\pi/2}$$

or $$2I_1 = I_2 - \frac{\pi}{2}\log 2 \qquad \text{. . . (5)}$$

where $$I_2 = \int_0^{\pi/2} \log(\sin 2x)\, dx$$

Let $2x = t \Rightarrow 2dx = dt$ and when $x = 0$ then $t = 0$, when $x = \pi/2$ then $t = \pi$

$$\therefore \quad I_2 = \frac{1}{2}\int_0^{\pi} \log(\sin t)\, dt = \frac{1}{2}\int_0^{\pi} \log(\sin x)\, dx \qquad \text{(property I)}$$

or, $$I_2 = \frac{1}{2} \times 2\int_0^{\pi/2} \log(\sin x)\, dx \qquad \text{(property VII)}$$

or, $$I_2 = \int_0^{\pi/2} \log \sin x\, dx = I_1$$

putting the value of I_2 in equation (5)

$$2I_1 = I_1 - \frac{\pi}{2}\log 2$$

or $$I_1 = \frac{\pi}{2}\log\frac{1}{2}$$

$$\therefore \quad I = 2I_1 = 2 \times \frac{\pi}{2}\log\frac{1}{2} = \pi \log\frac{1}{2}$$

or $$\int_0^{\pi/2} \log(1 + \cos x)\, dx = \pi \log\frac{1}{2}$$

Example 22. Prove that

$$\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx = \pi\left[(\pi/2) - 1\right]$$

Solution : $$\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx = \int_0^{\pi} x.\left(\frac{\sin x}{1 + \sin x}\right) dx$$

Here, $$f(x) = \frac{\sin x}{1 + \sin x}$$

then, $f(\pi - x) = \dfrac{\sin(\pi - x)}{1+\sin(\pi - x)} = \dfrac{\sin x}{1+\sin x} = f(x)$

$\therefore$ Eliminating x rule, $\int_a^b x f(x)\,dx = \dfrac{a+b}{2}\int_a^b f(x)\,dx$

$$\int_0^{\pi} \frac{x\tan x}{\sec x + \tan x}dx = \frac{\pi}{2}\int_0^{\pi}\frac{\sin x}{1+\sin x}dx$$

$$= \frac{\pi}{2}\int_0^{\pi}\left(1 - \frac{1}{1+\sin x}\right)dx = \frac{\pi}{2}\int_0^{\pi}\left(1 - \frac{1-\sin x}{\cos^2 x}\right)dx$$

$$= \frac{\pi}{2}\int_0^{\pi}\left(1 - \sec^2 x + \sec x\tan x\right)dx$$

$$= \frac{\pi}{2}\left[x - \tan x + \sec x\right]_0^{\pi} = \frac{\pi}{2}\left[(\pi - 0 - 1) - (0 - 0 + 1)\right]$$

$$= \frac{\pi}{2}[\pi - 2] = \pi(\pi/2 - 1).$$ Hence Proved

Exercise 10.3

Evaluate the following definite integrals:

1. $\int_{-2}^{2} |2x+3|\,dx$

2. $\int_{-2}^{2} |1-x^2|\,dx$

3. $\int_1^4 f(x)\,dx$, where $f(x) = \begin{cases} 7x+3 & ; \quad 1 \le x \le 3 \\ 8x & ; \quad 3 \le x \le 4 \end{cases}$

4. $\int_0^3 [x]\,dx$ when $[.]$ is the greatest integer function

5. $\int_{-\pi/4}^{\pi/4} x^5 \cos^2 x\,dx$

6. $\int_{-\pi}^{\pi} \dfrac{\sin x \cos x}{1+\cos^2 x}dx$

7. $\int_{-\pi/4}^{3\pi/4} \dfrac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx$

8. $\int_0^{\pi} \dfrac{e^{\cos x}}{e^{\cos x}+e^{-\cos x}}dx$

9. $\int_0^{\pi/2} \sin 2x.\log\tan x\,dx$

10. $\int_{-1}^{1} \log\left[\dfrac{2-x}{2+x}\right]dx$

11. $\int_0^1 \log\left(\dfrac{1}{x}-1\right)dx$

12. $\int_{\pi/6}^{\pi/3} \dfrac{dx}{1+\sqrt{\tan x}}$

13. $\int_0^{\pi/2} \dfrac{\sin x}{\sin x + \cos x}dx$

14. $\int_0^{\pi/2} \log\sin 2x\,dx$

15. $\int_{-\pi/4}^{\pi/4} \dfrac{\left(x+\dfrac{\pi}{4}\right)}{2-\cos 2x}dx$

16. $\int_0^{\pi} \log(1-\cos x)\,dx$

17. $\int_{-\pi/4}^{\pi/4} \sin^2 x\, dx$

18. $\int_0^{\pi} \frac{x}{1+\sin x} dx$

19. $\int_0^{\pi} x \sin^3 x\, dx$

20.- $\int_0^{\pi/2} \log(\tan x + \cot x)\, dx$

21. $\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^x} dx$

22. $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)} dx$

Miscellaneous Examples

Example 23. Prove that:

$$\int_0^{\pi} \frac{x\, dx}{1+\cos\alpha \sin x} = \frac{\pi\alpha}{\sin\alpha}$$

Solution : Let $f(x) = \frac{1}{1+\cos\alpha \sin x}$

$$\therefore \quad f(\pi - x) = \frac{1}{1+\cos\alpha \sin(\pi - x)} = \frac{1}{1+\cos\alpha \sin x} = f(x)$$

eliminating x rule

$$\int_0^{\pi} \frac{x}{1+\cos\alpha \sin x} dx = \frac{\pi}{2} \int_0^{\pi} \frac{1}{1+\cos\alpha \sin x} dx$$

$$= \frac{\pi}{2} \int_0^{\pi} \frac{1}{1+\cos\alpha \left(\frac{2\tan(x/2)}{1+\tan^2(x/2)} \right)} dx$$

$$= \frac{\pi}{2} \int_0^{\pi} \frac{\sec^2(x/2)}{1+\tan^2(x/2)+2\cos\alpha \tan(x/2)} dx$$

Let $\tan(x/2) = t \Rightarrow \frac{1}{2}\sec^2(x/2)\cdot dx = dt$

when $x=0$ then $t=0$ and when $x=\pi$ then $t=\infty$

$$\therefore \quad \int_0^{\pi} \frac{x}{1+\cos\alpha \sin x} dx = \frac{\pi}{2} \int_0^{\infty} \frac{2}{1+t^2+2t\cos x} dx$$

$$= \pi \int_0^{\infty} \frac{dt}{(t+\cos\alpha)^2 + (\sin\alpha)^2}$$

$$= \pi \times \frac{1}{\sin\alpha} \left[\tan^{-1}\left(\frac{t+\cos\alpha}{\sin\alpha} \right) \right]_0^{\infty}$$

$$= \frac{\pi}{\sin\alpha}\left[\tan^{-1}(\infty) - \tan^{-1}(\cot\alpha)\right]$$

$$= \frac{\pi}{\sin\alpha}\left[\pi/2 - (\pi/2 - \alpha)\right] \qquad \left[\because \cot\alpha = \tan(\pi/2 - \alpha)\right]$$

$$= \frac{\pi}{\sin\alpha}(\alpha) = \frac{\pi\alpha}{\sin\alpha}$$

Example 24. Evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$

Solution : Let $I = \int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$

$$= \frac{1}{a^2-b^2}\int_0^\infty \left(\frac{1}{x^2+b^2} - \frac{1}{x^2+a^2}\right)dx \qquad \text{(Partial fractions)}$$

$$= \frac{1}{(a^2-b^2)}\left[\frac{1}{b}\tan^{-1}\frac{x}{b} - \frac{1}{a}\tan^{-1}\frac{x}{a}\right]_0^\infty$$

$$= \frac{1}{(a^2-b^2)}\left[\left(\frac{1}{b}\tan^{-1}\infty - \frac{1}{a}\tan^{-1}\infty\right) - (0-0)\right]$$

$$= \frac{1}{(a^2-b^2)}\left[\frac{1}{b}.\frac{\pi}{2} - \frac{1}{a}.\frac{\pi}{2}\right]$$

$$= \frac{\pi}{2(a^2-b^2)}\left(\frac{a-b}{ab}\right) = \frac{\pi}{2(a+b)(a-b)} \times \frac{(a-b)}{ab} = \frac{\pi}{2ab(a+b)}$$

Example 25. Evaluate $\int_{\pi/4}^{\pi/2} \cos 2x \log \sin x \, dx$

Solution : Let $I = \int_{\pi/4}^{\pi/2} \underset{\text{II}}{\cos 2x} \, \underset{\text{I}}{\log \sin x} \, dx$

$$= \left[\log \sin x . \frac{\sin 2x}{2}\right]_{\pi/4}^{\pi/2} - \int_{\pi/4}^{\pi/2} \cot x \times \frac{\sin 2x}{2} dx$$

$$= \left[0 - \frac{1}{2}\log\frac{1}{\sqrt{2}}\right] - \int_{\pi/4}^{\pi/2} \cos^2 x \, dx$$

$$= -\frac{1}{2}\log\frac{1}{\sqrt{2}} - \frac{1}{2}\int_{\pi/4}^{\pi/2}(1+\cos 2x)\,dx$$

$$= \frac{1}{4}\log 2 - \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_{\pi/4}^{\pi/2}$$

$$= \frac{1}{4}\log 2 - \frac{1}{2}\left[\left(\frac{\pi}{2} + \frac{\sin \pi}{2}\right) - \left(\frac{\pi}{4} + \frac{\sin \pi/2}{2}\right)\right]$$

$$= \frac{1}{4}\log 2 - \frac{1}{2}\left[\frac{\pi}{4} - \frac{1}{2}\right]$$

$$= \frac{1}{4}\log 2 - \frac{\pi}{8} + \frac{1}{4}.$$

Example 26. Evaluate $\int_0^\infty \frac{\log(1+x^2)}{1+x^2}dx$.

Solution : Let $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

when $x = 0$ then $\theta = 0$ and $x = \infty$ then $\theta = \pi/2$

$\therefore$ $$I = \int_0^{\pi/2} \frac{\log\left(1+\tan^2\theta\right)}{\left(1+\tan^2\theta\right)}\sec^2\theta\, d\theta$$

$$= \int_0^{\pi/2} \log\left(1+\tan^2\theta\right)d\theta = \int_0^{\pi/2} \log\sec^2\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \log\sec\theta\, d\theta = -2\int_0^{\pi/2} \log\cos\theta\, d\theta$$

$$= -2\int_0^{\pi/2} \log\cos\left(\pi/2 - \theta\right)d\theta \qquad \text{(Property IV)}$$

$$= 2\int_0^{\pi/2} \log\sin\theta\, d\theta = -2\left(-\pi/2\log 2\right) \qquad \text{(standard integral)}$$

$$= \pi\log_e 2$$

Miscellaneous Exericse –10

1. The value of $\int_0^{\pi/4} \sqrt{1+\sin 2x}\, dx$ is

   (A) $2\int_0^a \sin^3 x.x\, dx$ (B) 0 (C) a^2 (D) 1

2. The value of $\int_2^5 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{7-x}}dx$ is

   (A) 3 (B) 2 (C) 3 / 2 (D) 1 / 2

3. The value of $\int_{a-c}^{b-c} f(x+c)\,dx$ is

(A) $\int_a^b f(x+c)\,dx$ (B) $\int_a^b f(x)\,dx$ (C) $\int_{a-2c}^{b-2c} f(x)\,dx$ (D) $\int_a^b f(x+2c)dx$

4. If $A(x)=\int_0^x \theta^2 d\theta$, then the value of $A(3)$ is

(A) 9 (B) 27 (C) 3 (D) 81

Evaluate the following definite integrals:-

5. $\int_1^2 \frac{(x+3)}{x(x+2)}dx$

6. $\int_1^2 \frac{xe^x}{(1+x)^2}dx$

7. $\int_0^{\pi/2} e^x\left(\frac{1+\sin x}{1+\cos x}\right)dx$

8. $\int_{1/3}^1 \frac{(x-x^3)^{1/3}}{x^4}dx$

9. $\int_0^{\pi/2} x^2\cos^2 x\,dx$

10. $\int_0^1 \tan^{-1} x\,dx$

11. $\int_0^{\pi/4} \sin 3x \sin 2x$

12. $\int_{-2}^2 |1-x^2|\,dx$

13. $\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{(1+\cos^2 x)}dx$

14. $\int_0^{1/\sqrt{2}} \frac{\sin^{-1} x}{(1-x^2)^{3/2}}dx$

15. $\int_0^1 (\cos^{-1} x)^2 dx$

16. $\int_0^{\pi} \frac{dx}{1-2a\cos x+a^2}, a>1$

17. Prove that $\int_0^{\pi} \frac{xdx}{a^2\cos^2 x+b^2\sin^2 x}=\frac{\pi^2}{2ab}$

IMPORTANT POINTS

1. The vlaue of definite integral is unique.

2. (i) $\int_a^b k\,f(x)dx=k\int_a^b f(x)\,dx$ (ii) $\int_a^b [f(x)\pm\phi(x)]\,dx=\int_a^b f(x)dx\pm\int_a^b \phi(x)dx$

(iii) $\int_a^a f(x)dx=0$

3. (i) $\int_a^{\infty} f(x)dx=\lim_{b\to\infty}\int_a^b f(x)dx$ (ii) $\int_{-\infty}^b f(x)dx=\lim_{a\to\infty}\int_{-a}^b f(x)dx$

(iii) $\int_{-\infty}^{\infty} f(x)dx=\lim_{a\to\infty}\int_{-a}^a f(x)dx$

4. **Properties of definite integral:**

(i) $\int_a^b f(x)\,dx=\int_a^b f(t)\,dt$ (ii) $\int_a^b f(x)\,dx=-\int_b^a f(x)dx$

(iii) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)dx,$ where $a<c<b$

Generalisation: $a < c_1 < c_2 < c_3 < ... < c_n < b$

$$\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)\,dx + \int_{c_2}^{c_3} f(x)\,dx + ... + \int_{c_n}^{b} f(x)\,dx$$

(iv) $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx \;\therefore\; \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

(v) $\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx$ if $f(a+x) = f(x)$ [$f(x)$ is a periodic function of period a]

(vi) $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{If is an even function i.e. } f(-x) = f(x) \\ 0, & \text{If } f(x) \text{ is an odd function i.e. } f(-x) = -f(x) \end{cases}$

(vii) $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{If } f(2a-x) = f(x) \\ 0, & \text{If } f(2a-x) = -f(x) \end{cases}$

5. **Rule of eliminating *x*** If $f(a+b-x) = f(x)$ then

$$\int_a^b x\,f(x)dx = \frac{a+b}{2}\int_a^b f(x)dx$$

6. $$\int_0^{\pi/2} \log\sin x\,dx = -\frac{\pi}{2}\log 2 = \int_0^{\pi/2} \log\cos x\,dx$$

and $$\int_0^{\pi/2} \log\operatorname{cosec} x\,dx = \frac{\pi}{2}\log 2 = \int_0^{\pi/2} \log\sec x\,dx$$

7. **Definite Integral as a limit of sum :** If $f(x)$ is continuous function in given interval $[a, b]$ then divide interval $[a, b]$ in n equal parts having width h.
To evaluate definite integral from this is called "Integration from first principal".

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\left[f(a+h) + f(a+2h) + ... + f(a+nh)\right], \text{where } n\to\infty, nh = b-a$$

Answers

Exericse 10.1

1. 4
2. $\frac{1}{3}(b^3 - a^3)$
3. 86 / 3
4. $e^{-a} - e^{-b}$
5. 10
6. 82 / 3

Exercise 10.2

1. 290
2. $\pi/4$
3. $\sin(\log 3)$
4. $2(e-1)$
5. 2
6. $\frac{2}{3}\left(2-\sqrt{2}\right)c^{3/2}$
7. $e^{\pi/2} - 1$
8. $\frac{1}{3}(1+\log 2)^3 - \frac{1}{3}$

9. π	10. $\frac{1}{20}\log_e 3$	11. 0	12. $\frac{3\sqrt{2}-4}{10}$
13. $(e^2/2)-e$	14. $2/3$	15. $\log(e/2)$	16. $\frac{1}{2\sqrt{5}}\tan^{-1}\frac{2}{\sqrt{5}}$
17. $\pi/4$	18. $\frac{\pi-2}{2}$	19. 1	20. $\frac{\pi}{2(a+b)}$
21. $\log(4/e)$	22. $\frac{\pi}{2}-\frac{\pi}{4\sqrt{2}}-\frac{1}{\sqrt{2}}$	23. $\log(9/8)$	24. $3\pi/2$
25. $1-\pi/4$	26. $\log(9/8)$		

Exercise 10.3

1. 25 / 2	2. 4	3. 62	4. 3
5. 0	6. 0	7. $p/2$	8. $p/2$
9. 0	10. 0	11. 0	12. $p/12$
13. $p/4$	14. $\frac{\pi}{2}\log\frac{1}{2}$	15. $\frac{\pi^2}{6\sqrt{3}}$	16. $\pi\log\frac{1}{2}$
17. $\frac{\pi}{4}-\frac{1}{2}$	18. π	19. $\frac{2\pi}{3}$	20. $\pi\log 2$
21. 1		22. $\frac{b-a}{2}$	

Miscellaneous Exercise 10

1. (B)	2. (C)	3. (B)	4. (A)
5. $\frac{1}{2}\log 6$	6. $\frac{e}{6}(2e-3)$	7. $e^{\pi/2}$	8. 4
9. $\frac{\pi}{48}(\pi^2-6)$	10. $\frac{\pi}{4}-\frac{1}{2}\log 2$	11. $\frac{3\sqrt{2}}{10}$	12. 4
13. π^2	14. $\frac{\pi}{4}-\frac{1}{2}\log 2$	15. $\pi-2$	16. $\frac{\pi}{a^2-1}, a>1$